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# Amalgams and -Boundedness

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# Amalgams and $\chi$-boundedness

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### Abstract

A class of graphs is *hereditary* if it is closed under isomorphism and induced subgraphs. A class $\mathcal{G}$ of graphs is $\chi$-*bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for all graphs $G \in \mathcal{G}$, and all induced subgraphs $H$ of $G$, we have that $\chi(H) \leq f(\omega(H))$. We prove that proper homogeneous sets, clique-cutsets, and amalgams together preserve $\chi$-boundedness. More precisely, we show that if $\mathcal{G}$ and $\mathcal{G}^*$ are hereditary classes of graphs such that $\mathcal{G}$ is $\chi$-bounded, and such that every graph in $\mathcal{G}^*$ either belongs to $\mathcal{G}$ or admits a proper homogeneous set, a clique-cutset, or an amalgam, then the class $\mathcal{G}^*$ is $\chi$-bounded. This generalizes a result of [*J. Combin. Theory Ser. B*, 103(5):567–586, 2013], which states that proper homogeneous sets and clique-cutsets together preserve $\chi$-boundedness, as well as a result of [*European J. Combin.*, 33(4):679–683, 2012], which states that 1-joins preserve $\chi$-boundedness. The *house* is the complement of the four-edge path. As an application of our result and of the decomposition theorem for "cap-free" graphs from [*J. Graph Theory*, 30(4):289–308, 1999], we obtain that if $G$ is a graph that does not contain any subdivision of the house as an induced subgraph, then $\chi(G) \leq 3^{\omega(G)-1}$.

**AMS Classification:** 05C15, 05C75

**Keywords:** hereditary classes, $\chi$-bounded classes, graph decompositions, amalgam, house

## 1 Introduction

All graphs in this paper are simple and finite (possibly null). We denote by $\mathbb{N}$ the set of all non-negative integers and by $\mathbb{N}^+$ the set of all positive

integers. The vertex-set and the edge-set of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. A *clique* of a graph $G$ is a (possibly empty) set of pairwise adjacent vertices of $G$, and a *stable set* of $G$ is a (possibly empty) set of pairwise non-adjacent vertices of $G$. We denote by $\chi(G)$ the chromatic number of $G$, and by $\omega(G)$ the clique number of $G$ (i.e. the maximum size of a clique of $G$). A class of graphs is *hereditary* if it is closed under isomorphism and induced subgraphs. A class $\mathcal{G}$ of graphs is said to be $\chi$-*bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for all graphs $G \in \mathcal{G}$, and all induced subgraphs $H$ of $G$, we have that $\chi(H) \leq f(\omega(H))$. Under these circumstances, we also say that $\mathcal{G}$ is $\chi$-bounded *by* the function $f$, and that $f$ is a $\chi$-*bounding function* for $\mathcal{G}$. Note that for every $\chi$-bounded class $\mathcal{G}$, there exists a non-decreasing $\chi$-bounding function for $\mathcal{G}$; indeed, if $\mathcal{G}$ is $\chi$-bounded by $f : \mathbb{N} \to \mathbb{N}$, then $\mathcal{G}$ is also $\chi$-bounded by the non-decreasing function $g : \mathbb{N} \to \mathbb{N}$ given by $n \mapsto \max\{f(m) \mid 0 \leq m \leq n\}$. Furthermore, note that if $\mathcal{G}$ is a hereditary class, then $\mathcal{G}$ is $\chi$-bounded by a function $f : \mathbb{N} \to \mathbb{N}$ if and only if all graphs $G \in \mathcal{G}$ satisfy $\chi(G) \leq f(\omega(G))$. Not all hereditary classes are $\chi$-bounded. For instance, it is well-known that triangle-free graphs can have an arbitrarily large chromatic number [20, 25], and consequently, the class of triangle-free graphs is not $\chi$-bounded. Note that this implies that the class of all graphs is not $\chi$-bounded.

$\chi$-Bounded classes were introduced by Gyárfás [16] as a generalization of the class of "perfect" graphs. A graph $G$ is *perfect* if all induced subgraphs $H$ of $G$ satisfy $\chi(H) = \omega(H)$. Thus, the class of perfect graphs is $\chi$-bounded by the identity function. Perfect graphs were introduced in the 1960s by Berge [2], who also made the famous Strong Perfect Graph Conjecture, which states that a graph $G$ is perfect if and only if neither $G$ nor its complement contains an induced odd cycle of length at least five. This conjecture was proven more than four decades later, and it is now known as the Strong Perfect Graph Theorem [8]. Over the course of those four decades, many theorems of the following form were proven: if all proper induced subgraphs of a graph $G$ are perfect, and $G$ admits a particular graph decomposition, then $G$ is also perfect. (Under these circumstances, the decomposition in question is said to *preserve perfection*.) Decompositions considered in this context include clique-cutsets [15], star-cutsets [9], homogeneous sets [19], homogeneous pairs [10], 2-joins [12], amalgams [3], and others. Some of these theorems were crucial for the proof of the Strong Perfect Graph Conjecture.

For many classes $\mathcal{G}^*$ defined by forbidding an induced subgraph or a family of induced subgraphs, there is a decomposition theorem of the following form: every graph in $\mathcal{G}^*$ either belongs to some well-understood "basic" class $\mathcal{G}$ or admits one of several graph "decompositions." This raises the following question: which graph decompositions preserve $\chi$-boundedness? Formally,

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let us say that a decomposition $D$ *preserves $\chi$-boundedness* provided that for all hereditary classes $\mathcal{G}$ and $\mathcal{G}^*$ such that $\mathcal{G}$ is $\chi$-bounded, and such that every graph in $\mathcal{G}^*$ either belongs to $\mathcal{G}$ or admits the decomposition $D$, we have that the class $\mathcal{G}^*$ is $\chi$-bounded (however, the optimal $\chi$-bounding functions for $\mathcal{G}$ and $\mathcal{G}^*$ need not be the same). It is a routine exercise to show that clique-cutsets preserve $\chi$-boundedness. It is also known that cutsets of size at most $k$ preserve $\chi$-boundedness [1, 7, 22] (note that this decomposition does not preserve perfection), as do proper homogeneous sets [7] and 1-joins [14]. (All these decompositions are defined formally later in this section.) Obvious "candidates" for further study in this context are decompositions that are known to preserve perfection. In the present paper, we add the "amalgam" decomposition to the list of decompositions known to preserve $\chi$-boundedness (see Theorem 1.2).

Given graph decompositions $D_1, \ldots, D_k$, we say that $D_1, \ldots, D_k$ *together preserve $\chi$-boundedness* provided that for all hereditary classes $\mathcal{G}$ and $\mathcal{G}^*$ such that $\mathcal{G}$ is $\chi$-bounded, and such that every graph in $\mathcal{G}^*$ either belongs to $\mathcal{G}$ or admits at least one of the decompositions $D_1, \ldots, D_k$, we have that the class $\mathcal{G}^*$ is $\chi$-bounded (but again, the optimal $\chi$-bounding functions for $\mathcal{G}$ and $\mathcal{G}^*$ need not be the same). Since the preservation of $\chi$-boundedness does not entail the preservation of $\chi$-boundedness *by the same $\chi$-bounding function*, the fact that each of the decompositions $D_1, \ldots, D_k$ individually preserves $\chi$-boundedness does not imply that they together preserve it. Indeed, suppose that $D_1$ and $D_2$ are graph decompositions that individually preserve $\chi$-boundedness, and let $\mathcal{G}$ be a hereditary, $\chi$-bounded class. Set $\mathcal{G}_0 = \mathcal{G}$, and for all $i \in \mathbb{N}^+$, let $\mathcal{G}_{2i-1}$ be a hereditary class such that every graph in $\mathcal{G}_{2i-1}$ either belongs to $\mathcal{G}_{2i-2}$ or admits the decomposition $D_1$, and let $\mathcal{G}_{2i}$ be a hereditary class such that every graph in $\mathcal{G}_{2i}$ either belongs to $\mathcal{G}_{2i-1}$ or admits the decomposition $D_2$. Set $\mathcal{G}^* = \bigcup\limits_{i \in \mathbb{N}} \mathcal{G}_i$; then $\mathcal{G}^*$ is a hereditary class, and every graph in $\mathcal{G}^*$ either belongs to $\mathcal{G}$ or admits one of the decompositions $D_1$ and $D_2$. Now, since $D_1$ and $D_2$ individually preserve $\chi$-boundedness, we know that each $\mathcal{G}_i$ is $\chi$-bounded by some function $f_i : \mathbb{N} \to \mathbb{N}$. However, the $f_i$'s need not be the same, and the sequence of functions $\{f_i\}_{i \in \mathbb{N}}$ need not have an upper bound. Consequently, we cannot in general guarantee that $\mathcal{G}^*$ is $\chi$-bounded. Proposition 1.1 (see below) gives an example of this sort of behavior. We remark that Proposition 1.1 is an adaptation of the construction given in section 4 of [7]. The results of [7] concern operations that preserve $\chi$-boundedness, and here, we "translate" the example from that paper into the language of graph decompositions.

**Proposition 1.1.** *Let $D_{\mathrm{odd}}$ be the property of having an odd chromatic number, and let $D_{\mathrm{even}}$ be the property of having an even chromatic number. Then $D_{\mathrm{odd}}$ and $D_{\mathrm{even}}$ individually preserve $\chi$-boundedness, but they do not preserve it together.*

*Proof.* The fact that $D_{\mathrm{odd}}$ and $D_{\mathrm{even}}$ do not together preserve $\chi$-boundedness follows immediately from the fact that every graph has either an odd or an even chromatic number, and the class of all graphs is not $\chi$-bounded. More formally, one can let $\mathcal{G}$ be the empty class and $\mathcal{G}^*$ be the class of all graphs. Then $\mathcal{G}$ and $\mathcal{G}^*$ are hereditary, $\mathcal{G}$ is $\chi$-bounded (by any function $f : \mathbb{N} \to \mathbb{N}$), and every graph in $\mathcal{G}^*$ either belongs to $\mathcal{G}$ or admits $D_{\mathrm{odd}}$ or $D_{\mathrm{even}}$. However, $\mathcal{G}^*$ is not $\chi$-bounded.

It remains to show that $D_{\mathrm{odd}}$ and $D_{\mathrm{even}}$ individually preserve $\chi$-boundedness. Let us prove this for $D_{\mathrm{odd}}$ (the proof for $D_{\mathrm{even}}$ is analogous). Fix hereditary classes $\mathcal{G}$ and $\mathcal{G}^{\mathrm{odd}}$ such that $\mathcal{G}$ is $\chi$-bounded, and every graph in $\mathcal{G}^{\mathrm{odd}}$ either belongs to $\mathcal{G}$ or admits $D_{\mathrm{odd}}$ (that is, has an odd chromatic number). Let $f : \mathbb{N} \to \mathbb{N}$ be a non-decreasing $\chi$-bounding function for $\mathcal{G}$; we claim that $\mathcal{G}^{\mathrm{odd}}$ is $\chi$-bounded by $f + 1$. Suppose that this is not the case, and fix some $G \in \mathcal{G}^{\mathrm{odd}}$ such that $\chi(G) \geq f(\omega(G)) + 2$. Since $\mathcal{G}$ is $\chi$-bounded by $f$ and $\chi(G) > f(\omega(G))$, we see that $G \notin \mathcal{G}$. Thus, $G$ has an odd chromatic number. Let $H$ be an induced subgraph of $G$ such that $\chi(H) = \chi(G) - 1$. Now, since $f$ is non-decreasing, we have that $\chi(H) = \chi(G) - 1 \geq f(\omega(G)) + 1 \geq f(\omega(H)) + 1$, and so since $\mathcal{G}$ is $\chi$-bounded by $f$, we see that $H \notin \mathcal{G}$. On the other hand, since $\chi(H) = \chi(G) - 1$ and $\chi(G)$ is odd, we see that $\chi(H)$ is even. Since $H \notin \mathcal{G}$ and $\chi(H)$ is even, we deduce that $H \notin \mathcal{G}^{\mathrm{odd}}$. But this is impossible because $H$ is an induced subgraph of $G \in \mathcal{G}^{\mathrm{odd}}$, and $\mathcal{G}^{\mathrm{odd}}$ is a hereditary class. This proves that $\mathcal{G}^{\mathrm{odd}}$ is indeed $\chi$-bounded by $f + 1$, and it follows that $D_{\mathrm{odd}}$ preserves $\chi$-boundedness. $\square$

One might object that the example from Proposition 1.1 is somewhat artificial, and that $D_{\mathrm{odd}}$ and $D_{\mathrm{even}}$ are not "true" graph decompositions. Nevertheless, Proposition 1.1 demonstrates that one should not assume that the fact that two or more decompositions individually preserve $\chi$-boundedness implies that they together preserve it. (We remark that this problem does not arise in the context of perfect graphs: if decompositions $D_1, \ldots, D_k$ individually preserve perfection, then it is easy to see that they together preserve it.) In [7], it was shown that cutsets of size at most $k$ and clique-cutsets together preserve $\chi$-boundedness, as well as that proper homogeneous sets and clique-cutsets together preserve $\chi$-boundedness. In the present paper, we prove that proper homogeneous sets, clique-cutsets, and amalgams together preserve $\chi$-boundedness (see Theorem 1.2). These three decompositions, as well as the standard ways of decomposing graphs that admit them into smaller "blocks of decomposition," are represented in Figures 1.1, 1.2, and 1.3; formal definitions are given below. (We remark that in our figures, shaded circles represent cliques, a straight line between two circles indicates that all possible edges between the sets of vertices represented by those circles are present, a wavy line between two circles indicates arbitrary ad-
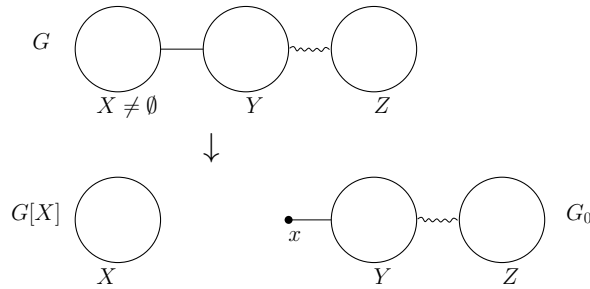
Figure 1.1: Graph $G$ and homogeneous set partition $(X, Y, Z)$ of $G$. The homogeneous set $X$ of $G$ is *proper* if $2 \leq |X| \leq |V(G)| - 1$. Graphs $G[X]$ and $G_0$ are the two blocks of decomposition; graph $G_0$ is obtained from $G$ by "shrinking" $X$ to a vertex $x$.
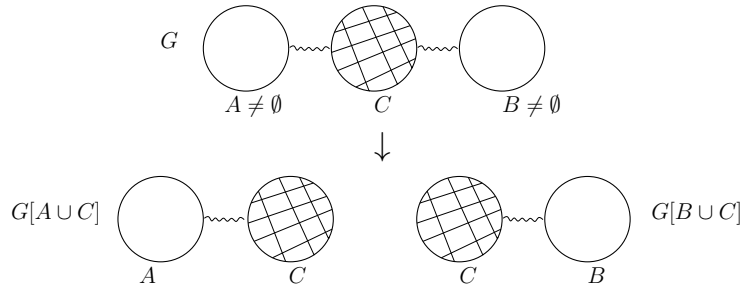


Figure 1.2: Graph $G$ and cut-partition $(A, B, C)$ of $G$, where $C$ is a clique-cutset of $G$. Graphs $G[A \cup C]$ and $G[B \cup C]$ are the two blocks of decomposition.

jacency between the corresponding sets of vertices, and the absence of any kind of line between two circles indicates that there are no edges between the sets of vertices represented by the two circles.)

Given a graph $G$, a set $S \subseteq V(G)$, and a vertex $v \in V(G) \smallsetminus S$, we say that $v$ is *complete* (respectively: *anti-complete*) to $S$ in $G$ provided that $v$ is adjacent (respectively: non-adjacent) to every vertex of $S$ in $G$; $v$ is said to be *mixed* on $S$ if $v$ is neither complete nor anti-complete to $S$ in $G$. Given disjoint sets $X, Y \subseteq V(G)$, we say that $X$ is *complete* (respectively: *anti-complete*) to $Y$ in $G$ provided that every vertex of $X$ is complete (respectively: anti-complete) to $Y$ in $G$. A *homogeneous set* of a graph $G$ (see Figure 1.1) is a non-empty set $X \subseteq V(G)$ such that no vertex in $V(G) \smallsetminus X$ is mixed on $X$; a homogeneous set $X$ of a graph $G$ is *proper* if $2 \leq |X| \leq |V(G)| - 1$. A *homogeneous set partition* of a graph $G$ is a partition $(X, Y, Z)$ of $V(G)$ such that $X$ is a non-empty set ($Y$ and $Z$ may

Figure 1.3: Graph $G$ and amalgam $(K, A, B, C, D)$ of $G$. Graphs $G_b$ and $G_c$ are the two blocks of decomposition. Graph $G_b$ (bottom right) is obtained 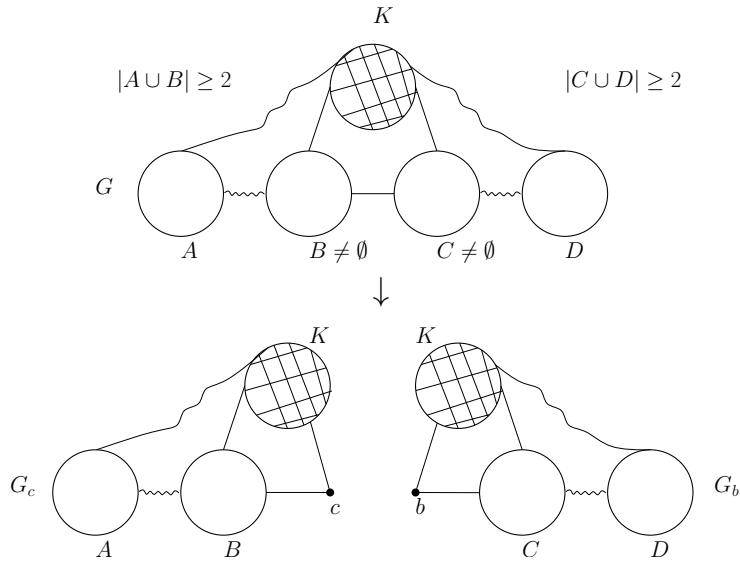from $G \smallsetminus A$ by "shrinking" $B$ to a vertex $b$, and graph $G_c$ (bottom left) is obtained from $G \smallsetminus D$ by "shrinking" $C$ to a vertex $c$.

possibly be empty), and $X$ is complete to $Y$ and anti-complete to $Z$ in $G$. Note that if $(X, Y, Z)$ is a homogeneous set partition of $G$, then $X$ is a (not necessarily proper) homogeneous set of $G$. Conversely, every homogeneous set of $G$ induces a unique homogeneous set partition of $G$. A *cutset* of a graph $G$ is a (possibly empty) set $C \subseteq V(G)$ such that $G \smallsetminus C$ is disconnected. A *cut-partition* of a graph $G$ is a partition $(A, B, C)$ of $V(G)$ such that $A$ and $B$ are non-empty ($C$ may possibly be empty), and $A$ is anti-complete to $B$ in $G$. Clearly, if $(A, B, C)$ is a cut-partition of a graph $G$, then $C$ is a cutset of $G$; conversely, every cutset of $G$ gives rise to at least one cut-partition of $G$. A *clique-cutset* of a graph $G$ (see Figure 1.2) is a cutset of $G$ that is a (possibly empty) clique of $G$. (In particular, the empty set is a clique-cutset of any disconnected graph.) A 1-*join* of a graph $G$ is a partition $(A, B, C, D)$ of $V(G)$ such that $B$ and $C$ are non-empty ($A$ and $D$ may possibly be empty), $B$ is complete to $C$, $A$ is anti-complete to $C \cup D$, $D$ is anti-complete to $A \cup B$, and $|A \cup B|, |C \cup D| \geq 2$. Note that if $(A, B, C, D)$ is a 1-join of $G$, then $(D, C, B, A)$ is also a 1-join of $G$. An *amalgam* of a graph $G$ (see Figure 1.3) is a partition $(K, A, B, C, D)$ of $V(G)$ such that $K$ is a (possibly empty) clique, $(A, B, C, D)$ is a 1-join of $G \smallsetminus K$, and $K$ is complete to $B \cup C$. Clearly, if $(K, A, B, C, D)$ is an amalgam of $G$, then $(K, D, C, B, A)$ is also an amalgam of $G$. Note also that if $(A, B, C, D)$ is a 1-join of $G$, then $(\emptyset, A, B, C, D)$ is an amalgam of $G$, and so the amalgam

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decomposition generalizes the 1-join decomposition.

As stated above, our main result is that proper homogeneous sets, clique-cutsets, and amalgams together preserve $\chi$-boundedness. More precisely, we prove the following theorem.

**Theorem 1.2.** *Let $\mathcal{G}$ and $\mathcal{G}^*$ be hereditary classes. Assume that $\mathcal{G}$ is $\chi$-bounded by a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$, and that every graph in $\mathcal{G}^*$ either belongs to $\mathcal{G}$, or admits a proper homogeneous set, a clique-cutset, or an amalgam. Let $L \in \mathbb{N} \cup \{\infty\}$ be such that all graphs $G \in \mathcal{G}$ satisfy $\omega(G) \leq L$. Then $\mathcal{G}^*$ is $\chi$-bounded by the function $\widetilde{f} : \mathbb{N} \to \mathbb{N}$ given by $\widetilde{f}(0) = 0$ and $\widetilde{f}(n) = (\sum\limits_{t=1}^{\min\{n,L\}} f(t))^{n-1}$ for all $n \in \mathbb{N}^+$.*

Let us now briefly discuss the idea of the proof of Theorem 1.2. First of all, note that $\chi$-boundedness is a property of graph classes, and not of individual graphs. To be sure, one can see $\chi$-boundedness *by a fixed function* as a property of graphs: given a function $f : \mathbb{N} \to \mathbb{N}$, a graph $G$ is $\chi$-*bounded by $f$* if $\chi(G) \leq f(\omega(G))$. (Thus, a hereditary class is $\chi$-bounded if and only if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph in the class is $\chi$-bounded by $f$.) Note, however, that every graph $G$ is $\chi$-bounded by the constant function $\chi(G)$, and so the concept of a $\chi$-bounded graph (without reference to a previously fixed function) is not useful. Furthermore, very few (natural) graph decompositions preserve $\chi$-boundedness by a fixed function, that is, there are not many graph decompositions $D$ for which the following statement is true: "if $f : \mathbb{N} \to \mathbb{N}$ is a function, and $G$ is a graph that admits the decomposition $D$ and has the property that all its proper induced subgraphs are $\chi$-bounded by $f$, then $G$ is also $\chi$-bounded by $f$." (It is easy to see that clique-cutsets preserve $\chi$-boundedness by a fixed non-decreasing function, but few other graph decompositions have this property.) For this reason, $\chi$-boundedness is typically inconvenient to work with directly if one wishes to show that a graph decomposition preserves $\chi$-boundedness (or that several graph decompositions together preserve it). One way around this problem (and this is the approach that we use to prove Theorem 1.2) is to find a graph property $P$ that is in a sense "equivalent" to $\chi$-boundedness, and that is also preserved by the graph decomposition(s) under consideration. More precisely, we need to be able to show that if a hereditary class is $\chi$-bounded, then every graph in the class has the property $P$; that if every graph in a hereditary class has the property $P$, then the class is $\chi$-bounded; and that if all proper induced subgraphs of a graph $G$ have the property $P$, and $G$ admits one of the decompositions under consideration, then $G$ is also $\chi$-bounded. It is important to note that if one wishes to use this approach to show that graph decompositions $D_1, \ldots, D_k$ together preserve $\chi$-bondedness, then one must find one graph property $P$ that is preserved by all $k$ decompositions. It would not be enough to find graph

properties $P_1, \ldots, P_k$ such that $D_i$ preserves $P_i$ for each $i \in \{1, \ldots, k\}$, for then one could not guarantee that the decompositions $D_1, \ldots, D_k$ do not exhibit the kind of behavior discussed in the paragraph preceding Proposition 1.1. Thus, even though it was shown in [7] that proper homogeneous sets and clique-cutsets together preserve $\chi$-boundedness, in order to prove Theorem 1.2, we must start from scratch and find a graph property that is preserved by all three decompositions from Theorem 1.2 (namely, proper homogeneous sets, clique-cutsets, and amalgams), and not just by the "new" decomposition (the amalgam decomposition).

In order to prove Theorem 1.2, we introduce a new graph property, which we call "$f$-colorability" (where $f : \mathbb{N} \to \mathbb{N}$ is a superadditive function), we show that the three decompositions from Theorem 1.2 preserve $f$-colorability (see Lemma 2.6), and we show that $f$-colorability is "equivalent" to $\chi$-boundedness in the sense discussed in the previous paragraph (see Lemmas 2.5 and 2.7). We remark, however, that the fact that a hereditary class $\mathcal{G}$ is $\chi$-bounded by a function $f : \mathbb{N} \to \mathbb{N}$ does *not* imply that every graph in $\mathcal{G}$ is $f$-colorable, but merely that there exists some function $\widetilde{f} : \mathbb{N} \to \mathbb{N}$ (which increases much faster than $f$) such that every graph in $\mathcal{G}$ is $\widetilde{f}$-colorable. The definition of $f$-colorability is somewhat technical, and we postpone it to section 2. In that section, we also prove some easy technical results concerning $f$-colorability, we state our three main technical lemmas (namely, Lemmas 2.5, 2.6, and 2.7), we derive Theorem 1.2 from these three lemmas, and we dereive two corollaries of Theorem 1.2 (see Corollaries 2.8 and 2.9; these corollaries are arguably easier to apply in practice than Theorem 1.2 iteself). One of the three technical lemmas (namely, Lemma 2.7) is proven in section 2. The other two (namely, Lemmas 2.5 and 2.6) are more difficult to prove, and their proofs are postponed to section 3.

In section 4 (the final section of this paper), we give an application of Theorem 1.2. We first need a few definitions. Given graphs $H$ and $G$, we say that $G$ is $H$-*free* if $G$ does not contain (an isomorphic copy of) $H$ as an induced subgraph. A subdivision of a graph $H$ is denoted by $H^*$ (in particular, $H$ itself is an $H^*$), and a graph $G$ is said to be $H^*$-*free* provided that $G$ does not contain any subdivision of $H$ as an induced subgraph. The class of all $H^*$-free graphs is denoted by $\mathrm{Forb}^*(H)$. Scott [23] showed that if $F$ is a forest, then $\mathrm{Forb}^*(F)$ is $\chi$-bounded, and he conjectured that $\mathrm{Forb}^*(H)$ is $\chi$-bounded for every graph $H$. Recently, Pawlik et al. [21] gave a counterexample to Scott's conjecture (see also [4]), but it remains an open problem to determine for which graphs $H$, the class $\mathrm{Forb}^*(H)$ is $\chi$-bounded. As already mentioned, $\mathrm{Forb}^*(H)$ is $\chi$-bounded if $H$ is a forest [23]. It is also known that $\mathrm{Forb}^*(H)$ is $\chi$-bounded if $H$ is a complete bipartite graph [17], if $H$ has at most four vertices (see [18] for the case when $H$ is the complete graph on four vertices; the other graphs on at most four vertices are easier to handle,
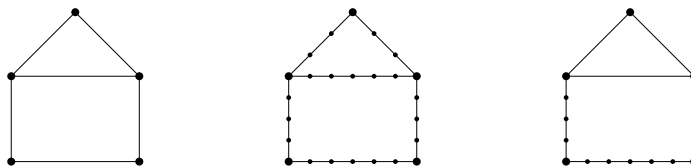
Figure 1.4: The *house* (left) is the complement of the four-edge path. A *house*\* (middle) is any subdivision of the house. A *cap* (right) is any graph obtained from the house by possibly subdividing the three edges of the house that do not belong to the unique triangle of the house.

and we refer the reader to the introduction of [6] for a summary), if $H$ is a cycle [5] (see [24] for the case when $H$ is a cycle of length five), and if $H$ is the bull (i.e. the five-vertex graph that consists of a triangle and two vertex-disjoint pendant edges) or a certain generalization of the bull called a "necklace" [6]. The *house* is the complement of the four-edge path, and a *cap* is any graph obtained from the house by possibly subdividing the three edges of the house that do not belong to the unique triangle of the house (see Figure 1.4). A graph is *cap-free* if it does not contain any cap as an induced subgraph. Thus, the house is a cap, every cap is a house\*, and every house\*-free graph is cap-free. Using Corollary 2.9 (which is an easy consequence of Theorem 1.2; see section 2) and a decomposition theorem for cap-free graphs from [11], one can show that all house\*-free graphs $G$ satisfy $\chi(G) \leq 4^{\omega(G)-1}$ (see Proposition 4.3), and so the class Forb\*(house) is $\chi$-bounded. However, if instead of using Corollary 2.9, we use certain technical results concerning $f$-colorability from section 2, we can obtain a better $\chi$-bounding function for the class Forb\*(house). In particular, we obtain the following theorem.

**Theorem 1.3.** *Every house\*-free graph $G$ satisfies $\chi(G) \leq 3^{\omega(G)-1}$.*

The proof of Theorem 1.3 is given in section 4. We remark that we do not know whether the bound from Theorem 1.3 is optimal.

## 2 $f$-Colorability and the proof of Theorem 1.2

The set of all finite subsets of a set $S$ is denoted by $\mathscr{P}_{\mathrm{fin}}(S)$. If $S$ is a set of sets, then we often write $\bigcup S$ instead of $\bigcup_{A \in S} A$. If $f$ is a function, and $S$ is a subset of the domain of $f$, then we denote by $f \restriction S$ the restriction of $f$ to $S$, and we denote by $f[S]$ the image of $S$ under $f$.

A function $f : \mathbb{N} \to \mathbb{N}$ is said to be *superadditive* if for all $m, n \in \mathbb{N}$, we have that $f(m) + f(n) \leq f(m+n)$. Note that every superadditive function

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$f : \mathbb{N} \to \mathbb{N}$ is non-decreasing and satisfies $f(0) = 0$. Note furthermore that if $f : \mathbb{N} \to \mathbb{N}$ is a superadditive function, then $f$ is strictly increasing if and only if $f(1) \geq 1$.

Given a positive integer $m$ and integers $a$ and $b$, we write $a \equiv_m b$ if $a$ and $b$ are congruent modulo $m$. For all $n \in \mathbb{N}$, we set $[n] = \{1, \ldots, n\}$ (in particular, $[0] = \emptyset$). For all $m, n \in \mathbb{N}$, we set $m + [n] = \{m + 1, \ldots, m + n\}$ (in particular, $m + [0] = \emptyset$ for all $m \in \mathbb{N}$, and $0 + [n] = [n]$ for all $n \in \mathbb{N}$).

If $G$ is a graph and $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of $G$ induced by $S$, and we denote by $G \smallsetminus S$ the subgraph of $G$ obtained by deleting $S$ (thus, $G \smallsetminus S = G[V(G) \smallsetminus S]$). If $v_1, \ldots, v_k \in V(G)$, we often write $G[v_1, \ldots, v_k]$ instead of $G[\{v_1, \ldots, v_k\}]$. A *proper* induced subgraph of $G$ is any induced subgraph of $G$ that has fewer vertices than $G$ (in particular, the null graph is a proper induced subgraph of every non-null graph).

A *weight function* for a graph $G$ is any function $w : V(G) \to \mathbb{N}^+$. Given a graph $G$ and a weight function $w : V(G) \to \mathbb{N}^+$ for $G$, we denote by $w(G)$ the maximum weight of a clique of $G$ with respect to $w$, that is,

$$w(G) \quad = \quad \max\{ \textstyle\sum_{v \in Q} w(v) \mid Q \text{ is a clique of } G\}.$$

Clearly, if $w : V(G) \to \mathbb{N}^+$ is a weight function for a graph $G$, and $S \subseteq V(G)$, then $w \restriction S$ is a weight function for $G[S]$. To simplify notation, we often refer to $w$ itself (and strictly speaking, we mean $w \restriction S$) as a weight function for $G[S]$. Consistently with the notation above, we denote by $w(G[S])$ the maximum weight of a clique of $G[S]$ with respect to $w$, that is,

$$w(G[S]) \quad = \quad \max\{ \textstyle\sum_{v \in Q} w(v) \mid Q \text{ is a clique of } G[S]\}.$$

Given a graph $G$, we denote by $K_G$ the set of all vertices of $G$ that do not have a non-neighbor in $G$, and we denote by $R_G$ the set of all vertices of $G$ that do have a non-neighbor in $G$ (see Figure 2.1); thus, $V(G) = K_G \cup R_G$, $K_G \cap R_G = \emptyset$, $K_G$ is a (possibly empty) clique, $K_G$ is complete to $R_G$, and every vertex of $R_G$ has a non-neighbor in $R_G$ (and consequently, either $R_G = \emptyset$ or $|R_G| \geq 2$). Note that if $H$ is an induced subgraph of a graph $G$, then $R_H \subseteq R_G$. Given a graph $G$ and a vertex $u \in V(G)$, we denote by $\Gamma_G(u)$ the set of all neighbors of $u$ in $G$, and we set $\Gamma_G[u] = \{u\} \cup \Gamma_G(u)$, $K_G(u) = K_{G[\Gamma_G(u)]}$, and $R_G(u) = R_{G[\Gamma_G(u)]}$ (see Figure 2.1).

If $G$ is a graph, we say that $Q$ is a *$G$-admissible clique* provided that $Q$ is a (possibly empty) clique of $G$ that satisfies the property that for all $u, u' \in Q$, either $\Gamma_G[u] \subseteq \Gamma_G[u']$ or $\Gamma_G[u'] \subseteq \Gamma_G[u]$. Note that if $Q$ is a $G$-admissible clique, then $Q$ can be ordered as $Q = \{u_1, \ldots, u_k\}$ (with
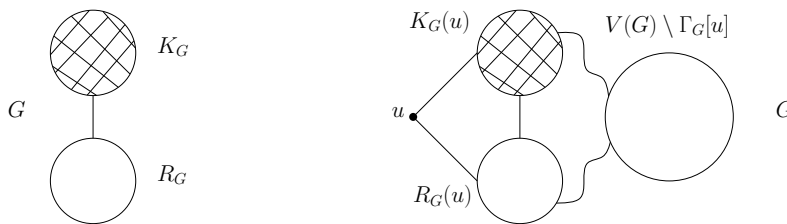
Figure 2.1: Left: Graph $G$ with $V(G)$ decomposed into $K_G$ and $R_G$; $K_G$ is a clique complete to $R_G$, and every vertex in $R_G$ has a non-neighbor in $R_G$. Right: Graph $G$ and vertex $u \in V(G)$ with $\Gamma_G(u)$ decomposed into $K_G(u)$ and $R_G(u)$; $K_G(u)$ is a clique complete to $R_G(u)$, and every vertex in $R_G(u)$ has a non-neighbor in $R_G(u)$.

$k = |Q| \geq 0$) so that for all $i, j \in \{1, \ldots, k\}$, if $i < j$, then $\Gamma_G[u_i] \subseteq \Gamma_G[u_j]$ (and consequently, $R_G(u_i) \subseteq R_G(u_j)$). Note also that if $Q$ is a $G$-admissible clique, and $H$ is an induced subgraph of $G$, then every subset of $Q \cap V(H)$ is an $H$-admissible clique. (In particular, every subset of a $G$-admissible clique is a $G$-admissible clique.) Clearly, the empty set is a $G$-admissible clique for every graph $G$. Furthermore, if $G$ is a non-null graph and $u \in V(G)$, then $\{u\}$ is a $G$-admissible clique.

We now define "$f$-colorability," the crucial concept of this paper. As explained in the introduction, the idea behind $f$-colorability is that it is in a certain sense "equivalent" to $\chi$-boundedness: roughly speaking, if all graphs in a hereditary class have the $f$-property (where $f$ is a suitable function), then the class is $\chi$-bounded, and conversely, if a hereditary class is $\chi$-bounded, then there is a suitable function $f$ such that every graph in the class is $f$-colorable. Furthermore, our three decompositions preserve $f$-colorability, that is, if all proper induced subgraphs of a graph $G$ are $f$-colorable, and $G$ admits one of our three decompositions (a proper homogeneous set, a clique-cutset, or an amalgam), then $G$ is $f$-colorable. All this is made formal in Lemmas 2.5, 2.6, and 2.7, which we state later in this section. But first, let us give the definition of $f$-colorability.

Given a superadditive function $f : \mathbb{N} \to \mathbb{N}$, a graph $G$, a weight function $w : V(G) \to \mathbb{N}^+$ for $G$, and a $G$-admissible clique $Q$, we define an $(f; w; Q)$-*valid coloring of $G$* to be any function $\phi : V(G) \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ that satisfies all the following:

(a) $\phi(v_1) \cap \phi(v_2) = \emptyset$ for all $v_1 v_2 \in E(G)$;

(b) $|\phi(v)| = f(w(v))$ for all $v \in V(G)$;

(c) $|\bigcup \phi[R_G]| \leq f(w(G[R_G]))$;

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(d) $|\bigcup \phi[R_G(u)]| \le f(w(G[R_G(u)]))$ for all $u \in Q$.

Given a superadditive function $f : \mathbb{N} \to \mathbb{N}$, we say that a graph $G$ is $f$-*colorable* provided that for every weight function $w : V(G) \to \mathbb{N}^+$ for $G$, and every $G$-admissible clique $Q$, there exists an $(f; w; Q)$-valid coloring of $G$.

Let us make a few remarks about the definition above. Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a superadditive function, $G$ a graph, $w : V(G) \to \mathbb{N}^+$ a weight function for $G$, $Q$ a $G$-admissible clique, and $\phi : V(G) \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ an $(f; w; Q)$-valid coloring of $G$. If we regard positive integers as colors, then the function $\phi$ can be seen as a variant of weighted coloring: by condition (a), $\phi$ assigns disjoint sets of colors to adjacent vertices of $G$, but by condition (b), rather than assigning $w(v)$ colors to each vertex $v$ of $G$ (as an ordinary weighted coloring would), $\phi$ assigns $f(w(v))$ colors to each vertex $v$; we will return to condition (b) and explain why we need it later in this section. Note that if $S \subseteq V(G)$, then $\bigcup \phi[S]$ is the set of all colors (positive integers) that $\phi$ uses on $S$. Note also that if $A$ and $B$ are disjoint subsets of $V(G)$, complete to each other of $G$, then condition (a) implies that $\bigcup \phi[A]$ and $\bigcup \phi[B]$ are disjoint (that is, $\phi$ uses disjoint color-sets on $A$ and $B$). Further, condition (c) places an upper bound on the number of colors that $\phi$ may use on $R_G$, and condition (d) places an upper bound on the number of colors that $\phi$ may use on $R_G(u)$ for $u \in Q$. We will return to conditions (c) and (d) and provide some intuition behind them later in this section. For now, we note that the definition of an $(f; w; Q)$-valid coloring of $G$ in fact places an upper bound on the number of colors that $\phi$ may use on $V(G)$, and not just on $R_G$. (This is not surprising given that $f$-colorability is supposed to be "equivalent" to $\chi$-boundedness.) More precisely, we have the following proposition.

**Proposition 2.1.** *Let $f : \mathbb{N} \to \mathbb{N}$ be a superadditive function, let $G$ be a graph, let $w : V(G) \to \mathbb{N}^+$ be a weight function for $G$, let $Q$ be a $G$-admissible clique, and let $\phi : V(G) \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ be an $(f; w; Q)$-valid coloring of $G$. Then for every set $S \subseteq V(G)$ such that either*

*(1) $R_G \subseteq S$, or*

*(2) $R_G(u) \subseteq S \subseteq \Gamma_G(u)$ for some $u \in Q$,*

*we have that $|\bigcup \phi[S]| \le f(w(G[S]))$. In particular, $|\bigcup \phi[V(G)]| \le f(w(G))$.*

*Proof.* The fact that $|\bigcup \phi[V(G)]| \le f(w(G))$ follows immediately from the preceding statement: we simply set $S = V(G)$, and we observe that $S$ satisfies (1).

To prove the first statement, we fix a set $S \subseteq V(G)$ that satisfies (1) or (2). First, we claim that $|\bigcup \phi[R_{G[S]}]| \le f(w(G[R_{G[S]}]))$. If $S$ satisfies (1),

then clearly, $R_{G[S]} = R_G$, and the claim follows from condition (c) of the definition of an $(f; w; Q)$-valid coloring of $G$. On the other hand, if $S$ satisfies (2), then we fix a vertex $u \in Q$ such that $R_G(u) \subseteq S \subseteq \Gamma_G(u)$, we observe that $R_{G[S]} = R_G(u)$, and we see that our claim follows from condition (d) of the definition of an $(f; w; Q)$-valid coloring of $G$.

Now, set $K_{G[S]} = \{x_1, \ldots, x_t\}$ (with $t = |K_{G[S]}| \geq 0$). By condition (b) of the definition of an $(f; w; Q)$-valid coloring of $G$, we have that $|\phi(x_i)| = f(w(x_i))$ for all $i \in \{1, \ldots, t\}$. Since $K_{G[S]} = \{x_1, \ldots, x_t\}$ is a clique, complete to $R_{G[S]}$ in $G$, we see that $w(G[S]) = w(G[K_{G[S]}]) + w(G[R_{G[S]}]) = (\sum_{i=1}^{t} w(x_i)) + w(G[R_{G[S]}])$; since $f$ is a superadditive function, it follows that $(\sum_{i=1}^{t} f(w(x_i))) + f(w(G[R_{G[S]}])) \leq f(w(G[S]))$. We now have that

$$
\begin{aligned}
|\bigcup \phi[S]| &= |(\bigcup_{i=1}^{t} \phi(x_i)) \cup (\bigcup \phi[R_{G[S]}])| \\[2mm]
&\leq (\sum_{i=1}^{t} |\phi(x_i)|) + |\bigcup \phi[R_{G[S]}]| \\[2mm]
&\leq (\sum_{i=1}^{t} f(w(x_i))) + f(w(G[R_{G[S]}])) \\[2mm]
&\leq f(w(G[S])),
\end{aligned}
$$

which is what we needed. $\qquad\square$

We next prove Proposition 2.2, which states that if $G[R_G]$ is $f$-colorable, then so is $G$. After that, we derive two easy corollaries of Proposition 2.2 (namely, Propositions 2.3 and 2.4), which we will use throughout the paper.

**Proposition 2.2.** *Let $f : \mathbb{N} \to \mathbb{N}$ be a superadditive function, and let $G$ be a graph such that $G[R_G]$ is $f$-colorable. Then $G$ is $f$-colorable.*

*Proof.* If $K_G = \emptyset$, the result is immediate. So assume that $K_G \neq \emptyset$, and set $K_G = \{x_1, \ldots, x_t\}$ (where $t = |K_G| \geq 1$). Fix a weight function $w : V(G) \to \mathbb{N}^+$ and a $G$-admissible clique $Q$. Clearly, $Q \cap R_G$ is a $G[R_G]$-admissible clique. By hypothesis, $G[R_G]$ is $f$-colorable; fix an $(f; w; Q \cap R_G)$-valid coloring $\phi_R : R_G \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ of $G[R_G]$. It is clear that $R_{G[R_G]} = R_G$, and it is also clear that $R_{G[R_G]}(u) = R_G(u)$ for all $u \in R_G$. Thus, the $(f; w; Q \cap R_G)$-valid coloring $\phi_R$ of $G[R_G]$ satisfies all the following:

(a') $\phi_R(v_1) \cap \phi_R(v_2) = \emptyset$ for all $v_1 v_2 \in E(G[R_G])$;

(b') $|\phi_R(v)| = f(w(v))$ for all $v \in R_G$;

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(c') $|\bigcup \phi_R[R_G]| \leq f(w(G[R_G]))$;

(d') $|\bigcup \phi_R[R_G(u)]| \leq f(w(G[R_G(u)]))$ for all $u \in Q \cap R_G$.

In view of (c'), we may assume that $\bigcup \phi_R[R_G] \subseteq [f(w(G[R_G]))]$ (we permute colors if necessary). We now define the function $\phi : V(G) \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ by setting $\phi \restriction R_G = \phi_R \restriction R_G$ and $\phi(x_i) = f(\omega(G[R_G])) + \sum_{j=1}^{i-1} f(w(x_j)) + [f(w(x_i))]$ for all $i \in \{1, \ldots, t\}$. Let us check that $\phi$ is an $(f; w; Q)$-valid coloring of $G$. We must verify that $\phi$ satisfies the following:

(a) $\phi(v_1) \cap \phi(v_2) = \emptyset$ for all $v_1 v_2 \in E(G)$;

(b) $|\phi(v)| = f(w(v))$ for all $v \in V(G)$;

(c) $|\bigcup \phi[R_G]| \leq f(w(G[R_G]))$;

(d) $|\bigcup \phi[R_G(u)]| \leq f(w(G[R_G(u)]))$ for all $u \in Q$.

The fact that $\phi$ satisfies (a) and (b) is immediate from the construction and the fact that $\phi_R$ satisfies (a') and (b'). The fact that $\phi$ satisfies (c) follows from the fact that $\phi_R$ satisfies (c'). It remains to check that $\phi$ satisfies (d). Fix $u \in Q$; we need to show that $|\bigcup \phi[R_G(u)]| \leq f(w(G[R_G(u)]))$. If $u \in R_G$, then this follows from the fact that $\phi_R$ satisfies (d'). On the other hand, if $u \in K_G$, then we have that $R_G(u) = R_G$, and the result follows from the fact that $\phi_R$ satisfies (c'). This proves that $\phi$ is an $(f; w; Q)$-valid coloring of $G$, and it follows that $G$ is $f$-colorable. $\square$

**Proposition 2.3.** *Let $f : \mathbb{N} \to \mathbb{N}$ be a superadditive function, and let $G$ be a graph such that $K_G \neq \emptyset$. If all proper induced subgraphs of $G$ are $f$-colorable, then so is $G$.*

*Proof.* Since $K_G \neq \emptyset$, we have that $G[R_G]$ is a proper induced subgraph of $G$. The result now follows immediately from Proposition 2.2. $\square$

**Proposition 2.4.** *Let $f : \mathbb{N} \to \mathbb{N}$ be a superadditive function. Then every complete graph is $f$-colorable.*

*Proof.* If $G$ is a complete graph, then $R_G = \emptyset$. Clearly, the null graph is $f$-colorable (the null function is a suitable coloring), and so the result follows from Proposition 2.2. $\square$

Now, our goal is to prove three lemmas (Lemmas 2.5, 2.6, and 2.7) about $f$-colorability and $\chi$-boundedness. We state these lemmas below, but roughly speaking, Lemma 2.5 states that "$\chi$-boundedness implies $f$-colorability," Lemma 2.6 states that "proper homogeneous sets, clique-cutsets, and amalgams preserve $f$-colorability," and Lemma 2.7 states that "$f$-colorability implies $\chi$-boundedness." Together, the three lemmas imply Theorem 1.2.

The proof of Lemma 2.7 is an easy exercise, and so we prove this lemma immediately. The proofs of Lemmas 2.5 and 2.6 are more involved, and we postpone them to section 3. We now state the three lemmas.

**Lemma 2.5.** *Let $\mathcal{G}$ be a hereditary class, $\chi$-bounded by a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ that satisfies $f(1) \geq 1$. Let $L \in \mathbb{N} \cup \{\infty\}$ be such that $L \geq 2$ and such that all graphs $G \in \mathcal{G}$ satisfy $\omega(G) \leq L$. Define $\widetilde{f} : \mathbb{N} \to \mathbb{N}$ by setting $\widetilde{f}(0) = 0$ and $\widetilde{f}(n) = ( \sum\limits_{t=1}^{\min\{n,L\}} f(t))^{n-1}$ for all $n \in \mathbb{N}^+$. Then $\widetilde{f}$ is a superadditive function that satisfies $\widetilde{f}(1) = 1$, and every graph in $\mathcal{G}$ is $\widetilde{f}$-colorable.*

**Lemma 2.6.** *Let $f$ be a superadditive function, and let $G$ be a graph that admits a proper homogeneous set, a clique-cutset, or an amalgam. Assume that all proper induced subgraphs of $G$ are $f$-colorable. Then $G$ is $f$-colorable.*

**Lemma 2.7.** *Let $f : \mathbb{N} \to \mathbb{N}$ be a superadditive function such that $f(1) = 1$, and let $\mathcal{G}$ be a hereditary class such that every graph in $G$ is $f$-colorable. Then $\mathcal{G}$ is $\chi$-bounded by $f$.*

*Proof.* Fix $G \in \mathcal{G}$; we need to show that $\chi(G) \leq f(\omega(G))$. Let $w : V(G) \to \mathbb{N}^+$ be given by $w(v) = 1$ for all $v \in V(G)$; clearly then, $w(G) = \omega(G)$. Further, it is clear that $\emptyset$ is a $G$-admissible clique. By hypothesis, $G$ is $f$-colorable; fix an $(f; w; \emptyset)$-valid coloring $\phi : V(G) \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ of $G$. Proposition 2.1 guarantees that $|\bigcup \phi[V(G)]| \leq f(w(G))$, and so after possibly permuting colors, we may assume that $\bigcup \phi[V(G)] \subseteq [f(w(G))]$. Since $\phi$ is an $(f; w; Q)$-valid coloring of $G$, we know that for all $v \in V(G)$, we have that $|\phi(v)| = f(w(v)) = f(1) = 1$ (and in particular, $\phi(v) \neq \emptyset$), and we also know that for all $v_1 v_2 \in E(G)$, we have that $\phi(v_1) \cap \phi(v_2) = \emptyset$. Thus, for all $v_1 v_2 \in E(G)$, we have that $\phi(v_1)$ and $\phi(v_2)$ are non-empty and disjoint, and consequently, $\phi(v_1) \neq \phi(v_2)$. This proves that $\phi$ is a proper coloring of $G$ (here, we consider finite subsets of $\mathbb{N}^+$ to be colors and not sets of colors). Now, since $\phi(v)$ is a one-element subset of $[f(w(G))]$ for all $v \in V(G)$, we see that $|\phi[V(G)]|$ is bounded above by the number of one-element subsets of $[f(w(G))]$, which is precisely $f(w(G)) = f(\omega(G))$. Thus, $G$ can be properly colored with $f(\omega(G))$ colors, and so $\chi(G) \leq f(\omega(G))$. This proves that $\mathcal{G}$ is $\chi$-bounded by $f$. $\square$

We now derive Theorem 1.2 from the three lemmas above. Theorem 1.2 is restated below for the reader's convenience.

**Theorem 1.2.** *Let $\mathcal{G}$ and $\mathcal{G}^*$ be hereditary classes. Assume that $\mathcal{G}$ is $\chi$-bounded by a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$, and that every graph in $\mathcal{G}^*$ either belongs to $\mathcal{G}$, or admits a proper homogeneous set, a clique-cutset, or an amalgam. Let $L \in \mathbb{N} \cup \{\infty\}$ be such that all graphs $G \in \mathcal{G}$ satisfy*

$\omega(G) \leq L$. *Then* $\mathcal{G}^*$ *is* $\chi$-*bounded by the function* $\widetilde{f} : \mathbb{N} \to \mathbb{N}$ *given by* $\widetilde{f}(0) = 0$ *and* $\widetilde{f}(n) = (\sum\limits_{t=1}^{\min\{n,L\}} f(t))^{n-1}$ *for all* $n \in \mathbb{N}^+$.

*Proof (assuming Lemmas 2.5, 2.6, and 2.7).* By construction, we have that $\widetilde{f}(0) = 0$ and $\widetilde{f}(1) = 1$. Thus, if every graph in $\mathcal{G}^*$ is edgeless, then $\mathcal{G}^*$ is $\chi$-bounded by $\widetilde{f}$, and we are done. So assume that $\mathcal{G}^*$ contains a graph that contains at least one edge. Since $\mathcal{G}^*$ is hereditary, it follows that $\mathcal{G}^*$ contains the complete graphs $K_1$ and $K_2$. Since neither $K_1$ nor $K_2$ admits a proper homogeneous set, a clique-cutset, or an amalgam, it follows that $K_1, K_2 \in \mathcal{G}$. Since $K_1 \in \mathcal{G}$ and $\mathcal{G}$ is $\chi$-bounded by $f$, it follows that $f(1) \geq 1$; and since $K_2 \in \mathcal{G}$ and $\omega(K_2) = 2$, it follows that $L \geq 2$. Then by Lemma 2.5, we know that $\widetilde{f}$ is a superadditive function that satisfies $\widetilde{f}(1) = 1$, and that every graph in $\mathcal{G}$ is $\widetilde{f}$-colorable. Lemma 2.6, together with an easy induction on the number of vertices, now implies that every graph in $\mathcal{G}^*$ is $\widetilde{f}$-colorable. Lemma 2.7 then guarantees that $\mathcal{G}^*$ is $\chi$-bounded by $\widetilde{f}$. $\qquad\square$

We now derive two easy corollaries of Theorem 1.2, which are arguably more convenient to use in practice than Theorem 1.2 itself. The first of these corollaries (Corollary 2.8) deals with the case when $L = \infty$, and the second (Corollary 2.9) deals with the case when $L$ is finite.

**Corollary 2.8.** *Let* $\mathcal{G}$ *and* $\mathcal{G}^*$ *be hereditary classes. Assume that* $\mathcal{G}$ *is* $\chi$-*bounded by a non-decreasing function* $f : \mathbb{N} \to \mathbb{N}$, *and that every graph in* $\mathcal{G}^*$ *either belongs to* $\mathcal{G}$, *or admits a proper homogeneous set, a clique-cutset, or an amalgam. Define* $\widetilde{f} : \mathbb{N} \to \mathbb{N}$ *by setting* $\widetilde{f}(0) = 0$ *and* $\widetilde{f}(n) = (\sum\limits_{t=1}^{n} f(t))^{n-1}$ *for all* $n \in \mathbb{N}^+$. *Then* $\mathcal{G}^*$ *is* $\chi$-*bounded by the function* $\widetilde{f}$.

*Proof (assuming Theorem 1.2).* We set $L = \infty$, and the result follows immediately from Theorem 1.2. $\qquad\square$

**Corollary 2.9.** *Let* $L$ *and* $c$ *be positive integers, and let* $\mathcal{G}$ *be a hereditary class such that for all graphs* $G \in \mathcal{G}$, *we have that either*

- $\omega(G) \leq L$ *and* $\chi(G) \leq c$, *or*

- $G$ *admits a proper homogeneous set, a clique-cutset, or an amalgam.*

*Then all graphs* $G \in \mathcal{G}$ *satisfy* $\chi(G) \leq (1 + (L-1)c)^{\omega(G)-1}$.

*Proof (assuming Theorem 1.2).* Let $\mathcal{G}_0$ be the class of all graphs $G$ such that $\omega(G) \leq L$ and $\chi(G) \leq c$, and let $f : \mathbb{N} \to \mathbb{N}$ be given by $f(0) = 0$, $f(1) = 1$, and $f(n) = c$ for all $n \geq 2$. Then $\mathcal{G}_0$ is a hereditary class, $\chi$-bounded by the non-decreasing function $f : \mathbb{N} \to \mathbb{N}$, and every graph in the hereditary class $\mathcal{G}$ either belongs to $\mathcal{G}_0$, or admits a proper homogeneous set, a clique-cutset, or an amalgam. By Theorem 1.2 then, we see that $\mathcal{G}$ is $\chi$-bounded by the

function $\widetilde{f} : \mathbb{N} \to \mathbb{N}$ given by $\widetilde{f}(0) = 0$ and $\widetilde{f}(n) = (\sum_{t=1}^{\min\{n,L\}} f(t))^{n-1}$ for all $n \in \mathbb{N}^+$. But clearly, for all $n \in \mathbb{N}$, we have that $\widetilde{f}(n) \leq (1 + (L-1)c)^{n-1}$. The result is now immediate. $\square$

We complete this section by giving some intuition behind the definition of $f$-colorability. Naturally, the definition of $f$-colorability was calibrated so as to allow us to prove Lemmas 2.5, 2.6, and 2.7. The most challenging part of coming up with a "correct" definition was the need for amalgams to preserve $f$-colorability. So let us give a brief outline of this fact (the fact that amalgams preserve $f$-colorability); in passing, we will comment on why various conditions from the definition of $f$-colorability were needed to make the proof go through.

Let $f : \mathbb{N} \to \mathbb{N}$ be a superadditive function, let $G$ be an graph, all of whose proper induced subgraphs are $f$-colorable, and assume that $G$ admits an amalgam $(K, A, B, C, D)$. We need to show that $G$ is $f$-colorable. Fix a weight function $w : V(G) \to \mathbb{N}^+$ of $G$ and a $G$-admissible clique $Q$; we need to exhibit an $(f; w; Q)$-valid coloring of $G$. After some "preprocessing," we can reduce the problem to the case when $K_G = \emptyset$, $R_{G[B]} = B$, $R_{G[C]} = C$, and $Q \subseteq K \cup A \cup B$. We now "decompose" $G$ into graphs $G_b$ and $G_c$ as in Figure 1.3 (thus, $G_b$ is obtained from $G \smallsetminus A$ by "shrinking" $B$ to a vertex $b$, and $G_c$ is obtained from $G \smallsetminus D$ by "shrinking" $C$ to a vertex $c$). Clearly, $G_b$ and $G_c$ are (isomorphic to) proper induced subgraphs of $G$, and so they are $f$-colorable. We define a weight function $w_b : V(G_b) \to \mathbb{N}^+$ by setting $w_b \restriction (K \cup C \cup D) = w \restriction (K \cup C \cup D)$ and $w_b(b) = w(G[B])$, and we define $w_c$ analogously; thus, $w_b(G_b), w_c(G_c) \leq w(G)$. Since $Q \subseteq K \cup A \cup B$, it is easy to see that $Q$ is a $G_c$-admissible clique, and that $Q \cap K$ is a $G_b$-admissible clique. Ideally, we would like to construct an $(f; w_b; Q \cap K)$-valid coloring of $G_b$ and an $(f; w_c, Q)$-valid coloring of $G_c$, and then put these two colorings together in order to obtain an $(f; w; Q)$-valid coloring of $G$. Since $K$ is a clique in both $G_b$ and $G_c$, and since $w_b \restriction K = w_c \restriction K$, it is easy to arrange (by permuting colors if necessary) that our $(f; w_b; Q \cap K)$-valid coloring of $G_b$ and our $(f; w_c; Q)$-valid coloring of $G_c$ agree on $K$. However, we run into a different problem: in order to be able to combine the colorings of $G_b$ and $G_c$ in a suitable way, we need to ensure that every color used on the set $C$ by the coloring of $G_b$ is used on the vertex $c$ by the coloring of $G_c$ (for otherwise, there is no obvious way to combine the two colorings so that the resulting coloring of $G$ uses disjoint color-sets on $B$ and $C$; disjoint color-sets must be used on $B$ and $C$ because $B$ and $C$ are disjoint and complete to each other in $G$). Consequently, we need a way to ensure that the number of colors used on the set $C$ is no greater than the number of colors used on the vertex $c$, and for this, we need that at most $f(w_c(c)) = f(w(G[C]))$ colors get used on $C$. (We remark that this is why it is essential that an

$(f; w; Q)$-valid coloring of $G$ should assign $f(w(v))$ colors, and not just $w(v)$ colors, to each vertex $v$ of $G$. In other words, this is why we need condition (b) in our definition.) In order to accomplish this, we use $b$ as a "marker vertex" in $G_b$, and we observe that $\Gamma_{G_b}(b) = K \cup C$. We now "add" $b$ to the $G_b$-admissible clique $Q \cap K$, and we observe that (thanks to our "preprocessing") we have that $R_{G_b}(b) = R_{G[K \cup C]} = R_{G[C]} = C$. We now apply condition (d) of our definition, and we conclude that the number of colors that our $(f; w_b; \{b\} \cup (Q \cap K))$-valid coloring of $G_b$ uses on the set $C$ is at most $f(w(G[C]))$, which is what we need. (We remark that the reason we need condition (d) in our definition is precisely so that we could "separate" $C$ from $K$ in the neighborhood of $b$ in $G_b$, and then ensure that the number of colors used on $C$ is not "too large" relative to $w(G[C])$.)

Our preceding comments clarify why we need conditions (b) and (d) in our definition of an $(f; w; Q)$-valid coloring of $G$. What about condition (c)? Could we not replace this condition with the (perhaps more natural) condition that $|\bigcup \phi[V(G)]| \le f(w(G))$? In fact, we need condition (c) so that we could prove that proper homogeneous sets preserve $f$-colorability. Indeed, suppose that $G$ is a graph that admits a proper homogeneous set $X$, and suppose that all proper induced subgraphs of $G$ are $f$-colorable. Let $w : V(G) \to \mathbb{N}^+$ be a weight function for $G$, and let $Q$ be a $G$-admissible clique. We decompose $G$ into $G[X]$ and the graph $G_0$ obtained from $G$ by "shrinking" $X$ to a vertex $x$ (see Figure 1.1). For the sake of simplicity, let us focus on the case when $\Gamma_G(u) = X$ for some vertex $u \in Q$. In this case, we have that $R_G(u) = R_{G[X]}$. Now, we wish to color $G_0$ and $G[X]$, and then put the two colorings together to produce an $(f; w; Q)$-valid coloring of $G$. Since $R_G(u) = R_{G[X]}$, the only way to ensure that condition (d) is satisfied for the vertex $u \in Q$ in our coloring of $G$ is to first ensure that our coloring of $G[X]$ satisfies condition (c).

Finally, one might ask why our definition of an $(f; w; Q)$-valid coloring involves a clique of "designated vertices" (the $G$-admissible clique $Q$) rather than just one designated vertex. The clique is necessary in order to make the induction (for amalgams) go through. For suppose that we were allowed just one designated vertex $v$ of $G$. If $v \in K$, then $v$ would be "inherited" by both $G_b$ and $G_c$, and we would not be able to "add" another designated vertex to either block. But as explained above, we need to be able to add $b$ to the collection of designated vertices of $G_b$. Thus, we need an unlimited number of designated vertices. It is not surprising that we do not allow our collection of designated vertices to be completely arbitrary, and that instead, we require it to form a clique. The fact that we require this clique to be $G$-admissible may seem a bit mysterious, though. However, this requirement is crucial for the proof of Lemma 2.5. There, we fix a graph $G \in \mathcal{G}$, a weight function $w : V(G) \to \mathbb{N}^+$ for $G$, and a $G$-admissible clique $Q$, and after

some "preprocessing," we construct a function $\phi^* : V(G) \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ that satisfies conditions (a), (b), and (c) of the definition of an $(\widetilde{f}; w; Q)$-valid coloring of $G$, but may fail to satisfy condition (d) (here, $\mathcal{G}$ and $\widetilde{f}$ are as in the statement of Lemma 2.5). Now, since $Q$ is a $G$-admissible clique, we see that the sets $R_G(u)$, with $u \in Q$, form a nested sequence, and this allows us to recursively "modify" $\phi^*$ on these sets until we obtain a function that satisfies condition (d). At each recursive step, we make sure that the new function still satisfies conditions (a), (b), and (c), and so when recursion is complete, we obtain an $(\widetilde{f}; w; Q)$-valid coloring of $G$. If the clique $Q$ were arbitrary (rather than $G$-admissible), then the sets $R_G(u)$, with $u \in Q$, could intersect in an essentially arbitrary fashion, and so this sort of recursive modification would not be possible. We remark that the fact that $Q$ is $G$-admissible is also used in the proof of Proposition 3.3 (which is in turn used to prove Lemma 2.6), but there, the "recursive modification" is far simpler, and we omit the details here.

# 3   Proof of Lemmas 2.5 and 2.6

In this section, we prove Lemmas 2.5 and 2.6. The proof of Lemma 2.5 relies on Lemma 2.6, and so we prove Lemma 2.6 first. We obtain Lemma 2.6 as a corollary of three lemmas: Lemma 3.2 (which states that proper homogeneous sets preserve $f$-colorability), Lemma 3.4 (which states clique-cutsets preserve $f$-colorability), and Lemma 3.5 (which states that amalgams preserve $f$-colorability). We begin with a technical proposition, which we will use in the proofs of Lemmas 3.2 and 3.5.

**Proposition 3.1.** *Let $G$ be a graph, and let $(X, Y, Z)$ be a homogeneous set partition of $G$. Let $x$ be a vertex that does not belong to $V(G)$, and let $G_0$ be the graph with vertex-set $V(G_0) = \{x\} \cup Y \cup Z$, and adjacency as follows: $G_0[Y \cup Z] = G[Y \cup Z]$, and $x$ is complete to $Y$ and anti-complete to $Z$ in $G$ (in other words, $G_0$ is obtained from $G$ by "shrinking" $X$ to the vertex $x$; see Figure 1.1). Let $w : V(G) \to \mathbb{N}^+$ be a weight function for $G$, and let $Q$ be a $G$-admissible clique. If $Q \cap X = \emptyset$, then set $Q_0 = Q$, and if $Q \cap X \neq \emptyset$, then set $Q_0 = (Q \smallsetminus X) \cup \{x\}$. Then $Q_0$ is a $G_0$-admissible clique, and $Q \cap X$ is a $G[X]$-admissible clique. Define $w_0 : V(G_0) \to \mathbb{N}^+$ by setting*

$$w_0(v) \;\; = \;\; \begin{cases} w(G[X]) & \text{if} \quad v = x \\[2mm] w(v) & \text{if} \quad v \neq x \end{cases}$$

*for all $v \in V(G_0)$. Then $w_0(G_0) = w(G)$. Let $\phi_0 : V(G_0) \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ be an $(f; w_0; Q_0)$-valid coloring of $G_0$, let $\phi_X : X \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ be an $(f; w; Q \cap X)$-valid coloring of $G[X]$, and assume that $\bigcup \phi_X[X] \subseteq \phi_0(x)$. Let $\phi : V(G) \to$*

$\mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ *be given by*

$$\phi(v) \quad = \quad \begin{cases} \phi_X(v) & \text{if} \quad v \in X \\[2mm] \phi_0(v) & \text{if} \quad v \in Y \cup Z \end{cases}$$

*for all $v \in V(G)$. Then $\phi$ is an $(f; w; Q)$-valid coloring of $G$.*

*Proof.* It is clear that $Q_0$ is a $G_0$-admissible clique, that $Q \cap X$ is a $G[X]$-admissible clique, and that $w_0(G_0) = w(G)$. It remains to show that $\phi$ is an $(f; w; Q)$-valid coloring of $G$. We must verify that $\phi$ satisfies all the following:

(a) $\phi(v_1) \cap \phi(v_2) = \emptyset$ for all $v_1 v_2 \in E(G)$;

(b) $|\phi(v)| = f(w(v))$ for all $v \in V(G)$;

(c) $|\bigcup \phi[R_G]| \leq f(w(G[R_G]))$;

(d) $|\bigcup \phi[R_G(u)]| \leq f(w(G[R_G(u)]))$ for all $u \in Q$.

We first prove that $\phi$ satisfies (a). Fix $v_1 v_2 \in E(G)$; we must show that $\phi(v_1) \cap \phi(v_2) = \emptyset$. If $v_1, v_2 \in X$, then this follows from the fact that $\phi_X$ is an $(f; w; Q \cap X)$-valid coloring of $G[X]$, and if $v_1, v_2 \in Y \cup Z$, then this follows from the fact that $\phi_0$ is an $(f; w_0; Q_0)$-valid coloring of $G_0$. So assume that one of $v_1$ and $v_2$ belongs to $X$, and the other one belongs to $Y \cup Z$; by symmetry, we may assume that $v_1 \in X$ and $v_2 \in Y \cup Z$. Since $X$ is anti-complete to $Z$ in $G$, it follows that $v_2 \in Y$, and so $xv_2 \in E(G_0)$. Since $\phi_0$ is an $(f; w_0; Q_0)$-valid coloring of $G_0$, it follows that $\phi_0(x) \cap \phi_0(v_2) = \emptyset$. Since $v_1 \in X$, $\phi \upharpoonright X = \phi_X$, and $\bigcup \phi_X[X] \subseteq \phi_0(x)$, we have that $\phi(v_1) = \phi_X(v_1) \subseteq \phi_0(x)$. Since $v_2 \in Y$ and $\phi \upharpoonright (Y \cup Z) = \phi_0 \upharpoonright (Y \cup Z)$, we have that $\phi(v_2) = \phi_0(v_2)$. It follows that $\phi(v_1) \cap \phi(v_2) \subseteq \phi_0(x) \cap \phi_0(v_2) = \emptyset$. This proves that $\phi$ satisfies (a).

The fact that $\phi$ satisfies (b) follows immediately from the fact that $\phi_0$ is an $(f; w_0; Q_0)$-valid coloring of $G_0$, and the fact that $\phi_X$ is an $(f; w; Q \cap X)$-valid coloring of $G[X]$.

We next prove that $\phi$ satisfies (c). We consider two cases: when $x \in K_{G_0}$, and when $x \in R_{G_0}$. Suppose first that $x \in K_{G_0}$. Then $R_G = R_{G_0} \cup R_{G[X]}$, and $R_{G_0}$ is complete to $R_{G[X]}$ in $G$; consequently, $w(G[R_G]) = w(G[R_{G_0}]) + w(G[R_{G[X]}]) = w_0(G_0[R_{G_0}]) + w(G[R_{G[X]}])$. Since $f$ is superadditive, it follows that $f(w_0(G_0[R_{G_0}])) + f(w(G[R_{G[X]}])) \leq f(w(G[R_G]))$. Further, it is clear that $\bigcup \phi[R_G] = (\bigcup \phi_0[R_{G_0}]) \cup (\bigcup \phi_X[R_{G[X]}])$. Since $\phi_0$ is an $(f; w_0; Q_0)$-valid coloring of $G_0$, we know that $|\bigcup \phi_0[R_{G_0}]| \leq f(w_0(G_0[R_{G_0}]))$, and since

$\phi_X$ is an $(f; w; Q \cap X)$-valid coloring of $G[X]$, we know that $|\bigcup \phi[R_{G[X]}]| \le f(w(G[R_{G[X]}]))$. It now follows that

$$
\begin{aligned}
|\bigcup \phi[R_G]| &\le |\bigcup \phi_0[R_{G_0}]| + |\bigcup \phi_X[R_{G[X]}]| \\[2mm]
&\le f(w_0(G_0[R_{G_0}])) + f(w(G[R_{G[X]}])) \\[2mm]
&\le f(w(G[R_G])),
\end{aligned}
$$

which is what we needed.

Suppose now that $x \in R_{G_0}$. Then $R_G = (R_{G_0} \smallsetminus \{x\}) \cup X$; furthermore, $G_0[R_{G_0}]$ is obtained from $G[R_G]$ by "shrinking" $X$ to the vertex $x$, and we easily deduce that $w_0(G_0[R_{G_0}]) = w(G[R_G])$. By construction, we have that $\bigcup \phi_X[X] \subseteq \phi_0(x)$, and consequently,

$$
\begin{aligned}
\bigcup \phi[R_G] &= (\bigcup \phi_0[R_{G_0} \smallsetminus \{x\}]) \cup (\bigcup \phi_X[X]) \\[2mm]
&\subseteq (\bigcup \phi_0[R_{G_0} \smallsetminus \{x\}]) \cup \phi_0(x) \\[2mm]
&= \bigcup \phi_0[R_{G_0}].
\end{aligned}
$$

Since $\phi_0$ is an $(f; w_0; Q_0)$-valid coloring of $G_0$, we know that $|\bigcup \phi_0[R_{G_0}]| \le f(w_0(G_0[R_{G_0}]))$. Since $w_0(G_0[R_{G_0}]) = w(G[R_G])$, it follows that $|\bigcup \phi[R_G]| \le |\bigcup \phi_0[R_{G_0}]| \le f(w_0(G_0[R_{G_0}])) = f(w(G[R_G]))$, which is what we needed. This proves that $\phi$ satisfies (c).

It remains to show that $\phi$ satisfies (d). Fix $u \in Q$; we must show that $|\bigcup \phi[R_G(u)]| \le f(w(G[R_G(u)]))$. We consider three cases: when $u \in X$, when $u \in Y$, and when $u \in Z$.

Suppose first that $u \in X$. Then $u \in Q \cap X$ and $x \in Q_0$. Note that $R_G(u) = R_{G_0}(x) \cup R_{G[X]}(u)$, and furthermore, $R_{G_0}(x)$ is complete to $R_{G[X]}(u)$ in $G$; consequently, $w(G[R_G(u)]) = w(G[R_{G_0}(x)]) + w(G[R_{G[X]}(u)])$. Since $f$ is superadditive, it follows that $f(w(G[R_{G_0}(x)])) + f(w(G[R_{G[X]}(u)])) \le f(w(G[R_G(u)]))$. Since $\phi_0$ is an $(f; w_0; Q_0)$-valid coloring of $G_0$, and $x \in Q_0$, we see that $|\bigcup \phi_0[R_{G_0}(x)]| \le f(w_0(G_0[R_{G_0}(x)])) = f(w(G[R_{G_0}(x)]))$; and since $\phi_X$ is an $(f; w; Q \cap X)$-valid coloring of $G[X]$ and $u \in Q \cap X$, we see that $|\bigcup \phi_X[R_{G[X]}(u)]| \le f(w(G[R_{G[X]}(u)]))$. Thus,

$$
\begin{aligned}
|\bigcup \phi[R_G(u)]| &= |(\bigcup \phi_0[R_{G_0}(x)]) \cup (\bigcup \phi_X[R_{G[X]}(u)])| \\[2mm]
&\le |\bigcup \phi_0[R_{G_0}(x)]| + |\bigcup \phi_X[R_{G[X]}(u)]| \\[2mm]
&\le f(w(G[R_{G_0}(x)])) + f(w(G[R_{G[X]}(u)])) \\[2mm]
&\le f(w(G[R_G(u)])),
\end{aligned}
$$

which is what we needed to show.

Suppose next that $u \in Z$. Then $u \in Q_0$, and $u$ is non-adjacent to $x$ in $G_0$ and anti-complete to $X$ in $G$. Thus, $R_G(u) = R_{G_0}(u)$ and $w(G[R_G(u)]) = w_0(G_0[R_{G_0}(u)])$. Since $\phi_0$ is an $(f; w_0; Q_0)$-valid coloring of $G$, we see that $|\bigcup \phi[R_G(u)]| = |\bigcup \phi_0[R_{G_0}(u)]| \leq f(w_0(G_0[R_{G_0}(u)])) = f(w(G[R_G(u)]))$, which is what we needed to show.

It remains to consider the case when $u \in Y$. Then $u \in Q_0$, and since $\phi_0$ is an $(f; w_0; Q_0)$-valid coloring of $G_0$, it follows that $|\bigcup \phi_0[R_{G_0}(u)]| \leq f(w_0(G_0[R_{G_0}(u)]))$. Further, we have that $x \in \Gamma_{G_0}(u)$, and so either $x \in K_{G_0}(u)$ or $x \in R_{G_0}(u)$.

Suppose first that $x \in K_{G_0}(u)$. Then $R_G(u) = R_{G_0}(u) \cup R_{G[X]}$, and furthermore, $R_{G_0}(u)$ is complete to $R_{G[X]}$ in $G$, so that $w(G[R_G(u)]) = w(G[R_{G_0}(u)]) + w(G[R_{G[X]}])$. Since $f$ is a superadditive function, it follows that $f(w(G[R_{G_0}(u)])) + f(w(G[R_{G[X]}])) \leq f(w(G[R_G(u)]))$. Since $\phi_X$ is an $(f; w; Q \cap X)$-valid coloring of $G[X]$, we know that $|\bigcup \phi_X[R_{G[X]}]| \leq f(w(G[R_{G[X]}]))$. We now have that

$$
\begin{aligned}
|\bigcup \phi[R_G(u)]| &= |(\bigcup \phi_0[R_{G_0}(u)]) \cup (\bigcup \phi_X[R_{G[X]}])| \\[2mm]
&\leq |\bigcup \phi_0[R_{G_0}(u)]| + |\bigcup \phi_X[R_{G[X]}]| \\[2mm]
&\leq f(w_0(G_0[R_{G_0}(u)])) + f(w(G[R_{G[X]}])) \\[2mm]
&= f(w(G[R_{G_0}(u)])) + f(w(G[R_{G[X]}])) \\[2mm]
&\leq f(w(G[R_G(u)])),
\end{aligned}
$$

which is what we needed.

Suppose now that $x \in R_{G_0}(u)$. Then $R_G(u) = (R_{G_0}(u) \smallsetminus \{x\}) \cup X$; furthermore, $G_0[R_{G_0}(u)]$ is obtained from $G[R_G(u)]$ by "shrinking" $X$ to the vertex $x$, and we easily deduce that $w_0(G_0[R_{G_0}(u)]) = w(G[R_G(u)])$. Further, by construction, we have that $\bigcup \phi_X[X] \subseteq \phi_0(x)$, and so we get the following:

$$
\begin{aligned}
\bigcup \phi[R_G(u)] &= \bigcup \phi[(R_{G_0}(u) \smallsetminus \{x\}) \cup X] \\[2mm]
&= (\bigcup \phi_0[R_{G_0}(u) \smallsetminus \{x\}]) \cup (\bigcup \phi_X[X]) \\[2mm]
&\subseteq (\bigcup \phi_0[R_{G_0}(u) \smallsetminus \{x\}]) \cup \phi_0(x) \\[2mm]
&= \bigcup \phi_0[R_{G_0}(u)].
\end{aligned}
$$

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It now follows that

$$
\begin{aligned}
|\bigcup \phi[R_G(u)]| &\leq |\bigcup \phi_0[R_{G_0}(u)]| \\
&\leq f(w_0(G_0[R_{G_0}(u)])) \\
&= f(w(G[R_G(u)])),
\end{aligned}
$$

which is what we needed to show. This proves that $\phi$ satisfies (d), and it follows that $\phi$ is an $(f; w; Q)$-valid coloring of $G$. This completes the argument. $\square$

We are now ready to prove Lemma 3.2, which states that proper homogeneous sets preserve $f$-colorability.

**Lemma 3.2.** *Let $f$ be a superadditive function, and let $G$ be a graph that admits a proper homogeneous set. Assume that all proper induced subgraphs of $G$ are $f$-colorable. Then $G$ is $f$-colorable.*

*Proof.* Let $w : V(G) \to \mathbb{N}^+$ be a weight function for $G$, and let $Q$ be a $G$-admissible clique. Let $X$ be a proper homogeneous set of $G$, and let $(X, Y, Z)$ be the associated homogeneous set partition of $G$. Let $x$ be a vertex that does not belong to $V(G)$, and let $G_0$, $w_0$, and $Q_0$ be as in the statement of Proposition 3.1. By Proposition 3.1, we have that $w_0(G_0) = w(G)$, that $Q_0$ is a $G_0$-admissible clique, and that $Q \cap X$ is a $G[X]$-admissible clique. Using the fact that $G_0$ is $f$-colorable (because $G_0$ is isomorphic to a proper induced subgraph of $G$), we fix an $(f; w_0; Q_0)$-valid coloring $\phi_0 : V(G_0) \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ of $G_0$. Then $|\phi_0(x)| = f(w_0(x)) = f(w(G[X]))$. Using the fact that $G[X]$ is $f$-colorable, we fix an $(f; w; Q \cap X)$-valid coloring $\phi_X : X \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ of $G[X]$. By Proposition 2.1, we have that $|\bigcup \phi_X[X]| \leq f(w(G[X])) = |\phi_0(x)|$, and so after possibly permuting colors, we may assume that $\bigcup \phi_X[X] \subseteq \phi_0(x)$. Proposition 3.1 now implies that the function $\phi : V(G) \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ given by

$$
\phi(v) = \begin{cases} \phi_X(v) & \text{if} \quad v \in X \\ \phi_0(v) & \text{if} \quad v \in Y \cup Z \end{cases}
$$

for all $v \in V(G)$ is an $(f; w; Q)$-valid coloring of $G$, and consequently, $G$ is $f$-colorable. $\square$

We now state and prove Proposition 3.3, a technical result that will be of use to us in the proof of Lemma 3.4 (which states that clique-cutsets preserve $f$-colorability), and also in the proof of Lemma 3.5 (which states that amalgams preserve $f$-colorability).

**Proposition 3.3.** *Let $G$ be a graph such that $K_G = \emptyset$, and let $(A, B, C)$ be a cut-partition of $G$. Let $w : V(G) \to \mathbb{N}^+$ be a weight function for $G$, and let*

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$Q$ be a $G$-admissible clique such that $Q \cap C \subseteq K_{G[C]}$. Set $Q_A = Q \smallsetminus B$ and $Q_B = Q \smallsetminus A$. Then $Q_A$ is a $G[A \cup C]$-admissible clique, and $Q_B$ is a $G[B \cup C]$-admissible clique. For each $X \in \{A, B\}$, let $\phi_X : X \cup C \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^+)$ be a function such that either

   *(1) $\phi_X$ is an $(f; w; Q_X)$-valid coloring of $G[X \cup C]$, or*

   *(2) $\phi_X$ satisfies all the following:*

      *(2.1) $\phi_X(v_1) \cap \phi_X(v_2) = \emptyset$ for all $v_1 v_2 \in E(G[X \cup C])$;*

      *(2.2) $|\phi_X(v)| = f(w(v))$ for all $v \in X \cup C$;*

      *(2.3) $|\bigcup \phi_X[X \cup C]| \leq f(w(G))$;*

      *(2.4) $|\bigcup \phi_X[R_G(u) \cap (X \cup C)]| \leq f(w(G[R_G(u)]))$ for all $u \in Q_X$.*

*Assume that $\phi_A \restriction C = \phi_B \restriction C$. Then there exists an $(f; w; Q)$-valid coloring of $G$.*

Before turning to its proof, let us first make a couple of remarks about Proposition 3.3. First of all, it is easy to see (and the details are given in the proof of Proposition 3.3) that if $\phi_X$ satisfies (1), then it also satisfies (2): conditions (2.1) and (2.2) from Proposition 3.3 are are identical to conditions (a) and (b), respectively, of the definition of an $(f; w; Q_X)$-valid coloring of $G[X \cup C]$, but conditions (3) and (4) are a bit weaker than conditions (c) and (d), respectively. Thus, Proposition 3.3 effectively gives us a way to "combine" functions $\phi_X$ (with $X \in \{A, B\}$) that are "almost" $(f; w; Q_X)$-valid colorings of $G[X \cup C]$ into an $(f; w; Q)$-valid coloring of $G$. The reason that we assume that $\phi_X$ satisfies (1) or (2), rather than simply assuming that it satisfies (2), is that this makes it easier to apply Proposition 3.3 directly (that is, without checking too many hypotheses) in the proofs of Lemmas 3.4 and 3.5. In the proof of Lemma 3.4 (which deals with clique-cutsets), our functions $\phi_A$ and $\phi_B$ both satisfy (1); in the proof of Lemma 3.5 (which deals with amalgams), one of the two functions satisfies (1), and the other one satisfies (2).

Let us now briefly discuss the idea of the proof of Proposition 3.3. As already stated, we first show that $\phi_A$ and $\phi_B$ both satisfy (2). Now, ideally, we would like to show that the function $\phi : V(G) \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^+)$ given by $\phi \restriction (A \cup C) = \phi_A$ and $\phi \restriction (B \cup C) = \phi_B$ (this is well-defined because $\phi_A \restriction C = \phi_B \restriction C$) is an $(f; w; Q)$-valid coloring of $G$. Since $\phi_A$ and $\phi_B$ satisfy conditions (2.1) and (2.2) from the statement of Proposition 3.3, the function $\phi$ defined in this way satisfies conditions (a) and (b) of the definition of an $(f; w; Q)$-valid coloring of $G$. Unfortunately, $\phi$ need not satisfy conditions (c) and (d). It may, for instance, be that $(\bigcup \phi_A[A]) \smallsetminus (\bigcup \phi_A[C])$ (the set of colors that $\phi_A$ uses on $A$ but not on $C$) and $(\bigcup \phi_B[B]) \smallsetminus (\bigcup \phi_B[C])$ (the set of colors that $\phi_B$ uses on $B$ but not on $C$) are non-empty and disjoint,

and so it is possible that $\phi$ uses "too many" colors on $G$, that is, more colors than condition (c) allows. Similar problems may occur with condition (d). In order to address this problem, we first "preprocess" $\phi_A$ and $\phi_B$ (that is, we rename colors if necessary) so that $\bigcup \phi_A[A \cup C], \bigcup \phi_B[B \cup C] \subseteq [f(w(G))]$ (here, we use the fact that $\phi_A$ and $\phi_B$ satisfy condition (2.3)); since $K_G = \emptyset$, this ensures that $\phi$ satisfies condition (c). To ensure that $\phi$ satisfies condition (d), we use the fact that the sets $R_G(u)$, with $u \in Q$, form a nested sequence, and if necessary, we permute colors so that for each $u \in Q$, one of $\bigcup \phi_A[R_G(u) \cap (A \cup C)]$ and $\bigcup \phi_B[R_G(u) \cap (B \cup C)]$ is included in the other. Using the fact that $\phi_A$ and $\phi_B$ satisfy condition (2.4), we then easily deduce that $\phi$ satisfies condition (d).

*Proof of Proposition 3.3.* It is clear that $Q_A$ is a $G[A \cup C]$-admissible clique, and that $Q_B$ is a $G[B \cup C]$-admissible clique. Now, let us show that the functions $\phi_X$ (with $X \in \{A, B\}$) satisfy (2); by symmetry, it suffices to prove this for $\phi_A$. By hypothesis, $\phi_A$ satisfies (1) or (2), and so we just need to show that if $\phi_A$ satisfies (1), then it also satisfies (2). So suppose that $\phi_A$ satisfies (1), that is, suppose that $\phi_A$ is an $(f; w; Q_A)$-valid coloring of $G[A \cup C]$. Then it is immediate from the definition that $\phi_A$ satisfies (2.1) and (2.2). Let us show that $\phi_A$ satisfies (2.3). Since $\phi_A$ is an $(f; w; Q_A)$-valid coloring of $G[A \cup C]$, Proposition 2.1 guarantees that $|\bigcup \phi_A[A \cup C]| \leq f(w(G[A \cup C]))$. Clearly, $w(G[A \cup C]) \leq w(G)$, and so since $f$ is superadditive (and therefore non-decreasing), it follows that $f(w(G[A \cup C])) \leq f(w(G))$. Thus, $|\bigcup \phi_A[A \cup C]| \leq f(w(G))$, that is, $\phi_A$ satisfies (2.3). It remains to show that $\phi_A$ satisfies (2.4). Fix $u \in Q$. Clearly, $R_{G[A \cup C]} \subseteq R_G(u) \cap (A \cup C) \leq \Gamma_{G[A \cup C]}(u)$, and so Proposition 2.1 guarantees that $|\bigcup \phi_A[R_G(u) \cap (A \cup C)]| \leq f(w(G[R_G(u) \cap (A \cup C)]))$. Clearly, $w(G[R_G(u) \cap (A \cup C)]) \leq w(G[R_G(u)])$, and so since $f$ is superadditive (and therefore non-decreasing), we have that $f(w(G[R_G(u) \cap (A \cup C)])) \leq f(w(G[R_G(u)]))$. Thus, $|\bigcup \phi_A[R_G(u) \cap (A \cup C)]| \leq f(w(G[R_G(u)]))$, that is, $\phi_A$ satisfies (2.4). This proves that $\phi_A$ satisfies (2), and similarly, $\phi_B$ satisfies (2).

Now, since $Q$ is a $G$-admissible clique, so is $Q \cap C$. Set $Q \cap C = \{u_1, \dots, u_k\}$ (with $k = |Q \cap C| \geq 0$) so that for all $i, j \in \{1, \dots, k\}$, if $i < j$, then $\Gamma_G[u_i] \subseteq \Gamma_G[u_j]$. For all $i \in \{1, \dots, k\}$, set $K_i = K_G(u_i)$, $R_i = R_G(u_i)$, $R_i^A = R_i \cap (A \cup C)$, and $R_i^B = R_i \cap (B \cup C)$. Clearly, for all $i, j \in \{1, \dots, k\}$, if $i < j$, then $R_i \subseteq R_j$, $R_i^A \subseteq R_j^A$, and $R_i^B \subseteq R_j^B$.

Since $\phi_A$ and $\phi_B$ satisfy (2.3), we have that $|\bigcup \phi_A[A \cup C]| \leq f(w(G))$ and $|\bigcup \phi_B[B \cup C]| \leq f(w(G))$. After possibly permuting colors, we may assume that $\bigcup \phi_A[A \cup C] \subseteq [f(w(G))]$ and $\bigcup \phi_B[B \cup C] \subseteq [f(w(G))]$.

Recall that $\phi_A \upharpoonright C = \phi_B \upharpoonright C$. Set $n_K = |\bigcup \phi_A[K_{G[C]}]| = |\bigcup \phi_B[K_{G[C]}]|$

and $n_R = |\bigcup \phi_A[R_{G[C]}]| = |\bigcup \phi_B[R_{G[C]}]|$. Since $K_{G[C]}$ is complete to $R_{G[C]}$ in $G$ (and therefore, in both $G[A \cup C]$ and $G[B \cup C]$), and since $\phi_A$ and $\phi_B$ both satisfy (2.1), we know that for each $X \in \{A, B\}$, the sets $\bigcup \phi_X[K_{G[C]}]$ and $\bigcup \phi_X[R_{G[C]}]$ are disjoint. Thus, after possibly permuting colors, we may assume that $\bigcup \phi_A[K_{G[C]}] = \bigcup \phi_B[K_{G[C]}] = [n_K]$ and $\bigcup \phi_A[R_{G[C]}] = \bigcup \phi_B[R_{G[C]}] = n_K + [n_R]$, and consequently, $\bigcup \phi_A[C] = \bigcup \phi_B[C] = [n_K + n_R]$. Further, we know that $R_1^A \subseteq \cdots \subseteq R_k^A$ and $R_1^B \subseteq \cdots \subseteq R_k^B$; thus, after possibly permuting colors, we may assume that for each $X \in \{A, B\}$, and all $i \in \{1, \ldots, k\}$, we have that $(\bigcup \phi_X[R_i^X]) \smallsetminus [n_K + n_R] = n_K + n_R + [|(\bigcup \phi_X[R_i^X]) \smallsetminus [n_K + n_R]|]$.

**Claim 1.** *For all $i \in \{1, \ldots, k\}$, one of $\bigcup \phi_A[R_i^A]$ and $\bigcup \phi_B[R_i^B]$ is included in the other.*

*Proof.* Fix $i \in \{1, \ldots, k\}$. By construction, one of $(\bigcup \phi_A[R_i^A]) \smallsetminus [n_K + n_R]$ and $(\bigcup \phi_B[R_i^B]) \smallsetminus [n_K + n_R]$ is included in the other. Thus, it suffices to show that $(\bigcup \phi_A[R_i^A]) \cap [n_K + n_R] = (\bigcup \phi_B[R_i^B]) \cap [n_K + n_R]$. We will prove the following stronger statement: $(\bigcup \phi_A[R_i^A]) \cap [n_K] = (\bigcup \phi_B[R_i^B]) \cap [n_K]$, $n_K + [n_R] \subseteq \bigcup \phi_A[R_i^A]$, and $n_K + [n_R] \subseteq \bigcup \phi_B[R_i^B]$.

We first show that $n_K + [n_R] \subseteq \bigcup \phi_A[R_i^A]$ and $n_K + [n_R] \subseteq \bigcup \phi_B[R_i^B]$. Since $u_i \in K_{G[C]}$, we see that $u_i$ is complete to $R_{G[C]}$ in $G$, and so $R_{G[C]} \subseteq R_i$; consequently, $R_{G[C]} \subseteq R_i^A$ and $R_{G[C]} \subseteq R_i^B$, and it follows that $n_K + [n_R] = \bigcup \phi_A[R_{G[C]}] \subseteq \bigcup \phi_A[R_i^A]$ and $n_K + [n_R] = \bigcup \phi_B[R_{G[C]}] \subseteq \bigcup \phi_B[R_i^B]$.

It remains to show that $(\bigcup \phi_A[R_i^A]) \cap [n_K] = (\bigcup \phi_B[R_i^A]) \cap [n_K]$. Fix $z \in [n_K]$; we need to show that $z \in \bigcup \phi_A[R_i^A]$ if and only if $z \in \bigcup \phi_B[R_i^B]$. By symmetry, it suffices to prove the "only if" part, and so we assume that $z \in \bigcup \phi_A[R_i^A]$, and we show that $z \in \bigcup \phi_B[R_i^B]$. Since $K_{G[C]}$ is a clique of $G[A \cup C]$, since $\phi_A$ satisfies (2.1), and since $z \in [n_K] = \bigcup \phi_A[K_{G[C]}]$, we know that there exists a unique vertex $u_z \in K_{G[C]}$ such that $z \in \phi_A(u_z)$. Since $u_i, u_z \in K_{G[C]}$, we see that $u_z \in \Gamma_G[u_i]$. Thus, $u_z \in \{u_i\} \cup (K_i \cap C) \cup (R_i \cap C)$. Now, $\{u_i\} \cup K_i$ is complete to $R_i$ in $G$, and therefore, $\{u_i\} \cup (K_i \cap C)$ is complete to $R_i^A$ in $G[A \cup C]$. Since $\phi_A$ satisfies (2.1), we deduce that $\bigcup \phi_A[\{u_i\} \cup (K_i \cap C)]$ and $\bigcup \phi_A[R_i^A]$ are disjoint. Since $z \in \bigcup \phi_A[R_i^A]$, we see that $z \notin \bigcup \phi_A[\{u_i\} \cup (K_i \cap C)]$. Thus, $u_z \notin \{u_i\} \cup (K_i \cap C)$, and consequently, $u_z \in R_i \cap C \subseteq R_i^B$. Since $\phi_A \restriction C = \phi_B \restriction C$, we see that $\phi_A(u_z) = \phi_B(u_z)$, and so $z \in \phi_A(u_z) = \phi_B(u_z) \subseteq \bigcup \phi_B[R_i^B]$. This proves the claim. $\square$

Using the fact that $\phi_A \restriction C = \phi_B \restriction C$, we now define the function $\phi : V(G) \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ by setting

$$\phi(v) \;\; = \;\; \begin{cases} \phi_A(v) & \text{if} \quad v \in A \cup C \\[2ex] \phi_B(v) & \text{if} \quad v \in B \cup C \end{cases}$$

for all $v \in V(G)$. We claim that $\phi$ is an $(f; w; Q)$-valid coloring of $G$. We must show that $\phi$ satisfies all the following:

(a) $\phi(v_1) \cap \phi(v_2) = \emptyset$ for all $v_1 v_2 \in E(G)$;

(b) $|\phi(v)| = f(w(v))$ for all $v \in V(G)$;

(c) $|\bigcup \phi[R_G]| \leq f(w(G[R_G]))$;

(d) $|\bigcup \phi[R_G(u)]| \leq f(w(G[R_G(u)]))$ for all $u \in Q$.

The fact that $\phi$ satisfies (a) follows from the fact that $(A, B, C)$ is a cut-partition of $G$, and so $E(G) = E(G[A \cup C]) \cup E(G[B \cup C])$, and from the fact that $\phi_A$ and $\phi_B$ satisfy (2.1). The fact that $\phi$ satisfies (b) follows from the fact that $\phi_A$ and $\phi_B$ satisfy (2.2). We next show that $\phi$ satisfies (c). We first observe that $\bigcup \phi[V(G)] = (\bigcup \phi_A[A \cup C]) \cup (\bigcup \phi_B[B \cup C])$, and we remind the reader that $\bigcup \phi_A[A \cup C], \bigcup \phi_B[B \cup C] \subseteq [f(w(G))]$; thus, $\bigcup \phi[V(G)] \subseteq [f(w(G))]$, and consequently, $|\bigcup \phi[V(G)]| \leq f(w(G))$. But by hypothesis, we have that $K_G = \emptyset$, and so $R_G = V(G)$. It now follows that $|\bigcup \phi[R_G]| \leq f(w(G[R_G]))$, that is, $\phi$ satisfies (c).

It remains to show that $\phi$ satisfies (d). Fix $u \in Q$; we must show that $|\bigcup \phi[R_G(u)]| \leq f(w(G[R_G(u)]))$. We consider two cases: when $u \in C$, and when $u \in A \cup B$. Suppose first that $u \in C$; then there exists some $i \in \{1, \ldots, k\}$ such that $u = u_i$, and so $R_G(u) = R_i = R_i^A \cup R_i^B$ and $\bigcup \phi[R_G(u)] = (\bigcup \phi_A[R_i^A]) \cup (\bigcup \phi_B[R_i^B])$. Since (by Claim 1) one of $\bigcup \phi_A[R_i^A]$ and $\bigcup \phi_B[R_i^B]$ is included in the other, it follows that either $\bigcup \phi[R_G(u)] = \bigcup \phi_A[R_i^A]$ or $\bigcup \phi[R_G(u)] = \bigcup \phi_B[R_i^B]$. Since $\phi_A$ and $\phi_B$ satisfy (2.4), we deduce that $|\bigcup \phi[R_G(u)]| \leq f(w(G[R_G(u)]))$, which is what we needed.

Suppose now that $u \in A \cup B$; by symmetry, we may assume that $u \in A$. Since $(A, B, C)$ is a cut-partition of $G$, we see that $R_G(u) = R_G(u) \cap (A \cup C)$, and it follows that $\bigcup \phi[R_G(u)] = \bigcup \phi_A[R_G(u) \cap (A \cup C)]$. Since $\phi_A$ satisfies (2.4), it follows that $|\bigcup \phi[R_G(u)]| \leq f(w(G[R_G(u)]))$, which is what we needed. Thus, $\phi$ satisfies (d), and it follows that $\phi$ is an $(f; w; Q)$-valid coloring of $G$. $\qed$

We are now ready to prove Lemmas 3.4 and 3.5, which state that clique-cutsets and amalgams, respectively, preserve $f$-colorability.

**Lemma 3.4.** *Let $f$ be a superadditive function, and let $G$ be a graph that admits a clique-cutset. Assume that all proper induced subgraphs of $G$ are $f$-colorable. Then $G$ is $f$-colorable.*

*Proof.* In view of Proposition 2.3, we may assume that $K_G = \emptyset$, and therefore, $R_G = V(G)$. Let $C$ be a clique-cutset of $G$, and let $(A, B, C)$ be an associated cut-partition of $G$; then $G[A \cup C]$ and $G[B \cup C]$ are proper induced

subgraphs of $G$, and therefore, $G[A \cup C]$ and $G[B \cup C]$ are $f$-colorable. To show that $G$ is $f$-colorable, we fix a weight function $w : V(G) \to \mathbb{N}^+$ for $G$ and a $G$-admissible clique $Q$; we need to show that there exists an $(f; w; Q)$-valid coloring of $G$.

Our goal is to apply Proposition 3.3. First, recall that $K_G = \emptyset$. Further, since $C$ is a clique of $G$, we see that $K_{G[C]} = C$, and consequently, $Q \cap C \subseteq K_{G[C]}$. Next, set $Q_A = Q \smallsetminus B$ and $Q_B = Q \smallsetminus A$; clearly, $Q_A$ is a $G[A \cup C]$-admissible clique, and $Q_B$ is a $G[B \cup C]$-admissible clique. Using the fact that $G[A \cup C]$ and $G[B \cup C]$ are both $f$-colorable, we fix an $(f; w; Q_A)$-valid coloring $\phi_A : A \cup C \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^+)$ of $G[A \cup C]$ and an $(f; w; Q_B)$-valid coloring $\phi_B : B \cup C \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^+)$ of $G[B \cup C]$; then $\phi_A$ and $\phi_B$ both satisfy condition (1) of Proposition 3.3. Since $C$ is a clique of $G$ (and therefore of both $G[A \cup C]$ and $G[B \cup C]$), we know that for all distinct $c_1, c_2 \in C$, we have that $\phi_A(c_1) \cap \phi_A(c_2) = \phi_B(c_1) \cap \phi_B(c_2) = \emptyset$, and we also know that for all $c \in C$, we have that $|\phi_A(c)| = |\phi_B(c)| = f(w(c))$. Thus, after possibly permuting colors, we may assume that $\phi_A \restriction C = \phi_B \restriction C$. We now see that the hypotheses of Proposition 3.3 are satisfied, and so Proposition 3.3 guarantees that there exists an $(f; w; Q)$-valid coloring of $G$, which is what we needed. $\qquad\square$

**Lemma 3.5.** *Let $f : \mathbb{N} \to \mathbb{N}$ be a superadditive function, and let $G$ be a graph that admits an amalgam. Assume that all proper induced subgraphs of $G$ are $f$-colorable. Then $G$ is $f$-colorable.*

*Proof.* In view of Proposition 2.3, we may assume that $K_G = \emptyset$, and therefore, $R_G = V(G)$. In view of Lemmas 3.2 and 3.4, we may assume that $G$ admits neither a proper homogeneous set nor a clique-cutset. By assumption, $G$ admits an amalgam. Choose an amalgam $(K, A, B, C, D)$ of $G$ that satisfies the property that for all amalgams $(K', A', B', C', D')$ of $G$, we have that $|K'| \leq |K|$.

**Claim 1.** *$A$ and $D$ are non-empty, $R_{G[K \cup B]} = R_{G[B]} = B$, and $R_{G[K \cup C]} = R_{G[C]} = C$.*

*Proof.* By symmetry, it suffices to show that $A \neq \emptyset$ and that $R_{G[K \cup B]} = R_{G[B]} = B$. By the definition of an amalgam, we know that $|A \cup B|, |C \cup D| \geq 2$. Thus, if $A = \emptyset$, then $2 \leq |B| \leq |V(G)| - 2$, and we deduce that $B$ is a proper homogeneous set of $G$, contrary to the fact that $G$ does not admit a proper homogeneous set. Thus, $A \neq \emptyset$. Next, since $K$ is a clique of $G$, complete to $B$ in $G$, we see that $K \subseteq K_{G[K \cup B]}$, and therefore, $R_{G[K \cup B]} \subseteq B$. In order to show that $R_{G[K \cup B]} = R_{G[B]} = B$, it now suffices to show that every vertex in $B$ has a non-neighbor in $B$. Suppose that some $b \in B$ is complete to $B \smallsetminus \{b\}$. If $B = \{b\}$, then $K \cup B$ is a clique-cutset of $G$, which contradicts the fact that $G$ does not admit a clique-cutset, and if

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$B \setminus \{b\} \neq \emptyset$, then $(K \cup \{b\}, A, B \setminus \{b\}, C, D)$ is an amalgam of $G$, contrary to the maximality of $K$. Thus, every vertex in $B$ has a non-neighbor in $B$, and it follows that $R_{G[K \cup B]} = R_{G[B]} = B$. This proves the claim. $\qquad\square$

Now, let $w : V(G) \rightarrow \mathbb{N}^+$ be a weight function for $G$, and let $Q$ be a $G$-admissible clique. We need to show that there exists an $(f; w; Q)$-valid coloring of $G$.

**Claim 2.** *Either $Q \subseteq K \cup A \cup B$ or $Q \subseteq K \cup C \cup D$.*

*Proof.* If $Q \cap A \neq \emptyset$, then using the fact that $A$ is anti-complete to $C \cup D$ and that $Q$ is a clique, we see that $Q \subseteq K \cup A \cup B$, and we are done. Similarly, if $Q \cap D \neq \emptyset$, then $Q \subseteq K \cup C \cup D$, and we are done. So assume that $Q \subseteq K \cup B \cup C$. If $Q \cap B = \emptyset$ or $Q \cap C = \emptyset$, then we are done. So suppose that $Q$ intersects both $B$ and $C$, and fix some $b \in Q \cap B$ and $c \in Q \cap C$. Since $Q$ is a $G$-admissible clique, it follows that one of $\Gamma_G[b]$ and $\Gamma_G[c]$ is included in the other; by symmetry, we may assume that $\Gamma_G[b] \subseteq \Gamma_G[c]$. Since $B$ is complete to $C$, we know that $C \subseteq \Gamma_G[b]$, and consequently, $C \subseteq \Gamma_G[c]$. But this is impossible because by Claim 1, $R_{G[C]} = C$, and so $c$ has a non-neighbor in $C$. This proves the claim. $\qquad\square$

By Claim 2 and by symmetry, we may assume that $Q \subseteq K \cup A \cup B$. We now construct graphs $G_b$ and $G_c$ as in Figure 1.3. Formally, let $b$ and $c$ be distinct vertices that do not belong to $V(G)$. Let $G_b$ be the graph with vertex-set $V(G_b) = K \cup C \cup D \cup \{b\}$, with adjacency as follows: $G_b[K \cup C \cup D] = G[K \cup C \cup D]$, and $b$ is complete to $K \cup C$ and anti-complete to $D$. Similarly, let $G_c$ be the graph with vertex-set $V(G_c) = K \cup A \cup B \cup \{c\}$, with adjacency as follows: $G_c[K \cup A \cup B] = G[K \cup A \cup B]$, and $c$ is complete to $K \cup B$ and anti-complete to $A$. By the definition of an amalgam, we know that $B$ and $C$ are non-empty and that $|A \cup B|, |C \cup D| \geq 2$; it is then easy to see that $G_b$ and $G_c$ are both (isomorphic to) proper induced subgraphs of $G$, and consequently, $G_b$ and $G_c$ are $f$-colorable. Now, we define $w_b : V(G_b) \rightarrow \mathbb{N}^+$ by setting

$$w_b(v) \;\; = \;\; \begin{cases} w(G[B]) & \text{if} \quad v = b \\[2mm] w(v) & \text{if} \quad v \neq b \end{cases}$$

for all $v \in V(G_b)$, and we define $w_c : V(G_c) \rightarrow \mathbb{N}^+$ by setting

$$w_c(v) \;\; = \;\; \begin{cases} w(G[C]) & \text{if} \quad v = c \\[2mm] w(v) & \text{if} \quad v \neq c \end{cases}$$

for all $v \in V(G_c)$. By construction, we have that $w_b(G_b) = w(G \setminus A)$ and $w_c(G_c) = w(G \setminus D)$; in particular then, $w_b(G_b), w_c(G_c) \leq w(G)$. Next,
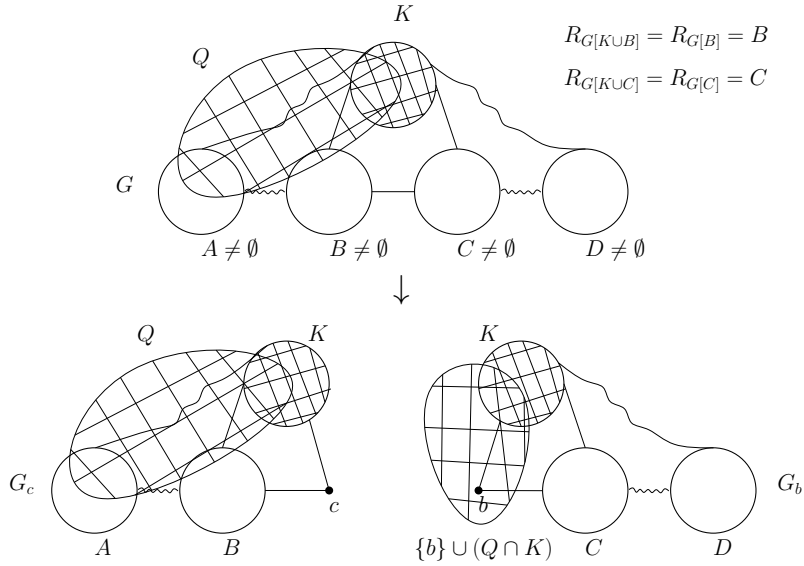
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Figure 3.1: Top: Graph $G$, amalgam $(K, A, B, C, D)$ of $G$, and $G$-admissible clique $Q$. Bottom left: Graph $G_c$ and $G_c$-admissible clique $Q$. Bottom right: Graph $G_b$ and $G_b$-admissible clique $\{b\} \cup (Q \cap K)$.

since $Q \subseteq A \cup B \cup K$ is a $G$-admissible clique, we see that $Q$ is a $G_c$-admissible clique, and that $Q \cap K$ is a $G_b$-admissible clique. Further, note that $\{b\} \cup (Q \cap K)$ is a clique of $G_b$, and that $\Gamma_{G_b}[b] = \{b\} \cup K \cup C$, whereas for all $u \in K$, we have that $\{b\} \cup K \cup C \subseteq \Gamma_G[u]$, and consequently, $\Gamma_G[b] \subseteq \Gamma_G[u]$. Thus, $\{b\} \cup (Q \cap K)$ is a $G_b$-admissible clique. Using the fact that $G_b$ and $G_c$ are $f$-colorable, we now fix an $(f; w_b; \{b\} \cup (Q \cap K))$-valid coloring $\phi_b : V(G_b) \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^+)$ of $G_b$ and an $(f; w_c; Q)$-valid coloring $\phi_c : V(G_c) \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^+)$ of $G_c$.

The situation to which we have reduced our problem is represented in Figure 3.1. Before continuing with the technical details, we give a brief outline of the remainder of the proof. We first "preprocess" $\phi_b$ and $\phi_c$ (by permuting colors if necessary) so as to ensure that $\phi_b \upharpoonright K = \phi_c \upharpoonright K$ and $\bigcup \phi_b[C] \subseteq \phi_c(c)$. We then apply Proposition 3.1 to $\phi_c$ and $\phi_b \upharpoonright C$ in order to obtain an $(f; w; Q)$-valid coloring $\phi_c^*$ of $G \smallsetminus D$. We then apply Proposition 3.3 to the cut-partition $(A \cup B, D, K \cup C)$ of $G$ and the functions $\phi_c^*$ and $\phi_b \upharpoonright (K \cup C \cup D)$, and we obtain an $(f; w; Q)$-valid coloring of $G$, which is what we need. We remark that the function $\phi_b \upharpoonright (K \cup C \cup D)$ need not be an $(f; w; Q \cap K)$-valid coloring of $G[K \cup C \cup D]$; however, we will see that the function $\phi_b \upharpoonright (K \cup C \cup D)$ (playing the role of $\phi_B$ from Proposition 3.3) does satisfy the hypotheses of Proposition 3.3.

Since $\phi_b$ is an $(f; w_b; \{b\} \cup (Q \cap K))$-valid coloring of $G_b$, we see that for all $u \in K$, we have that $|\phi_b(u)| = f(w_b(u)) = f(w(u))$, and since $K$ is a clique of $G_b$, we see that for all distinct $u_1, u_2 \in K$, we have that $\phi_b(u_1) \cap \phi_b(u_2) = \emptyset$. Similarly, since $\phi_c$ is an $(f; w_c; Q)$-valid coloring of $G_c$, we see that for all $u \in K$, we have that $|\phi_c(u)| = f(w_c(u)) = f(w(u))$, and that for all distinct $u_1, u_2 \in K$, we have that $\phi_c(u_1) \cap \phi_c(u_2) = \emptyset$. Thus, after possibly permuting colors, we may assume that $\phi_b \restriction K = \phi_c \restriction K$. Set $n_K = \sum_{u \in K} f(w(u))$. Then $|\bigcup \phi_b[K]| = |\bigcup \phi_c[K]| = n_K$, and by symmetry, we may assume that $\bigcup \phi_b[K] = \bigcup \phi_c[K] = [n_K]$.

Set $n_C = |\phi_b[C]|$. By Claim 1, we have that $R_{G_b}(b) = R_{G[\Gamma_{G_b}(b)]} = R_{G[K \cup C]} = R_{G[C]} = C$; since $\phi_b$ is an $(f; w_b; \{b\} \cup (Q \cap K))$-valid coloring of $G_b$, it follows that $n_C = |\bigcup \phi_b[C]| \leq f(w_b(G[C])) = f(w(G[C]))$. Further, since $\phi_b$ is an $(f; w_b; \{b\} \cup (Q \cap K))$-valid coloring of $G_b$, and since $C$ is complete to $K$ in $G_b$, we know that $\bigcup \phi_b[C]$ and $\bigcup \phi_b[K]$ are disjoint, and so by symmetry, we may assume that $\bigcup \phi_b[C] = n_K + [n_C]$. Next, since $\phi_c$ is an $(f; w_c; Q)$-valid coloring of $G_c$, we know that $|\phi_c(c)| = f(w_c(c)) = f(w(G[C])) \geq n_C$. Since $c$ is complete to $K$ in $G_c$, we know that $\phi_c(c)$ and $\bigcup \phi_c[K]$ are disjoint, and so by symmetry, we may assume that $n_K + [n_C] \subseteq \phi_c(c)$. We now have that $\bigcup \phi_b[C] = n_K + [n_C] \subseteq \phi_c(c)$.

We now define $\phi_c^* : V(G) \smallsetminus D \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ by setting

$$
\phi_c^*(v) \;\; = \;\; \begin{cases} \phi_c(v) & \text{if} \quad v \in K \cup A \cup B \\[2mm] \phi_b(v) & \text{if} \quad v \in C \end{cases}
$$

for all $v \in V(G) \smallsetminus D$. Since $\phi_c \restriction K = \phi_b \restriction K$, we see that $\phi_c^* \restriction (K \cup C) = \phi_b \restriction (K \cup C)$. Further, since $Q \subseteq K \cup A \cup B$ is a $G$-admissible clique, we see that $Q$ is a $(G \smallsetminus D)$-admissible clique.

**Claim 3.** $\phi_c^*$ is an $(f; w; Q)$-valid coloring of $G \smallsetminus D$.

*Proof.* Our goal is to apply Proposition 3.1 to the graphs $G \smallsetminus D$, $G_c$, and $G[C]$ (playing the roles of $G$, $G_0$, and $G[X]$, respectively, from Proposition 3.1), the vertex $c$ (playing the role of $x$ from Proposition 3.1), and the functions $\phi_c$, $\phi_b \restriction C$, and $\phi_c^*$ (playing the roles of $\phi_0$, $\phi_X$, and $\phi$, respectively, from Proposition 3.1). Recall that $\bigcup \phi_b[C] \subseteq \phi_c(c)$ and $Q \cap C = \emptyset$. Thus, the only thing left to show in order to verify that the hypotheses of Proposition 3.1 are satisfied is that $\phi_b \restriction C$ is an $(f; w; \emptyset)$-valid coloring of $G[C]$. For this, we must show that $\phi_b \restriction C$ satisfies all the following:

(a) $\phi_b(v_1) \cap \phi_b(v_2) = \emptyset$ for all $v_1 v_2 \in E(G[C])$;

(b) $|\phi_b(v)| = f(w(v))$ for all $v \in C$;

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(c) $|\bigcup \phi_b[R_{G[C]}]| \le f(w(G[R_{G[C]}]))$;

(d) $|\bigcup \phi_b[R_{G[C]}(u)]| \le f(w(G[R_{G[C]}(u)]))$ for all $u \in \emptyset$.

The fact that $\phi_b \restriction C$ satisfies (a) and (b) follows immediately from the fact that $\phi_b$ is an $(f; w_b; \{b\} \cup (Q \cap K))$-valid coloring of $G_b$. For condition (d), there is nothing to show. It remains to prove that $\phi_b \restriction C$ satisfies (c). Since $\phi_b$ is an $(f; w_b; \{b\} \cup (Q \cap K))$-valid coloring of $G_b$, we know that $|\bigcup \phi_b[R_{G_b}(b)]| \le f(w_b(G_b[R_{G_b}(b)]))$. But now recall that $R_{G_b}(b) = R_{G[K \cup C]} = R_{G[C]} = C$. Thus, $|\bigcup \phi_b[R_{G[C]}]| \le f(w(G[R_{G[C]}]))$, and so $\phi_b \restriction C$ satisfies (c). Thus, the hypotheses of Proposition 3.1 are satisfied, and it follows that $\phi_c^*$ is an $(f; w; Q)$-valid coloring of $G \smallsetminus D$. This proves the claim. $\qquad \square$

Since (by Claim 1) $D \ne \emptyset$, we see that $(A \cup B, D, K \cup C)$ is a cut-partition of $G$. Our goal is to apply Proposition 3.3 to the cut-partition $(A \cup B, D, K \cup C)$, and the functions $\phi_c^*$ and $\phi_b \restriction (K \cup C \cup D)$. First, recall that $K_G = \emptyset$. Next, by construction, we have that $\phi_c^* \restriction (K \cup C) = \phi_b \restriction (K \cup C)$. Further, since $Q \subseteq K \cup A \cup B$, Claim 1 implies that $Q \cap (K \cup C) \subseteq K = K_{G[K \cup C]}$. By Claim 3, we know that $\phi_c^*$ is an $(f; w; Q)$-valid coloring of $G \smallsetminus D = G[K \cup A \cup B \cup C]$, and so $\phi_c^*$ satisfies condition (1) of Proposition 3.3. Thus, to show that the hypotheses of Proposition 3.3 are satisfied, it only remains to show that $\phi_b \restriction (K \cup C \cup D)$ satisfies condition (1) or (2) from the statement of Proposition 3.3; in our next claim (Claim 4), we prove that $\phi_b \restriction (K \cup C \cup D)$ satisfies condition (2).

**Claim 4.** *The function $\phi_b \restriction (K \cup C \cup D)$ satisfies all the following:*

*(1) $\phi_b(v_1) \cap \phi_b(v_2) = \emptyset$ for all $v_1 v_2 \in E(G[K \cup C \cup D])$;*

*(2) $|\phi_b(v)| = f(w(v))$ for all $v \in K \cup C \cup D$;*

*(3) $|\bigcup \phi_b[K \cup C \cup D]| \le f(w(G))$;*

*(4) $|\bigcup \phi_b[R_G(u) \cap (K \cup C \cup D)]| \le f(w(G[R_G(u)]))$ for all $u \in Q \cap (K \cup C \cup D)$.*

*Proof.* The fact that $\phi_b \restriction (K \cup C \cup D)$ satisfies (1) and (2) follows immediately from the fact that $\phi_b$ is an $(f; w_b; \{b\} \cup (Q \cap K))$-valid coloring of $G_b$, and the fact that $w_b \restriction (K \cup C \cup D) = w \restriction (K \cup C \cup D)$. Next, since $\phi_b$ is an $(f; w_b; \{b\} \cup (Q \cap K))$-valid coloring of $G_b$, Proposition 2.1 guarantees that $|\bigcup \phi_b[V(G_b)]| \le f(w_b(G_b))$. We know that $w_b(G_b) = w(G \smallsetminus A) \le w(G)$, and so since $f$ is superadditive (and therefore non-decreasing), we have that $f(w_b(G_b)) \le f(w(G))$. It follows that $|\bigcup \phi_b[K \cup C \cup D]| \le |\bigcup \phi_b[V(G_b)]| \le f(w_b(G_b)) \le f(w(G))$, and so $\phi_b \restriction (K \cup C \cup D)$ satisfies (3).

It remains to show that $\phi_b \restriction (K \cup C \cup D)$ satisfies (4). Fix $u \in Q \cap (K \cup C \cup D)$;

we must show that $|\bigcup \phi_b[R_G(u) \cap (K \cup C \cup D)]| \leq f(w(G[R_G(u)]))$. Recall that $Q \subseteq K \cup A \cup B$, and so $u \in Q \cap K$. Since $u \in K$, we have that $B \subseteq \Gamma_G(u)$, and so by Claim 1, we have that $B \subseteq R_G(u)$. Now, set $R_b = (R_G(u) \cap (K \cup C \cup D)) \cup \{b\}$; then $G_b[R_b]$ is obtained from $G[R_G(u) \smallsetminus A]$ by "shrinking" $B$ to the vertex $b$, and we easily deduce that $w_b(G_b[R_b]) = w(G[R_G(u) \smallsetminus A]) \leq w(G[R_G(u)])$. Since $f$ is a superadditive (and therefore non-decreasing) function, we deduce that $f(w_b(G_b[R_b])) \leq f(w(G[R_G(u)]))$. Next, it is easy to see that $R_{G_b}(u) \subseteq R_b \subseteq \Gamma_{G_b}(u)$, and so since $\phi_b$ is an $(f; w; \{b\} \cup (Q \cap K))$-valid coloring of $G_b$, Proposition 2.1 guarantees that $|\bigcup \phi_b[R_b]| \leq f(w_b(G_b[R_b]))$; since $f(w_b(G_b[R_b])) \leq f(w(G[R_G(u)]))$, it follows that $|\bigcup \phi_b[R_b]| \leq f(w(G[R_G(u)]))$. But by construction, $R_G(u) \cap (K \cup C \cup D) \subseteq R_b$, and so we deduce that $|\bigcup \phi_b[R_G(u) \cap (K \cup C \cup D)]| \leq f(w(G[R_G(u)]))$, which is what we needed. This proves the claim. $\qquad \square$

Using Claim 4, the paragraph that precedes it, and Proposition 3.3, we deduce that $G$ admits an $(f; w; Q)$-valid coloring. This completes the argument. $\qquad \square$

Clearly, Lemmas 3.2, 3.4, and 3.5 imply Lemma 2.6. It now only remains to prove Lemma 2.5, restated below for the reader's convenience.

**Lemma 2.5.** *Let $\mathcal{G}$ be a hereditary class, $\chi$-bounded by a non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ that satisfies $f(1) \geq 1$. Let $L \in \mathbb{N} \cup \{\infty\}$ be such that $L \geq 2$ and such that all graphs $G \in \mathcal{G}$ satisfy $\omega(G) \leq L$. Define $\widetilde{f} : \mathbb{N} \to \mathbb{N}$ by setting $\widetilde{f}(0) = 0$ and $\widetilde{f}(n) = (\sum\limits_{t=1}^{\min\{n,L\}} f(t))^{n-1}$ for all $n \in \mathbb{N}^+$. Then $\widetilde{f}$ is a superadditive function that satisfies $\widetilde{f}(1) = 1$, and every graph in $\mathcal{G}$ is $\widetilde{f}$-colorable.*

*Proof.* To simplify notation, for all $n \in \mathbb{N}$, we set $L_n = \min\{n, L\}$. Thus, $\widetilde{f}(0) = 0$ and $\widetilde{f}(n) = (\sum\limits_{t=1}^{L_n} f(t))^{n-1}$ for all $n \in \mathbb{N}^+$. By construction, we have that $\widetilde{f}(1) = 1$. Next, fix some $m, n \in \mathbb{N}$; we need to show that $\widetilde{f}(m) + \widetilde{f}(n) \leq \widetilde{f}(m+n)$. If $m = 0$ or $n = 0$, then this follows from the fact that $\widetilde{f}(0) = 0$. So assume that $m, n \geq 1$. Since $L \geq 2$, we have that $L_{m+n} \geq 2$, and since $f$ is non-decreasing with $f(1) \geq 1$, we have that $f(1), f(2), \widetilde{f}(m), \widetilde{f}(n) \geq 1$. It

now follows that

$$
\begin{aligned}
\widetilde{f}(m+n) &= \left(\sum_{t=1}^{L_{m+n}} f(t)\right)^{m+n-1} \\[2mm]
&= \left(\sum_{t=1}^{L_{m+n}} f(t)\right)\left(\sum_{t=1}^{L_{m+n}} f(t)\right)^{m-1}\left(\sum_{t=1}^{L_{m+n}} f(t)\right)^{n-1} \\[2mm]
&\geq (f(1)+f(2))\left(\sum_{t=1}^{L_m} f(t)\right)^{m-1}\left(\sum_{t=1}^{L_n} f(t)\right)^{n-1} \\[2mm]
&\geq 2\widetilde{f}(m)\widetilde{f}(n) \\[2mm]
&\geq \widetilde{f}(m)+\widetilde{f}(n).
\end{aligned}
$$

This proves that $\widetilde{f}$ is superadditive.

It remains to show that every graph in $\mathcal{G}$ is $\widetilde{f}$-colorable. Fix $G \in \mathcal{G}$, and assume inductively that every graph in $\mathcal{G}$ on fewer than $|V(G)|$ vertices is $\widetilde{f}$-colorable. Since $\mathcal{G}$ is hereditary, it follows that all proper induced subgraphs of $G$ are $\widetilde{f}$-colorable. In view of Proposition 2.4, we may assume that $G$ is not a complete graph (in particular, $G$ is non-null). In view of Proposition 2.3, we may assume that $K_G = \emptyset$, and therefore, $R_G = V(G)$. Further, in view of Lemmas 3.2 and 3.4 (or alternatively, in view of Lemma 2.6), we may assume that $G$ admits neither a proper homogeneous set nor a clique-cutset. Note that for all $v \in V(G)$, we have that $R_G(v) \neq \emptyset$, for otherwise, either $G$ would be a complete graph, or $\Gamma_G(v)$ would be a clique-cutset of $G$, neither of which is possible. In particular then, $G$ has no isolated vertices (where an *isolated vertex* is a vertex of degree zero). Note also that for all distinct $v, v' \in V(G)$, we have that $\Gamma_G[v] \neq \Gamma_G[v']$, for otherwise, either $G$ would be a complete graph on the vertex-set $\{v, v'\}$, or $\{v, v'\}$ would be a proper homogeneous set of $G$, neither of which is possible. Now, to show that $G$ is $\widetilde{f}$-colorable, we fix a weight function $w : V(G) \to \mathbb{N}^+$ for $G$, and a $G$-admissible clique $Q$. We must exhibit an $(\widetilde{f}; w; Q)$-valid coloring of $G$.

Note that for all $v \in V(G)$, we have that $\{v\}$ is a $G$-admissible clique, and that any $(\widetilde{f}; w; \{v\})$-valid coloring of $G$ is also an $(\widetilde{f}; w; \emptyset)$-valid coloring of $G$. Thus, we may assume that $Q \neq \emptyset$. Since $Q$ is a $G$-admissible clique, we know that it can be ordered as $Q = \{u_1, \ldots, u_k\}$ (with $k = |Q| \geq 1$) so that for all $i, j \in \{1, \ldots, k\}$, if $i < j$, then $\Gamma_G[u_i] \subseteq \Gamma_G[u_j]$. We saw above that for all distinct $v, v' \in V(G)$, we have that $\Gamma_G[v] \neq \Gamma_G[v']$; consequently, we have that for all $i, j \in \{1, \ldots, k\}$, if $i < j$, then $\Gamma_G[u_i] \subsetneq \Gamma_G[u_j]$.

To simplify notation, we set $R_i = R_G(u_i)$ for all $i \in \{1, \ldots, k\}$. We also set $R_0 = \emptyset$ and $R_{k+1} = R_G = V(G)$. For all $i \in \{0, \ldots, k+1\}$, we set

$\omega_i = \omega(G[R_i])$, $w_i = w(G[R_i])$, and $\chi_i = \chi(G[R_i])$. (Thus, $\omega_0 = w_0 = \chi_0 = 0$, $\omega_{k+1} = \omega(G)$, $w_{k+1} = w(G)$, and $\chi_{k+1} = \chi(G)$.) Finally, we set $M = \sum\limits_{t=1}^{L_{w(G)}} f(t)$, so that $\widetilde{f}(w(G)) = M^{w(G)-1}$. Since $G$ has no isolated vertices, we know that all $v \in V(G)$ satisfy $w(v) \leq w(G) - 1$, and consequently, $\widetilde{f}(w(v)) = (\sum\limits_{t=1}^{L_{w(v)}} f(t))^{w(v)-1} \leq M^{w(G)-2}$.

**Claim 1.** *For all $i \in \{0, \dots, k\}$, we have that $R_i \subsetneq R_{i+1}$, $\omega_i < \omega_{i+1}$, and $w_i < w_{i+1}$.*

*Proof.* Recall that $R_G(v) \neq \emptyset$ for all $v \in V(G)$. In particular then, $R_1 \neq \emptyset$, and it follows immediately that $R_0 \subsetneq R_1$, $\omega_0 < \omega_1$, and $w_0 < w_1$.

Next, fix some $i \in \{1, \dots, k-1\}$. We know that $\Gamma_G[u_i] \subsetneq \Gamma_G[u_{i+1}]$, and it follows that $R_i \subseteq R_{i+1}$. Furthermore, we know that $u_i \in \Gamma_G[u_{i+1}]$, and that $u_i$ has a non-neighbor in $\Gamma_G[u_{i+1}]$, and so it follows that $u_i \in R_{i+1} \smallsetminus R_i$. Thus, $R_i \subsetneq R_{i+1}$. Since $u_i$ is complete to $R_i$ in $G$, we see that $\omega_{i+1} \geq \omega_i + 1$ and $w_{i+1} \geq w_i + w(u_i)$, and consequently, $\omega_i < \omega_{i+1}$ and $w_i < w_{i+1}$.

It remains to show that $R_k \subsetneq R_{k+1}$, $\omega_k < \omega_{k+1}$, and $w_k < w_{k+1}$. By construction, we have that $R_{k+1} = R_G = V(G)$; since $u_k \notin R_k$, it follows that $R_k \subsetneq R_{k+1}$. Further, since $u_k$ is complete to $R_k$, we deduce that $\omega_{k+1} \geq \omega_k + 1$ and $w_{k+1} \geq w_k + w(u_k)$, and consequently, $\omega_k < \omega_{k+1}$ and $w_k < w_{k+1}$. This proves the claim. $\square$

**Claim 2.** *For all $i \in \{0, \dots, k+1\}$, we have that $\chi_i \leq f(L_{w_i})$ and $\sum\limits_{t=1}^{i} \chi_t \leq \sum\limits_{t=1}^{L_{w_i}} f(t)$. In particular, $\sum\limits_{t=1}^{k+1} \chi_t \leq M$.*

*Proof.* The second statement (that is, the statement that $\sum\limits_{t=1}^{k+1} \chi_t \leq M$) follows from the first because $M = \sum\limits_{t=1}^{L_{w(G)}} f(t)$ and $w(G) = w_{k+1}$. It remains to prove the first statement. First, since $G \in \mathcal{G}$, we see that $\omega_i = \omega(G[R_i]) \leq \omega(G) \leq L$ for all $i \in \{0, \dots, k+1\}$. Since the codomain of the function $w$ is $\mathbb{N}^+$, we see that for all $i \in \{0, \dots, k+1\}$, we have that $\omega_i \leq w_i$, and it follows that $\omega_i \leq \min\{w_i, L\} = L_{w_i}$. Since $f$ is non-decreasing, we deduce that $f(\omega_i) \leq f(L_{w_i})$. Thus, it suffices to prove the following claim: for all $i \in \{0, \dots, k+1\}$, we have that $\chi_i \leq f(\omega_i)$ and $\sum\limits_{t=1}^{i} \chi_t \leq \sum\limits_{t=1}^{\omega_i} f(t)$. We proceed by induction on $i$. Since $\omega_0 = \chi_0 = 0$, the claim clearly holds for $i = 0$. We now fix some $i \in \{0, \dots, k\}$, and we assume that the claim holds for $i$; we need to show that it holds for $i + 1$. The fact that $\chi_{i+1} \leq f(\omega_{i+1})$

follows from the fact that $G[R_{i+1}] \in \mathcal{G}$ (because $G \in \mathcal{G}$, and $\mathcal{G}$ is hereditary), and the fact that $\mathcal{G}$ is $\chi$-bounded by $f$. Further, by Claim 1, we know that $\omega_i < \omega_{i+1}$, and so by what we just showed, and by the induction hypothesis, we have that $\sum\limits_{t=1}^{i+1} \chi_t = (\sum\limits_{t=1}^{i} \chi_t) + \chi_{i+1} \leq (\sum\limits_{t=1}^{\omega_i} f(t)) + f(\omega_{i+1}) \leq \sum\limits_{t=1}^{\omega_{i+1}} f(t)$. This completes the induction, and we are done. $\qquad\square$

For all $i \in \{0, \ldots, k\}$, fix a proper coloring $\psi_{i+1} : R_{i+1} \smallsetminus R_i \to (\sum\limits_{t=1}^{i} \chi_t) + [\chi_{i+1}]$ of $G[R_{i+1} \smallsetminus R_i]$. Using Claim 1 and the fact that $R_0 = \emptyset$ and $R_{k+1} = V(G)$, we see that the domains of the functions $\psi_1, \ldots, \psi_{k+1}$ are pairwise disjoint, and that their union is $V(G)$. Further, by Claim 2, the codomains of the functions $\psi_1, \ldots, \psi_{k+1}$ are all included in $[M]$. We now define the function $\psi : V(G) \to [M]$ by setting $\psi(v) = \psi_{i+1}(v)$ for all $i \in \{0, \ldots, k\}$ and $v \in R_{i+1} \smallsetminus R_i$. By construction, the codomains of $\psi_1, \ldots, \psi_{k+1}$ are pairwise disjoint, and so $\psi$ is a proper coloring of $G$.

We now define the function $\phi^* : V(G) \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^+)$ by setting

$$\phi^*(v) \quad = \quad \{\psi(v) + rM \mid 0 \leq r \leq \widetilde{f}(w(v)) - 1\}$$

for all $v \in V(G)$. Before continuing with the technical details, let us briefly outline the remainder of the proof and discuss the role that the function $\phi^*$ will play. The function $\phi^*$ is a "preliminary candidate" for an $(\widetilde{f}; w; Q)$-valid coloring of $G$: it is not hard to see that it satisfies conditions (a), (b), and (c) of the definition of an $(\widetilde{f}; w; Q)$-valid coloring of $G$ (this follows from Claim 3 below, together with the fact that $R_G = V(G)$), but unfortunately, it need not satisfy condition (d). The reason why $\phi^*$ may fail to satisfy condition (d) is that there may be an index $i \in \{1, \ldots, k\}$ and an isolated vertex $v_i$ of $G[R_i]$ such that $w(v_i) = w_i$. Now, conditions (b) and (d) of the definition of an $(\widetilde{f}; w; Q)$-valid coloring of $G$ imply that any $(\widetilde{f}; w; Q)$-valid coloring $\phi : V(G) \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^+)$ of $G$ satisfies $|\phi(v_i)| = \widetilde{f}(w(v_i)) = \widetilde{f}(w_i)$ and $|\bigcup \phi[R_i]| \leq \widetilde{f}(w_i)$, and consequently, $\bigcup \phi[R_i] = \phi(v_i)$. Unfortunately, $\phi^*$ need not have this property, and so $\phi^*$ may fail to satisfy condition (d). (We remark that this problem does not arise with condition (c). This is because $G$ contains no isolated vertices, and so all vertices $v \in V(G)$ satisfy $w(v) \leq w(G) - 1$, and consequently, $\widetilde{f}(w(v)) \leq M^{w(G)-2} = \frac{1}{M}\widetilde{f}(w(G))$.) To rectify the problem, we recursively modify $\phi^*$ on the nested sequence of sets $R_1, \ldots, R_k$ in order to eliminate this "anomaly" for one index $i$ at a time. At each recursive step, we make sure that the function that we construct still satisfies conditions (a), (b), and (c), and when recursion is complete, we obtain a function that satisfies condition (d) as well. We thus obtain an $(\widetilde{f}; w; Q)$-valid coloring of $G$, which is what we need. We now turn to the technical details.

**Claim 3.** *The function $\phi^*$ satisfies all the following:*

*(1)* $\phi^*(v_1) \cap \phi^*(v_2)$ *for all* $v_1v_2 \in E(G)$;

*(2)* $|\phi^*(v)| = \widetilde{f}(w(v))$ *for all* $v \in V(G)$;

*(3)* $\bigcup \phi^*[V(G)] \subseteq [\widetilde{f}(w(G))]$.

*Proof.* The fact that $\phi^*$ satisfies (2) is immediate from the construction of $\phi^*$. We next show that $\phi^*$ satisfies (1). Fix $v_1v_2 \in E(G)$; we must show that $\phi^*(v_1) \cap \phi^*(v_2) = \emptyset$. Since $\psi$ is a proper coloring of $G$, we know that $\psi(v_1) \neq \psi(v_2)$; since the codomain of $\psi$ is $[M]$, it follows that $\psi(v_1) \not\equiv_M \psi(v_2)$. By construction then, for all $x_1 \in \phi^*(v_1)$ and $x_2 \in \phi^*(v_2)$, we have that $x_1 \equiv_M \psi(v_1) \not\equiv_M \psi(v_2) \equiv_M x_2$, and consequently, $x_1 \neq x_2$. It follows that $\phi_0(v_1) \cap \phi_0(v_2) = \emptyset$, and so $\phi^*$ satisfies (1).

It remains to show that $\phi^*$ satisfies (3). We must check that for all $v \in V(G)$, we have that $\phi^*(v) \subseteq [\widetilde{f}(w(G))]$. Fix $v \in V(G)$, and recall that $\widetilde{f}(w(v)) \leq M^{w(G)-2}$. We then have that

$$
\begin{aligned}
\max \phi^*(v) &= \psi(v) + (\widetilde{f}(w(v)) - 1)M \\[2mm]
&\leq M + (M^{w(G)-2} - 1)M \\[2mm]
&= M^{w(G)-1} \\[2mm]
&= \widetilde{f}(w(G)),
\end{aligned}
$$

and consequently, $\phi^*(v) \subseteq [\widetilde{f}(w(G))]$. It follows that $\bigcup \phi^*[V(G)] \subseteq [\widetilde{f}(w(G))]$, and so $\phi^*$ satisfies (3). This proves the claim. $\square$

Set $S_0 = \emptyset$, and for all $i \in \{1, \ldots, k\}$, set

$$
S_i = \{(\sum_{t=1}^{i} \chi_t) + rM \mid 0 \leq r \leq \widetilde{f}(w_i) - 1\}.
$$

By Claim 1, we have that $w_i \geq 1$ for all $i \in \{1, \ldots, k\}$. Since $\widetilde{f}$ is superadditive (and therefore non-decreasing) and satisfies $\widetilde{f}(1) = 1$, it follows that for all $i \in \{1, \ldots, k\}$, we have that $\widetilde{f}(w_i) \geq 1$, and consequently, $S_i \neq \emptyset$.

**Claim 4.** *For all* $i \in \{0, \ldots, k\}$, *we have that* $|S_i| = \widetilde{f}(w_i)$ *and* $S_i \subseteq [\widetilde{f}(w(G))]$.

*Proof.* Fix $i \in \{0, \ldots, k\}$. If $i = 0$, then $S_i = \emptyset$, and the result is immediate. So assume that $i \geq 1$, so that $S_i \neq \emptyset$. It is clear that $|S_i| = \widetilde{f}(w_i)$, and we just need to show that $S_i \subseteq [\widetilde{f}(w(G))]$. By Claim 2, we know that $(\sum_{t=1}^{i} \chi_t) \leq M$,

and by Claim 1, we have that $w_i \leq w(G) - 1$, and so by the definition of $\widetilde{f}$ and $M$, we have that $\widetilde{f}(w_i) \leq M^{w_i-1} \leq M^{w(G)-2}$. Consequently,

$$
\begin{aligned}
\max S_i &= (\sum_{t=1}^{i} \chi_t) + (\widetilde{f}(w_i) - 1)M \\[2mm]
&\leq M + (M^{w(G)-2} - 1)M \\[2mm]
&= M^{w(G)-1} \\[2mm]
&= \widetilde{f}(w(G)),
\end{aligned}
$$

and it follows that $S_i \subseteq [\widetilde{f}(w(G))]$. This proves the claim. $\qquad\square$

Our goal now is to recursively define a sequence of functions $\phi_0, \ldots, \phi_k : V(G) \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^+)$ such for all $i \in \{0, \ldots, k\}$, $\phi_i$ satisfies all the following:

(a-$i$) $\phi_i(v_1) \cap \phi_i(v_2) = \emptyset$ for all $v_1 v_2 \in E(G)$;

(b-$i$) $|\phi_i(v)| = \widetilde{f}(w(v))$ for all $v \in V(G)$;

(c-$i$) $|\bigcup \phi_i[R_j]| \leq \widetilde{f}(w_j)$ for all $j \in \{1, \ldots, i\}$;

(d-$i$) $\bigcup \phi_i[R_i] \subseteq S_i$;

(e-$i$) $\phi_i \upharpoonright (V(G) \smallsetminus R_i) = \phi^* \upharpoonright (V(G) \smallsetminus R_i)$.

Once we have constructed functions these functions, we can easily show that $\phi_k$ is an $(\widetilde{f}; w; Q)$-valid coloring of $G$ (see Claim 6), from which we immediately deduce that $G$ is $\widetilde{f}$-colorable. We now proceed to construct functions $\phi_0, \ldots, \phi_k$.

First, set $\phi_0 = \phi^*$. Claim 3 guarantees that $\phi_0$ satisfies (a-0) and (b-0). Further, $\phi_0$ vacuously satisfies (c-0), it satisfies (d-0) because $R_0 = \emptyset$, and it satisfies (e-0) because $\phi_0 = \phi^*$.

Now, fix $i \in \{0, \ldots, k-1\}$, and suppose that we have defined a function $\phi_i : V(G) \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^+)$ that satisfies (a-$i$)-(e-$i$). Our goal is to construct a function $\phi_{i+1} : V(G) \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^+)$ that satisfies (a-$(i+1)$)-(e-$(i+1)$). Set

$$
U_{i+1} = \{v \in R_{i+1} \mid w(v) = w_{i+1}\}.
$$

By Claim 1, we have that $w_{i+1} > w_i$, and consequently, for all $v \in U_{i+1}$, we have that $w(v) > w_i$. Thus, $U_{i+1} \subseteq R_{i+1} \smallsetminus R_i$ and $R_i \subseteq R_{i+1} \smallsetminus U_{i+1}$. Furthermore, each vertex in $U_{i+1}$ is an isolated vertex of $G[R_{i+1}]$.

**Claim 5.** $|\bigcup \phi_i[R_{i+1} \smallsetminus U_{i+1}]| \leq \widetilde{f}(w_{i+1})$.

*Proof.* If $U_{i+1} = R_{i+1}$, then the result is immediate. So assume that $U_{i+1} \subsetneq R_{i+1}$. Note that this implies that $w_{i+1} \geq 2$. By (e-$i$), we have that $\phi_i \upharpoonright (R_{i+1} \smallsetminus R_i) = \phi^* \upharpoonright (R_{i+1} \smallsetminus R_i)$. Further, for all $v \in R_{i+1} \smallsetminus U_{i+1}$, we have that $w(v) \leq w_{i+1} - 1$, and so $\widetilde{f}(w(v)) \leq \widetilde{f}(w_{i+1} - 1)$. It now follows that

$$\bigcup \phi_i[R_{i+1} \smallsetminus (R_i \cup U_{i+1})]$$

$$= \bigcup \phi^*[R_{i+1} \smallsetminus (R_i \cup U_{i+1})]$$

$$= \bigcup\{\psi(v) + rM \mid v \in R_{i+1} \smallsetminus (R_i \cup U_{i+1}), 0 \leq r \leq \widetilde{f}(w(v)) - 1\}$$

$$\subseteq \{a + rM \mid a \in (\sum_{t=1}^{i} \chi_t) + [\chi_{i+1}], 0 \leq r \leq \widetilde{f}(w_{i+1} - 1) - 1\},$$

and consequently, $|\bigcup \phi_i[R_{i+1} \smallsetminus (R_i \cup U_{i+1})]| \leq \chi_{i+1}\widetilde{f}(w_{i+1} - 1)$. Further, by Claim 2, we have that $\chi_{i+1} \leq f(L_{w_{i+1}})$, and so if $i = 0$ (so that $R_0 = \emptyset$), then we have that $R_{i+1} \smallsetminus U_{i+1} = R_{i+1} \smallsetminus (R_i \cup U_{i+1})$, and consequently,

$$|\bigcup \phi_i[R_{i+1} \smallsetminus U_{i+1}]| \leq \chi_{i+1}\widetilde{f}(w_{i+1} - 1)$$

$$\leq f(L_{w_{i+1}})(\sum_{t=1}^{L_{w_{i+1}}-1} f(t))^{w_{i+1}-2}$$

$$\leq (\sum_{t=1}^{L_{w_{i+1}}} f(t))^{w_{i+1}-1}$$

$$= \widetilde{f}(w_{i+1}),$$

and we are done. So assume that $i \geq 1$. Then by Claim 1, we have that $w_{i+1} > w_i \geq 1$ and $\chi_i \geq 1$, and so by Claim 2, $\chi_{i+1} + 1 \leq \sum_{t=1}^{i+1} \chi_t \leq \sum_{t=1}^{L_{w_{i+1}}} f(t)$.

Further, by (c-$i$), we have that $|\bigcup \phi_i[R_i]| \leq \widetilde{f}(w_i)$. Thus,

$$
\begin{aligned}
\left|\bigcup \phi_i[R_{i+1} \setminus U_{i+1}]\right| \;\leq\; & \left|\bigcup \phi_i[R_{i+1} \setminus (R_i \cup U_{i+1})]\right| + \left|\bigcup \phi_i[R_i]\right| \\[2mm]
\leq\; & \chi_{i+1}\widetilde{f}(w_{i+1}-1) + \widetilde{f}(w_i) \\[2mm]
=\; & \chi_{i+1}\Big(\sum_{t=1}^{L_{w_{i+1}}-1} f(t)\Big)^{w_{i+1}-2} + \Big(\sum_{t=1}^{L_{w_i}} f(t)\Big)^{w_i-1} \\[2mm]
\leq\; & (\chi_{i+1}+1)\Big(\sum_{t=1}^{L_{w_{i+1}}} f(t)\Big)^{w_{i+1}-2} \\[2mm]
\leq\; & \Big(\sum_{t=1}^{L_{w_{i+1}}} f(t)\Big)\Big(\sum_{t=1}^{L_{w_{i+1}}} f(t)\Big)^{w_{i+1}-2} \\[2mm]
=\; & \widetilde{f}(w_{i+1}).
\end{aligned}
$$

This proves the claim. $\qquad\square$

In view of Claims 4 and 5, we have that $|\bigcup \phi_i[R_{i+1} \setminus U_{i+1}]| \leq |S_{i+1}|$. Fix an injective function $\sigma : \bigcup \phi_i[R_{i+1} \setminus U_{i+1}] \to S_{i+1}$, and define the function $\phi_{i+1} : V(G) \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^+)$ by setting

$$
\phi_{i+1}(v) \;=\; \begin{cases} S_{i+1} & \text{if } v \in U_{i+1} \\[2mm] \sigma[\phi_i(v)] & \text{if } v \in R_{i+1} \setminus U_{i+1} \\[2mm] \phi_i(v) & \text{if } v \notin R_{i+1} \end{cases}
$$

for all $v \in V(G)$.

Let us check that the function $\phi_{i+1}$ satisfies (a-$(i+1)$)-(e-$(i+1)$). The fact that $\phi_{i+1}$ satisfies (d-$(i+1)$) is immediate from the construction of $\phi_{i+1}$. The fact that $\phi_{i+1}$ satisfies (e-$(i+1)$) follows from the fact that $\phi_i$ satisfies (e-$i$). Next, by Claim 4 and (d-$(i+1)$), we have that $|\bigcup \phi_{i+1}[R_{i+1}]| \leq |S_{i+1}| = \widetilde{f}(w_{i+1})$; and for all $j \in \{1, \dots, i\}$, the fact that $\phi_i$ satisfies (c-$i$), that $\sigma$ is an injection, and that $R_j \subseteq R_i \subseteq R_{i+1} \setminus U_{i+1}$, implies that $|\bigcup \phi_{i+1}[R_j]| = |\sigma[\bigcup \phi_i[R_j]]| = |\bigcup \phi_i[R_j]| \leq \widetilde{f}(w_j)$. Thus, $\phi_{i+1}$ satisfies (c-$(i+1)$). For (b-$(i+1)$), we fix some $v \in V(G)$, and we show that $|\phi_{i+1}(v)| = \widetilde{f}(w(v))$. If $v \in U_{i+1}$, then $w(v) = w_{i+1}$, and so by Claim 4 and the construction of $\phi_{i+1}$, we have that $|\phi_{i+1}(v)| = |S_{i+1}| = \widetilde{f}(w_{i+1}) = \widetilde{f}(w(v))$. On the other hand, if $v \notin U_{i+1}$, then using the fact that $\sigma$ is an injection, and the fact that $\phi_i$ satisfies (b-$i$), we see that $|\phi_{i+1}(v)| = |\phi_i(v)| = \widetilde{f}(w(v))$. Thus, $\phi_{i+1}$ satisfies (b-$(i+1)$).

It remains to show that $\phi_{i+1}$ satisfies (a-$(i+1)$). Fix $v_1v_2 \in E(G)$; we must show that $\phi_{i+1}(v_1) \cap \phi_{i+1}(v_2) = \emptyset$. If $v_1, v_2 \notin R_{i+1}$, then this follows from the fact that $\phi_i$ satisfies (a-$i$). Suppose next that $v_1, v_2 \in R_{i+1}$. Since $v_1v_2 \in E(G[R_{i+1}])$, and since every vertex of $U_{i+1}$ is an isolated vertex of $G[R_{i+1}]$, we see that $v_1, v_2 \in R_{i+1} \setminus U_{i+1}$. Then since $\sigma$ is an injection, and since $\phi_i$ satisfies (a-$i$), we have that $\phi_{i+1}(v_1) \cap \phi_{i+1}(v_2) = \sigma[\phi_i(v_1)] \cap \sigma[\phi_i(v_2)] = \sigma[\phi_i(v_1) \cap \phi_i(v_2)] = \sigma[\emptyset] = \emptyset$. It remains to consider the case when exactly one of $v_1$ and $v_2$ belongs to $R_{i+1}$; by symmetry, we may assume that $v_1 \in R_{i+1}$ and $v_2 \notin R_{i+1}$. By (d-$(i+1)$), we know that $\phi_{i+1}(v_1) \subseteq S_{i+1}$, and consequently, all $x_1 \in \phi_{i+1}(v_1)$ satisfy $x_1 \equiv_M (\sum_{t=1}^{i+1} \chi_t)$. On the other hand, by (e-$(i+1)$), we have that $\phi_{i+1}(v_2) = \phi^*(v_2)$, and consequently, for all $x_2 \in \phi_{i+1}(v_2)$, we have that $x_2 \equiv_M \psi(v_2) \in [M] \setminus [\sum_{t=1}^{i+1} \chi_t]$ and therefore $x_2 \not\equiv_M \sum_{t=1}^{i+1} \chi_t$. Thus, for all $x_1 \in \phi_{i+1}(v_1)$ and $x_2 \in \phi_{i+1}(v_2)$, we have that $x_1 \equiv_M (\sum_{t=1}^{i+1} \chi_t) \not\equiv_M x_2$, and consequently, $x_1 \neq x_2$. It follows that $\phi_{i+1}(v_1) \cap \phi_{i+1}(v_2) = \emptyset$, and so $\phi_{i+1}$ satisfies (a-$(i+1)$). This completes the induction.

**Claim 6.** *The function* $\phi_k : V(G) \to \mathscr{P}_{\mathrm{fin}}(\mathbb{N}^+)$ *is an* $(\widetilde{f}; w; Q)$*-valid coloring of* $G$.

*Proof.* We need to show that $\phi_k$ satisfies all the following:

(a) $\phi_k(v_1) \cap \phi_k(v_2) = \emptyset$ for all $v_1v_2 \in E(G)$;

(b) $|\phi_k(v)| = \widetilde{f}(w(v))$ for all $v \in V(G)$;

(c) $|\bigcup \phi_k[R_G]| \leq \widetilde{f}(w(G[R_G]))$;

(d) $|\bigcup \phi_k[R_G(u)]| \leq \widetilde{f}(w(G[R_G(u)]))$ for all $u \in Q$.

The fact that $\phi_k$ satisfies (a), (b), and (d) follows immediately from the fact that it satisfies (a-$k$), (b-$k$), and (c-$k$), respectively. It remains to show that $\phi_k$ satisfies (c). Since $V(G) = R_G$, it suffices to show that $|\bigcup \phi_k[V(G)]| \leq \widetilde{f}(w(G))$. First, since $\phi_k$ satisfies (d-$k$), we know that $\bigcup \phi_k[R_k] \subseteq S_k$, and so by Claim 4, we have that $\bigcup \phi_k[R_k] \subseteq [\widetilde{f}(w(G))]$. On the other hand, since $\phi_k$ satisfies (e-$k$), we know that $\phi_k \restriction (V(G) \setminus R_k) = \phi^* \restriction (V(G) \setminus R_i)$, and so Claim 3 implies that $\bigcup \phi_k[V(G) \setminus R_k] \subseteq [\widetilde{f}(w(G))]$. We now have that $\bigcup \phi_k[V(G)] = (\bigcup \phi_k[R_k]) \cup (\bigcup \phi_k[V(G) \setminus R_k]) \subseteq [\widetilde{f}(w(G))]$, and consequently, $|\bigcup \phi_k[V(G)]| \leq \widetilde{f}(w(G))$. Thus, $\phi_k$ satisfies (c). This proves the claim. $\square$

Claim 6 immediately implies that $G$ is $\widetilde{f}$-colorable. This completes the argument. $\square$

# 4 House*-free graphs and the proof of Theorem 1.3

We remind the reader that the *house* is the complement of the four-edge path, a *house** is any subdivision of the house, and a *cap* is any graph obtained from the house by possibly subdividing the three edges of the house that do not belong to the unique triangle of the house (see Figure 1.4). Equivalently, a *cap* is a graph that consists of a cycle of length at least four, together with a vertex that is adjacent to two adjacent vertices of the cycle, and non-adjacent to all the remaining vertices of the cycle. Thus, the house is a cap, and any cap is a house*. Furthermore, every house*-free graph is cap-free (but not every cap-free graph is house*-free). The class of cap-free graphs is not $\chi$-bounded because every triangle-free graph is cap-free, and triangle-free graphs can have an arbitrarily large chromatic number [20, 25], while their clique number is at most two. However, as we show in this section, the class of house*-free graphs is $\chi$-bounded. To prove this, we rely on a decomposition theorem for cap-free graphs from [11], and the fact that every house*-free graph is cap-free. Before stating the decomposition theorem for cap-free graphs, we need a definition. A *chordal graph* is a graph that does not contain any induced cycles of length greater than three. It is well-known (and easy to prove) that every chordal graph either is a complete graph or admits a clique-cutset [13]. (We remark that Proposition 2.4 and Lemma 3.4 imply that if $f : \mathbb{N} \to \mathbb{N}$ is a superadditive function, then every chordal graph is $f$-colorable.) We now state the decomposition theorem for cap-free graphs from [11].

**Theorem 4.1.** *[11] Let $G$ be a cap-free graph. Then $G$ satisfies at least one of the following:*

- *$G$ is a chordal graph;*

- *$G$ is a 2-connected, triangle-free graph, together with at most one additional vertex, which is adjacent to all other vertices of $G$;*

- *$G$ admits an amalgam.*

As a corollary, we easily obtain the following.

**Corollary 4.2.** *Let $G$ be a house*-free graph. Then either $G$ is a bipartite graph or an odd cycle of length at least five, or $G$ admits a proper homogeneous set, a clique-cutset, or an amalgam.*

*Proof.* Since $G$ is house*-free, it is cap-free, and so we may apply Theorem 4.1 to $G$. If $G$ admits an amalgam, then we are done. If $G$ is chordal, then either $G$ is a complete graph (in which case either it contains at most two vertices and is therefore bipartite, or it admits a proper homogeneous set), or $G$ admits a clique-cutset, and in either case, we are done. Further, if $G$ is a 2-connected, triangle-free graph, together with exactly one additional

vertex, adjacent to all other vertices of $G$, then $G$ admits a proper homogeneous set, and again we are done. So by Theorem 4.1, we may assume that $G$ is a 2-connected, triangle-free graph. If $G$ contains no induced odd cycles of length at least five, then the fact that $G$ is triangle-free implies that $G$ is bipartite, and we are done. So assume that $G$ does contain such a cycle, and let $c_0 - c_1 - \cdots - c_{2k-1} - c_{2k} - c_0$ (with indices in $\mathbb{Z}_{2k+1}$, $k \geq 2$) be an induced odd cycle of $G$.

**Claim 1.** *No vertex in $V(G) \smallsetminus \{c_0, \ldots, c_{2k}\}$ has more than one neighbor in $\{c_0, \ldots, c_{2k}\}$.*

*Proof.* Fix $v \in V(G) \smallsetminus \{c_0, \ldots, c_{2k}\}$, and suppose that $v$ has at least two neighbors in $\{c_0, \ldots, c_{2k}\}$. If $v$ is adjacent to precisely two vertices in $\{c_0, \ldots, c_{2k}\}$, then $G[v, c_0, \ldots, c_{2k}]$ is a house*, contrary to the fact that $G$ is house*-free. So assume that $v$ has at least three neighbors in $\{c_0, \ldots, c_{2k}\}$. Fix distinct indices $r, t \in \mathbb{Z}_{2k+1}$ such that $v$ is adjacent to $c_r$ and $c_t$, and to precisely one vertex, say $c_s$, in $\{c_{r+1}, \ldots, c_{t-1}\}$. Since $v$ is complete to $\{c_r, c_s, c_t\}$, and since $G$ is triangle-free, we know that $\{c_r, c_s, c_t\}$ is a stable set, and we easily deduce that $G[v, c_r, c_{r+1}, \ldots, c_s, c_{s+1}, \ldots, c_t]$ is a house*, contrary to the fact that $G$ is house*-free. This proves the claim. $\square$

**Claim 2.** *For every component $C$ of $G \smallsetminus \{c_0, \ldots, c_{2k}\}$, at most one vertex in $\{c_0, \ldots, c_{2k}\}$ has a neighbor in $C$.*

*Proof.* Fix a component $C$ of $G \smallsetminus \{c_0, \ldots, c_{2k}\}$, and suppose that at least two vertices in $\{c_0, \ldots, c_{2k}\}$ have a neighbor in $C$. By symmetry, we may assume that $c_0$ has a neighbor in $C$. Set $C_0 = \Gamma_G(c_0) \cap V(C)$; by construction, $C_0$ is non-empty, and by Claim 1, we know $C_0$ is anti-complete to $\{c_1, \ldots, c_{2k}\}$. Since at least two vertices in $\{c_0, \ldots, c_{2k}\}$ have a neighbor in $C$, we know that some vertex in $V(C) \smallsetminus C_0$ has a neighbor in $\{c_1, \ldots, c_{2k}\}$. Using the fact that $C$ is connected, we fix a minimal (and therefore induced) path $v_0 - v_1 - \cdots - v_t$ in $C$ that has the property that $v_0 \in C_0$ and $v_t$ has a neighbor in $\{c_1, \ldots, c_{2k}\}$. By the minimality of $v_0 - v_1 - \cdots - v_t$, we know that $\{v_1, \ldots, v_{t-1}\}$ is anti-complete to $\{c_0, \ldots, c_{2k}\}$. Fix $r \in \mathbb{Z}_{2k+1} \smallsetminus \{0\}$ such that $v_t$ is adjacent to $c_r$; by Claim 1, we see that $c_r$ is the only neighbor of $v_t$ in $\{c_0, \ldots, c_{2k}\}$. But now we see that $G[v_0, v_1, \ldots, v_t, c_0, c_1, \ldots, c_{2k}]$ is a house*, contrary to the fact that $G$ is house*-free. This proves the claim. $\square$

If $\{c_0, c_1, \ldots, c_{2k}\} \subsetneq V(G)$, then Claim 2 implies that $G$ admits a cutset of size at most one, contrary to the fact that $G$ is 2-connected. It follows that $V(G) = \{c_0, c_1, \ldots, c_{2k}\}$, and so $G$ is an odd cycle of length at least five. This completes the argument. $\square$

Using Corollaries 2.9 and 4.2, we easily obtain the following.

**Proposition 4.3.** *Every house*-free graph $G$ satisfies $\chi(G) \leq 4^{\omega(G)-1}$.*

*Proof.* Corollary 4.2 implies that if $G$ is house*-free graph, then either $G$ is a triangle-free, 3-colorable graph, or $G$ admits a proper homogeneous set, a clique-cutset, or an amalgam. The result now follows from Corollary 2.9 (with $L = 2$ and $c = 3$). □

As stated in the introduction, we can improve the bound from Proposition 4.3 by relying on technical results concerning $f$-colorability from section 2 (rather than relying on Corollary 2.9). In particular, we can prove Theorem 1.3, stated in the introduction and restated below for the reader's convenience.

**Theorem 1.3.** *Every house*-free graph $G$ satisfies $\chi(G) \leq 3^{\omega(G)-1}$.*

*Proof.* Let $f : \mathbb{N} \to \mathbb{N}$ be given by $f(0) = 0$ and $f(n) = 3^{n-1}$ for all $n \in \mathbb{N}^+$. Clearly, $f$ is a superadditive function that satisfies $f(1) = 1$. Furthermore, it is clear that the class of house*-free graphs is hereditary. Thus, it suffices to show that every house*-free graph is $f$-colorable, for Lemma 2.7 will then imply that the class of house*-free graphs is $\chi$-bounded by $f$, and then we are done. Fix a house*-free graph $G$, and assume inductively that all proper induced subgraphs of $G$ are $f$-colorable; we must show that $G$ is $f$-colorable. In view of Proposition 2.4, we may assume that $G$ is not a complete graph (and in particular, $G$ is non-null), and in view of Proposition 2.3, we may assume that $K_G = \emptyset$ (and so $R_G = V(G)$). Further, in view of Lemma 2.6, we may assume that $G$ does not admit a proper homogeneous set, a clique-cutset, or an amalgam. Thus, Corollary 4.2 implies that $G$ is either a bipartite graph or an odd cycle of length at least five (and in particular, $G$ is triangle-free). Furthermore, we have that all $v \in V(G)$ satisfy $R_G(v) \neq \emptyset$, for otherwise, either $G$ would be a complete graph, or $\Gamma_G(v)$ would be a clique-cutset of $G$, contrary to the fact that $G$ is not a complete graph and does not admit a clique-cutset.

Fix a weight function $w : V(G) \to \mathbb{N}^+$ for $G$, and a $G$-admissible clique $Q$; we must exhibit an $(f; w; Q)$-valid coloring of $G$. Let us first show that $|Q| \leq 1$. Suppose that $|Q| \geq 2$; then there exist distinct $u_1, u_2 \in Q$ such that $\Gamma_G[u_1] \subseteq \Gamma_G[u_2]$. Clearly, $u_2 \in K_G(u_1)$. Using the fact that $R_G(u_1) \neq \emptyset$, we fix some $v \in R_G(u_1)$, and we observe that $\{v, u_1, u_2\}$ is a triangle in $G$, contrary to the fact that $G$ is triangle-free. Thus, $|Q| \leq 1$. Next, note that for every vertex $v \in V(G)$, we have that $\{v\}$ is a $G$-admissible clique, and that every $(f; w; \{v\})$-valid coloring of $G$ is also an $(f; w; \emptyset)$-valid coloring of $G$. Thus, we may assume that $Q \neq \emptyset$, and so $|Q| = 1$. Set $Q = \{u_0\}$.

Since $G$ is triangle-free, we see that $\Gamma_G(u_0)$ is a stable set. Since $G$ is either a bipartite graph or an odd cycle of length at least five, we see that there exists a proper coloring $\psi : V(G) \to [3]$ of $G$ such that for all $v \in \Gamma_G(u_0)$, we have that $\psi(v) = 1$. Now, define $\phi : V(G) \to \mathscr{P}_{\text{fin}}(\mathbb{N}^+)$ by setting

$\phi(v) = \{\psi(v) + 3r \mid 0 \leq r \leq f(w(v)) - 1\}$. We claim that $\phi$ is an $(f; w; Q)$-valid coloring of $G$. We must show that $\phi$ satisfies all the following:

(a) $\phi(v_1) \cap \phi(v_2) = \emptyset$ for all $v_1 v_2 \in E(G)$;

(b) $|\phi(v)| = f(w(v))$ for all $v \in V(G)$;

(c) $|\bigcup \phi[R_G]| \leq f(w(G[R_G]))$;

(d) $|\bigcup \phi[R_G(u)]| \leq f(w(G[R_G(u)]))$ for all $u \in Q$.

The fact that $\phi$ satisfies (b) is immediate from the construction of $\phi$. We next show that $\phi$ satisfies (a). Fix $v_1 v_2 \in E(G)$; we must show that $\phi(v_1) \cap \phi(v_2) = \emptyset$. Since $\psi$ is a proper coloring of $G$, we know that $\psi(v_1) \neq \psi(v_2)$; since the codomain of $\psi$ is [3], it follows that $\psi(v_1) \not\equiv_3 \psi(v_2)$. It now follows from the construction of $\phi$ that for all $x_1 \in \phi(v_1)$ and $x_2 \in \phi(v_2)$, we have that $x_1 \equiv_3 \psi(v_1) \not\equiv_3 \psi(v_2) \equiv_3 x_2$, and consequently, $x_1 \neq x_2$. Thus, $\phi(v_1) \cap \phi(v_2) = \emptyset$, and $\phi$ satisfies (a).

Let us now show that $\phi$ satisfies (c). Since $R_G = V(G)$, we just need to show that $|\bigcup \phi[V(G)]| \leq f(w(G))$. We prove this by showing that for all $v \in V(G)$, we have that $\phi(v) \subseteq [f(w(G))]$; clearly, it suffices to show that $\max \phi(v) \leq f(w(G))$. Fix $v \in V(G)$. Since $R_G(v) \neq \emptyset$, we know that $v$ has a neighbor in $G$, and so $w(v) \leq w(G) - 1$. Thus,

$$
\begin{aligned}
\max \phi(v) \quad &= \quad \psi(v) + 3(f(w(v)) - 1) \\[2mm]
&\leq \quad 3 + 3(3^{w(G)-2} - 1) \\[2mm]
&= \quad 3^{w(G)-1} \\[2mm]
&= \quad f(w(G)).
\end{aligned}
$$

This proves that $\phi$ satisfies (c).

It remains to show that $\phi$ satisfies (d). Since $Q = \{u_0\}$, we just need to show that $|\bigcup \phi[R_G(u_0)]| \leq f(w(G[R_G(u_0)]))$. Recall that for all $v \in \Gamma_G(u_0)$, we have that $\psi(v) = 1$. It then follows that

$$
\begin{aligned}
\bigcup \phi[R_G(u_0)] \quad &= \quad \{\psi(v) + 3r \mid v \in R_G(u_0), 0 \leq r \leq f(w(v)) - 1\} \\[2mm]
&\subseteq \quad \{1 + 3r \mid 0 \leq r \leq f(w(G[R_G(u_0)])) - 1\},
\end{aligned}
$$

and consequently, $|\bigcup \phi[R_G(u_0)]| \leq f(w(G[R_G(u_0)]))$. This proves that $\phi$ satisfies (d).

We now have that $\phi$ is an $(f; w; Q)$-valid coloring of $G$, and consequently, $G$ is $f$-colorable. This completes the argument. $\qquad \square$

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