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Liftings in finite graphs and linkages in infinite graphs with prescribed edge-connectivity*

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Abstract

Let G be a graph and let s be a vertex of G. We consider the structure of the set of all lifts of two edges incident with s that preserve edge-connectivity. Mader proved that two mild hypotheses imply there is at least one pair that lifts, while Frank showed (with the same hypotheses) that there are at least $(\deg(s)-1)/2$ disjoint pairs that lift. We consider the *lifting graph*: its vertices are the edges incident with s, two being adjacent if they form a liftable pair. We have three main results, the first two with the same hypotheses as for Mader's Theorem.
(i) Let F be a subset of the edges incident with s. We show that F is independent in the lifting graph of G if and only if there is a single edge-cut C in G of size at most $r+1$ containing all the edges in F, where r is the maximum number of edge-disjoint paths from a vertex (not s) in one component of $G-C$ to a vertex (not s) in another component of $G-C$.
(ii) In the k-lifting graph, two edges incident with s are adjacent if their lifting leaves the resulting graph with the property that any two vertices different from s are joined by k pairwise edge-disjoint paths. If both $\deg(s)$ and k are even, then the k-lifting graph is a connected complete multipartite graph. In all other cases, there are at most two components. If there are exactly two components, then each component is a complete multipartite graph. If $\deg(s)$ is odd and there are two components, then one component is a single vertex.
(iii) Huck proved that if k is odd and G is $(k+1)$-edge-connected, then G is weakly k-linked (that is, for any k pairs $\{x_i, y_i\}$, there are k edge-disjoint paths P_i, with P_i joining x_i and y_i). We use our results to extend a slight weakening of Huck's theorem to some infinite graphs: if k is odd, every $(k+2)$-edge-connected, locally finite, 1-ended, infinite graph is weakly k-linked.
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1 Introduction

For distinct vertices x and y in a graph G, $\lambda_G(x,y)$ denotes the maximum number of pairwise edge-disjoint xy-paths in G. We shall assume that x and y have a *target connectivity* $\tau_G(x,y) \le \lambda_G(x,y)$. In the cases of immediate interest, either $\tau_G \equiv \lambda_G$ or τ_G is constant, but the target unifies and generalizes both these particular cases.

Let s be a vertex of G and let sv and sw be two edges incident with s. The *lift of G at sv and sw* is the graph $G_{v,w}$ obtained from $G - \{sv, sw\}$ by adding the edge vw.

The lift of G at sv and sw is *τ_G-feasible* if, for every pair x, y of distinct vertices in $G-s$, $\lambda_{G_{v,w}}(x,y) \ge \tau_G(x,y)$. We will just say *feasible*, since τ_G will always be understood.

Let s be a vertex in a graph G that does not have degree 3 and is not incident with an isthmus. (An *isthmus* is an edge whose deletion from G increases the number of components.) Mader [5] proved (for target λ_G and therefore for any target) that there is always a feasible lift in G using two edges incident with s. Frank [3] extended this to show that there are $\lfloor \deg(s)/2 \rfloor$ pairwise disjoint such feasible pairs.

For any subset A of $V(G)$, we set $\delta_G(A)$ to be the set of edges of G having one end in A and one end not in A. By Menger's Theorem, the obstruction to sv and sw yielding a feasible lift is that there is a pair a, b of vertices and a set A of vertices so that $a \in A$, $b, s \notin A$, and $|\delta_{G_{v,w}}(A)| < \tau_G(a,b)$. Since obviously $|\delta_G(A)| \ge \tau_G(a,b)$ and $|\delta_{G_{v,w}}(A)| \ge |\delta_G(A)| - 2$, we see that $|\delta_G(A)| \le |\tau_G(a,b)| + 1$. Thus motivates the following important notion.

Let A be a subset of $V(G) \setminus \{s\}$. Then $r(A)$ is defined to be $\max\{\tau_G(a,b) \mid a \in A,\ b \notin A \cup \{s\}\}$. Also, A is a *dangerous set* if $|\delta_G(A)| \le r(A) + 1$. The preceding paragraph readily implies the observation that sv and sw do not have a feasible lift if and only if there is a dangerous set A such that $v, w \in A$.

Henceforth, all considerations are in G, so we write $\delta(A)$ instead of $\delta_G(A)$.

The first of our three main results is the following. The "if" part of the statement is trivial; the "only if" is proved in the next section.

Theorem 1.1 *Let G be a graph and let s be a vertex of G that does not have degree 3 and is not incident with an isthmus. Let F be any set of at least two edges, all incident with s. Then no pair of edges in F yields a feasible lift if and only if there is a dangerous set A so that, for every $sv \in F$, $v \in A$.*

Let G be a graph, let s be a vertex of G, and let τ be the edge-connectivity target function. The *lifting graph* $L(G, s, \tau)$ has as its vertices the edges of G incident with s and two edges are adjacent in $L(G, s, \tau)$ if they form a τ-feasible pair. If there is a positive integer k so that $\tau \equiv k$, then we write $L(G,s,k)$ for $L(G,s,\tau)$; $L(G,s,k)$ is the *k-lifting graph*.

Thomassen [8] proved that the k-lifting graph of an Eulerian graph has a disconnected complement. This was used to prove a decomposition theorem for infinite graphs that implies, among other things, a conjecture from 1989: every $8k$-edge-connected infinite graph has a k-arc-connected orientation.

Part (1.2.4) of our second main result generalizes Thomassen's Eulerian result to the k-lifting graph when $\deg(s)$ and k are both even.

Theorem 1.2 *Let G be a graph with a vertex s and let k be a positive integer such that any distinct vertices different from s are joined by k pairwise edge-disjoint paths. If s is not incident with an isthmus and* $\deg(s) \geq 4$*, then:*

(1.2.1) the k-lifting graph $L(G,s,k)$ has at most two components;

(1.2.2) if $\deg(s)$ *is odd and $L(G,s,k)$ has two components, then one has only one vertex and the other component is complete multipartite;*

(1.2.3) if $\deg(s)$ *is even and $L(G,s,k)$ has two components, then each component is complete multipartite with an even number of vertices; and*

(1.2.4) if $\deg(s)$ *and k are both even, then $L(G,s,k)$ is a connected, complete multipartite graph (in particular, it has a disconnected complement).*

If either $L(G,s,k)$ is not connected or both $\deg(s)$ *and k are even, then any component of $L(G,s,k)$ with at least 4 vertices is not a star $K_{1,r}$.*

A graph G is *weakly k-linked* if, for any sequences $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_k$ of (not necessarily distinct) vertices of G, there are k edge-disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i has ends x_i and y_i. By choosing all the x_i to be the same vertex and all the y_i to be the same vertex, we see that any weakly k-linked graph is k-edge-connected. Thomassen [7] conjectured that, when k is odd, the converse holds. Okamura [6] obtained the first significant result about this conjecture (roughly: if G is $\frac{4}{3}k$-edge-connected, then G is weakly k-linked). Then Huck [4] proved that, if k is odd and G is $(k+1)$-edge-connected, then G is weakly k-linked.

We use Huck's Theorem and Theorem 1.2 (1.2.4) to prove the following. Recall that an infinite graph G is *locally finite* if, for every vertex v of G, $\deg(v)$ is finite. Also, a graph G is *1-ended* if, for every finite set S of vertices, $G - S$ has at most one infinite component.

Theorem 1.3 *Let k be an odd positive integer. If G is a $(k+2)$-edge-connected, 1-ended, locally finite graph, then G is weakly k-linked.*

We remark that we can prove that the hypothesis of Theorem 1.3 implies that any $(k+2)$-edge-connected, infinite, locally finite graph with only finitely many ends is weakly k-linked. There are some technicalities that are not germane to the application of Huck's Theorem and Theorem 1.2. We believe the following much stronger statement is true and so choose not to include this intermediate result.

Conjecture 1.4 *Let k be an odd positive integer. If G is a $(k+2)$-edge-connected (infinite) graph, then G is weakly k-linked.*

2 Characterizing independent sets in the lifting graph

Our goal in this section is to prove Theorem 1.1. It is evident that, if there is a dangerous set A such that, for every $sv \in F$, $v \in A$, then no two edges in F give a feasible lift. It was the converse that attracted us.

Chan et al [2] give a very closely related argument, presented very efficiently. Our theorem is used significantly in the next section, so we include our slightly modified version of their proof.

For the proof, it will be helpful to set $\sigma(A) = |\delta(A)| - r(A)$ and $\delta(A, B)$ as the set of edges with one end in A and other end in B. We note that A is dangerous if and only if $\sigma(A) \leq 1$. The following observation is due to Frank.

Lemma 2.1 *[3, Prop. 2.3] Let s be a vertex in a graph G and let A and B be subsets of $V(G) \setminus \{s\}$. Then either*

(2.1.1) $\sigma(A \cup B) + \sigma(A \cap B) + 2|\delta(A \setminus B, B \setminus A)| \leq \sigma(A) + \sigma(B)$ *or*

(2.1.2) $\sigma(A \setminus B) + \sigma(B \setminus A) + 2|\delta(A \cap B, V(G) \setminus (A \cup B))| \leq \sigma(A) + \sigma(B)$. ■

The key lemma for our proof is the following variant of [2, Lemma 2.7]. The proof requires only very minor modifications from that in [2].

Lemma 2.2 *Let G be a graph and s a vertex of G. Suppose sa, sb, and sc are three edges incident with s so that none of the lifts of $\{sa, sb\}$, $\{sa, sc\}$, and $\{sb, sc\}$ is τ-feasible. For $\{x, y, z\} = \{a, b, c\}$, let D_x be a dangerous set containing y and z. Then either s has degree 3, or s is incident with an isthmus, or there is a dangerous subset of $D_a \cup D_b \cup D_c$ containing all three of a, b, and c and at least one of D_a, D_b, and D_c.*

Proof. If any two of a, b, c are the same, then the result is trivial, so we assume a, b, and c are all distinct. We consider two cases.

Case 1: *For at least one of the pairs (A, B) from (D_a, D_b), (D_a, D_c), or (D_b, D_c), (2.1.1) holds in Lemma 2.1.*

We may choose the labelling of a, b, and c, so that

$$\sigma(D_a \cup D_b) + \sigma(D_a \cap D_b) + 2|\delta(D_a \setminus D_b, D_b \setminus D_a)| \leq \sigma(D_a) + \sigma(D_b)\,.$$

As each term on the right side is at most 1, the left-hand side is at most 2. If $D_a \cup D_b$ is dangerous, then we are done, so we may assume $\sigma(D_a \cup D_b) \geq 2$. Therefore, the right-hand side is exactly 2, $\sigma(D_a \cup D_b) = 2$, $\sigma(D_a \cap D_b) = 0$, and $|\delta(D_a \setminus D_b, D_b \setminus D_a)| = 0$.

Suppose Lemma 2.1 (2.1.1) holds for $A = D_a \cap D_b$ and $B = D_c$; that is,

$$\begin{aligned}\sigma((D_a \cap D_b) \cup D_c) + \sigma((D_a \cap D_b) \cap D_c) + 2|\delta((D_a \cap D_b) \setminus D_c, D_c \setminus (D_a \cap D_b))| \\ \leq \quad \sigma(D_a \cap D_b) + \sigma(D_c)\,.\end{aligned}$$

Since $\sigma(D_a \cap D_b) = 0$, the right side is at most 1 and, therefore, $(D_a \cap D_b) \cup D_c$ is dangerous, and we are done. Therefore, we may assume Lemma 2.1 (2.1.2) applies to $A = D_a \cap D_b$ and $B = D_c$. In particular, $\sigma(D_c \setminus (D_a \cap D_b)) \leq \sigma(D_a \cap D_b) + \sigma(D_c)$, showing $D_c \setminus (D_a \cap D_b)$ is dangerous. (It is evidently not empty, as it contains a and b.)

Set $D'_c = D_c \setminus (D_a \cap D_b)$. The edges sa and sb show that $|\delta((D_a \cup D_b) \cap D'_c, V(G) \setminus (D_a \cup D_b \cup D'_c))| \geq 2$. On the other hand, the labelling for this case shows $\sigma(D_a \cup D_b) \leq \sigma(D_a) + \sigma(D_b) \leq 2$ and the preceding paragraph shows $\sigma(D'_c) \leq 1$. Thus,

$$2|\delta((D_a \cup D_b) \cap D'_c, V(G) \setminus (D_a \cup D_b \cup D'_c))| \geq 4 > 3 \geq \sigma(D_a \cup D_b) + \sigma(D'_c)\,.$$

Consequently, Lemma 2.1 implies

$$\sigma((D_a \cup D_b) \cup D'_c) + \sigma((D_a \cup D_b) \cap D'_c) + 2|\delta((D_a \cup D_b) \setminus D'_c, D'_c \setminus (D_a \cup D_b))| \leq \sigma(D_a \cup D_b) + \sigma(D'_c)\,.$$

If $(D_a \cup D_b) \cup D'_c$ is dangerous, then we are done, so we may assume $\sigma((D_a \cup D_b) \cup D'_c) \geq 2$. As $\sigma(D_a \cup D_b) = 2$ and $\sigma(D'_c) \leq 1$, we conclude that $\sigma((D_a \cup D_b) \cap D'_c) \leq 1$ and $|\delta((D_a \cup D_b) \setminus D'_c, D'_c \setminus (D_a \cup D_b))| = 0$. The inequality shows $(D_a \cup D_b) \cap D'_c$ is dangerous, while $|\delta(D_a \setminus D_b, D_b \setminus D_a)| = 0$ implies $|\delta((D_a \cap D'_c) \setminus (D_b \cap D'_c), (D_b \cap D'_c) \setminus (D_a \cap D'_c))| = 0$.

We claim that either sa or sb is an isthmus of G. We have just seen that $(D_a \cup D_b) \cap D'_c$ is dangerous, so,

$$\begin{aligned}1 &\geq \sigma((D_a \cup D_b) \cap D'_c) \\ &= |\delta((D_a \cup D_b) \cap D'_c)| - r((D_a \cup D_b) \cap D'_c)| \\ &\geq |\delta(D_a \cap D'_c)| + |\delta(D_b \cap D'_c)| - \max\{r(D_a \cap D'_c), r(D_b \cap D'_c)\} \\ &\geq \min\{|\delta(D_a \cap D'_c)|, |\delta(D_b \cap D'_c)|\}\,.\end{aligned}$$

Therefore, either $|\delta(D_a \cap D'_c)| \leq 1$ or $|\delta(D_b \cap D'_c)| \leq 1$. We may choose the labelling of a and b so that the former holds. Since $b \in D_a \cap D'_c$, sb shows $|\delta(D_a \cap D'_c)| \geq 1$, so we have $|\delta(D_a \cap D'_c)| = 1$. Therefore, sb is an isthmus, completing the proof in Case 1.

Case 2: *For every one of the pairs (D_a, D_b), (D_a, D_c), and (D_b, D_c), (2.1.2) holds in Lemma 2.1.*

The assumption of the case implies that, for example,

$$\sigma(D_a \setminus D_b) + \sigma(D_b \setminus D_a) + 2|\delta(D_a \cap D_b, V(G) \setminus (D_a \cup D_b))| \leq \sigma(D_a) + \sigma(D_b) \leq 2\,.$$

Since $c \in D_a \cap D_b$ and $s \in V(G) \setminus (D_a \cup D_b)$, $|\delta(D_a \cap D_b, V(G) \setminus (D_a \cup D_b))| \geq 1$. We conclude that $|\delta(D_a \cap D_b, V(G) \setminus (D_a \cup D_b))| = 1$, $\sigma(D_a \setminus D_b) = 0$, and $\sigma(D_b \setminus D_a) = 0$.

As in the preceding paragraph, since $b \in (D_a \setminus D_b) \cap D_c$, we see that $|\delta((D_a \setminus D_b) \cap D_c, V(G) \setminus ((D_a \setminus D_b) \cup D_c))| \geq 1$. Also, $\sigma(D_a \setminus D_b) = 0$ and $\sigma(D_c) \leq 1$. Thus, Lemma 2.1 (2.1.2) does not hold for $A = D_a \setminus D_b$ and $B = D_c$. Therefore (2.1.1) holds in Lemma 2.1; in particular, $\sigma((D_a \setminus D_b) \cup D_c) \leq \sigma(D_a \setminus D_b) + \sigma(D_c) \leq 1$. That is, $(D_a \setminus D_b) \cup D_c$ is dangerous. Since this does not contain c, we could set $D'_c = (D_a \setminus D_b) \cup D_c$ and conduct this argument over again. When we do this, $D_a \setminus D_b \subseteq D'_c$, so we may assume this happens in the first place. That is, we may assume $D_a \setminus D_b \subseteq D_c$; likewise, we may assume $D_c \setminus D_a \subseteq D_b$, and $D_b \setminus D_c \subseteq D_a$.

We still have $|\delta(D_a \cap D_b, V(G) \setminus (D_a \cup D_b))| = 1$. Likewise both $|\delta(D_a \cap D_c, V(G) \setminus (D_a \cup D_c))| = 1$ and $|\delta(D_b \cap D_c, V(G) \setminus (D_b \cup D_c))| = 1$ hold. In particular, we know there is only one edge from s to each of a, b, and c. Also, it follows that $|\delta(D_a \cup D_b \cup D_c, V(G) \setminus (\{s\} \cup D_a \cup D_b \cup D_c))| = 0$.

If s is not incident with an isthmus, then, for every component K of $G - s$, $|\delta(V(K))| \geq 2$. Since $|\delta(D_a \cup D_b \cup D_c)| = 3$ and all edges in $\delta(D_a \cup D_b \cup D_c)$ are also incident with s, we conclude that $G[D_a \cup D_b \cup D_c]$ is connected and is a component of $G - s$. Therefore, there are two edge-disjoint as-paths in $G[\{s\} \cup D_a \cup D_b \cup D_c]$.

If the degree of s is not 3, then we conclude that $G - s$ has at least two components. If K is a component of $G - s$ other than $G[D_a \cup D_b \cup D_c]$ and s is not incident with an isthmus, then, for any neighbour t of s in K, there are two edge-disjoint ts-paths in $G[\{s\} \cup V(K)]$. It follows that there are two edge-disjoint at-paths in G, showing that $r(D_a \cup D_b \cup D_c) \geq 2$.

Since $|\delta(D_a \cup D_b \cup D_c)| = 3$, we conclude that $\sigma(D_a \cup D_b \cup D_c) \leq 1$. Thus, $D_a \cup D_b \cup D_c$ is dangerous, as required. ∎

The proof of Theorem 1.1 is now quite simple.

Proof of Theorem 1.1. We proceed by induction on $|F|$, with the cases $|F| = 2$ and 3 being, respectively, trivial and an immediate consequence of Lemma 2.2. So assume $|F| \geq 4$, with $F = \{su_1, su_2, \ldots, su_k\}$. By induction, there are dangerous sets A_{k-1} and A_k containing, respectively, all of $F \setminus \{su_{k-1}\}$ and $F \setminus \{su_k\}$. If either $u_{k-1} \in A_{k-1}$ or $u_k \in A_k$, then we are done, so we may assume neither of these containments occurs.

Because su_{k-1} and su_k do not make a feasible lift, there is a dangerous set A containing both u_{k-1} and u_k; among all such dangerous sets, we choose A to be maximal. If, for every $i \in \{1, 2, \ldots, k-2\}$, $u_i \in A$, then we are done. Otherwise, there is some $i \in \{1, 2, \ldots, k-2\}$ such that $u_i \notin A$.

We apply Lemma 2.2 to the pairs $\{u_i, u_{k-1}\}$, $\{u_i, u_k\}$, and $\{u_{k-1}, u_k\}$ and the sets A, A_{k-1}, and A_k. We conclude that there is a dangerous set A^* containing all of u_i, u_{k-1}, and u_k and also containing one of A, A_{k-1}, and A_k.

If $A \subseteq A^*$, then, since $u_i \in A^* \setminus A$, we contradict the maximality of A. Therefore, either A_{k-1} or A_k is contained in A^*, from which we conclude that every u_j is in A^*, as required. ■

3 Connection in the lifting graph

In this section, we prove Theorem 1.2 dealing with the structure of the k-lifting graph $L(G, s, k)$.

The proofs are inductive and the base cases $\deg(s) = 4$ or 5 require some effort. There is one special argument needed for $\deg(s) = 6$ when k is odd. The inductive arguments are based on the following simple observation and its contrapositive.

Observation 3.1 *If, after lifting the feasible pair $\{e_1, e_2\}$, the pair $\{e_3, e_4\}$ is feasible, then $\{e_3, e_4\}$ is feasible in the original graph.* □

3.1 Some general arguments

In this subsection, we give a few elementary general arguments used later for describing the lifting graph. The first arguments are based on standard methods for "crossing cuts". Let A_1 and A_2 be two subsets of $V(G)$. It is an easy exercise to verify that, where $\overline{A} = V(G) \setminus A$,

$$2\Big[|\delta(A_1)| + |\delta(A_2)| - \big(|\delta(A_1 \cap A_2, \overline{A_1 \cup A_2})| + |\delta(A_2 \setminus A_1, A_1 \setminus A_2)|\big)\Big] = \tag{3.1}$$

$$|\delta(A_1 \cap A_2)| + |\delta(A_2 \setminus A_1)| + |\delta(A_1 \setminus A_2)| + |\delta(\overline{A_1 \cup A_2})|\,.$$

A typical application will be when all four sets $A_1 \cap A_2$, $A_2 \setminus A_1$, $A_1 \setminus A_2$, and $\overline{A_1 \cup A_2}$ are non-empty and G is k-edge-connected. In that case, the right-hand side is at least $4k$. If, for example, both $\delta(A_1)$ and $\delta(A_2)$ have size k, we deduce that $\delta(A_1 \cap A_2, \overline{A_1 \cup A_2})$ and $\delta(A_2 \setminus A_1, A_1 \setminus A_2)$ are both empty. Furthermore, it is a routine exercise to verify that this extreme case can only occur with k even.

We will apply a slightly more sophisticated consequence of Equation 3.1.

Lemma 3.2 *Let k be a natural number, and let G be a graph with a vertex s such that any two vertices in $G - s$ are joined by k pairwise edge-disjoint paths in G. For $i = 1, 2$, let F_i be an independent set in $L(G, s, k)$ of size r_i and suppose there is a dangerous set A_i so that $F_i = \delta(\{s\}) \cap \delta_G(A_i)$. Set $\alpha = |F_1 \cap F_2|$. If $\alpha > 0$, $r_1 > \alpha$, $r_2 > \alpha$, and $\overline{A_1 \cup A_2 \cup \{s\}} \neq \varnothing$, then $r_1 + r_2 \leq \lfloor \deg(s)/2 \rfloor + 2$.*

Proof. Observe that: $|\delta_{G-s}(A_1)| \leq k + 1 - r_1$; $|\delta_{G-s}(A_2)| \leq k + 1 - r_2$; $|\delta_{G-s}(A_1 \cap A_2)| \geq k - \alpha$; $|\delta_{G-s}(A_2 \setminus A_1)| \geq k - (r_2 - \alpha)$; $|\delta_{G-s}(A_1 \setminus A_2)| \geq k - (r_1 - \alpha)$; and $|\delta_{G-s}(V(G - s) \setminus (A_1 \cup A_2))| \geq k - (\deg(s) - (r_1 + r_2 - \alpha))$.

From Equation 3.1, we deduce that

$$2(k+1-r_1+k+1-r_2) \geq (k-\alpha)+(k-(r_2-\alpha))+(k-(r_1-\alpha))+(k-(\deg(s)-(r_1+r_2-\alpha)))\,.$$

Rearranging, we see that $\deg(s) + 4 \geq 2(r_1 + r_2)$. Since every term except possibly $\deg(s)$ is even, $\lfloor \deg(s)/2 \rfloor + 2 \geq r_1 + r_2$, as required. ∎

Our final preliminary result gives our first glimpse of some structure in $L(G, s, k)$.

Lemma 3.3 *Let k be a natural number, and let G be a graph with a vertex s such that any two vertices in $G - s$ are joined by k pairwise edge-disjoint paths in G. If $\deg(s)$ is at least 4, then:*

(3.3.1) every independent set in $L(G, s, k)$ has size at most $\lceil \frac{1}{2} \deg(s) \rceil$; and

(3.3.2) if $\deg(s)$ is even and at least 6, then any two distinct independent sets in $L(G, s, k)$ of size $\frac{1}{2} \deg(s)$ are disjoint.

Proof. By Theorem 1.1, an independent set F corresponds to a dangerous set A containing all the non-s ends of the edges in F, so $|\delta(A)| \leq k + 1$. If $|\delta(\{s\}) \setminus F| < |F| - 1$, then $\delta(A \cup \{s\})$ has size at most $k - 1$, a contradiction. Thus, $|\delta(\{s\}) \setminus F| \geq |F| - 1$, as required for (3.3.1).

Suppose F_1 and F_2 are non-disjoint independent sets of size $\frac{1}{2} \deg(s)$, with corresponding dangerous sets A_1 and A_2. At most $\deg(s) - 1$ of the edges of $\delta(\{s\})$ have one end in

$A_1 \cup A_2$, so $\overline{A_1 \cup A_2 \cup \{s\}} \neq \varnothing$. Also, each of $A_1 \cap A_2$, $A_2 \setminus A_1$, and $A_1 \setminus A_2$ has an end of an edge in $F_1 \cup F_2$. Since, for $i = 1, 2$, Lemma 3.3 (3.3.1) implies $F_i = \delta(\{s\}) \cap \delta(A_i)$, the hypotheses of Lemma 3.2 are satisfied. However, $r_1 = \frac{1}{2}\deg(s) = r_2$, showing the conclusion of Lemma 3.2 fails, a contradiction that proves (3.3.2). ■

3.2 $\deg(s) = 4$

In this subsection, we treat the case $\deg(s) = 4$. Let e_1, e_2, e_3, e_4 be the four edges incident with s. It is a triviality that if some pair, say e_1, e_2 is feasible, then so is the complementary pair e_3, e_4. It follows that $L(G, s, k)$ is a union of perfect matchings; Mader's Theorem already shows there is at least one such matching in $L(G, s, k)$. Since it has only four vertices, it can only be one of: a perfect matching; a 4-cycle C_4; and K_4. These are all realizable. However, when k is even, the perfect matching is not achievable, as we show next.

Proposition 3.4 *Let k be a natural number, and let G be a graph with a vertex s such that any two vertices in $G - s$ are joined by k pairwise edge-disjoint paths in G. If $\deg(s) = 4$, then $L(G, s, k)$ is one of: a perfect matching; C_4; and K_4. If k is even, then $L(G, s, k)$ is not a perfect matching.*

Proof. We only prove the second assertion. Suppose both pairs e_1, e_2 and e_1, e_3 are not feasible. Then there are dangerous sets A_2 and A_3 so that the non-s ends of e_1, e_2 are in A_2 and the non-s ends of e_1, e_3 are in A_3.

By definition, $|\delta_G(A_2)| \leq k + 1$, while the hypothesis implies $|\delta_G(A_2 \cup \{s\})| \geq k$. Therefore, e_3 and e_4 have their non-s ends in $\overline{A}_2 = V(G) \setminus (A_2 \cup \{s\})$. The analogous statement holds for A_3.

It follows that $|\delta_{G-s}(A_2 \cap A_3)|$, $|\delta_{G-s}(A_2 \setminus A_3)|$, $|\delta_{G-s}(A_3 \setminus A_2)|$, and $|\delta_{G-s}(\overline{A_2 \cup A_3})|$ are all at least $k-1$, while $|\delta_{G-s}(A_2)|$ and $|\delta_{G-s}(A_3)|$ are both at most $k-1$. But $k-1$ is odd, so Equation 3.1 cannot be realized (as mentioned in the paragraph following Equation 3.1). ■

We comment that the proofs of Proposition 3.4 and Equation 3.1 also imply that, when k is odd, there is only one pattern for G for which $L(G, s, k)$ is a perfect matching; this is illustrated in Figure 3.5, where there are four edges incident with s and the thick edges represent $(k-1)/2$ edges. No two edges consecutive in the illustrated cyclic rotation at s form a feasible pair.

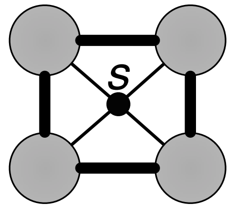

Figure 3.5: Each thick edge represents $(k-1)/2$ edges.

3.3 $\deg(s) = 5$

In this subsection, we prove the following, dealing with the case $\deg(s) = 5$.

Proposition 3.6 *Let k be a natural number, and let G be a graph with a vertex s such that any two vertices in $G - s$ are joined by k pairwise edge-disjoint paths in G. If $\deg(s) = 5$, then $L(G, s, k)$ is either an isolated vertex plus a 4-cycle or a connected graph. If k is even and $L(G, s, k)$ is connected, then G is a complete multipartite graph.*

Proof. Lemma 3.3 (3.3.1) implies the largest independent set in $L(G, s, k)$ has size at most 3. We break the proof into two cases.

Case 1: *$L(G, s, k)$ contains an independent set of size 3.*

Let F be an independent set in $L(G, s, k)$ of size 3 and let A_1 be a dangerous set in G so that the non-s ends of the edges in F are all in A_1. As there are only two edges incident with s and not in F, they both have their non-s ends in $\overline{A}_1 = V(G) \setminus (A_1 \cup \{s\})$. In particular, $|\delta_G(A_1)| = k + 1$ and $|\delta_G(A_1 \cup \{s\})| = k$, so the two edges in $\delta(\{s\}) \setminus F$ are also independent in $L(G, s, k)$.

Suppose $e_1 \in F$ and $e_2 \in \delta(\{s\}) \setminus F$ do not form a feasible pair and let A_2 be a dangerous set that witnesses this. As in the preceding paragraph, there are at least two edges in $\delta(\{s\}) \setminus \{e_1, e_2\}$ having their non-s ends in $\overline{A}_2$; at least one of these is in $F \setminus \{e_1\}$.

Thus, there is at least one edge from s to each of $A_1 \cap A_2$ (namely, e_1), $A_2 \setminus A_1$ (e_2), and $A_1 \setminus A_2$ (the one at the end of the preceding paragraph).

If $\overline{A_1 \cup A_2 \cup \{s\}} \neq \varnothing$, then Lemma 3.2 implies $3 + |\delta(\{s\}) \cap \delta(A_2)| \leq 4$. But $e_1, e_2 \in \delta(\{s\}) \cap \delta(A_2)$, so we deduce that $\overline{A_1 \cup A_2 \cup \{s\}} = \varnothing$.

It follows that both edges in $\delta(\{s\}) \setminus F$ have their non-s ends in $A_2 \setminus A_1$. Thus, $|\delta(\{s\}) \cap \delta(A_2)| \geq 3$. Since A_2 is dangerous, Lemma 3.3 implies $|\delta(\{s\}) \cap \delta(A_2)| \leq 3$. Therefore there are also two edges in $\delta(\{s\})$ with ends in $A_1 \setminus A_2$.

An immediate consequence of the preceding is that e_1 has no feasible lift with any other edge in $\delta(\{s\})$. Frank's Theorem implies that there is at most one edge incident with s that is not in any feasible pair. It follows that e_1 is the only such edge; now applying the above argument to another edge e_1' in $F \setminus \{e_1\}$ and an edge e_2 in $\delta(\{s\}) \setminus F$ shows e_1', e_2 is a feasible pair.

We conclude that, in the event there is an independent set of size 3 in $L(G, s, k)$, $L(G, s, k)$ is either $K_{2,3}$ or an isolated vertex plus C_4.

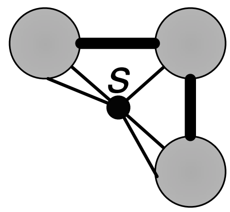

Figure 3.7: If each thick edge represents $k - 2$ edges, then $L(G, s, k)$ is an isolated vertex and C_4. Changing one thick edge to $k - 1$ edges turns $L(G, s, k)$ into $K_{2,3}$.

Case 2: *every independent set in $L(G, s, k)$ has size at most 2.*

Suppose there are three edges e_0, e_1, e_2 in $\delta(\{s\})$ such that neither e_0, e_1 nor e_0, e_2 is a feasible pair.

(F1) *The assumption of this case implies e_1, e_2 is a feasible pair.*

For $i = 1, 2$, let A_i be a dangerous set containing the non-s ends of both e_0 and e_i. Because we are in Case 2, none of the three edges in $\delta(\{s\}) \setminus \{e_0, e_i\}$ has an end in A_i. Thus, each of these three edges has an end in $\overline{A_i \cup \{s\}}$. Since these three edges do not make an independent set in $L(G, s, k)$, $|\delta(\overline{A_i \cup \{s\}})| > k + 1$. Evidently, $|\delta(A_i)| \leq k + 1$, so $|\delta(A_i)| = k + 1$.

Moreover, there is precisely one edge from $\delta(\{s\})$ having an end in each of $A_1 \cap A_2$ (e_0), $A_2 \setminus A_1$ (e_2), and $A_1 \setminus A_2$ (e_1). Therefore, the remaining two edges have their non-s ends in $\overline{A_1 \cup A_2 \cup \{s\}}$.

Since $\{e_0, e_1, e_2\}$ is not an independent set of size 3, $|\delta_G(A_1 \cup A_2)| \geq k + 2$. Thus, each of $\delta_{G-s}(A_1 \cap A_2)$, $\delta_{G-s}(A_2 \setminus A_1)$, $\delta_{G-s}(A_1 \setminus A_2)$, and $\delta_{G-s}(\overline{A_1 \cup A_2 \cup \{s\}})$ has size at least $k - 1$ (as this is trivially true for the first three). Since $\delta_{G-s}(A_1)$ and $\delta_{G-s}(A_2)$ have size precisely $k - 1$, as before from Equation 3.1, $k - 1$ is even.

It follows that, for k even, e_0, e_1, and e_2 do not exist, so $L(G, s, k)$ is complete multipartite.

In the case k is odd, $|\delta_G(\overline{A_1 \cup A_2 \cup \{s\}})| = k + 1$, showing the following.

(F2) *The pair e_3, e_4 of edges in $\delta(\{s\}) \setminus \{e_0, e_1, e_2\}$ is not feasible.*

Subcase 2.1: *e_1, e_3 is not feasible.*

Applying (F1) to e_1, e_0 and e_1, e_3, we see that e_0, e_3 is a feasible pair.

On the other hand, (F2) implies the pair of edges e_2, e_4 in $\delta(\{s\}) \setminus e_1, e_0, e_3$ is not feasible. Now using e_2, e_0 and e_2, e_4, we conclude from (F1) that e_0, e_4 is feasible.

Finally, (F1) and the infeasible pairs e_3, e_1 and e_3, e_4 show e_1, e_4 is feasible, and analogously e_2, e_3 is feasible. In this case, $L(G, s, k)$ is C_5.

Subcase 2.2: *no version of Subcase 2.1; that is, $\{e_1, e_2, e_3, e_4\}$ induces $K_4 - e_3e_4$ in $L(G, s, k)$.*

(We remark that this subcase occurs in the version of Figure 3.8 with one thick edge being $(k+1)/2$ edges.) Suppose e_0, e_3 is not a feasible pair. Then (F2) applied to e_0, e_1, e_3 yields the contradiction that e_2, e_4 is not feasible. Therefore, e_0, e_3 and, symmetrically, e_0, e_4, are feasible pairs. In this final case, $L(G, s, k)$ is $K_5 - \{e_0e_1, e_0e_2, e_3e_4\}$. ■

Figure 3.8 gives two examples for odd k. One has $L(G, s, k)$ being a 5-cycle, while, for the other, $L(G, s, k)$ is $K_5 - \{e_0e_1, e_0e_2, e_3e_4\}$.

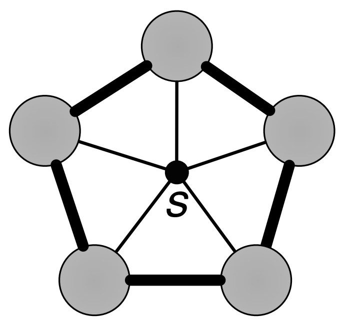

Figure 3.8: If each thick edge represents $(k-1)/2$ edges, then $L(G, s, k) = C_5$. Changing one thick edge to $(k+1)/2$ edges turns $L(G, s, k)$ into $K_5 - \{e_0e_1, e_0e_2, e_3e_4\}$.

3.4 The inductive step

In this subsection, we proceed with the induction to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. For (1.2.1), we observe that if $\deg(s) = 4$ or 5, then $L(G, s, k)$ has at most two components. For the induction, suppose $\deg(s) \geq 6$. If $L(G, s, k)$ has more than two components, then it is the union of three subgraphs J_1, J_2, J_3, with each J_i a union of components of $L(G, s, k)$.

Suppose, for some $i \in \{1, 2, 3\}$, J_i has at least three vertices. Frank's Theorem implies J_i has an edge e_1e_2. Lifting e_1e_2 produces a graph G' with $\deg_{G'}(s) = \deg_G(s) - 2$ and there is no edge of $L(G', s, k)$ between any two of the $J_j \cap L(G', s, k)$. This contradicts the inductive assumption that $L(G', s, k)$ has at most two components.

Therefore, each J_i has at most two vertices; since $\deg(s) \geq 6$, each J_i has precisely two vertices and $\deg(s) = 6$. However, in this case, there are 8 different independent sets of size 3, each consisting of one vertex from each of the J_i. This contradicts Lemma 3.3 (3.3.2), completing the proof of (1.2.1).

For (1.2.2), the claim holds for $\deg(s) = 5$, so suppose $\deg(s) \geq 7$. Let H and J be the components of $L(G, s, k)$ with $|V(H)| < |V(J)|$. Then $|V(J)| \geq 4$ and if we lift an edge from J to get the graph G', there is still no edge between $H \cap L(G', s, k)$ and $J \cap L(G', s, k)$ and the latter has at least two vertices. Thus, $H \cap L(G', s, k)$, and therefore H, has only one vertex, as required.

To see that J is complete multipartite, suppose there exist e_0, e_1, e_2 in $V(J)$ such that e_0 is not adjacent in J to either of e_1 and e_2, while $e_1e_2 \in E(J)$. Lift the pair e_1, e_2 to get the graph G'. Since J has at least 6 vertices, $J \cap L(G', s, k)$ is a component of $L(G', s, k)$ with at least 4 vertices. By the inductive assumption, it is not a star, so it has an edge e_3e_4 not incident with e_0. Then e_3e_4 is an edge of J.

Lift e_3, e_4 in G to get G''; the pair e_1, e_2 is feasible in G'' (the resulting graph is the same as first lifting e_1, e_2 and then lifting e_3, e_4), so e_1e_2 is an edge in $J \cap L(G'', s, k)$. But neither e_0e_1 nor e_0e_2 is an edge in $J \cap L(G'', s, k)$, contradicting the inductive assumption applied to $L(G'', s, k)$. Thus, J is both complete multipartite and not a star, as required.

For (1.2.3), we first prove that every component of $L(G, s, k)$ has an even number of vertices; this is trivial if there is only one component. This is known for $\deg(s) = 4$, so we suppose $\deg(s) \geq 6$. Let H and J be the two components with $|V(H)| \leq |V(J)|$. Let e_1e_2 be an edge of J and let G' be the result of lifting the pair e_1, e_2. Then $H \cap L(G', s, k)$ and $J \cap L(G', s, k)$ are the two components of $L(G', s, k)$. By induction they each have an even number of vertices, so this also holds for $L(G, s, k)$.

If $\deg(s) = 6$, then the induction and Lemma 3.3 (3.3.2) imply that $L(G, s, k)$ is the disjoint union of K_2 and either C_4 or K_4. Therefore, we may assume $\deg(s) \geq 8$.

Case 1: *both components of $L(G, s, k)$ have at least four vertices.*

Suppose by way of contradiction that there are vertices e_0, e_1, e_2 in the component K of $L(G, s, k)$ such that neither e_0e_1 nor e_0e_2 is an edge of K, while e_1e_2 is an edge of K. Let J be the other component of $L(G, s, k)$.

Lift e_1, e_2 to get G'. Then $K \cap L(G', s, k)$ and $J \cap L(G', s, k)$ are the two components of $L(G', s, k)$. Thus, there is an edge e_3e_4 in $J \cap L(G', s, k)$. Now lift e_3, e_4 in G to get G''. Then $K \cap L(G'', s, k)$ is a component of $L(G'', s, k)$. The edge e_1e_2 is in $K \cap L(G'', s, k)$, while neither e_0e_1 nor e_0e_2 is an edge of $K \cap L(G'', s, k)$. This contradicts the inductive assumption that $K \cap L(G'', s, k)$ is complete multipartite.

Case 2: *one component of $L(G, s, k)$ has precisely two vertices.*

Let J and K be the components of $L(G, s, k)$ so that J has precisely two vertices; thus K has at least six vertices. Suppose e_0, e_1, e_2 are vertices of K such that neither e_0e_1 nor e_0e_2 is an edge of K, yet e_1e_2 is an edge of K.

Lift e_1, e_2 to obtain the graph G'. By the induction, $K \cap L(G', s, k)$ is a component of $L(G', s, k)$, and it has at least 4 vertices, so it is not a star. Therefore, it has an edge e_3e_4 disjoint from e_0; we lift e_3, e_4 in G to obtain G''. Induction tells us that $K \cap L(G'', s, k)$ is complete multipartite, which contradicts the fact that e_0, e_1, e_2 are all in $K \cap L(G'', s, k)$, e_0e_1 and e_0e_2 are not edges, and e_1e_2 is an edge.

Lastly, we prove (1.2.4). Proposition 3.4 gives the result for $\deg(s) = 4$, so we assume $\deg(s) \geq 6$. Suppose e_0, e_1, e_2 are vertices in $L(G, s, k)$ such that e_0 is not adjacent to either e_1 or e_2, but e_1e_2 is an edge of $L(G, s, k)$. Lifting e_1, e_2 yields a graph G' for which $L(G', s, k)$ has at least 4 vertices. By induction, $L(G', s, k)$ is connected, complete multipartite, and not a star; in particular, it has an edge e_3e_4 disjoint from e_0.

Lifting e_3, e_4 in G produces a graph G''; by induction $L(G'', s, k)$ is complete multipartite. However, e_0 is still not adjacent to either e_1 or e_2, while e_1e_2 is an edge. This contradiction shows $L(G, s, k)$ is complete multipartite and Frank's Theorem [3] shows it is not a star, as required. ■

We conclude this section with Figure 3.9. This is an example having $\deg(s) = 6$ and $k = 5$ so that $L(G, s, k)$ is $K_{3,3}$ minus an edge; in particular, it is connected and not complete multipartite. The three edges incident with s on the left side are one independent set, the three on the right are a second, and the two going to the bottom are not feasible.

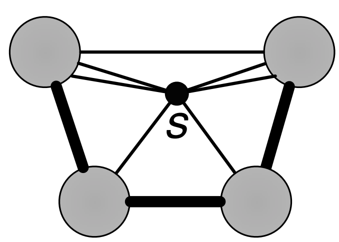

Figure 3.9: Each thick edge represents 2 edges and $k = 5$.

4 Weakly k-linked infinite graphs

In this section we prove Theorem 1.3: if k is odd, then a $(k+2)$-edge-connected, locally finite, 1-ended, infinite graph G is weakly k-linked.

If $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_k)$ are sequences of (not necessarily distinct) vertices in graph G, then an $\mathbf{xy}$-*linkage* is a set $\{P_1, P_2, \ldots, P_k\}$ of pairwise edge-disjoint paths in G such that, for $i = 1, 2, \ldots, k$, P_i is an x_iy_i-path.

Before we prove Theorem 1.3, we require extensions of the theorems of Mader and Frank and of our Theorem 1.2 to locally finite graphs. These extensions may all be proved as follows. Let G_d be the subgraph of a locally finite graph G consisting of those vertices at distance at most d from the specified vertex s. Let G'_d be the graph obtained from G by contracting each component of $G - V(G_d)$ to a vertex. For infinitely many d, the lifting graph $L(G'_d, s, \tau)$ is the same graph; this is the the lifting graph $L(G, s, \tau)$.

Proof of Theorem 1.3. Let $\mathbf{x}$ and $\mathbf{y}$ be any sequences of k (not necessarily distinct) vertices of G. Let A be the set of vertices that occur in $\mathbf{x}$ and $\mathbf{y}$.

Let S be a finite set of vertices containing A. There is a unique infinite component K of $G - S$. Let $\mathcal{P}$ be a largest set of pairwise edge-disjoint, 1-way infinite paths (or *rays*), that begin with an edge in $\delta(V(K))$ and are otherwise contained in K. It is a standard fact that there is a finite set S' containing S such that $|\delta(S')| = |\mathcal{P}|$. We are interested only in S', which we relabel as S, and restrict the rays in $\mathcal{P}$ to begin at their edge in $\delta(S')$.

Because G is $(k+2)$-edge-connected, $|\delta(S)| \geq k+2$. We consider three cases.

Case 1: $|\delta(S)| = k+2$.

Contract $G - S$ to a single vertex v_S, yielding a finite $(k+2)$-edge-connected graph $G/(G-S)$. Huck's Theorem shows there is a weak $\mathbf{xy}$-linkage $\mathcal{L}$ in $G/(G-S)$.

Let v be any vertex of $G - S$. There is a set $\mathcal{L}'$ of $(k+2)$ pairwise edge-disjoint paths with origin v whose other end is in S and incident with an edge of $\delta(S)$. Evidently, we can replace any passage of a path in $\mathcal{L}$ through v_S with an appropriate pair of paths in $\mathcal{L}'$. Simplifying the resulting walks as needed, we convert $\mathcal{L}$ into a weak $\mathbf{xy}$-linkage in G.

Case 2: $|\delta(S)|$ *is odd and at least* $k+4$.

In this case, let e be any edge of $\delta(S)$ and let $G' = G - e$. Now G' is $(k+1)$-edge-connected and $|\delta(S)|$ is even. We now proceed as in Case 3.

Case 3: $|\delta(S)|$ *is even.*

In this case, we need only that G is $(k+1)$-edge-connected (so Case 2 continues smoothly here). Contract $G - S$ to a single vertex v_S resulting in the finite graph G^S.

We claim that $\delta(S)$ partitions into $|\delta(S)|/2$ pairs $\{e_i, e'_i\}$, $i = 1, 2, \ldots, |\delta(S)|/2$, such that, letting $G^S_0 = G^S$ and, for $i = 1, 2, \ldots, |\delta(S)|/2$, G^S_i is the graph obtained from lifting $\{e_i, e'_i\}$ in G^S_{i-1}:

1. for $i \geq 1$, the pair $\{e_i, e'_i\}$ is $(k+1)$-liftable in G^S_{i-1}; and
2. for $i = 1, 2, \ldots, |\delta(S)|/2$, there is a path P_i joining e_i and e'_i with only its end vertices and e_i, e'_i not in $G - S$ such that P_i is edge-disjoint from $P_1 \cup \cdots \cup P_{i-1}$ and from all the rays in $\mathcal{P}$ containing $e_{i+1}, e'_{i+1}, \ldots, e_{|\delta(S)|/2}, e'_{|\delta(S)|/2}$.

Suppose we have the pairs $\{e_1, e'_1\}, \ldots, \{e_{i-1}, e'_{i-1}\}$ and paths $P_1, \ldots, P_{i-1}$. We show the existence of $\{e_i, e'_i\}$ and P_i.

Set $\delta_i(S)$ to be $\delta(S) \setminus \{e_1, e'_1, \ldots, e_{i-1}, e'_{i-1}\}$. These are the edges in $G - \{e_1, e'_1, \ldots, e_{i-1}, e'_{i-1}\}$ having precisely one end in S. Let $\mathcal{P}_i$ denote the paths in $\mathcal{P}$ that do not contain any of the edges in $\{e_1, e'_1, \ldots, e_{i-1}, e'_{i-1}\}$.

There are two graphs with vertex set $\delta_i(S)$ that are relevant to completing the proof.

In the *end graph* $\mathcal{E}_i$, distinct edges e, e' in $\delta_i(S)$ are adjacent if there are infinitely many vertex-disjoint paths in $G - S$ that: (i) join the two paths in $\mathcal{P}_i$ containing e and e'; and (ii) are edge-disjoint from all the other paths in $\mathcal{P}_i$. Since all the paths in $\mathcal{P}_i$ are in the same end, $\mathcal{E}_i$ is connected.

The other graph is the $(k+1)$-lifting graph $\mathcal{L}_i$ for v_S in G^S_{i-1}. By Theorem 1.2 (1.2.4), $\mathcal{L}_i$ is a complete multipartite graph. Therefore, its complement is disconnected.

Since $\mathcal{E}_i$ is connected, there is an edge $e_i e'_i$ of $\mathcal{E}_i$ that is not in the complement of $\mathcal{L}_i$; that is, $e_i e'_i$ is an edge of $\mathcal{L}_i$. This is the required next pair of edges.

Let Q and Q' be the rays in $\mathcal{P}$ containing e_i and e'_i, respectively. Because $e_i e'_i$ is an edge of $\mathcal{E}_i$, there are infinitely many vertex-disjoint paths in G joining Q and Q' that are edge-disjoint from the other rays in $\mathcal{P}_i$. Let P be one of these contained in $G - S$ that is also disjoint from all of the finitely many finite paths $P_1, \ldots, P_{i-1}$. Then $Q \cup P \cup Q'$ contains a path P_i containing e_i, and e'_i. This is the required next path.

The choices of the lifts $\{e_i, e'_i\}$ show that $G^S_{|\delta(S)|/2}$ is $(k+1)$-connected. Huck's Theorem shows that $G^S_{|\delta(S)|/2}$ has an **xy**-linkage $\mathcal{Q}$.

An occurrence of the lift of $\{e_i, e'_i\}$ in some path in $\mathcal{Q}$ can be replaced by P_i. This converts $\mathcal{Q}$ into an **xy**-linkage in G, as required. ■

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