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Characterizing Width Two for Variants of Treewidth

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Abstract

In this paper, we consider the notion of *special treewidth*, recently introduced by Courcelle [16]. In a special tree decomposition, for each vertex v in a given graph, the bags containing v form a rooted path. We show that the class of graphs of special treewidth at most two is closed under taking minors, and give the complete list of the six minor obstructions. As an intermediate result, we prove that every connected graph of special treewidth at most two can be constructed by arranging blocks of special treewidth at most two in a specific tree-like fashion.

Inspired from the notion of special treewidth, we introduce three natural variants of treewidth, namely *spaghetti treewidth*, *strongly*

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[1]An earlier paper on the characterization of special treewidth two appeared in the proceedings of WG 2013 [13].

chordal treewidth and *directed spaghetti treewidth*. All these parameters lie between pathwidth and treewidth, and we provide common structural properties on these parameters. For each parameter, we prove that the class of graphs having the parameter at most two is minor closed, and we characterize those classes in terms of a *tree of cycles* with additional conditions. Finally, we show that for each $k \geq 3$, the class of graphs with special treewidth, spaghetti treewidth, directed spaghetti treewidth, or strongly chordal treewidth, respectively at most k, is not closed under taking minors.

1 Introduction

Treewidth and *pathwidth* are one of the basic parameters in graph algorithms and they play an important role in structural graph theory. Numerous problems which are NP-hard on general graphs, have been shown to be solvable in polynomial time on graphs of bounded treewidth [2, 6]. Courcelle [15] provided a celebrated algorithmic meta-theorem which states that every graph property expressible in monadic second-order logic formulas (MSO_2) can be decided in linear time on graphs of bounded treewidth.

In this paper, we discuss a relatively new notion of *special treewidth*, introduced by Courcelle [16]. A special tree decomposition is a tree decomposition where for each vertex of a given graph, the bags containing this vertex form a rooted path in the tree. Courcelle developed this parameter to reduce the difficulty in representing tree decompositions algebraically. The monadic second-order logic (MSO_2) checking algorithm for treewidth in the meta-theorem is based on the constructions of finite automata, and he observed that these constructions become much simpler when working with special tree decompositions compared to standard tree decompositions.

Courcelle asked several questions on properties of special treewidth. One of the questions was how to characterize the class of graphs of special treewidth at most k by forbidden configurations. In this context, he showed that the graphs of special treewidth one are exactly the forests, but if $k \geq 5$, then the class of graphs of special treewidth at most k is not closed under taking minors.

In this paper, we prove that the class of graphs of special treewidth at most two is closed under taking minors, and provide the minor obstruction set. We also sharpen Coucelle's bound, and show that for $k \geq 3$, the class of graphs of special treewidth at most k is not closed under taking minors. The graph K_4 denotes the complete graph on four vertices, and the other five graphs are depicted in Figure 1 and Figure 2.

Parameter	Graph Class
treewidth	chordal graphs
pathwidth	interval graphs
special treewidth	RDV graphs
directed spaghetti treewidth	DV graphs
spaghetti treewidth	UV graphs
strongly chordal treewidth	strongly chordal graphs
treedepth	trivially perfect graphs

Table 1: Graph parameters which can be defined by the clique number of a supergraph from a class of graphs. Graph classes are defined in Section 2.

Theorem 5.8. *A graph has special treewidth at most two if and only if it has no minor isomorphic to* $K_4, D_3, S_3, G_1, G_2,$ *or* G_3*.*

To show this, we first prove that every block of special treewidth at most two must have pathwidth at most two. But it is not a sufficient condition for having special treewidth two, and we establish a precise condition how those blocks can be attached to obtain a graph of special treewidth two.

Inspired by special treewidth, we introduce new three variants of treewidth. From the results by Courcelle, we observe that having bounded special treewidth is a much stronger property than having bounded treewidth. We can naturally ask whether there exist elegant width parameters lying between special treewidth and treewidth, which establish a link from pathwidth to treewidth.

Two variants, *spaghetti treewidth* and *directed spaghetti treewidth*, are defined by taking different models of tree decompositions. While in the intersection model of special treewidth, we associate each vertex with a rooted path, in a spaghetti tree decomposition, the bags containing each vertex form a 'usual' path in a tree (that is, without the condition of being rooted), and in a directed spaghetti tree decomposition, the bags containing each vertex form a directed path in a tree with a given direction. The *strongly chordal treewidth* of a graph G is defined as the minimum of the clique number of H minus one over all strongly chordal supergraphs H of G. These parameters are at most the pathwidth and at least the treewidth of the graph.

Each of these new parameters can be alternatively defined as the minimum of the clique number of all supergraphs where the supergraphs belong to a certain graph class. Another related notion is treedepth [9, 10], and it can be defined as the minimum of the clique number of all trivially perfect supergraphs of a given graph. Table 1 gives an overview of the parameters and the corresponding classes.

Graph classes	Minor obstructions for 2-connected graphs	Minor obstructions for general graphs
tw ≤ 2	K_4 (see [3, 37])	K_4 (see [3, 37])
spghtw ≤ 2	K_4, D_3	K_4, D_3
sctw ≤ 2	K_4, S_3	K_4, S_3
dspghtw ≤ 2	K_4, D_3, S_3	K_4, D_3, S_3
spctw ≤ 2	K_4, D_3, S_3	6 graphs
pw ≤ 2	K_4, D_3, S_3 [4, 13]	110 graphs [28]
td ≤ 3	K_4, C_5 [19]	12 graphs [19]

Table 2: Summary of results. tw, spghtw, sctw, dspghtw, spctw, pw and td denote treewidth, spaghetti treewidth, strongly chordal treewidth, directed spaghetti treewidth, special treewidth, pathwidth, and treedepth respectively.

We expect that these new parameters can be used to provide a link between pathwidth and treewidth by yielding new structural or algorithmic results. As a similar approach, Fomin, Fraigniaud, and Nisse [23] introduced a parameterized variant of tree decompositions, called *q-branched tree decompositions*, and provided a unified method to compute pathwidth and treewidth. In this paper, we study common structural properties of our notions.

For each of the three parameters, we show that the class of graphs having width at most two is closed under taking minors. Moreover, we precisely describe how those graphs look like in terms of *trees of cycles* with specific conditions depending on the parameter. Trees of cycles were used to characterize treewidth two [11] and pathwidth two [8]. In Table 2, we see an overview of the different parameters and the minor obstruction sets for these classes. As 2-connected graphs play a special role in several proofs, the 2-connected graphs in the obstruction sets are given in the second column. In addition, for each of these parameters and each value $k \geq 3$, we show that the class of graphs with the parameter at most k is not closed under taking minors.

Our characterizations in terms of forbidden minors fit in a line of research, originated by the ground breaking results in the graph minor project by Robertson and Seymour [36]. From the results of Robertson and Seymour, for every minor-closed class $\mathcal{G}$ of graphs, there exists a finite obstruction set $ob(\mathcal{G})$ of graphs such that for each graph H, $H \in \mathcal{G}$ if and only if H has no minor isomorphic to a graph in $ob(\mathcal{G})$. For several minor-closed graph

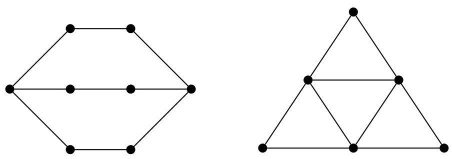

Figure 1: The graphs D_3 and S_3. The graph S_3 is called the 3-sun.

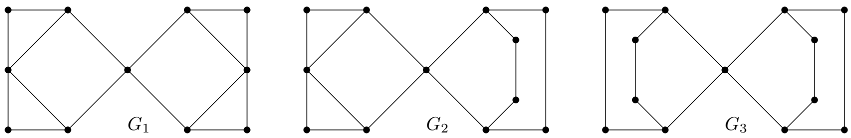

Figure 2: The graphs G_1, G_2, G_3 of the minor obstruction set for graphs of special treewidth two, which are not 2-connected.

classes, the obstruction set is known, for example, planar graphs ($\{K_5, K_{3,3}\}$ [38]), graphs embeddable in the projective plane [1], graphs of treewidth at most two ($\{K_4\}$, see [18, Proposition 12.4.2]), graphs of treewidth at most three (a set of four graphs [3]), graphs of pathwidth at most two (a set of 110 graphs [28]), and outerplanar graphs ($\{K_4, K_{2,3}\}$). The obstruction set of graphs of treedepth at most three (and smaller values) was given by Dvořák, Giannopoulou and Thilikos [19]; it contains exactly twelve graphs.

This paper is organized as follows. In Section 2, we give a number of preliminary definitions and results, including the *trees of cycles* and *paths of cycles* models for 2-connected graphs of treewidth and pathwidth two. In Section 3, we give the characterizations of graphs of spaghetti treewidth at most two. Section 4 discusses graphs with strongly chordal treewidth at most two. In Section 5, we discuss graphs of special treewidth at most two, and obtain similar results for graphs of directed spaghetti treewidth at most two. Section 6 considers classes with special treewidth, strongly chordal treewidth, spaghetti treewidth, or directed spaghetti treewidth, respectively, at most k, for $k \geq 3$. We show that none of these classes is closed under taking minors. Some final remarks are made in Section 7.

2 Preliminaries

Unless stated otherwise, graphs are considered to be undirected and simple. Let $G = (V, E)$ be a graph. For a vertex set $S \subseteq V$, we denote $G[S]$ as the subgraph of G induced on S. For $v \in V$ and $e \in E$, we denote $G - v$, $G - e$, G/e as the graphs obtained from G by removing v, removing e, and contracting e, respectively. For a pair of vertices $u, v \in V$ which are not adjacent in G, we denote $G + uv$ as the graph obtained from G by adding an edge uv. A subset S of V is a *clique* of G if all vertices in S are pairwise adjacent in G. The *clique number* of a graph, denoted by $\omega(G)$, is defined as the size of a maximum clique in the graph. A vertex v in a graph G is a *simplicial vertex* if the neighborhood of v forms a clique. The *length* of a path is the number of edges in the path.

A graph G is *connected* if for each pair of vertices $v, w \in V$, there exists a path from v to w in G. A graph G is 2-*connected*, if $|V| \geq 3$ and $G[V - X]$ is connected for every vertex set $X \subseteq V$ with $|X| \leq 1$. A vertex v of a connected graph G is a *cut vertex* if $G - v$ is not connected. A *block* of a graph G is a maximal connected subgraph of G without a cut vertex.

A graph H is a *minor* of a graph G, if H can be obtained from G by a series of deletion of a vertex, deletion of an edge, and contraction of an edge. A *subdivision* of a graph G is a graph obtained from G by replacing some edges of G with independent paths between their end vertices.

2.1 Graph Classes

Several of the notions we look at can be defined in terms of intersection graphs. Let $\mathcal{F}$ be a finite family of graphs. The *intersection graph* of $\mathcal{F}$ is the graph $G_{\mathcal{F}}$ whose vertices are the members of the family such that two distinct vertices f, f' of $G_{\mathcal{F}}$ are adjacent, if and only if the corresponding graphs have a common vertex.

A *chord* in a cycle of a graph, is a pair of adjacent vertices on the cycle that are not consecutive on the cycle. A graph is *chordal*, if each cycle with length at least four has a chord. Alternatively, a graph is a chordal graph, if and only if it is the intersection graph of subtrees of a tree [24]. (See also [14, 26].) A graph is an *interval* graph if it is the intersection graph of subpaths of a path.

For three variants of intersection graphs of paths on a tree, we follow the terms in the paper by Monma and Wei [31]. A graph is an *undirected vertex path graph (shortly, an UV graph)* if it is the intersection graph of a set of paths in a tree. UV graphs are also known as path graphs [25] or VPT graphs [27]. A *directed tree* is a directed graph whose underlying graph is a tree, and it is called a *rooted tree* if it has exactly one specified vertex called the root and every arc of it is directed to the root. A graph is a *directed vertex path graph (shortly, a DV graph)* if it is the intersection graph of a set of directed paths in a directed tree. A graph is a *rooted directed vertex path graph (shortly, an RDV graph)* if it is the intersection graph of directed paths in a rooted tree.

A graph G is *strongly chordal* if G is chordal and every even cycle of length at least six in G has a chord, called an *odd chord*, dividing the cycle into two odd paths of length at least three.

The following relations are well known [14].

$$\begin{aligned}&(\text{interval}) \subsetneq (\text{RDV}) \subsetneq (\text{strongly chordal}) \subsetneq (\text{chordal}),\\ &(\text{RDV}) \subsetneq (\text{DV}) \subsetneq (\text{UV}) \subsetneq (\text{chordal}).\end{aligned}$$

2.2 Tree Decompositions

The notions of pathwidth and treewidth were first introduced by Robertson and Seymour [34, 35].

Definition 2.1. A *tree decomposition* of a graph $G = (V, E)$ is a pair $(T, \mathcal{B} = \{B_x\}_{x \in V(T)})$ where T is a tree and for all $x \in V(T)$, $B_x \subseteq V$ which are called *bags*, satisfying the following three conditions:

(T1) $V = \bigcup_{x \in V(T)} B_x$.

(T2) For every edge uv of G, there exists a vertex x of T such that $u, v \in B_x$.

(T3) For every vertex v in G, the bags containing v induce a subtree in T.

The *width* of a tree decomposition $(T, \mathcal{B})$ is $\max\{|B_x| - 1 : x \in V(T)\}$. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum width of all tree decompositions of G. A *path decomposition* of a graph G is a tree decomposition $(T, \mathcal{B})$ where T is a path. The *pathwidth* of G, denoted by $\mathrm{pw}(G)$, is the minimum width of all path decompositions of G.

We observe the following relations.

Theorem 2.1 (folklore; see Bodlaender [7]). *Let k be a positive integer.*

1. *A graph has treewidth at most k if and only if it is a subgraph of a chordal graph with clique number at most $k + 1$.*

2. *A graph has pathwidth at most k if and only if it is a subgraph of an interval graph with clique number at most $k + 1$.*

For a tree decomposition $\mathcal{I} = (T, \{B_x\}_{x \in V(T)})$ of a graph $G = (V, E)$ and $S \subseteq V$, we denote $P(\mathcal{I}, S)$ as the set of the vertices x in T such that B_x contains a vertex in S. For $x \in V$, $P(\mathcal{I}, \{x\})$ is denoted shortly as $P(\mathcal{I}, x)$.

2.3 Special Treewidth, Directed Spaghetti Treewidth, Spaghetti Treewidth and Strongly Chordal Treewidth

Courcelle [16] introduced the notion of special treewidth. Directed spaghetti treewidth and spaghetti treewidth are natural variants of this notion.

A *special tree decomposition* of a graph G is a tree decomposition $(T, \mathcal{B})$ where T is a rooted tree and for every vertex v in G, the bags containing v induce a directed path in T. The *special treewidth* of G, denoted by $\mathrm{spctw}(G)$, is the minimum width of all special tree decompositions of G.

A *directed spaghetti tree decomposition* of a graph G is a tree decomposition $(T, \mathcal{B})$ where T is a directed tree (not necessarily rooted) and for every vertex v in G, the bags containing v induce a directed path in T. The *directed spaghetti treewidth* of G, denoted by $\mathrm{dspghtw}(G)$, is the minimum width of all directed spaghetti tree decompositions of G.

A *spaghetti tree decomposition* of a graph G is a tree decomposition $(T, \mathcal{B})$ where for every vertex v in G, the bags containing v induce a path in T. The *spaghetti treewidth* of G, denoted by $\mathrm{spghtw}(G)$, is the minimum width of all spaghetti tree decompositions of G.

From the definitions, we can easily deduce that

$$\mathrm{tw}(G) \leq \mathrm{spghtw}(G) \leq \mathrm{dspghtw}(G) \leq \mathrm{spctw}(G) \leq \mathrm{pw}(G).$$

For a positive integer k, we can observe the following from the definitions.

- A graph has special treewidth at most k if and only if it is a subgraph of an RDV graph with clique number at most $k + 1$ [16].
- A graph has spaghetti treewidth at most k if and only if it is a subgraph of an UV graph with clique number at most $k + 1$.
- A graph has directed spaghetti treewidth at most k if and only if it is a subgraph of a DV graph with clique number at most $k + 1$.

The *strongly chordal treewidth* of a graph G, denoted by $\mathrm{sctw}(G)$, is the minimum k such that G is a subgraph of a strongly chordal graph with clique number $k+1$. Farber [21] showed that every RDV graph is strongly chordal. (See also [14].) Since a strongly chordal graph is chordal, we have that

$$\mathrm{tw}(G) \leq \mathrm{sctw}(G) \leq \mathrm{spctw}(G).$$

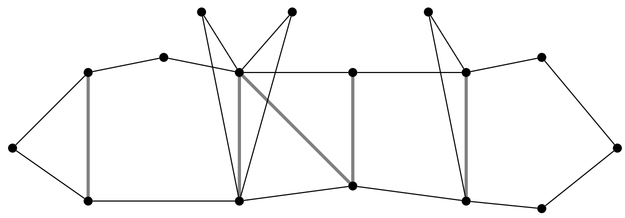

Figure 3: A path of cycles with four simplicial triangles and five edge separators.

2.4 Models for Treewidth Two and Pathwidth Two

2-connected graphs of treewidth two and of pathwidth two have characterizations in terms of *trees of cycles* [11] and *paths of cycles* [8], respectively. The *cell completion* $\widetilde{G}$ of a 2-connected graph $G = (V, E)$ is the graph, obtained from G by adding an edge vw for all pairs of nonadjacent vertices v, $w \in V$ such that $G[V - \{v, w\}]$ has at least three connected components.

Definition 2.2 (Bodlaender and Kloks [11])**.** The class of *trees of cycles* is the class of graphs recursively defined as follows.

- Each cycle is a tree of cycles.
- For each tree of cycles G and each cycle C, the graph obtained from G and C by taking the disjoint union and identifying an edge and its end vertices in G with an edge and its end vertices in C, is a tree of cycles.

Theorem 2.2 (Bodlaender and Kloks [11])**.** *Let G be a 2-connected graph. The graph G has treewidth two if and only if the cell completion $\widetilde{G}$ of G is a tree of cycles.*

An edge in a tree of cycles G is called an *edge separator* if it is contained in at least two distinct chordless cycles of G. We distinguish two different types of chordless cycles on a tree of cycles. A triangle of a tree of cycles G is called a *simplicial triangle* if it contains a simplicial vertex; all other chordless cycles are called *body cycles.* Every simplicial triangle of a tree of cycles contains at most one edge separators.

Definition 2.3 (Bodlaender and de Fluiter [8])**.** A *path of cycles* is a tree of cycles G for which the following holds.

1. Each chordless cycle of G has at most two edge separators.

2. If an edge $e \in E$ is contained in $m \geq 3$ chordless cycles of G, then at least $m-2$ of these cycles are simplicial triangles.

See Figure 3 for an example of a path of cycles. A triangulated path of cycles has been called 2-caterpillar [33]. Every path of cycles can be represented by a sequence of chordless cycles. This structure will be used to characterize special treewidth two in Section 5.

Definition 2.4 (Bodlaender and de Fluiter [8])**.** Let G be a path of cycles. Let $C = (C_1, \ldots, C_p)$ be a sequence of chordless cycles such that each chordless cycle in G appears exactly once in the sequence of cycles, and for $1 \leq i \leq p-1$, C_i shares exactly one edge e_i with C_{i+1}. Let $E = (e_1, \ldots, e_{p-1})$ be the corresponding set of common edges. The pair (C, E) is called a *cycle path model* for G.

Theorem 2.3 (Bodlaender and de Fluiter [8])**.** *Let G be a 2-connected graph. The graph G has pathwidth two if and only if $\widetilde{G}$ is a path of cycles.*

To obtain similar characterizations for spaghetti treewidth two and strongly chordal treewidth two, we will observe the structure of trees of cycles with exactly one of the conditions in Definition 2.3.

2.5 Simple Cases

The main body of our paper discusses the cases where the special treewidth, spaghetti treewidth, directed spaghetti treewidth or strongly chordal treewidth is at most two. We now briefly discuss the much simpler case when these parameters are at most one.

Proposition 2.4. *Let G be a graph. The following are equivalent.*

1. *G is a forest.*
2. *G has treewidth at most one.*
3. *G has spaghetti treewidth at most one.*
4. *G has strongly chordal treewidth at most one.*
5. *G has directed spaghetti treewidth at most one.*
6. *G has special treewidth at most one.*

Proof. Since the treewidth, spaghetti treewidth, special treewidth, directed spaghetti treewidth or strongly chordal treewidth of a graph equals the maximum of the parameter over the connected components of a graph, it is sufficient to show this proposition for connected graphs.

We assume that G is connected. It is well known that a connected graph has treewidth at most one, if and only if it is a tree. Courcelle [16] has shown that trees have special treewidth at most one. From the inequalities $\mathrm{tw}(G) \leq \mathrm{spghtw}(G) \leq \mathrm{dspghtw}(G) \leq \mathrm{spctw}(G)$ and $\mathrm{tw}(G) \leq \mathrm{sctw}(G) \leq \mathrm{spctw}(G)$, we conclude that all of the statements are equivalent. $\square$

The following result was observed in the case of special treewidth by Courcelle [16]. The same proof can be used to obtain this result for other width measures, as shown below.

Lemma 2.5. *Let $G = (V, E)$ be a graph and let $v \in V$ such that v is adjacent to all vertices of $V - \{v\}$ in G. Then*

$$\mathrm{spctw}(G) = \mathrm{spghtw}(G) = \mathrm{dspghtw}(G) = \mathrm{pw}(G) = \mathrm{pw}(G - v) + 1.$$

Proof. If we have a special tree decomposition, spaghetti tree decomposition, directed spaghetti tree decomposition or path decomposition of G, we may assume that all bags contain v because v is adjacent to all vertices of $V - \{v\}$. All other bags can be deleted. Such a special tree decomposition, spaghetti tree decomposition, or directed spaghetti tree decomposition is also a path decomposition, as the bags containing v form a path. From this observation, it follows that the first four terms are equal.

If we take a path decomposition of G with v belonging to each bag of width k, we obtain a path decomposition of $G - v$ of width $k - 1$ by removing v from all bags. If we have a path decomposition of $G - v$, we can obtain one of G by adding v to each bag. This shows that the last two terms are equal. $\square$

3 Characterizations of Spaghetti Treewidth Two

In this section, we characterize the class of graphs of spaghetti treewidth at most two. We first define a variant of the trees of cycles to characterize 2-connected graphs of spaghetti treewidth two.

Definition 3.1. A *chain tree of cycles* is a tree of cycles $G = (V, E)$ for which the following holds.

- If an edge $e \in E$ is contained in $m \geq 3$ chordless cycles of G, then at least $m-2$ of these cycles are simplicial triangles.

We show the following theorem. Let D_3 be the graph having two specified vertices and three internally vertex-disjoint paths of length three between those vertices; see Figure 1.

Theorem 3.1. *Let $G = (V, E)$ be a graph. The following are equivalent.*

1. *G has spaghetti treewidth at most two.*
2. *Each block of G is either a 2-connected subgraph whose cell completion is a chain tree of cycles, or a single edge, or an isolated vertex.*
3. *G has no minor isomorphic to K_4 or D_3.*

We need the following lemma.

Lemma 3.2. *The spaghetti treewidth of a subdivision of D_3 is three.*

Proof. Let H be a subdivision of D_3 consisting of two vertices, say a and b, of degree three and three independent paths $au_1u_2\ldots u_lb$, $av_1v_2\ldots v_mb$ and $aw_1w_2\ldots w_nb$, where $l, m, n \geq 2$. Let $U = \{u_i : 1 \leq i \leq l\}$, $V = \{v_i : 1 \leq i \leq m\}$, $W = \{w_i : 1 \leq i \leq n\}$. It is easy to see that $\mathrm{spghtw}(H) \leq 3$. We shall prove $\mathrm{spghtw}(H) \geq 3$.

Suppose that H has a spaghetti tree decomposition $\mathcal{I} = (T, \{B_x\}_{x \in V(T)})$ of width two. We may assume that none of B_x is a singleton.

If $P(\mathcal{I}, a) \cap P(\mathcal{I}, b) = \emptyset$, then there exists $y \in P(\mathcal{I}, a)$ which separates $P(\mathcal{I}, a) - \{y\}$ from $P(\mathcal{I}, b)$ in T. Since u_1 is contained in some bag of $P(\mathcal{I}, a)$ and u_l is contained in some bag of $P(\mathcal{I}, b)$, B_y contains some u_i. Similarly, B_y contains some v_j and w_k. As $a \in B_y$, the size of B_y is at least four. It contradicts to that the width of $\mathcal{I}$ is two.

Now we assume that $P(\mathcal{I}, a) \cap P(\mathcal{I}, b) \neq \emptyset$. Note that $P(\mathcal{I}, a) \cap P(\mathcal{I}, b)$ forms a path in T. By the same reason as above,

$$U' := (P(\mathcal{I}, a) \cap P(\mathcal{I}, b)) \cap P(\mathcal{I}, U) \neq \emptyset.$$

Similarly, we obtain

$$V' := (P(\mathcal{I}, a) \cap P(\mathcal{I}, b)) \cap P(\mathcal{I}, V) \neq \emptyset,$$

$$W' := (P(\mathcal{I}, a) \cap P(\mathcal{I}, b)) \cap P(\mathcal{I}, W) \neq \emptyset.$$

Since every bag of $P(\mathcal{I}, a) \cap P(\mathcal{I}, b)$ has two vertices a and b, no two of U', V', W' have a common vertex. We may assume that V' lies between U'

and W' in $P(\mathcal{I}, a) \cap P(\mathcal{I}, b)$. Since $H[V]$ has at least one edge, we must have $P(\mathcal{I}, V) - V' \neq \emptyset$. We choose $x \in V'$ and $y \in P(\mathcal{I}, V) - V'$ such that they are neighbors in T. Then $|B_x \cap B_y| \leq 1$. Since $|B_y| \geq 2$, it contradicts to the 2-connectedness of H. $\square$

The following lemma is a key lemma to obtain a subdivision of D_3 as a subgraph.

Lemma 3.3. *Let $k \geq 1$. Let $G = (V, E)$ be a 2-connected graph and let $u, v \in V$. If $\widetilde{G}[V - \{u, v\}]$ has k components of size at least two, then G has k internally vertex-disjoint paths of length at least three from u to v.*

Proof. We claim that if $\widetilde{G}[V - \{u, v\}]$ has a component H such that $|V(H)| \geq 2$, then $G[V(H)]$ has an edge. Suppose that $G[V(H)]$ has no edges. Since $|V(H)| \geq 2$, there are two vertices x and y of $G[V(H)]$ such that $xy \in E(\widetilde{G}) - E$. From the definition of a cell completion, $G[V - \{x, y\}]$ has at least three components. It leads a contradiction because every vertex in $V - \{x, y\}$ is connected to u and v in $G[V - \{x, y\}]$.

We assume that $\widetilde{G}[V - \{u, v\}]$ has k components $G_1, G_2, \ldots, G_k$ where each G_i has at least two vertices. By the claim, each $G[V(G_i)]$ has a component G'_i having at least one edge. Since G is 2-connected, G has k internally vertex-disjoint paths of length at least three from u to v along each G'_1, $G'_2, \ldots, G'_k$, as required. $\square$

3.1 Characterization with Cycle Model

We characterize 2-connected graphs of spaghetti treewidth two in terms of trees of cycles.

Theorem 3.4. *Let $G = (V, E)$ be a 2-connected graph. Then G has spaghetti treewidth two if and only if the cell completion $\widetilde{G}$ of G is a chain tree of cycles.*

We first show that if a 2-connected graph G has spaghetti treewidth two, then $\widetilde{G}$ is a chain tree of cycles.

Proposition 3.5. *Let $G = (V, E)$ be a 2-connected graph. If G has spaghetti treewidth two, then $\widetilde{G}$ is a chain tree of cycles.*

Proof. Suppose $\mathrm{spghtw}(G) = 2$. Since G is 2-connected and $\mathrm{tw}(G) \leq 2$, by Theorem 2.2, $\widetilde{G}$ is a tree of cycles. So, it is sufficient to check that every edge separator of $\widetilde{G}$ is contained in at most two body cycles of $\widetilde{G}$.

Suppose that an edge separator uv is contained in three body cycles in $\widetilde{G}$. So, $\widetilde{G}[V - \{u, v\}]$ has three components having at least two vertices.

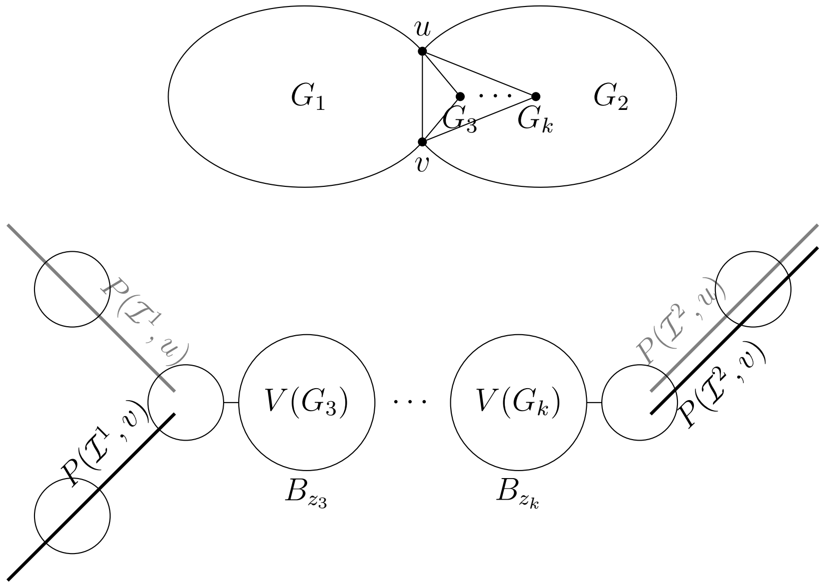

Figure 4: The induction step in Proposition 3.6.

By Lemma 3.3, G has three internally vertex-disjoint paths of length at least three from u to v, and therefore, G has a subgraph isomorphic to a subdivision of D_3. By Lemma 3.2, $\mathrm{spghtw}(G) \geq 3$, which contradicts to the assumption. □

We prove the other direction.

Proposition 3.6. *Every chain tree of cycles has spaghetti treewidth two.*

For a tree of cycles G, let $\mathcal{G}(G)$ be the set of body cycles of G and let $\mathcal{D}(G)$ be the set of simplicial triangles of G. A subset $\mathcal{P}$ of $\mathcal{D}(G)$ is called a *potential set* of G if each edge separator of G is contained in exactly two cycles of $\mathcal{G}(G) \cup \mathcal{P}$. Briefly, we will show that a chain tree of cycles G with a fixed potential set $\mathcal{P}$ admits a special type of a spaghetti tree decomposition. For a potential set $\mathcal{P}$ of G, let $F(G, \mathcal{P})$ be the set of all non-edge separator edges contained in cycles of $\mathcal{G}(G) \cup \mathcal{P}$.

Lemma 3.7. *Let $G = (V, E)$ be a chain tree of cycles with a potential set $\mathcal{P}$, and let uv be an edge separator of G. Let H be a component of $G[V - \{u, v\}]$ such that $H' := G[V(H) \cup \{u, v\}]$ and H' is not a chordless cycle. Let C' be the chordless cycle of H' containing the edge uv. Then*

1. *$\{C : C \in \mathcal{P}, V(C) \subseteq V(H')\} \cup \{C'\}$ is a potential set of H' if C' is a triangle having two edge separators in G, and*

2. $\{C : C \in \mathcal{P}, V(C) \subseteq V(H')\}$ *is a potential set of* H' *if otherwise.*

Proof. Since H' is not a chordless cycle, C' has at least two edge separators in G. If either C' is a cycle of length at least 4 or C' has three edge separators in G, then C' is still a body cycle of H', and there is nothing to prove. If C' is a triangle having only one edge separator f other than uv in G, then f is contained in exactly one cycle of $\{C : C \in \mathcal{P}, V(C) \subseteq V(H')\}$, and C' is a simplicial triangle of H'. Thus, $\{C : C \in \mathcal{P}, V(C) \subseteq V(H')\} \cup \{C'\}$ is a potential set of H'. □

Proof of Proposition 3.6. Let $G = (V, E)$ be a chain tree of cycles. Since G is 2-connected, it is sufficient to show that $\mathrm{spghtw}(G) \leq 2$. Let $\mathcal{P}$ be a potential set of G. We claim that G has a spaghetti tree decomposition $\mathcal{I} = (T, \{B_x\}_{x \in V(T)})$ of width two such that

(GC) there exists an injective function g from $F(G, \mathcal{P})$ to $V(T)$ where $P(\mathcal{I}, u)$ and $P(\mathcal{I}, v)$ have a common end vertex $g(uv)$ in T for $uv \in F(G, \mathcal{P})$.

We prove it by induction on the number of edge separators of G.

If G has no edge separators, then G is a chordless cycle. Let G be a chordless cycle $c_1c_2c_3 \cdots c_mc_1$ for some $m \geq 3$. We construct a tree decomposition $\mathcal{I} = (T, \{B_x\}_{x \in V(T)})$ of G and define a function g from E to $V(T)$ such that

- T is a path $p_0p_1p_2p_3 \cdots p_{m-1}$,
- $B_{p_0} = \{c_m, c_1\}$, $B_{p_{m-1}} = \{c_m, c_{m-1}\}$,
- for each $1 \leq i \leq m-2$, $B_{p_i} = \{c_m, c_i, c_{i+1}\}$, and
- g is the function from E to $V(T)$ such that $g(c_mc_1) = p_0$ and $g(c_ic_{i+1}) = p_i$ for all $1 \leq i \leq m-1$.

We can easily check that $\mathcal{I}$ is a spaghetti tree decomposition of G having width two and it satisfies the condition (GC).

Now suppose that G has an edge separator uv. Let $H_1, H_2, \ldots, H_k$ be the components of $G[V - \{u, v\}]$ where $k \geq 2$ and let $G_i = G[V(H_i) \cup \{u, v\}]$. Note that exactly two graphs of $\{G_i\}_{1 \leq i \leq k}$ have a cycle in $\mathcal{G}(G) \cup \mathcal{P}$. Without loss of generality, we assume that G_1 and G_2 have a cycle in $\mathcal{G}(G) \cup \mathcal{P}$.

We first check that for each $j \in \{1, 2\}$, G_j admits a spaghetti tree decomposition satisfying the condition (GC). We may assume that G_j is not a chordless cycle. For each $j \in \{1, 2\}$, let $\mathcal{P}_j = \{C : C \in \mathcal{P}, V(C) \subseteq V(G_j)\}$, and let C_j be the chordless cycle of G_j containing the edge uv. We define

- $\mathcal{P}'_j := \mathcal{P}_j \cup \{C_j\}$ if C_j is a triangle having two edge separators in G,
- $\mathcal{P}'_j := \mathcal{P}_j$ if otherwise.

In both cases, by Lemma 3.7, P'_j is a potential set of G_j and C_j is contained in $\mathcal{G}(G_j) \cup \mathcal{P}'_j$. By the induction hypothesis, G_j has a spaghetti tree decomposition $\mathcal{I}^j = (T^j, \{B^j_x\}_{x \in V(T^j)})$ of width two such that there is an injective function g_j from $F(G_j, \mathcal{P}'_j)$ to $V(T^j)$ which satisfies the condition (GC). Let $w_j = g_j(uv)$.

We construct a new tree decomposition $(T, \{B_x\}_{x \in V(T)})$ and define the function g from $F(G, \mathcal{P})$ to $V(T)$ such that

- T is obtained from the disjoint union of T^1, T^2 and the path $z_3 \cdots z_k$ by adding edges $z_3 w_1$, $z_k w_2$, and
- for all $1 \leq i \leq 2$ and $x \in V(T_i)$, $B_x = B^i_x$,
- for all $3 \leq i \leq k$, $B_{z_i} = V(G_i)$, and
- $g(e) = g_i(e)$ if $e \in F(G_i, \mathcal{P}'_i)$ for $i \in \{1, 2\}$.

This case is depicted in Figure 4. Clearly, $P(\mathcal{I}, u)$ and $P(\mathcal{I}, v)$ form paths in T. So, $(T, \{B_x\}_{x \in V(T)})$ is a spaghetti tree decomposition of G having width two. Because the only $P(\mathcal{I}^j, u)$ and $P(\mathcal{I}^j, v)$ are changed in each T^j and $uv \notin F(G, \mathcal{P})$, g is injective, as required. $\square$

Proof of Theorem 3.4. If $\operatorname{spghtw}(G) = 2$, then by Proposition 3.5, $\widetilde{G}$ is a chain tree of cycles. If $\widetilde{G}$ is a chain tree of cycles, then by Proposition 3.6, $\operatorname{spghtw}(\widetilde{G}) = 2$. Since G is 2-connected and spaghetti treewidth does not increase when taking a subgraph, $\operatorname{spghtw}(G) = 2$. $\square$

3.2 The Minor Obstruction Set for Spaghetti Treewidth Two

We provide the minor obstruction set for the class of 2-connected graphs of spaghetti treewidth two.

Theorem 3.8. *Let $G = (V, E)$ be a 2-connected graph. The graph G has spaghetti treewidth two if and only if it has no minor isomorphic to K_4 or D_3.*

Proof. Suppose G has spaghetti treewidth two. If G has a minor isomorphic to K_4, then $\operatorname{spghtw}(G) \geq \operatorname{tw}(G) \geq 3$. If G has a minor isomorphic to D_3, then G has a subgraph isomorphic to a subdivision of D_3. By Lemma 3.2, $\text{spghtw}(G) \geq 3$ and it contradicts to our assumption on G.

Now suppose that G has no minor isomorphic to K_4 and $\text{spghtw}(G) \geq 3$. Since G is 2-connected, by Theorem 2.2 and 3.4, $\widetilde{G}$ is a tree of cycles but not a chain tree of cycles. So, $\widetilde{G}$ has an edge separator uv such that uv is contained in three body cycles. Therefore, $\widetilde{G}[V - \{u, v\}]$ has three components having at least two vertices, and by Lemma 3.3, G has three internally vertex-disjoint paths of length at least three from u to v. Thus, G has a minor isomorphic to D_3. □

Using the following lemma, we have the results for general cases.

Lemma 3.9. *Let k be a positive integer. A graph has spaghetti treewidth at most k if and only if every block of it has spaghetti treewidth at most k.*

Proof. The forward direction is trivial. For the converse direction, suppose every block of a graph G has spaghetti treewidth at most k. We may assume that G is connected. We prove by induction on the number of cut vertices of G. We may assume that G has a cut vertex v. Let $H_1, H_2, \ldots, H_k$ be the components of $G - v$ and let $G_i = G[V(H_i) \cup \{v\}]$. By the induction hypothesis, there exists a spaghetti tree decomposition $\mathcal{I}_i$ of H_i having width at most k. We can obtain a new tree decomposition $\mathcal{I}$ from the disjoint union of the decompositions $\mathcal{I}_i$ by just connecting bags among the bags in $\bigcup_{1 \leq i \leq k} P(\mathcal{I}_i, v)$ so that $P(\mathcal{I}, v)$ forms a path. It shows that G has spaghetti treewidth at most k. □

Proof of Theorem 3.1. By Theorem 3.4, (1) implies (2), and with Lemma 3.9, (2) also implies (1). By Theorem 3.8 and Lemma 3.9, (1) and (3) are equivalent. □

4 Characterizations of Strongly Chordal Treewidth Two

In this section, we characterize the class of graphs of strongly chordal treewidth at most two with cycle model and we provide the minor obstruction set for the class. We introduce another variant of a tree of cycles, called a *tree of two-boundaried cycles*. The name 'two-boundaried' comes from the property that every chordless cycle of it may attach with other chordless cycles on at most two edges.

Definition 4.1. A *tree of two-boundaried cycles* is a tree of cycles G for which the following holds.

- Each chordless cycle of G has at most two edge separators.

We mainly show the following. The graph S_3 is depicted in Figure 1.

Theorem 4.1. *Let $G = (V, E)$ be a graph. The following are equivalent.*

1. *G has strongly chordal treewidth at most two.*
2. *Each block of G is either a 2-connected subgraph whose cell completion is a tree of two-boundaried cycles, or a single edge, or an isolated vertex.*
3. *G has no minor isomorphic to K_4 or S_3.*

Unlike D_3, it seems to be tedious to characterize the subgraph minimal graphs containing S_3 as a minor, because S_3 has maximum degree four. So, we first show that the class of graphs of strongly chordal treewidth at most two is closed under taking minors.

We use the following fact that S_3 has strongly chordal treewidth three.

Lemma 4.2. *The strongly chordal treewidth of S_3 is three.*

Proof. If we add one odd chord in the cycle of length six in S_3, then the resulting graph is a strongly chordal graph with clique number four. Therefore, $\mathrm{sctw}(S_3) \leq 3$. If there exists a strongly chordal graph H having S_3 as a subgraph, then the cycle of length six in $H[V(S_3)]$ must have an odd chord. Thus, $\omega(H) \geq 4$ and it implies that $\mathrm{sctw}(S_3) \geq 3$. $\square$

The following lemma will be used to find S_3 as a minor.

Lemma 4.3. *Let $G = (V, E)$ be a 2-connected graph having treewidth two and let uv be an edge separator of $\widetilde{G}$. If C is a chordless cycle of $\widetilde{G}$ containing uv, then G has two internally vertex-disjoint paths from u to v such that they have no vertices of C except u and v.*

Proof. We have two cases.
Case 1. $uv \in E$. The edge uv is one of the required paths. Since uv is an edge separator of $\widetilde{G}$, $\widetilde{G}[V - \{u, v\}]$ has at least one component having no vertices of C. Thus, $G[V - \{u, v\}]$ has a component H having no vertices of C, and G has a path from u to v in G along H.
Case 2. $uv \in E(\widetilde{G}) - E$. By the definition of a cell completion, $G[V - \{u, v\}]$ has at least three components. Therefore, $G[V - \{u, v\}]$ has two components H_1 and H_2 which contain no vertices in C. Clearly, there are two internally vertex-disjoint paths from u to v in G along H_1 and H_2. $\square$

4.1 Contractions on Graphs of Strongly Chordal Treewidth Two

We show that the class of graphs of strongly chordal treewidth at most two is closed under taking minors.

Proposition 4.4. *The class of graphs of strongly chordal treewidth at most two is closed under taking minors.*

We will use a known characterization of strongly chordal graphs. For an integer $n \geq 3$, a graph G is called an *n-sun* if G is a graph with $2n$ vertices which are partitioned into two parts $U = \{u_1, u_2, \ldots, u_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ such that U induces a clique, W induces an independent set and each vertex w_i in W is adjacent to u_j if and only if $i \equiv j$ or $i \equiv j + 1 \pmod{n}$. Note that S_3 is the 3-sun.

Theorem 4.5 (Farber [22])**.** *A graph G is strongly chordal if and only if G is chordal and it has no induced subgraph isomorphic to a sun.*

Since taking subgraphs does not increase strongly chordal treewidth, it is enough to show the following.

Proposition 4.6. *Let $G = (V, E)$ be a strongly chordal graph of $\omega(G) \leq 3$ and let $e \in E$. Then G/e is strongly chordal.*

We will prove, by induction on the size of odd cycles C in G which contain the contracted edge e, that C/e has an odd chord. For the base case, we need a lemma.

Lemma 4.7. *Let $G = (V, E)$ be a chordal graph and let $e \in E$. If $G/e = S_3$, then either G is not strongly chordal or $\omega(G) = 4$.*

Proof. Let $V = \{w, v_1, v_2, \ldots, v_6\}$ and let us assume that for some $i \in \{1, 2, \ldots, 6\}$, $e = wv_i$, and after contracting wv_i in G, the contracted vertex is again labeled by v_i. Suppose $G/e = S_3$ where $v_1v_2 \cdots v_6v_1$ is the cycle of length six and $v_2v_4v_6$ is the triangle in the middle. By symmetry, we may assume $i = 1$ or 2. We may also assume that both w and v_i have degree at least two in G, otherwise one of $G - w$ and $G - v_i$ is isomorphic to S_3.

If $i = 1$, then the number of edges between $\{w, v_1\}$ and $\{v_2, v_6\}$ is at least three because G is chordal. So, one of $G - w$ and $G - v_1$ must be isomorphic to S_3. Therefore, G is not strongly chordal.

Now we assume that $i = 2$. We first claim that each of w and v_2 is adjacent to exactly one of v_1 and v_3, and the neighbors of w and v_2 are distinct. If v_2 is adjacent to both v_1 and v_3, then $v_1v_2v_3v_4v_5v_6v_1$ is an even

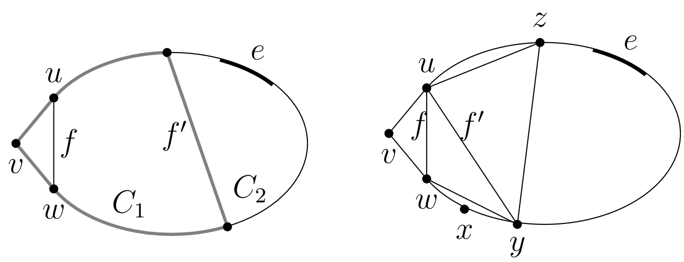

Figure 5: The cycle C of length at least nine in Proposition 4.6 and two even cycles C_1 and C_2. The second picture depicts the last case that C_1 has four edges and f and f' meet at u.

cycle of length six in G without chords v_1v_4, v_2v_5, v_3v_6. So, G is not strongly chordal. By the same reason, w cannot be adjacent to both v_1 and v_3. If v_2 is adjacent to neither v_1 nor v_3, then w must be adjacent to both v_1 and v_3, and we already see that it is impossible. Thus, each of w and v_2 is adjacent to exactly one of v_1 and v_3 and the neighbors are distinct. By symmetry, we may assume that w and v_2 are adjacent to v_1 and v_3, respectively.

Since G is chordal and v_1 is only adjacent to w and v_6 which form a cycle with other vertices in G, wv_6 must be an edge of G. By the same reason, $v_2v_4 \in E$. Since $wv_2v_4v_6$ is a cycle of length 4, we have either $v_2v_6 \in E$ or $wv_4 \in E$. If $v_2v_6 \notin E$ (or $wv_4 \notin E$), then $G - v_3$ (or $G - v_1$) is isomorphic to S_3, and therefore G is not strongly chordal. If $v_2v_6, wv_4 \in E$, then $w(G) = 4$, as required. □

Proof of Proposition 4.6. Note that the cycles affected by the contraction of e are the cycles containing e. As G/e is again chordal, we shall only consider the odd cycles in G of length at least seven which contain e. Let C be one of such cycles. We shall show, by induction on the length of C, that

– for every edge $e \in E$, C/e has an odd chord in G/e.

Suppose the length of C is seven. Since G/e is chordal, if C/e has no odd chord, then $V(C/e)$ induces a graph isomorphic to S_3 in G/e. By Lemma 4.7, either $G[V(C)]$ is not strongly chordal or $\omega(G[V(C)]) = 4$, contradicting to our assumption on G. Thus, C/e must have an odd chord.

Now suppose that C has length at least nine and the assertion holds for all odd cycles shorter than C. Since G is chordal, C has a chord, say $f = uw$, connecting two vertices at distance two on C. See Figure 5. Let v be the common neighbor of u and w in C, and let C' be the cycle $(C - v) + f$. Since C' is an even cycle of length at least eight in G, it has an odd chord of C', say f'. Let C_1, C_2 be the two distinct cycles in $C' + f'$ containing f' such that C_1 contains the edge f. If $e \notin E(C_2)$ then f' is an odd chord of C/e. Thus we may assume that $e \in E(C_2)$. We consider two cases.

Case 1. C_1 has length at least six. Here, the cycle $(C_1 - f) + uv + vw$ has length at least seven in G. So by the induction hypothesis, $((C_1 - f) + uv + vw)/f'$ has an odd chord h in G/f'. This chord was a chord in $(C_1 - f) + uv + vw$ where the part avoiding f' has odd edges with at least three edges. Therefore, h is an odd chord of C/e.

Case 2. C_1 has length four. See the second picture in Figure 5. If f does not meet f', then any chord of the cycle C_1 is an odd chord in C/e. So, we may assume that C contains the path $u - v - w - x - y$ and $f' = uy \in E$. If $ux \in E$, then it is an odd chord for C/e. We may assume that $ux \notin E$ and $wy \in E$.

Since C_2 is a cycle of G, there exists a vertex z in C_2 other than u and y such that uyz is a triangle in G. We claim that $G[\{u, v, w, x, y, z\}]$ is isomorphic to S_3. Since $\omega(G) \leq 3$, G has no edges ux, yv, wz. If G has one of edges xv, vz, zx, then one of the sets $\{x, v, u, y\}$, $\{v, z, w, y\}$, $\{z, x, u, w\}$ form a cycle of length four in G, and this cycle forces one of the edges ux, yv, wz. Therefore, G also has no edges xv, vz, zx. So, $G[\{u, v, w, x, y, z\}]$ is isomorphic to S_3, contradicting to the assumption that G is strongly chordal.

We conclude that for every even cycle of length at least six in G/e has an odd chord. Therefore, G/e is strongly chordal. □

Proof of Proposition 4.4. Let G be a graph and suppose that there exists a strongly chordal graph H of $\omega(H) \leq 3$ such that G is a subgraph of H. Clearly, taking a subgraph does not increase strongly chordal treewidth. Also, for $e \in E$, G/e is a subgraph of H/e and by Proposition 4.6, H/e is also strongly chordal. Therefore, $\mathrm{sctw}(G/e) \leq 2$. □

4.2 Characterization with Cycle Model

We characterize the class of strongly chordal treewidth two in terms of a cycle model.

Theorem 4.8. *Let $G = (V, E)$ be a 2-connected graph. Then G has strongly chordal treewidth two if and only if the cell completion $\widetilde{G}$ of G is a tree of two-boundaried cycles.*

Proposition 4.9. *Let $G = (V, E)$ be a 2-connected graph. If G has strongly chordal treewidth two, then $\widetilde{G}$ is a tree of two-boundaried cycles.*

Proof. Suppose G has strongly chordal treewidth two. Since G is 2-connected and has treewidth two, by Theorem 2.2, $\widetilde{G}$ is a tree of cycles. Suppose that

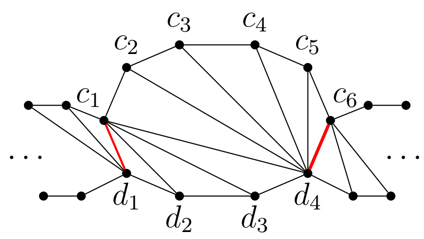

Figure 6: Triangulating each chordless cycle in Proposition 4.10.

a chordless cycle C of $\widetilde{G}$ has edge separators $\{v_i w_i\}_{1\le i\le k}$ where $k \ge 3$. By Lemma 4.3, for each $1 \le i \le k$, G has two internally vertex-disjoint paths P_1^i, P_2^i from v_i to w_i in G such that they have no vertices of C except v_i and w_i. So, $C - v_1w_1 - v_2w_2 \cdots - v_kw_k$ with the paths $\bigcup_{1\le i\le k}\{P_1^i, P_2^i\}$ in G has a minor isomorphic to S_3. By Lemma 4.2 and Proposition 4.4, $\mathrm{sctw}(G) \ge 3$ and it contradicts to the assumption on G. $\square$

For the opposite direction, we prove the following.

Proposition 4.10. *A tree of two-boundaried cycles has strongly chordal tree-width two.*

Proof. Let $G = (V, E)$ be a tree of two-boundaried cycles. We will construct a graph G' from G by triangulating each chordless cycle such that G' is a strongly chordal graph with $\omega(G') = 3$.

By the definition of a tree of two-boundaried cycles, each chordless cycle of G has at most two edge separators. For convenience, we choose up to two non-edge separator edges in each chordless cycle and call them also edge separators so that each chordless cycle has exactly two edge separators. Note that for any chordless cycle C of G and edges u_1v_1 and u_2v_2 in C, we can triangulate C into C' with maximum clique size at most three so that every triangle in C' has an edge other than u_1v_1, u_2v_2, which is not contained in any other triangles of C'. Let G' be the graph obtained from G by triangulating each chordless cycle as described. See Figure 6.

Since $\omega(G') = 3$, it remains to prove that G' is strongly chordal. Suppose that G' is not strongly chordal. Since G' is a chordal graph and $\omega(G') = 3$, by Theorem 4.5, G' has an induced subgraph isomorphic to S_3. However, since every triangle of G' has an edge which is not contained in other triangles of G', it leads a contradiction. $\square$

4.3 The Minor Obstruction Set for Strongly Chordal Treewidth Two

We provide the minor obstruction set for the class of graphs of strongly chordal treewidth at most two.

Theorem 4.11. *Let G be a 2-connected graph. The graph G has strongly chordal treewidth two if and only if it has no minor isomorphic to K_4 or S_3.*

Proof. If G has a minor isomorphic to K_4 or S_3, then by Lemma 4.2 and Proposition 4.4, $\mathrm{sctw}(G) \geq 3$. Suppose G has no minor isomorphic to K_4 and $\mathrm{sctw}(G) \geq 3$. Then, by Theorem 4.8, $\widetilde{G}$ is a tree of cycles but not a tree of two-boundaried cycles. Therefore, $\widetilde{G}$ has a chordless cycle having at least three edge separators. As we already observed in Proposition 4.9, by Lemma 4.3, we can easily show that G has a minor isomorphic to S_3. □

Proof of Theorem 4.1. From the definition of strongly chordal treewidth, the strongly chordal treewidth of a graph is the maximum of this parameter over all blocks of it. With this observation, by Theorem 4.8, the statements (1) and (2) are equivalent. Also, Theorem 4.11 implies that (1) and (3) are equivalent. □

5 Characterizations of Directed Spaghetti Treewidth Two and Special Treewidth Two

In this section, we mainly characterize the class of graphs having special treewidth at most two, and the class of graphs having directed spaghetti treewidth.

A graph is called a *mamba*[2] if it is either a 2-connected graph of pathwidth two, or a single edge, or an isolated vertex. The notion of mambas reflects the linear structure of them, and we will define head vertices of mambas which have a key role in our characterization.

We first show that every block of a graph of directed spaghetti treewidth at most two, or special treewidth at most two is a mamba. For directed spaghetti treewidth, we directly obtain a characterization of width at most two for general cases. For special treewidth, we will see how the different mambas are glued to make graphs of special treewidth at most two.

[2]Mambas are a type of snakes.

5.1 Mambas

We obtain the following characterization of 2-connected mambas as a corollary of the results in Section 3 and Section 4.

Corollary 5.1. *Let G be a 2-connected graph. The following are equivalent.*

1. *G has pathwidth two (equivalently, G is a mamba).*
2. *G has special treewidth two.*
3. *G has directed spaghetti treewidth two.*
4. *G has no minor isomorphic to K_4, D_3 or S_3.*
5. *The cell completion $\widetilde{G}$ of G is a path of cycles.*

For the direction (3) $\Rightarrow$ (4), we prove that every DV graph with clique number at most three is strongly chordal. This gives a relation between directed spaghetti treewidth and strongly chordal treewidth when the parameter is at most two. A sun is called *even* (or *odd*) if the size of the central clique is even (or odd).

Theorem 5.2 (Panda [32])**.** *Every DV graph has no induced subgraph isomorphic to an odd sun.*

Lemma 5.3. *Every DV graph with clique number at most three is strongly chordal.*

Proof. Let G be a DV graph with clique number three. Since G is a DV graph, by Theorem 5.2, G has no induced subgraph isomorphic to an odd sun. Since $\omega(G) \leq 3$, G has no induced subgraph isomorphic to an even sun. Therefore, by Theorem 4.5, G is strongly chordal. □

Proof of Corollary 5.1. By Theorem 2.3, (5) implies (1), and from the inequalities between the parameters, (1) implies (2) and (2) implies (3).

(3) $\Rightarrow$ (4) : If G has directed spaghetti treewidth two, then G has spaghetti treewidth at most two. Also, by Lemma 5.3, if G has directed spaghetti treewidth two, then G has strongly chordal treewidth at most two. Since G is 2-connected, by Theorem 3.8 and 4.11, G has no minor isomorphic to K_4, D_3 or S_3.

(4) $\Rightarrow$ (5) : Suppose G has no minor isomorphic to K_4, D_3 or S_3. Since G is 2-connected, by Proposition 3.5 and 4.9, $\widetilde{G}$ is both a chain tree of cycles and a tree of two-boundaried cycles. By the definition of a path of cycles, $\widetilde{G}$ is a path of cycles. □

The graphs of directed spaghetti treewidth at most two are exactly the graphs whose block is a mamba.

Theorem 5.4. *Let $G = (V, E)$ be a graph. The following are equivalent.*

1. *G has directed spaghetti treewidth at most two.*

2. *Each block of G is a mamba.*

3. *G has no minor isomorphic to K_4, D_3 or S_3.*

Proof. Similarly in the proof of Lemma 3.9, we can easily verify that G has directed spaghetti treewidth at most two if and only if every block of it has directed spaghetti treewidth at most two. So, all directions are easily obtained from Corollary 5.1. □

5.2 Characterizing Graphs of Special Treewidth Two

A result like Theorem 5.4 does not hold for special treewidth two: we can have a graph with special treewidth at least three, where each block has special treewidth at most two. For instance, the graphs G_1, G_2, and G_3 in Figure 2 have special treewidth three but one can easily observe that each block is a mamba. Thus, for a graph to have special treewidth at most two, it is necessary but not sufficient that each block is a mamba. An additional condition, expressing how the different mambas are attached to each other, is given below; adding this condition gives a full characterization.

Head vertices of mambas play a central role in the characterizations.

Definition 5.1. Let $G = (V, E)$ be a mamba. A vertex $v \in V$ is a *head vertex* of G, if there is a path decomposition $(T, \{B_x\}_{x \in V(T)})$ of G having width at most two such that $T = p_1 p_2 \cdots p_r$ and $v \in B_{p_1}$.

We define *mamba trees*, which are recursively constructed by attaching mambas at head vertices. We will show that mamba trees precisely characterize the connected graphs of special treewidth at most two. In the next section, we characterize this class using forbidden minors.

Definition 5.2. The class of *mamba trees* is the class of graphs recursively defined as follows.

- Each mamba is a mamba tree.

- For each mamba tree G and each mamba M, the graph obtained from a disjoint union of G and M by identifying a vertex of G with a head vertex of M is a mamba tree.

Theorem 5.5. *A graph has special treewidth at most two, if and only if it is a disjoint union of mamba trees.*

Let $\mathcal{I}$ be a special tree decomposition with a root bag R. A bag B_y is a *descendant* of B_x in $\mathcal{I}$, if B_x belongs to the path of $\mathcal{I}$ linking B_y to the root R. If B_y is a descendant of B_x and they are neighbor bags, then B_x is called a *parent* of B_y, and B_y is a *child* of B_x.

Proposition 5.6. *The special treewidth of a mamba tree is at most two.*

Proof. Let $G = (V, E)$ be a mamba tree. If G is a mamba, then by Corollary 5.1, G has special treewidth at most two.

Otherwise, we know from Definition 5.2 that G is constructed by taking a mamba tree G' and a mamba M, and identifying a vertex in G' with a head vertex in M. Let v be this vertex. From Definition 5.1 we know that since v is a head vertex, there is a path decomposition $\mathcal{I}^v = (T^v, \{B^v_x\}_{x\in V(T^v)})$ of M of width at most two such that v is in the first bag B_1. Now iteratively we get a special tree decomposition $\mathcal{I}$ of G of width at most two from a special tree decomposition $\mathcal{I}' = (T', \{B'_x\}_{x\in V(T')})$ of G' by attaching T^v to T' and making B_1 the child of the lowest bag of $P(\mathcal{I}', v)$ in $\mathcal{I}'$. So, we conclude that every mamba tree has special treewidth at most two. □

Proposition 5.7. *Every connected graph having special treewidth at most two is a mamba tree.*

Proof. Let $G = (V, E)$ be a connected graph of special treewidth at most two. We prove by induction on the number of vertices in G. If $|V| \leq 3$, then this is always true. We may assume that $|V| \geq 4$.

We choose a special tree decomposition $\mathcal{I} = (T, \{B_x\}_{x\in V(T)})$ of G having width at most two such that $\sum_{x\in V(T)}|B_x|$ is minimum. Note that T is a rooted tree and if B is a bag of $\mathcal{I}$, then B has at most one child B' such that $|B \cap B'| \geq 2$, otherwise, there must be a vertex of G where the bags containing it do not form a rooted path.

We choose a maximal rooted path $P = B_1 - B_2 - \cdots - B_n$ in T such that for all $1 \leq i \leq n-1$, $|B_i \cap B_{i+1}| = 2$ and B_i is a child of B_{i+1} in T. We show that $G[\bigcup_{1\leq i\leq n} B_i]$ is 2-connected, and so, it is a mamba. To show this, we analyze some cases forcing edges in the graph.

Claim 5.7.1. Let $t \in V(T)$ and let B' be the parent of B_t such that $B_t \cap B' = \{v_1, v_2\}$ and $B_t - B' = \{w\}$. If there is no child B'' of B_t such that $|B_t \cap B''| = 2$, then w is adjacent to both v_1 and v_2.

Proof. Suppose v_i is not adjacent to w in G for some $i \in \{1, 2\}$ and let v_j be the vertex in $B_t \cap B'$ other than v_i. If B_t has no child containing v_i, we can simply remove v_i from B_t. In the below of B_t in T, if there exists a component T' of $T - t$ containing a bag with v_i, then we cut the branch T' from T, and attach this on B', and remove v_i from B_t. Since T' has no bag containing w or v_j, the modified decomposition is a special tree decomposition and $\sum_{x \in V(T)} |B_x|$ is decreased by one, contradiction. □

Claim 5.7.2. Let B_t be a bag of $\mathcal{I}$ and let B' be a child of B_t such that $B_t \cap B' = \{v_1, v_2\}$ and $B_t - B' = \{w\}$. If B_t is a non-root bag with the parent B'' such that $|B_t \cap B''| = 1$, then w is adjacent to both v_1 and v_2.

Proof. Suppose v_i is not adjacent to w in G for some $i \in \{1, 2\}$ and let v_j be the vertex in $B_t \cap B'$ other than v_i. If $B_t \cap B'' \neq \{v_i\}$, then we can remove v_i from the bag B_t. Thus, we may assume that $B_t \cap B'' = \{v_i\}$. Let L be the bag of $P(\mathcal{I}, v_j)$ where the distance from L to the root is maximum. If B_t has a child B_z containing w, let T_z be the subtree of $T - t$ containing z.

Let $\mathcal{I}' = (T', \{B_x\}_{x \in V(T')})$ be a decomposition obtained by removing B_t, and connecting B' and B'', and adding a new bag $\{w, v_j\}$ on the bag L, and attaching T_z on the new bag so that B_z is a child of the bag $\{w, v_j\}$, if exists. The resulting decomposition is a special tree decomposition and $\sum_{x \in V(T)} |B_x|$ is decreased by one, which leads a contradiction. □

Claim 5.7.3. Let B_t be a non-root bag of $\mathcal{I}$ with a child B' and the parent B'' such that $B_t \cap B' = B_t \cap B'' = \{v_1, v_2\}$. If $B_t - B' = \{w\}$, then w is adjacent to both v_1 and v_2.

Proof. Suppose v_i is not adjacent to w for some $i \in \{1, 2\}$ and let v_j be the vertex of $\{v_1, v_2\}$ other than v_i. Similarly in Claim 5.7.2, we first remove the bag B_t, and connect B' and B'', and add a new bag $\{w, v_j\}$ on the below of the path $P(\mathcal{I}, v_j)$, and if there is a component of $T - t$ containing a bag with the vertex w, then cut and attach it on the new bag $\{w, v_j\}$. The resulting decomposition is a special tree decomposition and $\sum_{x \in V(T)} |B_x|$ is decreased by one, which leads a contradiction. □

Claim 5.7.4. Let B be a non-root bag of $\mathcal{I}$ with a child B' and the parent B'' such that $B \cap B' = \{w, v_1\}$ and $B \cap B'' = \{w, v_2\}$. Then v_1 is adjacent to v_2.

Proof. Suppose v_1 is not adjacent to v_2. Note that v_1 is contained in neither B'' nor any child of B other than B'. So, we can remove v_1 from the bag B and reduce $\sum_{x \in V(T)} |B_x|$. It contradicts to the minimality of $\sum_{x \in V(T)} |B_x|$. □

Now we show that $G[\bigcup_{1\leq i\leq n} B_i]$ is 2-connected. Since $\sum_{x\in V(T)}|B_x|$ is minimum, $B_1 - B_2 \neq \emptyset$ and $B_n - B_{n-1} \neq \emptyset$. If $B_1 - B_2 = \{x\}$, then by Claim 5.7.1, x is adjacent to both vertices in $B_1 \cap B_2$. If $B_n - B_{n-1} = \{x\}$, then by Claim 5.7.2, x is adjacent to both vertices in $B_{n-1} \cap B_n$.

Suppose B_{i-1}, B_i, B_{i+1} are three consecutive bags in P. We have two cases. If $B_{i-1}\cap B_i = B_i\cap B_{i+1}$, then by Claim 5.7.3, the vertex w in $B_i - B_{i-1}$ is adjacent to both vertices in $B_{i-1} \cap B_i$. Suppose $B_{i-1} \cap B_i \neq B_i \cap B_{i+1}$. Let $B_{i-1} - B_i = \{y\}$ and $B_{i+1} - B_i = \{z\}$. In this case, by Claim 5.7.4, y is adjacent to z in G. From these analysis, it is easy to verify that $G[\bigcup_{1\leq i\leq n} B_i]$ is 2-connected, and therefore $G[\bigcup_{1\leq i\leq n} B_i]$ is a mamba.

Now we show that G is a mamba tree.

We may assume that there exists a non-root bag B of $\mathcal{I}$ having the parent B' such that $|B \cap B'| = 1$, otherwise G is a mamba. We choose such a bag B so that the distance from B to the root is maximum and assume that $B \cap B' = \{v\}$. Let P be the union of B and all descendants of B. If $|P| \leq 3$, then $G[P]$ consists of either one or two blocks of size two, or a triangle. If $|P| \geq 4$, then $G[P]$ is a mamba and since $v \in B$, v is a head vertex of this mamba. By the induction hypothesis, $G[(V - P) \cup \{v\}]$ is a mamba tree. In all cases, we conclude that G is a mamba tree. $\square$

Proof of Theorem 5.5. As the special treewidth of a graph is the maximum of the special treewidth of its connected components, by Proposition 5.6 it follows that any disjoint union of mamba trees has special treewidth at most two. If a graph has special treewidth at most two, then by Proposition 5.7, it is a disjoint union of mamba trees. $\square$

5.3 The Minor Obstruction Set for Special Treewidth Two

This section is devoted to the proof of Theorem 5.8, given below.

Theorem 5.8. *A graph has special treewidth at most two if and only if it has no minor isomorphic to K_4, S_3, D_3, G_1, G_2, or G_3.*

In Figure 2, the graphs G_1, G_2, G_3 in the obstruction set are displayed.

From our structural characterization of graphs of special treewidth at most two of the previous sections, we can easily check that the class is minor closed.

Also, Proposition 5.7 immediately follows that every graph of $\{G_1, G_2, G_3\}$ has special treewidth at least three, and a tedious case analysis shows that each proper minor of a graph in $\{K_4, S_3, D_3, G_1, G_2, G_3\}$ has special treewidth at most two. So $\{K_4, S_3, D_3, G_1, G_2, G_3\}$ is a subset of the obstruction set

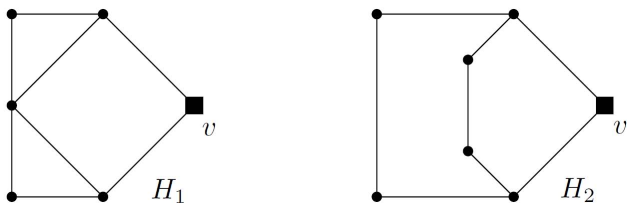

Figure 7: Blocks of obstructions that are not 2-connected

for the class of graphs of special treewidth at most two. Thus, Theorem 5.8 follows from the next lemma.

Lemma 5.9. *If a graph $G = (V, E)$ contains no minor isomorphic to K_4, S_3, D_3, G_1, G_2 or G_3, then the special treewidth of G is at most two.*

To show Lemma 5.9, we extend the standard minor notion to pairs of a graph and a vertex. For pairs (G, v) and (H, v), with $G = (V, E)$, $H = (W, F)$, $v \in V$, $v \in W$, we say that (H, v) is a *minor* of (G, v), if we can obtain H from G by a series of the following operations: deletion of a vertex other than v, deletion of an edge, and contraction of an edge, such that whenever we contract an edge with v as an endpoint, the contracted vertex is named v. For pairs (G, v) and (H, w), we say (G, v) and (H, w) are *isomorphic* if there is a graph isomorphism f from G to H with $f(v) = w$.

The following lemma is a key lemma to find a minor isomorphic to G_1, G_2 or G_3. The pairs (H_1, v) and (H_2, v) are depicted in Figure 7, with v the marked vertex.

Lemma 5.10. *Let B be a 2-connected mamba and let z be a vertex which is not a head vertex of B. Then (B, z) has a minor isomorphic to either (H_1, v) or (H_2, v).*

Proof. Since B is a 2-connected mamba, by Theorem 2.3, $\widetilde{B}$ is a path of cycles. Let (U, F) be a cycle path model of $\widetilde{B}$ with $U = (C_1, \ldots, C_p)$ and $F = (f_1, \ldots, f_{p-1})$. We may assume that

1. $C_1 = C$ if $\widetilde{B}$ has a chordless cycle C of length at least four containing exactly one edge separator f_1, and

2. $C_p = C'$ if $\widetilde{B}$ has a chordless cycle C' of length at least four containing exactly one edge separator f_{p-1}.

By removing repeated edges from F, we can obtain a linear ordering $e_1, \ldots, e_r$ of all edge separators of $\widetilde{B}$. If $r = 1$, then we can easily observe that either

(B, z) has a minor isomorphic to (H_2, v), or there is a path decomposition of $\widetilde{B}$ having width two such that the first bag contains z. Therefore, we may assume that $r \geq 2$.

For each $i \in \{1, r\}$, let $\mathcal{C}_i$ be the set of all chordless cycles of $\widetilde{B}$ containing exactly one edge separator e_i. We observe the following.

1. If C_i is not a simplicial triangle, then $z \notin V(C_i)$ but z can be a vertex of degree two in a simplicial triangle in $\mathcal{C}_i$.

2. If C_i is a simplicial triangle, then $z \notin \bigcup_{C \in \mathcal{C}_i} V(C)$.

We fix $i \in \{1, r\}$ and $e_i = u_i v_i$. If C_i is a simplicial triangle, then all cycles in $\mathcal{C}_i$ are simplicial triangles. So, the second statement is true because if $z \in \bigcup_{C \in \mathcal{C}_i} V(C)$, then it is not hard to construct a path decomposition of $\widetilde{B}$ where the first bag contains z. Also, in the first statement, if $z \in V(C_i)$, we can easily construct a path decomposition where the first bag contains z.

Suppose C_i is not a simplicial triangle and z is a vertex of degree two in a simplicial triangle in $\mathcal{C}_i$. In this case, since $\widetilde{B}[V(B) - \{u_i, v_i\}]$ has two components of size at least two not containing v, using Lemma 3.3, B has two internaly vertex-disjoint paths of length at least three from u_i to v_i. Then (B, z) has a minor isomorphic to (H_2, v). Since there is no path decomposition of H_2 having width two where the first bag contains v, we conclude that z is not a head vertex.

Now we show that for each case, (B, z) has a minor isomorphic to either (H_1, v) or (H_2, v). We may assume that there exist two vertex-disjoint paths P_1, P_2 in $\widetilde{B}$ where P_1 links u_1 to u_r and P_2 links v_1 to v_r.

We have two cases.

Case 1. z is the vertex of degree two in a simplicial triangle of $\widetilde{B}$ having an edge separator xy. If $xy = e_1$ (or $xy = e_r$), then from the above observation, C_1 (or C_r) is a cycle of length at least four. So, in any cases, we have that $\widetilde{B}[V(B) - \{x, y\}]$ has two components of size at least two, which have the vertices of C_1 and C_p, respectively. By Lemma 3.3, B has two internally vertex-disjoint paths of length at least three from x to y. Therefore, (B, z) has a minor isomorphic to (H_2, v).

Case 2. $z \in (V(P_1) \cup V(P_2)) \setminus \{u_1, u_r, v_1, v_r\}$. If we use a chordless cycle which contains e_1 and e_2, then by Lemma 4.3, B has two internally vertex-disjoint paths from u_1 to v_1 such that they have no vertices of $(V(P_1) \cup V(P_2)) \setminus \{u_1, v_1, u_r, v_r\}$. By the same reason, B has two internally vertex-disjoint paths from u_r to v_r such that they have no vertices of $(V(P_1) \cup V(P_2)) \setminus \{u_1, v_1, u_r, v_r\}$.

By symmetry, we may assume that $z \in V(P_1) \setminus \{u_1, u_r\}$. On the path P_1, the distance between z and u_1 (or u_r) is at least one. So, by contracting all edges of P_2, we get a minor isomorphic to (H_1, v) together with the paths which we obtained before. □

Proof of Lemma 5.9. Suppose that the lemma does not hold. Let G be a minimal counterexample such that no minor of G is a counterexample. Since the special treewidth of a graph is the maximum of the special treewidth of its connected components, we may assume that G is connected. Since G has no minor isomorphic to K_4, S_3, or D_3, by Corollary 5.1, each block of G is a mamba. We may also assume that G has at least two blocks.

We use the well known fact that the blocks of a connected graph form a tree, called the block tree. We choose a block B of G corresponding to a leaf of the block tree, and let B' be the block having an intersection v with B. If v is a head vertex of B, then from the minimality of G, $G[(V - B) \cup \{v\}]$ is a mamba tree, and G is again a mamba tree. So, we may assume that every block of G corresponding to a leaf of the block tree is not attached to the remaining graph with a head vertex.

Since G has at least two blocks, G has at least two blocks B_1 and B_2 corresponding to leaves of the block tree, with cut vertices z_1 and z_2, respectively. By Lemma 5.10, each (B_i, z_i) has a minor isomorphic to either (H_1, v) or (H_2, v). Since there exists a path from z_1 to z_2 in G, it implies that G has a minor isomorphic to either G_1, G_2, or G_3, which is contradiction. □

6 Classes of Graphs having Width at most k where $k \geq 3$

In this section, we show that for each $k \geq 3$, none of the classes of graphs with special treewidth, spaghetti treewidth, directed spaghetti treewidth and strongly chordal treewidth at most k is closed under taking minors.

Proposition 6.1. *Let $k \geq 3$. Each of the following classes of graphs is not closed under taking minors.*

1. *The graphs of special treewidth at most k.*
2. *The graphs of spaghetti treewidth at most k.*
3. *The graphs of directed spaghetti treewidth at most k.*

Proof. Note that trees can have arbitrary large pathwidth [20]. Let $T = (V, E)$ be a tree with pathwidth k. Let $G_T = (V', E')$ be a graph obtained by

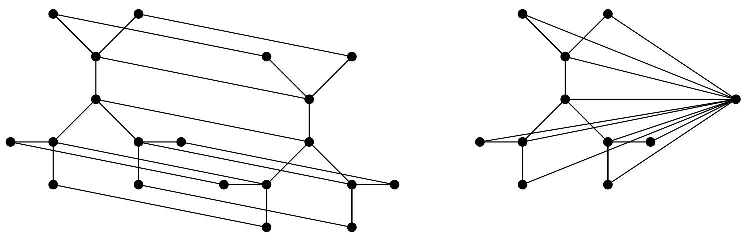

Figure 8: G_T and G'_T: an example of the construction in the proof of Proposition 6.1.

taking two copies of T and adding edges between copies of vertices, that is, $V' = V_1 \cup V_2$ with $V_i = \{v_i : v \in V\}$, for $i \in \{1, 2\}$, $E = \{\{v_i, w_i\} : \{v, w\} \in E, i \in \{1, 2\}\} \cup \{\{v_1, v_2\} : v \in V\}$. See Figure 8 for an example.

The special treewidth of G_T is at most three. With induction to the size of T, we show that G_T has a special tree decomposition of width at most three such that for each $v \in V$, all bags that contain v_1 also contain v_2 and vice versa. This clearly holds when T consists of a single vertex. Let $x \in V$ be a leaf of T, with the parent y. By induction, we assume we have a special tree decomposition of width at most three of G_{T-x}, such that for each $v \in V - \{x\}$, all bags that contain v_1 also contain v_2 and vice versa. Let i be a bag of maximal depth in the tree decomposition with $\{y_1, y_2\} \subseteq X_i$. By assumption, no descendant of i contains y_1 or y_2. Now add a new bag j to the tree decomposition with i the parent of j and $X_j = \{y_1, y_2, x_1, x_2\}$. This is a special tree decomposition of G_T of width three and for each $v \in V$, all bags that contain v_1 also contain v_2 and vice versa.

Now, consider the graph G'_T obtained from G_T by contracting all vertices in $\{v_2 : v \in V\}$ to a single vertex w. Clearly, w is adjacent to all vertices of $V(G'_T) - \{w\}$ in G'_T. Hence, by Lemma 2.5, the special treewidth, spaghetti treewidth, and directed spaghetti treewidth of G'_T equal one plus the pathwidth of G. So, G'_T is a minor of G_T and has special treewidth, spaghetti treewidth, and directed spaghetti treewidth exactly $k+1$. $\square$

Now, we prove that the graphs of strongly chordal treewidth at most k are not closed under taking minors.

Proposition 6.2. *Let $k \geq 3$. The class of graphs of strongly chordal treewidth at most k is not closed under taking minors.*

For each $k \geq 4$, we define SC_k as follows. For each $1 \leq i \leq 3$, let K_k^i be the complete graph on the vertex set $\{v_1^i, v_2^i, \ldots, v_k^i\}$. The graph SC_k is obtained from the disjoint union of K_k^1, K_k^2, K_k^3 and the complete graph on the vertex set $\{w_1, w_2, w_3, w_4\}$ by

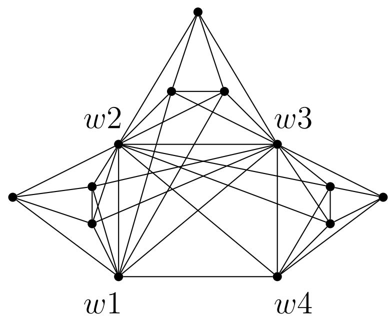

Figure 9: The graph SC_5.

– identifying $v_1^1v_2^1$ with w_1w_2 (with identifying each end vertex), and $v_1^2v_2^2$ with w_2w_3, and $v_1^3v_2^3$ with w_3w_4,

– adding edges $w_3v_i^1$ for all $3 \leq i \leq n-1$,

– adding edges $w_1v_i^2$ for all $3 \leq i \leq n-1$,

– adding edges $w_2v_i^3$ for all $3 \leq i \leq n-1$.

See Figure 9. Note that $\omega(SC_k) = k$.

We first show that SC_{k+1} is strongly chordal. Let T be a tree. For a vertex $c \in V(T)$ and $r \geq 0$, we define $T(c,r)$ as the subtree of T which induces on the vertices v such that the distance between v and c is at most r. For a positive integer k, a graph $G = (V,E)$ is called a *neighborhood subtree tolerance graph with tolerance* k if there exists a tree T, and a set $S = \{T(c_v, r_v) : v \in V\}$ of subtrees of T such that $xy \in E$ if and only if $|V(T(c_x,r_x)) \cap V(T(c_y,r_y))| \geq k$.

Bibelnieks and Dearing showed the following.

Theorem 6.3 (Bibelnieks and Dearing [5])**.** *Let k be a positive integer. If G is a neighborhood subtree tolerance graph with tolerance k, then G is strongly chordal.*

Lemma 6.4. *Let $k \geq 3$. The graph SC_{k+1} is a neighborhood subtree tolerance graph with tolerance* 1*. Hence SC_{k+1} is strongly chordal.*

Proof. Let $A = a_1a_2\cdots a_6$, $B = b_1b_2\cdots b_5$ and $C = c_1c_2\cdots c_7$ be paths. Let T be the tree obtained from the disjoint union of A, B, C and a new vertex v by adding edges va_1, vb_1 and vc_1. We define

1. $T_{v_{k+1}^1} = T(a_6, 0)$, $T_{v_{k+1}^2} = T(b_5, 0)$ and $T_{v_{k+1}^3} = T(c_7, 0)$,

2. for each $3 \leq j \leq k$, $T_{v_j^1} = T(a_5, 2)$, $T_{v_j^2} = T(b_4, 2)$ and $v_j^3 = T(c_6, 2)$,

3. $T_{w_1} = T(a_2, 4)$, $T_{w_2} = T(a_1, 6)$, $T_{w_3} = T(c_1, 6)$ and $T_{w_4} = T(c_4, 3)$.

We can easily check that the set of subtrees $\{T_v\}_{v \in V(SC_{k+1})}$ on the tree T indeed forms a neighborhood subtree models with tolerance 1. Therefore, by Theorem 6.3, SC_{k+1} is strongly chordal. $\square$

Lemma 6.5. *Let $k \geq 3$. The graph SC_{k+1}/w_1w_4 has strongly chordal treewidth at least $k+1$.*

Proof. We say w_1 for the contracted vertex of SC_{k+1}/w_1w_4. Then clearly, $w_1v^1_{k+1}w_2v^2_{k+1}w_3v^3_{k+1}w_1$ is a cycle of length six in SC_{k+1}/w_1w_4 and it does not have an odd chord. So we should add an odd chord, to make it as a subgraph of a strongly chordal graph. We can verify that as soon as we add $v^1_{k+1}w_3$, $\{w_1, w_2, w_3, v^1_3, v^1_4, \ldots, v^1_{k+1}\}$ becomes a clique of size $k+2$ in SC_{k+1}/w_1w_4. The same result appears when adding $v^2_{k+1}w_1$ or $v^3_{k+1}w_2$. Therefore, SC_{k+1}/w_1w_4 has a strongly chordal treewidth at least $k+1$. $\square$

Proof of Proposition 6.2. Since $\omega(SC_{k+1}) = k+1$, by Lemma 6.4, SC_{k+1} has strongly chordal treewidth k. By Lemma 6.5, SC_{k+1}/w_1w_4 has strongly chordal treewidth at least $k+1$. Therefore, the class of graphs of strongly chordal treewidth at most k is not closed under taking minors. $\square$

7 Conclusions

In this paper, we consider the graphs of special treewidth, spaghetti treewidth, directed spaghetti treewidth, or strongly chordal treewidth two. Similar to treewidth, pathwidth and treedepth, these graph parameters can be defined as the minimum of the maximum clique size over all supergraphs in some graph class $\mathcal{G}$, with $\mathcal{G}$ the class of chordal graphs (in case of treewidth) or a subclass of the chordal graphs. (See the discussion in Section 1 and Table 1.)

Our main results are twofold: for each of the four parameters, we give the obstruction set of the graphs with this parameter at most two. These obstruction sets are summarized in Table 2 in Section 1. Secondly, we give characterizations in terms of (special types of) trees of cycles of the cell completion. A 2-connected graph has treewidth two, if and only if its cell completion is a tree of cycles (see Section 2); for each of the other parameters, a similar result with additional conditions on the tree of cycles exists. We summarize these in Table 3. We have that the treewidth, spaghetti treewidth, and strongly chordal treewidth of a graph equals the maximum of this parameter over the blocks of the graph. This is not the case for pathwidth and for special treewidth. For special treewidth two, we have established a precise condition (building upon the notion of head vertices) how blocks of special treewidth at most two can be connected to obtain a graph of special treewidth two (see Section 5).

parameter	cycle tree model	connecting blocks
treewidth	tree of cycles [11, 29]	everywhere
pathwidth	path of cycles [17, 8]	!(not simple) [17]
spaghetti tw	chain tree of cycles (Th. 3.4)	everywhere
strongly chordal tw	tree of 2-boundaried cycles (Th. 4.8)	everywhere
dir. spaghetti	path of cycles (Cor. 5.1)	everywhere
special tw	path of cycles (Cor. 5.1)	head vertices

Table 3: Models for cell completions of graphs with value of parameter at most two. The second column gives the characterization for 2-connected graphs; the last column shows how blocks of width at most 2 can be connected: everywhere = each block has width at most 2 is sufficient; ! = no simple characterization exist; head vertices = see Section 5.2.

We expect that similar characterizations are hard or impossible to obtain for values larger than two. For instance, we see in Section 6 that the classes of graphs of special treewidth, spaghetti treewidth, directed spaghetti treewidth, or strongly chordal treewidth at most k, for $k \geq 3$ are not closed under taking minors.

It may be interesting to pursue a similar investigation for parameters that are defined in a similar way by other subclasses of chordal graphs. Bodlaender, Kratsch and Kreuzen [13] showed that special treewidth and spaghetti treewidth are fixed parameter tractable; an adaptation of the algorithms by Bodlaender and Kloks [12] or Lagergren and Arnborg [30] gives linear time decision algorithms for each fixed bound on the width. We conjecture that in a similar way, it can be shown that directed spaghetti treewidth is fixed parameter tractable. Whether strongly chordal treewidth is fixed parameter tractable, we leave as an open problem.

References

[1] D. Archdeacon. A Kuratowski theorem for the projective plane. *Journal of Graph Theory*, 7:325–334, 1983.

[2] S. Arnborg and A. Proskurowski. Linear time algorithms for NP-hard problems restricted to partial k-trees. *Discrete Applied Mathematics*, 23:11–24, 1989.

[3] S. Arnborg, A. Proskurowski, and D. G. Corneil. Forbidden minors characterization of partial 3-trees. *Discrete Mathematics*, 80:1–19, 1990.

[4] J. Barát, P. Hajnal, Y. Lin, and A. Yang. On the structure of graphs with path-width at most two. *Studia Scientiarum Mathematicarum Hungarica*, 49(2):211–222, 2012.

[5] E. Bibelnieks and P. M. Dearing. Neighborhood subtree tolerance graphs. *Discrete Applied Mathematics*, 43(1):13–26, 1993.

[6] H. L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoretical Computer Science*, 209:1–45, 1998.

[7] H. L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoret. Comput. Sci.*, 209(1-2):1–45, 1998.

[8] H. L. Bodlaender and B. de Fluiter. On intervalizing k-colored graphs for DNA physical mapping. *Discrete Applied Mathematics*, 71:55–77, 1996.

[9] H. L. Bodlaender, J. S. Deogun, K. Kansen, T. Kloks, D. Kratsch, H. Müller, and Z. Tuza. Rankings of graphs. *SIAM Journal on Discrete Mathematics*, 11:168–181, 1998.

[10] H. L. Bodlaender, J. R. Gilbert, H. Hafsteinsson, and T. Kloks. Approximating treewidth, pathwidth, frontsize, and minimum elimination tree height. *Journal of Algorithms*, 18:238–255, 1995.

[11] H. L. Bodlaender and T. Kloks. A simple linear time algorithm for triangulating three-colored graphs. *Journal of Algorithms*, 15:160–172, 1993.

[12] H. L. Bodlaender and T. Kloks. Efficient and constructive algorithms for the pathwidth and treewidth of graphs. *Journal of Algorithms*, 21:358–402, 1996.

[13] H. L. Bodlaender, S. Kratsch, and V. J. C. Kreuzen. Fixed-parameter tractability and characterizations of small special treewidth. In A. Brandstädt, K. Jansen, and R. Reischuk, editors, *Proceedings of the 39th International Workshop on Graph-Theoretic Concepts in Computer Science, WG 2013*, volume 8165 of *Lecture Notes in Computer Science*, pages 88–99. Springer Verlag, 2013.

[14] A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph Classes: A survey.* SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[15] B. Courcelle. The monadic second-order logic of graphs I: Recognizable sets of finite graphs. *Information and Computation*, 85:12–75, 1990.

[16] B. Courcelle. On the model-checking of monadic second-order formulas with edge set quantifications. *Discrete Applied Mathematics*, 160(6):866–887, 2012.

[17] B. de Fluiter. *Algorithms for Graphs of Small Treewidth.* PhD thesis, Utrecht University, 1997.

[18] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics.* Springer, Heidelberg, fourth edition, 2010.

[19] Z. Dvořák, A. C. Giannopoulou, and D. M. Thilikos. Forbidden graphs for tree-depth. *European Journal of Combinatorics*, 33(5):969–979, 2012.

[20] J. A. Ellis, I. H. Sudborough, and J. Turner. The vertex separation and search number of a graph. *Information and Computation*, 113:50–79, 1994.

[21] M. Farber. *Applications of Linear Programming Duality to Problems Involving Independence and Domination.* PhD thesis, Rutgers University, 1982.

[22] M. Farber. Characterizations of strongly chordal graphs. *Discrete Math.*, 43(2-3):173–189, 1983.

[23] F. V. Fomin, P. Fraigniaud, and N. Nisse. Nondeterministic graph searching: From pathwidth to treewidth. *Algorithmica*, 53:358–373, 2009.

[24] F. Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *Journal of Combinatorial Theory, Series B*, 16:47–56, 1974.

[25] F. Gavril. A recognition algorithm for the intersection graphs of paths in trees. *Discrete Mathematics*, 23:211–227, 1978.

[26] M. C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs.* Academic Press, New York, 1980.

[27] M. C. Golumbic and R. E. Jamison. Edge and vertex intersections of paths in a tree. *Discrete Mathematics*, 55:151–159, 1985.

[28] N. G. Kinnersley and M. A. Langston. Obstruction set isolation for the gate matrix layout problem. *Discrete Applied Mathematics*, 54:169–213, 1994.

[29] T. Kloks. *Treewidth. Computations and Approximations*, volume 842 of *Lecture Notes in Computer Science*. Springer Verlag, Berlin, 1994.

[30] J. Lagergren and S. Arnborg. Finding minimal forbidden minors using a finite congruence. In J. L. Albert, B. Monien, and M. Rodríguez-Artalejo, editors, *Proceedings of the 18th International Colloquium on Automata, Languages and Programming, ICALP'91*, pages 532–543. Springer Verlag, Lecture Notes in Computer Science, vol. 510, 1991.

[31] C. L. Monma and V. K. Wei. Intersection graphs of paths in a tree. *Journal of Combinatorial Theory, Series B*, 41:141–181, 1986.

[32] B. S. Panda. The forbidden subgraph characterization of directed vertex graphs. *Discrete Mathematics*, 196(1-3):239–256, 1999.

[33] A. Proskurowski. Maximal graphs of pathwidth k or searching a partial k-caterpillar. Technical Report CIS-TR-89-17, Dept. of Computer and Information Science, University of Oregon, 1989.

[34] N. Robertson and P. D. Seymour. Graph minors. I. Excluding a forest. *Journal of Combinatorial Theory, Series B*, 35:39–61, 1983.

[35] N. Robertson and P. D. Seymour. Graph minors. II. Algorithmic aspects of tree-width. *Journal of Algorithms*, 7:309–322, 1986.

[36] N. Robertson and P. D. Seymour. Graph minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92:325–357, 2004.

[37] A. Satyanarayana and L. Tung. A characterization of partial 3-trees. *Networks*, 20:299–322, 1990.

[38] K. Wagner. Über eine Eigenshaft der ebenen Complexe. *Math. Ann.*, 14:570–590, 1937.