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Highlights:

- This paper characterizes equilibrium bidding behavior in a multi-unit uniform-price auction.
- When bidders demand two units and valuations are independent, equilibrium bidding behavior entails bidding their values for the first unit, and bidding below their valuations for the second unit.
- We provide a new characterization of equilibrium bidding behavior in this environment.

A new characterization of equilibrium in multiple-object uniform-price auctions

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Abstract

This paper characterizes equilibrium bidding behavior in a multi-unit uniform-price auction. As posited by Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998), when bidders demand two units and valuations are independent, equilibrium bidding behaviour entails bidding their values for the first unit, and bidding below their valuations for the second unit. We identify some errors in the analysis of Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998), and provide a characterization of equilibrium behavior which involves a threshold value that separates bidders who bid zero for the second unit from those who shade their bids for the second unit.

Keywords: multiple-objects; uniform-price; auctions; equilibrium.

JEL Classification: D44.

1 Introduction

This paper provides a new characterization of equilibrium bidding behavior in a multi-unit uniform-price auction. It corrects and complements Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998), which were published nearly two decades

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ago, at time where multiple-object auctions began to be used to solve complex allocation problems such as the sale of the spectrum of frequency for mobile telephony (Cramton *et al.* (2006), Milgrom (2004)) and the privatization of government-owned companies through the sale of shares (Menezes (1995)).

While the last two decades have seen substantial progress in our understanding of multiple-object auctions[1], a number of papers in the 1990s show that the bidders in a multi-unit auction may bid below their values in a uniform-price auction. This phenomenon become known as 'demand reduction'.

In particular, several papers demonstrated that demand reduction could be severe. For example, Menezes (1995) showed the existence of an equilibrium in an ascending price auction where bidders were characterized by a demand for a divisible good, and where the auction never got off the ground, with the object being sold at the reserve price. Noussair (1995) (hereafter referred to as Noussair) and Engelbrecht-Wiggans and Kahn (1998) (hereafter referred to as EWK) examined uniform-price, simultaneous auctions, where bidders demand two objects each and claimed that there exist an equilibrium where bidders bid their values for the fist unit, but they bid less than their values for the second unit, with some bidders bidding zero for the second unit.

We identify the errors in the analysis of Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998), and provide a characterization of equilibrium behavior which involves a threshold value that separates bidders who bid zero for the second unit from those who shade their bids for the second unit.

2 Model

A seller has $k > 1$ identical objects for sale to $n+1$ potential buyers. Each buyer demands two units of the good. We assume that $2(n+1) > k$ so that there is excess demand. We denote by v_1^i and v_2^i the values that bidder i places on, respectively, the first and second units, with $v_1^i \geq v_2^i$. The pair of values are independently and identically drawn from a distribution function $G(.,.)$. Further, we assume that $G_1(v_1)$ and $G_2(v_2)$ represent the marginal distributions of v_1 and v_2, with densities g_1 and g_2, respectively, and support $[0, \bar{v}]$.

We consider uniform-price auctions where bidder submit two non-negative bids, one for each unit, the goods are awarded to the kth highest bids, and winners pay

[1]For an introduction to multiple object auctions, see for example Krishna (2002) (chapters 12 to 17) and Menezes and Monteiro (2005) (chapter 7).

the equivalent to the highest losing bid. The reserve price is set to zero so if there are fewer than k non-zero bidders, all winners pay a price of zero.

3 Equilibrium characterization in EWK and Noussair

We first compare the set-up and the results in Noussair and EWK. Noussair analyses a more restrictive set of equilibria compared to EWK. In particular, Noussair considers what he refers to as a type M bidding function $B(\cdot,\cdot)$, which is characterized by an initial condition such that $B(0,0) = (0,0)$, it is continuous, monotone and it takes a value of zero for the second unit for valuations below a threshold.

EWK focus on a set of strategies, which they refer to as set S, where the higher of the two bids equals the higher of the two valuations, and the lower of the two bids is non-negative and no greater than the lower of the two valuations. They show this set is the exact set of strategies that are not dominated. Note that in EWK examples there are equilibria for cases in which no equilibrium involving type M bidding functions exists.

Our starting point is EWK's Theorem 4.2, which characterizes a threshold value v^* such that all bidders with values less than v^* for the second unit, bid zero on the second units as long as $n + 1 \geq k$. EWK's threshold value is determined by the following expression:

$$(k-1)(v^* - b)\frac{g_1(b)}{1 - G_1(b)} \leq 1, \tag{1}$$

for all $b \in [0, v^*)$.

However, as we show in the next paragraph, this condition fails to characterize a threshold value $v^* < \bar{v}$ such that bidders with values lower than that threshold bid zero for the second unit. In fact, this condition only applies to a case where $v^* = \bar{v}$. Therefore, Theorem 4.2 in EWK only provides a condition such that all bidders bid zero for the second unit. In other words, Theorem 4.2 fails to characterize equilibria where some bidders bid zero for the second unit, and others bid below their valuations. This failure has further implications for EWK's analysis, which we will discuss further below.

To see why Theorem 4.2 of EWK fails for any $v^* < \bar{v}$, we follow EWK and define $F_1(x)$ and $F_2(x)$ as the $k-1$th and kth order statistics of n independent draws from $G_1(.)$ draws respectively as follows:

$$F_1(x) = \sum_{r=n-k+2}^{n} \binom{n}{r} [G_1(x)]^r [1 - G_1(x)]^{n-r} \tag{2}$$

$$F_2(x) = \sum_{r=n-k+1}^{n} \binom{n}{r} [G_1(x)]^r [1 - G_1(x)]^{n-r} \tag{3}$$

EWK acknowledge that the above equations represent the $k-1$th and kth order bid statistics – bidders bid their valuations for the first object in EWK's equilibrium candidate – when all bidders submit zero bids for the second unit. However, in the proof of Theorem 4.2, they use the above equations for all bids in $[0, v^*)$.

Now suppose that some bidders do not bid zero for the second unit, that is, $v^* < \bar{v}$. In this case (2) and (3) no longer represent the $k-1$th and kth order statistics. Therefore, the condition in Theorem 4.2 is only true when $v^* = \bar{v}$.

In Corollary 3, EWK posit that if $n+1 \geq k$, and condition (1) holds for $v^* = \bar{v}$, then there is an equilibrium where the sales price is equal to zero. The analysis above shows that Corollary 3 becomes trivial as the only value of v^* that satisfies (2) and (3) is $\bar{v}$.

In summary, EWK's condition (1), alongside the assumption that $n+1 \geq k$, imply that there is a zero price equilibrium, whereby every bidder bids zero for the second unit, rather than an equilibrium where some bidders may have non-zero bids for the second unit. By inspecting condition (1), one can readily see that it is a strong condition, that will become harder to be satisfied as the number of object for sale increases, regardless of the number of bidders.

We now focus on Noussair's Theorem 1 (p. 340), which establishes sufficient conditions for a M type equilibrium to exist. According to the initial conditions of the Theorem 1 on page 340, if $n+1 > k$, then v^* is equal to zero. As we have seen above, this contradicts Theorem 4.2 of EWK, which states $v^* = \bar{v}$. In the next section we correctly characterize v^* and extend the result of Theorem 4.2 in EWK to allow for some bidders to submit non-negative bids for the second object in equilibrium.

Noussair's Theorem 1 also claims that if $n+1 < k$, then $v^* > 0$, that is, bidders with values less than v^* bid zero for the second unit. The following Theorem provides a simple proof to show that this claim is incorrect.

Theorem 1. *When $n+1 < k$, bidding equal to zero for the second unit cannot be an equilibrium.*

Proof. Define c_1 and c_2 as the $k-1$th and kth highest competing bids facing bidder i. Also H_1 and H_2 as their marginal distributions with density h_1 and h_2 respectively.

Suppose $v^* > 0$ and bidders with realized values below v^* bid zero for the second unit. We analyze three scenarios with respect to the competing bids as follows.

$$\begin{cases} (1) & c_1 \geq c_2 > 0 \\ (2) & c_1 > 0, c_2 = 0 \\ (3) & c_1 = c_2 = 0 \end{cases} \tag{4}$$

Suppose bidder i with realized value $v_2 < v^*$, deviates from this strategy and bids ϵ for the second unit where $0 < \epsilon < v_2$. In scenario (1), for an ϵ arbitrary close to zero, the payoff of player i would remain the same. In scenario (2), ϵ becomes the highest losing bid and this has a negative effect on player i's payoff equal to ϵ. Suppose the probability that scenario (2) happens is $\gamma > 0$. Therefore, the expected loss is $\gamma\epsilon$. In scenario (3), player i wins the second unit and pays a price equal to zero. Suppose the probability of the third scenario is $\kappa > 0$. Then there exist an $\epsilon > 0$ such that, $\gamma\epsilon < \kappa v_2$. Therefore, a bid equal to ϵ for the second unit dominates a bid equal to zero.

□

To see why an argument similar to the proof of Theorem 1 is not applicable to a case where $n + 1 \geq k$ remember that bidders bid their values for the first unit. Thus when the number of bidders is at least as large as the number of units, then only scenario (1) is possible.

4 The Main Result

What remains is the characterization of v^* when $n + 1 \geq k$. The expected payoff of a buyer who submits b_1 and b_2 with $b_2 < b_1$ is equal to[2],

$$\begin{aligned} \Pi(v_1, v_2, b_1, b_2) = H_1(b_2)(v_1 + v_2) - 2\int_0^{b_2} c_1 h_1(c_1) dc_1 + (H_2(b_1) - H_1(b_2))v_1 \\ - (H_2(b_2) - H_1(b_2))b_2 - \int_{b_2}^{b_1} c_2 h_2(c_2) dc_2 \end{aligned} \tag{5}$$

Then the first-order condition with respect to b_2 is,

[2]For further explanation regarding the derivation of this payoff see Krishna (2002), page 192.

$$\frac{\partial \Pi}{\partial b_2} = h_1(b_2)(v_2 - b_2) - H_2(b_2) + H_1(b_2) = 0 \tag{6}$$

This would result in the following bidding function.

$$b_2 = v_2 - \frac{H_2(b_2) - H_1(b_2)}{h_1(b_2)} \tag{7}$$

If the second-order condition of maximization is satisfied, then the bidding function in (7) is monotone and increasing in v. Now imagine there exist a v^* such that bidders with realized values equal or below it bid zero for the second unit. If this is the case it must be such that bidders with value v^* are indifferent between bidding according to (7) or bidding equal to zero, that is,

$$\Pi(v_1, v_2^*, b_2(v^*)) = \Pi(v_1, v_2^*, 0) \tag{8}$$

We use this fact to characterize v^*. From (5) we have,

$$\begin{aligned} &H_1(b_2(v^*))(v_1 + v_2) - 2\int_0^{b_2(v^*)} c_1 h_1(c_1) dc_1 + (H_2(b_1) - H_1(b_2(v^*)))v_1 \\ &- (H_2(b_2(v^*)) - H_1(b_2(v^*)))b_2(v^*) - \int_{b_2(v^*)}^{b_1} c_2 h_2(c_2) dc_2 \\ &= H_2(b_1)v_1 - \int_0^{b_1} c_2 h_2(c_2) dc_2 \end{aligned} \tag{9}$$

After some manipulation we have,

$$\begin{aligned} &H_1(b_2(v^*))(v^*) - 2\int_0^{b_2(v^*)} c_1 h_1(c_1) dc_1 \\ &- (H_2(b_2(v^*)) - H_1(b_2(v^*)))b_2(v^*) + \int_0^{b_2(v^*)} c_2 h_2(c_2) dc_2 = 0 \end{aligned} \tag{10}$$

Integration by part yields,

$$\begin{aligned} &H_1(b_2(v^*))(v^*) - 2b_2(v^*)H_1(b_2(v^*)) + 2\int_0^{b_2(v^*)} H_1(c_1) dc_1 \\ &- [H_2(b_2(v^*)) - H_1(b_2(v^*))]b_2(v^*) + b_2(v^*)H_2(b_2(v^*)) - \int_0^{b_2(v^*)} H_2(c_2) dc_2 = 0 \end{aligned} \tag{11}$$

After simplifying we get,

$$H_1(b_2(v^*))(v^* - b_2(v^*)) + 2\int_0^{b_2(v^*)} H_1(c_1)dc_1 - \int_0^{b_2(v^*)} H_2(c_2)dc_2 = 0 \tag{12}$$

Substituting v^* from the bidding function in (7) yields,

$$H_1(b_2(v^*))(\frac{H_2(b_2(v^*)) - H_1(b_2(v^*))}{h_1(b_2(v^*))}) + 2\int_0^{b_2(v^*)} H_1(c_1)dc_1 - \int_0^{b_2(v^*)} H_2(c_2)dc_2 = 0 \tag{13}$$

The following Theorem summarizes the above result.

Theorem 2. *When the number of bidders is at least as large as the number of units, the threshold value v^* such that bidders with $0 < v_2 < v^*$ bid zero for the second unit is given by,*

$$v^* = b_2(v^*) + \frac{H_2(b_2(v^*)) - H_1(b_2(v^*))}{h_1(b_2(v^*))} \tag{14}$$

where $b_2(v^*)$ is the solution to (13).

5 Conclusion

This paper characterizes equilibrium behavior in a multiple-object, uniform-price auction. When bidders demand two objects, and their values for the two objects are drawn independently from the a common distribution, we show the existence of an equilibrium where bidders bid their true valuations for the first unit, but reduce their demand for the second unit. In particular, in the equilibrium we identified, bidders with a value above a certain threshold, make a non-zero bid for the second unit. The threshold value, and their non-negative bids for the second unit, is a function of the distribution of the kth and k-1th highest values. This new equilibrium characterization complements the analysis in Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998).

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