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On some Hermite series identities and their applications to Gabor analysis

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Abstract: We prove some infinite series identities for the Hermite functions. From these identities we disprove the Gabor frame set conjecture for Hermite functions of order $4m+2$ and $4m+3$ for $m \in \{0\} \cup \mathbb{N}$. The results hold not only for Hermite functions, but for two large classes of eigenfunctions of the Fourier transform associated with the eigenvalues -1 and i, and the results indicate that the Gabor frame set of all such functions must have a rather complicated structure.

1 Introduction

Since John von Neumann's claim of completeness of the coherent state subsystems generated by the Gaussian in his work on quantum mechanics [16], it has been of interest in mathematical physics and analysis to determine when the set of coherent states $\mathcal{G}(g,a,b) := \left\{ \mathrm{e}^{2\pi i b m \cdot}\, g(\cdot - ak) \right\}_{k,m\in\mathbb{Z}}$ is complete in various function spaces, e.g., $L^2(\mathbb{R})$. In engineering, $\mathcal{G}(g,a,b)$ is the so-called Gabor system generated by the window function $g \in L^2(\mathbb{R})$ with time-frequency shifts along the lattice $a\mathbb{Z} \times b\mathbb{Z}$ in phase space. For most applications in signal processing and functional analysis, completeness of $\mathcal{G}(g,a,b)$ is nowadays not considered to be sufficient; for instance, to guarantee unconditionally L^2-convergent and stable expansions of functions in $L^2(\mathbb{R})$ and to provide characterizations of classical function spaces, one needs a stronger property of $\mathcal{G}(g,a,b)$, namely that the Gabor system constitutes a frame for $L^2(\mathbb{R})$, i.e, existence of constants $A, B > 0$, termed frame bounds, such that

$$A \left\| f \right\|^2 \le \sum_{k,m\in\mathbb{Z}} \left| \left\langle f, \mathrm{e}^{2\pi i b m \cdot}\, g(\cdot - ak) \right\rangle \right|^2 \le B \left\| f \right\|^2 \quad \text{for all } f \in L^2(\mathbb{R}). \tag{1.1}$$

In this work we are interested in the frame properties of Gabor systems generated by Hermite functions. We define the nth Hermite function h_n by

$$h_n(x) = (c_n)^{-1/2}\, \mathrm{e}^{\pi x^2} \left(\frac{d^n}{dx^n}\, \mathrm{e}^{-2\pi x^2} \right),$$

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where $c_n = (2\pi)^n 2^{n-1/2} n!$ for $n \in \mathbb{N} \cup \{0\}$. The class of Hermite functions forms a natural continuation of the study of von Neumann [16] and Gabor [2] as it contains the Gaussian as a special case, $n = 0$. The *frame set* of a window function $g \in L^2(\mathbb{R})$, denoted by $\mathscr{F}(g)$, is the parameter values $(a, b) \in \mathbb{R}^2_+$ for which the associated Gabor system $\mathcal{G}(g, a, b)$ is a frame for $L^2(\mathbb{R})$. Hence, we will study the set $\mathscr{F}(h_n)$, or to be more precise, properties of its compliment. That is, following [5], we will ask what prevents $\mathcal{G}(g, a, b)$ from generating a frame? Our answers will show that the Gabor frame set of Hermite functions must have a rather complicated structure. Indeed, we will derive new obstructions of the frame property for two classes of eigenfunctions of the Fourier transform associated with the eigenvalue -1 and i, respectively, which, in particular, disproves a conjecture on Hermite functions by Gröchenig [5].

To understand Gröchenig's conjecture, let us recall what is known about $\mathscr{F}(h_n)$. Since Hermite functions have exponential decay in time and frequency domain, it is known, see e.g., [5], that the upper frame bound holds, that the set $\mathscr{F}(h_n)$ is open in $\mathbb{R}^2$ and that $\mathscr{F}(h_n) \subset \left\{(a, b) \in \mathbb{R}^2_+ : ab < 1\right\}$. For the Gaussian h_0, the necessary condition $ab < 1$ for the frame property is also sufficient. This important result was conjectured by Daubechies and Grossmann [1] and proved by Lyubarskii [13] and by Seip and Wallstén [14, 15]. The proof relies on analytic properties of the short-time Fourier transform of the Gaussian and the fact that the Bargmann transform of an L^2-function is analytic. In [7, 8] Gröchenig and Lyubarskii obtained the following generalization: for any pair (a, b) in $\mathbb{R}^2_+$ with $ab < \frac{1}{n+1}$, the Gabor family $\mathcal{G}(h_n, a, b)$ is a frame. Finally, Lyubarskii and Nes [12] proved that the frame set of any sufficiently nice, *odd* window function, in particular, h_{2m+1}, $m \in \mathbb{N} \cup \{0\}$, cannot contain the hyperbolas $ab = \frac{p}{p+1}$ for any $p \in \mathbb{N}$. As no other obstructions for the frame property of h_n was known, this led Gröchenig [5] to conjecture that the frame set for the even Hermite functions is the largest possible set $\mathscr{F}(h_{2m}) = \left\{(a, b) \in \mathbb{R}^2_+ : ab < 1\right\}$, and that the frame set for the odd Hermite functions is $\mathscr{F}(h_{2m+1}) = \{(a, b) \in \mathbb{R}^2_+ : ab < 1, ab \neq \frac{p}{p+1}, p \in \mathbb{N}\}$, $m \in \mathbb{N} \cup \{0\}$. The conjecture is true for h_0 by the above mentioned results. The conjecture for h_1 is due to Lyubarskii and Nes [12], and this paper will not shed new light on this case. However, our results show that the conjecture is false for h_n with $n = 4m + 2$ and $n = 4m + 3$, $m \in \mathbb{N} \cup \{0\}$. We also give numerical evidence in Section 5 that it is false for $n = 4$ and $n = 5$ which leads us to believe that the conjecture is also false for $n = 4m$ and $n = 4m + 1$, whenever $m > 0$.

Our proofs are based on Zak transform methods and certain infinite series identities which are of independent interest. As an example, we will show that h_{4m+2}, $m \in \mathbb{N} \cup \{0\}$, satisfies

$$\sum_{k \in \mathbb{Z}} (-1)^k h_{4m+2}(\sqrt{2}(k + \tfrac{p}{4})) = 0 \quad \text{for } p \in \{1, 3\}. \tag{1.2}$$

For $m = 0$ the identity concerns h_2, and it reads, for $p = 1$,

$$\sum_{k \in \mathbb{Z}} (-1)^k (8\pi(k + \tfrac{1}{4})^2 - 1)\, \mathrm{e}^{-2\pi\left(k+\frac{1}{4}\right)^2} = 0, \tag{1.3}$$

which is illustrated in Figure 1. As we shall see in Section 3, the identities in (1.2) are even true for any sufficiently nice function that is an eigenfunction of the Fourier transform with eigenvalue -1. From the identity (1.2) it follows that the Zak transform $Z_{\sqrt{2}}$ of h_{4m+2} has

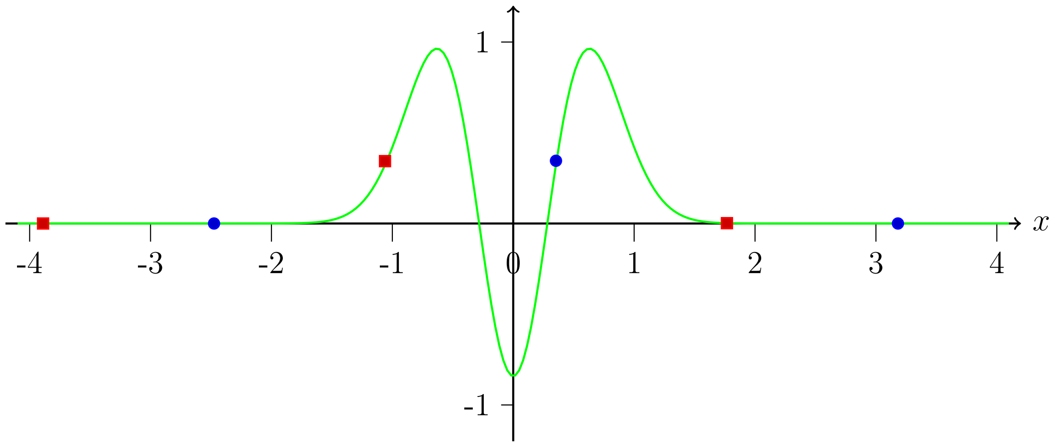

Figure 1: The graph of h_2 and an illustration of the identity (1.3), where the samples for even and odd $k \in \mathbb{Z}$ are marked with blue circles and red squares, respectively. Note that the sampling has no simple symmetries, e.g., $h_2(-\sqrt{2}\,3/4) \neq h_2(\sqrt{2}/4)$.

two zeros in $[0,1)^2$, located one-half apart on a horizontal line. By standard Zak transform methods in Gabor analysis, detailed in Section 4, it follows that $\mathcal{G}(h_{4m+2}, 1/\sqrt{2}, 1/\sqrt{2})$ is not a frame. Note that it is not our focus to give a detailed analysis of the frame set of specific Hermite functions, e.g., $\mathscr{F}(h_2)$. Instead, we are interested in determining values of a and b for which $\mathcal{G}(g,a,b)$ fails to be a frame for every nice window g in, e.g., the class of eigenfunctions of the Fourier transform associated with the eigenvalue -1 to which all Hermite functions of the form h_{4m+2}, $m \in \mathbb{N} \cup \{0\}$, belong. Previously, not a single obstruction for the frame property was known for any of the functions in this class.

2 Preliminaries

We begin by recalling some properties of the Hermite functions and the Zak transform.

2.1 Hermite functions

Hermite functions arise in many different contexts, e.g., as eigenfunctions of the Hermite operator $H = -\frac{d^2}{dx^2} + (2\pi x)^2$. What is more important for us is that the Hermite functions are also eigenfunctions for the Fourier transform:

$$\hat{h}_n(\gamma) = (-i)^n h_n(\gamma) \quad a.e.\ \gamma \in \mathbb{R}.$$

Here, the Fourier transform is defined for $f \in L^1(\mathbb{R})$ by

$$\mathcal{F} f(\xi) = \hat{f}(\gamma) = \int_{\mathbb{R}} f(x)\, \mathrm{e}^{-2\pi i \gamma x}\, \mathrm{d}x$$

with the usual extension to $L^2(\mathbb{R})$. We let H_j, $j = 0,1,2,3$, denote the eigenspace of the Fourier transform corresponding to the eigenvalue $(-i)^j$. More specifically, since $\{h_n\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R})$,

$$H_j = \ker(\mathcal{F} - (-i)^j I) = \overline{\operatorname{span}\{h_{4m+j} : m \in \mathbb{N}\}} = \left\{ \sum_{m\in\mathbb{N}} c_m h_{4m+j} : (c_m) \in \ell^2(\mathbb{N}) \right\}.$$

By $\mathcal{F}^2\{f(x)\} = f(-x)$, it follows that any function in H_j, $j = 0, 2$, is even and that any function in H_j, $j = 1, 3$, is odd.

Since the Fourier transform is a unitary operator, it preserves the frame property, that is, the system $\mathcal{G}(g, a, b)$ is a frame if and only if the Fourier transform of the system $\mathcal{G}(\hat{g}, b, a)$ is a frame. Since the eigenvalue of the Hermite functions is of modulus one, we immediately have the following simple result. It implies that the frame set of Hermite functions is symmetric about the line $a = b$, i.e., $(a, b) \in \mathscr{F}(h_n)$ if and only if $(b, a) \in \mathscr{F}(h_n)$.

Lemma 1. *Let $a, b > 0$, $A, B > 0$, and let $g \in H_j$ for some $j = 0, 1, 2, 3$. Then the following are equivalent:*

(i) $\mathcal{G}(g, a, b)$ is a frame with bounds A and B,

(ii) $\mathcal{G}(g, b, a)$ is a frame with bounds A and B.

2.2 The Zak transform

For any $\lambda > 0$, the Zak transform of a function $f \in L^2(\mathbb{R})$ is defined as

$$(Z_\lambda f)(x, \gamma) = \sqrt{\lambda} \sum_{k \in \mathbb{Z}} f(\lambda(x + k))\, \mathrm{e}^{-2\pi i k \gamma}, \quad a.e.\ x, \gamma \in \mathbb{R}, \tag{2.1}$$

with convergence in $L^2_{\mathrm{loc}}(\mathbb{R})$. The Zak transform Z_λ is a unitary map of $L^2(\mathbb{R})$ onto $L^2([0,1)^2)$, and it has the following quasi-periodicity:

$$Z_\lambda f(x + 1, \gamma) = \mathrm{e}^{2\pi i \gamma}\, Z_\lambda f(x, \gamma), \quad Z_\lambda f(x, \gamma + 1) = Z_\lambda f(x, \gamma) \quad \text{for a.e. } x, \gamma \in \mathbb{R}.$$

The Zak transform has been used by Weil [17] in harmonic analysis on locally compact abelian groups, by Gel'fand [3] in the study of Schrödinger's equation, and by Zak [18] in solid state physics. For a systematic treatment of the Zak transform and its use in applied mathematics, we refer to the paper by Janssen [9]. Recent applications in Gabor analysis include [6, 10, 11].

The Zak transform inherits symmetries of the function f. The following basic lemma will be used several times in the later sections. The Wiener space $W(\mathbb{R})$ consists of functions $g \in L^\infty(\mathbb{R})$ for which $\sum_{k \in \mathbb{Z}} \operatorname{ess\,sup}_{x \in [0,1]} |g(x + k)| < \infty$. The assumption that f belongs to $W(\mathbb{R})$ and is continuous in Lemma 2 implies that $Z_\lambda f$ is continuous which guarantees that the identities in the lemma hold pointwise.

Lemma 2. *Let $m \in \mathbb{Z}$ and $\lambda > 0$. Assume that $f \in W(\mathbb{R})$ is continuous.*

(i) If f is an even function, then

$$Z_\lambda f(x, \gamma) = Z_\lambda f(-x, -\gamma) \qquad \text{for all } x, \gamma \in \mathbb{R}.$$

In particular, $Z_\lambda f(x, \frac{m}{2}) = (-1)^m Z_\lambda f(1 - x, \frac{m}{2})$ and

$$Z_\lambda f(x, \gamma) = 0 \qquad (x, \gamma) \in \mathbb{Z}^2 + (\tfrac{1}{2}, \tfrac{1}{2}).$$

(ii) If f is an odd function, then

$$Z_\lambda f(x,\gamma) = -Z_\lambda f(-x,-\gamma) \qquad \text{for all } x,\gamma \in \mathbb{R}.$$

In particular, $Z_\lambda f(x, \frac{m}{2}) = (-1)^{m+1} Z_\lambda f(1-x, \frac{m}{2})$ *and*

$$Z_\lambda f(x,\gamma) = 0 \qquad (x,\gamma) \in \tfrac{1}{2}\mathbb{Z}^2 \setminus \left(\mathbb{Z}^2 + (\tfrac{1}{2},\tfrac{1}{2})\right).$$

By quasi-periodicity, the function $Z_\lambda f$ on $\mathbb{R}^2$ is determined by its values on $[0,1)^2$. Hence, if $Z_\lambda f(x_0,\gamma_0) = 0$ for some $(x_0,\gamma_0) \in \mathbb{R}^2$, then $Z_\lambda f(x,\gamma) = 0$ for all $(x,\gamma) \in \mathbb{Z}^2 + (x_0,\gamma_0)$. For this reason we will often only explicitly mention the zeros of $Z_\lambda f$ on $[0,1)^2$.

If $f \in W(\mathbb{R})$ and $\hat{f} \in W(\mathbb{R})$, it follows by an application of Poisson summation formula, see e.g., [9] or [4, Proposition 8.2.2], that

$$Z_\lambda f(x,\gamma) = \mathrm{e}^{2\pi i x\gamma}\, Z_{1/\lambda}\hat{f}(\gamma,-x) \quad \text{for all } x,\gamma \in \mathbb{R}, \tag{2.2}$$

with absolute convergence of the series . In particular, this relation holds for any function f in $H_j \cap W(\mathbb{R})$ for $j = 0,1,2,3$. Note that any function f in $H_j \cap W(\mathbb{R})$ is continuous since $\hat{f} \in W(\mathbb{R}) \subset L^1(\mathbb{R})$.

3 Some infinite series identities

The infinite series identities for Hermite functions derived in this section will play a crucial role in the counterexamples in Section 4. The identities are of independent interest and can be formulated as multiple zeros of the Zak transform. We remark that it is not difficult to find a single zero of the Zak transform of Hermite functions, see, e.g., Lemma 2. We will find k zeros of $Z_\lambda h_n(x,\gamma)$ for a fixed value of γ, each $1/(k+1)$ apart with respect to the x variable, which is a much harder task that depends delicately on the parameter λ.

Lemma 3. *Let $n = 4m+2$ for some $m \in \mathbb{N} \cup \{0\}$. Then*

$$Z_{\sqrt{2}} h_n(\tfrac{p}{4},\tfrac{1}{2}) \overset{\text{def}}{=} 2^{1/4} \sum_{k\in\mathbb{Z}} (-1)^k h_n(\sqrt{2}(k+\tfrac{p}{4})) = 0 \quad \text{for } p \in \{1,3\}, \tag{3.1}$$

and

$$Z_{\sqrt{3}} h_n(\tfrac{p}{6},\tfrac{1}{2}) \overset{\text{def}}{=} 3^{1/4} \sum_{k\in\mathbb{Z}} (-1)^k h_n(\sqrt{3}(k+\tfrac{p}{6})) = 0 \quad \text{for } p \in \{1,5\}. \tag{3.2}$$

Proof. We first prove the assertions in (3.1). Let $p = 1$. Since the sum in (3.1) converges absolutely, we can split the sum in even and odd indices $k \in \mathbb{Z}$. Hence, proving (3.1) is equivalent to proving:

$$\sum_{k\in\mathbb{Z}} h_n(\sqrt{2}(2k+\tfrac{1}{4})) = \sum_{k\in\mathbb{Z}} h_n(\sqrt{2}(2k+\tfrac{5}{4})).$$

In terms of the Zak transform, we need to prove that

$$Z_{2^{3/2}} h_n(\tfrac{1}{8},0) = Z_{2^{3/2}} h_n(\tfrac{5}{8},0). \tag{3.3}$$

We first consider the left hand side. By (2.2) and the fact that $\hat{h}_n = -h_n$, we obtain

$$Z_{2^{3/2}} h_n(\tfrac{1}{8},0) = Z_{2^{-3/2}} \hat{h}_n(0,-\tfrac{1}{8}) \overset{\text{def}}{=} -2^{-3/4} \sum_{k\in\mathbb{Z}} h_n(2^{-3/2}k)\, \mathrm{e}^{2\pi i k/8}.$$

Substituting $k \in \mathbb{Z}$ for $8m+\ell$, where $m \in \mathbb{Z}$ and $\ell = 0,1,\dots,7$, we find that

$$\begin{aligned} Z_{2^{3/2}} h_n(\tfrac{1}{8},0) &= -2^{-3/4} \sum_{\ell=0}^{7} \sum_{m\in\mathbb{Z}} h_n(2^{3/2}(m+\frac{\ell}{8}))\, \mathrm{e}^{2\pi i \ell/8} \\ &= -2^{-3/2} \sum_{\ell=0}^{7} Z_{2^{3/2}} h_n(\tfrac{\ell}{8},0)\, \mathrm{e}^{2\pi i \ell/8} \end{aligned} \tag{3.4}$$

The odd terms over ℓ sum to:

$$\begin{aligned} \sum_{\ell\in\{1,3,5,7\}} Z_{2^{3/2}} h_n(\tfrac{\ell}{8},0)\, \mathrm{e}^{2\pi i \ell/8} =& Z_{2^{3/2}} h_n(\tfrac{1}{8},0)(\mathrm{e}^{2\pi i/8} + \mathrm{e}^{2\pi i 7/8}) + Z_{2^{3/2}} h_n(\tfrac{5}{8},0)(\mathrm{e}^{2\pi i 3/8} + \mathrm{e}^{2\pi i 5/8}) \\ =& \sqrt{2} Z_{2^{3/2}} h_n(\tfrac{1}{8},0) - \sqrt{2} Z_{2^{3/2}} h_n(\tfrac{5}{8},0), \end{aligned}$$

where we have used Lemma 2. Similarly, we find that

$$Z_{2^{3/2}} h_n(\tfrac{5}{8},0) = -2^{-3/2} \sum_{\ell=0}^{7} Z_{2^{3/2}} h_n(\tfrac{\ell}{8},0)\, \mathrm{e}^{2\pi i 5\ell/8} \tag{3.5}$$

where the odd terms over ℓ sum to:

$$\begin{aligned} \sum_{\ell\in\{1,3,5,7\}} Z_{2^{3/2}} h_n(\tfrac{\ell}{8},0)\, \mathrm{e}^{2\pi i 5\ell/8} =& Z_{2^{3/2}} h_n(\tfrac{5}{8},0)(\mathrm{e}^{2\pi i/8} + \mathrm{e}^{2\pi i 7/8}) + Z_{2^{3/2}} h_n(\tfrac{1}{8},0)(\mathrm{e}^{2\pi i 3/8} + \mathrm{e}^{2\pi i 5/8}) \\ =& \sqrt{2} Z_{2^{3/2}} h_n(\tfrac{5}{8},0) - \sqrt{2} Z_{2^{3/2}} h_n(\tfrac{1}{8},0). \end{aligned}$$

Note that $\ell \equiv 5\ell \pmod 8$ for even $\ell \in 2\mathbb{Z}$. Thus, if we subtract the two right hand sides of (3.4) and (3.5), the even terms over $\ell = 0,2,4,6$ cancel out. Hence,

$$Z_{2^{3/2}} h_n(\tfrac{1}{8},0) - Z_{2^{3/2}} h_n(\tfrac{5}{8},0) = -(Z_{2^{3/2}} h_n(\tfrac{1}{8},0) - Z_{2^{3/2}} h_n(\tfrac{5}{8},0)).$$

However, this is only possible if (3.3) holds which was what we had to prove. This completes the proof of the case $p = 1$.

For the case $p = 3$, note that, by Lemma 2,

$$Z_{\sqrt{2}} h_n(\tfrac{1}{4},\tfrac{1}{2}) = -Z_{\sqrt{2}} h_n(\tfrac{3}{4},\tfrac{1}{2}),$$

hence the identity follows from the case $p = 1$.

The proof of (3.2) goes along the same lines as the proof of (3.1); the details are left for the reader. □

Lemma 4. *Let $n = 4m+3$ for some $m \in \mathbb{N} \cup \{0\}$ and let $s \in \{2,3,4\}$. Then*

$$Z_{\sqrt{s}} h_n(\tfrac{p}{s},0) \overset{\text{def}}{=} s^{1/4} \sum_{k\in\mathbb{Z}} h_n(\sqrt{s}(k+\tfrac{p}{s})) = 0 \quad \textit{for } p \in \{0,1,\dots,s-1\}.$$

Proof. We will only prove the case $s=3$ as the other cases are similar. For $p=0$ the identity follows from the fact that h_n is an odd function. For $p=1$ we have, using (2.2) and $\hat{h}_n = ih_n$,

$$\begin{aligned}3^{-1/4} Z_{\sqrt{3}} h_n(\tfrac{1}{3},0) &= 3^{-1/4} Z_{\frac{1}{\sqrt{3}}} \hat{h}_n(0,-\tfrac{1}{3}) = 3^{-1/2} \sum_{k\in\mathbb{Z}} ih_n(\tfrac{1}{\sqrt{3}}k)\, \mathrm{e}^{2\pi ik/3} \\ &= 3^{-1/2} i \left(\sum_{m\in\mathbb{Z}} h_n(\sqrt{3}m) + \sum_{m\in\mathbb{Z}} h_n(\sqrt{3}(m+\tfrac{1}{3}))\, \mathrm{e}^{2\pi i/3} + \sum_{m\in\mathbb{Z}} h_n(\sqrt{3}(m-\tfrac{1}{3}))\, \mathrm{e}^{-2\pi i/3} \right),\end{aligned} \tag{3.6}$$

where we have substituted k for $3m+\ell$ with $m\in\mathbb{Z}$ and $\ell\in\{-1,0,1\}$, Since h_n is odd, it follows directly that $\sum_{m\in\mathbb{Z}} h_n(\sqrt{3}m) = 0$. By yet another symmetry argument (e.g., Lemma 2), we also see that

$$\sum_{m\in\mathbb{Z}} h_n(\sqrt{3}(m-\tfrac{1}{3})) = \sum_{m\in\mathbb{Z}} h_n(\sqrt{3}(m+\tfrac{2}{3})) = -\sum_{m\in\mathbb{Z}} h_n(\sqrt{3}(m+\tfrac{1}{3})).$$

Continuing the computation in (3.6) yields

$$\begin{aligned}\sum_{k\in\mathbb{Z}} h_n(\sqrt{3}(k+\tfrac{1}{3})) &\overset{\text{def}}{=} 3^{-1/4} Z_{\sqrt{3}} h_n(\tfrac{1}{3},0) = 3^{-1/2} i (\mathrm{e}^{2\pi i/3} - \mathrm{e}^{-2\pi i/3}) \sum_{m\in\mathbb{Z}} h_n(\sqrt{3}(m+\tfrac{1}{3})) \\ &= -\sum_{m\in\mathbb{Z}} h_n(\sqrt{3}(m+\tfrac{1}{3})),\end{aligned}$$

where we use that $\mathrm{e}^{2\pi i/3} - \mathrm{e}^{-2\pi i/3} = i\sqrt{3}$. Thus $\sum_{m\in\mathbb{Z}} h_n(\sqrt{3}(m+\tfrac{1}{3})) = 0$ which completes the case $p=1$.

Consider now $p=2$. By Lemma 2 we have

$$Z_{\sqrt{3}} h_n(\tfrac{1}{3},0) = -Z_{\sqrt{3}} h_n(\tfrac{2}{3},0),$$

hence the assertion for $p=2$ follows from the case $p=1$. □

Note that the only property of h_n used in the proof of the above two lemmas is that h_n is an eigenfunction of the Fourier transform associated with the eigenvalue -1 and i, respectively, for which Poisson summation formula (2.2) holds pointwise with absolute convergence. Recall that functions in $H_2 \cap W(\mathbb{R})$ are even and continuous, while functions in $H_3 \cap W(\mathbb{R})$ are odd and continuous. Therefore, we can formulate the following extension of the results in this section using Lemma 2.

Lemma 5. *(i) For $g\in H_2\cap W(\mathbb{R})$, we have:*

$$Z_{\sqrt{2}} g(x,\gamma) = 0 \quad \textit{for } (x,\gamma) \in (\tfrac{1}{4}\mathbb{Z}\setminus\mathbb{Z}) \times (\mathbb{Z}+\tfrac{1}{2}),$$

and

$$Z_{\sqrt{3}} g(x,\gamma) = 0 \quad \textit{for } (x,\gamma) \in (\tfrac{1}{3}\mathbb{Z}+\tfrac{1}{6}) \times (\mathbb{Z}+\tfrac{1}{2}).$$

(ii) For $g\in H_3\cap W(\mathbb{R})$ and $s\in\{2,3,4\}$, we have:

$$Z_{\sqrt{s}} g(x,\gamma) = 0 \quad \textit{for } (x,\gamma) \in \tfrac{1}{s}\mathbb{Z}\times\mathbb{Z}.$$

4 New obstructions of the frame property

For rationally oversampled Gabor systems, i.e., $\mathcal{G}(g,a,b)$ with

$$ab \in \mathbb{Q}, \quad ab = \frac{p}{q} \quad \gcd(p,q)=1,$$

we define column vectors $\phi_\ell^g(x,\gamma) \in \mathbb{C}^p$ for $\ell \in \{0,1,\dots,q-1\}$ by

$$\phi_\ell^g(x,\gamma) = \left(p^{-\frac{1}{2}}(Z_{\frac{1}{b}}g)(x-\ell\frac{p}{q},\gamma+\frac{k}{p})\right)_{k=0}^{p-1} \quad a.e.\ x,\gamma \in \mathbb{R}.$$

The following characterization of rationally oversampled Gabor frames is due to Zibulski and Zeevi [19].

Theorem 6. *Let $A,B>0$, and let $g \in L^2(\mathbb{R})$. Suppose $\mathcal{G}(g,a,b)$ is a rationally oversampled Gabor system. Then the following assertions are equivalent:*

(i) $\mathcal{G}(g,a,b)$ is a Gabor frame for $L^2(\mathbb{R})$ with bounds A and B,

(ii) $\{\phi_\ell^g(x,\gamma)\}_{\ell=0}^q$ is a frame for $\mathbb{C}^p$ with uniform bounds A and B for a.e. $(x,\gamma)\in[0,1)^2$.

If $p=1$, i.e., $ab=1/q$, the Gabor system $\mathcal{G}(g,a,b)$ is said to be integer oversampled. By Theorem 6 it is a frame with bounds A and B if and only if

$$A \le \left(\sum_{\ell=0}^{q-1}\left|Z_{\frac{1}{b}}g(x-\ell/q,\gamma)\right|^2\right)^{1/2} \le B \quad \text{for a.e. } x,\gamma \in [0,1)^2. \tag{4.1}$$

If $g \in W(\mathbb{R})$ is odd and continuous, then, by Lemma 2(ii), $Z_{1/b}g(0,0)=Z_{1/b}g(\frac{1}{2},0)=0$ for any $b>0$, which by (4.1) immediately implies that $\mathcal{G}(g,a,b)$ is not a frame along the hyperbola $ab=\frac{1}{2}$. Lyubarskii and Nes [12] showed that this assertion extends to any of the hyperbolas $ab=\frac{p}{p+1}$ for $p\in\mathbb{N}$ for any such odd window function. The results in the remainder of this section show that the frame property also must fail for certain (a,b)-values for window functions with other symmetries formulated in terms of the Fourier transform. We denote the new "failure" points in $\{(a,b)\in\mathbb{R}_+^2 : ab<1\}$ by (a_i,b_i), $i=0,1,2,3,4$, where

$$a_i=b_i=\frac{1}{\sqrt{i+2}}\ (i=0,1,2), \quad a_3=\frac{2}{\sqrt{3}}, b_3=\frac{1}{\sqrt{3}} \quad a_4=\frac{1}{\sqrt{3}}, b_4=\frac{2}{\sqrt{3}}. \tag{4.2}$$

Theorem 7. *Let $g \in H_2 \cap W(\mathbb{R})$. For any point (a_i,b_i), $i\in\{0,1,3,4\}$, as defined in (4.2), the Gabor system $\mathcal{G}(g,a_i,b_i)$ is not a frame for $L^2(\mathbb{R})$, in particular, $\mathcal{G}(h_n,a_i,b_i)$ is not a frame for $n=4m+2$, $m\in\mathbb{N}\cup\{0\}$.*

Proof. We consider first the assertion for $i=0$. Note that $a_0b_0=1/2$, hence $\mathcal{G}(g,a_0,b_0)$ is an integer oversampled Gabor system with $p=1$ and $q=2$. By (3.1) in Lemma 5(i), it follows that $Z_{1/b_0}g(x-\ell/q,\gamma)=0$ for $\ell=0,1$ for $(x,\gamma)=(\frac{3}{4},\frac{1}{2})$. Since the Zak transform is continuous for $g\in H_2\cap W(\mathbb{R})$, we see that the lower bound in (4.1) cannot hold. Thus, $\mathcal{G}(g,a_0,b_0)$ is not a frame.

For the case $i = 1$, we have $a_1 b_1 = 1/3$, hence $p = 1$ and $q = 3$. By (3.2) in Lemma 5(i), it follows that $Z_{1/b_1} g(x - \ell/q, \gamma) = 0$ for $\ell = 0, 1, 2$ for $(x, \gamma) = (\frac{5}{6}, \frac{1}{2})$. As before, this violates the frame property of $\mathcal{G}(g, a_1, b_1)$.

For the case $i = 3$, we have $a_3 b_3 = 1/3$, hence $p = 2$ and $q = 3$. From case $i = 1$, we see that the matrix $\Phi^g = \{\phi_\ell^g(x, \gamma)\}_{\ell=0}^q$ has a row of zeros. It follows from Theorem 6 that $\mathcal{G}(g, a_3, b_3)$ is not a frame.

The assertion for $i = 4$ follows from case $i = 3$ by symmetry using Lemma 2. □

Theorem 8. *Let $g \in H_3 \cap W(\mathbb{R})$. For any point (a_i, b_i), $i \in \{1, 2\}$, as defined in (4.2), the Gabor system $\mathcal{G}(g, a_i, b_i)$ is not a frame for $L^2(\mathbb{R})$, in particular, $\mathcal{G}(h_n, a_i, b_i)$ is not a frame for $n = 4m + 3$, $m \in \mathbb{N} \cup \{0\}$.*

Proof. We consider first the assertion for $i = 1$. In this case $a_1 b_1 = 1/3$ and $\mathcal{G}(g, a_1, b_1)$ is an integer oversampled Gabor system with $p = 1$ and $q = 3$. By Lemma 5(ii), it follows that $Z_{1/b_1} g(x - \ell/q, \gamma) = 0$ for $\ell = 0, 1, 2$ for $(x, \gamma) = (\frac{2}{3}, 0)$. As in the proof of Theorem 7, this shows that $\mathcal{G}(g, a_1, b_1)$ cannot be a frame. The proof for $i = 2$ we note that $Z_{1/b_2} g(x - \ell/q, \gamma) = 0$ for $\ell = 0, 1, 2, 3$ for $(x, \gamma) = (\frac{3}{4}, 0)$ □

Note that it also follows from Lemma 5(ii) that $(a_0, b_0), (a_3, b_3)$ and (a_4, b_4) fall outside $\mathscr{F}(g)$ for $g \in H_3 \cap W(\mathbb{R})$. However, these obstructions are already known by the results in [12] since functions in $H_3 \cap W(\mathbb{R})$ are odd.

From Theorem 7 we have four obstruction points for the window class $H_2 \cap W(\mathbb{R})$. Theorem 8 provides us with two new obstruction points for the window class $H_3 \cap W(\mathbb{R})$, not already covered by the hyperbolic obstructions $ab = p/(p+1)$, $p \in \mathbb{N}$. On the other hand, in general, no obstruction points can exist for the class $H_0 \cap W(\mathbb{R})$ since it contains the Gaussian h_0. If the conjecture by Lyubarskii and Nes [12] holds true, then there are no general obstructions for the class $H_1 \cap W(\mathbb{R})$ in addition to $ab = p/(p+1)$, $p \in \mathbb{N}$.

One might ask how badly the Gabor system fails to be a frame in the obstruction points. From the proofs above, it is clear that it is the lower frame bound that fails. In fact, any window in $W(\mathbb{R})$ satisfy the upper frame bound. The lower frame bound is a strong condition that is equivalent to injectivity and closedness of the range of the analysis operator $C_{g,a,b} : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2)$ defined by $C_{g,a,b} f = \left\{ \left\langle f, \mathrm{e}^{2\pi i b m \cdot} g(\cdot - ak) \right\rangle \right\}_{k,m \in \mathbb{Z}}$. Note that injectivity of $C_{g,a,b}$ is equivalent with the Gabor system $\mathcal{G}(g, a, b)$ being complete in $L^2(\mathbb{R})$. For Hermite windows, Gröchenig, Haimi, and Romero [6] recently showed that, at least, completeness is guaranteed. To be precise, they proved as part of a more general result that, for any $n \in \mathbb{N}$, the system $\mathcal{G}(h_n, a, b)$ is complete in $L^2(\mathbb{R})$ for any rational $ab \le 1$. Hence, for each (a_i, b_i), $i \in \{0, 1, 2, 3, 4\}$, given in (4.2), the Gabor system $\mathcal{G}(h_n, a_i, b_i)$, for $n = 4m + 2$ or $n = 4m + 3$, is a complete Bessel system for which the lower frame bound is not satisfied because the range of C_{h_n, a_i, b_i} fails to be closed.

Even though both $(1/\sqrt{2}, 1/\sqrt{2}) \notin \mathscr{F}(g)$ and $(1/\sqrt{3}, 1/\sqrt{3}) \notin \mathscr{F}(g)$ for $g \in H_2 \cap W(\mathbb{R})$, no other points of the form $(1/\sqrt{k}, 1/\sqrt{k})$ can be obstruction points for the frame property for the window class $H_2 \cap W(\mathbb{R})$. In fact, by [7, 8] we know that $ab < 1/(n+1)$ is sufficient for the frame property of $\mathcal{G}(h_n, a, b)$, hence, in particular, that $\mathcal{G}(h_2, a, b)$ is a frame for $ab < 1/3$. Moreover, the obstruction point $(a_1, b_1) = (1/\sqrt{3}, 1/\sqrt{3})$ shows that the region $ab < 1/3$ is sharp for h_2 in the sense that the smallest constant c such that $\{(a, b) : ab < c\} \subset \mathscr{F}(h_2)$ is $c = 1/3$. A similar observation holds for $H_3 \cap W(\mathbb{R})$. In this case, the obstruction point $(a_2, b_2) = (1/2, 1/2)$ shows that the region $ab < 1/4$ is sharp for h_3. Since $ab < 1$ and $ab < 1/2$ are sharp for h_0 and h_1, respectively, it is natural to ask if $ab < \frac{1}{n+1}$ is sharp for h_n for all $n \in \mathbb{N}$.

We have here focused on finding (a, b)-values that serve as obstructions of the frame property simultaneously for an entire class of window functions. For a specific choice of a Hermite function h_n, $n \geq 2$, one can most likely find many more new obstructions; this is indeed indicated by the numerical experiments in the next section.

5 Numerical experiments

The numerical experiments in Matlab below use double precision floating-point numbers. We truncate the Hermite functions to obtain compactly supported functions whenever the function value drops sufficiently low. This way a close approximation to the Zak transform $Z_{1/b}h_n$ can be computed as a finite sum. We then discretize the Zak transform domain on a uniform sampling grid, e.g., 51×51. As we only consider integer oversampled Gabor systems, close approximations to the frame bounds are easily computed for given values of a and b using the formula (4.1). The approximated bounds A_{apx} and B_{apx} will (up to machine precision) be larger and smaller, respectively, than the true optimal frame bounds from (4.1), i.e., $A_{\text{opt}} \leq A_{\text{apx}} \leq B_{\text{apx}} \leq B_{\text{opt}}$.

Example 1. Let us first illustrate Theorem 7 for h_2. Figure 2 shows that the upper and lower frame bound of $\mathcal{G}(h_2, a, b)$ along $ab = 1/2$ for $b \in \left[\frac{1}{8}, 4\right]$. We first remark that

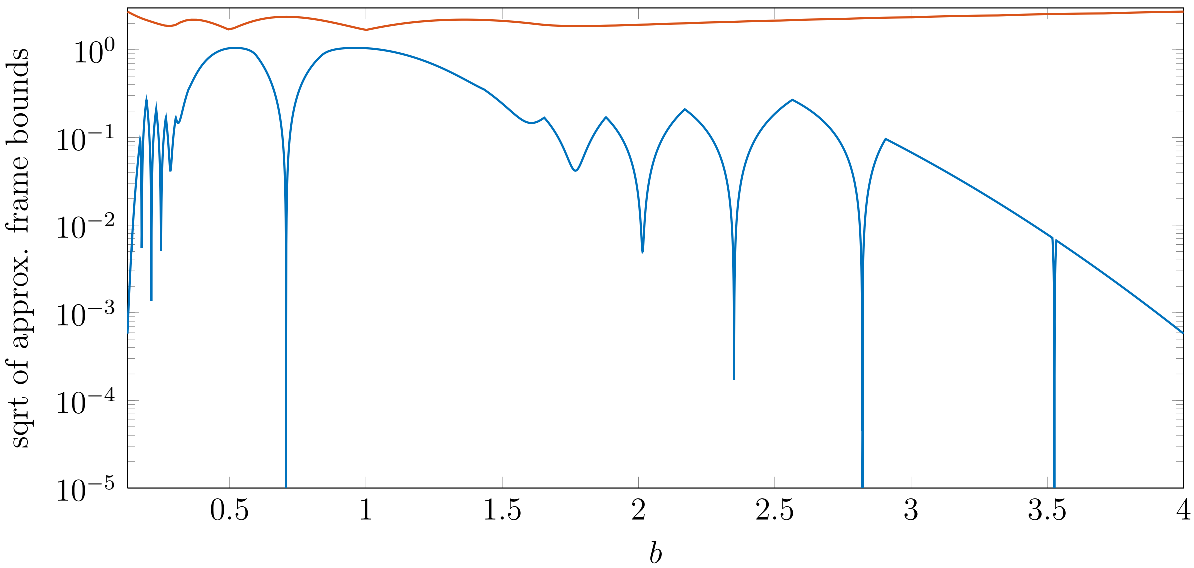

Figure 2: Numerical approximations of the upper (red) and lower (blue) frame bound for $\mathcal{G}(h_2, a, b)$ along $ab = 1/2$. At the point $(a_0, b_0) = (1/\sqrt{2}, 1/\sqrt{2})$ the estimate of $\sqrt{A}$ essentially drops to machine precision $\approx 7 \cdot 10^{-16}$ (not shown).

$A_{\text{apx}}^{1/2}$ drops to machine precision at $(a_0, b_0) = (1/\sqrt{2}, 1/\sqrt{2})$. Note also that the frame bounds are symmetric about $b = 1/\sqrt{2}$ according to Lemma 2, that is, $\mathcal{G}(h_2, 1/(2b), b)$ and $\mathcal{G}(h_2, b, 1/(2b))$ have the same frame bounds.

The behavior of $A_{\text{apx}}^{1/2}$ is rather complicated. The drops of $A_{\text{apx}}^{1/2}$ below, say, 10^{-3}, are very narrow and therefore difficult to resolve due to the discretization of the b range.

Moreover, it is unclear if $(a_0, b_0) = (1/\sqrt{2}, 1/\sqrt{2})$ is the only point along $ab = 1/2$ that does not belong to $\mathscr{F}(h_2)$. Around $b = 2.35$ and $b = 2.82$ the values of $A_{\rm apx}^{1/2}$ are in the order of 10^{-4} and 10^{-7}, respectively. At $b = 3.5261848971734$ the value of $A_{\rm apx}^{1/2}$ even drops to $1.6 \cdot 10^{-12}$, however, it does not drop below this value even when the discretization is refined. There may very well exist a (a, b)-pair near the point $(1/(2b), b)$, where $b = 3.5261848971734$, for which $\mathcal{G}(h_2, a, b)$ is not a frame. In any event, since $A_{\rm apx}^{1/2} \approx 10^{-12}$, i.e., $A_{\rm apx} \approx 10^{-24}$, such a Gabor system is badly conditioned and should not be used for numerical purposes.

Let us end this paper with two examples not covered by the results in Section 4.

Example 2. In this example we consider Gabor systems generated by h_4 and h_5. Note that these functions belong to H_0 and H_1, respectively. Recall that no obstructions of the frame property is known of h_4, while the hyperbolas $ab = \frac{p}{p+1}$, $p \in \mathbb{N}$, are the only known obstructions for h_5. Figure 3 shows the approximated frame bounds of $\mathcal{G}(h_4, a, b)$ along $ab = 1/2$ and of $\mathcal{G}(h_5, a, b)$ along $ab = 1/3$. The general behavior is similar to that of h_2 in Figure 2. For $\mathcal{G}(h_4, a, b)$ the lower frame bound $A_{\rm apx}^{1/2}$ drops to machine precision four times in the considered b range. This behavior can be explained as follows. In Maple one can verify with arbitrary precision that

$$Z_{\sqrt[4]{3}} h_4(0, \tfrac{1}{2}) \stackrel{\text{def}}{=} 3^{1/8} \sum_{k\in\mathbb{Z}} (-1)^k h_4(3^{1/4} k) = 0 \tag{5.1}$$

holds. Recall that the Zak transform also has a zero at $(\frac{1}{2}, \frac{1}{2})$ since h_4 is even. Hence, equation (5.1) implies that the lower bound in (4.1) is violated for $(x, \gamma) = (\frac{1}{2}, \frac{1}{2})$. Therefore, $\mathcal{G}(h_4, a, b)$ is not frame for $(a, b) = (3^{1/4}/2, 3^{-1/4})$, and by symmetry using Lemma 2, also not for $(a, b) = (3^{-1/4}, 3^{1/4}/2)$. Similarly, one can verify in Maple with arbitrary precision that

$$Z_{\frac{1}{\sqrt[4]{3}}} h_4(0, \tfrac{1}{2}) \stackrel{\text{def}}{=} 3^{-1/8} \sum_{k\in\mathbb{Z}} (-1)^k h_4(3^{-1/4} k) = 0 \tag{5.2}$$

holds. Equation (5.2) implies that $\mathcal{G}(h_4, a, b)$ is not frame for $(a, b) = (3^{-1/4}/2, 3^{1/4})$, and by symmetry, also not for $(a, b) = (3^{1/4}, 3^{-1/4}/2)$. A proof of the identities (5.1) and (5.2) must rely on other methods that used in Section 3 since the Gaussian h_0 does not satisfy the identities and since both h_0 and h_4 belong to H_0.

For the 5th Hermite function h_5 the lower frame bound $A_{\rm apx}^{1/2}$ drops to machine precision seven times along $ab = 1/3$ in the considered range in Figure 3. Here, similar arguments as for h_4 can be used to show that $(a, b) = (\sqrt[4]{27}/3, 1/\sqrt[4]{27})$ and $(a, b) = (1/\sqrt[4]{27}, \sqrt[4]{27}/3)$ do not belong to $\mathscr{F}(h_5)$. Indeed, one can verify in Maple with arbitrary precision that

$$Z_{\sqrt[4]{27}} h_5(\tfrac{p}{3}, \tfrac{1}{2}) = 0 \quad \text{for } p \in \{0, 1, 2\}\,.$$

Similar identities that explain the other five drops of $A_{\rm apx}^{1/2}$ for $\mathcal{G}(h_5, a, b)$ most likely exist.

In Example 2 above we briefly considered obstructions of the frame property for Hermite functions outside the two classes H_2 and H_3. The methods developed in this paper for Wiener space functions in the eigenspaces H_2 and H_3 rely only on the corresponding eigenvalue of the Fourier transform. However, since both $h_0 \in H_0$ and $h_4 \in H_0$ have the same eigenvalue, namely 1, it is obvious that other methods are needed if one attempts to disprove Gröchenig's conjecture for, say, all functions in $\{h_{4m} : m \in \mathbb{N}\}$.

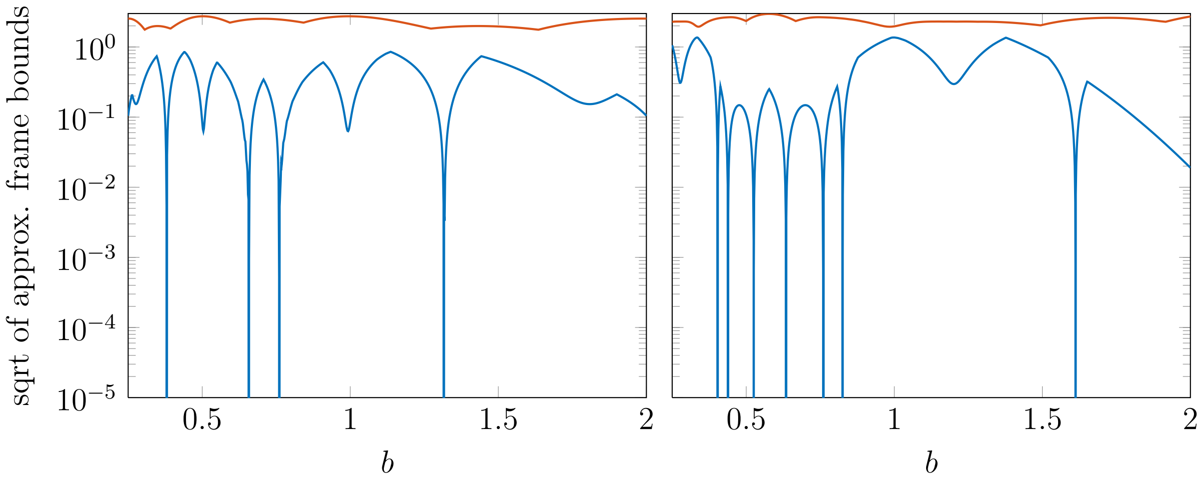

Figure 3: Numerical approximations of the square root of the upper (red) and lower (blue) frame bound for $\mathcal{G}(h_4, a, b)$ along $ab = 1/2$ (left) and $\mathcal{G}(h_5, a, b)$ along $ab = 1/3$ (right). In all instances, where $A_{\text{apx}}^{1/2}$ drops below 10^{-5}, it drops to a value in the order of machine precision $\approx 10^{-16}$ (not shown).

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