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Warranty Menu Design for a Two-Dimensional Warranty

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ABSTRACT

Fierce competitions in the commercial product market have forced manufacturers to provide customer-friendly warranties with a view to achieving higher customer satisfaction and increasing the market share. This study proposes a strategy that offers customers a two-dimensional warranty menu with a number of warranty choices, called a flexible warranty policy. We investigate the design of a flexible two-dimensional warranty policy that contains a number of rectangular regions. This warranty policy is obtained by dividing customers into several groups according to their use rates and providing each group a germane warranty region. Consumers choose a favorable one from the menu according to their usage behaviors. Evidently, this flexible warranty policy is attractive to users of different usage behaviors, and thus, it gives the manufacturer a good position in advertising the product. When consumers are unaware about their use rates upon purchase, we consider a fixed two-dimensional warranty policy with a stair-case warranty region and show that it is equivalent to the flexible policy. Such an equivalence reveals the inherent relationship between the rectangular warranty policy, the L-shape warranty policy, the step-stair warranty policy and the iso-probability of failure warranty policy that were extensively discussed in the literature.

Key words: Flexible warranty policy, stair-case warranty, accelerated failure time model, reliability.

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1 INTRODUCTION

A product warranty is an assurance provided by a manufacturer who commits to rectifying product failures within the warranty coverage. Existing research on warranty has revealed multiple roles a warranty is playing in marketing: It provides insurance to risk-averse customers, signals the quality of a product (Liu et al. 2013) and works as an incentive mechanism for manufacturers to improve product quality (Murthy and Blischke 2006, Wu 2013). Many companies thus use warranties as a marketing tool for their products by highlighting their attractive product warranties, such as the Templer Systems in the test and measurement equipment market (http://www.templersystems.com/about-templer-systems.asp), Jaguar and Hyundai in the automobile market, Samsung, Sears and Panasonic in the home electronics market (Park and Pham 2012)

The warranty coverage can be regarded as an important product attribute that affects both buyers' decisions and manufacturers' profit (Chattopadhyay and Rahman 2008, Huang et al. 2007, Liu et al. 2007). However, customer behaviors are heterogeneous, e.g., customer use rates (Lawless et al. 2009, Lawless et al. 1995, Padmanabhan 1995, Ye et al. 2013) and customer risk preferences (Gallego et al. 2014, Hartman and Laksana 2009), and hence customers value a warranty coverage differently (Liu et al. 2013, Xie and Liao 2013). This creates variations in insurance demands and in the willingness to pay for the warranty from customers' end. As such, having only one warranty coverage implies a degree of unfairness for the customer. From the manufacturer's end, nevertheless, a single warranty coverage is not able to attract "extreme" customers such as users with low use rates. To cater to these variations, many firms would prefer to provide a flexible warranty that offers a customer a warranty menu with several different choices. A customer can then choose the best formfitting one from the menu. Evidently, a flexible policy gives the manufacturers a better way to market their product as it achieves a higher degree of fairness (from the customers' feelings) for the customers, and thus a higher degree of satisfaction (Kurata and Nam 2013). As a typical form of the flexible warranty, the extended warranty has received wide applications in many firms, e.g., Sears, Apple, GM, etc. (Padmanabhan 1995). An extended warranty is an optional purchase of service contract for the consumer and provides additional coverage for an item after the expiry of the manufacturers' base warranty. The rationality of providing an extended warranty can be explained by heterogeneous customer risk preferences (Wu and Longhurst 2011), heterogeneous customer use rates (Shafiee and Chukova 2013), and both of these two factors (Wu 2014). Alternatively, the retailers' service plan can be regarded as another form of optional warranty contract. The variations to the base warranty can be regarded as service customization for cost reduction (Jiang 2015).

Traditionally, many products are sold with one-dimensional warranties, which expire after a specific time period. Under this circumstance, the above extended warranty serves as a flexible warranty by providing the customers an option to buy an extended period of warranty coverage with some expense. Most extended warranties and service plans are provided by retailors. Lam and Lam (2001) studied optimal decision-making on extended warranties for manufacturers and consumers. The model was extended by Jack and Murthy (2007) and Gallego et al. (2014) to allow the customers to choose the starting date and length of the extended warranty. Hartman and Laksana (2009) used dynamic programming to derive a menu of extended warranties for the customers. On the other hand, the past two decades have witnessed a substantial growth in both research and applications on two-dimensional (2-D) warranties. A 2-D warranty is defined by a region in a 2-D plane with the age and usage representing the two axes. The key tenet of implementing the 2-D warranties is the heterogeneity in customer use rates (Wang et al. 2015). For overviews of the related research, readers are referred Murthy and Blischke (2006). Most products sold with a 2-D warranty have a rectangular region with two parameters – W(age limit) and U (usage limit), being the same for all customers. The most common example is the rectangular warranty for cars offered by automobile manufacturers. Other products sold with this policy include locomotive traction motor (Eliashberg et al. 1997), aircrafts, jet engines (Gertsbakh and Kordonsky 1998), motors (Pal and Murthy 2003), printers and excavators. For products sold with 2-D warranties, many studies, e.g., Murthy et al. (1995), have claimed that the warranty policy with a single rectangular region favors users with medium use rates yet is inequitable to users with either low or high use rates. It is thus of interest to provide a flexible 2-D warranty to achieve higher satisfaction among all users.

The literature on the 2-D warranty focuses on failure modelling and warranty cost estimation under a base warranty and a given maintenance policy, e.g., Murthy et al. (1995), Ye et al. (2013) and Gupta et al. (2014). Some studies further optimized the warranty coverage and the maintenance setting by minimizing the servicing cost, e.g., Chen and Popova (2002), Huang et al.

(2013) and Wang et al. (2015). However, few studies on flexible 2-D warranties are found in the literature. Shahanaghi et al. (2013) suggested a fixed extended 2-D warranty region on top of the base warranty, under which the optimal maintenance policy that minimizes a manufacturer's servicing cost was derived. Optimal pricing of this extended warranty was studied by Tong et al. (2014). However, considering the different usage behaviors, offering a fixed extended 2-D warranty may not be attractive to most customers. In addition, a 2-D warranty is often provided by the original equipment manufacturer directly. For example, direct manufacturer auto sales are in vogue in Japan and Korea. Under this scenario, an extended warranty given a 2-D base warranty seems to be cumbersome, as the manufacturers can provide a flexible warranty in a more direct manner. A real world example is the building machines sold by the KUHN group. Its machines such as excavators are sold with 2-D warranties based on the accumulated operating hours and age. It is worth noting that the company directly provides a flexible warranty with a warranty menu, so that customers can choose the one that matches their needs. The motivation behind the flexible warranty policy is to ensure a fair shake for all customers, as customers do not use machines in the same way. This flexible warranty policy is highlighted in the company's website as an attractive feature for customers. However, optimal design of such a flexible warranty menu is still an open question.

This study is aimed at proposing and designing a warranty menu for a flexible 2-D warranty policy. Our objective is to achieve a degree of fairness for users with different use rates by providing the users with a warranty menu with a collection of rectangular regions. The design is done by classifying customers into n different groups in terms of use rates, each of which is designated with a different $(w; u)$ combination such that some fairness criterion is met. Determination of the parameters for each combination is the focus of the study. We first consider the case in which consumers know their use rates. We show that under certain realistic conditions, it is able to design a flexible 2-D warranty menu such that customers will choose the right region designated for them. This warranty menu gives the manufacturers a better way to promote their products to both light and heavy users. An extreme case happens when n tends to infinity, meaning that the manufacturer customizes warranty for each customer. A difficulty faced by the customization is that customers may lie to the manufacturer. We find that when the marginal effect of use rates on product failures is not very significant, customers are willing to tell their true use rates to the manufacturer. In the literature, Padmanabhan and Rao (1993) also considered the design of flexible warranty policies under usage heterogeneity. Our study differs from theirs in that we consider 2-D warranties and use a dynamic failure model. The dynamic model enables us to obtain an explicit warranty menu.

We then consider the scenario where customers are not sure of their use rates. A rational way to ensure fairness is to merge all warranty regions in the original menu and form a fixed policy with a stair-case warranty region. This approach extends the conventional wisdom in the following ways. Competition among manufacturers to increase the market share forces the manufacturers to offer customers friendly warranties policies. This motivates researchers to propose 2-D warranties of different shapes. Blischke and Murthy (1992), Singpurwalla and Wilson (1993) and Murthy et al. (1995) studied the L-shape warranty region which results in better coverage for both light and heavy users. To achieve a higher fairness, Chun and Tang (1999) even suggested the iso-cost region, under which the expected cost for each customer, regardless of his/her use rate, is the same. When the repair assumed for each failure is minimal, the iso-cost region is the same as the iso-probability-of-failure (iso-PF) region, under which the probability of failure for each user is the same. Our approach reveals that the L-shape warranty can be readily obtained by dividing customers into two groups. If customers are divided into n groups, then we are able to obtain an n-stair stair-case warranty policy, which is a natural generalization of the L-shape policy. We also show that when n tends to infinity, the stair-case policy converges to the iso-PF policy. Therefore, this study reveals the relations between the rectangular, the limited L-shape, the stair-case and the iso-PF policies to some extent. In addition, our research naturally leads to design of these policies, which is important yet not discussed in literature. In this paper, we use the failure probability as the planning criterion because of its simplicity. When there are other factors such as imperfect repairs and random repair costs for different failure modes, one can readily derive the expected cost and use it as planning criterion. The procedure is almost the same but the notation will be much more complicated.

The rest of this paper is organized as follows. Section 2 formulates the problem with some assumptions about product failures and customer behaviors. Section 3 is devoted to design of a flexible policy with a warranty menu under the assumption that consumers know their use rates. Section 4 investigates design of a fixed policy with a stair-case warranty region by assuming that consumers are not sure about their usage behavior. Section 5 illustrates the design procedures. Section 6 concludes the paper and points out some possible topics for future research.

2 MODEL FORMULATION

2.1. Customer Behavior

Consider a manufacturer marketing a single product of known quality to a population of customers with heterogeneous usage behaviors. We assume that each customer has a constant use rate over the use period while the rate varies across the population. Rationality of this assumption is supported by warranty data analysis, which reveals that the cumulative usage for a customer is approximately linear over the age of a product (Jung and Bai 2007, Lawless et al. 2009, Lawless et al. 1995). Variation in the rate is modeled by a random variable with a cumulative distribution function (CDF) $G(z)$ and a probability density function (PDF) $g(z)$. Assume the manufacturer knows the distribution, either through a customer survey, or from previous products. This is reasonable as products with 2-D warranties are often expensive and are sold by famous manufacturers who have sufficient experience on the market.

Most products sold with 2-D warranties are expensive. So users may carefully consider every aspect, including their use rates, before making the buying decision. Their use rates can be known from a back-of-the-envelope estimate or from past usage records. When one wants to buy a new car, for example, one may readily determine the use rate based on the record of old cars, if any. Therefore, customers usually have a general idea about their own use rates. We will consider design of a self-selecting warranty menu for this scenario in Section 3. In some cases, consumers may not be aware of their use rates. Under this case, the manufacturer can combine the warranty menu to form a stair-case warranty policy, as discussed in Section 4.

We focus on the usage heterogeneity, and do not consider the consumer moral hazard problem and risk reversion. Therefore, we assume that the population is homogenous in terms of risk attitudes, and that customers will not alter their use rates after purchasing an item, implying that they reveal their true use rates at selection.

2.2. Modeling Failures

A product is designed for some nominal use rate z_0, conditional on which the distribution for the time-to-first-failure T_0 is $F_0(t)$ and the associated failure rate is $\lambda_0(t)$. When an item is sold, the use rate Z of a customer differs from the nominal z_0. We model the effect of the rate on the first failure time through the accelerated failure time (AFT) model. Denote by z a realization of Z and T_z the first failure time under z. Conditional on $Z = z$, T_z is linked to T_0 through the AFT relationship (Jack et al. 2009, Lawless et al. 2009, Shahanaghi et al. 2013)

$$T_z = T_0 (z_0/z)^{\gamma}, \tag{1}$$

where $\gamma > 0$ is the acceleration coefficient. Let $F_z(t)$ and $\lambda_z(t)$ be the CDF and failure rate for T_z. They are linked to $F_0(t)$ and $\lambda_0(t)$ through (1).

Here, we assume that T_z is decreasing with z. In practical situations undue underuse may also increase the failure rate. However, products sold with 2-D warranties are usually expensive and so there is little incentive for underuse. In addition, quantifying the effect of underuse on product failures may be difficult as acceleration cannot be used. These two reasons might explain that models for quantifying underuse on 2-D product failures are not found. Therefore, this article does not consider underuse as well.

2.3. Warranty Policy and Criterion for Fairness

The strategy is based on dividing customers into n groups according to their use rates, each taking up $1/n$ of the whole population. Therefore, the sizes of the groups are approximately constant. The k-th group is provided with a rectangular warranty $\Omega_{(W_k, U_k)}$ defined by the vertex (W_k, U_k). As such, the manufacturer is able to come up with a warranty menu of n different choices. Customers choose a warranty from this menu according to their use rates. Because customers do not change their use rates after purchase, the rational choice for a customer is to choose from the n options a warranty with the longest warranty period. An extreme case arises when n tends to infinity, meaning that the manufacturer designs warranty to meet the needs of each customer. It is noted that Shahanaghi et al. (2013) also discretized the use rate, but their purpose is to simplify computation when marginalizing over the random use rate.

Design of such a warranty menu boils down to the determination of parameters of (W_k, U_k), $k = 1, 2, \cdots, n$. The objective is to ensure that the flexible policy is fairer to all customer groups. This requires a criterion for fairness. The criterion is usually based on expected warranty costs for all groups. The warranty costs depend on the type of repair assumed, e.g., minimal, imperfect or perfect. Since a 2-D product is usually repaired rather than replaced upon failure, and because the effect of imperfect repair is hard to quantify, this study adopts the minimal repair strategy. The minimal repair strategy for products with 2-D warranties is often seen in the literature, e.g., Majeske (2007) and Lawless et al. (2009). If minimal repair is assumed, this criterion is equivalent to that based on the equal probability of failures. When the expected probability of failure for each group is the same, say p, the probability of failure for the whole population is also equal to p, meaning that the manufacturer is able to control the fraction failings. In addition, The failure time distribution can be easily estimated from lab tests or from warranty data analysis, e.g. see Ye et al. (2013).

3 WARRANTY DESIGN WHEN USERS KNOW THEIR RATES – A FLEXIBLE WARRANTY

We first consider the scenario where customers know their use rates. A manufacturer can thus provide a warranty menu $\{\Omega_{(W_1,U_1)}, \Omega_{(W_2,U_2)}, \ldots, \Omega_{(W_n,U_n)}\}$ consisting of n rectangular warranty choices. A consumer chooses a favorable warranty from this menu upon purchase. Design of the menu is the focus of this section.

3.1. Expected Probability of Failure

We have uniformly divided customers into n groups based on the use rate. The range of rates for the k-th group is denoted by $[z_{k-1}, z_k),\ 1 \le k \le n$; with

$$0 = z_0 < z_1 < \cdots < z_{k-1} < z_k < z_{k+1} < \cdots < z_n = \infty,$$

where $z_k = G^{-1}(k/n)$, $G^{-1}(\cdot)$ being the quantile function of Z. We use a conditional approach to evaluate the probability of failure by first conditioning on the use rate and then taking the un-conditioning probability of failure for each group. Consider an item sold to a user with rate z, $z_{k-1} < z \le z_k$. The failure rate and CDF of T_z are obtained via the accelerated failure time model (1) as

$$\lambda_z(t) \equiv \lambda(t \mid Z = z) = (z/z_0)^{\gamma} \cdot \lambda_0(t(z/z_0)^{\gamma}),$$

and

$$F_z(t) = F_0((z/z_0)^{\alpha} t).$$

Recall that the k-th group is provided with $\Omega_{(W_k,U_k)}$. Conditional on $Z = z$, warranty ceases at W_z, which is given by

$$W_k(z) = \begin{cases} W_k, & z \le U_k/W_k; \\ U_k/z, & z > U_k/W_k. \end{cases}$$

The probability of warranty failures for this user is $F_z(W_k(z))$. Therefore, the probability of failure for users in the k-th group is given by

$$P_k\left(\Omega_{(W_k,U_k)}\right) = n \int_{z_{k-1}}^{z_k} F_z(W_k(z))\, dG(z)$$

The integrals can be evaluated through numerical procedures, e.g., the Riemann-Stietjes sums method (Ye et al. 2012).

3.2. Design of the Warranty Policy

A necessary condition for successful implementation of the flexible warranty policy is that customers in the k-th group will choose $\Omega_{(W_k,U_k)}$, the one designated for them. Theorem 1 shows the equivalent statements that customers will choose the designated warranty.

THEOREM 1. Consumers are classified into n groups as described above. Suppose that $W_1 > W_2 > \dots > W_n$ and $U_1 < U_2 < \dots < U_n$. The following three statements are equivalent.

(a) A collection of rectangular warranty regions $\left\{\Omega_{(W_1,U_1)}, \Omega_{(W_2,U_2)}, \cdots, \Omega_{(W_n,U_n)}\right\}$ serves as a flexible warranty policy, where users in the k-th group choose $\Omega_{(W_k,U_k)}$, $k = 1, 2, \cdots, n$.

(b) The user with rate $z_{k/n}$ is indifferent between $\Omega_{(W_k,U_k)}$ and $\Omega_{(W_{k+1},U_{k+1})}$ for $k = 1, 2, \cdots, n-1$.

(c) Denote by $\Omega_{\left(W_k^{(c)},U_k^{(c)}\right)} = \Omega_{(W_k,U_k)} \cap \Omega_{(W_{k+1},U_{k+1})}$, $k = 1, 2, \cdots, n-1$. Then $U_k^{(c)}/W_k^{(c)} = z_{k/n}$.

■

This theorem establishes basic properties of the flexible policy. It tells that if we draw all the warranty regions on the $W - U$ plane, the line $u = z_{k/n} w$ should pass through the point $(W_k^{(c)}, U_k^{(c)})$. This property is useful when we use the graphical technique to design the flexible warranty. Before introducing the graphical technique, the following theorem is useful.

Given a failure probability p, we can find a series of points (w_k, u_k) such that $P_k\{\Omega_{(w_k,u_k)}\} = p$ for the k-th group. Denote by Π_k the collection of all these points and C_k the curve consisting of all these points on the $W - U$ plane. This means that the curve C_k is obtained such that a rectangular warranty defined by any point on C_k would give the same probability of failure p for this group. The next theorem states the relationship between C_k and C_{k+1}.

THEOREM 2. Given a probability of failure p, we have the following results:

(a) If $\gamma < 1$, then C_k and C_{k+1} crossover at some point $(W_k^{(c)}, U_k^{(c)})$. Moreover, when $W > W_k^{(c)}$, C_k is above C_{k+1}. When $W < W_k^{(c)}$, C_k is below C_{k+1}.

(b) If $\gamma > 1$, then C_k is always above C_{k+1} for any $W > 0$.

■

Theorem 2 lays the foundation for the existence of such a warranty menu that fulfills our objective. The acceleration coefficient β reflects the marginal effect of the rate on product failures. A small value of β implies a mild effect while a large value represents a significant effect. Theorem 2 implies that when $\gamma > 1$, meaning that the use rate has a significant effect, the fairness of equal probability of failure cannot be achieved. This result is elucidated in Corollary 1.

Corollary 1. When $\gamma > 1$, it is not possible to provide a flexible warranty policy $\{\Omega_{(W_1,U_1)}, \Omega_{(W_2,U_2)}, \ldots, \Omega_{(W_n,U_n)}\}$ such that the expected probability of failure is the same for all groups.

■

Most 2-D warranty data analyses using the AFT model reveal that $\gamma < 1$, e.g., Lawless et al. (1995) and Lawless et al. (2009). For example, the dataset in Lawless et al. (1995) gives $\gamma = 0.9$, and the other in Lawless et al. (2009) gives $\gamma = 0.58$. This is a reasonable result because of the concept of robust design used in the automobile industry. The parameter β reflects the sensitivity of the failure process to the use rate. An automobile should be robust to the use rate in the sense that the failure rate should not change too dramatically with the use rate. If $\gamma > 1$, the effect of the use rate on the failure process would be too significant. The change of the γ value from 0.9 in the 1995 dataset to 0.58 in the 2009 dataset is a strong indicator of robustness of modern automobiles.

When $\gamma < 1$, we are able to design a flexible warranty policy $\{\Pi_{(W_1,U_1)}, \Pi_{(W_2,U_2)}, \cdots, \Pi_{(W_n,U_n)}\}$ for each customer group based on the results in Theorems 1 and 2. The design procedure is provided as follows.

Procedure 1: Design of a flexible warranty policy

- The manufacturer specifies the target value for the probability of failure p.
- On the $W - U$ plane, draw C_i for the i-th group, $i = 1, 2, \cdots, n$. Then draw the lines $L_{k/n}$: $u = z_{k/n} w$ for $k = 1, 2, \ldots, n - 1$.
- On the right-hand side of $L_{1/n}$, choose a point (W_1^*, U_1^*) on C_1 such that the manufacture would like to offer $\Pi_{(W_1^*,U_1^*)}$ to the first group.
- Draw the horizontal line $u = U_1^*$ and determine its crossover point with $L_{1/n}$, which is denoted as (W_2^*, U_1^*). Draw the vertical line $w = W_2^*$ and determine its crossover point with C_2, which is denoted by (W_2^*, U_2^*).
- Following a similar procedure as above, we determine (W_3^*, U_3^*), …, (W_n^*, U_n^*).
- A flexible warranty policy determined from this procedure is

$$\left\{\Pi_{(W_1^*,U_1^*)}, \Pi_{(W_2^*,U_2^*)}, \ldots, \Pi_{(W_n^*,U_n^*)}\right\}.$$

■

A straightforward way to determine (W_1^*, U_1^*) is that we can draw the line $L_{1/2n}$: $u = z_{1/2n} w$, and use the crossover point of $L_{1/2n}$ and C_1 as (W_1^*, U_1^*). In practical applications when a warranty menu is provided, n should not be too large. Otherwise customer will be confused to select the appropriate warranty coverage. Our suggestion is to use $n = 2$ or 3.

3.3. Special Case [$n = 2$]

When $n = 2$, we classify the customers into two groups, i.e., light and heavy users. The principal purpose of a flexible warranty policy is to balance the interest between these two groups. When $\gamma < 1$, we can determine a flexible warranty policy $\left\{\Omega_{(W_1^*;U_1^*)};\Omega_{(W_2^*;U_2^*)}\right\}$ following Procedure 1. Note that we need to subjectively specify $\Omega_{(W_1^*;U_1^*)}$ in this procedure. If we denote the crossover point of C_1 and C_2 as $\left(W_1^{(c)};U_1^{(c)}\right)$ and set $(W_2^*;U_2^*) = \left(W_1^{(c)};U_1^{(c)}\right)$, then the flexible policy degenerates to a simple 2-D warranty with warranty region of $\left(W_1^{(c)};U_1^{(c)}\right)$. If a simple 2-D warranty has to be provided, then the above degenerated one is recommended. Using this degenerated warranty region, the expected probabilities of failure for the whole population, and the groups with low and high use rates are the same. Another use of this degenerated region is to determine $(W_1^*;U_1^*)$ for Procedure 1. We can subdivide the first group into two groups and determine the crossover point of their C-curves. This point can be used for $(W_1^*;U_1^*)$.

Some remarks are made here. Customers often have a good idea about their use rates. This is especially true for products sold with 2-D warranties, which are most often expensive. Often, the manufacturer knows the use-rate distribution of the whole customer population but is not aware of the individual usage behavior. As such, there is information asymmetry. Under the flexible warranty policy the customers choose an appropriate warranty that gives them the maximum coverage. In a sense, the manufacturer has achieved a separated solution for different customers as opposed to a pooled solution that all customers choose the same, with some subsidizing others. Therefore, a higher customer satisfaction is achieved through application of the flexible policy.

3.4. Limiting Case $[n \to \infty]$

When $n \to \infty$, the manufacturer customizes warranty for each customer. Each customer reports the use rate z, and then the manufacturer provides a rectangular region $(W_z; zW_z)$ to the customer such that the probability of failure equals p. Therefore, $W_z = F_z^{-1}(p)$. A problem of interest under this circumstance is the inherent asymmetric information. The manufacturer may know the overall use rate distribution but does not know the rate for each customer. Therefore, the customer may have motivation to falsely declare the rate. We find that this happens when $\gamma > 1$. The result is summarized as follows.

Theorem 3. Suppose a customer with use rate z reports a rate z' to the manufacturer and the manufacturer customizes a rectangular region $(W_{z'}, z'W_{z'})$ such that $W_{z'} = F_{z'}^{-1}(p)$.

(a) When $\gamma > 1$, customers have motivation to report use rates lower than their true rates, i.e., $z' < z$.

(b) When $\gamma < 1$, customers will report their true use rates to the manufacturer, i.e., $z' = z$.

■

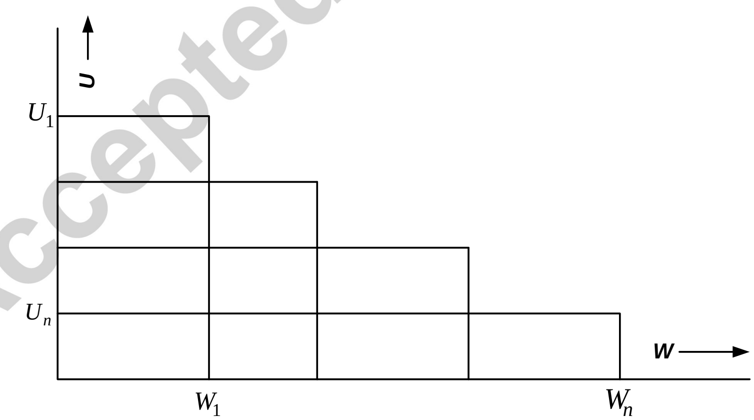
The proof is analogous to that of Theorem 2, and thus it is omitted. Theorem 3 states that it is impossible to customize warranty for each customer when $\gamma > 1$. This does make sense, because when $\gamma > 1$, meaning that the product is sensitive to the use rate, a small decrease in the ue rate would greatly decease the inclination of failure. Given the same probability of failure p, reporting a use rate slightly smaller than the true use rate would lead to a warranty region that subsumes the region associated with the true use rate. Therefore, the consumer has incentives to report a smaller use rate in a bid to prolong his warranty length. This provides some management insights. If manufacturers would like to provide personalized warranty service to customers, they need to make sure that the product is robustly designed such that it is relatively insensitive to the use rate. In addition, this theorem implies that if the product is designed to be sensitive to the use rate, then it may not be attractive to the heavy users. This is because the manufacturer has to provide a warranty with a small region, i.e., small coverage, to the heavy user for the sake of controlling the overall probability of warranty failure.

4 WARRANTY DESIGN WHEN CUSTOMERS ARE UNCERTAIN ABOUT THEIR RATES – STAIR-CASE WARRANTIES

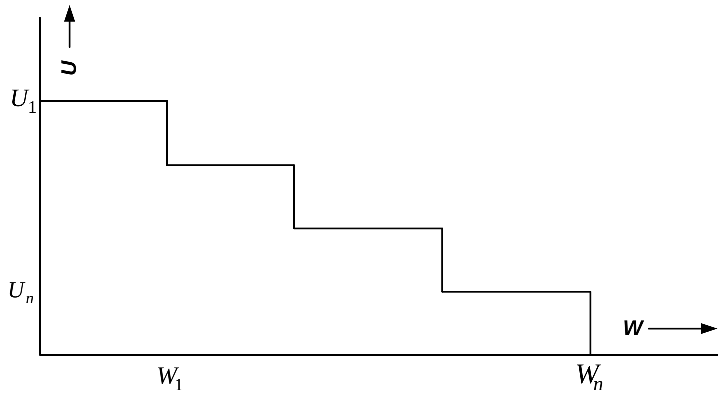
A manufacturer may know the users' use rate distribution from a customer survey or from the analysis of warranty data. However, customers might not be aware of their use rate, especially when the product is not so expensive or important to them. Under this circumstance, a flexible warranty (i.e., a warranty menu) that asks the users to self-reveal their use rates upon purchase seems imprudent. This section considers a fixed policy under which the manufacturer provides a fixed warranty with a stair-case region to the customers. This fixed policy achieves the same efficacy as the flexible policy and does not require customers to know their rates. But a deficiency is that it is unwieldy to explain to customers compared to the flexible warranty.

4.1. Converting a Flexible Warranty to a Fixed Warranty

Based on the estimated use rate distribution, the manufacturer is always able to work out a flexible warranty policy $\{\Omega_{(W_1,U_1)}, \Omega_{(W_2,U_2)}, \cdots, \Omega_{(W_n,U_n)}\}$ following Procedure 1 in the previous section, as long as $\gamma < 1$. When the consumers are unsure about their rates, however, they are not able to select the correct warranty region designated for them. Under this circumstance, we can combine the flexible warranty with menu $\{\Omega_{(W_1,U_1)}, \Omega_{(W_2,U_2)}, \cdots, \Omega_{(W_n,U_n)}\}$ to form a fixed policy with a stair-case warranty region as $\Omega_{(W_1,U_1)} \cup \Omega_{(W_2,U_2)} \cup \cdots \cup \Omega_{(W_n,U_n)}$. A schematic diagram of combining a flexible warranty to form an L-shape warranty is given in Figure 1. If such a stair-case warranty is offered to customers, we can achieve the same degree of fairness as the flexible warranty, as the warranty expiry date for each customer does not alter compared to the flexible counterpart. In addition, an advantage of offering such a policy is that customers are not required to specify their use rate. But compared with the flexible warranty policy, it may be less convenient to explain the stair-case policy to the buyers.



(a)



(b)

Figure 1. Converting a flexible warranty with n warranty choices to a stair-case warranty: (a) a flexible policy with a menu containing n rectangular warranty choices, and (b) a fixed policy with stair-case warranty.

When $n = 2$, the resulting stair-case region boils down into an L-shape, which has been studied by a number of researchers, e.g., see Blischke and Murthy (1992), Singpurwalla and Wilson (1993), Murthy et al. (1995) and Chun and Tang (1999), among others. When $n > 2$, the resulting warranty region has n steps, which is a class of new shapes in the literature. This type of warranty region is friendly to users with low, medium and high use rates. An interesting question is what the step-shape looks like when n tends to infinity, which is answered in the next sub-section.

4.2. Limiting Case $[n \to \infty]$

Before proceeding to the main results, we need the definition of the iso-probability of failure (iso-PF) region determined by the iso-PF curve. The iso-PF region is obtained such that users with different use rates would have the same probability of failures. The frontier of this region is called the iso-PF curve. This definition is similar to the iso-cost region defined in Chun and Tang (1999). We find that under the limiting case when $n \to \infty$, the n-stair stair-case region converges to the iso-PF region. The result is elucidated in Theorem 4.

Theorem 4. Suppose that the n-step stair-case region is determined as described above. If $\gamma < 1$, then when $n \to \infty$, the stair-case region converges to the iso-PF region.

■

This theorem expounds the relationship between the stair-case regions and the iso-PF region. Without any doubt, the iso-PF policy achieves the highest degree of fairness in the sense that the probability of warranty failures for each customer is the same. The stair-case warranty can be regarded as a compromise policy in that instead of striving for an absolutely fair shake to all customers, we classify the users into several groups and aim to be equitable to each group. Compared to the iso-PF policy, the compromised version (i.e., the stair-case warranty) is much easier to implement.

5 NUMERICAL EXAMPLE

We illustrate how to use the graphical approaches proposed in Sections 3 and 4 for warranty design. Warranty data analysis reveals that the power law process is appropriate as the baseline failure rate (Jung and Bai 2007, Lawless et al. 2009, Majeske 2007). Following Lawless et al. (2009), we adopt the baseline failure rate as

$$\lambda_0(t) = \beta t^{\beta-1}/\eta^{\beta}; \ t > 0, \text{ where } \beta = 1.1 \text{ and } \eta = 47.2.$$

Lawless et al. (1995) and Lawless et al. (2009) mentioned that the log-normal distribution provides a good fit to mileage accumulation data for vehicles. Here, we use the lognormal distribution with PDF

$$g(z) = \frac{1}{z\sqrt{2\pi}\sigma}\exp\left(-\frac{(\ln z - \mu)^2}{2\sigma^2}\right); \ z > 0, \text{ where } \mu = 1.4 \text{ and } \sigma = 0.58.$$

Wu further assume that the baseline use rate is $z_0 = 1$, the acceleration coefficient is $\gamma = 0.8$, and the probability of failure targeted by the manufacturer is $p = 0.05$.

Consider $n = 2$, i.e., we divide customers into two groups. First off, we draw $L_{1/2}$, C_1 and C_2 on the $W - U$ plane. Set $(W_1^*; U_1^*)$ as the crossover point of $L_{1/4}$ and C_1. $(W_2^*; U_2^*)$ is then determined using Procedure 1. This process is illustrated in Figure 2. The manufacturer can provide a warranty menu with two choices $\Omega_{(W_1^*;U_1^*)}$ and $\Omega_{(W_2^*;U_2^*)}$. If customers are not sure about their use rate, manufacturer can provide a fixed warranty with an L-shape region of $\Omega_{(W_1^*;U_1^*)} \cup \Omega_{(W_2^*;U_2^*)}$.

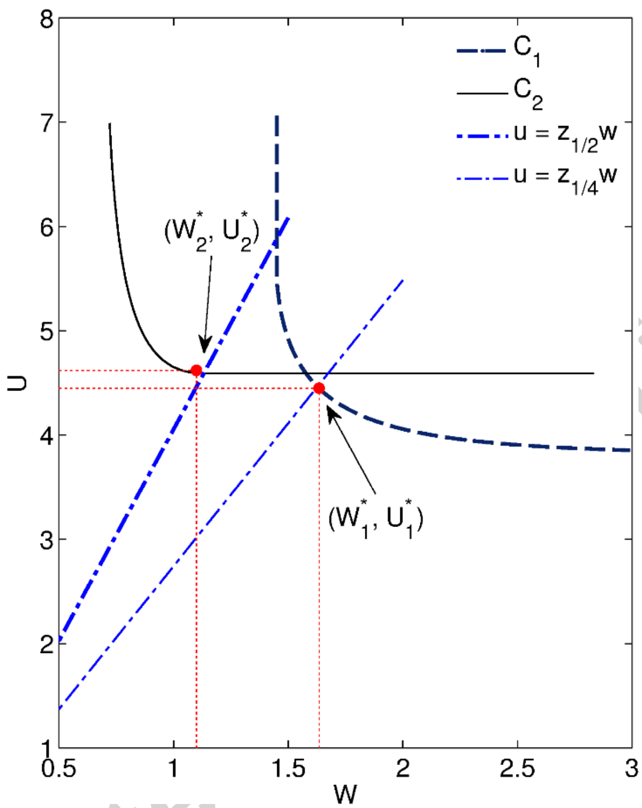


Figure 2. Design of a warranty menu with two rectangular warranty choices.

On the other hand, we can use Procedure 1 to obtain the degenerated policy. As illustrated in Figure 3, warranty region of this degenerated policy is defined by the crossover point of C_1 and C_2. If a fixed policy with a simple rectangle has to be offered, this degenerated warranty is recommended, because it is fairer to users of both high and low use rates.

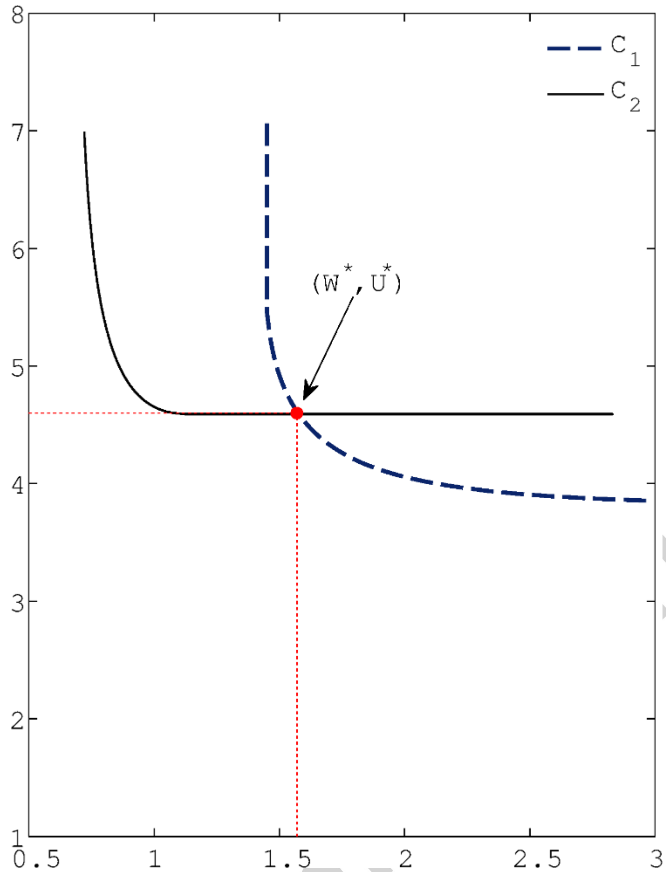


Figure 3. The degenerated case.

To obtain a higher degree of fairness, we may divide customers into 3 or more groups according to their use rates. As illustrated in Figure 4, we start with the crossover point $(W_1^*; U_1^*)$, which is chosen by further dividing the light use rate group into two subgroups and determine the degenerated point for these two subgroups. Afterwards, $(W_2^*; U_2^*)$ and $(W_3^*; U_3^*)$ can be sequentially determined. The manufacturer can provide a warranty menu with three choices to achieve a higher degree of fairness. Or alternatively, the manufacturer can provide a 3-step staircase warranty when customers are not aware of their use rates. As we classify customers into more groups, we achieve a higher degree of fairness by providing a warranty menu with more choices. But this will have to increase the size of the menu, especially when n is large. Therefore, there is a trade-off between simplicity and fairness.

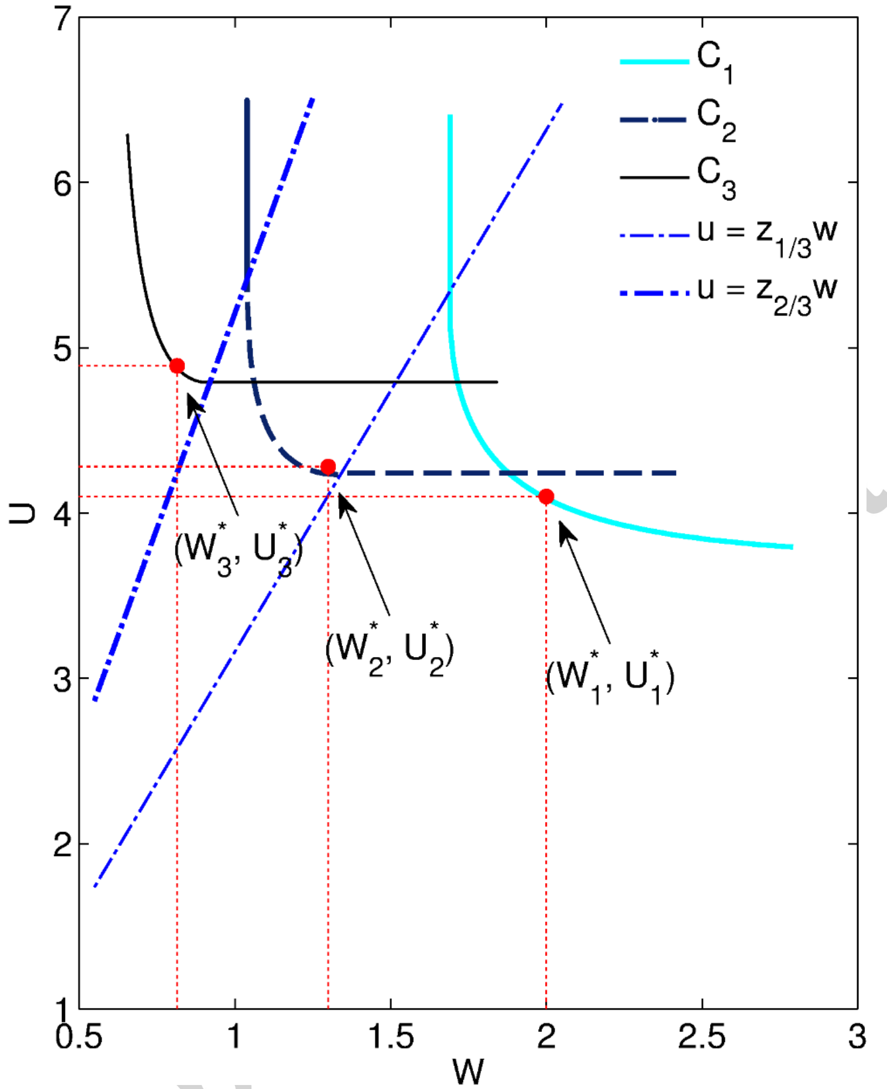


Figure 4. Design of a flexible warranty menu with three rectangular choices.

When η tends to infinity, the flexible policy is to customize a warranty region for each customer. If the manufacturer uses a fixed policy, then when η tends to infinity, the warranty tends to the iso-PF policy, as shown in Figure 5. Because the acceleration coefficient ˇ may have significant effect on the warranty design, we further investigate how the iso-PF curve changes over ˇ. As can be seen from Figure 5, ˇ has a significant effect on users with high use rates. If the product is not robust to the use rate, i.e., ˇ is large, then the manufacturer has to offer a shorter warranty period to high-use-rate users to control the failure probability. As a result, the product may not be attractive to these users. This result reveals the importance of robust product design.

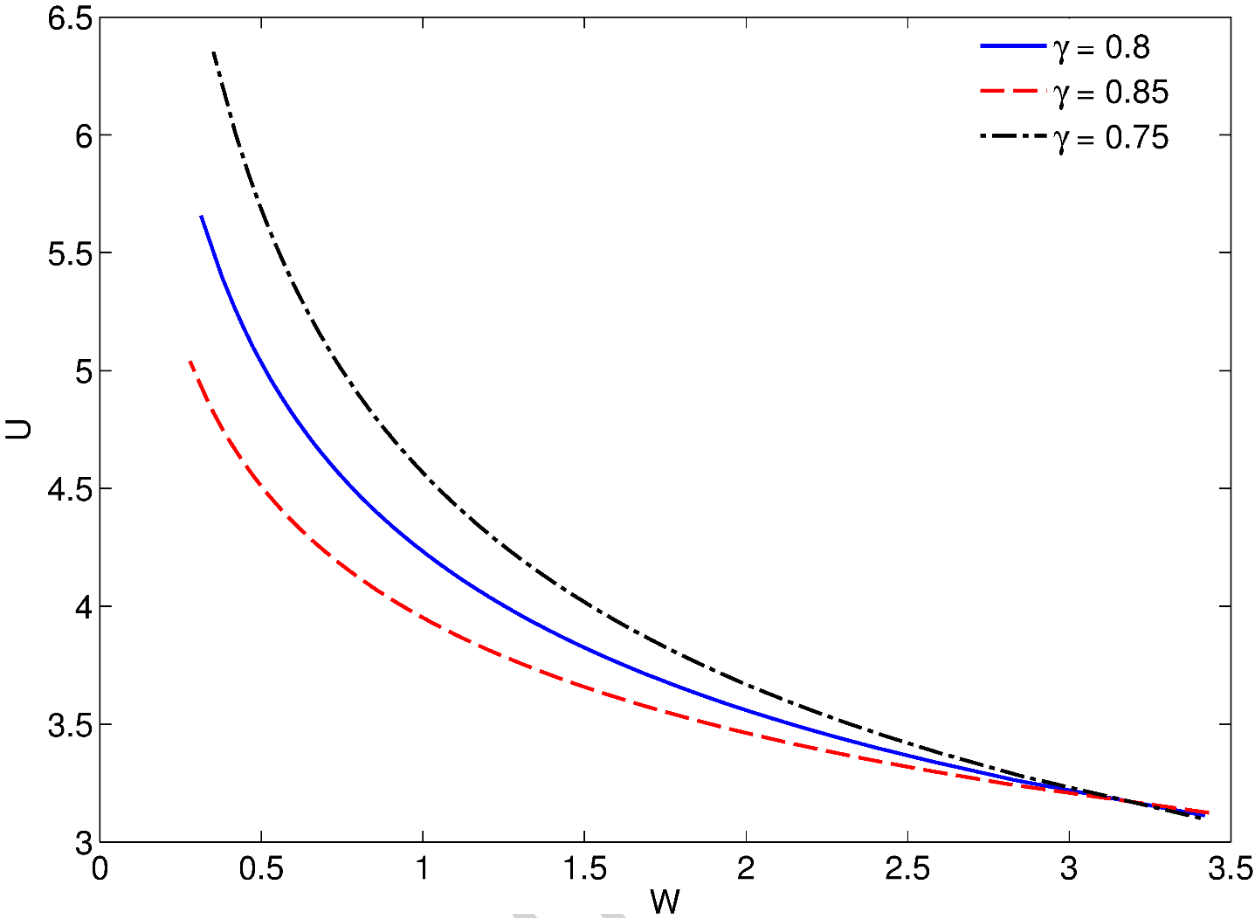


Figure 5. Design of the iso-PF warranty policy.

6 CONCLUSIONS

This study has investigated the design of a flexible 2-D warranty policy with a view to achieving a higher degree of satisfaction for customers of different usage behaviors. We showed that when the product failure is not very sensitive to customer use rates, the manufacturer is able to offer customers a warranty menu with a number of rectangular warranty choices such that each group of users has the same probability of warranty failures. In addition, we also found that it is impossible to customize warranty for a product that is sensitive to the usage behaviors. This gives some backing to the important research area of robust product design. In the case that consumers are unaware of their use rates, this study developed design of a fixed 2-D warranty policy with a stair-case warranty region. This design method successfully establishes the intimate relationship between the rectangular warranty, the L-shape warranty, the stair-case warranty and the iso-PF warranty. Design of such warranty policies can be easily determined through the graphical technique, and is potentially very important in practice. If a simple 2-D warranty with a rectangular region $(W; U)$ is provided, customers are apt to change their use rate towards $U = W$ (Singpurwalla and Wilson 1993). The flexible policy as well as the fixed warranty with stair-case region is able to mitigate this moral hazard problem. Moreover, these policies are able to encourage light and heavy users to buy the product, which put the manufacturer at an competitive edge in the advertisement and promotion.

The above findings have elaborated on the importance of the flexible 2-D warranty with a warranty menu consisting of a number of rectangular choices. Many factors are not considered in our design process, and thus we call for more attention to this interesting and important topic. Some possible topics for further research are as follows.

- Customers in different usage groups may have different risk attitude and different perceived values for a product. The manufacturer can thus use the warranty to screen customers. Price discrimination can be done by endowing each choice in the warranty menu a different price. Using the warranty menu to simultaneously provide fairer warranty and screening customers is an interesting topic of future research.
- We have assumed known product quality in this study. If the customers are not sure about the product quality, then the adverse selection, i.e., a customer chooses an inappropriate warranty from the menu, may happen. Remedies of such issue should be investigated in the future.
- Minimal repair has been assumed in this study. In practice, imperfect maintenance is also common. The design of a warranty menu in the presence of imperfect repair is also an interesting topic.

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APPENDIX

Proof of Theorem 1.

(a) $\Rightarrow$ (b): It follows directly from the fact that Z is a continuous random variable.

(b) $\Rightarrow$ (c): When the user with use rate $z_{k/n}$ is indifferent between $\eth_{(W_k;U_k)}$ and $\eth_{(W_{k+1};U_{k+1})}$, these two warranties should yield the same warranty period for him. Based on the condition $W_1 > W_2 > \ldots > W_n$ and $U_1 < U_2 < \ldots < U_n$, the warranty expiry date under $\eth_{(W_k;U_k)}$ is $U_k/z_{k/n}$ while the warranty expiry date under $\eth_{(W_{k+1};U_{k+1})}$ is W_{k+1}. Setting these two expiry dates to be equal yields the desiring results.

(c) $\Rightarrow$ (a): Consider a customer with use rate z in the k-th group.

If he chooses $\eth_{(W_k;U_k)}$, his warranty expiry date $W_k(z)$ is within the interval $[U_k/z; W_k]$. If he chooses $\eth_{(A_j;U_j)}$, $j > k$, then his warranty expiry date is $W_j(z) = W_j$. If he chooses $\eth_{(A_j;U_j)}$, $j < k$, then his warranty expiry date $W_j(z)$ is within the interval $[U_j/z_k; U_j/z_{k+1}]$

Applying the condition $W_1 > W_2 > \ldots > W_n$ and $U_1 < U_2 < \ldots < U_n$, we can verify that $W_j(z) \leq W_k(z)$ when $j \neq k$. Recall that the objective of the customer is to maximize his warranty period. Therefore, this customer will choose $\eth_{(W_k;U_k)}$.

Proof of Theorem 2.

Combine the k-th and the $(k+1)$-st groups to form a new group, which we call the K-th group. We can define Π_K and C_K similar to Π_k and C_k. For the elements in Π_K, when $U \to \infty$, $W \to \underline{W}$ where $\underline{W}$ satisfies

$$p = \frac{n}{2}\int_{z_{k-1}}^{z_{k+1}} F_z(\underline{W})\, dG(z).$$

When $W \to \infty$, $U \to \underline{U}$ where $\underline{U}$ satisfies $p = \int_{z_{k-1}}^{z_{k+1}} F_z(\underline{U}/z)\, dG(z)$. That is, $(\underline{W}, \infty)$ and $(\infty, \underline{U}) \in \Gamma_K$.

For the warranty region $\Omega_{(\underline{W},\infty)}$, it is easy to see that

$$P_k\left(\Omega_{(\underline{W},\infty)}\right) < P_{k+1}\left(\Omega_{(\underline{W},\infty)}\right).$$

Therefore, we have $P_k\left(\Omega_{(\underline{W},\infty)}\right) < p$ and $P_{k+1}\left(\Omega_{(\underline{W},\infty)}\right) > p$.

For the warranty region $\Omega_{(\infty,\underline{U})}$,

$$P_k\left(\Omega_{(\infty,\underline{U})}\right) = \frac{n}{2}\int_{z_{k-1}}^{z_k} F_z(\underline{U}/z)\, dG(z),$$

$$P_{k+1}\left(\Omega_{(\infty,\underline{U})}\right) = \frac{n}{2}\int_{z_k}^{z_{k+1}} F_z(\underline{U}/z)\, dG(z).$$

According to the mean value theorem of integrals, we can know that there exist x_1, x_2, $z_{k-1} < x_1 < z_k < x_2 < z_{k+1}$, such that

$$P_k\left(\Omega_{(\infty,\underline{U})}\right) = F_0\left[\frac{\underline{U}}{x_1}(x_1/z_0)^{\gamma}\right]$$

and

$$P_{k+1}(\Omega_{(\infty,\underline{U})}) = F_0\left[\frac{\underline{U}}{x_2}(x_2/z_0)^{\gamma}\right].$$

There exists a point $(W_m, U_m) \in \Gamma_k$ such that $U_m/W_m = z_{k/n}$. When $W < W_m$, $P_k\left(\Omega_{(W,U)}\right)$ is increasing in W. When $W > W_m$, $P_{k+1}\left(\Omega_{(W,U)}\right)$ is decreasing in W, and thus $P_k\left(\Omega_{(W,U)}\right)$ is increasing in W. In sum, $P_k\left(\Omega_{(W,U)}\right)$ is increasing in W, where $(W,U) \in \Gamma_p$. If $\gamma \geq 1$, then $\frac{U}{x_1}(x_1/z_0)^{\gamma} \leq \frac{U}{x_2}(x_2/z_0)^{\gamma}$ and thus

$$P_k\left(\Omega_{(\infty,\underline{U})}\right) \leq P_{k+1}\left(\Omega_{(\infty,\underline{U})}\right).$$

Then C_k is always on top of C_{k+1} for any $t > 0$. Therefore, the first part of this theorem holds.

When $\gamma < 1$,

$$\frac{U}{x_1}(x_1/z_0)^{\gamma} > \frac{U}{x_2}(x_2/z_0)^{\gamma},$$

and thus

$$P_k\left(\pi_{(1,\underline{U})}\right) > P_{k+1}\left(\pi_{(1,\underline{U})}\right).$$

According to the first part of this theorem, we can infer that there is an $(W^*, U^*) \in \Pi_p$ such that $P_l\left(\pi_{(W^*,U^*)}\right) = P_h\left(\pi_{(W^*,U^*)}\right) = p$ When $(W, U) \in \Pi_p$ and $W < W^*$, $P_k\left(\pi_{(W,U)}\right) < p < P_{k+1}\left(\pi_{(W,U)}\right)$. If there exist U_1 and U_2 such that $P_k\left(\pi_{(W,U_1)}\right) = P_{k+1}\left(\pi_{(W,U_2)}\right) = p$, then $U_1 > U_2$. Similarly when $(W, U) \in \Pi_p$ and $W > W^*$, $P_k\left(\pi_{(W,U)}\right) > p > P_{k+1}\left(\pi_{(W,U)}\right)$. If there exist U_1 and U_2 such that $P_k\left(\pi_{(W,U_1)}\right) = P_{k+1}\left(\pi_{(W,U_2)}\right) = p$, then $U_1 < U_2$.

In sum, the third part follows.

■

Proof of Theorem 4.

For a given n, Suppose the range of use rate for the i-th group is $\left[y_{(i-1):n}, y_{i:n}\right)$, $i = 1, 2, \cdots, n$, where $y_0 = 0$ and $y_n = \infty$. Between the lines $L_{(i-1):n} : u = y_{(i-1):n}t$ and $L_{i:n} : u = y_{i:n}t$, we can obtain a curve segment from $C_{i:n}$, which is denoted as $S_{i:n}$. With a somewhat abuse of definition, the curve composed of $S_{i:n}$ is called Envelope-n. An example of Envelop-3 is given in Figure 6. In this figure, we also show the iso-PF curve. If we can prove that the Envelope-n curve converge to the iso-PF curve when $n \to \infty$, then Theorem 4 is obvious.

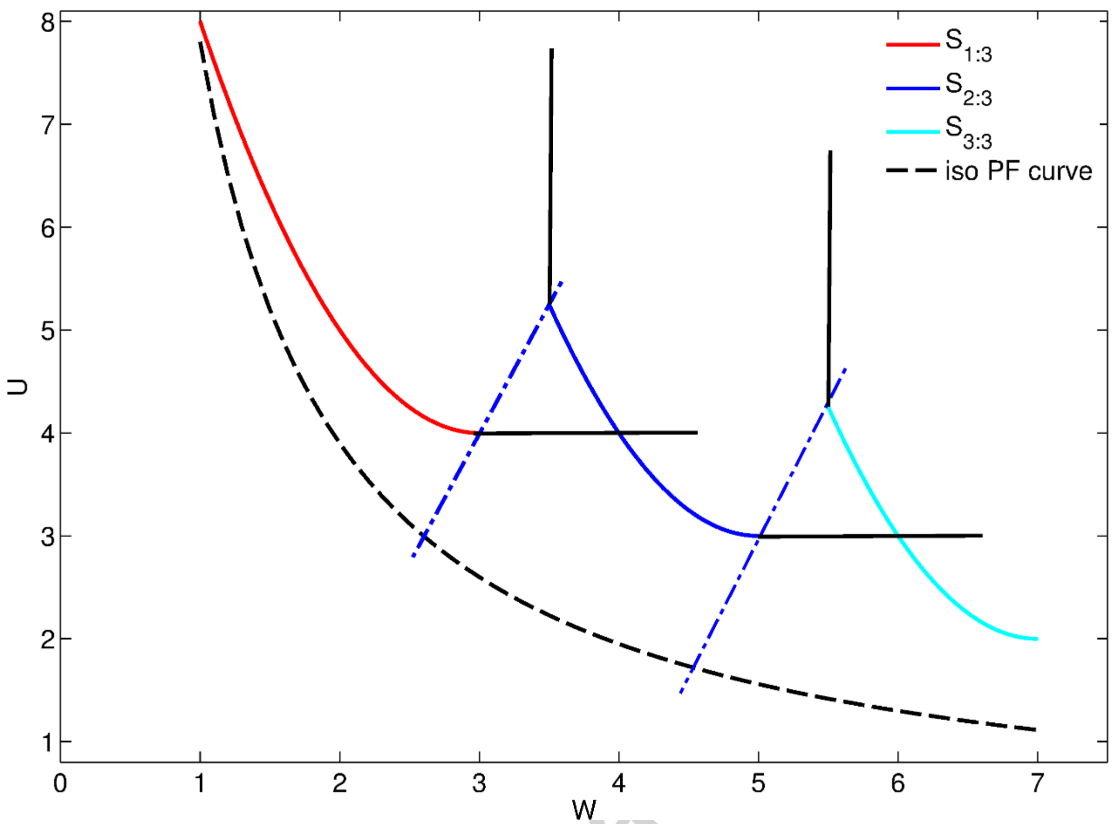

Figure 6. An illustrative example of the envelop curve: the red line is the envelope-3 curve, the dash-dotted line is the iso-PFC curve.

Consider a specific use rate x. Draw the curve $L_x : u = xt$ and determine the crossover point of L_x and the iso-PF curve, which is denoted as (W, xW). For a given n, there exists an i such that $x \in (y_{(i-1):n}, y_{i:n})$. Denote the crossover point of L_x and $C_{i:n}$ by (W_n, xW_n). From the proof of Theorem 1 we can know that

$$F\left(\frac{xW_n}{y}\middle| y\right) \le F(W|x) \text{ for any } y > x,$$

$$F\left(\frac{W_n}{y}\middle| y\right) \le F(W|x) \text{ for any } y < x.$$

Therefore,

$$\min\left\{F\left(\frac{U_{i:n}}{y_{i:n}}\middle| y_{i:n}\right), F\left(W_{i:n} | y_{(i-1):n}\right)\right\} \le F(W|y) \le F(W_{i:n}|y)$$

When $n \to \infty$,

$$F\left(\frac{U_{i:n}}{y_{i:n}} \Big| y_{i:n}\right) \to F(W_{i:n} | y) \text{ and } F\left(W_{i:n} | y_{(i-1):n}\right) \to F(W_{i:n} | y)$$

.

Therefore, $F(W|y) = F(W_{i:n}|y)$ when $n \to \infty$. When the PDF of $F(\cdot)$ is greater than 0 almost surely, we can know that $W_{i:n} \to W$ when $n \to \infty$. This means that the envelop-n curve converges to the iso-PF curve pointwisely when $n \to \infty$. Therefore, the theorem follows.

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Highlights

- We design a two-dimensional warranty menu with a number of warranty choices.
- Consumers can choose a favorable one from the menu as per their usage behavior.
- We further consider a fixed 2D warranty policy with a stair-case warranty region.
- We show the equivalence of the two warranty policies.