The Interactive Sum Choice Number of Graphs

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Abstract

We introduce a variant of the well-studied sum choice number of graphs, which we call the *interactive sum choice number*. In this variant, we request colours to be added to the vertices' colour-lists one at a time, and so we are able to make use of information about the colours assigned so far to determine our future choices. The interactive sum choice number cannot exceed the sum choice number and we conjecture that, except in the case of complete graphs, the interactive sum choice number is always strictly smaller than the sum choice number. In this paper we provide evidence in support of this conjecture, demonstrating that it holds for a number of graph classes, and indeed that in many cases the difference between the two quantities grows as a linear function of the number of vertices.

Keywords: List coloring, sum-choice number.

1 Introduction

The *choice number* of a graph G is the minimum length of colour-list that must be assigned to each vertex of G so that, regardless of the choice of colours in these lists, there is certain to be a proper colouring of G in which every vertex is coloured with a colour from its list. A small subgraph of G which is, in some sense, "hard" to colour, can therefore force the choice number for G to be large, even if most of the graph is "easy" to colour. The *sum choice number* of G (written $\chi_{\mathrm{SC}}(G)$), introduced by Isaak [9], captures the "average difficulty" of colouring a graph: each vertex can now be assigned a different length of colour-list, and the aim is to minimise the sum of the list lengths (while still guaranteeing that there will be a proper list colouring for any choice of lists). A long odd cycle is an example of a graph where most of the graph is "easier" to colour than the choice number indicates.

For any graph $G = (V, E)$, we have $\chi_{\mathrm{SC}}(G) \leq |V| + |E|$: we can order the vertices arbitrarily and assign to each vertex $d^-(v) + 1$ colors, where $d^-(v)$ is the number of neighbors of v that are before it in the order, and colour greedily in this order. Graphs for which this so-called greedy bound is in fact equal to the sum choice number are said to be *sc-greedy*, and one of the main topics for research into the sum choice number has been the identification of families of graphs which are (or are not) sc-greedy; a lot of work has been done on the sum choice number of graphs (see, for example [2,6,7,8,10,11]), but relatively little is known.

In this paper we introduce a variation of the sum choice number, called the *interactive sum choice number* of G (written $\chi_{\mathrm{ISC}}(G)$), in which we do not have to determine in advance all of the lengths of the colour lists: at each step we ask for a new colour to be added to the colour list for some vertex of our choosing and, depending on what colours have been added to lists so far, we can adapt our strategy. It is clear that $\chi_{\mathrm{ISC}}(G) \leq \chi_{\mathrm{SC}}(G)$ for any graph G, as we can simply ask for the appropriate number of colours to be added to the list for each vertex without paying any attention to the colours that have been added so far. The natural question is then whether we are in fact able to improve on the sum choice number of G by exploiting partial information about the colour lists.

If $G = (V, E)$ is a complete graph, the answer to this question is no. To

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see this, suppose that for every vertex $v \in V$, the first time we ask to add a vertex to the colour list for v it will be given colour 1, the second time it will be given colour 2, and so on. Then, whatever order we request to add colours, we know that there can then be at most one vertex for which we never request a second colour (otherwise two adjacent vertices would have to be assigned colour 1), at most one vertex for which we request exactly two colours, and more generally for each $1 \leq i \leq n = |V|$ there can be at most 1 vertex for which we request exactly i colours in total. Thus we see that

$$\chi_{\mathrm{ISC}}(G) \geq \sum_{i=1}^{n} i = |V| + |E| \geq \chi_{\mathrm{SC}}(G).$$

However, we believe that this may be the only case in which there is equality:

Conjecture 1.1 *If G is not a complete graph, then $\chi_{ISC}(G) < \chi_{SC}(G)$.*

The main purpose of this paper is to give evidence for Conjecture 1.1. We confirm it for sc-greedy graphs, and prove more strongly that the gap between $\chi_{\mathrm{ISC}}(G)$ and $\chi_{\mathrm{SC}}(G)$ is an increasing function of the number of vertices for various graph classes including trees, unbalanced complete bipartite graphs and grids (the latter two being classes which are known not to be sc-greedy). A particular challenge in proving special cases of our conjecture is that, for many graphs G, $\chi_{\mathrm{SC}}(G)$ is only lower and upper bounded, not fully determined.

Two other variants of sum-choosability have also been introduced recently. In the *sum-paintability* variant [4,12], the *painter* decides a budget for each vertex in advance (as in sum list colouring), then in each round the *lister* reveals a subset of vertices which have colour c in the list and the painter must decide immediately which of these vertices to paint with colour c. Thus, there is less information available than in the standard setting of sum-choosability since painter must fix the colour of some vertices before knowing the entire contents of the colour lists. The relationship between the interactive sum choice number and the second of these variants, the *slow-colouring game* [13,14] is less clear. In this variant, at each round, lister reveals a nonempty subset M of the remaining vertices (scoring $|M|$ points), from which painter chooses an independent set to delete; painter seeks to maximise the total score when all vertices have been deleted, while lister seeks to minimise this quantity. Compared with our setting, lister has the advantage of discovering at the same time all vertices which are permitted to use colour c, but on the other hand he must decide immediately which of these to colour with c.

2 Graphs that are sc-greedy

In this section we consider the interactive sum choice number for graphs that are known to be sc-greedy. We begin with the result that Conjecture 1.1 holds for all sc-greedy graphs.

Theorem 2.1 *Let G be a connected graph on $n \geq 3$ vertices, which is not isomorphic to the complete graph K_n. If G is sc-greedy, then $\chi_{ISC}(G) < \chi_{SC}(G)$; in particular, if $\omega(G)$ is the cardinality of the largest clique in G, then $\chi_{ISC}(G) \leq \chi_{SC}(G) - \frac{n-\omega(G)}{2}$.*

The key observation we use in this proof is that, if P_3 denotes the path on 3 vertices, we have $\chi_{\mathrm{ISC}}(P_3) = 4 < \chi_{\mathrm{SC}}(P_3)$. Any graph that is not complete must contain an induced copy of P_3; by the assumption that the graph is sc-greedy we can first colour the induced P_3 and then extend this colouring greedily to a colouring of the whole graph, still beating the greedy bound for G. Repeating this process gives the result.

As an immediate corollary of this result, we obtain an upper bound on the interactive sum choice number of trees. As trees are known to be sc-greedy [10], we have that $\chi_{\mathrm{SC}}(T) = 2n - 1$ for any tree T on n vertices, so the difference between the two quantities grows linearly in the number of vertices.

Corollary 2.2 *Let T be a tree on n vertices. Then $\chi_{ISC}(T) \leq \lfloor \frac{3n}{2} \rfloor$.*

We observe that the bound in Theorem 2.2 is tight for paths; however, for the case of stars there is in fact an even larger difference between the two quantities.

Lemma 2.3 *$\chi_{ISC}(K_{1,p}) = p + q + 1$, where $q = max\{q \in \mathbb{N} | \frac{q*(q+1)}{2} \leq p\}$.*

We can also prove that there is a linear gap between the sum-choice and interactive sum-choice number for cycles, which were shown to be sc-greedy in [2].

Theorem 2.4 *Let C_n be the cycle on $n \geq 3$ vertices. Then $\chi_{ISC}(C_n) = \lfloor \frac{3(n+1)}{2} \rfloor$.*

Surprisingly, removing a single edge can make a relatively big difference.

Theorem 2.5 *For every $p \geq 10$, we have $\chi_{ISC}(K_p - e) \leq \chi_{ISC}(K_p) - \frac{p-2}{3}$.*

3 Graphs that are not sc-greedy

Since we know that Conjecture 1.1 holds for all sc-greedy graphs, the next challenge is to try to extend our results to deal with graphs that are not sc-greedy. The most natural family of non-sc-greedy graphs to consider first is perhaps that of complete bipartite graphs: $K_{2,3}$ is the smallest graph which is *not* sc-greedy. For the case of complete bipartite graphs, where one side of the bipartition is much larger than the other, we can show that our conjecture holds.

Theorem 3.1 *If $p \ll q$ then $\chi_{ISC}(K_{p,q}) < \chi_{SC}(G)$.*

We prove this result in two phases, showing first that for any $p, q \geq 2$ we have $\chi_{\mathrm{SC}}(K_{p,q}) \geq 2(p+q)$, and demonstrating on the other hand that $\chi_{\mathrm{ISC}}(K_{p,q}) \leq p + q + p^2\sqrt{2q}$. We therefore have that $\chi_{\mathrm{ISC}}(K_{p,q}) < \chi_{\mathrm{SC}}(K_{p,q})$ so long as q is sufficiently large compared with p that $p^2\sqrt{2q} < p+q$. In fact, the methods we use to prove this result can be generalised to deal with a slightly more general family of graphs: any graph that contains a very (very) large subgraph with small maximum degree satisfies Conjecture 1.1.

Theorem 3.2 *Let $G = (A \cup B, E)$ be a graph. If $(max(\Delta(G[A]), \Delta(G[B]) + 1) \cdot |A|^2\sqrt{2|B|} < |A| + |B|$, then $\chi_{ISC}(G) < \chi_{SC}(G)$.*

Another family of bipartite graphs that are not, in general, sc-greedy is that of grid graphs. In this case, we can show that the interactive sum choice number is strictly smaller than the sum choice number of the $k \times \ell$ grid $G_{k,\ell}$ for any positive integers k and ℓ; in fact we prove the following result.

Theorem 3.3 *Let $G_{k,\ell}$ denote the $k \times \ell$ grid, where $k \leq \ell$, and suppose that $\ell \geq 3$. Then*

$$\chi_{SC}(G_{k,\ell}) - \chi_{ISC}(G_{k,\ell}) \geq \frac{1}{18}k\ell.$$

To prove this theorem, we first bound from below the sum-choice number of $G_{k,\ell}$, using the formula for $\chi_{\mathrm{SC}}(G_{3,\ell})$ derived by Heinold [8]; we then prove an upper bound on $\chi_{\mathrm{ISC}}(G_{k,\ell})$ by decomposing the grid into a number of paths.

4 Conclusions and open problems

We have introduced the interactive sum choice number of graphs, a variation of the sum choice number in which we are able to exploit partial information about the contents of colour lists in order to inform our strategy. We demonstrated that in many cases this additional information allows us to guarantee a proper list colouring when the sum of list lengths over all the vertices is strictly smaller than the sum choice number of the graph, and for several families of graphs we were in fact able to prove the existence of a large gap between the sum choice number and the interactive sum choice number. Proofs and additional statements can be found in [3].

As is often the case when a new problem is introduced, this paper raises more questions than it solves. The key open question arising from this work is to prove Conjecture 1.1, namely that if G is not a complete graph then $\chi_{\mathrm{ISC}}(G) < \chi_{\mathrm{SC}}(G)$; a first step would be to attempt to prove the conjecture for further graph classes, for example k-degenerate graphs, chordal graphs, planar graphs, cographs or graphs of bounded treewidth. Since graphs with high degeneracy are known to have high choice number [1] with a proof that only really uses arguments around one arbitrary vertex, it might be worth trying to prove similarly that they have (very) high sum choice number. In turn, that would be a step towards Conjecture 1.1 for graphs with high degeneracy.

It would also be interesting to investigate further just how much these two quantities can differ; in particular, the upper bounds on the interactive sum choice number that we have obtained for unbalanced bipartite graphs and grids are unlikely to be tight, so it seems natural to seek better bounds for these graph classes. A natural next step is to attempt to find further classes of graphs for which the difference between the sum choice number and the interactive sum choice number is a growing function of the number of vertices. Our proof that Conjecture 1.1 holds for sc-greedy graphs actually seems to indicate that sc-greedy graphs may well form such a class. On the other hand, what can we say about the structure of graphs for which the difference between the sum choice number and interactive sum choice number is bounded by some constant independent of the number of vertices?

In addressing any of these questions, it would be extremely helpful to understand how to use cut-edges, cut-vertices, modules, joins, and similar decompositions of graphs.

References

[1] Alon, N., *Degrees and choice numbers*, Random Structures and Algorithms **16** (2000), no. 4, 364–368.

[2] Berliner, A., U. Bostelmann, A.R. Brualdi, and L. Deaett, *Sum list coloring*

graphs, Graphs and Combinatorics **22** (2006), no. 2, 173–183.

[3] Bonamy, M. and K. Meeks, *The interactive sum choice number of graphs*, arXiv:1703.05380 [cs.DM], 2017.

[4] Carraher, J.M., T. Mahoney, G.J. Puleo, and D.B. West, *Sum-paintability of generalized theta-graphs*, Graphs and Combinatorics **31** (2015), no. 5, 1325–1334.

[5] Diestel, R., *Graph theory*, third ed., Graduate Texts in Mathematics, vol. 173, Springer-Verlag, Heidelberg, 2005.

[6] Füredi, Z. and I. Kantor, *List colorings with distinct list sizes, the case of complete bipartite graphs*, Journal of Graph Theory **82** (2016), no. 2, 218–227.

[7] Heinold, B., *Sum list coloring and choosability*, Ph.D. thesis, Lehigh University, 2006.

[8] Heinold, B., *The sum choice number of* $P_3\Box P_n$, Discrete Applied Mathematics **160** (2012), no. 78, 1126 – 1136.

[9] Isaak, G., *Sum list coloring 2 by n arrays*, Electronic Journal of Combinatorics **9** (2002), Note 8.

[10] Isaak, G., *Sum list coloring block graphs*, Graphs and Combinatorics **20** (2004), no. 4, 499–506.

[11] Lastrina, M. and M. Young, *Sum list coloring, the sum choice number, and sc-greedy graphs*, arXiv:1305.1962 [math.CO], 2013.

[12] Mahoney, T., C. Tomlinson, and J.I. Wise, *Families of online sum-choice-greedy graphs*, Graphs and Combinatorics **31** (2015), no. 6, 2309–2317.

[13] Mahoney, T., G.J. Puleo, and D.B. West, *Online Paintability: The Slow-Coloring Game*, arXiv:1507.06513 [math.CO], 2015.

[14] Puleo, G.J. and D.B. West, *Online Sum-Paintability: Slow-Coloring of Trees*, arXiv:1612.04702 [math.CO], 2016.