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Orientations of infinite graphs with prescribed edge-connectivity

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Abstract

We prove a decomposition result for locally finite graphs which can be used to extend results on edge-connectivity from finite to infinite graphs. It implies that every $4k$-edge-connected graph G contains an immersion of some finite $2k$-edge-connected Eulerian graph containing any prescribed vertex set (while planar graphs show that G need not contain a subdivision of a simple finite graph of large edge-connectivity). Also, every $8k$-edge connected infinite graph has a k-arc-connected orientation, as conjectured in 1989.

Keywords: infinite graphs, orientations, connectivity
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1 Introduction.

Many basic statements on finite graphs extend easily to the infinite case using some variant of the Axiom of Choice such as König's Infinity Lemma, Zorn's Lemma, Rado's Selection Principle, or compactness. Some of these are discussed in [18].

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A striking example of a basic result that does not extend easily is the result of Nash-Williams, see [14], that the edge set of every graph with no finite odd cut is the union of pairwise edge-disjoint cycles. In the finite or countably infinite case this result is trivial because every edge is in a cycle and the deletion of the edge set of that cycle leaves a graph with no finite odd cut. However, the general case is surprisingly difficult. If G has no finite odd cut and edge-connectivity at least $2k$ (where k is a natural number) and we make each of the cycles in the decomposition into a directed cycle, then the resulting digraph has arc-connectivity at least k. Nash-Williams [12] proved that every finite $2k$-edge-connected graph has a k-arc-connected orientation, and Mader [10] also obtained this result from his general lifting theorem. It is not known if every infinite $2k$-edge-connected graph has a k-arc-connected orientation. As pointed out in [2], this was stated by Nash-Williams in 1967 [14], but not in [15] and apparently never published. Unaware of this, the present author conjectured in 1989 [19] the weakened version where the edge-connectivity $2k$ is replaced by a function $f(k)$. We prove, among other things, that weaker conjecture in the present paper.

There are also fundamental results on finite graphs that do not extend to the infinite case. One such example is the result of Edmonds [6], Nash-Williams [13] and Tutte [21] that every $2k$ edge-connected graph has k pairwise edge-disjoint spanning trees. In fact, there are graphs of arbitrarily large edge-connectivity that have no two edge-disjoint spanning trees, as proved in [1]. The construction in [1] also shows that several other basic results on finite graphs do not extend to the infinite case. As many edge-disjoint spanning trees imply high edge-connectivity and also orientations with large arc-connectivity, one might say that the counterexamples in [1] and the orientation results in the present paper are on the border line of what does not extend and what does extend to the infinite case.

It is possible to extend infinite graphs to topological spaces so that large edge-connectivity implies many edge-disjoint spanning (topological) trees, as discussed in Chapter 8 in [4]. However, orientations of these trees need not result in large arc-connectivity.

We prove a general decomposition result for a connected, locally finite graph into a finite number of sets, one of which is a prescribed set A_0. The decomposition has the following property: If a vertex set S in the decomposition has more than one vertex, then the edges in the boundary of the set S are the first edges in a collection of pairwise edge-disjoint paths in $G(S)$ all belonging to the same end of G, that is, no two of them can be separated by a finite edge-set. As an application, every $4k$-edge-connected graph G contains an immersion of a finite $2k$-edge-connected Eulerian graph such that the immersion contains a prescribed vertex set in G (while planar graphs show that G need not contain a subdivision of a finite graph with no multiple edges and of large edge-connectivity). We use that to obtain the main application of the decomposition result, namely that every $8k$-edge-connected graph admits a k-arc-connected orientation. This proves Conjecture 20 in [19].

2 Decomposing an infinite, locally finite, connected graph into finitely many boundary-linked subgraphs.

The graphs in this paper are allowed to contain loops and multiple edges. However, each edge-multiplicity is finite. And, an edge is not a loop, unless explicitly said so. (The graphs in the theorems are loopless. However, loops may arise in a proof when we identify some vertices.) A graph is *locally finite* if every vertex has finite degree. If G is a graph and A is a set of vertices in G, then $G(A)$ is the subgraph of G induced by A, that is, $G(A)$ has vertex set A and contains all those edges of G which join two vertices in A. If A is a set of vertices in G, then the edges with precisely one end in A is called a *cut* and is also called the *boundary* of A and the boundary of $G(A)$.

We say that a vertex set A (and the subgraph $G(A)$) are *boundary-linked* if $G(A)$ together with its boundary has a collection of pairwise edge-disjoint one-way infinite paths $P_1, P_2, \ldots$ such that each edge in the boundary is the first edge of one of $P_1, P_2, \ldots$, and such that the paths $P_1, P_2, \ldots$ belong to the same *end* of $G(A)$, that is, for any finite edge set E in $G(A)$, and any two paths P_i, P_j, $G(A) - E$ has a path joining P_i, P_j. (Another way of saying this is that $G(A)$ has infinitely many pairwise edge-disjoint paths joining P_i, P_j. As G is locally finite, these paths can even be chosen to be vertex-disjoint.)

Analogous path systems were investigated by Halin [7].

Theorem 1 *Let G be a connected, locally finite graph, and let A_0 be a vertex set with finite boundary. Then $V(G) \setminus A_0$ can be divided into finitely many pairwise disjoint vertex sets each of which is either a singleton or a boundary-linked vertex set with finite boundary.*

Before we prove Theorem 1, we explain the idea behind the proof. We consider a maximal connected subgraph in $G - A_0$ among those infinite connected subgraphs that have smallest boundary. We contract that subgraph into a single vertex. We repeat this operation, possibly infinitely many times. Then we repeat this procedure for maximal infinite subgraphs with second smallest boundary, and then for maximal infinite subgraphs with third smallest boundary, etc. When we obtain a finite graph (except that A_0 may be infinite), then the vertices of that graph correspond to the desired vertex partition of $G - A_0$.

We now turn this argument into a formal proof of Theorem 1. We may assume that $G - A_0$ is infinite since otherwise, there is nothing to prove.

Let $V(G) \setminus A_0 = \{v_1, v_2, \ldots\}$. Let $e_1, e_2, \ldots, e_k$ be the edges leaving A_0. If $k = 1$, the result follows by König's Infinity Lemma. So assume that $k \geq 2$.

Every component of $G - A_0$ has boundary of size at most k. At least one of these components is infinite. Let k' be the size of a smallest boundary of an infinite subgraph of $G - A_0$. The proof is by induction on $k - k'$.

Consider first the case where $k - k' = 0$.

By König's Infinity Lemma, $G(A)$ contains a one-way infinite path P_0. We define the subgrahs $H_1, H_2, \ldots$ as follows: H_1 is the unique infinite component of $G - A_0$ containing infinitely many edges of P_0. Having defined H_i we delete all vertices of H_i incident with the boundary, and we define H_{i+1} as the unique infinite component of the resulting subgraph of H_i containing infinitely many edges of P_0. Then $H_1, H_2, \ldots$ is a decreasing sequence of connected subgraphs of G each with finite boundary and each containing an infinite subpath of P_0. As $k' = k$, the boundary of H_i has at least k edges. By Menger's theorem, G has k pairwise edge-disjoint paths $P_{i,1}, P_{i,2}, \ldots, P_{i,k}$ such that $P(i,j)$ starts with e_j and terminates with an edge in the boundary of H_i for $j = 1, 2, \ldots, k$ and $i = 1, 2, \ldots$. We choose the paths $P_{i,1}, P_{i,2}, \ldots, P_{i,k}$ such that their total number of edges is minimum.

We now define a limit of these path systems as follows. For infinitely many i, the paths $P_{i,1}$ have the same second edge. For infinitely many of those i, the paths $P_{i,2}$ have the same second edge. Repeating this argument, the same holds for $P_{i,3}, \ldots, P_{i,k}$. For infinitely many of those i, the paths $P_{i,1}$ have the same third edge. We repeat this argument and obtain an infinite path P_j from the paths $P_{i,j}$ for each $j = 1, 2, \ldots, k$. The paths $P_1, P_2, \ldots, P_k$ are pairwise edge-disjoint.

We claim that the paths $P_1, P_2, \ldots, P_k$ belong to the same end of G as P_0, that is, each of them is joined by infinitely many pairwise disjoint paths (possibly of length zero) to P_0. Suppose (reductio ad absurdum) that this claim is false. Then for at least one of the paths $P_1, P_2, \ldots, P_k$, say P_1, there is a natural number i such that P_1 is disjoint from H_i. Then $G - A_0$ has a connected subgraph M disjoint from H_i and also disjoint from the vertices incident with the boundary of A_0 such that M has finite boundary and such that M contains an infinite subpath of P_1. We now consider the paths $P_{j,1}, P_{j,2}, \ldots, P_{j,k}$ where $j > i$. Some of these paths may intersect M and there are infinitely many possibilities for that intersection. However, if we merely keep track of the sequences of boundary edges of M where the paths $P_{j,1}, P_{j,2}, \ldots, P_{j,k}$ enter and leave M (possibly several times), then the number of those possibilities is finite because the boundary of M is finite. This means that there is a fixed natural number m such that, for each $j > i$, the paths $P_{j,1}, P_{j,2}, \ldots, P_{j,k}$ have at most m edges in M (by the minimality property of these paths). But then also P_1 has at most m edges in M, a contradiction.

This proves Theorem 1 in the case where $k' = k$.

Consider next the case where $k' < k$. Then $G - A_0$ has an infinite vertex set whose boundary has cardinality k'. Let A_1 be a maximal such set (which exists by Zorn's Lemma). If possible, we choose A_1 such that it contains v_1. We contract A_1 into a single vertex a_1 and call the resulting graph G_1.

If $G_1 - A_0$ has an infinite vertex set whose boundary has cardinality k', then we let A_2 be a maximal such set. If possible, we choose A_2 such that it contains v_2. We contract A_2 into a single vertex a_2 and call the resulting graph G_2. Note that A_2 does not contain a_1 because of the maximality of A_1. We continue like this defining $A_1, a_1, G_1, A_2, a_2, G_2, \ldots$. By the first part of the proof, each of the graphs $G(A_1), G(A_2), \ldots$ is boundary-linked.

Consider first the case where $G_n - A_0$ is finite, for some n. The sets $A_1, A_2, \ldots, A_n$ together with the vertex-singletons not contained in $A_0 \cup A_1 \cup \ldots \cup A_n$ is a partition of $V(G) \setminus A_0$ into boundary-linked subsets and singletons.

Consider next the case where each $G_n - A_0$ is infinite. (The sequence $G_1, G_2, \ldots$ may be finite or infinite.) Let G' be obtained from G by contracting each A_n into the vertex a_n. Then G' is connected and locally finite. We claim that we can apply induction to the pair G', A_0. For otherwise, there would be an infinite connected subgraph H' in $G' - A_0$ whose boundary has at most k' edges and hence precisely k' edges because the boundary of H' is also a boundary in G. By the maximality of A_m, it follows that H' cannot contain a_m. And, if H' contains v_m, then we obtain a contradiction to the way A_m is defined. (If possible, A_m should contain v_m.) So H' cannot exist, and this implies that we can apply induction to the pair G', A_0. We claim that the partition of $G' - A_0$ into finitely many boundary-linked subgraphs (and singletons) also results in a partition of $G - A_0$ into boundary-linked subgraphs and singletons. To see this, let us consider a boundary-linked vertex set B in the partition of $G' - A_0$. Let q denote the number of edges in the boundary of B, and let $P_1, P_2, \ldots, P_q$ be pairwise edge-disjoint paths in $G'(B)$ such that no two of these are separated by a finite edge set in $G'(B)$. Consider a set A_i in G which is contracted into a single vertex a_i in G' such that at least two of the paths $P_1, P_2, \ldots, P_q$ contain a_i. Let $e'_1, e'_2, \ldots, e'_s$ be their incoming edges, and let $e''_1, e''_2, \ldots, e''_s$ be their outgoing edges. Let $P'_1, P'_2, \ldots, P'_s, P''_1, P''_2, \ldots, P''_s$ be one-way infinite paths in $G(A_i)$ starting with the edges $e'_1, e'_2, \ldots, e'_s, e''_1, e''_2, \ldots, e''_s$ and belonging to the same end. We shall now find pairwise edge-disjoint paths in $G(A_i)$ joining $e'_1, e'_2, \ldots, e'_s$ to (a permutation of) $e''_1, e''_2, \ldots, e''_s$ as follows. $G(A_i)$ contains a path $P_{\alpha,\beta}$ which joins some P'_α to some P''_β and which has no edge in common with any of $P'_1, P'_2, \ldots, P'_s, P''_1, P''_2, \ldots, P''_s$. We let $P'_{\alpha,\beta}$ be a path joining e'_α, e''_β contained in the union $P_{\alpha,\beta} \cup P_\alpha \cup P_\beta$. We continue like this joining edges in $e'_1, e'_2, \ldots, e'_s$ to edges in $e''_1, e''_2, \ldots, e''_s$ by pairwise disjoint paths in $G(A_i)$. In this way we determine how the paths $P_1, P_2, \ldots, P_q$ should be continued when they hit a_i. The resulting infinite walks may have repetitions of vertices (but not edges), and they contain one-way infinite paths, and these paths belong to the same end.

This completes the proof of Theorem 1.

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3 Liftings of finite graphs.

In this section k is a fixed natural number, and all graphs are finite. We consider a finite Eulerian graph G (that is, G is connected, and all vertices in G have even degree), a vertex set A in G, and a special vertex x_0 not contained in A. The edges incident with x_0 are denoted $e_1, e_2, \ldots, e_{2q}$. We assume that any two vertices of A are joined by $2k$ pairwise edge-disjoint paths. Equivalently, there is no edge-cut with fewer than $2k$ edges which separates two vertices of A. *Lifting edges* e_i, e_j means that we first delete the edges e_i, e_j and then add instead a new edge joining the two ends of e_i, e_j distinct from x_0. (If those two ends are identical we do not add the corresponding loop, as we do not allow loops in the present paper.) We say that the lifting is *admissible* if, in the lifted graph, any two vertices of A are joined by $2k$ pairwise edge-disjoint paths. Maders lifting theorem [10], and also the special case for Eulerian graphs by Lovász [9], implies that some two edges incident with x_0 form an admissible pair. (In the general version of Mader's theorem it is assumed that x_0 is not a cutvertex. In the Eulerian case we may allow x_0 to be a cutvertex, as e.g. the proof of Theorem 2 below shows.) The resulting graph also contains an admissible pair, so we can lift all edges incident with x_0 and preserve the connectivity property. We define the *lifting graph* $L(G, A, x_0)$ as the graph whose vertices are $e_1, e_2, \ldots, e_{2q}$, and two vertices e_i, e_j (possibly part of a multiple edge in G) are neighbors if e_1, e_2 is an admissible pair. The *bad graph* $B(G, A, x_0)$ is the complement of $L(G, A, x_0)$. With this notation we have

Theorem 2 *$B(G, A, x_0)$ is disconnected. Moreover, if $B(G, A, x_0)$ has four vertices, then it either has no edges, or precisely two edges forming a perfect matching.*

In the proof we shall make use of the following which we formulate as a theorem, although the proof is very short.

Theorem 3 *Let G be a finite connected graph. Assume that G is not complete, and G is not a cycle. Then G contains two non-neighbors x, y such that $G - x - y$ is connected.*

Proof of Theorem 3:

Consider a spanning tree T in G. If T has two end-vertices (that is, vertices of degree 1 in T) which are non-neighbors in G, then we are done. So assume that all end-vertices in T induce a complete graph. In particular, G has no cut-vertex.

We now apply Theorem 2 in [5] which says that every graph which is not a complete graph or a cycle or a complete regular bipartite graph has a path which cannot be extended to a cycle.

The assumption of the theorem says that G is not complete, and G is not a cycle. If G is a complete bipartite graph $K_{q,q}$ where $q \geq 3$, then any two non-neighbors in G can play the role of x, y. So assume that G is not of the form $K_{q,q}$.

Hence G has a path P between the vertices x, y, say, which cannot be extended to a cycle. As G has no cut-vertex, P has at least four vertices, and hence x, y are non-neighbors. Also $G - x - y$ is connected. For otherwise $P - x - y$ belongs to a connected component of $G - x - y$, and x, y have a neighbor in another component of $G - x - y$ which contradicts the assumption that P cannot be extended to a cycle.

■

Proof of Theorem 2:

The proof is by induction on the degree of x_0, that is, the number of vertices of $B(G, A, x_0)$.

If $B(G, A, x_0)$ has two vertices, then $B(G, A, x_0)$ has no edge.

We now consider the case where $B(G, A, x_0)$ has four vertices e_1, e_2, e_3, e_4. If x_0 has only one neighbor, then $B(G, A, x_0)$ has no edge. If x_0 has two neighbors x_1, x_2, and three of the edges e_1, e_2, e_3, e_4 are incident with the same vertex x_1, say, then again, it is easy to see that $B(G, A, x_0)$ has no edge. If x_0 has two neighbors x_1, x_2 each joined to x_0 by two edges, then each of the two edges joining x_0, x_1 are neighbors in $L(G, A, x_0)$ to each of the two edges joining x_0, x_2. If x_0 has three neighbors x_1, x_2, x_3, and two of the edges e_1, e_2, e_3, e_4, say e_1, e_2, are incident with the same vertex x_1, say, then it is easy to see that each of e_1, e_2 is joined to each of e_3, e_4 in $L(G, A, x_0)$. For, if we consider $2k$ pairwise edge-disjoint paths joining two vertices of A before the lifting, then it is easy to transform these paths into $2k$ pairwise edge-disjoint paths joining the same two vertices of A after the lifting.

We now consider the case where $B(G, A, x_0)$ has four vertices e_1, e_2, e_3, e_4, and these edges have distinct ends in $G - x_0$. Suppose (reductio ad absurdum) that e_1 is joined to each of e_2, e_3 in $B(G, A, x_0)$. When we lift the pair e_1, e_2 we create an edge-cut with fewer than $2k$ edges separating at least two vertices in A. As G is Eulerian, this cut has at most $2k - 2$ edges. A similar edge-cut arises when we lift the pair e_1, e_3. Thus the vertex set of G can be divided into four sets A_1, A_2, A_3, A_4 such that one of the above cuts consists of the edges between $A_1 \cup A_2$ and $A_3 \cup A_4$, and the other cut consists of the edges between $A_1 \cup A_4$ and $A_2 \cup A_3$. We may assume that x_0 is in A_1 and that e_1 is in both of the cuts, that is, e_1 has an end in A_3. Moreover, both of e_2, e_3 are in one of the cuts. If we contract each of A_2, A_3, A_4 into a single vertex, then the liftings of e_1, e_2 and e_1, e_3 are still non-admissible in the resulting graph H. This implies that x_0 has four distinct neighbors in H because we have already disposed of the case where x_0 in incident with a multiple edge. This implies that e_1, e_2, e_3 go to distinct sets A_2, A_3, A_4, and e_4 goes to A_1. In particular, all sets A_1, A_2, A_3, A_4 intersect A. This implies that there are at least $2k$ edges leaving each of the sets A_1, A_2, A_3, A_4, and hence the total number of edges joining two of the sets A_1, A_2, A_3, A_4 is at least $4k$. But the two above edge-cuts contain at most $2k$ edges each, and e_1 is contained in both of the cuts, so the union of the two cuts has less than $4k$ edges. This contradiction disposes of the case where $B(G, A, x_0)$ has four vertices.

Assume now that $B(G, A, x_0)$ has at least six vertices. Suppose (reductio ad absurdum) that $B(G, A, x_0)$ is connected.

We claim that $B(G, A, x_0)$ is not a complete graph. If x_0 is not a cutvertex, then this follows from Mader's lifting theorem. If x_0 is a cutvertex, then it is easy to see that any two edges joining x_0 to distinct components of $G - x_0$ form an admissible pair. (It is here important that G is Eulerian so that no edge incident with x_0 is a bridge.) This proves that claim that $B(G, A, x_0)$ is not a complete graph.

Consider now the case where $B(G, A, x_0)$ is not a cycle. Now we apply Theorem 3 to $B(G, A, x_0)$. Let e_1, e_2 be edges incident with x_0 such that e_1, e_2 are non-neighbors in $B(G, A, x_0)$ and such that $B(G, A, x_0) - e_1 - e_2$ is connected. Now we lift e_1, e_2 in G and call the resulting graph G'. As $B(G', A, x_0)$ contains $B(G, A, x_0) - e_1 - e_2$ which is connected, this contradicts the induction hypothesis.

There remains only the case that $B(G, A, x_0)$ is a cycle, say $e_1 e_2 \ldots e_{2q} e_1$. If $2q = 6$, we lift e_1, e_3. In the resulting graph the pairs e_5, e_4 and e_5, e_6 are non-admissible contradicting the induction hypothesis. If $2q \geq 8$, then again, we lift e_1, e_3. If e_2, e_4 is non-admissible after the lifting, then we obtain a contradiction to the induction hypothesis. If e_2, e_4 is admissible after the lifting, then we also lift this pair, and then we again obtain a connected bad graph, contradicting the induction hypothesis.

This completes the proof of Theorem 2.

■

4 Finite immersions with large edge-connectivity.

If G is a graph and H is a graph with vertices $x_1, x_2, \ldots$, then an *immersion* of H in G is a subgraph consisting of vertices $y_1, y_2, \ldots$ in G and a collection of pairwise edge-disjoint paths in G such that, for each edge $x_i x_j$ in H, there is a corresponding path in the collection joining y_i, y_j.

It is well known that there are planar, locally finite graphs P_k with arbitrarily large finite connectivity k and with no multiple edges. A finite subgraph of a planar graph with no multiple edges has minimum degree and hence edge-connectivity at most 5, by Euler's formula. So, if H' is a subdivision of a finite graph H with no multiple edges, and H' is a subgraph of P_k, then H has edge-connectivity at most 5. This changes if we consider immersions rather than subdivisions.

Theorem 4 *Let k be a natural number, let G be an $4k$-edge-connected graph, and let A_0 be a finite vertex set in G.*

Then G contains an immersion of a finite Eulerian $2k$-edge-connected graph with vertex set A_0.

Proof of Theorem 4 for locally finite graphs:

We apply Theorem 1. Let G' be the finite graph obtained by contracting each of the boundary-linked sets into a single vertex. Then G' is $4k$-edge-connected and contains therefore, by the result of Edmonds [6] , Nash-Williams [13] and Tutte [21] $2k$ pairwise edge-disjoint spanning trees. The union of any two edge-disjoint spanning trees contains a spanning Eulerian subgraph which is connected and hence 2-edge-connected. (To see this, just delete an appropriate edge-set from one of the trees.) Hence G' contains a subgraph G'' which is the union of k pairwise edge-disjoint spanning Eulerian subgraphs. We shall modify G'' into the desired immersion.

First observe that in G'', no two vertices of A_0 are separated by fewer than $2k$ edges. Then consider a vertex v in G'' but not in A_0. If v is a singleton in the decomposition, then we use Mader's lifting theorem to lift all edges incident with v such that in the resulting Eulerian graph it is still true that no two vertices of A_0 are separated by fewer than $2k$ edges.

Next we consider a vertex v in G'' which in the decomposition corresponds to a boundary-linked set A. Again, we shall lift the edges incident with v, but not using Mader's lifting theorem. Instead we focus on the lifting graph $L(G'', A_0, v)$ which we know has a disconnected complement, by Theorem 2. The vertices of this graph $L(G'', A_0, v)$ are the edges $e_1, e_2, \ldots, e_{2q}$ incident with v. We now define another graph M defined on this vertex set. We consider the one-way infinite paths $P_1, P_2, \ldots, P_{2q}$ in $G(A)$ starting with the edges $e_1, e_2, \ldots, e_{2q}$ in the boundary of A, that is, the edges incident with v. We say that two vertices e_i, e_j are neighbors in M if $G(A)$ has a collection of infinitely many pairwise disjoint paths joining P_i, P_j having only the ends in common with $P_1 \cup P_2, \ldots \cup P_{2q}$. Since any two of $P_1, P_2, \ldots\ , P_{2q}$ are joined by infinitely many pairwise disjoint paths in $G(A)$, it follows easily that M

is connected. As $L(G'', A_0, v)$ has a disconnected complement, by Theorem 2, it follows that $L(G'', A_0, v)$ and M have a common edge joining e_α, e_β, say. Let P' be a path in $G(A)$ joining P_α, P_β with only its ends in common with $P_1, P_2, \dots, P_{2q}$. Let $P_{\alpha,\beta}$ be a path in $P_\alpha \cup P_\beta \cup P_{\alpha,\beta}$ starting and terminating with e_α, e_β. Now delete the edges of $P_{\alpha,\beta}$ from $G(A)$, lift e_α, e_β in G'' and define a new graph M and a new lifting graph where we now ignore P_α, P_β although these two paths P_α, P_β are still present. The new M and the new lifting graph have a common edge, and we repeat the above argument to find a new path in $G(A)$ and lift the corresponding edges in the new G''. Doing this for each vertex v in G'' not in A_0 results in an Eulerian $2k$-edge-connected graph with vertex set A_0. When we reverse the liftings we modify this graph to an immersion in G.

■

We discuss in Sections 7,8 how to extend Theorem 4 to the general case (allowing vertices to have infinite degree).

5 Orientations of finite graphs with large edge-connectivity.

In this section we establish an orientation result for finite graphs, to be used in the main application of Theorem 1. An edge with a direction is called a *directed edge* or an *arc*. A path in which all edges have a direction is called a *mixed path*. It is called a *directed path* if all edges have the same direction when we traverse the path. A *directed cycle* is defined analogously. We say that a directed graph is *k-arc-connected* if the deletion of any set of fewer than k arcs results in a strongly connected directed graph. By Menger's theorem this is equivalent to the statement: For any two vertices x, y, the directed graph has k pairwise arc-disjoint directed paths from x to y (and also from y to x).

Let k be a natural number. The result of Nash-Williams [12] implies that every finite $2k$-edge-connected graph has a k-arc-connected orientation. This does not follow from the result of Edmonds [6] , Nash-Williams [13] and Tutte [21] that every finite $2k$-edge-connected graph has k pairwise edge-disjoint spanning trees. But the following weakening does: Every finite $4k$-edge-connected graph has a k-arc-connected orientation. To see this, we consider a collection of $2k$ pairwise edge-disjoint spanning trees. We select a vertex v

in G, we direct half of the trees away from v and the other half towards v.

The main idea in this section is a simple algorithmic proof of this weakening of Nash-Williams' orientation result. We consider a finite graph G and we perform alternately the following two operations. (That is, we first perform O_1, then O_2, then O_1 on the resulting graph, then O_2 on the resulting graph etc.)

O_1: Select a maximal collection of pairwise edge-disjoint cycles such that no edge has a direction and make each of them into a directed cycle.

O_2: Select two vertices u, v joined by the maximum number of edge-disjoint mixed paths, and identify u, v into one vertex.

It turns out, perhaps surprisingly, that if G is $4k$-edge-connected, then the resulting oriented graph is k-arc-connected. To prove this, we use the following well-known lemma.

Lemma 1 *Let k be a natural number, and let G be a graph with $n \geq 2$ vertices and more than $(k-1)(n-1)$ edges.*

Then G contains two distinct vertices joined by k pairwise edge-disjoint paths.

Proof of Lemma 1: The proof is by induction on n. For $n = 2$ there is nothing to prove (as an edge is not a loop), so we proceed to the induction step. If G is k-edge-connected, we use Menger's theorem. So assume the vertex set of G can be divided into nonempty sets A, B such that there are at most $k-1$ edges between A, B. Then one of $G(A), G(B)$ satisfies the induction hypothesis. ∎

Theorem 5 *Let k be a natural number, and let G be a finite $(4k-2)$-edge-connected graph.*

Successively perform either of the following two operations:

O_1': Select a cycle in which no edge has a direction and make it into a directed cycle.

O_2': Select two vertices u, v joined by $2k-1$ pairwise edge-disjoint mixed paths, and identify u, v into one vertex.

When none of these operations can be performed the resulting oriented graph has only one vertex. The edge-orientations of G obtained by O_1' result in a k-arc-connected directed graph.

Proof of Theorem 5:

Suppose we end up with a graph G' with n' vertices. Assume (reductio ad absurdum) that $n' \geq 2$. As we cannot perform operation O_1' on G', it follows that G' has at most $n' - 1$ undirected edges. As we cannot perform operation O_2' on G', it follows that G' has at most $(2k-2)(n'-1)$ directed edges, by Lemma 1. So G' has at most $(2k-1)(n'-1)$ edges. However, as G' is $(4k-2)$-edge-connected, it has at least $(2k-1)n'$ edges, a contradiction which shows that $n' = 1$, and hence every edge has received a direction.

We now prove, by induction on the number of vertices of G, that the orientation of G is k-arc-connected. If operation O_2' is never used, then each cut is balanced, that is, G is $(2k-1)$-arc-connected. So assume that operation O_2' is used, and let G' be the graph resulting from the first use of O_2'. When G' is formed, then some edges received a direction in G, but those directions can be thought of as directions obtained in G' as well. By the induction hypothesis, G' becomes a k-arc-connected directed graph when all edges have been given a direction. G is obtained from G' by splitting a vertex up into two vertices u, v. Just before the vertex identification u, v are joined by $2k-1$ pairwise arc-disjoint mixed paths. This implies that there are k arc-disjoint directed paths from u to v and also k arc-disjoint directed paths from v to u. (For, if there is a cut separating u from v such that there are at most $k-1$ arcs from one side to the other, then there are also at most $k-1$ arcs in the other direction in the cut, because we have only used operation O_1' so far. Then the cut has at most $2k-2$ arcs, a contradiction.) Because of these directed paths between u, v, the k-arc-connectedness of G' implies k-arc-connectedness of G.

■

Theorem 6 *Let k be a natural number, and let G be a finite $(4k-2)$-edge-connected graph. Let H be an orientation of a subgraph of G such that the indegree of every vertex of H equals the outdegree. Then the edge-orientation of H can be extended to an orientation of G which is a k-arc-connected directed graph.*

Proof of Theorem 6: The orientation of H can be obtained using operation O_1'. Now Theorem 6 follows from Theorem 5.

■

6 Orientations of infinite graphs with large edge-connectivity.

We now turn to the main application of Theorem 1.

Theorem 7 *Let k be a natural number, and let G be an $8k$-edge-connected graph.*

Then G has a k-arc-connected orientation.

Proof of Theorem 7 for locally finite graphs:

Let $e_0, e_1, \ldots$ be the edges of G. We construct a sequence of finite oriented subgraphs using the operations O'_1, O'_2. After n steps in this sequence of operations we have a vertex v_n (to be explained below) and a directed Eulerian oriented subgraph W_n containing v_n. First, we let W_0 be a directed cycle containing e_0, and we let v_0 be any vertex in W_0. We assume that we have constructed W_n and explain how to obtain W_{n+1}. We apply Theorem 4 with $8k$ instead $4k$ and with A_0 consisting of $V(W_n)$ and the two ends of e_n. Let H denote the $4k$-edge-connected graph resulting in the application of Theorem 4. We now apply Theorem 5 to H. (The edges which already have an orientation form an Eulerian subgraph of H, so their orientations can be regarded as a result of Operation O'_1.) After the operations O'_1 and O'_2, H (or, more precisely, the vertex set in G corresponding to the vertex set of H) becomes a single vertex v_{n+1} and some loops. Those loops correspond to edge-disjoint directed paths in G (because H is an immersion rather than a subgraph), and hence they correspond to directed cycles after H has been transformed into v_{n+1}. We define the union of those directed cycles to be the oriented graph W_{n+1}.

In this way each edge of G gets an orientation. If x, y are vertices of G, then there exists an n such that x, y are part of the vertex v_n. But, this means that in the subgraph induced by the vertices that form v_n, there are k arc-disjoint directed paths from x to y and k arc-disjoint directed paths from y to x. As this holds for any x, y in G, the orientation of G is k-arc-connected.

■

We discuss in Sections 7,8 how to extend Theorem 7 to the general case.

7 From locally finite to countably infinite.

Let G be a graph, and let A be a vertex set in G. Then an *A-pairing* is a collection of pairwise edge-disjoint paths in G joining vertices in A such that each vertex in A is the end-vertex of precisely one such path. An *A-near-pairing* is a collection of pairwise edge-disjoint paths in G joining vertices in A such that each vertex in A, except one, is the end of precisely one such path.

Theorem 8 *Let T be a tree, and let A be a vertex set in T. Then T has an A-pairing or an A-near-pairing.*

Proof of Theorem 8: If A is finite, the statement is an easy exercise. We consider the case where A is a countable set with vertices $v_1, v_2, \ldots$ (This is merely for notational convenience. If A is uncountable we consider a well-ordering instead.) If possible, we consider a path P_1 joining two vertices v_i, v_j in A such that $T - E(P_1)$ has only one component T_1 containing vertices of $A \setminus \{v_i, v_j\}$. We consider that tree T_1 instead of T and we replace A by $A_1 = A \setminus \{v_i, v_j\}$. If possible, we consider a path P_2 joining two vertices v_p, v_q in A_1 such that $T_1 - E(P_2)$ has only one component T_2 containing vertices of $A_1 \setminus \{v_p, v_q\}$. We proceed like this pairing vertices of A. Since these pairings are inductively ordered by inclusion, we use Zorn's lemma to find a maximal such pairing. (To see that the pairings are inductively ordered, we consider a chain of pairings. If v_i, v_j are vertices of A which are not part of these pairings, then the path in T between v_i, v_j consists of edges none of which are part of any of the pairings in the chain and hence also not in the union of the pairings.)

After we have used Zorn's lemma we have a tree containing the non-paired vertices A' in A. The maximality property of the paired vertices implies that T', A' has the following property which we call *property p*: it is not possible to find a path P' in T' joining two vertices v_i, v_j in A' such that $T' - E(P')$ has only one component containing vertices of $A' \setminus \{v_i, v_j\}$. If A' consists of one vertex we have obtained an A-near-pairing. So assume that A' has at least two vertices. Then A' has infinitely many vertices by the maximality property and the argument at the beginning of the proof. We claim that in this case T' has an A'-pairing. Let i be the smallest number such that v_i is in A', that is, v_i is not paired. Let P be a path in T' from v_i to another vertex a of A' such that no intermediate vertex of P is in A' and such that as few components of $T' - E(P)$ as possible contain precisely one vertex of A'. We add P to the pairing and delete the edges of P from T'.

We claim that no component of $T' - E(P)$ contains precisely one vertex b of A'. For, if there were such a component, then we would replace P by the path P' from v_i to b. That path does not contain a because of the maximality of the pairing. We might thereby create a new component with precisely one vertex of A'. That vertex must be the other end a of P. But then the path between a and b has the property that the deletion of its edges creates a forest with only one component containing vertices of $A' \setminus \{a, b\}$, a contradiction to the maximality of the pairing.

Since no component of $T' - E(P)$ contains precisely one vertex of A', it follows that every component of $T' - E(P)$ contains either none or infinitely many vertices of A'. (For, if it contains a finite number ≥ 2 of vertices in A', then we can pair all these vertices except possibly one well-chosen vertex, and add that finite pairing to our maximal pairing and thereby obtain a contradiction to the maximality.) Consider one, say T'', which contains infinitely many vertices of A'. If T'', A'' has property p, then we repeat the argument we applied to T', A'. So assume that T'', A'' does not have property p. We now let P_1' be a path joining two vertices v_p, v_q in A'' such that $T'' - E(P_1')$ has only one component containing vertices of $A'' \setminus \{v_p, v_q\}$. We repeat this until we either obtain a pairing of A'' or obtain T''', A''' having property p, in which case we repeat the argument we applied to T', A'.

We now argue that we indeed reach one of those two possibilities. (Note that we need an argument since we are not satisfied with a near-pairing.) Let us therefore assume that the pairing procedure of A'' results in a partial pairing which does not include the vertex x in A'', say. Let P' be the path in T'' from x to the path P (from v_i to a). As T', A' has property p, it follows that P_1' contains an edge of P'. The same applies to $P_2', P_3', \ldots$, so this sequence must be finite. So after having deleted the edges of the paths in that sequence, we get a tree with property p.

Repeating this argument completes the proof.

■

A *splitting* of a graph G is a graph G' which is obtained from G by blowing each vertex up into a set of vertices. Formally, a splitting G' of G is a graph with the same edges as G. Each vertex v in G corresponds to a vertex set V_v in G' such that G' has no edge joining two vertices in V_v and such that the identification of all vertices of V_v into a single vertex (for each vertex v in G) results in G.

Theorem 9 *Let k be a natural number, and let G be a countably infinite k-edge-connected graph. Then G has a splitting such that the resulting graph is k-edge-connected, and each block of the resulting graph is locally finite.*

Proof of Theorem 9: Let B be a block of G and let v be a vertex in B of infinite degree in B. Let $A = \{v_1, v_2, \ldots\}$ be the set of neighbors of v in B. Let T be a spanning tree of $B-v$. Consider an A-pairing or an A-near-pairing in T. Assume that the pairing is an A-pairing, and the notation is chosen such that v_{2i-1}, v_{2i} are paired by the path P_i for $i = 1, 2, \ldots$. Let H be a locally finite k-edge connected graph with countably infinite edge set $e_1, e_2, \ldots$ and countably infinite vertex set $u_1, u_2, \ldots$. For notational convenience we give each path P_i a direction, and we give each edge e_i a direction. Note that we have a one-to-one correspondence between the edges in H and the paths P_i in the pairing. We split v up into vertices $w_1, w_2, \ldots$ as follows: In order to decide which neighbors w_i should have, we consider the vertex u_i in H. We let w_i be joined to the first vertices of those paths in the pairing which correspond to the edges leaving u_i in H and also to the last vertices of those paths in the pairing which correspond to the edges entering u_i in H. We also let w_1 be joined to the unpaired vertex v_1 in case we have an A-near-pairing. As H is k-edge connected, it follows that the resulting graph has k pairwise edge-disjoint paths between any two of the new vertices $w_1, w_2, \ldots$. Hence the resulting graph obtained by splitting v into new vertices is k-edge-connected. If v is a vertex of other blocks in G, then we let w_1 be part of those blocks.

We have now shown how to split one vertex of infinite degree such that the edge-connectivity is preserved. But, we have to dispose of all such vertices. To do this we enumerate all pairs of vertices of finite degree in G, say $L_1, L_2, \ldots$. (When we split a vertex of infinite degree we obtain new vertices of finite degree. The pairs containing those new vertices will be inserted in the sequence $L_1, L_2, \ldots$. The sequence $L_1, L_2, \ldots$ is now renamed $L_{1,1}, L_{2,1}, \ldots$ and the new sequence is called $L_{1,2}, L_{2,2}, \ldots$. We then enumerate the pairs $L_{i,j}$ as we enumerate the rational numbers.) Before we split the vertex v, we select k pairwise edge-disjoint paths between the two vertices of L_1. When we split v, we insist that the k paths between the two vertices of L_1 are preserved as paths. If they are destroyed after the splitting, we restore them by a finite number of vertex identifications. We still have a splitting of v into vertices of finite degree. Then we select k pairwise edge-disjoint paths joining the two vertices of L_2. When we split the next vertex of infinite degree, we insist that the k paths between the two vertices of L_1 are preserved as paths and also the k paths between the two vertices of L_2 are preserved as paths. Continuing like this results in the desired vertex splitting of G.

■

Using Theorem 9, we can now extend Theorems 4, 7 to the countable case. A graph is k-edge-connected if and only if every block is k-edge-connected. A directed graph is k-arc-connected if and only if every block is k-arc-connected. As vertex-identifications preserve edge-connectivity and arc-connectivity, Theorem 9 immediately extends Theorem 7 to the countable case.

Theorem 4 also extend easily to the countable case. In this theorem there are some prescribed vertices involved. If such a vertex u has infinite degree, then u is split up into vertices of finite degree. We just select one of them and let that vertex play the role of a new u. Each block is locally finite, but cutvertices may have infinite degree. However, if we wish to include two vertices x, y in an immersion, and x, y are separated by a cutvertex z, then we just add z to the vertex set of the immersion (and later we lift its edges because we want the vertex set of the immersed graph to be precisely A_0). So, the countable version of the result in Theorem 4 reduces to the locally finite case.

8 From countable to uncountable.

We now extend Theorems 4, 7 to the uncountable case.

Theorem 10 *Let k be a natural number, and let G be an infinite k-edge-connected graph, and let A_0 be a finite vertex set of G. Then G contains a countable subgraph which contains A_0 and which is k-edge-connected.*

Proof of Theorem 10: Let G_1 be obtained from A_0 by adding k pairwise edge-disjoint paths between any two pairs of vertices of A_0. Suppose we have constructed the finite graph G_n. Let G_{n+1} be obtained from G_n by adding k pairwise edge-disjoint paths between any two pairs of vertices of G_n. Then the union of the graphs $G_1, G_2, \ldots$ is countable and k-edge-connected.

■

Theorem 10 reduces immediately Theorem 4 to the countable case. To extend Theorem 7 to the uncountable case, let us consider any $8k$-edge-connected graph G. By Theorem 10, G has a finite or countably infinite

$8k$-edge-connected subgraph G' with at least two vertices. By the countable version of Theorem 7, G' has a k-arc-connected orientation. By Zorn's Lemma, G has a maximal oriented k-arc-connected subgraph containing G'. (When we apply Zorn's lemma, we consider subgraphs that are actually oriented. Not just some that have a k-arc-connected orientation.) Clearly, G'' is an induced subgraph. We claim that $G'' = G$. For if this were not the case, then we contract G'' into a single vertex v_0. By Theorem 10, the resulting graph H contains a finite or countably infinite subgraph G''' which is $8k$-edge-connected and which contains v_0 and has at least one more vertex. By the countable version of Theorem 7, G''' has a k-arc-connected orientation. The edges in $G'' \cup G'''$ (which have an orientation) now form a k-arc-connected directed subgraph of G contradicting the maximality of G''.

■

There is another way of extending Theorem 7 from the countable case to the uncountable case by using the result of Laviolette [8] that every infinite k-edge-connected graph is the union of pairwise edge-disjoint k-edge-connected countable subgraphs.

9 Open problems: Connectivity and $(2+\epsilon)$-flow.

We repeat the original question of Nash-Williams [14].

Problem 1 *Let k be a natural number. Does every $2k$-edge-connected graph admit a k-arc-connected orientation?*

Problem 2 *Let ϵ be a positive real number. Does there exist a natural number $f(\epsilon)$ such that every $f(\epsilon)$-edge-connected graph admits an orientation and a flow with flow values in the interval between 1 and $1+\epsilon$ such that every cut is balanced, that is, the sum of flow values in one direction of the cut equals the sum of flow values in the other direction?*

For finite graphs this problem became known as the $(2+\epsilon)$-flow conjecture by Goddyn and Seymour. For finite graphs it is now a theorem as it follows from the weak circular flow conjecture proved in [19].

Problem 3 *Let ϵ be a positive real number. Does there exist a natural number $f(\epsilon)$ such that every $f(\epsilon)$-edge-connected graph admits an orientation such* *that, for each cut, the number of edges directed in one direction is at least* $1 - \epsilon$ *times the number of edges directed in the other direction?*

An affirmative answer to Problem 2 implies an affirmative answer to Problem 3 which is also a theorem in the finite case. Prior to the proof in [19] it was known that Problems 2, 3 were equivalent for finite graphs.

Problems 2, 3 have affirmative answers if we focus only on finite cuts. It is the infinite cuts that are problematic. As a first step towards an investigation of infinite cuts we now characterize the graphs which admit an orientation such that each infinite cut has infinitely many edges in both directions.

10 Robbins' theorem extended to infinite cuts.

Robbins [16] proved that every finite 2-edge-connected graph has a strongly connected orientation. It is easy to extend this to infinite graphs using Zorn's lemma. It is also easy to prove that, for each finite vertex set A in an 2-edge-connected graph G, there is a strongly connected orientation of a finite subgraph containing A.

Theorem 11 *Let* G *be an infinite* 2*-edge-connected graph. Then* G *has a strongly connected orientation such that every infinite cut has infinitely many edges in both directions.*

Proof of Theorem 11: We consider first the locally finite case. Let $e_1, e_2, \ldots$ be the edges of G. Let G_0 be a single vertex of G. Suppose we have defined a finite subgraph G_n and given each edge of G_n an orientation such that G_n is strongly connected. Then $G - V(G_n)$ has only finitely many components, and each component contains only finitely many bridges and only finitely many maximal 2-edge-connected subgraphs. Consider a maximal 2-edge-connected subgraph H of $G - V(G_n)$. By a remark above, H has a finite subgraph H' which contains those vertices of H having neighbors outside H such that H' has a strongly connected orientation. We also give the bridges of $G - V(G_n)$ and all edges between G_n and $G - V(G_n)$ an orientation such that the edges with an orientation form a finite strongly connected subgraph G_{n+1}. We may assume that G_{n+1} contains e_{n+1}. Then the union of $G_1, G_2, \ldots$ is a strongly connected orientation of G.

We claim that all infinite cuts are balanced. To prove this consider an edge e. Let n be the smallest number such that e is in G_n. We select a directed cycle C_e in G_n containing e. We say that each edge in $C_e - e$ is *demanded by* e. For each edge e in a maximal 2-edge-connected subgraph H of $G - V(G_n)$ it is possible to choose C_e such that it is contained in H. This means that every edge is demanded by only a finite number of edges. Consider now an infinite cut consisting of the edges between A, B, say. As G is strongly connected, there is at least one edge from A to B and at least one edge from B to A. Every edge from A to B demands an edge from B to A. As each edge in the cut is demanded by only finitely many edges, there are infinitely many edges from B to A and, similarly, infinitely many edges from A to B.

Using Theorem 9 the argument extends to the countable case. Extending to the uncountable case is routine but a little tedious so we leave it for the reader.

■

For each property p_1, p_2, p_3 below we can now characterize the connected graphs with that property.

p_1: G has an orientation with no infinite directed cut (that is, a cut in which all arcs have the same direction).

p_2: G has an orientation in which each infinite cut has infinitely many edges in either direction.

p_3: G has an orientation in which each infinite cut is balanced, that is, the cardinality of arcs in one direction equals the cardinality of arcs in the other direction.

It turns out that the properties p_1, p_2, p_3 are equivalent. To see this, consider a connected graph G. Let T be the tree obtained by contracting each maximal 2-edge-connected subgraph into a single vertex. If T has a vertex of infinite degree or infinitely many vertices of degree at least 3, then it is easy to see that every orientation of G results in a graph where there is an infinite directed cut. In other words, G does not satisfy any of p_1, p_2, p_3 in this case.

Conversely, if T is locally finite and has only finitely many vertices of degree at least 3, then we can make every path in T with intermediate vertices of degree 2 into a directed path, and we can apply Theorem 11 to each maximal 2-edge-connected subgraph. In the resulting orientation each infinite cut has infinitely many arcs in either direction of the cut. If some maximal 2-edge-connected subgraph H is uncountable, then we use the result of Laviolette [8] to first decompose H into countable 2-edge-connected graphs and then we apply Theorem 11 to each of those. It is easy to see that every infinite cut of G is balanced, that is, G satisfies each of p_1, p_2, p_3.

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