# RECIPROCAL RELATIONS OF RELATIVE SIZE IN THE INSTRUCTIONAL CONTEXT OF FRACTIONS AS MEASURES

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*The presented study is part of a bigger design and research enterprise in the teaching of fractions as measures. We analyze extracts of a teaching session with a single fifth grade student, in which he flexibly compared the relative sizes of the lengths of three drinking straws, skillfully using unitary, proper, and improper fractions. We identify aspects of his prior instructional experiences that supported the emergence of his relatively sophisticated ways of reasoning. Findings suggest that supporting students' reasoning about reciprocal relations of relative size can be a viable goal in an instructional agenda on fractions as measures.*

In this paper, we explore how to instructionally support the emergence of a form of mathematical reasoning that involves reciprocally quantifying the size of two magnitude values (Ramful, 2013; Thompson & Saldanha, 2003). Quantifying in this way is central to many scientific and everyday practices. For instance, in money exchange, the relative value of two currencies is determined in a reciprocal way, so that if the value of an Australian dollar is 16 Mexican pesos, the value of a Mexican peso is 1/16 of an Australian dollar (or 0.0625 AUD).

A key aspect of reciprocal quantitative comparisons is that they always involve the use of rational numbers. In the example above, although the value of an Australian dollar relative to a Mexican peso can be quantified using only natural numbers, quantifying the reciprocal relation necessarily requires the use of fractions (or an equivalent rational expression). The relatively sophisticated fraction understanding that is required to conduct this kind of comparisons, makes them difficult to study in elementary years, where such levels of fraction understanding are seldom developed (e.g., Hannula, 2003).

We analyze the reasoning of a fifth grade student, Pedro, who became skillful in using fractions to determine the relative size of up to three different magnitude values in a reciprocal way. In the analysis, we also identify aspects of Pedro's reasoning that can be directly linked to specific instructional experiences. Based on the results of the analysis, we then consider the resources for classroom teaching that would need to be developed so that proficient teachers could support the development of similar ways of reasoning in regular classrooms.

## BACKGROUND

Pedro attended an urban public school in Mexico, and was experiencing difficulties with learning mathematics. The first author worked with him, after school, in one hour weekly sessions, both to help him overcome his difficulties, and as part of a broader

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research effort (Cortina, Visnovska, & Zúñiga, 2014a). This effort has centered on developing an instructional sequence on fractions as measures (Kieren, 1980), using the general methodological approach of *design research* (Gravemeijer & Cobb, 2006).

In our instructional design, we follow the tenets of Realistic Mathematics Education (Gravemeijer, 1994), as well as the adaptations of this theory proposed by Cobb and colleagues (Cobb, Zhao, & Visnovska, 2008). We aim to (a) clarify the progression of forms of student reasoning that are likely to emerge as students engage in specific instructional activities, and (b) provide guidance for the teacher about how the emergence of these forms of reasoning can be proactively supported in classrooms.

It is worth clarifying that we have developed the instructional sequence by working with groups of students in their regular classrooms. The first author's involvement in Pedro's remedial education was used as an opportunity to explore the continuation of the sequence, prior to its classroom testing. In this sense, the presented analysis belongs to the 'preparing for a classroom experiment' phase of design research and will be used to formulate the rationale of the extended sequence.

The samples of Pedro's reasoning that we analyze in this paper come from the 21st weekly remedial session. In sessions 8-20, the first author guided Pedro through a previously developed sequence of instructional activities on fractions as measures (Cortina et al., 2014a). The progression of Pedro's reasoning during these weeks was consistent with prior findings. We include the overview of these developments to clarify the instructional approach taken, particularly with respect to how Pedro was supported to understand unitary and common fractions. Against this background we then introduce the activity of week 21, in which Pedro engaged in instructional activity that involved making reciprocal quantitative comparisons.

## SUPPORTING PEDRO TO REASON ABOUT FRACTIONS AS MEASURES

Pedro was first asked to use parts of his body (e.g., hand span) to measure lengths and reflected on the advantages and disadvantages of gauging the lengths of objects in this way. He was then asked to measure lengths using a standardized unit, a wooden stick with no marks on it, about 24 cm long. In doing so, the issue of the remainder became a concern, as he realized that many objects did not measure a whole number of iterations of the stick. For instance, a table would measure three sticks and a bit more.

In order to quantify the lengths of the remainders, Pedro was oriented to produce subunits of measure, the lengths of which corresponded to unit fractions of the length of the stick. Importantly, he was not led to construe unit fractions as quotients of a partitive division (i.e., the result of equally partitioning a whole into a certain number of equal size parts). Instead, he was supported to construe them as divisors in a measurement division, in which the reference unit is divided a whole number of times, with no remainder.

To clarify the distinction we are making, asking a student to fold a plastic drinking straw of the length equal to a stick, twice, and think about the length of the resulting four equal size parts, would correspond to construing 1/4 as a quotient of a partitive division. In contrast, Pedro was given a plastic drinking straw shorter than the stick and asked to cut it (i.e., to create the divisor) so that when using it to measure the stick, it would fit in exactly four times (see Figure 1). He did this through a process of trial and error in which he would cut the straw and test it to see if its length met the specified condition. If the straw turned out to be too long, he would cut it shorter. If it turned out to be too short, he would start afresh with a new straw. We believe that the distinction described made the difference to Pedro's reasoning and return to it in the discussion section of this paper.

Figure 1: 1/4 as the length of a subunit (a plastic drinking straw) that fits exactly four times in the length of a reference unit (a white wooden stick).

Using the described procedure, Pedro produced straws of the following lengths: 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9 and 1/10, in this order. Consistent with findings from prior experiments (Cortina et al., 2014a), by reflecting on the process of producing unit fractions in this way, Pedro came to develop a comprehensive understanding of the inverse order relation of unit fractions (i.e., the bigger the number in the denominator, the smaller the fraction size).

Next, Pedro was oriented to interpret common fractions as measures that accounted for a length that corresponded to the iteration of a subunit a certain number of times. For instance, the fraction 7/4 would account for a length that corresponded to seven iterations of a subunit of length 1/4.

Pedro was then supported to recognize how subunits, when iterated a specific whole number of times, rendered a length identical to that of the reference unit (e.g., $4 \times 1/4 = 1$). Also consistent with prior findings, developing such an understanding allowed him to correctly judge any fraction as representing a length smaller than, as big as, or bigger than one (e.g., $2/3 < 1$; $3/3 = 1$; $4/3 > 1$). It also allowed him to soundly and correctly convert improper fractions into mixed fractions and vice versa.

When Pedro began to engage with the instructional activities that had not been previously tested in classrooms, the first author took photographs of his work, created detailed field notes after each teaching session, and debriefed Pedro's learning with the second author weekly. The first problem related to assessing reciprocal relations was presented to Pedro in session 20, and his relatively seamless response motivated video recording of the subsequent session, which we analyze here.

## REASONING ABOUT RECIPOCAL RELATIONS OF RELATIVE SIZE

The 21st session started by asking Pedro to cut a small straw of about the same length as his little finger. This straw was initially construed as having the measure *one*. He was then asked to produce two more straws, one that measured four (small) straws, and another seven. Pedro chose to label the shortest straw as “*A*”, the middle one as “*B*”, and the longest one as “*C*” (see Figure 2).

The teacher then asked Pedro to decide which of the two longer straws would now play the role of being “the stick” (i.e., the reference unit) and, thus, being attributed the measure *one*. The conversation below (translated from Spanish) then took place, where P and T refer to Pedro and teacher respectively.

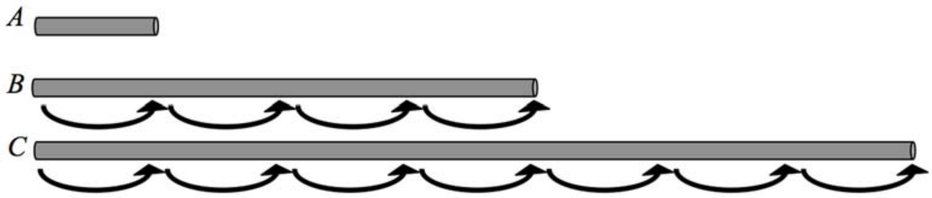


Figure 2: Straws *A*, *B* (4 times as long as *A*), and *C* (seven times as long as *A*).

A1 P: *B* will be one so *C* will be two (chuckling)?

A2 T: Let’s focus on *A* first.

A3 P: *A* will be two.

A4 T: Let’s see, why two?

A5 P: No, *A* is going to be four (showing four fingers).

A6 T: And what is bigger, one or four?

A7 P: Four.

A8 T: So is this longer than this (placing straw *A* next to straw *B*).

A9 P: No. (Pause). Then it would be smaller? No? (Looking at the teacher).

A10 T: Let’s see. If this is your stick (pointing at straw *B*), what is this (holding straw *A*)?

A11 P: A fourth.

A12 T: Ok. Why a fourth?

A13 P: Because *B* is divided into four (gesturing with his hand along straw *B*), and since I have a fourth, then it is one, two, three, four (taking straw *A* and iterating it along straw *B* as he counted).

A14 T: Ok, write it in the table (Pedro’s record of reciprocal comparisons).

This first extract illustrates how Pedro came to reason quantitatively about the reciprocal of iterating the length of a unit, a whole number of times. With relatively little support from the teacher (see utterance A10), he seemed to have readily reinterpreted the situation as one in which the length of a subunit was to be quantified. He now interpreted the length of straw *A* in the way he had been oriented to construe the quantitative meaning of a unit fraction—in this case *one fourth*—as being the length of a subunit that would fit four times, exactly, in the length of the reference unit (see utterance A13 and Figure 1). Evidently, his reasoning was consistent with his prior instructional experiences in the remedial teaching sessions.

Pedro was next asked to determine the length of *C*, relative to the length of *B*.

B1 P: Then, *C* would be bigger than *B* (looking at straws *B* and *C*)

B2 T: Ok.

B3 P: Four (likely meaning the length of straw *B*), they would be (touching straw *C*, closing his eyes and pausing to think) a seventh?

B4 T: A seventh? What is bigger, a whole (referring to straw B) or a seventh (referring to straw C)?

B5 P: A whole.

B6 T: So this one (touching straw *B*) is longer than this one (touching straw *C*)?

B7 P: Oh, no.

B8 T: So how can *C* be a seventh of *B?*

B9 P: Oh no. Then it would be one whole (closing one eye and pausing to think) three (short pause) fourths?

B10 T: Why?

B11 P: Because there are four here (gesturing with his hand along straw *B*), and there are four here (gesturing with his hand in the same way along part of straw *C*), but there are three more here (pointing at the rest of the length of straw *C*), so a whole has been formed, with three fourths (added to it).

This second extract illustrates how Pedro reasoned about the size of a length (*C*) relative to size of a new reference unit (*B*). He initially seemed to focus on both the lengths of straws *B* and *C* as being a product of iterating the length of straw *A* a certain number of times. This might have led him to think first about the inverse of producing a length seven times as long as *A* (i.e., a seventh). However, prompted by the realization that *C* could not be one seventh of *B*, he seemed to have then realized that straw *C*, by being the product of iterating *A* seven times, would be three iterations of one fourth of *B* longer than *B*. Pedro seemed to have relied on his prior realization that the length of straw *A* was one fourth of the length of *B,* and on the fact that *C* was three iterations of *A* longer than *B* (see utterance B11).

Here too, Pedro's reasoning was consistent with his instructional history in the remedial teaching sessions. More specifically, it was consistent with how he had been supported to interpret the meaning of common fractions in the teaching sessions—as lengths produced by iterating a subunit a certain number of times.

Finally in this problem, Pedro was asked to determine the lengths of *A* and *B* relative to the length of *C*.

C1 P: Then, if *C* is one, it (meaning *A*) would be (pause) one seventh?

C2 T: Are you just guessing?

C3 P: No. That one would actually be a seventh (pointing at *A*).

C4 T: A seventh, why?

C5 P: Because it fits seven times in *C*. And *B* would have four sevenths.

C6 T: Ok. Why?

C7 P: Because here (aligning straws *B* and *C*) if you measure it (meaning "with *A*"), there are four here (touching the *B* straw) and seven here (touching the *C* straw). But if you join them, there are four sevenths here (touching the *B* straw). So it is four sevenths.

In this final interaction, Pedro engaged in the same kind of reasoning as earlier. He seemed to have readily reconceptualised *C* from being a length seven times as long as *A*, to being a reference unit into which the subunit *A* would fit seven times; and *B*, from being of a length four times as long as *A*, to being a straw as long as four iterations of a seventh of the length of straw *C* (see utterance C7).

In the remainder of the session, Pedro engaged in similar reasoning with relative ease. He correctly compared the lengths of three other straws (1, 3, and 10) without physically creating them. Afterwards, he succeeded in establishing that his age (10 years) was 10/13 of his sister's, and his sister's age was his plus 3/10 of his age. Finally, he determined the fraction of the student population of his school, in his classroom (32/407), as well as the size of the school population relative to the number of students in his classroom (407/32).

## CONCLUSION AND DISCUSION

The analysis of Pedro's reasoning provides a justification for extending the instructional sequence on fractions as measures, towards a goal of supporting students' reasoning about reciprocal relations of relative size. As the findings indicate, Pedro's relatively sophisticated ways of reasoning were tightly linked to his instructional experiences, and reflected how unitary and common fractions were conceptualised within the sequence. It is reasonable to expect that when using this sequence in a classroom, some of the students' reasoning elicited by the reciprocal comparison tasks would be similar to Pedro's. According to the theory of Realistic Mathematics Education, a teacher could then aim to advance the instructional agenda by making such reasoning the focus of collective analysis and discussion, while proactively supporting the sense making of all students.

A classroom design experiment is required not only to trial this process, but to design appropriate resources for classroom teaching on which a teacher could build. Such resources would include (but are not limited to) accounts of the diversity of student reasoning in specific instructional activities, and symbolic and other means of supporting mathematical conversations in the classroom.

Regarding a broader agenda on the teaching of fractions, the analysis illustrates the developmental advantages of supporting students to make sense of the quantitative meaning of unitary fractions, primarily as divisors in a measurement division. In a recent paper, Beckmann and Izsák (2015) propose a distinction between two quantitatively different ways of construing ratios in instruction, which they convincingly argue can have significant implications for how students come to understand this idea. The distinction they make is closely related to the difference between the measurement and partitive meanings of division, as well as between two meanings for multiplication tightly linked to those two of division. We believe that a similar distinction could be important to consider in fractions instruction.

In terms of multiplicative relationships (Beckmann & Izsák, 2015), our analysis shows that central to Pedro's success in reasoning about reciprocal relations of relative size was his reconceptualization of *the iteration* of a given unquantified magnitude value $A$, from being a multiplication ($n\times A=B$) to being a measurement division ($B\div A=n$). Such a reconceptualization entailed the reinterpretation of the produced magnitude value $B$, from being a product $n$ times as big as the original reference unit (when $A$=1, $B=n\times 1$), to being a reference unit in its own right ($B$=1). In addition, it entailed reinterpreting the original magnitude value $A$, from being a multiplied reference unit (i.e., a multiplicand in $B=n\times A$), to being a divisor, of a measurement division, that would divide the new reference unit, $B$, $n$-times exactly ($1\div A=n$), and would thus be one $n$-th as big as the new reference unit ($A=1/n \times 1$).

Despite the apparent complexity of the reasoning just described, it seems to have been readily available to Pedro. For us, this is unsurprising given his prior instructional experiences in the remedial teaching sessions. As we explain above, the image of a unit fraction Pedro was purposefully supported to develop was not the typical one used in initial fraction instruction—namely, that of the size of a part of an equally partitioned whole (Cortina et al., 2014a). Instead, it was that of the length of an object (a plastic drinking straw) that would exactly fit a whole number of times into the length of a rather arbitrarily defined reference unit of measure (a wooden stick; see Figure 1). Such an image is consistent with regarding a unit fraction as a divisor, in a measurement division, that divides a reference unit a whole number of times, with no remainder. Once Pedro conceived the originally iterated magnitude value $A$ as having the value of $1/n$, it seems to have also been rather easy for him to reinterpret the iterations of this magnitude value as being iterations of a unit fraction. Thus, he could then soundly construe the iterations of the original magnitude value as iterations of one $n$-th of a new reference unit.

The specialized literature is abundant with descriptions of how students struggle to make sense of fraction-related ideas, as well as with evidence that shows that very few children develop sound understandings of fraction notions that they are expected to master during their elementary education. By and large, fractions have been portrayed in the literature as conceptually complex, and making sense of them as quite difficult for most students. Elsewhere (Cortina, Visnovska, & Zúñiga, 2014b) we have advanced the conjecture that much of the documented difficulties in fraction learning can be regarded as a function of the trajectories that instructional designers, teachers, and researchers have expected students to follow in learning fractions. More specifically, we have conjectured that those difficulties can be a function of expecting students to develop an initial and basic quantitative meaning of a unit fraction as the size of a part of an equally partitioned whole.

Pedro's case illustrates how helping students to make sense of unitary fractions in an alternative way can make a difference. For Pedro, this alternative was very helpful when he came to reason quantitatively about the complex and challenging notion (Ramful, 2013) of reciprocal relations of relative size. Whether the proposed instructional sequence on fractions as measures would result in trajectories for students' fractions learning that do not run into the detours of traditional difficulties is the focus of our ongoing research agenda.

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