Technical University of Denmark

DTU

Approximation by planar elastic curves

Brander, David; Gravesen, Jens; Nørbjerg, Toke Bjerge

DTU Library
Technical Information Center of Denmark

CrossMark

Approximation by planar elastic curves

David Brander[1] (iD) · **Jens Gravesen[1]** ·
Toke Bjerge Nørbjerg[1]

Abstract We give an algorithm for approximating a given plane curve segment by a planar elastic curve. The method depends on an analytic representation of the space of elastic curve segments, together with a geometric method for obtaining a good initial guess for the approximating curve. A gradient-driven optimization is then used to find the approximating elastic curve.

1 Introduction

An *Euler elastica* or *elastic curve* is the solution to the variational problem of minimizing the bending energy, the integral of the curvature squared $\int \kappa(s)^2 \mathrm{d}s$, among curves of a given length with fixed endpoints and with the tangents prescribed at the endpoints. All solutions to this problem were described by Euler [4] in 1744, and the curves can be parameterized in terms of elliptic functions.

Communicated by: Helmut Pottmann.

✉ David Brander
dbra@dtu.dk

Jens Gravesen
jgra@dtu.dk

Toke Bjerge Nørbjerg
tokebn@gmail.com

[1] Department of Applied Mathematics and Computer Science, Technical University of Denmark,
Matematiktorvet, Building 303 B, DK-2800 Kgs. Lyngby, Denmark

 Springer

The energy minimizing property qualifies the elastica as a mathematical model for the shape assumed by a thin inextensible rod when constraints are placed only at the endpoints. This shape appears naturally in certain manufacturing scenarios: for example in a construction made from thin, flexible strips of wood or similar, the shape of each strip, between the fixed points, is an elastica. In another application, a thin metal blade can be heated and used to cut polystyrene for architectural formwork, the blade changing its shape during the motion. This permits the construction of curved geometries potentially much more cheaply than with the alternative of numerically controlled milling; this article is part of a larger project dedicated to the development of this so-called *robotic hot-blade cutting* technology [3, 15]. Here, too, the shape of the blade is an elastica.

The present standard representation for curves and surfaces in computer-aided design (CAD) systems is *rational* splines, that is, piecewise rational functions. In architecture there is typically a second step after the initial conceptual design, whereby the CAD model is adapted slightly for practical realization with respect to the chosen manufacturing process, and this is called *rationalization*. The problem of rationalizing a standard CAD design for a construction method involving elastic curves entails the approximation of a rational or polynomial spline curve by an elastica, and it is this problem that motivates us here. It is worth noting that the term "spline", in pre-CAD years, referred to a thin strip of wood used to draw curves that interpolate a given set of points smoothly. Between the interpolation points, these strips assumed the shapes of elastica, or rather *planar* elastica, because the strips were laid on a flat surface. Thus, for the type of rationalization referred to above, one could say that the task is, somewhat ironically, to approximate a digital spline with an analogue spline.

With this in mind, we consider here the problem of approximating an arbitrary curve by a planar elastica.

Precursors Quite distinctly from the applications mentioned above, elastic curves are a good candidate for the choice of functions used in geometric modeling because their curvature minimizing property makes them optimally faired. The idea of using them as mathematical splines – here meaning piecewise smooth functions – has been discussed by many authors: for example the references [1, 2, 5, 6, 11–13] constitute a representative, though not an exhaustive, list. These works are concerned either with the problem of how to compute the elastic curve satisfying various constraints respecting placement of endpoints, end-tangents, lengths etc., or with such problems as the existence of an elastic curve interpolating a given set of points. As a testament to the importance of these curves, one finds that many aspects of the theory are re-derived independently several times.

For our purpose, however, the main point is that previous work on elastica and their applications in spline theory does not consider the problem of approximating an arbitrary curve by an elastica.

Outline of this article We first make use of the analytic solutions, which were derived in 1880 by L. Saalschütz [14], to give a representation of the space of elastic curve segments. The space depends real analytically on seven control parameters, and
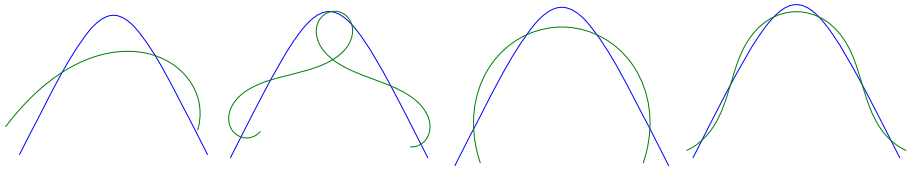
Fig. 1 The *blue curve* is to be approximated by an elastica segment. The *green curves* show the result of IPOPT optimization with different initial guesses. In the third case the optimization terminated before an extremum was reached

one can therefore, in principle, use gradient driven optimization software to achieve an approximation. The challenge is that the result of this non-convex nonlinear optimization depends very much on the initial guess, as illustrated in Fig. 1.

The optimization approach can only deliver a useable approximation algorithm if a means of choosing a good *initial* elastic curve is found. Our solution is to use the fact that the curvature function for an arc-length parameterized elastica is *affine* in a certain direction. We first show, in Section 3, how to use this fact to recover the seven control parameters – in a numerically stable manner – from a given elastic curve segment. We can then (Section 4) apply essentially the same procedure to an *arbitrary C^2* plane curve to obtain a canonical elastic curve segment that, with respect to the global geometric characteristics selected for in the procedure, is close to the given curve.

The canonical initial guess can then be taken as input for an optimization package to compute a good approximating elastic curve for a given curve, provided a good solution exists. We illustrate the results of the overall algorithm on a sample of Bézier curves – some of which are close to elastica and some of which are not – with endpoints free (Fig. 7) and fixed (Fig. 8).

Finally, in Section 5, we discuss applications of this method to the problem of approximating curves by *piecewise* elastic spline curves and ongoing work on applications in manufacturing.

2 Planar elastica

2.1 Euler-Lagrange equation

Here is a brief sketch of the equations defining planar elastica. More details, background and references can be found in [8]. Let $\gamma : [0, \ell] \to \mathbb{R}^2$ be a plane curve segment parameterized by arclength. Let $\theta(s)$ denote the *tangent angle*, defined by the equation $\dot{\gamma}(s) = (\cos\theta(s), \sin\theta(s))$. Then a curve segment of length ℓ starting at (x_0, y_0) and ending at (x_ℓ, y_ℓ) satisfies $x_\ell = x_0 + \int_0^\ell \cos\theta \, ds$ and $y_\ell = y_0 + \int_0^\ell \sin\theta \, ds$.

Let κ denote the curvature $\dot{\theta}(s)$. An *elastica* is a minimizer, among curves with the same endpoints and end tangents, of the *bending energy* $\frac{1}{2}\int_0^\ell \kappa(s)^2 ds$. Suppose γ is an elastica from (x_0, y_0) to (x_ℓ, y_ℓ) with angle function θ, and consider the perturbed curve γ_t with angle function $\theta_t(s) = \theta(s) + t\psi(s)$, where ψ is a differentiable

function with $\psi(0) = \psi(\ell) = 0$. Applying the method of Lagrange multipliers to the bending energy, we set:

$$\mathcal{E}(\boldsymbol{\gamma}) = \tfrac{1}{2} \int_0^\ell \left(\frac{\mathrm{d}\theta}{\mathrm{d}s} \right)^2 \mathrm{d}s + \lambda_1 \left(x_0 + \int_0^\ell \cos\theta \, \mathrm{d}s - x_\ell \right) + \lambda_2 \left(y_0 + \int_0^\ell \sin\theta \, \mathrm{d}s - y_\ell \right),$$

and we require that

$$0 = \frac{\mathrm{d}\mathcal{E}(\boldsymbol{\gamma}_t)}{\mathrm{d}t} \bigg|_{t=0} = - \int_0^\ell \psi \left(\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \lambda_1 \sin\theta - \lambda_2 \cos\theta \right) \mathrm{d}s.$$

Since ψ was arbitrary, it follows that θ satisfies the Euler-Lagrange equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + \lambda_1 \sin\theta - \lambda_2 \cos\theta = 0. \tag{1}$$

Setting $(\lambda_1, \lambda_2) = \lambda(\cos\phi, \sin\phi)$, with $\lambda \geq 0$, this becomes $\ddot{\theta} + \lambda \sin(\theta - \phi) = 0$. Note that $\lambda = 0$ if and only if κ is constant, i.e. the curve $\boldsymbol{\gamma}$ is either a straight line segment or a piece of a circle. If $\lambda \neq 0$, set $\tilde{\boldsymbol{\gamma}}(s) = \sqrt{\lambda} R_{-\phi} \boldsymbol{\gamma}(s/\sqrt{\lambda})$, where R_ϕ is the rotation by angle ϕ. Then $\tilde{\boldsymbol{\gamma}}$ is also an elastica with tangent angle $\tilde{\theta}(s) = \theta(s/\sqrt{\lambda}) - \phi$ satisfying the normalized pendulum equation $\tilde{\theta}'' = -\sin\tilde{\theta}$. In summary:

Theorem 1 *Up to a scaling and rotation of the ambient space, all arclength parameterized elastica $\boldsymbol{\gamma} : [0, 1] \to \mathbb{R}^2$, with non-constant curvature κ, can be expressed as: $\boldsymbol{\gamma}(s) = \boldsymbol{\gamma}(0) + \int_0^s (\cos\theta(t), \sin\theta(t)) \, \mathrm{d}t$, where*

$$\ddot{\theta} = -\sin\theta. \tag{2}$$

2.2 Parameterization of the space of elastica

We now want to define some suitable control parameters to describe an arbitrary elastic curve segment. Essentially, the parameters need to specify which solution to Eq. 2 is involved, the start and endpoints on the solution curve in question, and a rotation and scaling.

The elastic curves can be expressed in closed form via the elliptic functions (see Appendix A). The formulas can be found in Love [10]. There are two classes of elastica: curves with inflection points (i.e. points where $\dot{\theta} = 0$) and curves without inflections.

2.2.1 Basic elastica

The solution to Eq. 2 starting at $(0, 0)$ with initial angle $\theta(0) = 0$ and $\dot{\theta}(0) \geq 0$ is

$$\boldsymbol{\zeta}_k(s) = (2E(s, k) - s, \; 2k(1 - \mathrm{cn}(s, k))),$$

where $k = \dot{\theta}(0)/2$. For $k \in [0, 1)$ we get inflectional elastica; for $k \geq 1$, we use the *extended* elliptic functions defined in Appendix A to obtain elastic curves without inflections. We reserve the name $\boldsymbol{\zeta}_k$ for these basic elastica. Figure 2 shows elastic

$k=0$	$k=0.99$
$k=0.3$	$k=1$
$k=0.7$	$k=1.05$
$k=0.83$	$k=1.2$
$k=0.87$	$k=1.6$
$k=0.9089...$	$k=10$
$k=0.937$	$k\to\infty$
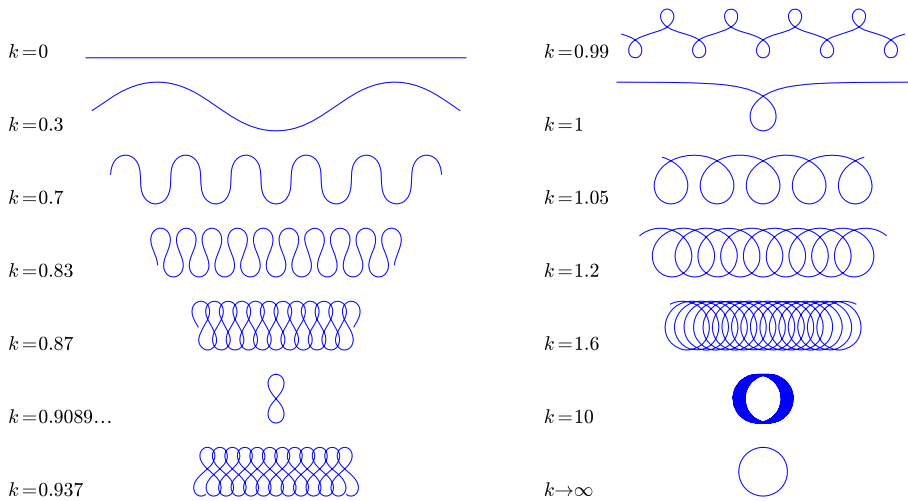
Fig. 2 Elastica for different values of k. The curves are scaled to a uniform "height"

curves for different values of k. All elastica are obtained by scaling and rotating these curves. The periodicity of the curves is given by:

$$\boldsymbol{\zeta}_k(s + 4K) = \boldsymbol{\zeta}_k(s) + (2E(4K, k) - 4K, \ 0),$$

where K is the quarter-period defined in Appendix A.

2.2.2 General elastica

To parameterize an arbitrary segment of an elastica, we can choose a segment of a basic elastica by choosing k, a starting point s_0 and an endpoint $s_0 + \ell$, where $\ell \in \mathbb{R} \setminus \{0\}$. It will be convenient to remove the dependence on s_0 and ℓ from the domain and include them in the parameterization. We parameterize the elastica segments on the unit interval, setting $s = s_0 + \ell t$, with $t \in [0, 1]$. The new curve parameter t is not unit speed. Finally, any elastica segment can be obtained by introducing a scaling factor $S > 0$, a rotation by an angle $\phi \in (-\pi, \pi]$ and translation by a vector (x_0, y_0). We thus have a standard elastic segment parameterization

$$\boldsymbol{\gamma}_{(k,s_0,\ell,S,\phi,x_0,y_0)}(t) = SR_\phi \boldsymbol{\zeta}_k(s_0 + \ell t) + (x_0, \ y_0), \quad t \in [0, 1].$$

It depends on seven control parameters, but we will usually omit the subscript. Such a curve has constant speed $|\ell|S$ and length $L = |\ell|S$.

Note 1 If $\ell < 0$, the orientation of the curve is changed. In the inflectional case, the elastica with opposite orientation can be obtained by a rotation by π. In this case we may therefore assume $\ell > 0$ without loss of generality. For elastica *without* inflections, however, a segment with $\ell < 0$ cannot be described as a segment with $\ell > 0$. One can instead reverse the direction of the parameterization for that case, and so all cases can be handled with the assumption $\ell > 0$.

For any elastic curve $\boldsymbol{\gamma}$ of the above type, the curve $\tilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}(\frac{t}{\ell S})$ is unit speed. Letting θ and $\tilde{\theta}$ denote the angle functions of $\boldsymbol{\zeta}_k$ and $\tilde{\boldsymbol{\gamma}}$, respectively, we have $\tilde{\theta}(t) = \theta(s_0 + \frac{t}{S}) + \phi$, and thus

$$\tilde{\theta}''(t) = \tfrac{1}{S^2}\ddot{\theta}(s_0 + \tfrac{t}{S}) = -\tfrac{1}{S^2}\sin\theta\left(s_0 + \tfrac{t}{S}\right) = -\tfrac{1}{S^2}\sin\left(\tilde{\theta}(t) - \phi\right),$$

so the angle function for $\tilde{\boldsymbol{\gamma}}$ satisfies (1) with

$$(\lambda_1, \lambda_2) = \tfrac{1}{S^2}(\cos\phi, \sin\phi). \tag{3}$$

We will also use the fact that the curvature for the elastica $\boldsymbol{\gamma}_{(k,s_0,\ell,S,\phi,x_0,y_0)}$ is

$$\kappa(t) = \tfrac{2k}{S}\,\mathrm{cn}(s_0 + \ell t). \tag{4}$$

3 Finding the control parameters of an elastic curve segment

We first describe a way to calculate numerically the control parameters of a given planar elastic curve segment. In the next section the same recipe will be applied to an arbitrary planar curve to obtain a canonical first guess for an approximating elastic curve. The main idea is to exploit the fact that the curvature of an elastica is an affine function of the distance along a special direction.

Let $\mathbf{x} : [a, b] \to \mathbb{R}^2$ be an elastic curve parameterized by arclength. As for any planar curve, we can write the tangent and the normal as $\mathbf{t} = (\cos\theta, \sin\theta)$, $\mathbf{n} = (-\sin\theta, \cos\theta)$, and we have the Frenet-Serret equations

$$\frac{d\mathbf{t}}{ds} = \frac{d\theta}{ds}\mathbf{n} = \kappa\,\mathbf{n}, \qquad \frac{d^2\mathbf{t}}{ds^2} = \frac{d^2\theta}{ds^2}\mathbf{n} - \kappa^2\,\mathbf{t}.$$

The tangent angle θ must satisfy the Euler-Lagrange Eq. 1 for some Lagrangian multipliers λ_1, λ_2 to be found.

Let u denote the projection of $\mathbf{x}$ onto the line spanned by $(\lambda_2, -\lambda_1)$, i.e.,

$$u = \frac{1}{\lambda}(\lambda_2, -\lambda_1) \cdot (x, y) = \frac{\lambda_2 x - \lambda_1 y}{\lambda},$$

where $\lambda = \|(\lambda_1, \lambda_2)\| = S^{-2}$. Setting $\phi = 0$ in Eq. 3, we find that the vector $(\lambda_2, -\lambda_1)$ points in the downward direction in Fig. 2. It follows that u is bounded and periodic in s. Moreover, we can write the Euler-Lagrange equation as $\ddot{\theta} = \lambda\,\dot{u}$, so we have

$$\kappa = \frac{d\theta}{ds} = \lambda u + \alpha = \lambda_2 x - \lambda_1 y + \alpha, \tag{5}$$

which is to say that the curvature is an affine function of u.

In order to find λ_1, λ_2 and α in a numerically stable manner, we solve the above equation in the least squares sense, i.e., we consider the quadratic minimization problem

$$\underset{\lambda_1,\lambda_2,\alpha}{\text{minimize}} \int_a^b (\kappa + \lambda_1\,y - \lambda_2\,x - \alpha)^2\,ds\,,$$

which leads to the following linear system

$$
\begin{pmatrix}
\int_a^b y^2\, ds & -\int_a^b x\, y\, ds & -\int_a^b y\, ds \\
-\int_a^b x\, y\, ds & \int_a^b x^2\, ds & \int_a^b x\, ds \\
-\int_a^b y\, ds & \int_a^b x\, ds & \int_a^b 1\, ds
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\alpha
\end{pmatrix}
=
\begin{pmatrix}
-\int_a^b y\,\kappa\, ds \\
\int_a^b x\kappa\, ds \\
\int_a^b \kappa\, ds
\end{pmatrix}.
\tag{6}
$$

Let θ_u denote the angle between the tangent vector $\mathbf{t}$ and the u-axis (see Fig. 3). We have

$$
\begin{aligned}
\cos\theta_u &= \tfrac{1}{\lambda}(\lambda_2,\ -\lambda_1)\cdot\mathbf{t} = \tfrac{1}{\lambda}(\lambda_2,\ -\lambda_1)\cdot\frac{d\mathbf{x}}{ds} = \frac{du}{ds}, \\
\sin\theta_u &= \tfrac{1}{\lambda}(\lambda_1,\ \lambda_2)\cdot\mathbf{t},
\end{aligned}
\tag{7}
$$

and hence

$$
\frac{d\sin\theta_u}{du} = \frac{ds}{du}\frac{d\sin\theta_u}{ds} = \frac{1}{\cos\theta_u}\cos\theta_u\frac{d\theta_u}{ds} = \kappa = \lambda\, u + \alpha,
$$

or equivalently

$$
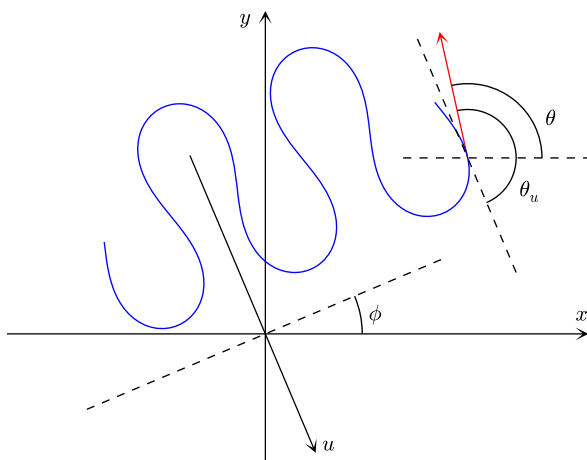P(u) := \sin\theta_u = \frac{1}{2}\lambda\, u^2 + \alpha\, u + \beta.
\tag{8}
$$

We solve this equation with respect to β in the least squares sense and obtain

$$
\beta = \frac{1}{L}\int_a^b \left(\sin\theta_u - \frac{1}{2}\lambda\, u^2 - \alpha\, u\right) ds,
\tag{9}
$$

where $L = b - a$ is the length of the curve $\mathbf{x}$.

For an elastica $\mathbf{x}(s) = SR_\phi\boldsymbol{\zeta}_k(s/S) + (x_0, y_0)$ we have $(\lambda_1, \lambda_2) = S^{-2}(\cos\phi, \sin\phi)$ and $\kappa(s) = (2k/S)\,\mathrm{cn}(s/S)$. Substituting these into the definitions $u = (\lambda_2 x - \lambda_1 y)/\lambda$, $\alpha = \kappa - \lambda u$ and $\sin\theta_u = (1/\lambda)(\lambda_1, \lambda_2)\cdot\mathbf{t}$ we have

$$
\begin{aligned}
u &= -2Sk(1 - \mathrm{cn}(s/S)) + x_0\sin\phi - y_0\cos\phi, \\
\sin\theta_u &= 2\,\mathrm{dn}^2(s/S) - 1, \\
\alpha &= 2kS^{-1}\,\mathrm{cn}(s/S) - \lambda u = 2k/S + (y_0\cos\phi - x_0\sin\phi)/S^2
\end{aligned}
\tag{10}
$$

Fig. 3 A segment of an elastica in the plane, showing ϕ, θ, the u-axis and the angle θ_u

Then the equation $\beta = \sin\theta_u - \lambda u^2/2 - \alpha u$ becomes:

$$\beta = 1 + \frac{x_0 \sin\phi - y_0 \cos\phi}{2S^2}(x_0 \sin\phi - y_0 \cos\phi - 4kS). \tag{11}$$

It follows from Eq. 8 that all points on the elastica correspond to u-values where the value of the polynomial P is between -1 and 1, and hence

$$u \in \left[\frac{-\alpha - \delta_-}{\lambda}, \frac{-\alpha + \delta_-}{\lambda}\right],$$

where $\delta_- = \sqrt{\alpha^2 - 2\lambda(\beta - 1)}$.

If the elastica has an inflection, there must be some u_*, such that $\kappa = \lambda u_* + \alpha = 0$, but this means that u_* is a minimizer for $P(u)$, and thus its minimum must lie in $[-1, 1]$. Moreover, the inflectional elastica has points where $\sin\theta_u = 1$ (which happens twice per period), but no points where $\sin\theta_u = -1$ (see Fig. 4). Hence u runs through all of the interval where $P(u)$ is less that 1; in other words, $u_{\min}$ and $u_{\max}$ are exactly the endpoints of the above interval.

For the elastica without inflections, the tangent makes full rotations, so $\sin\theta_u$ takes both of the values ± 1. Hence, in this case, we must have

$$[u_{\min}, u_{\max}] = \left[\frac{-\alpha - \delta_-}{\lambda}, \frac{-\alpha - \delta_+}{\lambda}\right] \quad \text{or} \quad [u_{\min}, u_{\max}] = \left[\frac{-\alpha + \delta_+}{\lambda}, \frac{-\alpha + \delta_-}{\lambda}\right],$$
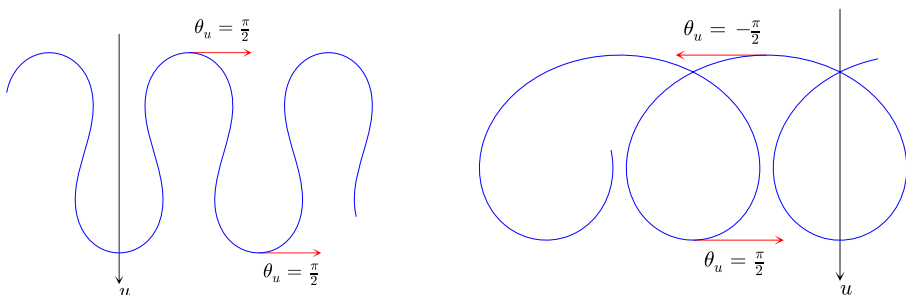
where $\delta_+ = \sqrt{\alpha^2 - 2\lambda(\beta + 1)}$; these are the two cases corresponding to $\ell < 0$ and $\ell > 0$, respectively. We can thus determine whether the elastica has inflection points based on whether the minimum for the polynomial $P(u)$ is smaller or greater than -1, see Fig. 5. In fact, from Eqs. 10 and 11 we have

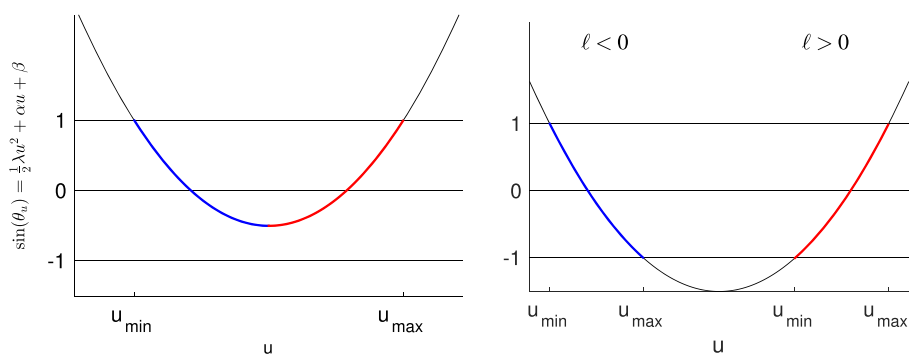$$\alpha^2 - 2\lambda(\beta - 1) = \frac{4k^2}{S^2},$$

or equivalently

$$k = \frac{\sqrt{\alpha^2 - 2\lambda(\beta - 1)}}{2\sqrt{\lambda}}, \tag{12}$$

so we can find S, ϕ and k from $\lambda_1, \lambda_2, \alpha$ and β.

Fig. 4 The inflectional elastica have points where $\sin\theta_u = 1$, but not -1. For the non-inflectional elastica $\sin\theta_u$ takes both the values ± 1

Fig. 5 The parabola (8). To the *left* in the case of an elastica with inflection points, to the *right* without. The *blue* and *red* part corresponds to points with negative and positive curvature, respectively

Note 2 The above formula also holds if $\ell < 0$ and so does the expression for β in control parameters. The expressions for κ, λ_1, λ_2 and α simply change sign in this case.

We still need to recover s_0 and ℓ. We have

$$u = -2kS\left(1 - \mathrm{cn}\left(s_0 + \tfrac{t}{S}\right)\right) + x_0 \sin\phi - y_0 \cos\phi,$$

and since

$$u_{\max} = \frac{-\alpha + \delta_-}{\lambda} = x_0 \sin(\phi) - y_0 \cos(\phi),$$

we get

$$\Delta(u) = u_{\max} - u = 2kS\left(1 - \mathrm{cn}\, s\right),$$

so

$$\mathrm{cn}(s, k) = 1 - \frac{\Delta(u)}{2kS}. \tag{13}$$

If we consider the unbounded complete elastica, then u oscillates between $u_{\min}$ and $u_{\max}$ and we can divide the elastica into segments where u is monotone, each with length equal to a half period $2KS$.

We first consider the case of an elastica with inflection points (i.e. $k < 1$). Here we have $\mathrm{cn}(s, k) = \cos(\mathrm{am}(s, k))$. If the start point $\mathbf{x}_0 = \mathbf{x}(a)$ is on segment number 1 and u is decreasing here, then

$$\mathrm{am}(s_0, k) = \arccos\left(1 - \frac{\Delta(u_0)}{2kS}\right),$$

and if the end point $\mathbf{x}_1 = \mathbf{x}(b)$ is on segment number n, then

$$\mathrm{am}(s_1, k) = \begin{cases} (n-1)\pi + \arccos\left(1 - \frac{\Delta(u_1)}{2kS}\right), & \text{if } n \text{ is odd,} \\ n\pi - \arccos\left(1 - \frac{\Delta(u_1)}{2kS}\right), & \text{if } n \text{ is even.} \end{cases}$$

If u is increasing on segment number 1, then

$$\mathrm{am}(s_0, k) = 2\pi - \arccos\left(1 - \frac{\Delta(u_0)}{2kS}\right),$$

and

$$\mathrm{am}(s_1, k) = \begin{cases} (n+1)\pi - \arccos\left(1 - \frac{\Delta(u_1)}{2kS}\right), & \text{if } n \text{ is odd,} \\ n\pi + \arccos\left(1 - \frac{\Delta(u_1)}{2kS}\right), & \text{if } n \text{ is even.} \end{cases}$$

In all cases we have

$$s_i = F(\mathrm{am}(s_i, k), k), \qquad i = 0, 1,$$

and $\ell = s_1 - s_0$.

In the case of an elastica without inflections points (i.e. $k \geq 1$) we need a little work to find am. We have

$$\mathrm{sn}(s, k) = \frac{1}{k}\mathrm{sn}\left(ks, \tfrac{1}{k}\right) = \frac{1}{k}\sin\left(\mathrm{am}\left(ks, \tfrac{1}{k}\right)\right)$$

and

$$\mathrm{sn}(s, k) = \sqrt{1 - \mathrm{cn}^2(s, k)} = \sqrt{\frac{\Delta(u)}{kS}\left(1 - \frac{\Delta(u)}{4kS}\right)}.$$

If u is decreasing on segment 1 then

$$\mathrm{am}\left(ks_0, \tfrac{1}{k}\right) = \arcsin\sqrt{\frac{\Delta(u_0)}{S}\left(k - \frac{\Delta(u_0)}{4S}\right)},$$

and if we have n segments

$$\mathrm{am}\left(\frac{s_1}{k}, k\right) = \begin{cases} \frac{n-1}{2}\pi + \arcsin\sqrt{\frac{\Delta(u_1)}{S}\left(k - \frac{\Delta(u_1)}{4S}\right)}, & \text{if } n \text{ is odd,} \\ \frac{n}{2}\pi - \arcsin\sqrt{\frac{\Delta(u_1)}{S}\left(k - \frac{\Delta(u_1)}{4S}\right)}, & \text{if } n \text{ is even.} \end{cases}$$

If u is increasing on segment 1 then

$$\mathrm{am}\left(ks_0, \frac{1}{k}\right) = \pi - \arcsin\sqrt{\frac{\Delta(u_0)}{S}\left(k - \frac{\Delta(u_0)}{4S}\right)},$$

and if we have n segments

$$\mathrm{am}\left(ks_1, \frac{1}{k}\right) = \begin{cases} \frac{n+1}{2}\pi - \arcsin\sqrt{\frac{\Delta(u_1)}{S}\left(k - \frac{\Delta(u_1)}{4S}\right)}, & \text{if } n \text{ is odd,} \\ \frac{n}{2}\pi + \arcsin\sqrt{\frac{\Delta(u_1)}{S}\left(k - \frac{\Delta(u_1)}{4S}\right)}, & \text{if } n \text{ is even.} \end{cases}$$

Finally, we find the s-values using the incomplete elliptic integral

$$s_i = \tfrac{1}{k}F\left(\mathrm{am}\left(ks_i, \tfrac{1}{k}\right), \tfrac{1}{k}\right), \qquad i = 0, 1,$$

and $\ell = s_1 - s_0$.

Note 3 If we have a negatively curved noninflectional elastica (i.e. $\ell < 0$), we can reverse the parameterization, find the elastica, and interchange (s_0, s_1).

We now have a scaled and rotated elastica segment, $\boldsymbol{\gamma}_0 = \boldsymbol{\gamma}_{(k,s_0,\ell,S,\phi,0,0)}$, and all that is left is to find the final translation (x_0, y_0). This is done by solving the equation

$$\mathbf{x}(s) = \boldsymbol{\gamma}_0(s) + (x_0, y_0),$$

in the least squares sense. The solution is

$$(x_0, \ y_0) = \frac{1}{L} \int_a^b (\mathbf{x}(s) - \boldsymbol{\gamma}_0(s)) \, \mathrm{d}s. \tag{14}$$

4 Approximating a plane curve by a planar elastica

We are now given a curve $\mathbf{x} \colon [0, 1] \to \mathbb{R}^2$, not necessarily elastic and not necessarily parameterized by arclength. The arclength is given by

$$s(t) = \int_0^t \|\mathbf{x}'(\tau)\| \mathrm{d}\tau \,,$$

and the length of the curve is $L = s(1)$. We want to approximate this curve by a piece of an elastica. We do this by minimizing a suitable distance, such as the L^2, H^1, or H^2 distance, over the control parameters $\mathbf{p} = (k, s_0, \ell, S, \phi, x_0, y_0)$. In the case of L^2 the problem is

$$\underset{k, s_0, \ell, S, \phi, x_0, y_0}{\text{minimize}} \ \mathcal{F}(k, s_0, \ell, S, \phi, x_0, y_0),$$

where

$$\mathcal{F}(\mathbf{p}) = \frac{1}{2} \int_0^1 \left\| \boldsymbol{\gamma}_{\mathbf{p}} \left(\frac{s(t)}{L} \right) - \mathbf{x}(t) \right\|^2 \|\mathbf{x}'(t)\| \, \mathrm{d}t \,.$$

If we want the elastic curve to satisfy further conditions, such as having the same endpoints and/or end tangents as the original curve, we can include these in the optimization problem.

For the optimization, we have used the gradient driven tool IPOPT [16], so we need the first and second order partial derivatives of $\mathcal{F}$ with respect to the control parameters, which are

$$\frac{\partial \mathcal{F}}{\partial p_i} = \int_0^1 \left(\boldsymbol{\gamma}_{\mathbf{p}} \left(\frac{s(t)}{L} \right) - \mathbf{x}(t) \right) \cdot \frac{\partial \boldsymbol{\gamma}_{\mathbf{p}}}{\partial p_i} \left(\frac{s(t)}{L} \right) \|\mathbf{x}'(t)\| \mathrm{d}t \,,$$

$$\frac{\partial^2 \mathcal{F}}{\partial p_i \partial p_j} = \int_0^1 \left(\frac{\partial \boldsymbol{\gamma}_{\mathbf{p}}}{\partial p_i} \left(\frac{s(t)}{L} \right) \cdot \frac{\partial \boldsymbol{\gamma}_{\mathbf{p}}}{\partial p_j} \left(\frac{s(t)}{L} \right) \right.$$
$$\left. + \left(\boldsymbol{\gamma}_{\mathbf{p}} \left(\frac{s(t)}{L} \right) - \mathbf{x}(t) \right) \cdot \frac{\partial^2 \boldsymbol{\gamma}_{\mathbf{p}}}{\partial p_i \partial p_j} \left(\frac{s(t)}{L} \right) \right) \|\mathbf{x}'(t)\| \mathrm{d}t \,.$$

See Appendix B for a list of specific derivatives.

The optimization problem is non convex and the result depends on the initial guess (see Fig. 1). A canonical geometrically plausible guess is obtained from a generalization of the procedure of Section 3 to the case of an arbitrary input curve, which we will now describe.

We find λ_1, λ_2, α as before, by solving (6), noting that $\int_{s(0)}^{s(1)} f \, \mathrm{d}s = \int_0^1 f(t) \frac{\mathrm{d}s}{\mathrm{d}t} \, \mathrm{d}t$. This gives us the scaling and rotation of the elastica. We can judge the success by calculating the normalized residual

$$R_1 = \sqrt{\int_0^1 (\kappa(t) + \lambda_1 \, y(t) - \lambda_2 \, x(t) - \alpha)^2 \, \frac{\mathrm{d}s}{\mathrm{d}t} \, \mathrm{d}t} \Big/ \sqrt{\int_0^1 \kappa^2(t) \, \frac{\mathrm{d}s}{\mathrm{d}t} \, \mathrm{d}t} \,.$$
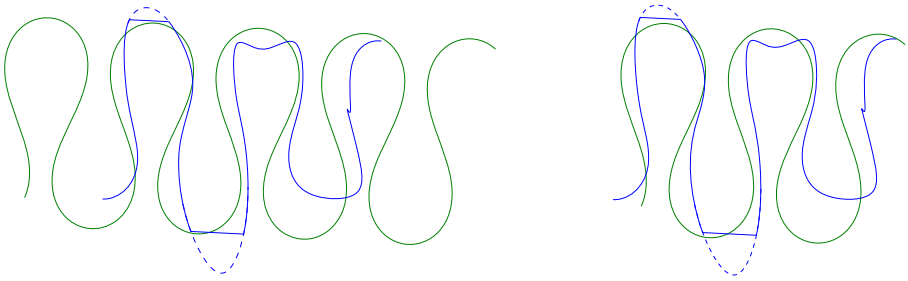
Fig. 6 If the given curve moves outside the interval $[u_{\min}, u_{\max}]$ (*dotted segment*), it is simply cut off in these regions. The resulting elastica is shown in *green*. On the left all oscillations of the input curve are counted, on the *right* the two very small ones are ignored

Similarly β can be found by Eq. 9, where $\sin \theta_u$ is given by Eq. 7, and we can calculate the normalized residual

$$R_2 = \sqrt{\frac{1}{L} \int_0^1 \left(\sin \theta_u(t) - \tfrac{1}{2} \lambda\, u^2(t) - \alpha\, u(t) - \beta \right)^2 \frac{ds}{dt}\, dt}\;.$$
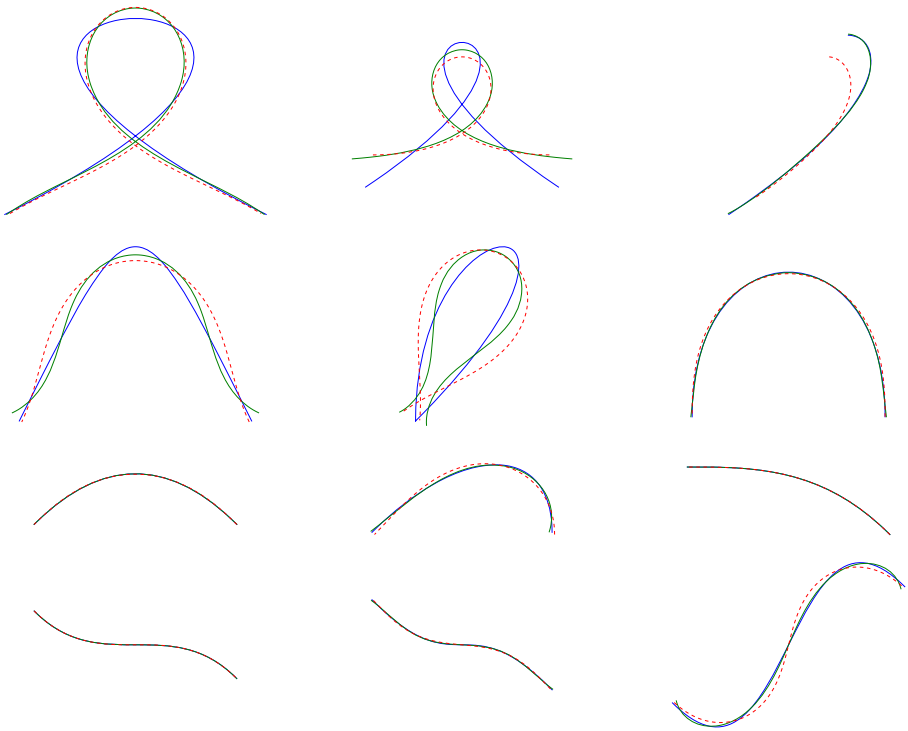
Fig. 7 Examples of cubic Bézier curves approximated with elastica. The *solid blue line* is the Bézier curve and the *dashed red line* is the initial guess for an approximating elastica. The *solid green curve* is the best approximating elastica found with IPOPT optimization

Note 4 Another possibility is to forget that we know λ and α and solve (8) with respect to λ, α, and β, but in the few cases we tried this, the results got worse.

We know that $\sin \theta_u$ takes values in $[-1, 1]$, and since β is chosen to minimize the distance between $\sin \theta_u$ and the polynomial $P(u) = \frac{1}{2}\lambda u^2 + \alpha u + \beta$, the latter must be less than 1 for some u-values, so the number $\delta_- = \sqrt{\alpha^2 - 2\lambda(\beta - 1)}$ is well-defined. We can thus determine whether the elastica has inflection points and we can determine the parameter k from Eq. 12.

At this point we need to take into account the fact that the input curve is not necessarily an elastica. For an elastica, we could easily count the oscillations, but for an arbitrary curve there may be oscillations of different sizes. We find the curve segments where u is monotone, but we only count such a segment as an oscillation if it has some minimal height: we have used half of the difference $u_{\max} - u_{\min}$ (as defined in Section 3) as this minimum. Moreover, the right hand side of Eq. 13 need not be between -1 and 1, or, in the noninflectional case, between $\sqrt{1 - 1/k^2}$ and 1. We have circumvented this problem by replacing too small values by -1 (or $\sqrt{1 - 1/k^2}$) and too large values by 1. The two issues are illustrated in Fig. 6.

We can thus find s_0 and ℓ. We can judge the validity by calculating

$$R_3 = \frac{1}{L} \int_{u(t)\notin[u_{\min}, u_{\max}]} \frac{ds}{dt} \, dt \, .$$
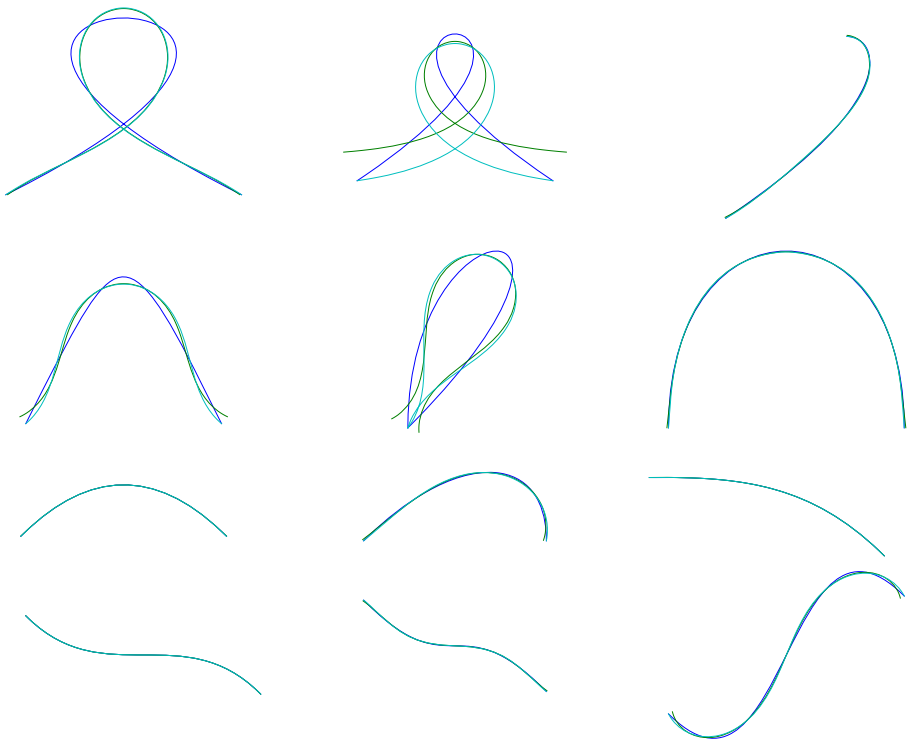
Fig. 8 The *solid* cyan curve is the best approximating elastica having the same endpoints as the original curve (*blue*). The *green curve* is the output curve with free endpoints, as in Fig. 7

Table 1 The first column refers to the examples in Fig. 7, the next three report the residuals R_1, R_2, and R_3, in the approximation process

	R_1	R_2	R_3	$R_4(\mathbf{p}_0)$	$R_4(\mathbf{p}_{\mathrm{opt}})$	$\|\nabla\mathcal{F}(\mathbf{p}_{\mathrm{opt}})\|$	♯ iter	$R_4(\mathbf{p}_{\mathrm{opt}}^*)$	♯ iter*
1	0.46	0.14	0.0	0.0097	0.0080	2.2e-08	35	0.0081	10
2	0.45	0.23	0.52	0.048	0.038	5.1	1000[a]	0.063	1000[a]
3	0.14	0.020	0.19	0.077	0.0018	6.0e-09	26	0.0023	16
4	0.68	0.17	0.14	0.022	0.012	5.9e-09	8	0.014	9
5	0.65	0.27	0.14	0.031	0.018	5.2e-09	9	0.021	8
6	0.099	0.025	0.048	0.0036	0.0011	8.1e-11	24	0.0015	14
7	0.064	0.0044	0.048	0.0010	0.00032	1.7e-10	20	0.00041	104
8	0.27	0.069	0.14	0.011	0.0032	1.5e-09	199	0.0046	100
9	0.017	0.00033	0.0	0.00013	0.0012	0.053	1000	5.1e-05	15
10	0.020	0.00062	0.0	0.00012	9.9e-05	1.2e-09	165	0.00011	29
11	0.33	0.014	0.0	0.0050	0.0017	8.3e-11	76	0.0020	11
12	0.36	0.10	0.19	0.015	0.0041	7.3e-09	178	0.0053	83

Next, we have the normalized L^2-distance, $R_4 = \sqrt{2\mathcal{F}/L^3}$, for the initial guess and then R_4, the gradient norm $\|\nabla\mathcal{F}\|$ and the number of iterations for the optimized elastica without and with endpoint constraints. We use the same initial guess, whether we constrain the endpoints or not

[a] IPOPT terminated because iteration count reached maximum (which was set to 1000)

We finally determine the translation by Eq. 14 and define the residual as

$$R_4(\mathbf{p}_0) = \sqrt{\frac{2}{L^3}\mathcal{F}(\mathbf{p}_0)}\,,$$

where $\mathbf{p}_0$ is the vector of control parameters found by the above procedure.

We have tested the procedure on a selection of cubic Bézier curves, displayed in Figs. 7 and 8. In Table 1 we have reported the residuals.

5 Conclusions, discussion and related work

When combined with a suitable segmentation, we have found that the method described here gives an effective algorithm for approximating curves by piecewise elastic curves. The degrees of freedom allow piecewise elastica, with C^1 continuity at the joins if end-points are fixed, and with C^2 continuity if end-points are allowed to move. The method is incorporated in on-going work on approximating *surfaces* by segmented surfaces, with the segments swept out by elastic curves (Fig. 9), for the purpose of manufacturing architectural formwork by robotic hot-blade cutting [3, 15].

Our choice of parameters allows us to work with analytic expressions of elastic curves. The advantage is that when the seven parameters are known any subsequent calculation is accurate and easy to perform. The disadvantage is that the geometric meaning is not obvious for all seven parameters. A more geometric set of parameters
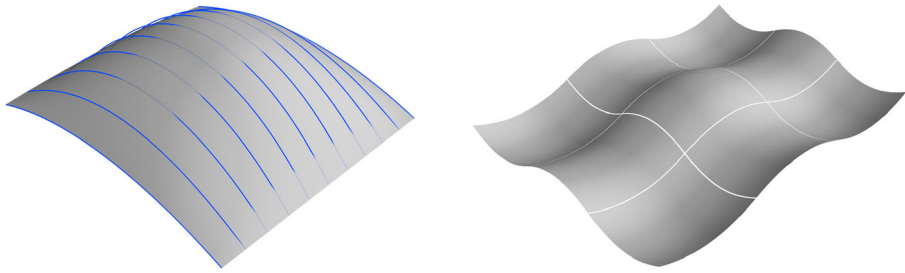
Fig. 9 Surface rationalization for hot-blade cutting. *Left* a CAD surface patch is approximated by a family of elastic curves. *Right* a surface segmented into elastica-foliated pieces

is the length and the end points and tangents; the problem with that alternative is that we would have to solve a nonlinear boundary problem to determine the curve and, more severely, the solution is not unique. For example, if we imagine that we rotate one of the tangents through 2π and follow a continuous family of elastic curves then the curve at the end will be different from the curve at the start.

Note that all the examples in Section 4 have, for convenience, been computed using the L^2 distance. Given that our initial guess method is based on the curvature, an H^2 norm may be a more natural choice, but in practice more complicated.

Another approach to the problem is to minimize $\int \|\gamma - x\| \, ds + \beta \int \kappa^2 \, ds$. Here it is not clear how to choose β. If $\beta = 0$ then we obtain $x = \gamma$ and in the limit where $\beta \to \infty$ we obtain the best line segment approximation to γ. In order to make a purely numerical approach work we need to have a good approximation to an elastic curve at all times while we minimize the distance to the target curve γ. This means we will have to solve a nonlinear equation at each step of the optimization and also find the sensitivities of this solution.

It should be pointed out that the method for obtaining the initial guess (Section 3) is only useful for a curve segment that is not too far from some elastic curve segment. Therefore, the practical use of this algorithm requires that a curve first be segmented into suitable pieces. There are several ways to approach this: for example (a) apply the initial guess method to the curve, (b) measure the distance between the resulting elastic segment and the original curve, (c) if the distance is too large, divide the curve into two and repeat. An example with various segmentations is given in Fig. 10.
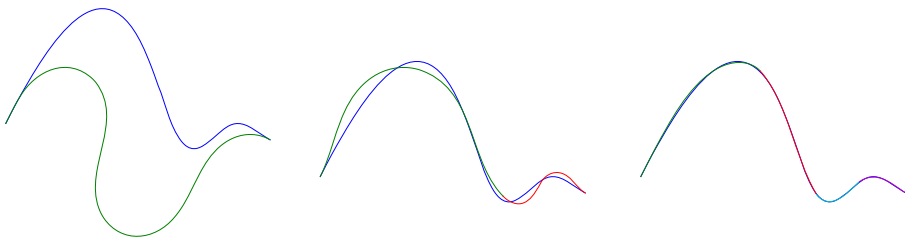
Fig. 10 Approximations of a more complex curve (*blue*) by, in order, one, two and four tangent continuous elastica segments. Both the endpoints and endtangents of the target curve are matched

In physical applications such as hot-blade cutting, it will also be necessary to consider the *stability* of the elastic curve segment obtained from this method; that is, small perturbations of the length, endpoints and tangents of the curve should correspond to small changes in the solution shape. See, for example, [9] for a study of stability. Constraints such as demanding that the curve segments have no inflection points, or an upper bound on the curvature can be added to the procedure to ensure stability.

Appendix A: Elliptic functions

We list some details of the Jacobi elliptic functions for convenience and to fix conventions. Let $k \in (0, 1)$. The *elliptic functions* sn, cn and dn with *(elliptic) modulus* k are defined as the solutions to the system of differential equations:

$$
\begin{aligned}
\text{sn}'(u) &= \text{cn}(u)\,\text{dn}(u),\ \text{sn}(0) = 0, \\
\text{cn}'(u) &= -\,\text{sn}(u)\,\text{dn}(u),\ \text{cn}(0) = 1, \\
\text{dn}'(u) &= -k^2\,\text{sn}(u)\,\text{cn}(u),\ \text{dn}(0) = 1.
\end{aligned}
$$

The *complementary modulus* $k' \in [0, 1]$ is defined by $k^2 + k'^2 = 1$. We have the identities:

$$
\text{sn}^2\,u + \text{cn}^2\,u = 1, \qquad \text{dn}^2\,u + k^2\,\text{sn}^2\,u = 1, \qquad \text{dn}^2\,u - k^2\,\text{cn}^2\,u = k'^2. \quad (15)
$$

The elliptic functions can be expressed in terms of integrals of trigonometric functions as follows. Define the *(elliptic) amplitude* am as:

$$
\text{am}(t) = F^{-1}(t), \qquad F(\phi) = \int_0^\phi \frac{1}{\sqrt{1 - k^2\sin^2 u}}\,\text{d}u.
$$

Then

$$
\text{sn}(u) = \sin(\text{am}\,u), \quad \text{cn}(u) = \cos(\text{am}\,u), \quad \text{dn}(u) = \sqrt{1 - k^2\sin^2(\text{am}\,u)}.
$$

Elliptic integrals

The integral $F(\phi, k)$ given, for each k, by the formula $F(\phi)$ above, is called *the incomplete elliptic integral of the first kind*. We define *the incomplete elliptic integral of the second kind* by

$$
E(\phi, k) = \int_0^\phi \text{dn}^2(u, k)\,\text{d}u.
$$

Addition formulas and periodicity

The elliptic functions satisfy the addition formulae:

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

$$\operatorname{cn}(u+v) = \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

$$\operatorname{dn}(u+v) = \frac{\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$$

We define the *quarter period* K by:

$$\operatorname{am}(K) = \frac{\pi}{2}, \quad \text{i.e.} \quad K = F\left(\frac{\pi}{2}\right),$$

so that $\operatorname{sn} K = 1$, $\operatorname{cn} K = 0$ and $\operatorname{dn} K = k'$. Then one obtains the periodicity:

$$\operatorname{sn}(u+2K) = -\operatorname{sn} u, \qquad \operatorname{cn}(u+2K) = -\operatorname{cn} u, \qquad \operatorname{dn}(u+2K) = \operatorname{dn} u.$$

Extension of k-domain

The elliptic functions, as we have defined them, are only valid for $k \in [0, 1]$. However, by analytic continuation (see e.g. Lawden [7]), the domain of k may be extended. For $k > 1$, the following identities hold for all $u \in \mathbb{R}$:

$$\operatorname{sn}(u, k) = \tfrac{1}{k} \operatorname{sn}(ku, \tfrac{1}{k}),$$

$$\operatorname{cn}(u, k) = \operatorname{dn}(ku, \tfrac{1}{k}),$$

$$\operatorname{dn}(u, k) = \operatorname{cn}(ku, \tfrac{1}{k}),$$

$$E(u, k) = kE(ku, \tfrac{1}{k}) + u(1 - k^2).$$

Observe that $K(k) \to \infty$ as $k \to 1$, so we cannot extend K continuously. We choose the extension

$$K(k) = \tfrac{1}{2k} K\left(\tfrac{1}{k}\right), \quad k > 1,$$

which ensures that the period of cn is always $4K$. We stress that this is *not* the analytic continuation of K, which in fact takes non-real values for $k > 1$.

Appendix B: Derivatives

In this section, we list the derivatives of the basic elastica ζ, using the shorthand notation

$$S = \operatorname{sn}(s, k), \ C = \operatorname{cn}(s, k), \ D = \operatorname{dn}(s, k), \ E = E(s, k).$$

We have

$$\zeta(s, k) = \begin{pmatrix} 2E - s \\ 2k(1 - C) \end{pmatrix}.$$

The derivatives with respect to s follow from the definitions of the elliptic functions.

$$\frac{\partial}{\partial s}\boldsymbol{\zeta}(s,k) = \begin{pmatrix} 2D^2 - 1 \\ 2kSD \end{pmatrix},$$

$$\frac{\partial^2}{\partial s^2}\boldsymbol{\zeta}(s,k) = 2kC\begin{pmatrix} -2kSD \\ D^2 - k^2S^2 \end{pmatrix} = 2kC\begin{pmatrix} -2kSD \\ 2D^2 - 1 \end{pmatrix}.$$

The derivatives with respect to k of sn, cn, dn and E can be found in [7]. From these, with repeated use of Eq. 15, one can find

$$\frac{\partial}{\partial k}\boldsymbol{\zeta}(s,k) = \frac{2}{k'^2}\begin{pmatrix} k\left(SCD - EC^2 - sk'^2S^2\right) \\ k'^2 + C(k^2 - D^2) - SD(E - sk'^2) \end{pmatrix},$$

$$\frac{\partial^2}{\partial s\,\partial k}\boldsymbol{\zeta}(s,k) = \frac{2}{k'^2}\left(SD - C(E - sk'^2)\right)\begin{pmatrix} -2kSD \\ 2D^2 - 1 \end{pmatrix},$$

$$\frac{k'^4}{2}\frac{\partial^2}{\partial k^2}\boldsymbol{\zeta}(s,k) = \begin{pmatrix} 2SDC\left(D^2 - k^2E^2 + k'^2\left(s^2k^2 - (E-s)^2 - \frac{1}{2}\right)\right) \\ \frac{1}{k}\left((1 - 2k^2S^2)(E-s)(2sk^2 + E - s)C + DS(sk'^2 - E)(4k^2C^2 + k'^2)\right) \end{pmatrix}$$
$$+ \begin{pmatrix} (E-s)(C^2 + D^2 - 4C^2D^2) + 2sk^2(2S^2 - 1)D^2 - sk'^2 \\ -skk'^2DS + s^2k^3C + kCS^2(2 - 2s^2k^4 - 2k^2S^2 + k^2) \end{pmatrix}$$

The derivatives of $\boldsymbol{\gamma}_{(k,s_0,\ell,S,\phi,x_0,y_0)}$ with respect to the control parameters can be found by straightforward calculations using the above.

References

1. Birkhoff, G., Boor, C.D.: Piecewise polynomial interpolation and approximation. In: Garabedian, H.L. (ed.) Approximation of functions, Proceedings general motors symposium 1964, pp. 164–190. Elsevier Publ. Co., Amsterdam (1965)
2. Borbély, A., Johnson, M.: Elastic splines I: existence. Constr Approx **40**, 189–218 (2014)
3. Brander, D., Bærentzen, A., Evgrafov, A., Gravesen, J., Markvorsen, S., Nørbjerg, T., Nørtoft, P., Steenstrup, K.: Hot blade cuttings for the building industry. Preprint
4. Euler, L.: Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes; Additamentum I: de curvis elasticis (1744). Translation: [Oldfather, W.A., Ellis, C.A., Brown, D.M.: Leonhard Euler's elastic curves. Isis 20(1), pp. 72–160 (1933). http://www.jstor.org/stable/224885]
5. Golomb, M., Jerome, J.: Equilibria of the curvature functional and manifolds of nonlinear interpolating spline curves. SIAM J. Math. Anal. **13**, 421–458 (1982)
6. Horn, B.: The curve of least energy. ACM Trans. Math. Software **9**, 441–460 (1983)
7. Lawden, D.: Elliptic functions and applications applied mathematical sciences, vol. 80. Springer, New York (1989)
8. Levien, R.: From spiral to spline; optimal techniques for interactive curve design. Ph.D. thesis, UC Berkeley (2009)
9. Levyakov, S.V., Kuznetsov, V.V.: Stability analysis of planar equilibrium configurations of elastic rods subjected to end loads. Acta Mech. **211**, 73–87 (2010)
10. Love, A.: A treatise on the mathematical theory of elasticity. Cambridge University Press (1906)
11. Malcolm, M.: On the computation of nonlinear spline functions. SIAM J. Numer. Anal. **14**, 254–282 (1977)
12. Mehlum, E.: Nonlinear splines. In: Computer aided geometric design (Proc. Conf., Univ. Utah, Salt Lake City, Utah, 1974), 173–207 (1974)
13. Mumford, D.: Elastica and computer vision. In: Algebraic Geometry and Its Applications (West Lafayette, IN, 1990), pp. 491–506. Springer, New York (1994)

14. Saalschütz, L.: Der Belastete Stab Unter Einwirkung Einer Seitlichen Kraft. B.G. Teubner, Leipzig (1880)
15. Søndergaard, A., Feringa, J., Nøbjerg, T., Steenstrup, K., Brander, D., Gravesen, J., Markvorsen, S., Bærentzen, A., Petkov, K., Hattel, J., Clausen, K., Jensen, K., Knudsen, L., Kortbek, J.: Robotic hot-blade cutting. In: Robotic fabrication in architecture, art and design 2016, pp. 150–164. Springer International Publishing
16. Wächter, A., Biegler, L.T.: On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. Math. Program., Ser. A **106**, 25–57 (2006)