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A Novel Equivalent Definition of Modified Bessel Functions for Performance Analysis of Multi-Hop Wireless Communication Systems

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ABSTRACT A statistical model is derived for the equivalent signal-to-noise ratio of the Source-to-Relay-to-Destination (S-R-D) link for Amplify-and-Forward (AF) relaying systems that are subject to block Rayleigh-fading. The probability density function and the cumulated density function of the S-R-D link SNR involve modified Bessel functions of the second kind. Using fractional-calculus mathematics, a novel approach is introduced to rewrite those Bessel functions (and the statistical model of the S-R-D link SNR) in series form using simple elementary functions. Moreover, a statistical characterization of the total receive-SNR at the destination, corresponding to the S-R-D and the S-D link SNR, is provided for a more general relaying scenario in which the destination receives signals from both the relay and the source and processes them using maximum ratio combining (MRC). Using the novel statistical model for the total receive SNR at the destination, accurate and simple analytical expressions for the outage probability, the bit error probability, and the ergodic capacity are obtained. The analytical results presented in this paper provide a theoretical framework to analyze the performance of the AF cooperative systems with an MRC receiver.

INDEX TERMS Modified bessel function of second kind, amplify and forward (AF), maximum ratio combining (MRC).

I. INTRODUCTION

Cooperative communication to enhance the transmission rate of a communication system was first introduced in [1], and "distributed spatial diversity" turns out to be a promising method that exploits the antennas of several distributed user terminals to achieve transmit diversity in space.

Several relaying protocols have been proposed in the literature, such as Amplify-and-Forward (AF), Decode-and-Forward (DF), Soft-DF, and Compress-and-Forward (CF) (see e.g. [2]), where, depending on the parameters of the network, each of them can be the method of choice. There is currently a lot of interest in AF relaying because of its low complexity compared to other relaying protocols; hence, AF is also the focus of this work.

For an analytical performance investigation of AF relaying with maximum-ratio-combining (MRC) at the destination, a statistical model of the total receive signal-to-noise (SNR) ratio, SNR_{tot}, is required, which is equivalent to the sum of SNRs corresponding to the Source-Destination (S-D) and the Source-Relay-Destination (S-R-D) link SNRs (i.e. $SNR_{\text{tot}} = SNR_{\text{sd}} + SNR_{\text{srd}}$). To the best of our knowledge, a theoretical statistical model of SNR_{tot} is yet unknown (but will be presented in this paper). Numerous studies, however, consider the problem of finding a statistical model of SNR_{srd}. There are two main trends in the literature: *either* a single-branch system model is assumed, in which the receiver operates only on the relay transmission and, hence, no MRC is employed at the destination, *or* upper and/or lower performance bounds are considered assuming an MRC receiver at the destination.

In [3]–[8] statistical models for SNR_{srd} have been proposed for Rayleigh and Nakagami-m channels assuming that the direct S-D channel is in a deep fade, so the effect of the S-D link can be ignored, and the problem to find a statistical model for $SNR_{\text{tot}} = SNR_{\text{srd}} + SNR_{\text{sd}}$ is not further investigated. The work reported in [9] deals with the outage performance of an AF relaying system assuming Nakagami-m fading and a Selection Combining (SC) receiver at the destination. However, an SC receiver is, essentially, a single-branch system model, in which the receiver operates *either* on the S-D link *or* on the S-R-D link, depending on the receive SNRs on the two links. Since the SC receivers are outperformed by MRC receivers, we are interested in the latter in this paper. For comparison, simulation results are presented in the forthcoming sections to show the superiority of MRC receivers over SC receivers. In [10] an exact closed-form expression is obtained for the ergodic capacity of a single-branch AF relaying system over Rayleigh fading channels, when the direct S-D link is not available, but no results are given for a scenario with an existing S-D link when the destination employs an MRC receiver. In [11]–[14] several performance measures for AF relaying systems are investigated, but in all of them a single-branch system model is assumed, in which the destination operates only based on the relay transmission. In fact, in all the aforementioned contributions, performance analysis is based on a system model in which the destination does *not* employ an MRC receiver.

The need for a simple analytical model of SNR_{tot} is evident when evaluating the performance of variable-gain AF cooperative systems with an MRC receiver at the destination. Due to the lack of such a statistical model, bounds are usually used to characterize performance: e.g. in [15] and [16] the ergodic capacity of AF relaying systems is investigated for the general case of an available S-D link and a multiple relay scenario, but only an upper bound of ergodic capacity is obtained. In [17] and [18] an approximation for the PDF of SNR_{tot} is used in order to avoid the use of Bessel functions (details to follow) by taking the worst of the S-R link and the R-D link to approximate the whole S-R-D link, leading to an upper bound because, albeit less so, the better link will also cause a loss of information. This way, theoretical performance *bounds* of the bit error rate, the outage probability and the ergodic capacity of multiple-relay AF cooperative systems over various fading models are obtained when the destination employs an MRC receiver. Several methods are provided in [19]–[26] for calculating the Symbol Error Rate (SER) of AF relaying systems but in all of them, deriving a statistical model for SNR_{tot} is avoided. Consequently, those approaches do not allow to obtain exact results on other performance measures such as ergodic capacity or outage probability.

A careful study of the literature reveals that, in spite of several attempts to deal with the original problem of finding the PDF of SNR_{tot}, mathematical complexity will not allow for explicit and practically useful analytical results. To the best of the our knowledge, no exact closed-form solution for the PDF of SNR_{tot} has been reported so far in the literature.

To cope with the computations involving complicated mathematical functions, a feasible solution is to use an equivalent series representation of the functions (e.g. [27]–[30]). In fact, the appearance of the modified Bessel function of the second kind, $K_\nu(\cdot)$, in the PDF of SNR_{srd} is the main source of intractability in AF-related calculations. We aim to substitute $K_\nu(\cdot)$ with an equivalent series representation in this paper. However, the specific choice of an *appropriate* equivalent representation is crucial: although a series representation of $K_\nu(\cdot)$ is available from [31, 8.446], this representation is much more complicated than the Bessel function itself. In [32] an equivalent representation for $K_\nu(\cdot)$ is also introduced, but this formulation again is not helpful for the performance analysis of AF relaying, because a "0" appears in the denominator of the series expression at functions of interest, therefore the formulation can't be used for a numerical evaluation.

Since there are several well known simple series representations of $I_\nu(\cdot)$ (modified Bessel functions of the first kind, ν-th order), one might propose to exploit the relation $K_\nu(\cdot) = \pi\left(I_{-\nu}(\cdot) - I_\nu(\cdot)\right)/2\sin(\pi\nu)$ from [33, 10.27.4], but note that $\sin(\pi\nu) = 0$ for $\nu = 0, 1, 2, \cdots$. As will be demonstrated in following sections, an equivalent series representation of $K_\nu(\cdot)$ is required particularly for $\nu = 0$ and 1; hence, employing this formulation for $K_\nu(\cdot)$ is not helpful to obtain the desired results.

As there is no appropriate and simple series representation of $K_\nu(\cdot)$ available in the literature, we derive a novel series representation of $K_\nu(\cdot)$ in terms of simple elementary functions (such as $x^n e^{-x}$) using fractional calculus mathematics. With this result, the complex statistical model of SNR_{srd} turns out to be simple and easily tractable. Thereafter, the Cumulative Distribution Function (CDF) and the Probability Density Function (PDF) of SNR_{tot} will be derived at high SNRs. Using the PDF of SNR_{tot}, closed-form expressions for the outage probability, the bit-error probability (BEP) and the ergodic capacity will be derived, which, to the best of our knowledge, are the first analytical results on variable-gain AF cooperative systems with an MRC receiver at the destination. Note that we mainly consider a single relay scheme in the paper but as will be observed throughout the paper, the proposed method can be applied to multiple relay schemes as well. Some simulation results are also provided to show the usefulness of the proposed approach for more complicated multiple-relay scenarios.

The remainder of the paper is organized as follows: in Section II the system model is introduced. In Section III the fractional-calculus method is exploited to derive an equivalent series representation of $K_\nu(x)$. Based on the results derived in Section III, the PDF and CDF of $|h_{\text{eq}}|^2$ (equivalently the PDF and CDF of SNR_{tot}) are derived in Section IV, and in Section V novel closed-form expressions are provided for some performance measures of variable-gain AF cooperative systems, including outage probability, BEP and ergodic capacity.

II. SYSTEM MODEL

We consider a two-hop variable-gain AF cooperative system as illustrated by Fig. 1. The source (S) sends data to the destination (D) by the help of an intermediate relay node (R). The destination "hears" both the source and the relay transmissions and employs a maximum-ratio-combining (MRC) receiver to jointly exploit all information available at the destination.

Motivated by practical hardware constraints, the condition is posed that the relay operates in half-duplex mode, i.e. the

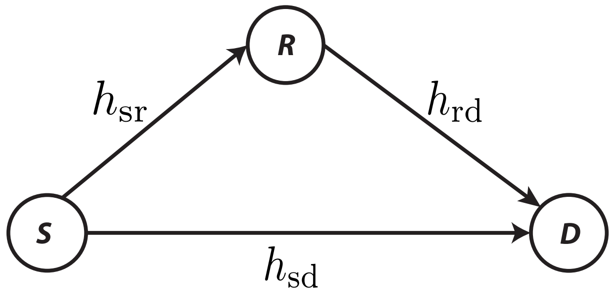

FIGURE 1. System model.

relay can't receive and transmit simultaneously at the same carrier frequency. Moreover, the overall system is assumed to be orthogonal in time, i.e. a repetitive "time-slotted" setting is used, in which one slot is reserved for the transmission by the source and the other slot is used for cooperative transmission by the relay; this is a very common system setting in the literature.

The signals corresponding to the source transmission received at the destination ($\mathbf{y}_{\mathrm{sd}}$) and the relay ($\mathbf{y}_{\mathrm{sr}}$) are

$$\mathbf{y}_{\mathrm{sd}} = \sqrt{P_{\mathrm{s}}} h_{\mathrm{sd}} \mathbf{s} + \mathbf{n}_{\mathrm{d}} \quad (1)$$

$$\mathbf{y}_{\mathrm{sr}} = \sqrt{P_{\mathrm{s}}} h_{\mathrm{sr}} \mathbf{s} + \mathbf{n}_{\mathrm{r}} \quad (2)$$

and the signal received at the destination corresponding to the relay transmission is

$$\mathbf{y}_{\mathrm{rd}} = \underbrace{\sqrt{\frac{P_{\mathrm{r}} P_{\mathrm{s}}}{P_{\mathrm{s}}|h_{\mathrm{sr}}|^2 + N_0}} h_{\mathrm{sr}} h_{\mathrm{rd}} \mathbf{s}}_{\text{``useful'' signal}} + \underbrace{\sqrt{\frac{P_{\mathrm{r}}}{P_{\mathrm{s}}|h_{\mathrm{sr}}|^2 + N_0}} h_{\mathrm{rd}} \mathbf{n}_{\mathrm{r}} + \mathbf{n}_{\mathrm{d}}}_{\text{noise}} \quad (3)$$

where $\mathbf{s}$ is transmit symbol vector with unit average power. The parameter P_{s} and P_{r} are the source and the relay power constraints, respectively. h_{sd}, h_{sr} and h_{rd} represent the channel coefficients corresponding to the S-D, the S-R and R-D links, respectively.

The channel coefficients, which capture the effects of path-loss and block fading, are assumed to be zero-mean, white complex Gaussian processes with variances σ^2_{ij} with $i \in \{\mathrm{s}, \mathrm{r}\}$ and $j \in \{\mathrm{r}, \mathrm{d}\}$. The additive receiver noise is modelled by $\mathbf{n}_{\mathrm{d}}$ and $\mathbf{n}_{\mathrm{r}}$, which are sample-vectors from zero-mean, white complex Gaussian processes. For simplicity of notation, the same noise power N_0 is assumed at the receivers of the relay and the destination. The SNR at the output of the MRC receiver is $SNR_{\mathrm{tot}} = \gamma(|h_{\mathrm{sd}}|^2 + |h_{\mathrm{srd}}|^2) = \gamma |h_{\mathrm{eq}}|^2$ with $\gamma = P_{\mathrm{s}}/N_0$. The channel power $|h_{\mathrm{sd}}|^2$ is exponentially distributed (i.e. $f_{|h_{\mathrm{sd}}|^2}(x) = \lambda_{\mathrm{sd}} \mathrm{e}^{-\lambda_{\mathrm{sd}} x}$) with parameter $\lambda_{\mathrm{sd}} = 1/\sigma^2_{\mathrm{sd}}$. The PDF of $|h_{\mathrm{srd}}|^2$ reads

$$f_{|h_{\mathrm{srd}}|^2}(x) = 2\mathrm{e}^{-\lambda_{\mathrm{S}} x} \Big[\eta \lambda_{\mathrm{P}} K_0(2\sqrt{\lambda_{\mathrm{P}} x (x + 1/\gamma)}) + \lambda_{\mathrm{S}} \sqrt{\lambda_{\mathrm{P}} x (x + 1/\gamma)} K_1(2\sqrt{\lambda_{\mathrm{P}} x (x + 1/\gamma)}) \Big] \quad (4)$$

with $K_\nu(\cdot)$ the modified Bessel function of the second kind and ν-th order, $\eta = 2x + 1/\gamma$, $\lambda_{\mathrm{P}} \doteq \lambda_{\mathrm{sr}} \lambda_{\mathrm{rd}}$ and $\lambda_{\mathrm{S}} \doteq \lambda_{\mathrm{sr}} + \lambda_{\mathrm{rd}}$. The CDF of $|h_{\mathrm{sd}}|^2$ is

$$F_{|h_{\mathrm{srd}}|^2}(x) = 1 - 2\mathrm{e}^{-\lambda_{\mathrm{S}} x} \sqrt{\lambda_{\mathrm{P}} x (x + 1/\gamma)} \times K_1\left(2\sqrt{\lambda_{\mathrm{P}} x (x + 1/\gamma)}\right). \quad (5)$$

A proof of (4) and (5) can be found in [34]; however, in order to keep the consistency of the notation, an alternative proof of (4) and (5) is provided in Appendix A.

The results in (4) and (5) do not easily lend themselves to further mathematical calculations (e.g. integration) as modified Bessel functions $K_\nu(\cdot)$ appear. Hence, no statistical model for SNR_{tot} is available in the literature so far.

In what follows, an equivalent representation of $K_\nu(\cdot)$ is derived that is based on a series-representation involving simple mathematical functions of the form $x^n \mathrm{e}^{-x}$. This novel equivalent representation of $K_\nu(\cdot)$ paves the way for further theoretical analysis of AF relaying systems.

III. NOVEL SERIES REPRESENTATION OF MODIFIED BESSEL FUNCTIONS OF SECOND KIND

The mathematical concept of integration and differentiation of arbitrary (non-integer) order is called "fractional calculus"; foundations of the theory are discussed e.g. in [35], [36]. It will be used below to derive a simple novel equivalent representation of $K_\nu(\beta x)$.

Theorem 1: Equivalent Representation of $K_\nu(\beta x)$
A modified Bessel function, $K_\nu(\beta x)$, of the second kind and ν-th order, with $\nu > 0$, can be represented by the infinite series

$$K_\nu(\beta x) = \mathrm{e}^{-\beta x} \sum_{n=0}^{\infty} \sum_{i=0}^{n} \Lambda(\nu, n, i) \cdot (\beta x)^{i-\nu}, \quad (6)$$

with the coefficients[1]

$$\Lambda(\nu, n, i) \doteq \frac{(-1)^i \sqrt{\pi}\, \Gamma(2\nu) \Gamma(\frac{1}{2} + n - \nu) L(n, i)}{2^{\nu - i} \Gamma(\frac{1}{2} - \nu) \Gamma(\frac{1}{2} + n + \nu) n!} \quad (7)$$

that involve the Lah numbers (e.g. [38])

$$L(n, i) \doteq \binom{n-1}{i-1} \frac{n!}{i!} \quad \text{for } n, i > 0, \quad (8)$$

and the conventions $L(0, 0) \doteq 1$; $L(n, 0) \doteq 0$ and $L(n, 1) \doteq n!$ for $n > 0$.

Proof: Let s be a real non-negative number, i.e. $s > 0$ and $s \in \mathbb{R}$. Let $f(x)$ be continuous on $x \in [0, \infty)$ and integrable on any finite subinterval of $x \geqslant 0$. Then the

[1] For the computation of the coefficients, some results for the Gamma-function are useful: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$, $\Gamma(1) = 1$, $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(n+1) = n!$ for natural numbers $n > 0$. For $n = 0, -1, -2, \ldots$ $\Gamma(n)$ is not defined (see e.g. [37]).

Riemann-Liouville operator (e.g. [36]) of fractional integration is defined as

$$I^s\{f(x)\} \doteq \frac{1}{\Gamma(s)}\int_0^x (x-t)^{s-1}f(t)dt. \tag{9}$$

On the other hand, from [31, 3.471.4] we have

$$\int_0^x (x-t)^{s-1}t^{-2s}\mathrm{e}^{-\beta/t}dt = \frac{\Gamma(s)\beta^{\frac{1}{2}-s}}{\sqrt{\pi x}}\mathrm{e}^{\frac{-\beta}{2x}}K_{s-\frac{1}{2}}(\frac{\beta}{2x}). \tag{10}$$

Assuming $f(t) = t^{-2s}\mathrm{e}^{-\beta/t}$, the two integrals in (9) and (10) are identical: this motivates the novel approach to derive an equivalent expression for $K_\nu(\beta x)$ by use of fractional integration.

It follows from (9) and (10) that

$$I^s\left\{x^{-2s}\mathrm{e}^{-\beta/x}\right\} = \frac{\beta^{\frac{1}{2}-s}}{\sqrt{\pi x}}\mathrm{e}^{\frac{-\beta}{2x}}K_{s-\frac{1}{2}}(\frac{\beta}{2x}). \tag{11}$$

The Leibniz rule for the Riemann-Liouville operator (see Appendix B for a proof) is given by

$$I^s\{h(x)g(x)\} = \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+s)}{n!\Gamma(s)}I^{(s+n)}\{h(x)\}D^n\{g(x)\} \tag{12}$$

where n is a non-negative integer, $s+n$ is a non-negative fractional number and $D^n \doteq \frac{d^n}{dx^n}$. By solving $I^{(s+n)}\{h(x)\}$ for $h(x) = x^{-2s}$ and $D^n\{g(x)\}$ for $g(x) = \mathrm{e}^{-\beta/x}$, the equivalent Bessel model (6) will be derived.

Let $h(x) = x^p$, then according to (9)

$$I^\alpha\{x^p\} = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^p dt, \quad (\alpha > 0) \tag{13}$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^x (1-\frac{t}{x})^{\alpha-1}x^{\alpha-1}t^p dt \tag{14}$$

$$= \frac{x^{\alpha+p}}{\Gamma(\alpha)}\int_0^1 u^p(1-u)^{\alpha-1}du, \tag{15}$$

$$= \frac{\Gamma(1+p)}{\Gamma(1+p+\alpha)}x^{p+\alpha}. \tag{16}$$

With the substitution $u = \frac{t}{x}$, (15) follows from (14); (16) follows from the definite integral $\int_0^1 x^a(1-x)^b dx = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}$ in [37, p. 70], with $\Gamma(\cdot)$ the Gamma-function and the real-valued constants $a, b \notin \{-1, -2, -3, ...\}$. Suppose that $p = -2s$ and $\alpha = s+n$, then

$$I^{(s+n)}\left\{x^{-2s}\right\} = \frac{\Gamma(1-2s)}{\Gamma(1-s+n)}x^{n-s}. \tag{17}$$

With $g(x) = \mathrm{e}^{-\beta/x}$ in (12), from [39] we have

$$D^n\left\{\mathrm{e}^{-\beta/x}\right\} = \mathrm{e}^{-\beta/x}\frac{(-1)^n}{x^n}\sum_{i=0}^{n}(-1)^i L(n,i)(\beta/x)^i, \tag{18}$$

with $L(n, i)$ defined in (8).

By substituting (17) and (18) into (11) and (12) it is straightforward to obtain

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(-1)^n\Gamma(n+s)}{n!\Gamma(s)}\frac{\Gamma(1-2s)}{\Gamma(1+n-s)}x^{n-s}\\ &\times \mathrm{e}^{-\beta/x}\frac{(-1)^n}{x^n}\sum_{i=0}^{n}(-1)^i L(n,i)(\beta/x)^i\\ &= \frac{\beta^{\frac{1}{2}-s}}{\sqrt{\pi x}}\mathrm{e}^{-\frac{\beta}{2x}}K_{s-\frac{1}{2}}(\frac{\beta}{2x})\end{aligned} \tag{19}$$

and so

$$\begin{aligned}K_{s-\frac{1}{2}}(\frac{\beta}{2x}) &= \mathrm{e}^{-\beta/(2x)}\sqrt{\pi}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\\ &\times\frac{\Gamma(n+s)\Gamma(1-2s)\cdot(-1)^i L(n,i)}{n!\Gamma(s)\Gamma(1+n-s)}\left(\frac{\beta}{x}\right)^{s+i-1/2}.\end{aligned} \tag{20}$$

Changing the variable $x \rightarrow \frac{1}{2x}$, assuming $1-2s = 2\nu$, exploiting $K_{-\nu} = K_\nu$ the result is the infinite series

$$K_\nu(\beta x) = \mathrm{e}^{-\beta x}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\Lambda(\nu,n,i)\cdot(\beta x)^{i-\nu}, \tag{21}$$

where $\Lambda(\nu, n, i)$ is given by (7). ■

It should be made clear that the above representation of $K_\nu(\beta x)$ is *not* valid for $\nu = \left\{0, \frac{1}{2}, \frac{3}{2}, \cdots\right\}$. That is because $\Gamma(2\nu)$ and $\Gamma(\frac{1}{2}+n-\nu)$ in (7) diverge to $\pm\infty$. However, for the case of $\nu = 0$, one can compute $K_0(\beta x)$ using the equivalent representation of $K_1(\beta x)$ and $K_2(\beta x)$ by $K_\nu(\beta x) = K_{\nu-2}(\beta x) + \frac{2(\nu-1)}{\beta x}K_{\nu-1}(\beta x)$ that is obtained from [33, 10.38.4].

A. FINITE SERIES REPRESENTATION OF $K_\nu(\beta x)$

The equivalent representation of $K_\nu(\beta x)$ may significantly simplify computations involving $K_\nu(\beta x)$, as the series in (6) contains the variable x only in the simple function-template $x^{i-\nu}\mathrm{e}^{-\beta x}$ that can, e.g., be easily integrated. The series representation contains, however, an infinite number of terms that can't be computed in practical applications.

Fortunately, the series representation of $K_\nu(\beta x)$ is rather accurate for a finite number of terms as defined as follows:

$$K_\nu(\beta x) = \mathrm{e}^{-\beta x}\sum_{n=0}^{k}\sum_{i=0}^{n}\Lambda(\nu,n,i)\cdot(\beta x)^{i-\nu} + \epsilon \tag{22}$$

with

$$\epsilon = \mathrm{e}^{-\beta x}\sum_{n=k+1}^{\infty}\sum_{i=0}^{n}\Lambda(\nu,n,i)\cdot(\beta x)^{i-\nu}. \tag{23}$$

The first term on the right-hand side of (22) represents the actual function to approximate $K_\nu(\beta x)$, and ϵ represents the truncation error. Fig. 2 illustrates numerical values of the finite series representation of $K_1(\beta x)$ (with $k = 2$ in (22)) for various values of β (dashed lines) and also the theoretical fully accurate values of $K_1(\beta x)$ (solid lines). It is clear from

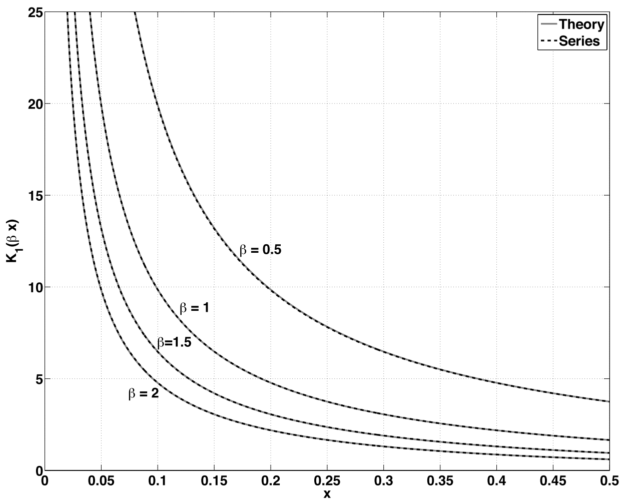

FIGURE 2. $K_1(\beta x)$ **and its finite series representation with** $k = 4$ **in (22).**

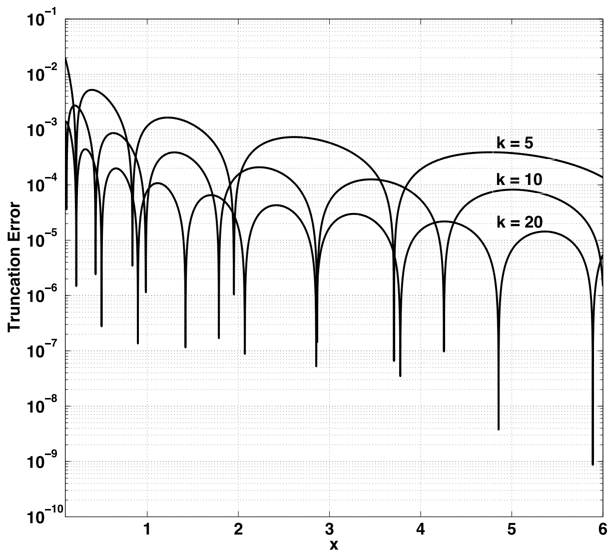

FIGURE 3. **Truncation error of** $K_1(x)$ **for various values of** k**.**

the figure that the finite series for $K_\nu(\beta x)$ with only $k = 4$ merges with theoretical $K_\nu(\beta x)$ with high accuracy.

1) TRUNCATION ERROR

In the Appendix B, it is proved that the Leibniz rule, (12), for the Riemann-Liouville operator is a direct result of a Taylor-series expansion of some function, say $h(t)$, at $t = x$. Consequently, the equivalent infinite series representation of $K_\nu(\beta x)$ in (21) is also a result of some Taylor expanssion at point x. Therefore, it is expected that the equivalent infinite series representation of $K_\nu(\beta x)$ can be truncated with high accuracy with only few terms. Fig. 3 shows the absolute value of the truncation error, i.e. $|\epsilon|$, for $k = 5, 10$ and 20. It is obvious from Fig. 3 that the error is as low as about 10^{-4} for $k = 10$ and as low as about 10^{-5} for $k = 20$. The truncation error is about 10^{-3} when $x \to 0$, but considering that $K_\nu(x) \to \infty$ as $x \to 0$, the truncation error of 10^{-3} is negligible. In the remainder of the paper we assume $k = 4$, although even much lower values of, e.g. $k = 2$, turn out to produce accurate results.

2) CONVENIENT REPRESENTATION FOR PRACTICAL USE OF THE TRUNCATED SERIES

For convenience we re-write (22) such that one of the sum-operators is included in the series coefficients. For this, the inner sum over i in (22) is evaluated row-wise, with the sum-index n counting the rows. This structure is then summed up column-wise and the result can be written as

$$K_\nu(\beta x) = \frac{\mathrm{e}^{-\beta x}}{(\beta x)^\nu} \sum_{q=0}^{k} \underbrace{\left(\sum_{l=q}^{k} \Lambda(\nu, l, q) \right)}_{\doteq a_{\nu,k,q}} \cdot (\beta x)^q + \epsilon. \quad (24)$$

As long as k is limited, the above series representation (24) can always be used to replace (22) without any convergence problems while the truncation error ϵ can be made arbitrarily small. Numerical values for the coefficients $a_{\nu,k,q}$ are given in Table 1 for the first-order ($\nu = 1$) modified Bessel function of the second kind $K_1(\cdot)$. It should be noted that the coefficient with q-index 1 is always found to equal $a_{1,k,1} \doteq \frac{2k}{2k+1}$. Using Table 1, a rather accurate approximation (see results in Fig. 3) of $K_1(\cdot)$ can be obtained by (24) with very few floating point operations, and this is also and particularly true for the order $\nu = 1$ where other series representations produce indeterminate results.

IV. DISTRIBUTION OF EQUIVALENT CHANNEL POWER

A. SINGLE RELAY

Assuming an MRC receiver at the destination, the total receive SNR is the sum

$$SNR_{\mathrm{tot}} = SNR_{\mathrm{sd}} + SNR_{\mathrm{srd}} = \gamma \underbrace{(|h_{\mathrm{sd}}|^2 + |h_{\mathrm{srd}}|^2)}_{|h_{\mathrm{eq}}|^2} \quad (25)$$

of SNRs corresponding to the S-D and the S-R-D links (this is also true when, as in the given case, the corresponding signals are received in different time slots, because they are jointly processed by the MRC receiver). For a performance analysis of the system, the PDF and the CDF of the equivalent channel power-gain

$$|h_{\mathrm{eq}}|^2 \doteq |h_{\mathrm{srd}}|^2 + |h_{\mathrm{sd}}|^2 \quad (26)$$

are derived below.

For the CDF we obtain

$$\begin{aligned} F_{|h_{\mathrm{eq}}|^2}(x) &= \mathbb{P}\left(|h_{\mathrm{sd}}|^2 + |h_{\mathrm{srd}}|^2 \leq x\right) \\ &= \int_0^x \int_0^{x-u} f_{|h_{\mathrm{sd}}|^2}(t) \cdot f_{|h_{\mathrm{srd}}|^2}(u) dt du \\ &= \int_0^x (1 - \mathrm{e}^{-\lambda_{\mathrm{sd}}(x-u)}) \cdot f_{|h_{\mathrm{srd}}|^2}(u) du \\ &= F_{|h_{\mathrm{srd}}|^2}(x) - \mathrm{e}^{-\lambda_{\mathrm{sd}} x} \int_0^x \mathrm{e}^{\lambda_{\mathrm{sd}} u} \cdot f_{|h_{\mathrm{srd}}|^2}(u) du \\ &= \lambda_{\mathrm{sd}} \mathrm{e}^{-\lambda_{\mathrm{sd}} x} \int_0^x \mathrm{e}^{\lambda_{\mathrm{sd}} u} \cdot F_{|h_{\mathrm{srd}}|^2}(u) du, \end{aligned} \quad (27)$$

TABLE 1. Coefficients $a_{\nu,k,q}$ in (24) for $\nu = 1$ with four digits of accuracy.

	$a_{1,k,0}$	$a_{1,k,1}$	$a_{1,k,2}$	$a_{1,k,3}$	$a_{1,k,4}$	$a_{1,k,5}$	$a_{1,k,6}$	$a_{1,k,7}$	$a_{1,k,8}$	$a_{1,k,9}$	$a_{1,k,10}$
$k=2$	1	4/5	-0.1333								
$k=3$	1	6/7	-0.2476	0.0381							
$k=4$	1	8/9	-0.3429	0.1016	-0.0106						
$k=5$	1	10/11	-0.4237	0.1824	-0.0375	2.693×10^{-3}					
$k=8$	1	16/17	-0.6100	0.4880	-0.2440	6.913×10^{-2}	-1.067×10^{-2}	8.274×10^{-4}	-2.489×10^{-5}		
$k=10$	1	20/21	-0.7047	0.7239	-0.5000	0.2111	-5.415×10^{-2}	8.375×10^{-3}	-7.55×10^{-4}	3.619×10^{-5}	-7.0724×10^{-7}

where the last equality follows from integration by parts with $\int_0^x p(u)q'(u)du = p(u)q(u)|_0^x - \int_0^x p'(u)q(u)du$ and $p(u) \doteq e^{\lambda_{sd}u}$ and $q'(u) \doteq f_{|h_{srd}|^2}(u)$.

The CDF $F_{|h_{srd}|^2}(\cdot)$ and the PDF $f_{|h_{srd}|^2}(\cdot)$ were derived in (5) and (4), respectively. For simplicity we will restrict calculations below to the high "transmit-SNR" regime but it will be demonstrated by Figs. 6,7,8 that this is justified because it leads to very accurate numerical results, even in the low-SNR region.

By using (5) in (27), and assuming "high SNR", (27) simplifies to

$$F_{|h_{eq}|^2}(r) = 1 - e^{-\lambda_{sd}r} - 2\lambda_{sd}\sqrt{\lambda_{sr}\lambda_{rd}}e^{-\lambda_{sd}r} \times \int_0^r x e^{-(\lambda_{sr}+\lambda_{rd}-\lambda_{sd})x} K_1(2\sqrt{\lambda_{sr}\lambda_{rd}x})dx. \quad (28)$$

The integral in (28) is non-trivial and does not seem to have closed-form solution. However, using the results from Section III, the integral can be rewritten as follows

$$\begin{aligned}\eta &\doteq \int_0^x 2\sqrt{\lambda_{sr}\lambda_{rd}}ue^{-(\lambda_{sr}+\lambda_{rd}-\lambda_{sd})u}K_1(2\sqrt{\lambda_{sr}\lambda_{rd}}u)dx \\ &\approx \sum_{q=0}^{k} a_{1,k,q}\int_0^x (2\sqrt{\lambda_{sr}\lambda_{rd}}u)^q e^{-(\lambda_{sr}+\lambda_{rd}+2\sqrt{\lambda_{sr}\lambda_{rd}}-\lambda_{sd})u}du \\ &\approx \sum_{q=0}^{k}\frac{(2\sqrt{\lambda_{sr}\lambda_{rd}})^q q! a_{1,k,q}}{(\lambda_{srd}-\lambda_{sd})^{q+1}} \\ &\quad \times \Big(1 - \sum_{c=0}^{q}\frac{(\lambda_{srd}-\lambda_{sd})^c}{c!}x^c e^{-(\lambda_{srd}-\lambda_{sd})x}\Big) \qquad (29)\end{aligned}$$

where the second step is obtained by using the series representation of $K_1(2\sqrt{\lambda_{sr}\lambda_{rd}}x)$ derived in (24). As the series is truncated (for k limited), this is an approximation, indicated by the use of "$\approx$" instead of strict equality; the truncation error can, however, be made arbitrarily small by choosing proper k. Assuming $\lambda_{srd} = (\sqrt{\lambda_{sr}} + \sqrt{\lambda_{rd}})^2$, the last equality follows from identity [31, 3.351.1] where

$$\int_0^x u^q e^{-\lambda u}du = \frac{q!}{\lambda^{q+1}}(1 - \sum_{c=0}^{q}\frac{\lambda^c}{c!}x^c e^{-\lambda x}). \quad (30)$$

By substituting (29) into (28) it is straightforward to obtain $F_{|h_{eq}|^2}(x)$ as

$$F_{|h_{eq}|^2}(x) \approx 1 - \mathcal{A}e^{-\lambda_{sd}x} + \sum_{q=0}^{k}\sum_{c=0}^{q}\mathcal{B}x^c e^{-\lambda_{srd}x} \quad (31)$$

where coefficients $\mathcal{A}$ and $\mathcal{B}$ are independent of x, defined as

$$\mathcal{A} \doteq 1 + \sum_{q=0}^{k}\frac{\lambda_{sd}(2\sqrt{\lambda_{sr}\lambda_{rd}})^q q! a_{1,k,q}}{(\lambda_{srd}-\lambda_{sd})^{q+1}} \quad (32)$$

$$\mathcal{B} \doteq \frac{\lambda_{sd}(2\sqrt{\lambda_{sr}\lambda_{rd}})^q q! a_{1,k,q}}{c!(\lambda_{srd}-\lambda_{sd})^{q-c+1}}. \quad (33)$$

The PDF of $|h_{eq}|^2$ is the derivative of $F_{|h_{eq}|^2}(x)$ in (31) w.r.t x, which is easy to calculate as the series representation involves simple elementary functions only:

$$f_{|h_{eq}|^2}(x) \approx \mathcal{A}\lambda_{sd}e^{-\lambda_{sd}x} + \sum_{q=0}^{k}\sum_{c=0}^{q}\mathcal{B}(cx^{c-1} - \lambda_{srd}x^c)e^{-\lambda_{srd}x}. \quad (34)$$

Fig. 4 illustrates the accuracy of $f_{|h_{eq}|^2}(x)$ using the expression derived in (34) (solid line) by a comparison with histogram-results obtained from Monte Carlo simulations.

In the literature (e.g. [17]–[20]), when considering relaying systems with MRC receiver at the destination, a method is proposed for estimating the statistics of SNR_{srd} based on the bound $SNR_{srd} \doteq \min(SNR_{sr}, SNR_{rd})$. Consequently, SNR_{srd} has as an exponential distribution with parameter $\lambda_{sr} + \lambda_{rd}$. Note that, although this method greatly simplifies the calculations by avoiding modified Bessel functions in the formulations, accuracy is compromised: Fig. 4 (dashed line) shows the result for $f_{|h_{eq}|^2}(x)$ when using the bound. It is clear that the approach presented in this paper, even with low truncation order k, leads to accurate results, while the accuracy of the bounding technique is much lower. Note that with the coefficients given in Table 1 for lower truncation orders k, the statistical model proposed in this paper is, indeed, no more complex than an exponential distribution (which on computing hardware would also be implemented by a series representation as any other transcendental function).

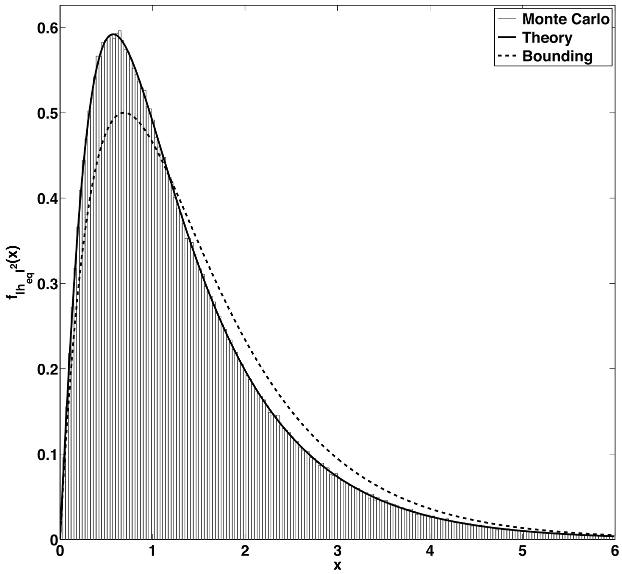

FIGURE 4. PDF $f_{|heq|^2}(x)$ using the theoretical model derived in (34); Monte Carlo simulations and the bounding technique proposed in the literature (e.g. [17]–[20]), all with $\lambda_{sr} = \lambda_{rd} = 1$.

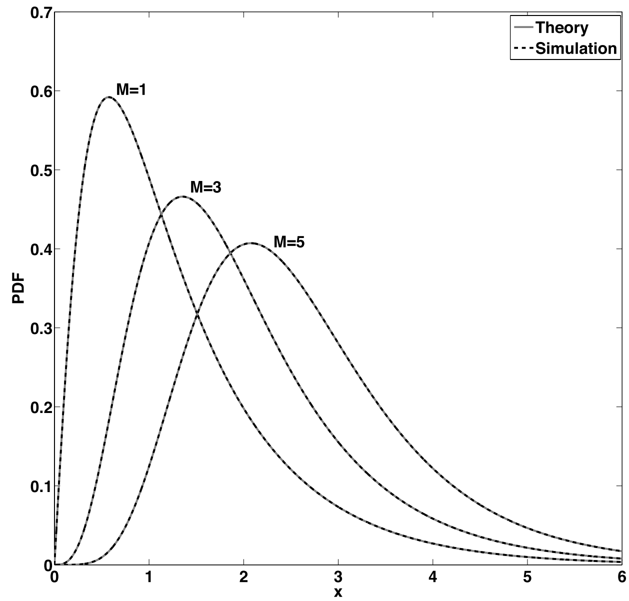

FIGURE 5. PDF $f_{|heq|^2}(x)$ for multiple relays; $\lambda_{sr} = \lambda_{rd} = 1$; series truncation at $k = 4$ for the theoretical results.

B. MULTIPLE RELAYS

So far, we have assumed a single-relay scenario where $SNR_{\text{tot}} = SNR_{\text{sd}} + SNR_{\text{srd}}$; the CDF and the PDF of SNR_{tot} corresponding to such a scenario have been derived in (31) and (34), respectively. However, the extension of such a system model to a scenario with multiple relays is straightforward. For a system with M relays, $SNR_{\text{tot}} = SNR_{\text{sd}} + \sum_{m=1}^{M} SNR_{\text{srd}_m}$. From [40], the PDF of SNR_{tot} can be written as

$$f_{\text{tot}} = f_{\gamma_{\text{sd}}} * f_{\gamma_{\text{srd}_1}} * \cdots f_{\gamma_{\text{srd}_M}} \tag{35}$$

where the symbol $*$ represents the convolution operation, and $f_{\gamma_{\text{srd}_m}}$ represents the PDF of the SNR received at the destination corresponding to relay m. Consequently, with the PDF of SNR_{srd} according to the series model explained above, the calculation of the convolution operations in (35) will reduces to the simple integration as in (30). Fig. 5 compares the PDF of SNR_{tot} with various numbers of relays using the theoretical approach explained above with the results of Monte-Carlo simulations. It is clear from the figure that the theoretical results excellently match the simulations, even with a truncation of the series at $k = 4$.

V. PERFORMANCE ANALYSIS

A closed-form expression for the CDF and the PDF of $|h_{\text{eq}}|^2$ (or equivalently the CDF and the PDF of SNR_{tot}) facilitates the calculation of exact[2] theoretical expressions for several performance measures of AF cooperative systems. In the sequel, outage probability, average bit-error probability (BEP) and ergodic capacity of the system are investigated. Note that the results in the last section were derived with the assumption of large γ, which means we consider the high transmit-SNR regime. As will become evident at the end of the section, the results are not only precise at high SNR but are rather accurate at low SNR too. Moreover, as there are extensive related studies in the literature (e.g. [9]) investigating selection combining (SC) receivers, some simulations are provided in this section that illustrate the superiority of MRC receivers to SC receivers and, hence, confirming the motivation for studying MRC receivers in this work.

A. OUTAGE PROBABILITY

The capacity of the Rayleigh faded system illustrated in Fig. 1 (assuming Gaussian transmit codebooks) is

$$I = \frac{1}{2}\log_2(1 + \gamma|h_{\text{eq}}|^2) \quad \text{bps/Hz}, \tag{36}$$

where the statistics of $|h_{\text{eq}}|^2$ were derived in (31) and (34). The factor $\frac{1}{2}$ reflects that the same information is conveyed to the destination in two time slots. The outage probability is defined as

$$\begin{aligned} p_{\text{out}}(R, SNR) &= \mathbb{P}(I < R) \\ &= \mathbb{P}\left(|h_{\text{eq}}|^2 \leq \frac{2^{2R} - 1}{\gamma}\right) \end{aligned} \tag{37}$$

Setting $x \doteq (2^{2R} - 1)/\gamma$, the outage probability, (37), is equivalent to $F_{|h_{\text{eq}}|^2}(x)$ that is derived in closed form in (31). Fig. 6 shows the outage probability plotted using Monte Carlo simulations and also exploiting the theoretical expression derived in (31) with the outer summation truncated at $k = 4$. The curves show that the truncated series representation, even with a small number k, provides accurate results across the whole range of SNRs (not only in the high-SNR region). It should be noted that the results of related studies in the

[2] "Exact" in the sense that for arbitrarily large but limited values for k in (24) the power series representation will have an arbitrarily small truncation error.

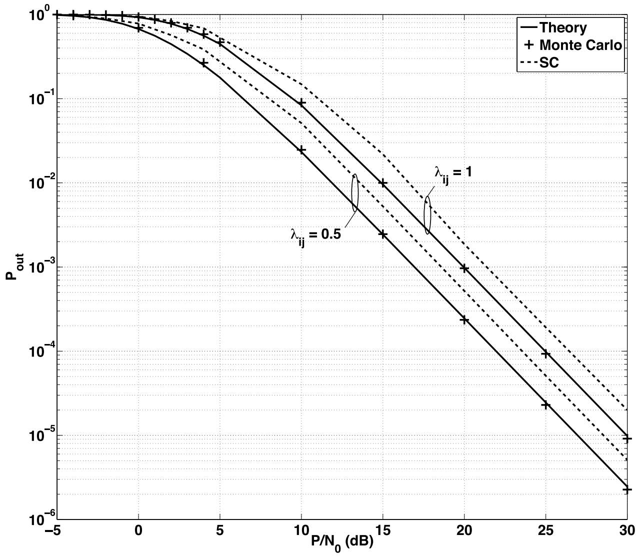

FIGURE 6. Outage probability for various λ_{ij}, $i \in \{s,r\}$ with $j \in \{r,d\}$ and $\alpha = 1$.

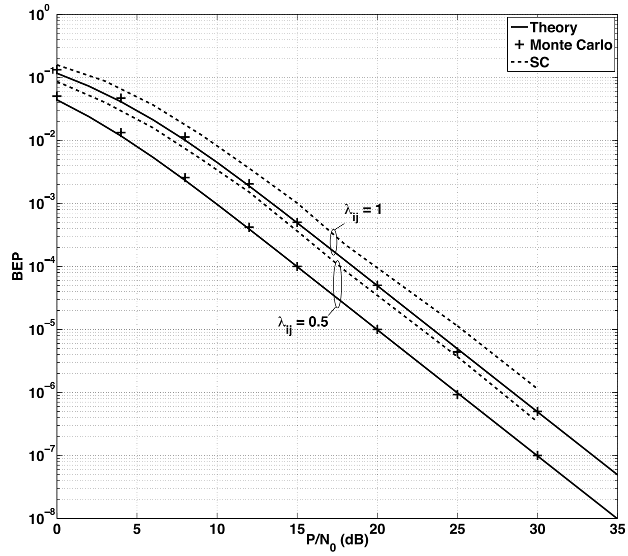

FIGURE 7. Bit Error Probability for various λ_{ij}, $i \in \{s,r\}$ with $j \in \{r,d\}$ and $\alpha = 1$.

literature (e.g. see [18, Fig. 3]) are usually based on bounding techniques that lead to much less accurate results than the approximation of the true results by a truncated power series we present in this paper. The dashed lines in Fig. 6 correspond to the outage probability of an equivalent system with an SC receiver at the destination. The comparison explicitly shows that MRC receivers outperform SC receivers.

B. AVERAGE BIT ERROR PROBABILITY

In this sequel, the average BEP of the AF relaying system in Fig. 1 is analytically investigated. The simplicity of channel model and the results in (31) and (34) allows for calculating the average BEP for arbitrary modulation schemes. For simplicity, however, we only evaluate the performance for BPSK modulation.

The instantaneous BEP of BPSK-modulated transmission over an AWGN channel, given channel coefficient h, is given by (e.g. [41])

$$p_b(e|h) = \frac{1}{2}\operatorname{erfc}(\sqrt{\gamma |h|^2}), \tag{38}$$

where $\gamma = P/N_0$ is transmit SNR. Therefore, assuming AF relaying with $h = h_{eq}$, the average BEP is

$$\begin{aligned} p_b(E) &= \frac{1}{2}\mathbb{E}_{h_{eq}}(\operatorname{erfc}(\sqrt{\gamma |h_{eq}|^2})) \\ &= \frac{1}{2}\int_0^\infty \operatorname{erfc}(\sqrt{\gamma x}) f_{|h_{eq}|^2}(x)dx. \end{aligned} \tag{39}$$

From [42, 7.1.19], $\frac{d}{dx}\operatorname{erfc}(\sqrt{\gamma x}) = -\sqrt{\frac{\gamma}{\pi}}x^{-1/2}e^{-\gamma x}$. Then, using integration by parts, $p_b(E)$ in (39) can be written as

$$p_b(E) = \frac{1}{2}\sqrt{\frac{\gamma}{\pi}}\int_0^\infty x^{-1/2}e^{-\gamma x}F_{|h_{eq}|^2}(x)dx. \tag{40}$$

By substituting $F_{|h_{eq}|^2}(x)$ from (31) into (40), then exploiting (30) and some basic manipulations, one can obtain the closed-form solution for the average BEP as

$$\begin{aligned} &p_b(E) \\ &= \frac{1}{2}\Big(1 - \mathcal{A}\sqrt{\frac{\gamma}{\gamma + \lambda_{sd}}} + \sum_{q=0}^{k}\sum_{c=0}^{q}\mathcal{B}\sqrt{\frac{\gamma}{\pi}}\frac{\Gamma(c+\frac{1}{2})}{(\gamma + \lambda_{srd})^{c+\frac{1}{2}}}\Big). \end{aligned} \tag{41}$$

Fig. 7 illustrates the BEP obtained from (41) with the outer summation truncated at $k = 4$. According to Fig. 7, regardless of the specific values of the fading parameters λ_{ij}, the theoretical BEP perfectly matches the Monte Carlo simulations at high *SNR*, whereas at low *SNR* the theoretical results only slightly deviate from the numerical results. Again it is clear from the comparison in Fig. 7 that the MRC receiver significantly outperforms the SC receiver (with the bounding technique applied to its analysis).

C. ERGODIC CAPACITY

As another example to show the usefulness of the proposed PDF and CDF for $|h_{eq}|^2$, the ergodic capacity of the relaying scheme depicted in Fig. 1 will be derived. The ergodic capacity is defined as

$$C_{av} = \frac{1}{2}\mathbb{E}_{h_{eq}}(\ln(1 + \gamma |h_{eq}|^2)) \tag{42}$$

where $\mathbb{E}_{h_{eq}}(\cdot)$ is defined as the expectation operator over h_{eq}, i.e.

$$C_{av} = \frac{1}{2}\int_0^\infty \ln(1+\gamma x) f_{|h_{eq}|^2}(x)dx \text{nats/s/Hz}, \tag{43}$$

where $\ln(\cdot)$ represents the natural logarithm. By substituting (34) into (43), we obtain a novel closed form expression for

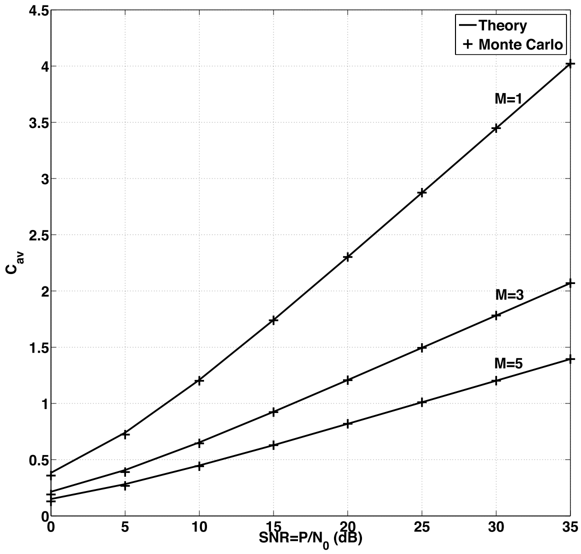

FIGURE 8. Ergodic Capacity (nats/sec/Hz) of the system with multiple relays and equal transmit power from the relays.

the exact ergodic capacity as

$$C_{av} = \frac{\mathcal{A}\lambda_{\rm sd}}{2\gamma} G_{2,3}^{3,1}\left(\frac{\lambda_{\rm sd}}{\gamma}\,\middle|\, \begin{matrix} -1, 0 \\ -1, -1, 0 \end{matrix}\right) + \sum_{q=0}^{k}\sum_{c=0}^{q} \frac{\mathcal{B}c}{2\gamma^c} G_{2,3}^{3,1}\left(\frac{\lambda_{\rm srd}}{\gamma}\,\middle|\, \begin{matrix} -c, -(c-1) \\ -c, -c, 0 \end{matrix}\right) - \frac{\mathcal{B}\lambda_{\rm srd}}{2\gamma^{c+1}} G_{2,3}^{3,1}\left(\frac{\lambda_{\rm srd}}{\gamma}\,\middle|\, \begin{matrix} -(c+1), -c \\ -(c+1), -(c+1), 0 \end{matrix}\right) \quad (44)$$

where $G_{p,q}^{m,n}(\cdot)$ is Meijer G-function [33, Chapter 16]. To solve (43), one can employ [31, 4.222.8] for integer c and some algebraic manipulations to arrive at (44); the details are omitted due to lack of space. Note that we exploit $f_{|h_{\rm eq}|^2}(x)$ in (34) to derive the C_{av} of a single relay cooperative system in (44). However, a simple approach based on the convolution of PDFs was explained in Section IV-B, in order to derive the PDF of $SNR_{\rm tot}$ for a multiple relay system. Using that, it is straightforward to derive the C_{av} of a multiple relay AF system. Fig. 8 illustrates the ergodic capacity of the system, theoretically and also using Monte Carlo simulations, assuming various numbers of relays. It is clear from Fig. 8 that, regardless of the number of the relays, the results are not only precise at high SNR but also are accurate at low SNR.

Again it should be noted that the results of related studies based on bounding techniques (see e.g. [16, Fig. 3] or [18, Fig. 4]) lead to much less accurate results.

VI. CONCLUSIONS

An analytical probabilistic description by the PDF and the CDF of the power-gain, $|h_{\rm srd}|^2$, of the equivalent source-relay-destination channel is derived for variable-gain Amplify-and-Forward relaying systems that are subject to block Rayleigh fading on the transmit channels. The PDF and CDF involve modified Bessel functions of the second kind, $K_\nu(\cdot)$. Based on fractional calculus mathematics, a novel power series representation of $K_\nu(\cdot)$ is introduced that has simple elementary functions of the form $x^n e^{-x}$ as its basic components. This allows to further obtain the PDF and the CDF of the power gain, $|h_{\rm eq}|^2$, of the overall channel observed by the destination, which consists of a source-relay-destination link and a direct source-destination link. For the analysis, a maximum-ratio-combining receiver is assumed at the destination.

The usefulness of the PDFs and CDFs expressed by the novel power series representation is demonstrated by theoretical analysis of the outage probability, the bit error probability and the ergodic capacity of a variable-gain AF cooperative system with an MRC receiver at the destination. In order to ease computation, the power series representations of the analytical formulas were truncated with a small number of power series coefficients, and the numerical results were demonstrated to perfectly match Monte Carlo simulations.

The analytical formulas presented in this paper (for the PDF and CDF as well those for the performance metrics) provide a theoretical framework to analyse the performance of the AF cooperative systems with an MRC receiver appear to be the first theoretical expressions derived so far that are accurate across the whole range of channel SNRs, in particular, the ones derived for outage probability and ergodic capacity.

APPENDIX A
STATISTICS OF THE S-R-D LINK SNR

Using (3), the $SNR_{\rm srd}$ is

$$SNR_{\rm srd} \doteq \frac{\frac{P_{\rm r}P_{\rm s}}{P_{\rm s}|h_{\rm sr}|^2+N_0}|h_{\rm sr}|^2|h_{\rm rd}|^2}{\frac{P_{\rm r}}{P_{\rm s}|h_{\rm sr}|^2+N_0}|h_{\rm rd}|^2N_0+N_0}. \quad (45)$$

where the notations are the same as defined in Section II. In what follows, without loss of generality, we set

$$P_{\rm s} \doteq P \quad \text{and} \quad P_{\rm r} \doteq \alpha P \quad \text{with} \quad \alpha > 0, P > 0 \quad (46)$$

to simplify notation. We obtain

$$SNR_{\rm srd} = \frac{(P/N_0)|h_{\rm sr}|^2\alpha|h_{\rm rd}|^2}{\alpha|h_{\rm rd}|^2+|h_{\rm sr}|^2+(N_0/P)}. \quad (47)$$

As the magnitudes of the channel coefficients are Rayleigh-distributed, the squared magnitudes $|h_{\rm sr}|^2$ and $|h_{\rm rd}|^2$ are exponentially distributed according to

$$f_{|h_{\rm sr}|^2}(x) = \lambda_{\rm sr}e^{-\lambda_{\rm sr}x} \text{ and } f_{|h_{\rm rd}|^2}(x) = \lambda_{\rm rd}e^{-\lambda_{\rm rd}x} \text{ with } x > 0 \quad (48)$$

and the parameters $\lambda_{\rm sr} \doteq 1/\sigma_{\rm sr}^2$ and $\lambda_{\rm rd} \doteq 1/\sigma_{\rm rd}^2$ (see [40, p. 190]. Note that $\alpha|h_{\rm rd}|^2$ is also exponentially distributed with parameter $\lambda_{\rm rd}/\alpha$. Therefore, to further simplify notation, we substitute $\alpha|h_{\rm rd}|^2$ by $|h_{\rm rd}|^2$ (i.e. $|h_{\rm rd}|^2 \rightarrow \alpha|h_{\rm rd}|^2$) but we assume that the new $|h_{\rm rd}|^2$ is exponentially distributed, now with parameter $\lambda_{\rm rd} \doteq 1/\alpha\sigma_{\rm rd}^2$. Moreover, we set $\gamma \doteq$

P/N_0, so (47) can be written as $SNR_{\text{srd}} = \gamma|h_{\text{srd}}|^2$ with

$$|h_{\text{srd}}|^2 \doteq \frac{|h_{\text{sr}}|^2|h_{\text{rd}}|^2}{|h_{\text{sr}}|^2 + |h_{\text{rd}}|^2 + 1/\gamma} \tag{49}$$

the power gain of the equivalent S-R-D channel. In the following *Lemma*, the CDF and the PDF of $|h_{\text{srd}}|^2$ is derived.

Lemma 1: The CDF and the PDF of $|h_{\text{srd}}|^2$

Let X_1 and X_2 be two independent exponentially distributed RVs with the PDFs $f_{X_i}(x_i) = \lambda_i \mathrm{e}^{-\lambda_i x_i}$, $x_i \geq 0$, $i \in \{1, 2\}$, and the parameters $\lambda_1, \lambda_2 > 0$, and let $\delta > 0$ be a real constant. Then, the CDF of the RV $X = \frac{X_1X_2}{X_1+X_2+\delta}$ is

$$\begin{aligned} F_X(x) &= 1 - 2\mathrm{e}^{-(\lambda_1+\lambda_2)x}\sqrt{\lambda_1\lambda_2 x(x+\delta)} \\ &\quad \times K_1\left(2\sqrt{\lambda_1\lambda_2 x(x+\delta)}\right) \end{aligned} \tag{50}$$

Proof: From the definition of the CDF we obtain

$$F_X(x) = \mathbb{P}(\frac{X_1X_2}{X_1+X_2+\delta} < x). \tag{51}$$

As long as $X_1 < x$ we have $X < x$ for any value of X_2, but for $X_1 > x$, we have $X < 0$ only for the range $0 < X_2 < \frac{(X_1+\delta)x}{X1-x}$. Hence, the probability in (51) is computed by two integrations as follows:

$$\begin{aligned} F_X(x) &= \int_{x_1=0}^{x}\int_{x_2=0}^{\infty} f_{X_1}(x_1)f_{X_2}(x_2)\mathrm{d}x_2\mathrm{d}x_1 \\ &\quad + \int_{x_1=x}^{\infty}\int_{x_2=0}^{\frac{(x_1+\delta)x}{(x_1-x)}} f_{X_1}(x_1)f_{X_2}(x_2)\mathrm{d}x_2\mathrm{d}x_1 \\ &= \int_{x_1=0}^{x} \lambda_1\mathrm{e}^{-\lambda_1x_1}\mathrm{d}x_1 \\ &\quad + \int_{x_1=x}^{\infty} \lambda_1\mathrm{e}^{-\lambda_1x_1}\int_{x_2=0}^{\frac{(x_1+\delta)x}{(x_1-x)}} \lambda_2\mathrm{e}^{-\lambda_2x_2}\mathrm{d}x_2\mathrm{d}x_1 \\ &= 1 - \mathrm{e}^{-\lambda_1x} \\ &\quad + \int_{x_1=x}^{\infty} \lambda_1\mathrm{e}^{-\lambda_1x_1}\left(1 - \mathrm{e}^{-\lambda_2\frac{(x_1+\delta)x}{(x_1-x)}}\right)\mathrm{d}x_1 \\ &= 1 - \lambda_1\int_{x_1=x}^{\infty} \mathrm{e}^{-\lambda_2\frac{(x_1+\delta)x}{x_1-x}} \cdot \mathrm{e}^{-\lambda_1x_1}\mathrm{d}x_1 \\ &= 1 - \lambda_1\int_{u=0}^{\infty} \mathrm{e}^{-\lambda_2\frac{(u+x+\delta)x}{u}} \cdot \mathrm{e}^{-\lambda_1(u+x)}\mathrm{d}u \\ &= 1 - \lambda_1\mathrm{e}^{-(\lambda_1+\lambda_2)x}\int_{u=0}^{\infty} \mathrm{e}^{-\lambda_2\frac{(x+\delta)x}{u}} \cdot \mathrm{e}^{-\lambda_1u}\mathrm{d}u \\ &= 1 - 2\mathrm{e}^{-(\lambda_1+\lambda_2)x}\sqrt{\lambda_1\lambda_2x(x+\delta)} \\ &\quad \times K_1\left(2\sqrt{\lambda_1\lambda_2x(x+\delta)}\right) \end{aligned} \tag{52}$$

where the last equality is obtained from [31, 3.471.9] with

$$\int_0^{\infty} u^{\nu-1}\mathrm{e}^{-\alpha u} \cdot \mathrm{e}^{-\frac{\beta}{u}}\mathrm{d}u = 2\left(\frac{\beta}{\alpha}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{\alpha\beta}\right), \tag{53}$$

where $\nu = 1$ and $\beta \doteq \lambda_2(x+\delta)x$, $\alpha \doteq \lambda_1$ are positive real values and $K_\nu(\cdot)$ is the modified Bessel function of the second kind and ν-th order.

The PDF is best computed as the derivative $f_X(x) = dF_X(x)/dx$ of (52); the result is given in (4). ■

APPENDIX B
PROOF OF THE LEIBNIZ RULE FOR THE RIEMMAN-LIOUVILLE INTEGRATION OPERATOR

Let $s > 0$. The Riemman-Liouville intagration operator is defined as $I^s\{h(x)g(x)\} = \frac{1}{\Gamma(s)}\int_0^x (x-t)^{s-1}h(t)g(t)\mathrm{d}t$. It is straightforward to derive the Leibniz rule by performing a Taylor series expansion of $h(t)$ at $t = x$, i.e. $h(t) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(x-t)^nD^n\{h(x)\}$. We obtain

$$\begin{aligned} &I^s\{h(x)g(x)\} \\ &\quad = \frac{1}{\Gamma(s)}\int_0^x (x-t)^{s-1}h(t)g(t)\mathrm{d}t \\ &\quad = \frac{1}{\Gamma(s)}\int_0^x (x-t)^{s-1} \\ &\qquad \times \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(x-t)^nD^n\{h(x)\}g(t)\mathrm{d}t \\ &\quad = \frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}D^n\{h(x)\}\int_0^x (x-t)^{n+s-1}g(t)\mathrm{d}t \\ &\quad = \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+s)}{n!\Gamma(s)}I^{n+s}\{g(x)\}D^n\{h(x)\} \end{aligned} \tag{54}$$

where $D^n\{h(x)\} = \frac{d^n}{dx^n}h(x)$.

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