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Optimal robust bilateral trade: Risk neutrality

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Abstract

A risk neutral seller and buyer with private information bargain over an indivisible item. We prove that optimal robust bilateral trade mechanisms are payoff equivalent to non-wasteful randomized posted prices.

1 Introduction

A seller and a buyer with private information bargaining over an indivisible item is a most fundamental market interaction. It leads to questions of pricing, aggregation of private information and efficiency. Ultimately, what is the set of pricing mechanisms, which are incentive compatible, feasible, robust to the details of traders' information, and, while satisfying these requirements, as efficient as possible (optimal)? A trading mechanism is robust if it satisfies *ex post* incentive compatibility, balanced budget and individual rationality, and it is optimal if it satisfies an appropriate Pareto criterion.[1] Under the assumption that the two traders are risk neutral, we answer this question: such pricing mechanisms are equivalent to non-wasteful randomized posted prices.

A randomized posted price is a common and intuitive mechanism. There is a fixed probability distribution over prices and a price is drawn from that distribution.

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[1] In their seminal study, [3] provide a foundation for using *ex post* incentive compatibility as a suitable notion for robustness. In environments with two agents, *ex post* incentive compatibility is equivalent to *interim* implementability on every type space. In private values environments, *ex post* incentive compatibility of a direct revelation mechanism is equivalent to incentive compatibility in dominant strategies. See also [14] and [7]. A suitable notion of Pareto optimality is the *ex post* constrained optimality, defined by [8], which is discussed in more detail below.

Each trader then announces whether or not he is willing to trade at that price, which presumably depends on the trader's private valuation of the item. Trade is effected if both traders agree to trade, otherwise no trade takes place. A randomized posted price is non-wasteful if prices at which at least one of the traders would never trade occur with zero probability. Posted prices are commonly used in the real world and albeit in more complex settings, [2] suggest that price dispersion can at least to some extent be viewed as pure randomization, see also, e.g., [13]. In the present context, randomized posted prices have been studied by [11], who gave some technical conditions (differentiability of prices and allocations, deterministic mechanisms, and mechanisms, which are step functions), under which robust trading mechanisms are characterized by (non-wasteful) randomized posted prices. As an auxilliary result, we use a different approach to prove their claim in general. This then allows us to apply a suitable notion of optimality and characterize the optimal robust trading mechanisms.[2,3]

It is quite evident that a randomized posted price satisfies *ex post* incentive compatibility, balanced budget and individual rationality, i.e., it is a robust trading mechanism. By misrepresenting his private valuation, for example by saying "no" to trade when the realized price would have made it profitable, a trader can only lose opportunities to make a profit while such misrepresentation brings no potential gains. But it is much less obvious that any pricing mechanism satisfying these properties is equivalent to a randomized posted price. Consider the mechanism whereupon reports $v = (v_1, v_2) \in (0,1)^2$, the traders trade at a price $p(v)$ with a probability $\varphi(v)$, and with probability $1 - \varphi(v)$ there is no trade,[4] where,

$$p(v) = \begin{cases} \frac{v_1+v_2}{2}, & \text{if } v_1 \le v_2 \\ 0, & \text{otherwise} \end{cases}; \quad \varphi(v) = \begin{cases} \alpha(v_2 - v_1), & \text{if } v_1 \le v_2 \\ 0, & \text{otherwise} \end{cases}; \quad \alpha \in (0,1]. \quad (1)$$

It is immediate to check that this mechanism is incentive compatible.[5] When $\alpha = 1$, this mechanism is equivalent to a randomized posted price whereby the price is

[2] Other related recent advances in bilateral trade concern weakly undominated strategies, see [18] and [5].

[3] In a very different setting of an exchange economy with a much richer domain of preferences, [1] prove that dominant-strategy incentive compatible social choice functions are characterized by trading at a finite number of pre-specified proportions, a result that is similar in spirit.

[4] Note that this parametrization is different from the parametrization in, e.g., [16], where a direct revelation mechanism is parametrized by the probability of trade and the expected transfer. That parametrization was then adopted by [11]. When one considers *ex post* individual rationality and budget balance, the price conditional on trade taking place seems more intuitive and suggests a different proof for the general characterization result.

[5] Suppose that the seller has a valuation v_1, reports $\tilde{v}_1$, and to keep this illustration simple, assume that the report of the buyer is $v_2 \ge \tilde{v}_1$. Then the seller's expected utility is $\alpha(v_2 - \tilde{v}_1)\left(\frac{\tilde{v}_1+v_2}{2} - v_1\right)$, which is a quadratic expression that is maximized at $\tilde{v}_1 = v_1$, for any $\alpha \in (0,1]$. It is quite obvious that this mechanism is individually rational and requires no external subsidies.

uniformly distributed over $(0, 1)$ – under each of the two mechanisms each trader obtains the same payoff for every realization of valuations. Indeed, $\varphi(v)$ is then the probability that the realized price is in the interval (v_1, v_2), when $v_2 > v_1$, and $p(v)$ is the conditional expected price, i.e., the expected price, conditional on trade taking place. When $\alpha < 1$, this mechanism is *wasteful*: since the traders' valuations come from the interval $(0, 1)$, $1 - \alpha$ represents the probability mass assigned to price 0, or any other price at which traders would never trade, and that is wasteful. One might be tempted to think that this example is somewhat special and that it might be easy to find robust trading mechanisms which cannot be represented in this way. In Proposition 1 we use a measure-theoretic approach to prove that any incentive compatible and feasible mechanism can be represented by a randomized (possibly wasteful) posted price.

The question looms whether the pricing mechanism in (1) is as efficient as a robust trading mechanism can be. Implicit in this question is a suitable criterion for optimality. When $\alpha < 1$, i.e., the mechanism is wasteful, the answer is certainly no – if instead $\alpha = 1$, both traders will be better off for any vector of valuations and strictly better off for any valuations in the set $v_2 > v_1$. Hence, the robust trading mechanism with $\alpha = 1$ *ex post Pareto dominates* the robust trading mechanism with $\alpha < 1$. This *ex post* Pareto dominance is what defines our efficiency criterion, i.e., a robust trading mechanism is optimal if there does not exist any robust trading mechanism which *ex post* Pareto dominates it. This notion of *ex post* constrained efficiency is relatively weak but that is not surprising: robustness requires that the details of the distribution of traders' private parameters should not matter much so that in order to dominate some other mechanism, a given mechanism must dominate it for a variety of different beliefs or type spaces, which then defines the notion of constrained optimality and durability for robust environments.[6] Of course, the argument above does not prove that with $\alpha = 1$ the mechanism (1) is optimal – there may in principle be some entirely different mechanism dominating it. The optimality of (1) is best illustrated by considering two different deterministic posted prices, $p, p' \in (0, 1)$. It is clear that neither *ex post* dominates the other as the two traders will trade at different valuations and the gains from trade will also be allocated differently. Along with Proposition 1 this argument yields our main result, Theorem 1, that a robust trading mechanism is optimal if and only if it is equivalent to a non-wasteful randomized posted price.

[6] See [8] for a detailed argument and further examples. This notion of optimality is also closely related to the *ex post* incentive efficiency defined by [12] for the case of *interim* (Byesian) incentive and feasibility constraints. [12] instantly discard the *ex post* incentive efficiency as unavailing on the grounds that at the *ex post* stage all the information is known so that there is presumably no longer any need to consider the informational constraints. However, if one is to compare *incentive-feasible allocations to other incentive-feasible allocations*, then the *ex post* incentive efficiency, or the *ex post* constrained optimality in the present environment, describes the set of allocation rules which are undominated when this comparison is effected at the *ex post* stage.

The questions addressed here have a clear parallel in a Bayesian setting, where the details of traders' information do matter. Under the Bayesian informational assumptions, specifically, the common prior, [16] provided a method to find the optimal *ex ante* incentive efficient mechanisms for bilateral trade and showed that in the specific example such a mechanism may arise as an equilibrium of a natural double-auction game. By deriving the minimal expected informational rent to each trader implied by the traders' incentive compatibility and feasibility constraints [16] showed that efficient bilateral trade is impossible: the social surplus does not suffice to incentivize the traders to reveal their private information; as a result trade is sometimes not effected when that would have been *ex post* efficient. This is of course also true in the present case as robustness is more difficult to satisfy. In an example when traders' priors are uniform and symmetric, [16] then showed that an equilibrium of a double-auction game studied by [6], which results in equal sharing of the surplus, attains the efficiency bound. In the present setting, a similar statement is true much more generally: the pricing mechanisms here arise naturally as equilibria of a continuous-time double-auction game, where a mediator only reveals bids to traders once they are compatible and trade is realized, see [10]. The efficiency measure considered by [16] is the ex ante incentive efficiency, which was proposed by [12]. Here, the efficiency measure is the *ex post* constrained optimality, proposed by [8] along the lines of [12].

When considering aggregation of information, one may view the results here as somewhat negative: in a posted price not much information gets aggregated beyond the fact that both traders were willing to trade at that price. One may also take a more descriptive perspective: posted prices are so common precisely because they are robust and optimal, at least in the static setting considered here. But the results here might perhaps be most useful from a normative perspective: the complete characterization of the *ex post* frontier of robust trading mechanisms may serve as a tool for applications to more specific *ex ante* welfare criteria consistent with robust analysis. Such is for example the regret-free criterion considered by [4] in a simpler setting with one-sided private information. Under any such *ex ante* welfare criterion, the optimal pricing mechanism must be some randomized (possibly deterministic) posted price.[7] Solving for an optimum within the set of randomized posted prices is a considerable simplification relative to optimizing the social welfare criterion subject to the incentive and feasibility constraints.

[7]In order to maximize any *ex ante* welfare criterion, a mechanism must be *ex post* constrained optimal.

2 Definitions and theorem

A seller, 1, and a buyer, 2, bargain over the price of an indivisible good. We denote by v_1 the seller's cost of producing the good, and by v_2 the buyer's value of the good. We assume that at the time of trade, each trader knows her own type (valuation) v_i, but doesn't know the type of the other trader. It is common knowledge that pairs of types $v = (v_1, v_2)$ are drawn from $(0,1)^2$ according to *some* joint probability distribution function F, with $support(F) = (0,1)^2$, and such that F is absolutely continuous with respect to the Lebesgue measure. While the support of F is common knowledge, F is not common knowledge so that the details of F are not known to the traders, or any other entity, such as a social planner or a mechanism designer.[8]

Denote by $\ell \in \{0,1\}$ the allocation of the good, where $\ell = 1$ if the object is transferred to the buyer – there is trade, and $\ell = 0$ if there is no trade. When the allocation is ℓ, the payment from the buyer to the seller is $p \in (0,1)$, and the valuations are $(v_1, v_2) \in (0,1)^2$, the payoff to trader i is given by a utility function $u_i(\ell, p, v_i)$, where for the seller, $u_1(p, v_1) = p - \ell v_1$, and for the buyer, $u_2(\ell, p, v_2) = \ell v_2 - p$. This is the standard case when traders are risk neutral and have additively separable utility for money and the object.

A direct revelation mechanism μ is a mapping from traders' reports $\tilde{v}_i \in (0,1)$ of their valuations into outcomes. An outcome is given by a lottery $\mu[\tilde{v}]$ over the possible allocations of the good and prices, $\{0,1\} \times (0,1)$. That is, $\mu[\tilde{v}]$ is a lottery that the traders face *ex post*, after they made reports $(\tilde{v}_1, \tilde{v}_2)$ of their valuations to a computer, or a broker. Therefore, a mechanism μ is a mapping,

$$\mu : (0,1)^2 \to \Delta\left(\{0,1\} \times (0,1)\right).$$

We assume that at the time of trade, each trader is allowed to step back if she finds the terms of trade unfavorable. For example, the buyer will not make a payment unless he obtains the good which he values at least as much as the payment. We also assume that trade is not subsidized from an external source. Therefore, a trading mechanism must satisfy *ex post* individual rationality and *ex post* budget balance. Additionally, a robust trading mechanism must satisfy *ex post* incentive compatibility, that is, reporting valuations truthfully must be an *ex post* Nash equilibrium.[9]

[8] For example, traders may have different beliefs about F, and different beliefs about the beliefs of the other trader and so on, i.e., any type space is allowed, see [3]. The analysis here easily generalizes to the case where the support of F is given by $(\underline{v}, \overline{v}) \times (\underline{v}, \overline{v})$, $\underline{v} < \overline{v}$, as well as to the case when the support of F is closed; the assumption that the support of F is open is made mainly for notational convenience: when the price of the good is 0 that signifies no trade.

[9] In a separable environment, *ex post* incentive compatibility is equivalent to requiring that the trading mechanism is *interim* (or Bayesian) incentive compatible on any type space, and in particular, for any common prior distribution over payoff types F, see [3] and also [14]. The revelation principle holds (see, e.g., [15]) so that the restriction to direct revelation mechanisms is without loss of generality.

A mechanism μ satisfies *ex post* budget balance and individual rationality if,[10]

$$support(\mu[v]) \subset \{(\ell, p) \mid v_1 \times \ell \leq p \leq v_2 \times \ell\}, \forall v \in (0,1)^2 .$$

Given a mechanism μ and reports $v = (v_1, v_2) \in (0,1)^2$, denote by $E_{\mu[v]}$ the expectation operator with respect to the probability measure $\mu[v]$. Denote by Ω the state space of realizations of lottery μ. By *ex post* individual rationality and budget balance, we can let $\Omega = [0, 1)$. The state space Ω then represents all possible prices, where price $p = 0$ signifies no trade. Throughout, j denotes the trader other than i.

A mechanism μ is *ex post* incentive compatible if,

$$E_{\mu[v_i,v_j]} u_i(\ell, p, v_i) \geq E_{\mu[\tilde{v}_i,v_j]} u_i(\ell, p, v_i), \forall \tilde{v}_i, \forall v_i, \forall v_j, i \in \{1, 2\} \tag{2}$$

Definition 1. A mechanism μ is a robust trading mechanism if it satisfies *ex post* budget balance, individual rationality, and incentive compatibility.

Our main question is: what are the efficient robust trading mechanisms? By [16] it is evident that *ex post* efficient mechanisms will in the present case not be possible – any mechanism satisfying *ex post* incentive and participation constraints also satisfies the *interim* incentive and participation constraints. Note also that under the present constraints only one type of *ex post* inefficiency is possible: namely, no trade occurring when it would be profitable.

The efficiency concept here cannot rely on the details of the distribution F over valuations since these details are not known. An appropriate concept of efficiency is the *ex post* constrained efficiency. It is intimately related to the *ex post* incentive efficiency of [12], adapted to the robust environment studied here, i.e., *ex post* Pareto optimality under the *ex post* incentive and feasibility constraints.[11] A robust

[10] A mechanism μ satisfies *ex post* budget balance, if, $\forall v \in (0,1)^2$, the payment received by the seller is at most as much as the payment made by the buyer (note that here it is equal), and the object is only transferred to the buyer if it has been produced by the seller, in any realization of the lottery $\mu[v]$. A mechanism μ satisfies *ex post* individual rationality, if, $\forall v \in (0,1)^2$, each trader obtains a non-negative utility,

$$support(\mu[v]) \subset \{(\ell, p) \mid p - v_1 \times \ell \geq 0, v_2 \times \ell - p \geq 0\}.$$

The interpretation of *ex post* budget balance and individual rationality here is literal, in the sense that these apply to any realization from the lottery $\mu[v]$, e.g., traders are allowed to walk away after the allocation of the good and the prices have been determined.

[11] [12] discard the *ex post* incentive efficiency as unavailing on the grounds that at the *ex post* stage all the information is known so that there is no longer any need to consider any informational constraints. However, if one is to compare *incentive-feasible allocations to other incentive-feasible allocations*, then the *ex post* incentive efficiency, or, in the present environment, the *ex post* constrained efficiency describes the set of allocation rules which are undominated under such a comparison at the *ex post* stage. See also [8].

trading mechanism is *ex post* constrained efficient if there is no other robust trading mechanism which (weakly) improves the payoffs to both players for all draws of their valuations. Therefore, an optimal robust trading mechanism lies on the *ex post* Pareto frontier of all robust trading mechanisms.

Given a robust trading mechanism μ, denote by $U_i^{\mu}(v)$ the *ex post* payoff to trader i when valuations are $v \in (0,1)^2$,

$$U_i^{\mu}(v) = E_{\mu[v]} u_i(\ell, p, v_i).$$

Given two robust trading mechanisms μ and μ', μ' *ex post* Pareto dominates μ if, $U_i^{\mu'}(v) \geq U_i^{\mu}(v)$, $\forall v \in (0,1)$, $i \in \{1,2\}$, and there is a trader $i \in \{1,2\}$ and a subset of types $O \subset (0,1)^2$, such that the Lebesgue measure of O is positive and,

$$U_i^{\mu'}(v) > U_i^{\mu}(v), \forall v \in O.$$

Definition 2. A robust trading mechanism μ is *optimal* if there does not exist a robust trading mechanism μ' which *ex post* Pareto dominates μ.

Consider a posted price $\bar{p}$. If the seller is willing to deliver the good to the buyer in exchange for the payment of $\bar{p}$, and the buyer is willing to pay $\bar{p}$ in exchange for the good, then trade is effected; otherwise the buyer does not obtain the good and no payment takes place. Formally,

$$\mu_{\bar{p}}[v] = \begin{cases} 1_{\{\ell=1, p=\bar{p}\}}, & \text{if } v_1 \leq \bar{p} \leq v_2, \\ 1_{\{\ell=0, p=0\}}, & \text{otherwise.} \end{cases}$$

More generally, a posted price can be randomized so that there is a predetermined lottery λ over prices in $[0,1)$, and a posted price is then randomly drawn according to λ. Such a randomized posted price is given by,

$$\mu_{\lambda}[v] = \begin{cases} \lambda(\bar{p}) 1_{\{\ell=1, p=\bar{p}\}}, & \text{if } v_1 \leq p \leq v_2, \\ \lambda(\bar{p}) 1_{\{\ell=0, p=0\}}, & \text{otherwise.} \end{cases}$$

A randomized posted price is *non wasteful* if $\lambda(\bar{p} = 0) = 0$. Given two mechanisms μ, μ', we say that μ and μ' are payoff equivalent if,

$$U_i^{\mu}(v) = U_i^{\mu'}(v), \forall v \in (0,1)^2.$$

Observe that two different posted prices $0 < \bar{p} < \bar{\bar{p}} < 1$ do not *ex post* dominate each other. It is also evident that two different probability distributions over posted prices do not dominate each other, unless either one assigns a non-zero probability to price 0.[12] The insight of our main Theorem 1 is that there are no robust trading mechanisms which are more efficient than such non-wasteful randomized posted prices.

Theorem 1. A mechanism is an optimal robust trading mechanism if and only if it is payoff equivalent to a non-wasteful randomized posted price.

3 Proof

Our proof proceeds along measure-theoretic lines. For a robust trading mechanism μ, we provide a payoff equivalent mechanism given by the probability of transferring the good, $\varphi(v)$, and the expected price, conditional on the good being transferred $\pi(v)$. This payoff-equivalent mechanism is also a robust trading mechanism, and φ and π are both weakly monotone. We then approximate φ and π by simple functions, construct the appropriate payoff equivalent randomized posted price, and apply the monotone convergence theorem (see, e.g., [17]) to prove that any robust trading mechanism is payoff equivalent to a randomized posted price. Finally, a randomized posted price is *ex post* Pareto dominated if a positive probability is assigned to price 0, i.e., to no trade.[13]

In what follows, denote by $U_i^\mu(v;\tilde{v}_i)$ the payoff to trader i in a mechanism μ when traders valuations are $v \in (0,1)^2$ but trader i reports $\tilde{v}_i$ (and trader j reports v_j truthfully),

$$U_i^\mu(v;\tilde{v}_i) = E_{\mu[\tilde{v}_i,v_j]} u_i(\ell, p, v_i)$$

Lemma 1. Let a mechanism μ be *ex post* incentive compatible. Then $U_1^\mu(v)$ is strictly decreasing in v_1 whenever $U_1^\mu(v) > 0$, and $U_2^\mu(v)$ is strictly increasing in v_2 whenever $U_2^\mu(v) > 0$.

Proof. We provide the proof for the seller. Let $U_1^\mu(v_1, v_2) > 0$, for some $0 < v_1 < v_2$ and let $\tilde{v}_1 < v_1$. Then $\mu[v]$ assigns a positive probability to some feasible prices, and by strict monotonicity of u_1 in v_1, we have $U_1^\mu(v_1, v_2; \tilde{v}_1) > U_1^\mu(v_1, v_2; v_1)$. By *ex post* incentive compatibility,

$$U^\mu(\tilde{v}_1, v_2; \tilde{v}_1) \geq U^\mu(v_1, v_2; \tilde{v}_1) > U^\mu(v_1, v_2; v_1).$$

[12] If we allowed valuations v_i to belong to the closed interval $[0,1]$ then we could model such wasteful randomized posted prices, for example, by assigning some probability to prices higher than 1 or lower than 0. Nothing in our theorem would change, and the proofs would have to be only slightly adapted.

[13] Note that in any randomized posted price there exist valuations v_1, v_2, $0 < v_1 < v_2 < 1$, such that the probability of no trade (price equal 0) is positive. The point is that in order for a randomized posted price to be on the *ex post* Pareto frontier, such a randomized posted price should be non-wasteful in that the probability of price 0 shouldn't be positive *a priori* but only as a result of positive probability of some unfortunate draws of valuations.

■

Lemma 2. Let a mechanism μ be *ex post* incentive compatible and let μ' be payoff-equivalent to μ. Then μ' is *ex post* incentive compatible.

Proof. Take a $v \in (0,1)^2$, $v_1 < v_2$, and by payoff equivalence of μ and μ',

$$E_{\mu[v]}p - v_1 E_{\mu[v]}\ell = E_{\mu'[v]}p - v_1 E_{\mu'[v]}\ell,$$
$$v_2 E_{\mu[v]}\ell - E_{\mu[v]}p = v_2 E_{\mu'[v]}\ell - E_{\mu'[v]}p,$$

so that, $E_{\mu[v]}\ell = E_{\mu'[v]}\ell$. Now take $\tilde{v}_1 > v_1$, and by *ex post* incentive compatibility of μ,

$$U_1^\mu(v) - U_1^\mu(v;\tilde{v}_1) = U_1^\mu(v) - U_1^\mu(\tilde{v}_1, v_2) - (\tilde{v}_1 - v_1)E_{\mu[\tilde{v}_1,v_2]}\ell \geq 0.$$

Since $U_1^{\mu'}(v) = U_1^\mu(v)$, $U_1^{\mu'}(\tilde{v}_1, v_2) = U_1^\mu(\tilde{v}_1, v_2)$, and $E_{\mu[\tilde{v}_1,v_2]}\ell = E_{\mu'[\tilde{v}_1,v_2]}\ell$, *ex post* incentive compatibility of μ' for the seller follows. Similarly, we prove *ex post* incentive compatibility of μ' for the buyer. ■

Given a robust trading mechanism μ, let $\varphi(v) = E_{\mu[v]}\ell = \mu[v](\{\ell = 1\})$, and $\pi(v) = E_{\mu[v]}p$, so that $\varphi(v)$ is the probability that the good is allocated to the buyer, and $\pi(v)$ is the expected price faced by trader i. Note that by *ex post* budget balance and individual rationality, the price can only be positive whenever the object is allocated, so that,

$$\pi(v) = \begin{cases} E_{\mu[v]}(p \mid \ell = 1), & \text{if } \mu(\{\ell = 1\}) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

We can therefore write,

$$U_i^\mu(v) = \mu[v](\{\ell = 1\})u_i(1, \pi(v), v_i) = \varphi(v)u_i(1, \pi(v), v_i),$$

that is,

$$U_1^\mu(v) = \varphi(v)(\pi(v) - v_1), \text{ and, } U_2^\mu(v) = \varphi(v)(v_2 - \pi(v)), \forall v \in (0,1)^2.$$

The mechanism μ is payoff equivalent to (φ, π). By Lemma 2, since μ is *ex post* incentive compatible, (φ, π) is also *ex post* incentive compatible. Moreover, since μ satisfies *ex post* budget balance and individual rationality, (φ, π) also satisfies *ex post* budget balance and individual rationality, so that (φ, π) is a robust trading mechanism. Note that the functions φ, π are measurable on $\Omega = [0, 1)$ (recall that Ω is the state space of all possible prices for the lottery μ). Note also that by Lemma 1, $\varphi(.,.)$ is weakly decreasing in v_1 and weakly increasing in v_2, and $\pi(.,.)$ is weakly increasing in v_1 and in v_2. The next Proposition 1 generalizes the main results in [11] and is key to our main theorem.

Proposition 1. Let (φ, π) be a robust trading mechanism. Then there exists a probability distribution λ over prices $[0,1)$, s.t.,

$$\varphi(v) = Pr_{\lambda}\left(\omega \in [v_1, v_2]\right), \tag{3}$$
$$\pi(v) = E_{\lambda}\left(\omega \mid \omega \in [v_1, v_2]\right). \tag{4}$$

Proof. Until further notice, we assume that (φ, π) is a robust trading mechanism, we fix a $v_2 \in (0,1)$, and denote,

$$\tilde{\varphi}^{v_2}(.) = \varphi(., v_2),$$
$$\tilde{\pi}_1^{v_2}(.) = \pi(., v_2).$$

We proceed by proving the following claims for the seller; analogous claims can be proven for the buyer.

1. $\tilde{\pi}_1^{v_2}(.)$ is a simple function if and only if $\tilde{\varphi}^{v_2}(.)$ is a simple function.

2. If $\tilde{\pi}_1^{v_2}(.)$ and $\tilde{\varphi}^{v_2}(.)$ are simple functions, then there exists a probability distribution λ^{v_2} on $\{0\} \times (0, v_2]$, such that,

$$\tilde{\varphi}^{v_2}(v_1) = Pr_{\lambda^{v_2}}\left(\omega \in [v_1, v_2]\right),$$
$$\tilde{\pi}_1^{v_2}(v_1) = E_{\lambda^{v_2}}\left(\omega \mid \omega \in [v_1, v_2]\right).$$

3. If $\tilde{\pi}_1^{v_2}(.)$ and $\tilde{\varphi}^{v_2}(.)$ are measurable on $(0,1)$, then there exists a probability distribution λ^{v_2} on $\{0\} \times (0, v_2]$, such that,

$$\tilde{\varphi}^{v_2}(v_1) = Pr_{\lambda^{v_2}}\left(\omega \in [v_1, v_2]\right),$$
$$\tilde{\pi}_1^{v_2}(v_1) = E_{\lambda^{v_2}}\left(\omega \mid \omega \in [v_1, v_2]\right).$$

Proof of Claims 1 and 2.
Step 1. $\tilde{\varphi}^{v_2}(.) \in \{0, \alpha\}$, for some $\alpha \in (0,1]$, if and only if, $\tilde{\pi}_1^{v_2}(.) \in \{0, \bar{p}\}$, for some $\bar{p} \in (0,1)$. Moreover, $\tilde{\varphi}^{v_2}(v_1) = \alpha, \tilde{\pi}_1^{v_2}(v_1) = \bar{p}$, if and only if, $v_1 \in (0, \bar{p}]$.
Proof. Suppose $\tilde{\pi}_1^{v_2}(.) \in \{0, \bar{p}\}$. If $\tilde{\pi}_1^{v_2}(v_1) = 0$, then by *ex post* individual rationality and budget balance $\tilde{\varphi}^{v_2}(v_1) = 0$. If $\tilde{\pi}_1^{v_2}(v_1) = \tilde{\pi}_1^{v_2}(\tilde{v}_1) = \bar{p}$, $v_1 > \tilde{v}_1$, then by *ex post* incentive compatibility, $\tilde{\varphi}^{v_2}(v_1) = \tilde{\varphi}^{v_2}(\tilde{v}_1)$, or v_1 would have incentives to misreport to $\tilde{v}_1$. By *ex post* individual rationality and budget balance $\tilde{\varphi}^{v_2}(v_1) = \alpha$, for some $\alpha \in (0,1]$.

If $\tilde{\varphi}^{v_2}(v_1) = 0$, then by *ex post* individual rationality and budget balance (taking into account the individual rationality of trader 2), $\tilde{\pi}_1^{v_2}(v_1) = 0$.
Suppose $\tilde{\varphi}^{v_2}(v_1) = \alpha$. Then $\tilde{\pi}_1^{v_2}(v_1) \geq v_1$, by *ex post* individual rationality. If $\tilde{\pi}_1^{v_2}(v_1) = v_1$, then, by *ex post* incentive compatibility, $\tilde{\varphi}^{v_2}(\tilde{v}_1) = \alpha$ and $\tilde{\pi}_1^{v_2}(\tilde{v}_1) = v_1$,

$\forall \tilde{v}_1 < v_1$, and $\tilde{\varphi}^{v_2}(\tilde{v}_1) = 0$ and $\tilde{\pi}_1^{v_2}(\tilde{v}_1, v_2) = 0$, $\forall \tilde{v}_1 > v_1$. If $\tilde{\pi}_1^{v_2}(v_1) > v_1$, then, by *ex post* incentive compatibility, $\tilde{\varphi}^{v_2}(\tilde{v}_1) = \alpha$ and $\tilde{\pi}_1^{v_2}(\tilde{v}_1) = \tilde{\pi}_1^{v_2}(v_1)$, $\forall \tilde{v}_1 \in (0, v_1]$.

Step 2. $\tilde{\varphi}^{v_2}(.) \in \{0, \alpha_1, ..., \alpha_K\}$, for some $\{\alpha_1, ..., \alpha_K\} \subset (0, 1], \alpha_k > \alpha_{k+1}, k < K$, if and only if, $\tilde{\pi}_1^{v_2}(.) \in \{0, p_1, ..., p_K\}$, for some $\{p_1, ..., p_K\} \subset (0, 1), p_k < p_{k+1}, k < K$. Moreover, $\tilde{\varphi}^{v_2}(v_1) = \alpha_{k+1}, \tilde{\pi}_1^{v_2}(v_1) = p_{k+1}$, if and only if, $v_1 \in (\bar{p}_k, \bar{p}_{k+1}]$, where,

$$p_k = \frac{\alpha_k - \alpha_{k+1}}{\alpha_k}\bar{p}_k + \frac{\alpha_{k+1}}{\alpha_k}p_{k+1},\ \ k < K, \text{ and } p_K = \bar{p}_K. \tag{5}$$

Proof. The first part follows as in Step 1, i.e., by *ex post* individual rationality and budget balance $\tilde{\varphi}^{v_2}(.) > 0$, if and only if $\tilde{\pi}_1^{v_2}(.) > 0$, and by *ex post* incentive compatibility, $\tilde{\varphi}^{v_2}(.)$ is constant, if and only if, $\tilde{\pi}_1^{v_2}(.)$ is constant.
That $p_K = \bar{p}_K$ follows from *ex post* incentive compatibility for type $v_1 = \bar{p}_K$. Now take types $v_1 = \bar{p}_k, k < K$, and $\tilde{v}_1 = v_1 + \epsilon, \epsilon > 0$. By *ex post* incentive compatibility for v_1,

$$\alpha_k(p_k - \bar{p}_k) \geq \alpha_{k+1}(p_{k+1} - \bar{p}_k),$$

and by *ex post* incentive compatibility for $\tilde{v}_1$,

$$\alpha_{k+1}(p_{k+1} - \bar{p}_k - \epsilon) \geq \alpha_k(p_k - \bar{p}_k - \epsilon).$$

Let $\epsilon \to 0$, and combine the last two inequalities to obtain,

$$\alpha_k(p_k - \bar{p}_k) = \alpha_{k+1}(p_{k+1} - \bar{p}_k),$$

which yields (5).

Step 3. If $\tilde{\varphi}^{v_2}(.)$ and $\tilde{\pi}_1^{v_2}(.)$ are simple functions, then there exist prices, $\bar{p}_k \leq v_2$, $0 < k \leq K$, $0 < \bar{p}_1 < ... < \bar{p}_k$, and weights $\lambda_k > 0$, $0 < k \leq K$, s.t., $\sum_1^K \lambda_k \leq 1$, and,

$$\tilde{\varphi}^{v_2}(v_1) = \sum_{k>0, v_1 \leq \bar{p}_k} \lambda_k,$$

$$\tilde{\pi}_1^{v_2}(v_1) = \frac{1}{\sum_{k>0, v_1 \leq \bar{p}_k} \lambda_k}\left(\sum_{k>0, v_1 \leq \bar{p}_k} \lambda_k \bar{p}_k\right).$$

Proof. Define $\alpha_{K+1} = 0$. By applying (5) from Step 2 recursively, we obtain,

$$p_k = \sum_{k'=k}^{k'=K} \bar{p}_{k'} \frac{\alpha_{k'} - \alpha_{k'+1}}{\alpha_k},$$

and by letting $\lambda_k = \alpha_k - \alpha_{k+1}, k \in \{1, ..., K\}$, we obtain the desired expressions.

Finally note that $\bar{p}_K \leq v_2$, by the *ex post* budget balance and individual rationality of trader 2. This concludes the proof of Claims 1 and 2.

Proof of Claim 3.
By *ex post* incentive compatibility, $\tilde{\varphi}^{v_2}(.)$ and $\tilde{\pi}_1^{v_2}(.)$ are monotone (Lemma 1). Next, for each $k > 0$ and $k' \in \{1, ..., k\}$, let $\alpha_{k'}^k = \tilde{\varphi}^{v_2}\left(\frac{k' v_2}{k}\right)$. Define a simple function $\tilde{\varphi}^{v_2,k}(.)$ by,

$$\tilde{\varphi}^{v_2,k}(v_1) = \alpha_{k'}^k, v_1 \in \left(\frac{(k'-1)v_2}{k}, \frac{k' v_2}{k}\right].$$

Therefore, $\{\tilde{\varphi}^{v_2,k}(.)\}_{k>0}$ is a monotone increasing sequence of functions, converging point-wise to $\tilde{\varphi}^{v_2}(.)$. For each $k > 0$ let $\tilde{\pi}_1^{v_2,k}(.)$ be such that $\tilde{\varphi}^{v_2,k}(.), \tilde{\pi}_1^{v_2,k}(.)$ satisfy *ex post* incentive compatibility, and $\tilde{\varphi}^{v_2,k}(.) > 0$ if and only if $\tilde{\pi}_1^{v_2,k}(.) > 0$. Then $\{\tilde{\pi}_1^{v_2,k}(.)\}_{k>0}$ is also a monotone increasing sequence, converging point-wise to $\tilde{\pi}_1^{v_2}(.)$. By Claim 2, for each $k > 0$, there is a probability distribution over $\lambda^{v_2,k}$ on $\{0\} \times (0, v_2]$, such that,

$$\tilde{\varphi}^{v_2,k}(v_1) = Pr_{\lambda^{v_2,k}}(\omega \in [v_1, v_2]),$$
$$\tilde{\pi}_1^{v_2,k}(v_1) = E_{\lambda^{v_2,k}}(\omega \mid \omega \in [v_1, v_2]).$$

Let λ^{v_2} be the point-wise limit of $\lambda^{v_2,k}$. By the monotone convergence theorem (see e.g., Rudin 1987),

$$\tilde{\varphi}^{v_2}(v_1) = Pr_{\lambda^{v_2}}(\omega \in [v_1, v_2]),$$
$$\tilde{\pi}_1^{v_2}(v_1) = E_{\lambda^{v_2}}(\omega \mid \omega \in [v_1, v_2]).$$

This concludes the proof of Claim 3.

Similarly, we prove analogous claims for the buyer, so that, for each $v_1 \in (0, 1)$, there exists a λ^{v_1}, such that,

$$\tilde{\varphi}^{v_1}(v_2) = Pr_{\lambda^{v_1}}(\omega \in [v_1, v_2])$$
$$\tilde{\pi}_2^{v_1}(v_2) = E_{\lambda^{v_1}}(\omega \mid \omega \in [v_1, v_2])$$

Now take a $v \in (0, 1)$. By above,

$$\varphi(v) = Pr_{\lambda^{v_2}}(\omega \in [v_1, v_2]) = Pr_{\lambda^{v_1}}(\omega \in [v_1, v_2]),$$

so that λ^{v_2} and λ^{v_1} must coincide almost everywhere and do not depend (respectively) on either v_2 or v_1. Let $\lambda \equiv \lambda^{v_2}$, which concludes the proof. ■

Our main theorem now follows from the next Corollary 2.

Corollary 2. A randomized posted price λ is optimal, if and only if, $Pr_\lambda((0, 1)) = 1$.

Proof. Take a randomized posted price λ, such that, $Pr_\lambda((0, 1)) = \alpha < 1$ and

$Pr_\lambda(\{0\}) = 1 - \alpha$, i.e., λ assigns a positive mass to price 0. Define a probability distribution over prices by $\bar{\lambda}(x) = \frac{1}{\alpha}\lambda(x) - 1 + \alpha$, $\forall x \in [0, 1)$. Evidently, $\bar{\lambda}(x)$ *ex post* Pareto dominates λ. The converse follows from Proposition 1. ■

4 Discussion

Why is it useful to know the set of optimal robust trading mechanisms? Consider a case in which the social planner knows the distribution of players' valuations F. Then the planner can find, for instance, the robust trading mechanism that maximizes the *ex ante* expected sum of traders' utilities. More generally, given F, what is the *ex ante* Pareto frontier of robust trading mechanism? The characterization from Theorem 1 is useful: every *ex ante* optimal robust trading mechanism must lie on the *ex post* Pareto frontier. In particular, for a given F and weights in the social welfare function, the *ex ante* optimal robust trading mechanism is a deterministic posted price.

Example 1. Suppose that the traders' valuations are identically and independently distributed according to a continuous probability distribution function F, with a density f. Then the *ex-ante* optimal robust trading mechanism which maximizes the sum of traders' utilities is given by a deterministic posted price p^*, where,

$$p^* = \arg \max_{\bar{p} \in (0,1)} \int_0^{\bar{p}} \int_{\bar{p}}^1 \left((v_2 - v_1) f(v_2) dv_2\right) f(v_1) dv_1.$$

For instance, when F is uniform on $(0, 1)$, $p^* = \frac{1}{2}$.

That p^* must be deterministic follows from the fact that the derivative of the expression on the right hand side is positive at 0, decreasing, and negative at 1. Therefore, there exists a unique posted price that maximizes the sum of traders' utilities *ex ante*, and randomizing over other prices would only decrease the sum of traders' expected payoffs.

One can similarly surmise that any weighted sum of traders' utilities is maximized by some deterministic posted price. Therefore, for a given F, the *ex-ante* Pareto frontier is given by a subset of deterministic posted prices, where the upper bound of this set is derived when the social welfare criterion assigns all the weight to the seller, and the lower bound when all the weight is assigned to the buyer. Our results can also be applied to other welfare criteria, for example, the minimax regret criterion. Under any such welfare criterion, the corresponding optimal mechanism must lie on the *ex post* Pareto frontier.[14]

[14] In the one-sided asymmetric information problem, [4] consider a seller who faces a buyer with an unknown valuation, and the buyer chooses a pricing rule to minimize regret, which results in a distribution over prices.

In a related point, our characterization allows for a comparison of *ex ante* welfare under the *interim* and the *ex post* constraints. Relative to the Bayesian setting, what is the loss in welfare due to the traders not knowing the details of distribution over the valuations? For example, suppose that F was in fact uniform. By [16], in a Bayesian setting, under the *interim* constraints, the expected gains from trade are maximized when traders trade with certainty as long as v_2 exceeds v_1 by at least $\frac{1}{4}$. Then the expected gains from trade are $\frac{9}{32}$. In contrast, the optimal robust trading mechanism (assuming that somehow the social planner knew F) that maximizes gains from trade under the uniform distribution is a deterministic posted price at $\frac{1}{2}$ and the corresponding expected gains from trade are $\frac{1}{8}$. Therefore $\frac{5}{32}$, or more than half of the expected surplus, is lost due to the traders' more limited knowledge.

As a final comment, if the support of traders valuations were not known, as might perhaps be a more realistic assumption, then randomized posted prices might have to be wasteful. We hope that this might be investigated in the future advances on the problem of bilateral trade.

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