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Singular indecomposable representations of sl(2,$\mathbb{C}$) and relativistic wave equations

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A detailed summary is given of the structure of singular indecomposable representations of sl(2,$\mathbb{C}$), as developed by Gel'fand and Ponomarev [Usp. Mat. Nauk **23**, 3 (1968); translated in Russ. Math. Surveys **23**, 1 (1968)]. A variety of four-vector operators Γ_μ is constructed, acting within direct sums of such representations, including some with nonsingular Γ_0. Associated wave equations of Gel'fand–Yaglom type are considered that admit timelike solutions and lead to mass-spin spectra of the Majorana type. A subclass of these equations is characterized in an invariant way by obtaining basis-independent expressions for the commutator and anticommutator of Γ_μ and Γ_ν. A brief discussion is given of possible applications to physics of these equations and of others in which nilpotent scalar operators appear.

I. INTRODUCTION

Many authors have studied first-order linear relativistically invariant wave equations of the type

$$(\Gamma_\mu \partial^\mu + i\kappa)\psi(x) = 0, \tag{1.1}$$

in which the wave function ψ takes its values in a vector space V carrying a representation π of the Lorentz group SL(2,$\mathbb{C}$), the Γ_μ (for $\mu = 0,1,2,3$) and κ are linear operators on V, and $\partial^\mu = \partial/\partial x_\mu$. The first systematic treatment was that of Gel'fand and Yaglom,[1] who gave detailed formulas for the structure of possible Γ_μ in the case where π is a direct sum of irreducible representations of SL(2,$\mathbb{C}$). These representations may or may not be infinite dimensional.

The results of Ref. 1 were obtained in a particular basis for V, but various authors have later emphasized the importance of invariant properties of wave equations. These are the properties that do not depend on the choice of basis in V or on the corresponding explicit form of the Γ_μ and κ. In particular, starting with the early work on wave equations (see, for example, Lubanski[2] and Harish-Chandra[3]), there has been great interest in what we shall refer to as their *algebraic structure*, by which we mean especially the algebras generated by the vector operator Γ_μ. This involves, in particular, a description of the commutator $[\Gamma_\mu,\Gamma_\nu]$ and the anticommutator $\{\Gamma_\mu,\Gamma_\nu\}$. Representations of such algebras lead to entire families of equations, giving us a systematic way of classifying some of the vast number of relativistic wave equations. Lubanski concentrated on the case where the commutator $[\Gamma_\mu,\Gamma_\nu]$ is a nonzero multiple of the generator $J_{\mu\nu}$ of SL(2,$\mathbb{C}$), so that the complex Lie algebra generated by the Γ_μ is just so(5,$\mathbb{C}$). An analysis of the possible Lie algebras generated by the Γ_μ was later carried out by Cant and Hurst,[4] who showed that arbitrarily large simple Lie algebras can be obtained. This contradicted earlier claims that had been made (see, for example, Refs. 5 and 6). It was also shown in Ref. 4 how a knowledge of the Lie algebra can help in deriving the mass and spin spectra associated with Eq. (1.1). Bracken[7] also used algebraic properties to characterize a class of wave equations, and to determine the associated mass and spin spectra, in a study of the family with κ a multiple of the identity operator on V, and

$$\pi = [\tfrac{1}{2},l_1] \oplus [\tfrac{1}{2}, -l_1], \quad l_1 \in \mathbb{C}. \tag{1.2}$$

[We use the standard notation[1] for the irreducible representations of SL(2,$\mathbb{C}$).] In fact, it has been shown[8] that an infinite-dimensional Lie algebra is generated in this case, except when $l_1 = 0$, where we recover one of the Majorana equations, or $2(l_1 - 1) \in \mathbb{N}$, where π is finite dimensional and the Lie algebra generated is sp$(2(l_1)^2 - \tfrac{1}{2})$. A further study of algebraic structure, concentrating on the role of real Lie algebras in wave equations, was carried out by Cant.[9]

Such algebraic properties will, we believe, be especially useful when one considers variations on the classical theme of wave equations based on direct sums of irreducible representations. For example, wave equations of the form (1.1), with the added property that V carries a representation of the larger group $\overline{\mathrm{SL}}(4,R)$, have been studied and applied to the problem of describing the gravitational interactions of hadrons.[10] Such equations are infinite dimensional and SL(2,$\mathbb{C}$) invariant, although not fully SL(4,R) invariant. The results of Ref. 9 were particularly useful in this work. Infinite-dimensional equations associated with representations of SO(4,2) have also been widely discussed in the literature.[11]

In this paper we shall be concerned with a different direction of generalization: we consider the case where π is a direct sum of reducible but *indecomposable* representations of the algebra[12] sl(2,$\mathbb{C}$). All such representations are infinite dimensional, and they can be very complicated objects. The first examples presented were the "expansors" described by Dirac,[13] who more recently emphasized the potential importance of indecomposable representations for physics.[14] A class of indecomposable representations was studied by Gel'fand and Ponomarev,[15] and divided into two subclasses, called singular and nonsingular. Bender and Griffiths,[16] in a study of the transformation properties of massless fields, examined the composition series for the tensor product of the

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four-vector representation of SL(2,ℂ) with an infinite-dimensional irreducible representation, and found that it can contain indecomposable representations. Hlavaty and Niederle[17] applied the results of Ref. 15 in constructing some examples of wave equations based on indecomposable representations. Their work describes the general structure of Γ_μ associated with a representation π, which is a direct sum of nonsingular indecomposable representations, but their results are slight in the more complicated, and possibly more interesting case, when singular indecomposable representations are involved: there they gave only one example of a wave equation (1.1), associated with a direct sum of particularly simple singular indecomposable representations that are in fact operator irreducible. Operators Γ_μ associated with such representations, sometimes called "integer-point" representations in the literature,[18] had been constructed earlier by Ruhl.[19] Hlavaty *et al.* showed for their example that no timelike solutions exist.

A different approach to indecomposable representations of sl(2,ℂ) and associated four-vector operators has been developed by Gruber and his associates.[20] To our knowledge, the relationship of the representations constructed there to those of Gel'fand and Ponomarev[15] has not been fully elucidated.

Our object in the present work is twofold. First, to summarize the results of Gel'fand and Ponomarev on singular indecomposable representations of sl(2,ℂ), in a form more readily accessible to physicists, and second, to give some examples of wave equations based on such representations, especially ones that do admit timelike solutions, unlike the example given in Ref. 17.

We shall show further that a subclass of these equations can be characterized in an invariant way, at least partly, by virtue of the simple form taken by the commutator and anticommutator of Γ_μ and Γ_ν. This subclass is a direct generalization of that considered in Ref. 7, which includes one of Majorana's equations,[21] and indeed the subclass of wave equations we discuss does lead to mass-spin spectra of the Majorana type.

II. SINGULAR INDECOMPOSABLE REPRESENTATIONS OF sl(2,ℂ)

Let $\mathscr{g} \cong \mathrm{sl}(2,\mathbb{C})$ denote the real Lie algebra of the homogeneous Lorentz group, and $\mathscr{k} \cong \mathrm{su}(2)$ its maximal compact subalgebra. We take the standard basis $\{h_1,h_2,h_3,f_1,f_2,f_3\}$ for (the complexification of) $\mathscr{g}$; $\{h_1,h_2,h_3\}$ for $\mathscr{k}$. The defining Lie product relations are

$$[h_p,h_q] = -[f_p,f_q] = i\epsilon_{pqr}h_r,$$
$$[h_p,f_q] = i\epsilon_{pqr}f_r, \tag{2.1}$$

where p,q,r run over 1,2,3, repeated subscripts are summed over those values, and ϵ_{pqr} is the usual alternating symbol. We work with h_3, f_3, $h_\pm = h_1 \pm ih_2$, and $f_\pm = f_1 \pm if_2$ in what follows. A representation τ of $\mathscr{g}$ is said to be $\mathscr{k}$ finite or a *Harish-Chandra representation* if in the direct sum decomposition of the restriction of τ to $\mathscr{k}$, equivalent irreducible representations of $\mathscr{k}$ occur with finite multiplicities only. Thus

$$\tau|_{\mathscr{k}} = \bigoplus_{l\in\mathscr{S}} \tau_l \tag{2.2}$$

where $\mathscr{S}$ is a subset of $\mathbb{N}/2 \equiv \{0,\frac{1}{2},1,\frac{3}{2},...\}$, and each τ_l is the direct sum of a finite number of copies of the $(2l+1)$-dimensional irreducible representation φ_l of $\mathscr{k}$. Such a τ is called *indecomposable* if it cannot be decomposed into a direct sum of representations of $\mathscr{g}$. The indecomposable Harish-Chandra representations of $\mathscr{g}$ have been classified by Gel'fand and Ponomarev,[15] and we now summarize their results.

Let τ be such a representation, V the associated $\mathscr{g}$ module, and $H_3 = \tau(h_3)$, $F_3 = \tau(f_3)$, etc. The Casimir operators are given, as usual,[1] by

$$\Delta_1 = \tfrac{1}{2}(H_-F_+ + F_-H_+) + H_3F_3,$$
$$\Delta_2 = H_-H_+ - F_-F_+$$
$$+ (H_3)^2 - (F_3)^2 + 2H_3, \tag{2.3}$$

and commute with H_3, $H_\pm$, F_3, and $F_\pm$ on V. If τ is in fact *irreducible*, then Δ_1 and Δ_2 are multiples of the identity operator on V. This is a necessary but not sufficient condition for subspace irreducibility. More generally, it is only true that Δ_1 and Δ_2 each have on V exactly one eigenvalue λ_1, λ_2, of the form

$$\lambda_1 = -il_0l_1, \quad \lambda_2 = l_0^2 + l_1^2 - 1, \tag{2.4}$$

with $l_0 \in \mathbb{N}/2$ and $l_1 \in \mathbb{C}$. If $l_1 - l_0$ is a nonzero integer, then τ is called singular; otherwise τ is nonsingular. The structure of nonsingular indecomposable representations of $\mathscr{g}$, as determined in Ref. 15, has been summarized by Hlavaty and Niederle.[17] In the present paper, we shall concentrate on the more complicated case of singular indecomposable representations.

Given such a representation τ then, with λ_1, λ_2, as in (2.4), we choose l_0 and l_1 without loss of generality, such that $0 \leqslant l_0 < |l_1|$. Then[15] the set $\mathscr{S}$ in (2.1) is given by $\mathscr{S} = \{l_0, l_0+1, l_0+2,...\}$. Corresponding to (2.2) we can write V as an algebraic direct sum

$$V \cong \bigoplus_{l\in\mathscr{S}} V_l, \tag{2.5}$$

where each V_l is an eigenspace of the Casimir operator $(H_3)^2 + H_-H_+ + H_3$ of $\mathscr{k}$, with eigenvalue $l(l+1)$. Each V_l can in turn be written as a direct sum

$$V_l \cong \bigoplus_{m=-l}^{l} V_{lm} \tag{2.6}$$

of eigenspaces V_{lm} of H_3 with eigenvalue $m \in \{l, l-1,..., -l\}$. The subspaces V_{lm} for $l = l_0, l_0+1,..., |l_1|-1$ all have dimension n_0, while the V_{lm} for $l = |l_1|, |l_1|+1,...$ all have dimension n_1, for some pair (n_0,n_1) of non-negative integers, not both of which are zero. If $n_1 = 0$ we must have $n_0 = 1$, in which case τ is the finite-dimensional irreducible representation labeled[1] $[l_0,l_1]$. If $n_0 = 0$ and $n_1 = 1$, τ is the infinite-dimensional irreducible representation $[|l_1|, \mathrm{sgn}(l_1)l_0]$, sometimes[18,19] called the "tail" of $[l_0,l_1]$. In all remaining cases, with $n_0 = 0, n_1 > 1$ or $n_0 > 0$, $n_1 > 0$, τ is subspace reducible, i.e., V contains a proper subspace invariant under the action of H_3, F_3, etc. Such a τ can loosely be thought of as n_0 copies of $[l_0,l_1]$ and n_1 copies of its tail "glued indecomposably" together.[16] However, τ is *not* in general determined to within equivalence by giving l_0, l_1, n_0 and n_1 alone. It is necessary to specify the action of H_3, F_3, etc., in a suitably chosen basis for V. In Ref. 15, it is shown that a basis of vectors $\xi_{lm\alpha}$ can be found, with $l=l_0,l_0+1,\ldots$; $m=l,l-1,\ldots,-l$; $\alpha=1,2,\ldots,n_0$ for $l_0\leqslant l<|l_1|$; and $\alpha=1,2,\ldots,n_1$ for $l\geqslant|l_1|$, such that (adapting the notation of Ref. 17)

$$\begin{aligned}
H_3\xi_{lm\alpha} &= m\xi_{lm\alpha},\\
H_\pm\xi_{lm\alpha} &= [(l\pm m+1)(l\mp m)]^{1/2}\xi_{lm\pm1\alpha},\\
F_3\xi_{lm\alpha} &= [l^2-m^2]^{1/2}(M_l^\tau)_{\alpha\beta}\xi_{l-1m\beta}-m(Z_l^\tau)_{\alpha\beta}\xi_{lm\beta}\\
&\quad-[(l+1)^2-m^2]^{1/2}(P_l^\tau)_{\alpha\beta}\xi_{l+1m\beta},\\
F_\pm\xi_{lm\alpha} &= \pm[(l\mp m)(l\mp m-1)]^{1/2}\\
&\quad\times(M_l^\tau)_{\alpha\beta}\xi_{l-1m\pm1\beta}\\
&\quad-[(l\mp m)(l\pm m+1)]^{1/2}(Z_l^\tau)_{\alpha\beta}\xi_{lm\pm1\beta}\\
&\quad\pm[(l\pm m+1)(l\pm m+2)]^{1/2}\\
&\quad\times(P_l^\tau)_{\alpha\beta}\xi_{l+1m\pm1\beta}.
\end{aligned}\tag{2.7}$$

Here repeated subscripts β are to be summed over the values 1 to n_0 or n_1, as appropriate. Note that certain vectors on the right-hand sides in (2.7) are undefined, e.g., $\xi_{l-1m\beta}$ when $m=l$, but these can be ignored because they always appear with vanishing coefficients. The matrices M_l^τ, Z_l^τ, and P_l^τ, whose elements are $(M_l^\tau)_{\alpha\beta}$, $(Z_l^\tau)_{\alpha\beta}$, and $(P_l^\tau)_{\alpha\beta}$, respectively, have the appropriate dimensions. Thus P_l^τ is $n_0\times n_0$ for $l_0\leqslant l<|l_1|-1$; $n_1\times n_0$ for $l=|l_1|-1$; and $n_1\times n_1$ for $l\geqslant|l_1|$. Similarly, M_l^τ is $n_0\times n_0$ for $l_0\leqslant l<|l_1|$; $n_0\times n_1$ for $l=|l_1|$; and $n_1\times n_1$ for $l>|l_1|$, while Z_l^τ is $n_0\times n_0$ for $l_0\leqslant l<|l_1|$ and $n_1\times n_1$ for $l\geqslant|l_1|$. These matrices have the following form:

$$P_l^\tau=\begin{cases}I_0, & l_0\leqslant l<|l_1|-1,\\ d_+, & l=|l_1|-1,\\ I_1, & l\geqslant|l_1|,\end{cases}\tag{2.8a}$$

$$M_l^\tau=\begin{cases}[(l^2-l_0^2)(l_1^2-l^2)/(4l^2-1)l^2][I_0+(l_1^2/(l_1^2-l^2))a_0], & l_0\leqslant l<|l_1|,\\ d_-, & l=|l_1|,\\ [(l^2-l_0^2)(l_1^2-l^2)/(4l^2-1)l^2][I_1+(l_1^2/(l_1^2-l^2))a_1+(l_0^2/(l_0^2-l^2))\delta], & l>|l_1|,\end{cases}\tag{2.8b}$$

$$Z_l^\tau=\begin{cases}[il_0l_1/l(l+1)]\sqrt{I_0+a_0}, & l_0\leqslant l<|l_1|,\\ [il_0l_1/l(l+1)]\sqrt{I_1+a_1+\delta}, & l\geqslant|l_1|,\end{cases}\tag{2.8c}$$

where I_0 and I_1 are the unit matrices of dimension $n_0\times n_0$ and $n_1\times n_1$, while d_+ is $n_1\times n_0$, d_- is $n_0\times n_1$, and δ is $n_1\times n_1$, such that

$$d_-\delta=\delta d_+=0$$
$$\delta \text{ and } d_+d_- \text{ (and hence } d_-d_+\text{) are nilpotent.}\tag{2.9}$$

(The matrices Z_l^τ and M_l^τ, in cases with $l_0=0$, should be interpreted as vanishing when $l=0$, as should M_l^τ when $l=l_0=\frac{1}{2}$.) In addition, we have set

$$\begin{aligned}a_0&=[(4l_1^2-1)/(l_1^2-l_0^2)]d_-d_+,\\ a_1&=[(4l_1^2-1)/(l_1^2-l_0^2)]d_+d_-.\end{aligned}\tag{2.10}$$

Each matrix square root in (2.8c) is of the form $\sqrt{I+A}$, with I a unit matrix and A nilpotent (say $A^{K+1}=0$, $A^K\neq0$ in a particular case), and is to be interpreted through the binomial expansion as[22]

$$\sqrt{I+A}=I+\tfrac{1}{2}A+\cdots+[(-1)^{K+1}(2K)!/(2K-1)(K!)^2 2^{2K}]A^K.\tag{2.11}$$

From the formulas (2.6)–(2.9) it can be deduced that the Casimir operators (2.3) leave each V_{lm} in (2.6) invariant, and act on V_{lm} as

$$\begin{aligned}\Delta_{1l}&=\begin{cases}-il_0l_1\sqrt{I_0+a_0}, & l_0\leqslant l<|l_1|,\\ -il_0l_1\sqrt{I_1+a_1+\delta}, & l\geqslant|l_1|,\end{cases}\\
\Delta_{2l}&=\begin{cases}(l_0^2+l_1^2-1)I_0+l_1^2a_0, & l_0\leqslant l<|l_1|,\\ (l_0^2+l_1^2-1)I_1+l_0^2\delta+l_1^2a_1, & l\geqslant|l_1|,\end{cases}\end{aligned}\tag{2.12}$$

and it can be seen that they do indeed have one eigenvalue each, of the form (2.4), because of (2.11) and the nilpotency of a_0, a_1, and δ.

To complete the description of a singular indecomposable representation τ, it remains to complete the description of the matrices d_+, d_-, and δ subject to (2.9). Gel'fand and Ponomarev[15] have shown that inequivalent indecomposable sets of such matrices are in one-to-one correspondence with certain diagrams, which are therefore also in one-to-one correspondence with inequivalent indecomposable representations τ having the same values of l_0, l_1, n_0, and n_1. In other words, each such diagram may be regarded as providing the remaining labels necessary to characterize a corresponding τ, up to equivalence.

There are two categories of diagrams, of so-called "open" and "closed" types, and, correspondingly, there are singular indecomposable representations of type I and type II.

Definition 2.1: An *open diagram* is a finite set M of points in the lattice $\mathbb{Z}^2$, arranged as an unbroken staircase descending from left to right. Thus M contains one point [which can be taken without loss of generality to be the origin (0,0) in $\mathbb{Z}^2$], starting from which we can generate all of M by going successively either right or down to the nearest neighboring lattice point. Each point is colored black or white, with the restriction that nearest neighbors in M are both black if they are vertically adjacent, and are opposite in color if they are horizontally adjacent. (This implies that the length of each horizontal part of the staircase must be an even integer, unless that part includes the first or last point, when its length may be even or odd.) ■

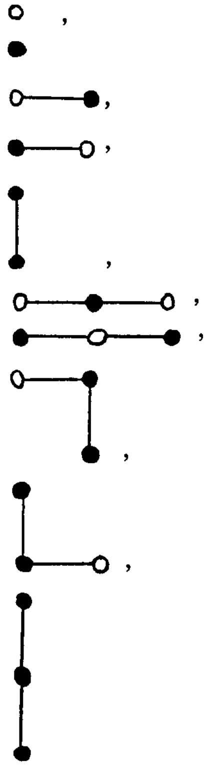
For the corresponding indecomposable representation τ of $\mathscr{G}$, n_0 equals the number of white points and n_1 the number of black points in the diagram.

The simplest diagrams are as follows:

$n_0 + n_1 = 1$: (2.13a) (2.13b)

$n_0 + n_1 = 2$: (2.13c) (2.13d) (2.13e)

$n_0 + n_1 = 3$: (2.13f) (2.13g) (2.13h) (2.13i) (2.13j)



To obtain from a given diagram the corresponding matrices $d_\pm, \delta$ of (2.7)–(2.9), we first associate with each of the points $(i,j) \in M$, a basis vector $e(i,j)$ in an (n_0+n_1)-dimensional complex vector space P. The basis vectors $e(i,j)$ corresponding to white points (i,j) span an n_0-dimensional subspace P_0 of P, while those corresponding to black points span an n_1-dimensional subspace P_1 of P; evidently

$$P = P_0 \oplus P_1. \tag{2.14}$$

Next we define linear operators a and b on P by

$$ae(i,j) = \begin{cases} e(i+1,j), & (i+1,j) \in M, \\ 0, & \text{otherwise}, \end{cases} \tag{2.15}$$

$$be(i,j) = \begin{cases} e(i,j+1), & (i,j+1) \in M, \\ 0, & \text{otherwise}. \end{cases} \tag{2.16}$$

Then a and b are nilpotent, with

$$\begin{aligned} &ab = ba = 0, \\ &aP_0 \subseteq P_1, \quad aP_1 \subseteq P_0 \\ &bP_0 = \{0\}, \quad bP_1 \subseteq P_1. \end{aligned} \tag{2.17}$$

If we now identify P with $\mathbb{C}^{n_0+n_1}$, choosing the $e(i,j)$ in such a way that vectors in P_0 have their bottom n_1 components zero, and vectors in P_1 have their top n_0 components zero, we obtain a matrix realization of a and b with the form

$$a = \begin{bmatrix} 0 & d_- \\ d_+ & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix}, \tag{2.18}$$

where d_+, d_-, and δ have dimension $n_1 \times n_0$, $n_0 \times n_1$, and $n_1 \times n_1$, respectively, and satisfy (2.9).

For example, for the diagram (2.13h) we can take the points to be at (0,0), (1,0), and (1, $-$1) in $\mathbb{Z}_2$, and set $e(0,0) = (1,0,0)^T$, $e(1,0) = (0,1,0)^T$, and $e(1,-1) = (0,0,1)^T$. Then (2.15) and (2.16) imply

$$a = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.19}$$

so that

$$d_+ = [1 \quad 0]^T, \quad d_- = [0 \quad 0], \quad \delta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{2.20}$$

In this way the matrices $d_\pm, \delta$ are determined by a given open diagram and, together with l_0 and l_1, complete the specification of a representation τ of type I. Note that for given l_0, l_1, the diagrams (2.13a) and (2.13b) correspond to the irreducible representations $[l_0, l_1]$ and $[|l_1|, \mathrm{sgn}(l_1) l_0]$.

Our treatment of an open diagram differs slightly from that of Ref. 15, in that we color the points to show explicitly the grading $P_0 \oplus P_1$. Apart from making the structure clearer, this is in fact necessary to distinguish between those inequivalent representations which would otherwise have the same "straight-row" diagram, consisting of n points in a horizontal line.

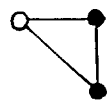
For example, (2.13c) and (2.13d) lead to

$$\begin{aligned} &d_+ = 1, \quad d_- = \delta = 0, \\ &d_- = 1, \quad d_+ = \delta = 0, \end{aligned} \tag{2.21}$$

respectively. For given l_0 and l_1, the corresponding inequivalent representations of $\mathscr{g}$ in this case are the well-known "operator-irreducible" indecomposable representations. For these the Casimir operators Δ_1 and Δ_2 of (2.3) are multiples of the identity by λ_1 and λ_2, as in (2.4), as follows from (2.21), (2.10), and (2.12), but the representations are nevertheless subspace reducible. In Ref. 23 they are denoted, respectively, by $\{l_0 \to l_1\}$ and $\{l_0 \leftarrow l_1\}$, this notation being intended to indicate that in the first case the subspace $V_{|l_1|} \oplus V_{|l_1|+1} \oplus \cdots$ of V is invariant, while in the second case $V_{l_0} \oplus V_{l_0+1} \oplus \cdots \oplus V_{|l_1|-1}$ is invariant. In Ref. 17 they are denoted by $(l_0, l_1, +)$ and $(l_0, l_1, -)$.

Definition 2.2: A *closed diagram* is obtained from any open diagram M that begins with a white point and ends with at least two successive black points. A line is drawn connecting the first and last points of M and the diagram is supplemented by a pair of labels (q,μ), $q \in \mathbb{N}$, $\mu \in \mathbb{C} \setminus 0$. ■

The simplest example has three points:



$$(q,\mu). \tag{2.22}$$

When $q = 1$, the procedure for constructing a and b, and subsequently $d_\pm$ and δ, is just as before, except that the definition (2.15) of a is supplemented by requiring

$$ae(k,l) = \mu e(0,0), \tag{2.23}$$

where (k,l) is the final and (0,0) the initial point of M. Again this leads to matrices a and b with the general form (2.18), from which d_+, d_-, and δ can be determined in order to complete the description of a singular indecomposable representation of $\mathscr{g}$ of type II. This representation is labeled (up to equivalence) by l_0, l_1, and the closed diagram [including the pair $(1,\mu)$].

For example, diagram (2.22) with $q = 1$ leads to

$$a = \begin{bmatrix} 0 & 0 & \mu \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.24}$$

and hence to

$$d_+ = [1 \quad 0]^T, \quad d_- = [0 \quad \mu], \quad \delta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad (2.25)$$

[Note that these matrices reduce to those in (2.20), corresponding to the open diagram (2.13h), if μ is set equal to zero. In fact, it is obviously true, in general, that the closed diagram with $q = 1$ and μ set equal to zero, reduces in this sense to the corresponding open diagram.]

More generally, for $q > 1$, a representation of type II is obtained by associating with each point (ij) of a closed diagram, a q-dimensional subspace $v(i,j)$ rather than a single vector $e(i,j)$ (as in the case $q = 1$). The definitions of a and b are generalized accordingly. Thus

$$av(i,j) \to v(i+1,j) \quad (2.26)$$

is an isomorphism if $(i+1,j) \in M$ (with a $q \times q$ matrix, which can be taken to the identity I_q); otherwise $av(i,j) = 0$, except that if (0,0) is the first and (k,l) the last point of M, then a maps $v(k,l)$ into $v(0,0)$ with a $q \times q$ matrix μ_q, which can be taken to be a single Jordan block with eigenvalue μ. The mapping

$$bv(i,j) \to v(i,j+1) \quad (2.27)$$

is an isomorphism (with matrix I_q) if $(i,j+1) \in M$, and $bv(i,j) = 0$ otherwise. In this case the subspaces P_0 and P_1 of P are of dimension $n_0 = qm_0$, $n_1 = qm_1$, respectively, where m_0 and m_1 are the numbers of white and black points in the diagrams. Associating P with $\mathbb{C}^{n_0+n_1}$ as before, we read off the matrices $d_\pm$ and δ from the matrices of a and b in the same way. The corresponding indecomposable representation of $\mathscr{g}$ of type II is labeled by l_0, l_1, and the closed diagram [including (q,μ)].

For example, taking the diagram (2.22) with $q = 2$, the matrices can be obtained from those for the case $q = 1$, as in (2.24), (2.25), by replacing each zero by a 2×2 block of zeros, each 1 by the 2×2 unit matrix, and each μ by the 2×2 matrix

$$\mu_2 = \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix}. \quad (2.28)$$

III. VECTOR OPERATORS: GENERALITIES

Suppose that $\psi(x)$ in (1.1) is, for each x, an element of the $\mathscr{g}$ module V_π of a Harish-Chandra representation π of $\mathscr{g}$,

$$\pi \cong \bigoplus_r \tau_r,$$

$$V_\pi \cong \bigoplus_r V^{\tau_r}, \quad (3.1)$$

where each τ_r is (singular) indecomposable. Suppose further that Γ_μ and κ in (1.1) are linear mappings (operators) from V_π into itself. Let $H_3 = \pi(h_3)$, $F_3 = \pi(f_3)$, etc. Generalizing well-known results,[1,24] we know that a sufficient condition for (1.1) to be locally[12] invariant under homogeneous (and, indeed, inhomogeneous) Lorentz transformations is that κ commutes with H_3, F_3, etc., on V_π and

$$[\Gamma_0, H_\pm] = [\Gamma_0, H_3] = 0, \quad (3.2a)$$

$$\Gamma_0 = [[F_3,\Gamma_0],F_3], \quad (3.2b)$$

$$\Gamma_3 = i[F_3,\Gamma_0], \quad \Gamma_1 \pm i\Gamma_2 = i[F_\pm,\Gamma_0], \quad (3.2c)$$

on V_π. Then we say that κ is a scalar operator and Γ_μ a four-vector operator on V_π. In searching for possible locally invariant equations, we therefore seek representations π for which κ and Γ_μ can be found with these properties. If we restrict attention to equations for which κ is invertible on V_π, then[24] there is no significant loss of generality in supposing κ to be a nonzero numerical multiple of the identity operator on V_π. Then the problem reduces to finding representations π for which a four-vector operator Γ_μ can be found. It is sufficient to search for Γ_0 satisfying (3.2a) and (3.2b), as Γ_1, Γ_2, and Γ_3 can then be *defined* by (3.2c). We shall concentrate on this problem here, but make some remarks about wave equations with noninvertible κ at the end of the paper.

Let τ and τ' denote any two of the indecomposable representations (or possibly one and the same representation) contained in π, and let P^τ, $P^{\tau'}$ be the operators projecting V_π onto the corresponding subspaces V^τ, $V^{\tau'}$ in (3.1). Define

$$\Gamma_\mu^{\tau'\tau} = P^{\tau'}\Gamma_\mu P^\tau \quad (3.3)$$

and note that, since P^τ and $P^{\tau'}$ commute on V_π with H_3, F_3, etc., $\Gamma_\mu^{\tau'\tau}$ is a four-vector operator whenever this is true of Γ_μ. We concentrate on the determination of $\Gamma_\mu^{\tau'\tau}$, in effect restricting attention to the case when $\pi = \tau \oplus \tau'$ (or $\pi = \tau$, if $\tau' = \tau$). The Γ_μ in a more general case can evidently be built up from such $\Gamma_\mu^{\tau'\tau}$.

Decomposing V^τ and $V^{\tau'}$ as in (2.5) and (2.6), and introducing bases, as in Sec. II, we see from (3.2a) and (3.3) that $\Gamma_0^{\tau'\tau}$ carries V_{lm}^τ into $V_{lm}^{\tau'}$. Let $X_l^{\tau'\tau}$ denote the corresponding matrix; it is independent of m, again because of (3.2a). Expressions for the matrices $\Gamma_{p;lm}^{\tau'\tau}$, defined by the action of the operators $\Gamma_p^{\tau'\tau}$, $p = 1,2,3$, on V_{lm}^τ, can then be written down. For example, we find, using (2.7) and (3.2c), that

$$\begin{aligned} i\Gamma_{3;lm}^{\tau'\tau} = {} & [l^2 - m^2]^{1/2}(X_{l-1}^{\tau'\tau}M_l^\tau - M_l^{\tau'}X_l^{\tau'\tau}) \\ & - m(X_l^{\tau'\tau}Z_l^\tau - Z_l^{\tau'}X_l^{\tau'\tau}) \\ & - [(l+1)^2 - m^2]^{1/2}(X_{l+1}^{\tau'\tau}P_l^\tau - P_l^{\tau'}X_l^{\tau'\tau}). \end{aligned} \quad (3.4)$$

The heart of the problem then is to determine the $X_l^{\tau'\tau}$ from the remaining condition (3.2b), for the appropriate range of l values in the representation τ. This condition leads to the following system of coupled matrix equations, essentially the same as Eqs. (3.6) of Ref. 17, in which Z_l^τ, M_l^τ, P_l^τ, and the corresponding primed variables are to be regarded as given, and the $X_l^{\tau'\tau}$ are unknowns:

$$2P_{l+1}^{\tau'}X_{l+1}^{\tau'\tau}P_l^\tau - P_{l+1}^{\tau'}P_l^{\tau'}X_l^{\tau'\tau} - X_{l+2}^{\tau'\tau}P_{l+1}^\tau P_l^\tau = 0, \quad (3.5a)$$

$$2M_{l-1}^{\tau'}X_{l-1}^{\tau'\tau}M_l^\tau - M_{l-1}^{\tau'}M_l^{\tau'}X_l^{\tau'\tau} - X_{l-2}^{\tau'\tau}M_{l-1}^\tau M_l^\tau = 0, \quad (3.5b)$$

$$\begin{aligned} & X_{l+1}^{\tau'\tau}[P_l^\tau Z_l^\tau + Z_{l+1}^\tau P_l^\tau] + [P_l^{\tau'}Z_l^{\tau'} + Z_{l+1}^{\tau'}P_l^{\tau'}]X_l^{\tau'\tau} \\ & \quad - 2P_l^{\tau'}X_l^{\tau'\tau}Z_l^\tau - 2Z_{l+1}^{\tau'}X_{l+1}^{\tau'\tau}P_l^\tau = 0, \end{aligned} \quad (3.5c)$$

$$\begin{aligned} & X_{l-1}^{\tau'\tau}[Z_{l-1}^\tau M_l^\tau + M_l^\tau Z_l^\tau] \\ & \quad + [Z_{l-1}^{\tau'}M_l^{\tau'} + M_l^{\tau'}Z_l^{\tau'}]X_l^{\tau'\tau} \end{aligned}$$

$$
\begin{aligned}
&- 2Z^{\tau'}_{l-1}X^{\tau'\tau}_{l-1}M^{\tau}_{l} - 2M^{\tau'}_{l}X^{\tau'\tau}_{l}Z^{\tau}_{l} = 0, \qquad (3.5\text{d})\\
&X^{\tau'\tau}_{l}[P^{\tau}_{l-1}M^{\tau}_{l} + M^{\tau}_{l+1}P^{\tau}_{l} + Z^{\tau}_{l}Z^{\tau}_{l}]\\
&\quad + [Z^{\tau'}_{l}Z^{\tau'}_{l} + P^{\tau'}_{l-1}M^{\tau'}_{l} + M^{\tau'}_{l+1}P^{\tau'}_{l}]X^{\tau'\tau}_{l}\\
&\quad - 2P^{\tau'}_{l-1}X^{\tau'\tau}_{l-1}M^{\tau}_{l} - 2M^{\tau'}_{l+1}X^{\tau'\tau}_{l+1}P^{\tau}_{l}\\
&\quad - 2Z^{\tau'}_{l}X^{\tau'\tau}_{l}Z^{\tau}_{l} = 0, \qquad (3.5\text{e})\\
&(l+1)^2[X^{\tau'\tau}_{l}M^{\tau}_{l+1}P^{\tau}_{l}\\
&\quad - 2M^{\tau'}_{l+1}X^{\tau'\tau}_{l+1}P^{\tau}_{l} + M^{\tau'}_{l+1}P^{\tau'}_{l}X^{\tau'\tau}_{l}]\\
&\quad + l^2[X^{\tau'\tau}_{l}P^{\tau}_{l-1}M^{\tau}_{l} - 2P^{\tau'}_{l-1}X^{\tau'\tau}_{l-1}M^{\tau}_{l}\\
&\quad + P^{\tau'}_{l-1}M^{\tau'}_{l}X^{\tau'\tau}_{l}] = X^{\tau'\tau}_{l}. \qquad (3.5\text{f})
\end{aligned}
$$

These equations must hold for each allowed l value in the representation τ, and $X^{\tau'\tau}_{l}$ must be set equal to zero unless both V^{τ}_{l} and $V^{\tau'}_{l}$ are non-null.

If τ and τ' are irreducible [so that each matrix in (3.5) is a single number], it is well known[1] that a necessary and sufficient condition for the existence of a nontrivial solution is that the labels (l_0,l_1) and (l'_0,l'_1) of τ and τ' (which serve to characterize the representations completely in such a case) satisfy the "interlocking" condition that one of the pairs (l'_0,l'_1), $(-l'_0,-l'_1)$ is equal to one of the pairs (l_0,l_1+1), (l_0,l_1-1), (l_0+1,l_1), or (l_0-1,l_1). Moreover, the structure of the solutions is known for all such cases.[1] Hlavaty and Niederle[17] have in fact extended these results to the case where τ and τ' are nonsingular indecomposable representations.

In the singular case, we expect that it is still true that the labels (l_0,l_1), (l'_0,l'_1) of τ and τ' (which now are not sufficient to characterize τ and τ' completely) must satisfy the interlocking condition if a nontrivial solution is to exist, although we have no general proof. On the other hand, it would be surprising if this condition is sufficient for existence, with no restriction on the diagrams for τ and τ', but this too remains an open question.

IV. VECTOR OPERATORS: EXAMPLES

For a given pair (τ,τ') of singular indecomposable representations,[25] it is not, in general, easy to determine if Eqs. (3.5) admit a nontrivial solution. The only solutions previously presented (apart of course from those corresponding to irreducible τ and τ') seem to have been those corresponding to the case of interlocking τ and τ' of the operator-irreducible type, with diagrams of the form (2.13c) or (2.13d). Then $n_0 = n_1 = 1$, so that all the matrices in (3.5) are simply numbers, and the problem of finding solutions is not substantially more difficult than in the irreducible case.

In Ref. 17, a nontrivial solution is found in a case of this type, where τ has labels $(\frac{1}{2},l_1)$ and τ' has labels $(\frac{1}{2},-l_1)$, with $l_1 \in \{\frac{3}{2},\frac{5}{2},...\}$, and the diagrams for τ and τ' are (2.13c) and (2.13d), respectively. (These are the operator-irreducible representations $[\frac{1}{2}\rightarrow l_1]$ and $[\frac{1}{2}\leftarrow -l_1]$ described earlier.) It was shown, however, that this leads to an operator Γ_0 with no nonzero eigenvalues, so that (1.1) has no timelike solutions in this case.[17] In fact, four-vector operators in the case of interlocked operator-irreducible τ and τ' were described earlier by Ruhl.[19]

In seeking to find, for representations with arbitrarily complicated diagrams, examples of four-vector operators that lead in at least some cases to wave equations with timelike solutions, we shall make the following simplifying assumptions: (1) τ and τ' have labels $(\frac{1}{2},l_1)$ and $(\frac{1}{2},-l_1)$, where $l_1 \in \{\frac{3}{2},\frac{5}{2},...\}$ (note that the interlocking condition is then satisfied); and (2) τ and τ' have the *same* diagram.

Note that condition (1) but not condition (2) is satisfied by the example of Ref. 17. The matrices $d_{\pm}$, δ (and hence a_0, a_1) can now be taken to be the same for τ' as for τ, and we obtain

$$Z^{\tau'}_{l} = -Z^{\tau}_{l};\quad P^{\tau'}_{l} = P^{\tau}_{l};\quad M^{\tau'}_{l} = M^{\tau}_{l}, \qquad (4.1)$$

and, for the matrices of the Casimir operators (2.12),

$$\Delta^{\tau}_{1l} = -\Delta^{\tau'}_{1l},$$
$$\Delta^{\tau}_{2l} = \Delta^{\tau'}_{2l}. \qquad (4.2)$$

Under these conditions, $X^{\tau'\tau}_{l}$ ($\frac{1}{2}\leqslant l<l_1$) and a_0 are $n_0\times n_0$ matrices, while $X^{\tau'\tau}_{l}$ ($l\geqslant l_1$), a_1, and δ are $n_1\times n_1$ matrices, where n_0 and n_1 are determined by the number of white and black points in the (common) diagram for τ and τ', as described in Sec. II. In order to simplify ordering problems in (3.5), we limit ourselves further by seeking only solutions such that (3) $X^{\tau'\tau}_{l}$ commutes with a_0 for $\frac{1}{2}\leqslant l<l_1$, and with a_1 and δ for $l\geqslant l_1$.

It follows from (2.12) and (4.2) that condition (3) is equivalent to requiring that $\Gamma^{\tau'\tau}_{\mu}$ satisfies

$$[\Gamma^{\tau'\tau}_{\mu},\Delta_2] = 0 = \{\Gamma^{\tau'\tau}_{\mu},\Delta_1\} \qquad (4.3)$$

on $V^{\tau}\oplus V^{\tau'}$. Equations (4.3) are known[7] to hold for all solutions in the case that τ and τ' are irreducible [given conditions (1) and (2)], when Δ_2 is a multiple of the identity operator (on $V^{\tau}\oplus V^{\tau'}$) and Δ_1 is a (generalized) Dirac γ_5 matrix, but it is not clear if the imposition of condition (3) places a nontrivial restriction on the solution of (3.5) in the present situation.

Having imposed conditions (1)–(3), we now consider (3.5a) and find using (2.8) that

$$X^{\tau'\tau}_{l+2} = 2X^{\tau'\tau}_{l+1} - X^{\tau'\tau}_{l} \qquad (4.4)$$

for $\frac{1}{2}\leqslant l<l_1-2$, and for $l\geqslant l_1$, so that

$$X^{\tau'\tau}_{l} = \begin{cases} lA + C, & \frac{1}{2}\leqslant l<l_1,\\ lB + D, & l\geqslant l_1,\end{cases} \qquad (4.5)$$

where A, B, C, and D are matrices independent of l. According to condition (3), A and C commute with a_0, and B and D commute with a_1 and δ. Use of (4.1) and (2.8) in (3.5c), with $l_0\leqslant l<l_1-1$, then gives

$$
\begin{aligned}
&[(2l-1)A + 2C]P^{\tau}_{l}Z^{\tau}_{l}\\
&\quad = [(2l+3)A + 2C]Z^{\tau}_{l+1}P^{\tau}_{l}. \qquad (4.6)
\end{aligned}
$$

But Eqs. (2.8) imply that $lP^{\tau}_{l}Z^{\tau}_{l} = (l+2)Z^{\tau}_{l+1}P^{\tau}_{l}$, so that

$$
\begin{aligned}
&(l+2)[(2l-1)A + 2C]P^{\tau}_{l}Z^{\tau}_{l}\\
&\quad = l[(2l+3)A + 2C]P^{\tau}_{l}Z^{\tau}_{l}, \qquad (4.7)
\end{aligned}
$$

and since $P^{\tau}_{l}Z^{\tau}_{l}$ is nonsingular, we obtain $C = \frac{1}{2}A$. In a similar way, using (3.5c) with $l\geqslant l_1$, we obtain $D = \frac{1}{2}B$. Thus

$$X_l^{\tau'\tau} = \begin{cases} (l+\frac{1}{2})A, & \frac{1}{2} \leqslant l < l_1, \\ (l+\frac{1}{2})B, & l \geqslant l_1. \end{cases} \tag{4.8}$$

Then (3.5a) with $l = l_1 - 2$ or $l = l_1 - 1$ implies

$$d_+ A = B d_+ . \tag{4.9}$$

Equation (3.5b) yields no new condition for $\frac{5}{2} \leqslant l < l_1$ or $l > l_1 + 2$, but for $l = l_1 + 1$ or $l = l_1$ we obtain

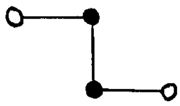
$$A d_- = d_- B. \tag{4.10}$$

The remaining equations (3.5) require just one more condition, that

$$B\delta = \delta B = 0. \tag{4.11}$$

Since (4.9) and (4.10) imply, with (2.10), that A commutes with a_0 and B commutes with a_1, the problem reduces to the following: for a given diagram and hence for given $d_\pm$, δ, find matrices A and B such that (4.9)–(4.11) are satisfied.

For small values of n_0 and n_1, we can now easily construct all solutions subject to the conditions (1)–(3). For example, the diagram



(4.12)

leads to

$$d_+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad d_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \delta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{4.13}$$

[and hence to $a_0 = a_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$], and it follows from (4.9)–(4.11) that

$$A = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}, \tag{4.14}$$

where α and β are arbitrary constants. Similarly, for diagram (2.22) with $q = 2$ and $\mu \neq 0$, we find that A equals the 2×2 zero matrix and B is a 4×4 matrix with an arbitrary 2×2 block in the upper right-hand corner, and zeros elsewhere.

Of more interest is the observation that a class of solutions (A,B) can now be determined as follows, whatever the common diagram of τ and τ'. Since (2.9) and (2.10) imply that

$$\begin{aligned} & d_+ a_0 = a_1 d_+, \quad a_0 d_- = d_- a_1, \\ & \delta a_1 = a_1 \delta = 0, \end{aligned} \tag{4.15}$$

we can satisfy (4.9) and (4.10) by taking

$$\begin{aligned} & A = \alpha_0 I_0 + \alpha_1 a_0 + \alpha_2 a_0^2 + \cdots + \alpha_N a_0^N, \\ & B = \alpha_0 I_1 + \alpha_1 a_1 + \alpha_2 a_1^2 + \cdots + \alpha_N a_1^N + c\delta^M, \end{aligned} \tag{4.16}$$

where c and the α_i, $i = 0,1,2,\ldots,N$ are arbitrary complex constants, N is the largest non-negative integer such that at least one of a_0^N, a_1^N is nonzero, and M is the largest non-negative integer such that δ^M is nonzero.

If $\delta = 0$, then (4.11) is satisfied trivially, and (4.16) defines a class of solutions (4.8) parametrized by $\alpha_0, \alpha_1, \ldots, \alpha_N$. The vanishing of δ is easily seen to require that the diagram of τ and τ' be a straight row: we discuss this case further in the next section .

If $\delta \neq 0$, we must set $\alpha_0 = 0$ in (4.16) in order to satisfy (4.11). We then still obtain a class of solutions (4.8), parametrized by $\alpha_1, \alpha_2, \ldots, \alpha_N$ and c, but note immediately that $X_l^{\tau'\tau}$ is then nilpotent for each l, and that the same will be true of $\Gamma_0^{\tau'\tau}$. The corresponding wave equation (1.1) with $\Gamma_\mu = \Gamma_\mu^{\tau'\tau}$ [or with $\Gamma_\mu = \Gamma_\mu^{\tau'\tau} + \Gamma_\mu^{\tau\tau'}$ as in (4.17) below] will not admit timelike solutions in such a case. The question arises as to whether or not (4.8) and (4.16) give in the way described, all solutions of (3.5) under conditions (1)–(3). That the answer is no, at least when $\delta \neq 0$, is shown by the counterexample (4.12)–(4.14), with $\alpha \neq 0$.

We attempted to find other solutions to (3.5), satisfying conditions (1) and (2), but not (3), with the help of the symbolic manipulation computer package[26] MUMATH, but were unsuccessful. Furthermore, for all diagrams leading to $\delta \neq 0$, we found only nilpotent solutions $X_l^{\tau'\tau}$. We conjecture that this is a general rule, at least when conditions (1) and (2) hold.

We also found only nilpotent solutions in cases like that considered in Ref. 17, where condition (2) does not hold. This was in fact our motivation for imposing that condition.

Note that if the roles of τ and τ' are interchanged in (3.5), then $X_l^{\tau\tau'}$ satisfies the same equations as $X_l^{\tau'\tau}$, as a consequence of (4.1). Therefore we have also found solutions $X_l^{\tau\tau'}$ of the same general form (4.8), (4.16), and we can consider

$$\Gamma_\mu = \Gamma_\mu^{\tau'\tau} + \Gamma_\mu^{\tau\tau'} \tag{4.17}$$

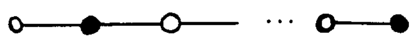
in (1.1). This will, in general, be necessary if equations possessing timelike solutions are to be obtained, as is familiar from the case of the Dirac matrices γ_μ, which couple the irreducible representations $\tau = [\frac{1}{2}, \frac{3}{2}]$, $\tau' = [\frac{1}{2}, -\frac{3}{2}]$; here

$$\begin{aligned} & \Gamma_\mu^{\tau'\tau} = \alpha\gamma_\mu(1+\gamma_5), \quad \Gamma_\mu^{\tau\tau'} = \beta\gamma_\mu(1-\gamma_5), \\ & \gamma_\mu = \Gamma_\mu^{\tau'\tau} + \Gamma_\mu^{\tau\tau'} \end{aligned} \tag{4.18}$$

(choosing $\alpha = \beta = \frac{1}{2}$).

V. A CLASS OF WAVE EQUATIONS

We consider the case when τ and τ' have the straight row diagram



(5.1)

with $2k$ points and, as in Sec. IV, condition (1) holds. Following the prescription outlined in Sec. II we obtain

$$d_+ = I_k, \quad \delta = 0, \quad \text{and}$$

$$d_- = \tfrac{1}{4} a_0 = \tfrac{1}{4} a_1 = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & 0 & \\ & 1 & 0 & & & \\ & & \cdots & & & \\ & & & \cdots & & \\ & 0 & & & \cdots & \\ & & & & 1 & 0 \end{bmatrix}, \tag{5.2}$$

each being a $k \times k$ matrix. Then from (4.8) and (4.16) we have

$$X_l^{\tau'\tau} = (l+\tfrac{1}{2}) \sum_{j=0}^{k-1} \zeta_j (d_-)^j, \tag{5.3}$$

where the ζ_j are arbitrary constants. Similarly,

$$X_l^{\tau\tau'} = (l+\tfrac{1}{2}) \sum_{j=0}^{k-1} \eta_j (d_-)^j, \tag{5.4}$$

with arbitrary η_j. We restrict our attention to cases where η_0 and ζ_0 are nonzero, so that $X_l^{\tau'\tau}$ and $X_l^{\tau\tau'}$ are not nilpotent; because we can multiply (1.1) throughout by an arbitrary constant, there is no significant loss of generality in assuming then that

$$\eta_0\zeta_0 = 1. \tag{5.5}$$

On each $2k$-dimensional subspace V_{lm} of V $(= V^\tau \oplus V^{\tau'})$,

$$V_{lm} = \begin{bmatrix} V_{lm}^\tau \\ V_{lm}^{\tau'} \end{bmatrix}, \quad m \in \{l, l-1, \ldots, -l\}, \tag{5.6}$$

the matrix of Γ_0 $(= \Gamma_0^{\tau'\tau} + \Gamma_0^{\tau\tau'})$ is

$$\Gamma_{0l} = \begin{bmatrix} 0 & X_l^{\tau\tau'} \\ X_l^{\tau'\tau} & 0 \end{bmatrix}, \tag{5.7}$$

where the zeros represent $k \times k$ blocks. It follows from (5.3)–(5.5) that Γ_{0l} has only $\pm (l+\frac{1}{2})$ as eigenvalues, but is not, in general, completely diagonalizable, depending on the values of the ζ_j and η_j.

For example, in the case that $k=3$, we can take $\zeta_0 = \eta_0 = 1$, $\zeta_1 = \eta_1 = \zeta_2 = \eta_2 = 0$, and find that for each eigenvalue $\pm (l+\frac{1}{2})$ of Γ_{0l} (and for each value of m) there are three linearly independent eigenvectors, and Γ_{0l} is diagonalizable; or take $\zeta_0 = \eta_0 = \zeta_2 = \eta_2 = 1$, $\zeta_1 = \eta_1 = 0$, and find only two linearly independent eigenvectors for each eigenvalue; or take $\zeta_0 = \eta_0 = \zeta_1 = \eta_1 = \zeta_2 = \eta_2 = 1$ and find only one eigenvector for each eigenvalue. In these last two cases, Γ_{0l} is not diagonalizable and does not have a complete set of eigenvectors.

It follows that (1.1) will admit timelike solutions corresponding to at least one set of positive and one set of negative energy particles with a Majorana-type mass-spin spectrum

$$m_l = \kappa/(l+\tfrac{1}{2}), \quad l = \tfrac{1}{2}, \tfrac{3}{2}, \ldots, \tag{5.8}$$

but that there will, in general, be enough linearly independent solutions to describe n such sets, $1 \leqslant n \leqslant k$. We can expect that, in general, there will also be lightlike and spacelike solutions of (1.1), as for the case[27] of infinite-dimensional irreducible representations τ and τ'.

Similar results hold in the case that the black and white points in (5.1) are interchanged. This simply leads to an interchange of d_- and d_+ in (5.2)–(5.4).

Slightly more complicated are the cases corresponding to the diagrams

○——●——○—— ··· ●——○ , (5.9a)

●——○——●—— ··· ○——● , (5.9b)

each with $2k+1$ points. For the diagrams (5.9a) we obtain

$$d_+ = \left[\begin{array}{c|c} I_k & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right], \quad d_- = \begin{bmatrix} 0 & 0 \cdots 0 \\ & I_k \end{bmatrix}, \quad \delta = 0; \tag{5.10}$$

d_+ is $k \times (k+1)$, d_- is $(k+1) \times k$, and δ is $k \times k$. Then $\frac{1}{4}a_0$ and $\frac{1}{4}a_1$ have the same form as the matrix d_- in (5.2), with a_0 being $(k+1) \times (k+1)$, and a_1 being $k \times k$. Our solution (4.8), (4.16) now gives

$$X_l^{\tau'\tau} = \begin{cases} \left(l+\dfrac{1}{2}\right) \displaystyle\sum_{j=0}^{k} \zeta_j \left(\dfrac{1}{4} a_0\right)^j, & \dfrac{1}{2} \leqslant l < l_1 \\ \left(l+\dfrac{1}{2}\right) \displaystyle\sum_{j=0}^{k-1} \zeta_j \left(\dfrac{1}{4} a_1\right)^j, & l \geqslant l_1. \end{cases} \tag{5.11}$$

We get a similar expression for $X_l^{\tau\tau'}$, with further arbitrary constants η_j replacing the ζ_j of (5.11). Again we suppose ζ_0, η_0 are nonzero, set $\eta_0\zeta_0 = 1$, and find that Γ_{0l} has only $\pm (l+\frac{1}{2})$ as eigenvalues. For some choices of arbitrary constants ζ_j and η_j, Γ_{0l} is diagonalizable (for every l), but for most it is not. A new feature that emerges is that for a given eigenvalue $\pm (l+\frac{1}{2})$, Γ_{0l} may have a different number of linearly independent eigenvectors for $l < l_1$ than for $l \geqslant l_1$. For example, if we set $\zeta_0 = \eta_0 = 1$, all other ζ_j and η_j being equal to zero, then Γ_{0l} is diagonalizable, with $(k+1)$ linearly independent eigenvectors for $l < l_1$, and k for $l \geqslant l_1$. The corresponding wave equation (1.1) would admit timelike solutions capable of describing $(k+1)$ positive (or negative) energy particles with spins $\frac{1}{2}, \frac{3}{2}, \ldots, l_1 - 1$, and k with spins $l_1, l_1+1, \ldots$. Again the mass-spin spectrum is of the Majorana type.

Similar remarks apply in the case of the diagram (5.9b); in this case Γ_{0l} will be $2k \times 2k$ for $l < l_1$, and $2(k+1) \times 2(k+1)$ for $l \geqslant l_1$.

For any of these straight row diagrams, we can restrict attention to the diagonalizable cases by requiring that (5.5) holds and

$$X_l^{\tau'\tau} X_l^{\tau\tau'} = (l+\tfrac{1}{2})^2 I_l, \tag{5.12}$$

where I_l is the unit matrix of the appropriate size. Then

$$(\Gamma_{0l})^2 = (l+\tfrac{1}{2})^2 (I_l \oplus I_l). \tag{5.13}$$

Some important algebraic properties of the corresponding operators Γ_μ can now be determined. It follows from (3.4) that, quite generally, the matrix of the operator $i[\Gamma_0^{\tau\tau'}\Gamma_3^{\tau'\tau} \pm \Gamma_3^{\tau\tau'}\Gamma_0^{\tau'\tau}]$ on the subspace V_{lm} of V is given by

$$\begin{aligned}&[l^2 - m^2]^{1/2} \{ X_{l-1}^{\tau\tau'} (X_{l-1}^{\tau'\tau} M_l^\tau - M_l^{\tau'} X_l^{\tau'\tau}) \pm (X_{l-1}^{\tau\tau'} M_l^{\tau'} - M_l^\tau X_l^{\tau\tau'}) X_l^{\tau'\tau} \} - m \{ X_l^{\tau\tau'} (X_l^{\tau'\tau} Z_l^\tau - Z_l^{\tau'} X_l^{\tau'\tau}) \\ &\quad \pm (X_l^{\tau\tau'} Z_l^{\tau'} - Z_l^\tau X_l^{\tau\tau'}) X_l^{\tau'\tau} \} - [(l+1)^2 - m^2]^{1/2} \{ X_{l+1}^{\tau\tau'} (X_{l+1}^{\tau'\tau} P_l^\tau - P_l^{\tau'} X_l^{\tau'\tau}) \pm (X_{l+1}^{\tau\tau'} P_l^{\tau'} - P_l^\tau X_l^{\tau\tau'}) X_l^{\tau'\tau} \}.\end{aligned} \tag{5.14}$$

In the present context, this reduces, with the help of (2.7), (2.8), and (5.12), to

$$i[\Gamma_0,\Gamma_3]_{lm} = (F_3 - 4H_3\,\Delta_1)_{lm},$$
$$i\{\Gamma_0,\Gamma_3\}_{lm} = (F_-H_+ - H_-F_+)_{lm}. \tag{5.15}$$

Since (5.13) also holds, it is then easy to verify from (3.2) that we have the following identities on V:

$$[\Gamma_\mu,\Gamma_\nu] = -iJ_{\mu\nu} + 4i\tilde{J}_{\mu\nu}\,\Delta_1, \tag{5.16a}$$

$$\{\Gamma_\mu,\Gamma_\nu\} = 2g_{\mu\nu}(\Delta_2 + \tfrac{1}{4}I) - \{J_{\mu\sigma},J_\nu{}^\sigma\}, \tag{5.16b}$$

where, as usual,

$$J_{pq} = \epsilon_{pqr}H_r, \quad J_{0p} = -J_{p0} = F_p,$$
$$\tilde{J}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\rho\sigma}. \tag{5.17}$$

(Here latin subscripts run over 1,2,3; Greek over 0,1,2,3. We use the summation convention and set $J^{\rho\sigma} = g^{\rho\alpha}g^{\sigma\beta}J_{\alpha\beta}$. The metric tensor $g_{\mu\nu} = g^{\mu\nu}$ is diagonal, with $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$; and the alternating tensor has $\epsilon_{0123} = -1$.)

It is noteworthy that the indentities (5.16) are exactly those proved by Bracken[7] for the family of four-vector operators based on the direct sum of the irreducible representations $[\frac{1}{2},l_1]$ and $[\frac{1}{2},-l_1]$, $l_1\in\mathbb{C}$. These identities have some interesting consequences. Since Δ_1 can never vanish in the present context (as $l_0l_1\neq 0$), it follows that we never obtain the so(5,$\mathbb{C}$) commutation relations

$$[\Gamma_\mu,\Gamma_\nu] = -iJ_{\mu\nu}. \tag{5.18}$$

In fact, an analysis similar to that of Cant[8] shows that an infinite-dimensional Lie algebra will be generated by the Γ_μ in the present situation.

Supposing that (1.1) holds (with κ a number), the identity (5.16b) implies that, for sufficiently smooth ψ,

$$(\tfrac{1}{4}\partial^\mu\partial_\mu + \omega^\mu\omega_\mu)\psi = -\kappa^2\psi, \tag{5.19}$$

where $\omega_\mu = i\tilde{J}_{\mu\nu}\,\partial^\nu$ is the Pauli–Lubanski vector operator, so that

$$\omega^\mu\omega_\mu = \tfrac{1}{2}J_{\mu\nu}J^{\mu\nu}\partial_\sigma\,\partial^\sigma - J_{\mu\sigma}J_\nu{}^\sigma\,\partial^\mu\partial^\nu$$
$$= \Delta_2\,\partial_\sigma\,\partial^\sigma - \tfrac{1}{2}\{J_{\mu\sigma},J_\nu{}^\sigma\}\partial^\mu\,\partial^\nu. \tag{5.20}$$

If ψ is a wave function for a particle with mass m_l and spin l, then we will also have

$$\partial^\mu\,\partial_\mu\psi = -m_l^2\psi, \tag{5.21a}$$

$$\omega_\mu\omega^\mu\psi = -l(l+1)m_l^2\psi, \tag{5.21b}$$

and (5.19) then implies that

$$m_l^2 = \kappa^2/(l+\tfrac{1}{2})^2, \tag{5.22}$$

in agreement with (5.8). Equation (5.19) also determines the nature of generalized mass-spin relations for lightlike and spacelike solutions of (1.1). In this connection we remark that it can be seen from (5.16b) that $(\Gamma_0 + \Gamma_p)$ and Γ_p $(p = 1,2,3)$ are not diagonalizable, unlike Γ_0.

VI. CONCLUDING REMARKS

The structure of indecomposable representations of sl(2,$\mathbb{C}$) is rich and interesting from a mathematical point of view. Because of the central role played by this Lie algebra and associated group in relativistic physics, we might expect that the theory of its indecomposable representations should be of relevance to applications as well. However, it must be said that, following the present work and that of Ref. 17, it is by no means clear that relativistic wave equations of the form (1.1), based on such representations, are likely to prove useful in physics.

In the case of singular indecomposable representations, we have shown that a variety of four-vector operators and corresponding wave equations can be constructed, corresponding to the great variety of such representations, labeled by ladder diagrams as in Sec. II, and we do not claim to have exhausted the possibilities, even under the restrictive conditions (1)–(3) imposed in Sec. IV. Our main objective has been to produce illustrative examples. Only for a very restricted subclass of representations (corresponding to straight row diagrams) did we find examples with Γ_0 not nilpotent, although even then there is a considerable variety of possibilities, as we have seen in Sec. V. However, these all lead to mass-spin spectra of the Majorana type, a dissappointing result from the point of view of potential applications.

It could be that the solutions of (1.1), in cases based on indecomposable representations, ought to be interpreted, in general, in a different way than in cases based on irreducible representations. For example, we could consider ψ to belong to the representation $[\frac{1}{2}\to\frac{3}{2}]\oplus[\frac{1}{2}\to -\frac{3}{2}]$. (See Sec. II.) The subspace $U\subset V$,

$$U = \bigoplus_{l\geqslant\frac{3}{2}} V_l,$$

is then invariant under the action of the sl(2,$\mathbb{C}$) algebra. Moreover, since Γ_0 leaves each V_l invariant, U is also invariant under the action of Γ_0 and therefore, by (3.2), of all Γ_μ. It is also invariant under the action of κ if that is a multiple of the identity operator on V. The component in U of each ψ satisfying (1.1) could then be "factored out" in order to construct an unusual "gauge description" of a massive spin-$\frac{1}{2}$ particle, i.e., we could regard as physically equivalent two ψ's that differed only on U. Whether this would lead to new physics would depend on how the infinite component wave function (field) could be coupled to other fields.

Another possibility is that the equations (1.1) of interest here are not those with κ nonsingular, as usually considered, but rather ones with singular κ, associated with gauge descriptions of *massless* particles. Singular scalar operators arise naturally in the present context. For example, it follows from (2.12) that the operator whose matrix on V_{lm} is a_0 for $l<|l_1|$ and a_1 for $l\geqslant|l_1|$, is a nilpotent sl(2,$\mathbb{C}$) scalar, as is the operator whose matrix is zero for $l<|l_1|$ and δ for $l\geqslant|l_1|$.

Alternatively, these interesting representations of sl(2,$\mathbb{C}$) may of course have applications to physics, not involving relativistic wave equations (1.1) at all.[13,14,16] In any event, we hope to have made their study more accessible.

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