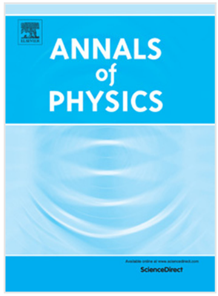
Accepted Manuscript



Hidden $sl(2)$-algebraic structure in Rabi model and its 2-photon and two-mode generalizations

Yao-Zhong Zhang

PII: S0003-4916(16)30236-6
DOI: http://dx.doi.org/10.1016/j.aop.2016.10.019
Reference: YAPHY 67237

To appear in: *Annals of Physics*

Received date: 5 September 2016
Accepted date: 17 October 2016

Please cite this article as: Y.-Z. Zhang, Hidden $sl(2)$-algebraic structure in Rabi model and its 2-photon and two-mode generalizations, *Annals of Physics* (2016), http://dx.doi.org/10.1016/j.aop.2016.10.019

Hidden $sl(2)$-algebraic structure in Rabi model and its 2-photon and two-mode generalizations

Yao-Zhong Zhang

School of Mathematics and Physics, The University of Queensland, Brisbane, Qld 4072, Australia
CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

Abstract

It is shown that the (driven) quantum Rabi model and its 2-photon and 2-mode generalizations possess a hidden $sl(2)$-algebraic structure which explains the origin of the quasi-exact solvability of these models. It manifests the first appearance of a hidden algebraic structure in quantum spin-boson systems without $U(1)$ symmetry.

PACS numbers: 03.65.Fd, 03.65.Ge, 02.30.Ik

1 Introduction

Quantum Rabi model and its multi-quantum and multi-mode generalizations constitute an important class of spin-boson systems without $U(1)$ symmetry. They are phenomenological or theoretical systems used to model the ubiquitous matter-light interactions in modern physics, and have applications in a variety of physical fields, including quantum optics [1], cavity and circuit quantum electrodynamics [2, 3], solid state semiconductor systems [4] and trapped ions [5].

The main difficulty in dealing with these models comes from the fact that not all their spectra seem algebraically accessible. Majority parts of the spectra (i.e. the so-called regular energies) are given by the zeros of transcendental functions which are either infinite power series or continued fractions with coefficients satisfying three-term recurrence relations [6–16]. The exact locations of the zeros and thus closed-form expressions for the regular energies can not be determined via algebraic means.

It is well-known that under certain circumstances the Rabi model and its 2-photon and 2-mode generalizations admit exact, analytic solutions [17–24], yielding closed-form expressions for parts of the energy spectra of the systems. These "exceptional" energies appear only when the model parameters satisfy some constraints. Thus the Rabi model and its 2-photon and 2-mode generalizations are quasi-exactly solvable [10, 21].

Quasi-exactly solvable systems are quantum mechanical problems for which only a finite part of their spectra can be found exactly [25–28]. They occupy an intermediate place between exactly solvable and non-solvable models. A typical feature of quasi-exact solvability is the existence of a hidden algebraic structure. The main purpose of this paper is to show that the Rabi model and its 2-photon and 2-mode generalizations possess a hidden $sl(2)$ structure, i.e. they allow for an $sl(2)$ algebraization. To our knowledge, this marks the first appearance of a hidden algebraic structure in quantum spin-boson models which do not have $U(1)$ symmetry.

2 General results

In this section we recall a general algebraic construction of quasi-exactly solvable differential equations [26], and prove that the 2nd-order differential operator (2.2) below has a hidden $sl(2)$ structure if its coefficients are algebraically dependent.

Let us take the $sl(2)$ algebra realized by the 1st-order differential operators in single variable z

$$J^+ = z^2\frac{d}{dz} - nz, \qquad J^0 = z\frac{d}{dz} - \frac{n}{2}, \qquad J^- = \frac{d}{dz}. \tag{2.1}$$

These differential operators satisfy the $sl(2)$ commutation relations for any value of the parameter n. If n is a non-negative integer, $n = 0, 1, 2, \cdots$, then (2.1) provide a $(n+1)$-dimensional irreducible representation $\mathcal{P}_{n+1}(z) = \text{span}\{1, z, z^2, \cdots, z^n\}$ of the $sl(2)$ algebra. It is evident that any differential operator which is a polynomial of the $sl(2)$ generators (2.1) with n being non-negative integer will have the space $\mathcal{P}_{n+1}(z)$ as its invariant subspace, i.e. possesses $(n+1)$ eigen-functions in the form of polynomial in z of degree n. This is the main idea in [26] behind quasi-exact solvability of a differential operator. Such differential operators are said to have a hidden $sl(2)$ algebraic structure or allow for an $sl(2)$ algebraization.

Now consider the 2nd order differential operator of the form

$$\mathcal{H} = X(z)\frac{d^2}{dz^2} + Y(z)\frac{d}{dz} + Z(z), \tag{2.2}$$

where $X(z), Y(z), Z(z)$ are polynomials of degree at most 4, 3, 2 respectively,

$$X(z) = \sum_{k=0}^{4} a_k z^k, \qquad Y(z) = \sum_{k=0}^{3} b_k z^k, \qquad Z(z) = \sum_{k=0}^{2} c_k z^k.$$

The differential operator (2.2) is usually called the Heun operator. Then we have

Proposition 2.1 *The differential operator $\mathcal{H}$ allows for an $sl(2)$ algebraization, i.e. has a hidden $sl(2)$ algebraic structure, if and only if*

$$b_3 = -2(n-1)a_4, \qquad c_2 = n(n-1)a_4, \qquad c_1 = -n[(n-1)a_3 + b_2]. \tag{2.3}$$

Proof. It suffices to prove that $\mathcal{H}$ is a quadratic combination of the $sl(2)$ generators (2.1) if and only if the relations (2.3) are satisfied.

Sufficiency. We have

$$\begin{aligned}\mathcal{H} &= X(z)\frac{d^2}{dz^2} + \left[-2(n-1)a_4z^3 + b_2z^2 + b_1z + b_0\right]\frac{d}{dz} \\ &\quad + n(n-1)a_4z^2 - n[(n-1)a_3 + b_2]z + c_0. \end{aligned} \tag{2.4}$$

It is easy to check that

$$\begin{aligned} & a_4J^+J^+ + a_3J^+J^0 + a_2J^0J^0 + a_1J^0J^- + a_0J^-J^- \\ &\quad = X(z)\frac{d^2}{dz^2} + \left[-2(n-1)a_4z^3 - \frac{3n-2}{2}a_3z^2 - (n-1)a_2z - \frac{n}{2}a_1\right]\frac{d}{dz} \\ &\qquad + n(n-1)a_4z^2 + \frac{n^2}{2}a_3z + \frac{n^2}{2}a_2, \\ & b_2J^+ + b_1J^0 + b_0J^- = (b_2z^2 + b_1z + b_0)\frac{d}{dz} - nb_2z - \frac{n}{2}b_1. \end{aligned} \tag{2.5}$$

Substituting into (2.4) gives rise to

$$\begin{aligned}\mathcal{H} &= a_4J^+J^+ + a_3J^+J^0 + a_2J^0J^0 + a_1J^0J^- + a_0J^-J^- + \left(\frac{3n-2}{2}a_3 + b_2\right)J^+ \\ &\quad + [(n-1)a_2 + b_1]J^0 + \left(\frac{n}{2}a_1 + b_0\right)J^- + \frac{n}{2}\left[\left(\frac{n}{2} - 1\right)a_2 + b_1\right] + c_0 \end{aligned} \tag{2.6}$$

Necessity. We take

$$\begin{aligned}\mathcal{H} &= A_{++}J^+J^+ + A_{+0}J^+J^0 + A_{00}J^0J^0 + A_{0-}J^0J^- \\ &\quad + A_{--}J^-J^- + A_+J^+ + A_0J^0 + A_-J^- + A_*, \end{aligned} \tag{2.7}$$

where A_{++} etc are constant coefficients to be determined. Then by means of the expressions (2.1),

$$\begin{aligned}\mathcal{H} &= \left(A_{++}z^4 + A_{+0}z^3 + A_{00}z^2 + A_{0-}z + A_{--}\right)\frac{d^2}{dz^2} + \Big[-2(n-1)A_{++}z^3 \\ &\quad + \left(A_+ - \frac{3n-2}{2}A_{+0}\right)z^2 + \left(A_0 - (n-1)A_{00}\right)z + A_- - \frac{n}{2}\Big]\frac{d}{dz} \\ &\quad + n(n-1)A_{++}z^2 + n\left(\frac{n}{2}A_{+0} - A_+\right)z + \frac{n}{2}\left(\frac{n}{2} - A_0\right) + A_*. \end{aligned} \tag{2.8}$$

The r.h.s. of (2.8) can be written as $X(z)\frac{d^2}{dz^2} + Y(z)\frac{d}{dz} + Z(z)$ provided that we make the identification

$$\begin{aligned} & a_4 = A_{++}, \quad a_3 = A_{+0}, \quad a_2 = A_{00}, \quad a_1 = A_{0-}, \quad a_0 = A_{--}, \\ & b_3 = -2(n-1)A_{++}, \quad b_2 = A_+ - \frac{3n-2}{2}A_{+0}, \quad b_1 = A_0 - (n-1)A_{00}, \\ & b_0 = A_- - \frac{n}{2}, \quad c_2 = n(n-1)A_{++}, \quad c_1 = n\left(\frac{n}{2}A_{+0} - A_+\right), \\ & c_0 = \frac{n}{2}\left(\frac{n}{2} - A_0\right) + A_*. \end{aligned} \tag{2.9}$$

It follows that

$$b_3 = -2(n-1)a_4, \quad c_2 = n(n-1)a_4, \quad c_1 = -n[(n-1)a_3 + b_2]. \tag{2.10}$$

This completes our proof. □

In the following sections, we will apply the general results in Proposition 2.1 to show that the (driven) Rabi model and its 2-photon and 2-mode generalizations possess a hidden $sl(2)$ algebraic structure.

3 Hidden $sl(2)$ structure in (driven) Rabi model

The Hamiltonian of the driven Rabi model is

$$H_R = \omega a^\dagger a + \Delta\sigma_z + g\,\sigma_x\left[a^\dagger + a\right] + \delta\,\sigma_x, \tag{3.1}$$

where g is the interaction strength, σ_z, σ_x are the Pauli matrices describing the two atomic levels separated by energy difference 2Δ, and $a^\dagger$ (a) are creation (annihilation) operators of a boson mode with frequency ω. Here $a^\dagger$ (a) satisfy the Heisenberg algebra relations $[a, a^\dagger] = 1, \quad [a, a] = 0 = a^\dagger, a^\dagger]$. The addition of the driving term $\delta\sigma_x$ breaks the Z_2 symmetry of the Rabi model. The driven Rabi model (3.1) is relevant to the description of some hybrid mechanical systems (see e.g. [13]).

By means of the Fock-Bargmann correspondence $a^\dagger \to z,\ a \to \frac{d}{dz}$, the Hamiltonian becomes a matrix differential operator

$$H_R = \omega z\frac{d}{dz} + \Delta\sigma_z + g\,\sigma_x\left(z + \frac{d}{dz}\right) + \delta\,\sigma_x. \tag{3.2}$$

Working in a representation defined by σ_x diagonal and in terms of the two-component wavefunction $\psi(z) = \begin{pmatrix}\psi_+(z)\\ \psi_-(z)\end{pmatrix}$, the time-independent Schrödinger equation $H_R\psi(z) = E\psi(z)$ gives rise to a coupled system of two 1st-order differential equations

$$\begin{aligned}
&(\omega z + g)\frac{d}{dz}\psi_+(z) + [gz - (E-\delta)]\psi_+(z) + \Delta\psi_-(z) = 0,\\
&(\omega z - g)\frac{d}{dz}\psi_-(z) - [gz + (E+\delta)]\psi_-(z) + \Delta\psi_+(z) = 0.
\end{aligned} \tag{3.3}$$

If $\Delta = 0$ these two equations decouple and reduce to the differential equations of two uncoupled displaced harmonic oscillators [29]. For this reason we will concentrate on the non-trivial $\Delta \neq 0$ case.

With the substitution $\psi_\pm(z) = e^{-gz/\omega}\phi_\pm(z)$, it follows

$$\begin{aligned}
&\left[(\omega z + g)\frac{d}{dz} - \left(\frac{g^2}{\omega} - \delta + E\right)\right]\phi_+(z) = -\Delta\phi_-(z),\\
&\left[(\omega z - g)\frac{d}{dz} - \left(2gz - \frac{g^2}{\omega} + \delta + E\right)\right]\phi_-(z) = -\Delta\phi_+(z).
\end{aligned} \tag{3.4}$$

Eliminating $\phi_-(z)$ from the system we obtain the uncoupled differential equation for $\phi_+(z)$,

$$\mathcal{H}_R\phi_+(z) = \Delta^2\phi_+(z), \tag{3.5}$$

where

$$\begin{aligned}\mathcal{H}_R &= (\omega z - g)(\omega z + g)\frac{d^2}{dz^2} + \big[-2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z - g\omega \\ &\quad +2g\left(\frac{g^2}{\omega} - \delta\right)\Big]\frac{d}{dz} + 2g\left(\frac{g^2}{\omega} - \delta + E\right)z + E^2 - \left(\delta - \frac{g^2}{\omega}\right)^2.\end{aligned} \tag{3.6}$$

By Proposition (2.1), $\mathcal{H}_R$ allows for an $sl(2)$ algebraization if

$$2g\left(E + \frac{g^2}{\omega} - \delta\right) \equiv c_1 = -n[(n-1)a_3 + b_2] \equiv 2g\omega n, \tag{3.7}$$

which gives one set of the exact (exceptional) energies of the driven Rabi model

$$E = \omega n + \delta - \frac{g^2}{\omega}, \qquad n = 0, 1, 2, \cdots. \tag{3.8}$$

Indeed, for such E values, $\mathcal{H}_R$ is dependent on the integer parameter n and can be expressed as the quadratic combination of the $sl(2)$ generators (2.1)

$$\begin{aligned}\mathcal{H}_R &= \omega^2 J^0 J^0 - g^2 J^- J^- - 2g\omega J^+ + (n\omega^2 - 2g^2 - 2\omega E)J^0 \\ &\quad -g\left[\omega + 2\left(\delta - \frac{g^2}{\omega}\right)\right]J^- + n\left(\frac{n}{4}\omega^2 - g - \omega E\right) + E^2 - \left(\delta - \frac{g^2}{\omega}\right)^2,\end{aligned} \tag{3.9}$$

where E is given by (3.8).

Similarly for the other set of solutions of the driven Rabi model, we set $\psi_\pm(z) = e^{gz/\omega}\varphi_\pm(z)$ and get from (3.3)

$$\begin{aligned}&\left[(\omega z + g)\frac{d}{dz} + \left(2gz + \frac{g^2}{\omega} + \delta - E\right)\right]\varphi_+(z) = -\Delta\varphi_-(z), \\ &\left[(\omega z - g)\frac{d}{dz} - \left(\frac{g^2}{\omega} + \delta + E\right)\right]\varphi_-(z) = -\Delta\varphi_+(z).\end{aligned} \tag{3.10}$$

Eliminating $\varphi_+(z)$ from the system we obtain the uncoupled differential equation for $\varphi_-(z)$,

$$\tilde{\mathcal{H}}_R\varphi_-(z) = \Delta^2\varphi_-(z), \tag{3.11}$$

where

$$\begin{aligned}\tilde{\mathcal{H}}_R &= (\omega z - g)(\omega z + g)\frac{d^2}{dz^2} + \big[2\omega g z^2 + (\omega^2 - 2g^2 - 2E\omega)z + g\omega \\ &\quad -2g\left(\frac{g^2}{\omega} + \delta\right)\Big]\frac{d}{dz} + 2g\left(\frac{g^2}{\omega} + \delta + E\right)z + E^2 - \left(\delta + \frac{g^2}{\omega}\right)^2.\end{aligned} \tag{3.12}$$

$\tilde{\mathcal{H}}_R$ allows for an $sl(2)$ algebraization if

$$-2g\left(E+\frac{g^2}{\omega}+\delta\right)\equiv c_1=-n[(n-1)a_3+b_2]\equiv -2g\omega n, \tag{3.13}$$

which gives the other set of the exact (exceptional) energies of the driven Rabi model

$$E=\omega n-\delta-\frac{g^2}{\omega}, \qquad n=0,1,2,\cdots. \tag{3.14}$$

For such E values, $\tilde{\mathcal{H}}_R$ is dependent on the integer parameter n and can be expressed as the quadratic combination of the $sl(2)$ generators (2.1)

$$\begin{aligned}\tilde{\mathcal{H}}_R &= \omega^2 J^0 J^0 - g^2 J^- J^- + 2g\omega J^+ + (n\omega^2 - 2g^2 - 2\omega E)J^0 \\ &\quad + g\left[\omega - 2\left(\delta+\frac{g^2}{\omega}\right)\right]J^- + n\left(\frac{n}{4}\omega^2 - g - \omega E\right) + E^2 - \left(\delta+\frac{g^2}{\omega}\right)^2, \end{aligned} \tag{3.15}$$

where E is given by (3.14).

Some remarks are in order. Exceptional energies (3.8) and (3.14) coincide with those obtained by other approaches (see e.g. the appendix of [11], and [13, 24]). The $sl(2)$ algebraizations (3.9) and (3.15) mean that the corresponding spectral problems (3.5) and (3.11) possess $(n+1)$ eigenfunctionns, respectively, in the form of polynomials of degree n. Other eigenfunctions are non-polynomial and are in general given by infinite power series with coefficients satisfying three-term recurrence relations [6, 7, 9, 11, 13].

4 Hidden $sl(2)$ algebraic structure in 2-photon Rabi model

The Hamiltonian of the 2-photon Rabi model reads

$$H_{2-p}=\omega a^\dagger a+\Delta\sigma_z+g\,\sigma_x\left[(a^\dagger)^2+a^2\right], \tag{4.1}$$

where g is the interaction strength. Introduce the operators $K_\pm, K_0$ by

$$K_+=\frac{1}{2}(a^\dagger)^2, \quad K_-=\frac{1}{2}a^2, \quad K_0=\frac{1}{2}\left(a^\dagger a+\frac{1}{2}\right). \tag{4.2}$$

Then the Hamiltonian (4.1) becomes

$$H_{2-p}=2\omega\left(K_0-\frac{1}{4}\right)+\Delta\sigma_z+2g\,\sigma_x(K_++K_-). \tag{4.3}$$

The operators $K_\pm, K_0$ form the usual $su(1,1)$ Lie algebra,

$$[K_0,K_\pm]=\pm K_\pm, \qquad [K_+,K_-]=-2K_0. \tag{4.4}$$

The quadratic Casimir operator C of the algebra is given by

$$C = K_+ K_- - K_0(K_0 - 1). \tag{4.5}$$

Consider the infinite-dimensional unitary irreducible representation of $su(1,1)$ known as the positive discrete series $\mathcal{D}^+(q)$, where the parameter q is the so-called Bargmann index. In this representation the basis states $\{|q,n\rangle\}$ diagonalize the operator K_0,

$$K_0|q,n\rangle = (n+q)|q,n\rangle \tag{4.6}$$

for $q > 0$ and $n = 0, 1, 2, \cdots$, and the Casimir operator C has the eigenvalue $q(1-q)$. The operators K_+ and K_- are hermitian to each other and act as raising and lowering operators respectively within $\mathcal{D}^+(q)$,

$$\begin{aligned} K_+|q,n\rangle &= \sqrt{(n+1)(n+2q)}\;|q,n+1\rangle, \\ K_-|q,n\rangle &= \sqrt{n(n+2q-1)}\;|q,n-1\rangle. \end{aligned} \tag{4.7}$$

It is well-known that the single-mode bosonic realization (4.2) provides a representation of $\mathcal{D}^+(q)$ with $C = \frac{3}{16}$ and $q = \frac{1}{4}, \frac{3}{4}$.

By means of the Fock-Bargmann correspondence the operators $K_\pm, K_0$ (4.2) are realized by single-variable 2nd-order differential operators

$$K_0 = z\frac{d}{dz} + q, \qquad K_+ = \frac{z}{2}, \qquad K_- = 2z\frac{d^2}{dz^2} + 4q\frac{d}{dz}, \tag{4.8}$$

and the 2-photon Rabi Hamiltonian becomes [21]

$$H_{2-p} = 2\omega\left(z\frac{d}{dz} + q - \frac{1}{4}\right) + \Delta\sigma_z + 2g\,\sigma_x\left(\frac{z}{2} + 2z\frac{d^2}{dz^2} + 4q\frac{d}{dz}\right). \tag{4.9}$$

Working in a representation defined by σ_x diagonal and in terms of the two component wavefunction, $\psi(z) = \begin{pmatrix} \psi_+(z) \\ \psi_-(z) \end{pmatrix}$, the time-independent Schrödinger equation $H_{2-p}\psi(z) = E\psi(z)$ leads to two coupled 2nd-order differential equations,

$$\begin{aligned} &4gz\frac{d^2}{dz^2}\psi_+(z) + (2\omega z + 8gq)\frac{d}{dz}\psi_+(z) + \left[gz + 2\omega\left(q - \frac{1}{4}\right) - E\right]\psi_+(z) + \Delta\psi_-(z) = 0, \\ &4gz\frac{d^2}{dz^2}\psi_-(z) + (-2\omega z + 8gq)\frac{d}{dz}\psi_-(z) + \left[gz - 2\omega\left(q - \frac{1}{4}\right) + E\right]\psi_-(z) - \Delta\psi_+(z) = 0. \end{aligned} \tag{4.10}$$

If $\Delta = 0$ these equations reduce to the differential equations of two uncoupled single-mode squeezed harmonic oscillators [29]. In the following we will concentrate on the $\Delta \neq 0$ case.

With the substitution

$$\psi_\pm(z) = e^{-\frac{\omega}{4g}(1-\Omega)z}\varphi_\pm(z), \qquad \Omega = \sqrt{1 - \frac{4g^2}{\omega^2}}, \tag{4.11}$$

where $\left|\frac{2g}{\omega}\right| < 1$, it follows [21]

$$\left\{4gz\frac{d^2}{dz^2} + [2\omega\Omega z + 8gq]\frac{d}{dz} + 2q\omega\Omega - \frac{1}{2}\omega - E\right\}\varphi_+ = -\Delta\varphi_-,$$
$$\left\{4gz\frac{d^2}{dz^2} + [2\omega(\Omega-2)z + 8gq]\frac{d}{dz} + \frac{\omega^2}{g}(1-\Omega)z + 2q\omega(\Omega-2) + \frac{1}{2}\omega + E\right\}\varphi_- = \Delta\varphi_+. \quad (4.12)$$

Eliminating $\varphi_-(z)$ from the system, we obtain the 4th-order differential equation for $\varphi_+(z)$

$$\mathcal{H}_{2-p}\varphi_+(z) = -\Delta^2\varphi_+(z), \quad (4.13)$$

where

$$\begin{aligned}\mathcal{H}_{2-p} \;=\;& 16g^2z^2\frac{d^4}{dz^4} + 64g^2\left[\frac{\omega}{4g}(\Omega-1)z^2 + \left(q+\frac{1}{2}\right)z\right]\frac{d^3}{dz^3}\\ &+\left\{4\omega^2(\Omega^2-3\Omega+1)z^2 + 16\omega g\left[3\left(q+\frac{1}{2}\right)\Omega - 3q - 1\right]z + 64g^2q\left(q+\frac{1}{2}\right)\right\}\frac{d^2}{dz^2}\\ &+\left\{2\frac{\omega^3}{g}\Omega(1-\Omega)z^2 + \left[8\omega^2q(1-\Omega) + 8\omega^2\left(q+\frac{1}{2}\right)(1-\Omega)^2\right.\right.\\ &\qquad\left.\left.+4\omega\left(E - 2\omega\left(q+\frac{1}{4}\right)\right)\right]z + 32\omega gq\left[\left(q+\frac{1}{2}\right)\Omega - q\right]\right\}\frac{d}{dz}\\ &+\frac{\omega^2}{g}(1-\Omega)\left(2q\omega\Omega - \frac{1}{2}\omega - E\right)z + 4\omega^2q^2(1-\Omega)^2 - \left[E - 2\omega\left(q-\frac{1}{4}\right)\right]^2. \quad (4.14)\end{aligned}$$

Using the identities

$$z^2\frac{d^4}{dz^4} = J^+(J^-)^3 + nz\frac{d^3}{dz^3}, \qquad z^2\frac{d^3}{dz^3} = J^+(J^-)^2 + nz\frac{d^2}{dz^2}, \qquad z\frac{d^3}{dz^3} = J^0(J^-)^2 + \frac{n}{2}\frac{d^2}{dz^2}, \quad (4.15)$$

we obtain

$$\mathcal{H}_{2-p} = 16g^2J^+(J^-)^3 + 16g\omega(\Omega-1)J^+(J^-)^2 + 16g^2[n+2(2q+1)]J^0(J^-)^2 + \mathcal{H}^{(2)}_{2-p}, \quad (4.16)$$

where

$$\begin{aligned}\mathcal{H}^{(2)}_{2-p} \;=\;& \left\{4\omega^2(\Omega^2-3\Omega+1)z^2 + 16\omega g\left[(\Omega-1)n + 3\left(q+\frac{1}{2}\right)\Omega - 3q - 1\right]z\right.\\ &\left.+8g^2n\left[n + 4\left(q+\frac{1}{2}\right)\right] + 64g^2q\left(q+\frac{1}{2}\right)\right\}\frac{d^2}{dz^2}\\ &+\left\{2\frac{\omega^3}{g}\Omega(1-\Omega)z^2 + \left[8\omega^2q(1-\Omega) + 8\omega^2\left(q+\frac{1}{2}\right)(1-\Omega)^2\right.\right.\\ &\qquad\left.\left.+4\omega\left(E - 2\omega\left(q+\frac{1}{4}\right)\right)\right]z + 32\omega gq\left[\left(q+\frac{1}{2}\right)\Omega - q\right]\right\}\frac{d}{dz}\\ &+\frac{\omega^2}{g}(1-\Omega)\left(2q\omega\Omega - \frac{1}{2}\omega - E\right)z + 4\omega^2q^2(1-\Omega)^2 - \left[E - 2\omega\left(q-\frac{1}{4}\right)\right]^2.\end{aligned} \quad (4.17)$$

$\mathcal{H}^{(2)}_{2-p}$ allows for an $sl(2)$ algebraization if

$$\begin{aligned}\frac{\omega^2}{g}(1-\Omega)\left(2q\omega\Omega-\frac{1}{2}\omega-E\right) &\equiv c_1=-n[(n-1)a_3+b_2]\\ &\equiv -2\frac{\omega^2}{g}(1-\Omega)\Omega n,\end{aligned} \tag{4.18}$$

which, for $\Omega\neq 1$ (the $\Omega=1$ case is trivial as it corresponds to $g=0$), gives the exact (exceptional) energies of the 2-photon Rabi model

$$E=-\frac{1}{2}\omega+\left[2n+2\left(q-\frac{1}{4}\right)+\frac{1}{2}\right]\omega\Omega,\qquad n=0,1,2,\cdots. \tag{4.19}$$

Indeed for such E values $\mathcal{H}_{2-p}$ depends on the integer parameter n and can be expressed as the quartic combination of the $sl(2)$ generators (2.1)

$$\begin{aligned}\mathcal{H}_{2-p} \;=\;& 16g^2J^+(J^-)^3+16g\omega(\Omega-1)J^+(J^-)^2+16g^2[n+2(2q+1)]J^0(J^-)^2\\ &+4\omega^2(\Omega^2-3\Omega+1)J^0J^0+16\omega g\left[(\Omega-1)n+3\left(q+\frac{1}{2}\right)\Omega-3q-1\right]J^0J^-\\ &\left[+8g^2n\left(n+4(q+\frac{1}{2})\right)+64g^2q\left(q+\frac{1}{2}\right)\right]J^-J^-+2\frac{\omega^3}{g}\Omega(1-\Omega)J^+\\ &+\big[4\omega^2(n-1)(\Omega^2-3\Omega+1)+8\omega^2q(1-\Omega)\\ &\quad+8\omega^2\left(q+\frac{1}{2}\right)(1-\Omega)^2+4\omega\left(E-2\omega(q+\frac{1}{4})\right)\bigg]J^0\\ &\left[+8g\omega n\left((\Omega-1)n+3(q+\frac{1}{2})\Omega-3q-1\right)+32\omega gq\left((q+\frac{1}{2})\Omega-q\right)\right]J^-\\ &+n(n-2)\omega^2(\Omega^2-3\Omega+1)+4n\omega^2q(1-\Omega)+4n\omega^2\left(q+\frac{1}{2}\right)(1-\Omega)^2\\ &+\omega n\left[E-2\omega\left(q+\frac{1}{4}\right)\right]+4\omega^2q^2(1-\Omega)^2-\left[E-2\omega\left(q-\frac{1}{4}\right)\right]^2.\end{aligned} \tag{4.20}$$

Here E is given by (4.19). This $sl(2)$ algebraization demonstrates that for each energy value E in (4.19) the 2-photon Rabi model has a hidden $sl(2)$ algebraic structure.

Notice that the exceptional energies (4.19) coincide with those obtained in [19,21] via different methods. It is clear that the corresponding spectral problem (4.13) has $n+1$ polynomial eigenfunctions in z of degree n. Other eigenfunctions are non-polynomial and can not be obtained in closed analytic form. We remark that as shown in [14] when $\Omega=0$, i.e. $|2g/\omega|=1$, the 2-photon Rabi model has no entire solutions.

5 Hidden $sl(2)$ algebraic structure in two-mode Rabi model

The Hamiltonian of the two-mode quantum Rabi model reads [21]

$$H_{2-m}=\omega(a_1^\dagger a_1+a_2^\dagger a_2)+\Delta\sigma_z+g\,\sigma_x(a_1^\dagger a_2^\dagger+a_1a_2), \tag{5.1}$$

where we assume that the boson modes are degenerate with the same frequency ω and g is the coupling constant. Introduce the operators $K_\pm, K_0$,

$$K_+ = a_1^\dagger a_2^\dagger, \quad K_- = a_1 a_2, \quad K_0 = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1). \tag{5.2}$$

Then the Hamiltonian (5.1) becomes [21]

$$H_{2-m} = 2\omega\left(K_0 - \frac{1}{2}\right) + \Delta\sigma_z + g\sigma_x(K_+ + K_-). \tag{5.3}$$

The operators $K_\pm, K_0$ form the $su(1,1)$ algebra (4.4). As in the previous section we shall use the unitary irreducible representation (i.e. the positive discrete series). However, to avoid confusion in this section we shall use κ to denote the Bargmann index of the representation. Using this notation the action of the operators $K_\pm, K_0$ and the Casimir C (4.5) on the basis states $|\kappa, n\rangle$ of the representation reads

$$\begin{aligned} K_0|\kappa,n\rangle &= (n+\kappa)\,|\kappa,n\rangle, \\ K_+|\kappa,n\rangle &= \sqrt{(n+2\kappa)(n+1)}\,|\kappa,n+1\rangle, \\ K_-|\kappa,n\rangle &= \sqrt{(n+2\kappa-1)n}\,|\kappa,n-1\rangle, \\ C|\kappa,n\rangle &= \kappa(1-\kappa)\,|\kappa,n\rangle, \end{aligned} \tag{5.4}$$

for $\kappa > 0$ and $n = 0, 1, 2, \cdots$. For the two-mode bosonic realization (5.2) of $su(1,1)$ that we require here the Casimir C takes the value $C = \kappa(1-\kappa)$ with the Bargmann index κ being any positive integers or half-integers, i.e. $\kappa = 1/2, 1, 3/2, \cdots$.

Using the Fock-Bargmann correspondence the operators $K_\pm, K_0$ (5.2) have the single-variable differential realization [29],

$$K_0 = z\frac{d}{dz} + \kappa, \quad K_+ = z, \quad K_- = z\frac{d^2}{dz^2} + 2\kappa\frac{d}{dz} \tag{5.5}$$

and the Hamiltonian (5.3) can be expressed as the matrix differential operator [21]

$$H_{2-m} = 2\omega\left(z\frac{d}{dz} + \kappa - \frac{1}{2}\right) + \Delta\sigma_z + g\,\sigma_x\left(z + z\frac{d^2}{dz^2} + 2\kappa\frac{d}{dz}\right). \tag{5.6}$$

Working in a representation defined by σ_x diagonal and in terms of the two-component wavefunction $\psi(z) = \begin{pmatrix} \psi_+(z) \\ \psi_-(z) \end{pmatrix}$, we see that the time-independent Schrödinger equation $H_{2-m}\psi(z) = E\psi(z)$ yields the two coupled differential equations,

$$\begin{aligned} &gz\frac{d^2}{dz^2}\psi_+(z) + 2(\omega z + g\kappa)\frac{d}{dz}\psi_+(z) + \left[gz + 2\omega\left(\kappa - \frac{1}{2}\right) - E\right]\psi_+(z) + \Delta\psi_-(z) = 0, \\ &gz\frac{d^2}{dz^2}\psi_-(z) + 2(-\omega z + g\kappa)\frac{d}{dz}\psi_-(z) + \left[gz - 2\omega\left(\kappa - \frac{1}{2}\right) + E\right]\psi_-(z) - \Delta\psi_+(z) = 0. \end{aligned} \tag{5.7}$$

If $\Delta = 0$ these reduce to the differential equations of two uncoupled two-mode squeezed harmonic oscillators [21]. So in the following we will concentrate on the $\Delta \neq 0$ case.

With the substitution

$$\psi_\pm(z) = e^{-\frac{\omega}{g}(1-\Lambda)z}\varphi_\pm(z), \qquad \Lambda = \sqrt{1-\frac{g^2}{\omega^2}}, \tag{5.8}$$

where $\left|\frac{g}{\omega}\right| < 1$, it follows [21]

$$\begin{aligned}&\left\{gz\frac{d^2}{dz^2} + 2[\omega\Lambda z + g\kappa]\frac{d}{dz} + 2\kappa\omega\Lambda - \omega - E\right\}\varphi_+ = -\Delta\varphi_-,\\ &\left\{gz\frac{d^2}{dz^2} + 2[\omega(\Lambda-2)z + g\kappa]\frac{d}{dz} + \frac{4\omega^2}{g}(1-\Lambda)z + 2\kappa\omega(\Lambda-2) + \omega + E\right\}\varphi_- = \Delta\varphi_+.\end{aligned} \tag{5.9}$$

Eliminating $\varphi_-(z)$ from the system, we obtain the 4th-order differential equation for $\varphi_+(z)$,

$$\mathcal{H}_{2-m}\varphi_+(z) = -\Delta^2\varphi_+(z), \tag{5.10}$$

where

$$\begin{aligned}\mathcal{H}_{2-m} \;=\;& g^2z^2\frac{d^4}{dz^4} + 4g^2\left[\frac{\omega}{g}(\Lambda-1)z^2 + \left(\kappa+\frac{1}{2}\right)z\right]\frac{d^3}{dz^3}\\ &+\left\{4\omega^2(\Lambda^2-3\Lambda+1)z^2 + 4\omega g\left[3\left(\kappa+\frac{1}{2}\right)\Lambda - 3\kappa - 1\right]z + 4g^2\kappa\left(\kappa+\frac{1}{2}\right)\right\}\frac{d^2}{dz^2}\\ &+\left\{8\frac{\omega^3}{g}\Lambda(1-\Lambda)z^2 + \left[8\omega^2\kappa(1-\Lambda) + 8\omega^2\left(\kappa+\frac{1}{2}\right)(1-\Lambda)^2\right.\right.\\ &\qquad\left.\left. +4\omega(E-2\omega\kappa)\right]z + 8\omega g\kappa\left[\left(\kappa+\frac{1}{2}\right)\Lambda - \kappa\right]\right\}\frac{d}{dz}\\ &+4\frac{\omega^2}{g}(1-\Lambda)\left(2\kappa\omega\Lambda - \omega - E\right)z + 4\omega^2\kappa^2(1-\Lambda)^2 - \left[E - 2\omega\left(\kappa-\frac{1}{2}\right)\right]^2.\end{aligned} \tag{5.11}$$

By means of the identities (4.15) we have

$$\mathcal{H}_{2-m} = gJ^+(J^-)^3 + 4\omega g(\Lambda-1)J^+(J^-)^2 + g^2\left[n + 4(\kappa+\frac{1}{2})\right]J^-(J^-)^2 + \mathcal{H}^{(2)}_{2-m}, \tag{5.12}$$

where

$$\begin{aligned}\mathcal{H}^{(2)}_{2-m} \;=\;& \left\{4\omega^2(\Lambda^2-3\Lambda+1)z^2 + 4\omega g\left[(\Lambda-1)n + 3\left(\kappa+\frac{1}{2}\right)\Lambda - 3\kappa - 1\right]z\right.\\ &\left. +g^2\frac{n}{2}\left[n + 4(\kappa+\frac{1}{2})\right] + 4g^2\kappa\left(\kappa+\frac{1}{2}\right)\right\}\frac{d^2}{dz^2}\\ &+\left\{8\frac{\omega^3}{g}\Lambda(1-\Lambda)z^2 + \left[8\omega^2\kappa(1-\Lambda) + 8\omega^2\left(\kappa+\frac{1}{2}\right)(1-\Lambda)^2\right.\right.\\ &\qquad\left.\left. +4\omega(E-2\omega\kappa)\right]z + 8\omega g\kappa\left[\left(\kappa+\frac{1}{2}\right)\Lambda - \kappa\right]\right\}\frac{d}{dz}\\ &+4\frac{\omega^2}{g}(1-\Lambda)\left(2\kappa\omega\Lambda - \omega - E\right)z + 4\omega^2\kappa^2(1-\Lambda)^2 - \left[E - 2\omega\left(\kappa-\frac{1}{2}\right)\right]^2.\end{aligned} \tag{5.13}$$

Similar to the 2-photon Rabi case, $\mathcal{H}^{(2)}_{2-m}$ allows for an $sl(2)$ algebraization if

$$\begin{aligned} 4\frac{\omega^2}{g}(1-\Lambda)\left(2\kappa\omega\Lambda-\omega-E\right) &\equiv c_1=-n[(n-1)a_3+b_2] \\ &\equiv -8\frac{\omega^3}{g}\Lambda(1-\Lambda)n, \end{aligned} \tag{5.14}$$

which, for $\Lambda \neq 1$ (the $\Lambda = 1$ case is trivial as it corresponds to $g = 0$), give the exceptional energies of the 2-mode Rabi model

$$E=-\omega+\left[2n+2\left(\kappa-\frac{1}{2}\right)+1\right]\omega\Lambda. \tag{5.15}$$

For such E values, $\mathcal{H}_{2-m}$ depends on integer parameter n and possesses an algebraization in terms of the $sl(2)$ generators (2.1)

$$\begin{aligned} \mathcal{H}_{2-m} &= gJ^+(J^-)^3+4\omega g(\Lambda-1)J^+(J^-)^2+g^2\left[n+4\left(\kappa+\frac{1}{2}\right)\right]J^-(J^-)^2 \\ &+4\omega^2(\Lambda^2-3\Lambda+1)J^0J^0+4\omega g\left[(\Lambda-1)n+3\left(\kappa+\frac{1}{2}\right)\Lambda-3\kappa-1\right]J^0J^- \\ &+\left\{g^2\frac{n}{2}\left[n+4\left(\kappa+\frac{1}{2}\right)\right]+4g^2\kappa\left(\kappa+\frac{1}{2}\right)\right\}J^-J^-+8\frac{\omega^3}{g}\Lambda(1-\Lambda)J^+ \\ &+\Big[4\omega^2(n-1)(\Lambda^2-3\Lambda+1)+8\omega^2\kappa(1-\Lambda) \\ &\quad+8\omega^2\left(\kappa+\frac{1}{2}\right)(1-\Lambda)^2+4\omega(E-2\omega\kappa)\Big]J^0 \\ &+\left\{2n\omega g\left[(\Lambda-1)n+3\left(\kappa+\frac{1}{2}\right)\Lambda-3\kappa-1\right]+8\omega g\kappa\left[\left(\kappa+\frac{1}{2}\right)\Lambda-\kappa\right]\right\}J^- \\ &+n\left(n-2\right)\omega^2(\Lambda^2-3\Lambda+1)+4n\omega^2\kappa(1-\Lambda)+4n\omega^2\left(\kappa+\frac{1}{2}\right)(1-\Lambda)^2 \\ &+2n\omega(E-2\omega\kappa)+4\omega^2\kappa^2(1-\Lambda)^2-\left[E-2\omega\left(\kappa-\frac{1}{2}\right)\right]^2. \end{aligned} \tag{5.16}$$

Here E is given by (5.15). Thus for each energy value E in (5.15) the 2-mode Rabi model has a hidden $sl(2)$ algebraic structure.

We remark that the exceptional energies (5.15) coincide with those presented in [21] via the Bethe ansatz method [30]. The $sl(2)$ algebraization of (5.16) implies that the corresponding spectral problem (5.10) possesses $n+1$ polynomial eigenfunctions of degree n. Other eigenfunctions are non-polynomial and can not be found in closed analytic form. Note that as shown in [14] when $\Lambda = 0$, i.e. $|g/\omega| = 1$, the two-mode Rabi model has no entire solutions.

Acknowledgments: I would like to thank Alexander Turbiner for encouraging me to publish the results in this paper. This work was partially supported by the Australian Research Council through Discovery Projects grant DP140101492.

References

[1] V. Vedral, Modern foundations of quantum optics, Imperial College Press, London, 2006.

[2] D. Englund, A. Faraon, I. Fushman, N. Stoltz, P. Petroff and J. Vucković, Nature **450**, 857 (2007).

[3] T. Niemczyk, F. Deppe, H. Huebl, E.P. Menzel, F. Hocke, M.J. Schwarz, J.J. Garcia-Ripoll, D. Zueco, T. Hümmer, E. Solano, A. Marx and R. Gross, Nature Phys. **6**, 772 (2010).

[4] G. Khitrova, H.M. Gibbs, M. Kira, S.W. Koch and A. Scherer, Nature Phys. **2**, 81 (2006).

[5] D. Leibfried, R. Blatt, C. Monroe and D. Wineland, Rev. Mod. Phys. **75**, 281 (2003).

[6] D. Braak, Phys. Rev. Lett. **107**, 100401 (2011).

[7] A. Moroz, Europhys. Lett. **100**, 60010 (2012).

[8] Q.H. Chen, C. Wang, S. He, T. Liu and K.L. Wang, Phys. Rev. A **86**, 023822 (2012).

[9] H. Zhong, Q. Xie, M.T. Batchelor and C. Lee, J. Phys. A: Math. Theor. **46**, 415302 (2013).

[10] A. Moroz, Ann. Phys. **338**, 319 (2013).

[11] Y.-Z. Zhang, Analytic solutions of 2-photon and two-mode Rabi models, arVix:1304.7827v2 [quant-ph].

[12] A. Moroz, Ann. Phys. **340**, 252 (2014).

[13] H. Zhong, Q. Xie, X.-W. Guan, M.T. Batchelor, K. Gao and C. Lee, J. Phys. A: Math. Theor. **47**, 045301 (2014).

[14] Y.-Z. Zhang, On the two-mode and k-photon quantum Rabi models, arXiv:1507.03863v2 [quant-ph].

[15] L. Duan, S. He, D. Braak and Q.-H. Chen, Europhys. Lett. **112**, 34003 (2015).

[16] L. Duan, Y.-F. Xie, D. Braak and Q.-H. Chen, Two-photon Rabi model: Analytic solutions and spectral collapse, arXiv:1603.04503v1 [quant-ph].

[17] H.G. Reik, H. Nusser and L.A. Ribeiro, J. Phys. A: Math. Gen. **15**, 3431 (1982).

[18] M. Kus, J. Math. Phys. **26**, 2792 (1985).

[19] C. Emary and R.F. Bishop, J. Phys. A: Math. Gen. **35**, 8231 (2002).

[20] C. Emary and R.F. Bishop, J. Math. Phys. **43**, 3916 (2002).

[21] Y.-Z. Zhang, J. Math. Phys. **54**, 102104 (2013).

[22] M. Tomka, O. El Araby, M. Pletyukhov and V. Gritsev, Phys. Rev. A **90**, 063839 (2014).

[23] A.F. Dossa and G.Y.H. Avossevou, J. Math. Phys. **55**, 102104 (2014).

[24] Z.-M. Li and M.T. Batchelor, J. Phys. A: Math. Theor. **48**, 454005 (2015).

[25] A. Turbiner, Comm. Math. Phys. **118**, 467 (1988).

[26] A. Turbiner, Quasi-exactly-solvable differential equations, in CRC Handbook of Lie Group Analysis of Differential Equations, Vol. **3**: New Trends in Theoretical Developments and Computational Methods, Chapter 12, CRC Press, N. Ibragimov (ed.), pp. 331-366 (1995).

[27] A.G. Ushveridze, Quasi-exactly solvable models in quantum mechanics, Institute of Physics Publishing, Bristol, 1994.

[28] A. Gonzárez-López, N. Kamran and P. Olver, Comm. Math. Phys. **153**, 117 (1993).

[29] Y.-Z. Zhang, J. Phys. A: Math. Theor. **46**, 455302 (2013).

[30] Y.-Z. Zhang, J. Phys. A: Math. Theor. **45**, 065206 (2012).

Research Highlights

- Practical condition for 2nd order differential operator to be *sl(2)* algebraic.
- Hidden *sl(2)* algebra structure in Rabi model and its generalizations.
- First hidden algebraic structure in spin-boson systems without *U(1)* symmetry.