RESTRICTED PERMUTATIONS, ANTICHAINS, ATOMIC CLASSES AND STACK SORTING

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Restricted Permutations, Antichains, Atomic Classes and Stack Sorting

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I would like to thank Michael Atkinson for having introduced me to the subject of closed classes and for having freely given plenty of help, information and open problems on the subject. I would also like to thank his family for tolerating me to come and stay with them during my trip to New Zealand.

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Zu den Übrigen, Ihr wisst wer Ihr seit. Besonders meiner Mutter will Ich danken. Auch Hans Otto Carmesin, Frances Williams, Quint Murphy, Tet Murphy und Ille Supremo. Und Pater mei, letztens weil meine Schuld zu ihm so offensichtlich ist.

Abstract

Involvement is a partial order on all finite permutations, of infinite dimension and having subsets isomorphic to every countable partial order with finite descending chains. It has attacted the attention of some celebrated mathematicians including Paul Erdős and, due to its close links with sorting devices, Donald Knuth.

We compare and contrast two presentations of closed classes that depend on the partial order of involvement: Basis or Avoidance Set, and Union of Atomic Classes. We examine how the basis is affected by a comprehensive list of closed class constructions and decompositions.

The partial order of involvement contains infinite antichains. We develop the concept of a fundamental antichain. We compare the concept of 'fundamental' with other definitions of minimality for antichains, and compare fundamental permutation antichains with fundamental antichains in graph theory. The justification for investigating fundamental antichains is the nice patterns they produce. We forward the case for classifying the fundamental permutation antichains.

Sorting devices have close links with closed classes. We consider two sorting devices, constructed from stacks in series, in detail.

We give a comment on an enumerative conjecture by Ira Gessel.

We demonstrate, with a remarkable example, that there exist two closed classes, equinumerous, one of which has a single basis element, the other infinitely many basis elements.

We present this paper as a comprehensive analysis of the partial order of permutation involvement. We regard the main research contributions offered here to be the examples that demonstrate what is, and what is not, possible; although there are numerous structure results that do not fall under this category. We propose the classification of fundamental permutation antichains as one of the principal problems for closed classes today, and consider this as a problem whose solution will have wide significance for the study of partial orders, and mathematics as a whole.

Contents

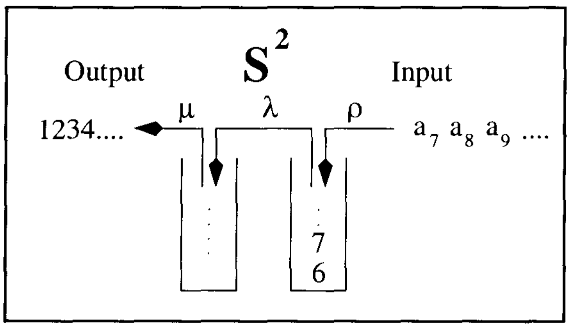
Chapter 1

Introduction

Origins, Sorting Machines, Partial Order of Involvement, Closed Classes, Representation by Sub and Basis, Atomic Classes, Symmetries, Summary of Chapters

1.1 Origins

The following is a typical example of the type of demi-mechanical sorting machine in which this subject has its roots.



A sequence is sorted by two stacks in series.

The purpose of machines of this sort is to take a finite numerical sequence as an input and process the terms in such a manner that when output, smaller terms precede greater ones. In this case the permissible moves are as follows:

ρ: Moves to the top of the right stack the first term still in the input.

λ: Moves to the top of the left stack the top term of the right stack.

μ: Moves to the output the top term of the left stack.

For example if the input sequence is 3764251 then a satisfactory sorting might consist of the following:

1. Perform a ρ operation to move the 3 from the input to the right stack.

2. Move the 7 from the input to the left stack by means of a ρ and a λ operation. Repeat to place the 6 on top of the 7, and the 4 on the 6.

3. Move the 3 onto the 4 by a λ operation.

4. Take the 2 from the input and place it on the left stack, on top of the 3.

5. Place the 5 in the now empty right stack.

6. Transfer the 1 from the input to the output by means of a ρ, a λ and a μ.

7. Move all the remaining terms to the output as follows: The 2, 3 and 4 may be output from the left stack by μ operations, the 5 may follow by a λ and a μ, and finally the 6 and 7 can be output by two μ operations.

As a sorting machine of arbitrary input the above is incompetent. Every term in an input may be acted upon only thrice and, in the entire system, there are only three different types of operation that may be performed. Thus any successful sorting of an input sequence of length n may be expressed as a word of length $3n$ over an alphabet of three letters. So there are at most 3^{3n} distinct sorting processes of inputs of length n, far fewer than the $n!$ permutations of length n, and every one of these permutations must be treated differently if it is to be sorted successfully. At least, we do require that n is large because only $n > 50$ implies that $27^n < n!$. Thus for sufficiently long input it may not be possible for the machine to sort the input. A similar analysis can be performed on all the machines that will appear in this thesis. It may fairly be said that to sort is their ambition, but not always their effect.

(To be a little more accurate, the number of distinct ways of passing n terms through S^2 is the three dimensional Catalan number, which is $2 * (3 * n)!/(n! * (n+1)! * (n+2)!)$, according to [31]. The bound this gives is barely an improvement though: The reader may be bemused to know that it reduces the earlier bound of 50 by one. In fact there can be several ways of sorting a given sequence in S^2 and the shortest permutation that S^2 cannot sort has length 7.)

We will concern ourselves not with the speed or the number of moves with which such a machine can sort a given input but rather with the question of what input the machine can sort. In "The art of computer programming" [17] D.E. Knuth considered the problem of rearranging railway carriages in a shunting station or by means of sections of parallel track. The problem remains active and relevant because of permutations that occur in general

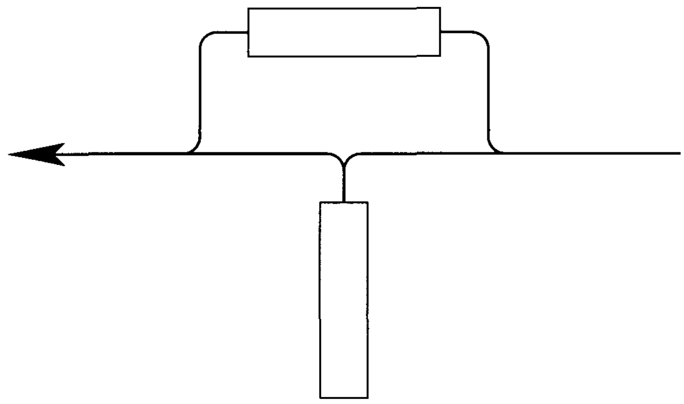


Figure 1.1: A queue (above) and stack (below) in parallel.

networks, some of which, specifically those involving stacks, queues and deques and double ended queues are considered in [13].

In spite of their limitations machines of this sort are remarkably common. Applications include sorting when the input is by nature somehow restricted to the set of permutations that *can* be sorted. Application can also be found in modelling disorder: A process of feeding a sequence into the right of the above machine and removing a sorted output on the left may be considered as a disordering performed in reverse, the set of sortable sequences being precisely those that may be produced as disorderings of ordered sequences fed into the left side of the above machine.

One specific application for restricted sorting devices arises in networking: A piece of information that must be sent from one node to another in a network may be split into smaller chunks, each of which makes its way independently from the one node to the other. The separation of the chunks can cause them to arrive in the wrong order but because of the limitations of the network only a restricted set of permutations will be possible. The duality between permuting through a finite network and sorting through another suggests that this is an ideal situation for applying a restricted sorting device, that may well operate in linear time and therefore be faster than any universal sorting machine can possibly be. This application is hypothetical, we have not investigated its practicality or how much of an improvement it might offer.

1.2 Closed Classes

We consider only two sorting devices in detail, M and S^2, and those only at the end of this thesis. Our main thrust is to establish a theoretical structure that is applicable to all sorting devices. To this end we select two easily mathematicised invariants of common sorting devices and develop a theory around them.

Closure in Sorting Machines

The first invariant, about to be presented, can be illustrated with the above sorting machine S^2: The machine can sort, for instance, the input 415623 into ascending order. It is evident that if we remove the term 4 from this input then we can still sort the resulting sequence, which is 15623, by the same sorting procedure that we used for the entire sequence but now omitting any move previously applied to the now missing term. Thus if the machine can sort a given input sequence then it can also sort every subsequence of that input. We call such a sorting machine *closed.*

Order Isomorphism

The second invariant, also illustrated, is that the actual values of input terms is irrelevant. We are concerned in producing an output in ascending order, thus we are concerned with which terms are greater than which other terms. Thus if we know that 1423 can be sorted then we also know that the sequences $-1\,10\,4\,5$ and 2534 and all other sequences like them can also be sorted. The sequences 1423 and $-1\,10\,4\,5$ are said to be *order isomorphic*, which is defined formally as follows:

Definition 1 Let $\alpha = a_1 \ldots a_m$ and $\beta = b_1 \ldots b_m$ be sequences of equal length. Then α and β are said to be *order isomorphic*, and we write $\alpha \cong \beta$, if for every $i, j \in \{1, 2, \ldots, m\}$ we have that $a_i < a_j$ if and only if $b_i < b_j$.

Involvement of one Sequence in Another

This leads to the next definition. We say that sequences such as 1423 that are order isomorphic to a subsequence 1523 of 415623, are *involved* in 415623. The full definition is as follows:

Definition 2 Let α and β be sequences. Then α is said to be *involved* in β if α is order isomorphic to some subsequence of β. If α is involved in β but is not order isomorphic to the entirety of β then α is said to be *properly involved* in β. If A and B are sets of sequences then we say that A is *involved* in B if every element of A is involved in an element of B. We may write $\alpha \preceq \beta$ or $\alpha \prec \beta$ if α is involved or, respectively, properly involved in β.

Closed Classes Defined

From this point on we discard from our consideration input sequences that have two or more terms of equal value. In sorting of such a sequence one of the equal terms must be output before the others. Thus this first term output might as well be given a smaller label — only in very few machines does re-labelling affect the sorting procedure. Thus the sequence 14243 can be sorted by, for instance, the above machine S^2 if and only if one of the sequences 15243 and 14253 can be sorted. Thus we can reasonably claim to 'understand' the set of arbitrary sequences that can be sorted by a machine if we already understand the set of all non-repeating sequences that might be sorted by the same.

We extend this input simplification a little further: Every finite non-repeating sequence of length n is order isomorphic to some permutation of the numbers $1 \dots n$. As the machines we will consider are all invariant under order isomorphism it suffices for our purposes to know which finite permutations can be sorted.

This leaves us with the objects of our attention:

Definition 3 A set X of permutations is said to be *closed* if every permutation involved in an element of X is itself also an element of X.

We warn the reader that if X is a set of permutations then we will occasionally state that some sequence is an element of X, meaning that the permutation order isomorphic to that sequence, which is always non-repeating, is an element of X. It might be preferable to regard order isomorphism as an equivalence relation and the set of finite permutations as representatives but the notational overhead this involves is forbidding.

1.2.1 The Partial Order

Involvement forms a partial order over the set of all permutations. The lower reaches of this partial order are shown in Figure 1.2. The partial order is rich in its complexity. The following should be noticed immediately:

- There are only countably many finite permutations, hence the number of nodes in the partial order is countable.
- The partial order has the property that every descending chain is finite. Indeed a permutation properly involved in another is shorter than it and sequences cannot become arbitrarily short.
- Maximal ascending chains in the set of all permutations are always infinite. Maximal ascending chains within a closed class may, in principle, be either infinite or finite.

As this is only a partial order there are permutations that are not involved in each other, and so we will have *antichains*, defined in the usual sense:

Definition 4 A set A of permutations is said to be an *antichain* if every two distinct elements of A are incomparable under the partial order of involvement.

An example of an antichain is the set of all permutations of some given length.

Chains and antichains will be of considerable importance throughout this thesis.

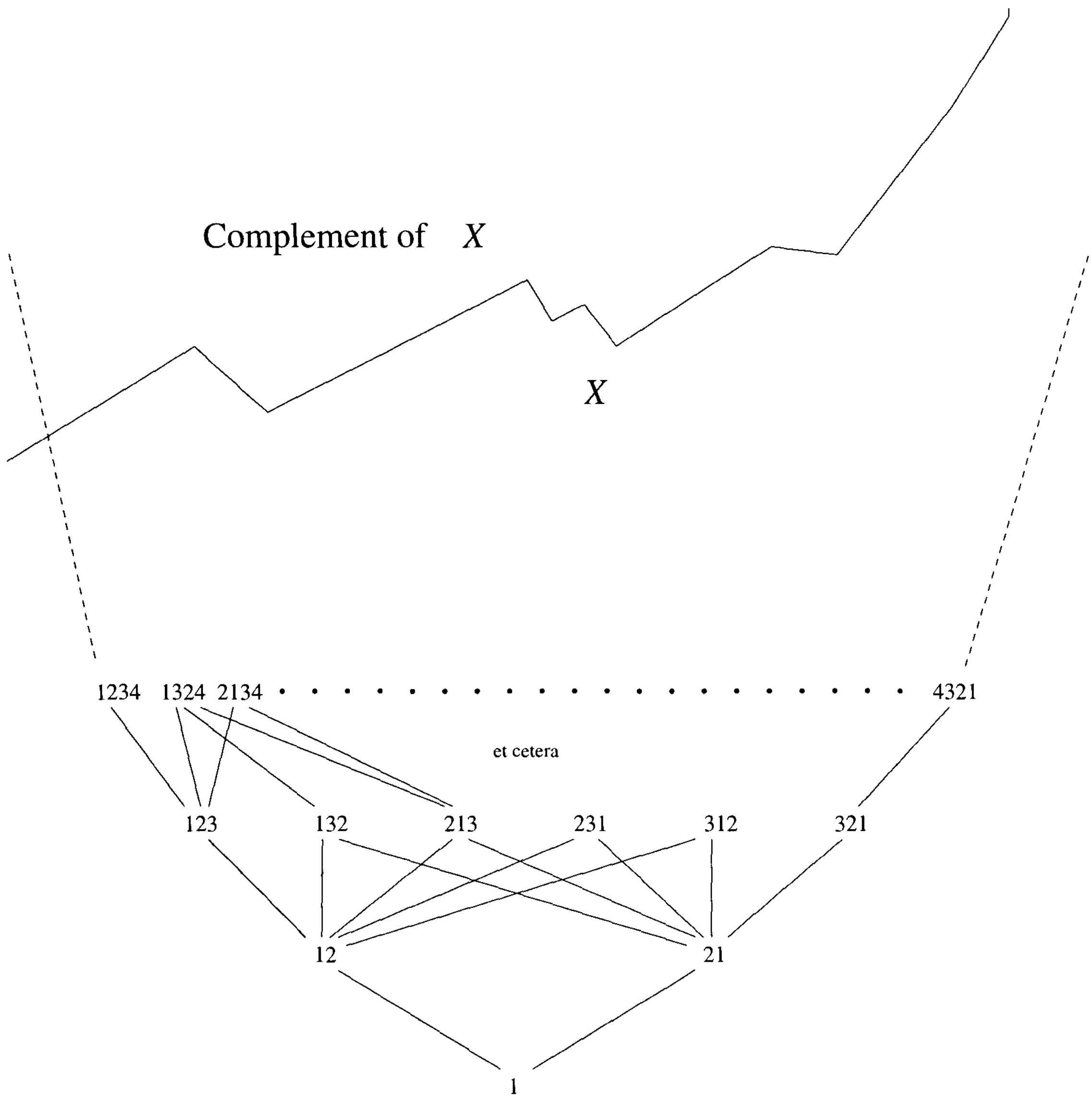


Figure 1.2: The partial order of involvement: In the ordering 1324 involves 123, 132, 213, 12, 21 and 1, but does not involve e.g. 321. Closed classes may be thought of as the set of all permutation lying below some "dividing line", as shown.

1.2.2 Representation: Sub and the Basis

We may give a closed class by listing all the elements of that class, however that is in general not a very efficient method. The intrinsic structure of closed classes permits us to abbreviate this mechanism. The simplest method of doing so is as follows: If a closed class contains the permutation 145263 then we also know that it contains all permutations involved in 145263, we do not need to add this information. Thus we introduce our first method of representing closed classes:

Definition 5 Let A be a set of sequences. The *closure* of A is the set of all finite permutations order isomorphic to an element of A or properly involved in such an element and is denoted by $Sub(A)$.

If a closed class X contains elements M maximal under involvement and if every element of the class is involved in one of those maximal elements then X can be described by the unique minimal expression $Sub(M)$. Note that the set of maximal permutations in a class forms an antichain. However in general the set of maximal elements in a class is not sufficient to define it.

For instance the set S of all permutations and the set I of all increasing sequences $(123 \ldots n)$ are both closed classes, neither has any maximal elements whatsoever, and yet these classes are undeniably distinct. Furthermore in the absence of defining maximal elements there is no minimal way of represening a closed class using the Sub notation, as used here. Consider I as a case study. Without the advantage of unique minimal representation the Sub notation loses some of its charm. Our second method of describing closed classes does not fail in this manner. We will describe X in terms of the complement of X:

By definition, if X is a closed class of permutations and α is an element of X, and if β is involved in α then we must have that β is an element of X. The contrapositive states that if β is not in the closed class X then neither is α. Moreover since all descending chains in involvement are finite this implies that there exists a shortest permutation involved in α and not in X. The set $\mathcal{B}(X)$ of all such minimal elements of the complement of X is called the *basis* of X. Every element in the complement of X involves an element of $\mathcal{B}(X)$ and no element of X involves an element of $\mathcal{B}(X)$. Indeed X can be described as the set of permutations that *avoid* the elements of B and we write $X = \mathcal{A}(\mathcal{B}(X))$. We define this notation formally as follows:

Definition 6 Let X be a closed class. Then the *basis* $\mathcal{B}(X)$ of X is the set of permutations minimal under involvement and not in X. If F is a set of permutations then the *avoidance set* $\mathcal{A}(F)$ is the set of all permutations that do not involve any element of F.

The principal advantage that the Sub notation maintains over the basis is that Sub gives examples of elements in the class being described. It is easier to imagine an arbitrary element of a class if one knows the properties of the elements of the class, rather than some properties that elements cannot have. This is especially important with relatively complicated classes of permutations but we will attempt to illustrate the principle:

Example 7 The finite closed class with the single maximal element 2134 is expressible as $Sub(2134)$ or as $\mathcal{A}(132, 231, 312, 321, 1234)$.

Example 8 The closed class that consists of permutations that a) begin with a decreasing sequence of length at most three, and b) continue to the end with an increasing sequence of terms all greater than the first decreasing set, e.g. 21345, can be written either as $Sub(3214567\ldots)$ or as $\mathcal{A}(132, 231, 312, 4321)$. The former is our first example of taking the closure of an infinite sequence.

Example 9 Let $I\, merge\, R$ be the set of permutations such as 912856743 that consist of a meld of an increasing and a decreasing subsequence, in this case 12567 and 9843. Then $I\, merge\, R$ is a closed class. (We permit the empty subsequence, thus the trivial permutation 1 consists of the subsequence 1 and the empty sequence.) The basis of $I\, merge\, R$ consists of the two permutations 2143 and 3412, as given in [10], so $I\, merge\, R = \mathcal{A}(2143, 3412)$.

Alternatively we may write

$$\begin{aligned} I\, merge\, R \quad = \quad & Sub\{14325,\ 183654729, \\ & 1\ 12\ 3\ 10\ 5\ 8\ 7\ 6\ 9\ 4\ 11\ 2\ 13, \\ & 1\ 16\ 3\ 14\ 5\ 12\ 7\ 10\ 9\ 8\ 11\ 6\ 13\ 4\ 15\ 2\ 17, \\ & \ldots\}. \end{aligned}$$

The latter may seem clumsy but at least if we plot one of these sequences then we will instantly have an idea of what we are dealing with, whereas with the basis notation one might well have to experiment a good deal before getting the same clear idea. Incidentally there are many different Sub representations of this class. Specifically here the reader should not be concerned that we use a sequence that oscillates strictly between the terms of increasing and decreasing subsequences. This is merely a convenient choice, it has no serious restricting effect.

Even the verbal description of this class is in character descriptive, not a set of restrictions. That in itself is a good reason for pursuing constructive representations for closed classes.

Note that the basis of a closed class is an antichain. The elements of the basis are minimal under involvement, thus they cannot be comparable.

1.3 Atomic Classes

The concept that we here introduce, of atomic classes, is designed to strengthen the *Sub* notation to overcome some of its weaknesses. The principal failing of the *Sub* notation is non-uniqueness. This problem occurs only when the class that is being described is not equal to the closure of the set of maximal elements. The situations where this occurs all have this one thing in common: The class contains an infinite ascending chain of permutations. Atomic classes as introduced here, it transpires, describe precisely closures of ascending chains. We first define the classes and then show this connection.

Definition 10 Let A, B be ordered sets and let π be a bijection from A to B. From every finite subset U of A a permutation may be obtained as follows:

Let $U = \{u_1, u_2, \ldots, u_n\}$ where $u_1 < u_2 < \ldots < u_n$.

Then $\pi(u_1)\pi(u_2)\ldots\pi(u_n)$ is a sequence of distinct elements of B and is order isomorphic to some permutation.

$\mathcal{B}(A, B, \pi)$ is the set of all such permutations. An *atomic class* is a set of permutations that can be expressed in the form $\mathcal{B}(A, B, \pi)$ for some A, B and π.

Essentially this is an extension of the notion of closure. We hereby define $Sub(\pi)$ to have the same meaning as $\mathcal{B}(A, B, \pi)$. This extension is consistent with our earler definition of Sub. By default we will use the Sub notation rather than $\mathcal{B}(A, B, \pi)$. The latter is more established and when the domain and range of a function are of interest we will use it, however in most cases that extra information is unnecessary.

Example 11 Let $A = B = [0, 1)$ and let π, a function from A to B, be defined by:

$$\pi(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 - x & \text{otherwise} \end{cases}$$

Then $\mathcal{B}(A, B, \pi)$ is the set of all permutations having the form $n\ (n-1)\ (n-2)\ \ldots\ 1$ or $1\ n\ (n-1)\ (n-2)\ \ldots\ 3\ 2$.

Example 12 The set I of all increasing permutations $123\ldots n$ is an atomic class, however the union of I and R, the set of all decreasing permutations is not atomic. Indeed suppose that $I \cup R$ is expressible as $\mathcal{B}(A, B, \pi)$ for some A, B and π. As 12 is an element of I there must be a pair of elements a_1, a_2 of A such that $a_1 < a_2$ and $\pi(a_1) < \pi(a_2)$. Similarly as 21 is an element of R there must exist two elements a_3, a_4, not necessarily both distinct from a_1, a_2, such that $a_3 < a_4$ and $\pi(a_3) > \pi(a_4)$. However if we consider the permutation in $\mathcal{B}(A, B, \pi)$ corresponding to $\{a_1, a_2, a_3, a_4\}$ then we rapidly reach a contradiction: for that permutation contains both an increasing pair of terms and a decreasing pair of terms.

Example 13 Let *I merge I* be the set of all permutations consisting of a meld of two increasing subsequences, such as 13246578 that consists of 134678 and 25. Then *I merge I* is an atomic class with representation $Sub(\pi)$ where π is a function on the real numbers defined by:

$$\pi(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x+1 & \text{otherwise} \end{cases}$$

To see that this is an accurate representation it may help to note that every element of *I merge I* can be expressed as a meld of a 'large' and a 'small' increasing subsequence, the terms of the large sequence being greater than all preceding terms of the small sequence. The representation follows from this. The basis of *I merge I* is $\{321\}$.[1]

Definition 14 Let X be a closed set of permutations. X is said to have the *join property* if for every two elements α and β of X, we have that X contains an element γ that involves both α and β.

Theorem 15 *Let X be a closed class. Then the following are equivalent:*

1. *X has the join property.*

2. *X can be expressed as $\cup_{i=1}^{\infty} Sub(\rho_i) = Sub(\rho_1) \cup Sub(\rho_2) \cup \ldots$ for some ascending chain of permutations $\rho_1 \preceq \rho_2 \preceq \rho_3 \preceq \ldots$.*

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[1]That $\mathcal{A}(321)$ is the set of all permutations that are the meld of an increasing and decreasing sequence is a well known and long established fact. The paper [16] is about the fact that this class and $\mathcal{A}(132)$ have the same number of permutations of each length; it was even then a well known fact that both are enumerated by the Catalan Numbers. D.E. Knuth in his earlier book, [18] also mentions the class $\mathcal{A}(321)$.

3. X is atomic.

4. X is not the union of two proper closed subclasses.

That 1 and 3 are equivalent is due to Mike Atkinson and Robert Beals in a private communication. We give a full proof none the less.

PROOF: $1 \Rightarrow 2$: Suppose that X has the join property. Let ρ_1 be any permutation in X. Then $Sub(\rho_1)$ is a subset of X. It is therefore clear that if $\rho_1, \rho_2, \ldots$ is a sequence of permutations in X then $Sub(\rho_1) \cup Sub(\rho_2) \cup \ldots$ is a subset of X.

Given that X has the join property we can ensure that if $\beta_1, \beta_2, \ldots$ is any sequence of permutations in X then each $Sub(\rho_i)$ contains β_i by the following mechanism: We can let $\rho_1 = \beta_1$, and for each integer $i \geq 2$ we can let ρ_i be some permutation in X that involves both ρ_{i-1} and β_i. This also ensures that $\rho_1 \preceq \rho_2 \preceq \ldots$.

Since every set of finite permutations is finite or countable this completes the proof.

$2 \Rightarrow 3$: First note that the closure of any finite permutation ρ_1 is expressible as $\mathcal{B}(A_1, B_1, \pi_1)$ for some function π_1 over the real numbers. Furthermore note that if ρ_2 involves ρ_1 then we can extend the function π_1 to obtain an atomic representation $\mathcal{B}(A_2, B_2, \pi_2)$ for $Sub(\rho_2)$. That is, there exists a function π_2 with range and domain $A_2 \supseteq A_1$, $B_2 \supseteq B_1$, both subsets of the real numbers, and with the properties that: i) π_2 restricted to A_1 is equal to π_1 and ii) if the set $A_2 = \{a_1, a_2, \ldots, a_n\}$ with $a_1 < a_2 < \ldots < a_n$ then $\pi_2(A_2) = \pi_2(a_1)\pi_2(a_2)\ldots\pi_2(a_n)$ is order isomorphic to ρ_2, which implies that $\mathcal{B}(A_2, B_2, \pi_2)$ is equal to $Sub(\rho_2)$.

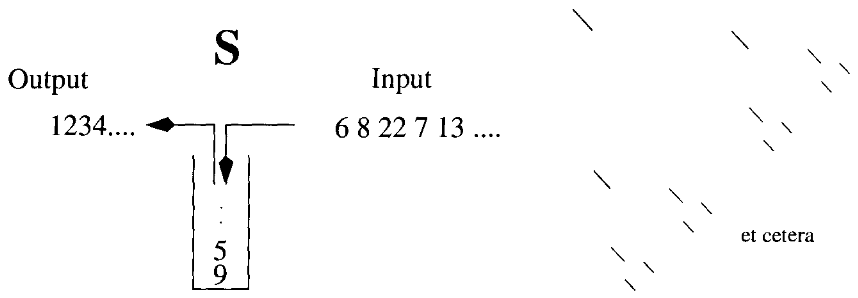


Figure 1.3: The archetypal sorting device: A single stack. *Observation:* If ever the contents of the stack become disordered by a large term lying on top of a smaller term then S cannot successfully complete sorting. *Basis:* A single stack can sort every input sequence except for those that contain a subsequence of the form 231. For instance 532416 cannot be sorted because of the subsequence 341 or 241. One stack sortable permutations are precisely defined by $\mathcal{A}(231)$. On the right is an atomic (and fractal) representation of $\mathcal{A}(231)$. Every one stack sortable permutation can be plotted on the black lines of the representation. Instantly we can see what elements of $\mathcal{A}(231)$ look like. *Enumeration:* It has been shown by Knuth [18] that the number of one-stack sortable permutations is the nth Catalan number, $\frac{2n!}{n!(n+1)!}$. One definition of the Catalan numbers is the number of ways of expressing n pairs of brackets in a balanced form. For instance $C_2 = 2$ because $(())$ and $()()$ are the only balanced expressions of two pairs of brackets. To contrast, $())($ is an unbalanced expression. Now note that there is a one-one correspondence between S sortable permutations and the sequences of push and pop operations used to sort them. And that there is a one-one correspondence between such push-pop sequences and balanced expressions of brackets. This case study illustrates all basic properties of interest.

Thus we can define a sequence of functions $\pi_1, \pi_2, \pi_3, \ldots$ on subsets of the real numbers, each function π_i generating an atomic representation of $Sub(\rho_i)$. As each of these functions is an extension of the preceding one this sequence converges to a function π, having the properties that: i) the domain A of π is the union of the domains $A_1, A_2, \ldots$ of the functions $\pi_1, \pi_2, \ldots$, and ii) for each domain A_i ($i \in \mathbb{Z}^+$), the function π restricted to A_i is equal to π_i. Then $\mathcal{B}(A, B, \pi)$ is equal to the union of all $\mathcal{B}(A_i, B_i, \pi_i)$, for:

It is clear that each $\mathcal{B}(A_i, B_i, \pi_i)$ is a subset of $\mathcal{B}(A, B, \pi)$. Secondly since each A_i is a subset of its successor A_{i+1} we have that every finite subset of A is also a finite subset of some A_i; therefore the permutation generated by π from such a subset is also generated by some π_i and is contained in the corresponding $\mathcal{B}(A_i, B_i, \pi_i)$.

Thus $\mathcal{B}(A, B, \pi)$ is an atomic representation of X, as required.

$3 \Rightarrow 1$: Suppose that $X = \mathcal{B}(A, B, \pi)$ and that α and β are some permutations in X. Then there exist subsets A_1 and A_2 of the domain of π that generate α and β. The union of A_1 and A_2 therefore generates a permutation in X involving both α and β, fulfilling the requirements of join.

$1 \Rightarrow 4$: Suppose that X has the join property and suppose that X is the union of two proper closed subclasses Y and Z. Let β and γ be permutations in $X \setminus Y$ and $X \setminus Z$ respectively. The class X has the join property, therefore there exists a permutation δ in X that involves both β and γ. X is equal to the union of Y and Z and therefore δ is contained in at least one of these. However if δ is in Y then by closure so is γ, which cannot be. By symmetry δ is not in Z, which yields a contradiction. Thus if X has the join property then it is not expressible as the union of two closed proper subclasses of X.

$\neg 1 \Rightarrow \neg 4$: Suppose that X does not have the join property, and denote the basis of X by $\mathcal{B}(X)$. There exist two permutations κ and λ in X such that every element of X involves at most one of κ and λ. Then X is equal to the union of $\mathcal{A}(\{\kappa\} \cup \mathcal{B}(X))$, which is the set of elements of X that do not involve κ, and of $\mathcal{A}(\{\lambda\} \cup \mathcal{B}(X))$, which is the set of elements of X that do not involve λ. Furthermore these are proper subclasses of X. This completes the proof.

■

1.4 Summary of Chapters

This thesis has three main thrusts.

First, it provides a comprehensive summary of the presently known structure theory of closed classes. It is intended to be an expanded version of [1] containing some significant developments and a substantial strengthening of existing theory. The Constructions chapter is dedicated to this purpose.

Secondly, it examines the decidability problem for whether an arbitrary closed class, given by its basis, is atomic. We solve this question in one case, the remaining cases are still open.

Thirdly, it attacks the most popular question in closed permutation classes – that of enumeration – the question of how many permutations of each length a given closed class has. We solve it for one class, M, of the permutations sortable by two ordered stacks in series[2]. We do not solve it for the general

[2] A full definition of what we mean by ordered is given in the introduction to the chapter on M. We are aware of work by other authors on more restricted ordered stacks.

two stacks in series, S^2, but we do conjecture a connection between permutations that can be sorted by S^2 and a set of coloured graphs. Finally we add a comment aimed at the ongoing debate of arbitrary closed classes.

To these ends there are certain qualities that can interest us in any closed class. We therefore are liable to ask the following:

- What is the basis of the class? Is the class finitely based?

- Is a class atomic? If not then can we express it as a union of atomic classes? (This corresponds to the question of "How is the class expressible by the Sub notation?". The tangible objects that are elements maximal under involvement correspond precisely to finite maximal atomic subclasses.)

- How many elements of each length does a class have? Is the class finite? Can we, for computational purposes, determine in polynomial time, or better, whether or not a given permutation is in the class?

- Does a class contain an infinite antichain? (A class that does not is said to be partially well ordered and has certain nice properties with regard to its atomic decomposition.)

Constructions

In Chapter 2 we introduce constructions and decompositions widely used within this field. It is its own best introduction and has, added to it, various theorems and counterexamples designed to show how how these costructions behave and relate to one another. Significant open questions are as follows:

- If two classes are defined by their respective bases then what is the basis of their merge?

- Is the merge of two finitely based classes necessarily finitely based?

- If two closed classes are given by their bases then what is the basis of their wreath product? If the two classes are finitely based is it decidable whether or not their wreath is finitely based? Is there a terminating mechanism that will deliver the basis of the wreath of two classes, providing that the wreath is finitely based?

- Is it true that the union of an infinitely based atomic class X and another class Y that does not contain X is necessarily infinitely based?

Antichains

As these appear everywhere within this subject it seems reasonable to add a chapter on them. The classification of all antichains is a favourite topic of mine and is as yet still incomplete.

Atomic Classes and Natural Classes

The most important single theme in Atomic classes is the following question:

- If a closed class has a finite basis is it possible to determine from the basis whether the class is atomic?

The question is still open in the general case. The set of all natural classes is a subset of the set of all atomic classes and we show that it is decidable whether a finitely based closed class, given by its basis, is natural.

Bibliothek

There is compiled library of antichains that the author regards as being in some sense 'nice' or 'significant', and a list of finitely based classes including properties known about them, especially enumeration, if known, is given, and if classes are known to be partially well ordered (contain no infinite antichain) then this is also stated.

Sorting Machines, and Conclusion

We consider two specifically, M and S^2. The machine M we here declare to be identical to S^2 except that both stacks are ordered so that no term is ever placed on top of a smaller term in either stack.

We draw conclusions relevant and perhaps even necessary for any analysis of machines composed of lesser machines placed in parallel or in series.

We also sketch out the open problems that we regard to be most important for the development of closed classes, and those that are most interesting in their own right. The two species of problem largely coincide.

Gessel's Conjecture

In the remaining chapter and appendices we present miscellanea and notes for programmers.

1.4.1 Notation

We denote *sequences, subsequences* and *permutations* by Greek letters. The only exceptions are made when dealing with established permutations, for example an antichain $U_{I_2}^{R_2}$ is defined in the library, and its elements, all permutations, are denoted $U_1, U_2, \ldots$.

We denote the *terms* of a sequence or permutation by lower case Latin letters. We avoid the use of *commas* and *brackets* wherever possible, so we will usually write $\mu = a_1a_2a_3$ instead of $\mu = \langle a_1, a_2, a_3 \rangle$ to represent the three terms of some one sequence μ.

We follow common tradition and denote sets by capital letters, either Latin or Greek.

We will frequently have to deal with subsequences of some given sequence, and we may modify these subsequences in an argument. For instance if η is a subsequence of a permutation π and if p_j is a term of π not in η then we may wish to consider the subsequence of π that consists of all the terms of η and the term p_j. We denote this by $\eta \cup p_j$. We may also wish to consider the subsequence that contains precisely those terms of π that do not lie in η. We denote that by $\pi \setminus \eta$. Essentially we treat sequences as sets of self organising material. We defend this as follows: A sequence is a function. A function is a set of ordered pairs, the first entry being an element of the domain, the second an element of the range. A restriction of a function is therefore a subset of that function, and we may use set notation to indicate the union, intersection and complement of such restricted functions. We are of the opinion that this yields the most natural and intuitively obvious notation for our purposes, and we will use it.

We call the permutation 1 the *trivial* permutation, and the permutation or sequence having no terms whatsoever the *empty* permutation.

Notation for Closed Sets

We denote the set of all permutations that *avoid* or *do not involve* any element of some set of permutations B by $\mathcal{A}(B)$.

We denote the *basis* of a closed class X by $\mathcal{B}(X)$. The letter $\mathcal{B}$ is also used for the representation of atomic classes, unfortunately both uses are established by convention. However the format of $\mathcal{B}(A, B, \pi)$ and $\mathcal{B}(X)$ is sufficiently different that there is little opportunity for confusion.

Some Closed Sets

We denote by I the set of all increasing sequences $123\ldots n$, and by R the set of all decreasing permutations $n\ (n-1)\ \ldots\ 3\ 2\ 1$. Accordingly we also denote by I_n and R_n the permutations $123\ldots n$ and $n\ldots 321$ respectively.

We follow broader mathematical usage when we let S denote the set of all permutations. There have been cases in the literature where S has been used to denote the set of *Separable* permutations, which is a special class in restricted permutations. When the latter appear we will conveniently not associate a letter with them to avoid confusion in so far as we are able.

1.5 Some Exercises

Exercise 16 Prove that the number of permutations of length $n+1$ involving a given permutation of length n is $(n+1)^2 - 2n = n^2 + 1$.

It is too long an exercise to verify that there are two permutations of length four that are involved in different numbers of permutations of length 6 but they do in fact exist. The implication of this is that counting the number of permutations of given length in a class with given basis is not a trivial task.

Exercise 17 Prove that for every non-negative integer $p \in \mathbb{N}$ there exists a closed class X and a number N such that for each integer $n \geq N$ there exist precisely p elements of X with length n.

Prove that there exists a class Y such that Y has precisely 2^{n-1} elements of length n.

Prove that there exists a class Z and an integer f such that Z contains precisely $3^f * c$ permutations of length c. Repeat for higher exponents.

It is open whether for any natural number $p > 2$ there exists a closed class K that has precisely p^u permutations of length u, for some fixed integer f and large enough u.

Exercise 18 Prove that the partial order of involvement has infinite dimension. Figure 1.4 contains a result that may help.

Exercise 19 (Harder)

Prove that every partial order with countably many elements and no infinite descending chains is contained in the partial order of involvement. This exercise may be easier after reading the chapter on antichains.

A Pleasant Result

Theorem 20 *Every permutation γ of length at least $mn + 1$ involves either I_{n+1} or R_{m+1} or both, where m, n are positive integers.*

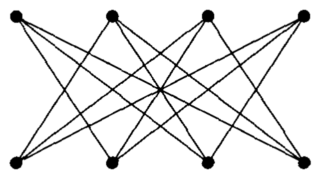


Figure 1.4: "The standard example (of an n dimensional partially ordered set) $\mathcal{S}_n$ is isomorphic to the poset formed by the 1-element and $(n-1)$-element subsets of the n-element set $[n]$ ordered by inclusion." - taken from [35]. It would appear that dimension theory was first introduced by Dushnick and Miller in [36] where this definition of $\mathcal{S}_n$ is extended so as to be able to generate 'transfinite' as well as finite dimensional partial orders. The reader is advised not to confuse the meaning of $\mathcal{S}_n$ used here with the permutation oriented meaning used elsewhere in the thesis.

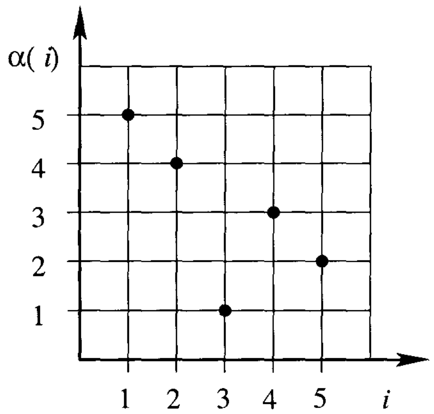
The proof that we give is attributed to Erdős and Szekeres on page 154 of [15].

PROOF: Suppose that γ contains no increasing subsequence of length greater than m.

For every $i \in \{1, 2, \ldots, m\}$ define L_i to be the set of terms $g \in \gamma$ such that every increasing subsequence of maximal length, subject to terminating with g, has length i.

By our supposition every term of γ is contained in at least one (and in fact precisely one) of these sets. Thus by the pigeonhole principle there exists at least one set L_k $k \in \{1, \ldots, m\}$ that has at least $n+1$ elements.

Finally, note that the elements of L_k form a decreasing subsequence of γ. Q.E.D. ■



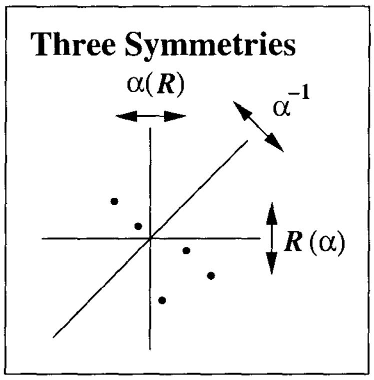


Figure 1.5: Plotting the permutation $\alpha = 54132$. The three symmetries, $\alpha^{-1} = 35421$, $\alpha(R) = \alpha^R = 23145$ and $R(\alpha) = 12534$ can be thought of as reflections, as shown on the right.

1.6 Permutation Symmetries

There are eight symmetries of a permutation, as there are for a square. We introduce them here so that we may use them freely later.

We clarify the analogy with the square: If α is a permutation of length n then α is a bijection on $\mathbb{Z}_n^+ = \{1, 2, \ldots, n\}$. Thus α may be plotted on the real plane with coordinates x and y, see Figure 1.5. The image of α will lie within the square with boundaries $x = 0$, $x = n+1$, $y = 0$ and $y = n+1$. Every symmetry of that square will transform the plotted points into the plot of another permutation of length n. We denote those symmetries that we will use in this text as follows:

- Inversion: We give α^{-1} the usual meaning of the permutation such that $\alpha(\alpha^{-1}) = \alpha^{-1}(\alpha) = I_n$, where I_n is the identity permutation on $\mathbb{Z}_6^+$. An inversion corresponds to the permutation obtained by reflecting the plot of α over the first diagonal, where $y = x$.

- Reversal: If $\alpha = a_1 a_2 \ldots a_n$ then the permutation $a_n a_{n-1} \ldots a_1$ obtained by listing the terms of α in reverse is denoted $\alpha(R)$ or α^R. This is logically justifiable in terms of composition of functions because R_n represents the permutation $n\ n-1\ \ldots 2\ 1$. Reversal is equivalent to reflection over the line $x = \frac{n+1}{2}$.

- Upturnment is equivalent to reflection over the horizontal axis $y = \frac{n+1}{2}$ and is denoted by $R(\alpha)$, as composition of mappings might demand. If the i^{th} term of α is a_i then that of $R(\alpha)$ is $n + 1 - a_i$.

- We will not reserve a specific notation for the reflection $(R(\alpha^R))^{-1}$ or any of the rotations.

Symmetries are useful when enumerating sets of permutations. If a permutation avoids 123 then its reverse will avoid 321, reversal is a bijection, thus there is no need to enumerate the number of permutations of given length avoiding 123 if we already have those statistics for 321.

Chapter 2

Constructions

Union, Intersection, Direct and Skew Sum, Direct and Skew Completion, Strong Completion, Sum Decidability, Direct and Skew Expansion, Wreath Product, Interval Free Permutations P, Juxtaposition, Differentiation, Merge

2.1 Union and Intersection of Closed Classes

Theorem 21 *[1] If X and Y are closed classes then $X \cap Y$ is closed. Furthermore if $X = \mathcal{A}(B_1)$ and $Y = \mathcal{A}(B_2)$ then $X \cap Y = \mathcal{A}(B_1 \cup B_2)$.*

PROOF: This is elementary and follows from the definitions of closed class and basis. ■

Definition 22 Let α and β be permutations. A *merge* of α and β is a permutation consisting of two not necessarily disjoint subsequences order isomorphic to α and β.

A *minimal merge* of some permutations α and β is a permutation that is minimal subject to involving both α and β. The definition extends to sets in a natural way: The merge of two sets of permutations is the set of all permutations that are a merge of a permutation in the one set and a permutation in the other. The *minimal merge* of two sets is however defined slightly differently:

Definition 23 Let A and B be sets of permutations. Then the *minimal merge* of A and B is the set of permutations, minimal under involvement, in the merge of A and B.

This does not necessarily include every minimal merge of an element of A and an element of B. It is in fact very easily possible to have two infinite sets, such as I and R, the sets of increasing and decreasing sequences, whose minimal merge consists of but a single permutation, in this case the permutation 1.

Theorem 24 *[1] If X and Y are closed classes then $X \cup Y$ is closed. Furthermore if $X = \mathcal{A}(B_1)$ and $Y = \mathcal{A}(B_2)$ then the basis of $X \cup Y$ is the minimal merge of B_1 and B_2.*

PROOF: To see that $X \cup Y$ is closed is elementary. Note that a permutation δ is not in $X \cup Y$ if and only if it involves elements of both B_1 and B_2. Thus the basis of $X \cup Y$ is the set of permutations such as δ, minimal under inclusion subject to involving elements of both B_1 and B_2, which is precisely the minimal merge of B_1 and B_2. Quad erat demonstrandum.

■

Corollary 25 *If X and Y are finitely based then so is $X \cup Y$. Indeed if X and Y have no basis elements of length greater than m and n respectively then $X \cup Y$ has no basis elements of length greater than $m + n$.*

The above results may all be derived from Theorem 2.1 on page 30 of [1].

Note: $\mathcal{A}(B_1 \cap B_2)$ is in general a larger set than and invariably contains $X \cup Y$, assuming that B_1 and B_2 are the bases of X and Y respectively.

2.1.1 Some More Advanced Material.

This section contains concepts that are only introduced in later chapters, the reader is not expected to comprehend it in first perusal. It contains some results, that the logician may be assured have not been used later in this thesis, and an open conjecture concerning unions of closed classes. The author is confident that similar questions regarding intersections of classes, as opposed to unions of classes, are sufficiently easily answered as not to warrant dedicated coverage. The author also awaits questions about intersections that *are* worthy of attention!

Theorem 26 *There exists an infinitely based class X and a finitely based class Y such that neither class is contained within the other and such that the union of X and Y is finitely based. Similarly there exist two infinitely based classes X and Y whose union is finitely based.*

PROOF: We present two infinitely based classes whose union is finitely based.

Let A and B be infinite antichains all of whose elements lie in $\mathcal{A}(321)$ and $\mathcal{A}(123)$ respectively. Furthermore let neither A nor B contain an element of length less than five. Such antichains exist and examples may be found in the Bibliothek. Let X be the set consisting of all permutations not in A but properly involved in an element of A, and additionally let X contain all elements of the set $\mathcal{A}(123)$. Thus:

$$X = \mathcal{A}(123) \cup (Sub(A) \setminus A).$$

X is infinitely based because every element of A is a basis element. (As every element of A has length at least five and avoids 321, every element of A must involve 123 and therefore is not an element of X. Hovever if any one term is removed from an element of A then we obtain a sequence order isomorphic to an element of $Sub(A) \setminus A$.)

Similarly let:

$$Y = \mathcal{A}(321) \cup (Sub(B) \setminus B).$$

Y also is infinitely based, however:

$$\begin{aligned} X \cup Y &= (\mathcal{A}(123) \cup (Sub(A) \setminus A)) \cup (\mathcal{A}(321) \cup (Sub(B) \setminus B)) \\ &= (\mathcal{A}(123) \cup (Sub(B) \setminus B)) \cup (\mathcal{A}(321) \cup (Sub(A) \setminus A)) \\ &= \mathcal{A}(123) \cup \mathcal{A}(321) \end{aligned}$$

which is finitely based.

If Y is instead defined to be $\mathcal{A}(321)$ then it is still true that neither set is contained within the other, and yet the union of X and Y is unaltered and has the same finite basis. ∎

Conjecture 27 *The union of an infinitely based atomic class X and any other closed class Y that is neither a subset nor a superset of X is by necessity infinitely based.*

The proof of this conjecture is not eased by the fact that if B is the basis of some infinitely based atomic class X then not every element of X is involved in an element of B, as is demonstrated in the following: Let A be an infinite, maximal and fundamental antichain in $\mathcal{A}(321)$. (There are only four such antichains.) Let $X = (Sub(A) \setminus A) \oplus Sub(1)$. Then only finitely many basis elements of X are not elements of A, and it is not true that every element of X is involved in a basis element of X. (The conjecture would be a straightforward corollary of this aforementioned fact *not* holding, however the fact stands.)

Maximal Elements: There exist two classes, both infinite and neither contained in the other, such that every element of each class is contained within a maximal element of that class, but where the intersection contains no maximal elements. The intersection is, by necessity, infinite.

Proof: Consider the closures of the fundamental antichains ${}^{I_2}_{I_2}U$ and $U^{I_2}_{I_2}$ listed in the library. Their intersection is the closure of an infinite increasing oscillating sequence. Q.E.D.

The entirety of the chapter on atomic classes concerns itself with writing closed classes as unions of lesser closed classes. Of especial interest is Theorem 182 which states that a finitely based non-atomic class can be written as a union of two incomparable finitely based closed classes, and Theorem 188 from which it follows that every partially well ordered class is expressible as the union of finitely many atomic classes.

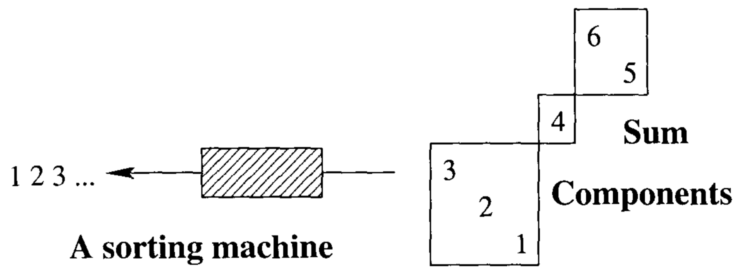


Figure 2.1: Sum decomposition occurs naturally with sorting devices that produce an increasing output: Smaller sum components are sorted first.

2.2 Direct and Skew Sum

Definition 28 The *direct sum* $\alpha \oplus \beta$ of two permutations α and β is the permutation $\gamma = g_1 g_2 \dots g_n$ where for some i we have that $g_1 \dots g_i$ is order isomorphic to α, that $g_{i+1} \dots g_n$ is order isomorphic to β and that for all j, k with $j \leq i < k$ we have that $g_j < g_k$.

Example 29 If $\alpha = 21$ and $\beta = 312$ then $\alpha \oplus \beta = 21534$, as $21 = \alpha$ and $534 \cong 312 = \beta$.

This definition extends to sets of permutations in a natural way:

$$X \oplus Y = \{\alpha \oplus \beta | \alpha \in X; \beta \in Y\}$$

The definition of skew sum is similar:

Definition 30 The *skew sum* $\alpha \ominus \beta$ of two permutations α and β is the permutation $\gamma = g_1 g_2 \dots g_n$ where for some i we have that $g_1 \dots g_i$ is order isomorphic to α, that $g_{i+1} \dots g_n$ is order isomorphic to β and that for all j, k with $j \leq i < k$ we have that $g_j > g_k$.

Example 31 If $\alpha = 231$ and $\beta = 312$ then $\alpha \ominus \beta = 564312$.

2.2.1 Sum Indecomposable: The Increasing Oscillating Sequence

Definition 32 A permutation γ that is expressible as $\alpha \oplus \beta$ or $\alpha \ominus \beta$ where α and β are non empty permutations is said to be *sum decomposable* or *skew decomposable* respectively.

Definition 33 Let $\alpha = a_1 \ldots a_n$ be a sequence. A contiguous subsequence $a_i a_{i+1} \ldots a_j$ of α is said to be a *sum component* if it is greater than all the terms it succeeds and less than all the terms it precedes, and is not order isomorphic to a sum decomposable permutation.

The definition of a *skew component* is analogous.

We introduce a sequence about which the structure of sum indecomposable sequences revolves, namely the *increasing oscillating sequence.* It is best first seen as an infinite sequence so that we may ignore end effects. It is a sequence that, in a sense adapted for infinite sequences, is sum indecomposable. Minimally so, in that if any term is removed then the result is sum decomposable. The sequence is defined by:

$$\ldots\ -3\ -6\ -1\ -4\ 1\ -2\ 3\ 0\ 5\ 2\ 7\ 4\ 9\ 6\ 11\ 8\ \ldots$$

A conventional increasing oscillating sequence is a finite sum indecomposable sequence order isomorphic to some subsequence of the increasing oscillating sequence. An example of such a sequence is 31527486. It is typical in that if any term is removed from this then, with two exceptions, the resulting sequence is sum decomposable. For instance if 5 is removed then $3127486 \cong 312 \oplus 3142$.

We do regard 1 and 21 to be increasing oscillating sequences. They are the simplest but we have no reason to exclude them.

Proposition 34 *Let a_i and a_j, with $i < j$, be any two terms of a permutation α. Suppose that there exists a sum indecomposable subsequence, $\alpha(A)$, of α containing both a_i and a_j. If $\alpha(A)$ is minimal then $\alpha(A)$ is an increasing oscillating sequence.*

PROOF: Throughout this proof we will consider only $\alpha(A)$, and if we choose a term satisfying certain conditions we will implicitly add the restriction that the term is in $\alpha(A)$. We proceed:

If all the terms preceding or equal to a_i are less than all the terms succeeding a_i, the latter including a_j, then $\alpha(A)$ is sum decomposable into two parts, one containing a_i, the other a_j. As this cannot be there exists a term to the right of a_i less than some term either preceding or equal to a_i. To be precise, if a_j is less than some term either preceding or equal to a_i then define a_{r_1} to be a_j, else define a_{r_1} to be the rightmost term less than some term either preceding or equal to a_i. Similarly, if $a_i > a_{r_1}$ then define a_{g_1} to be a_i, else let a_{g_1} be the greatest term preceding a_i. Now if $r_1 \geq j$ then we are done because we will find that the subsequence formed by the terms a_i, a_j, a_{r_1}, a_{g_1} is sum indecomposable and therefore constitutes all of $\alpha(A)$. There are four forms that this subsequence can take depending on whether $g_1 = i$ and on whether $r_1 = j$:

If $i = g_1$ and $j = r_2$ then because $a_{g_1} > a_{r_1}$ we have that $a_i a_j \cong 21$, which is sum indecomposable and an increasing oscillating sequence. If $i \neq g_1$ and $j \neq r_1$ then we have that $g_1 < i < j < r_1$ and that $a_{g_1} a_i a_j a_{r_1} \cong 3142$, which

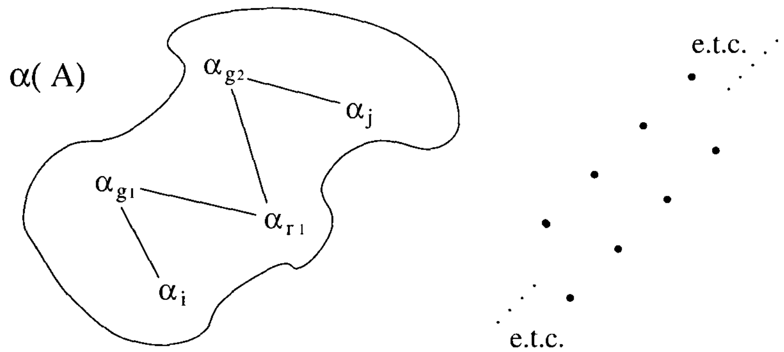


Figure 2.2: The terms of $\alpha(A)$ form an increasing oscillating sequence (right). We call a_{g_1} the *logical successor* of a_i, and a_{r_1} the logical successor of a_{g_1}, and so on.

is an increasing oscillating sequence. Similarly if $i = g_1$ but $j \neq r_1$ or if $j = r_1$ but $i \neq g_1$ then the sequence whose terms are in $\{a_i, a_j, a_{r_1}, a_{g_1}\}$ is order isomorphic to either 231 or 312 respectively, both of which are increasing oscillating sequences and so we are done.

If however $r_1 < j$ then we persevere: As $\alpha(A)$ is not sum decomposable the set of terms in $\alpha(A)$ preceding or equal to a_{r_1} must contain a term greater than or equal to some term succeeding a_{r_1}. Specifically, if a_j is less than some term preceding or equal to a_{r_1} then let $a_{r_2} = a_j$, else let a_{r_2} be the rightmost term less than some term preceding or equal to a_{r_1}. Define a_{g_2} to be the greatest term preceding or equal to a_{r_1}. (The definition of a_{g_2} is simpler than that of a_{g_1} and will remain so for a_{g_3} and succeeding terms of this form.) The order of appearance of the terms mentioned so far is $a_{g_1}a_ia_{g_2}a_{r_1}a_{r_2}$. This is because a_{r_2} lies to the right of a_{r_1}, hence a_{g_2} must be greater than a_{g_1} and must therefore lie to the right of a_i. The terms a_{g_1} and a_i may of course be equal. But finally we have that if $r_2 \geq j$ then we are done because the terms that we have mentioned specifically form a subsequence order isomorphic to one of 2 4 1 3, 3 1 5 2 4, 2 4 1 5 3, 3 1 5 2 6 4.

We continue defining a_{g_k} and a_{r_k} until we have an a_{r_k} that does not precede a_j and in every case we demonstrate that the terms we have selected together with a_i, a_j constitute all A, and that these terms form an increasing oscillating sequence. Q.E.D. To give one last example, in the case where the last terms that we generate are a_{r_3} and a_{g_3} we have that $\alpha(A)$ is order isomorphic to one of 2 4 1 6 3 5, 3 1 5 2 7 4 6, 2 4 1 6 3 7 5, 3 1 5 2 7 4 8 6. ■

Similarly we can prove that:

Proposition 35 *Let β and γ be sum indecomposable subsequences of a permutation α. If there exists a minimal subsequence Λ of α such that Λ, β and γ together form a sum indecomposable sequence then Λ is an increasing oscillating sequence.*

If we find the basis of the closure of the set of increasing oscillating sequences then we will obtain the following. This result is unproved and, in this text, unused.

Proposition 36 *Let π be the infinite Increasing Oscillating Sequence. Then $Sub(\pi) = \mathcal{A}(321, 3412, 2341, 4123)$.*

Increasing oscillating sequences may be interpreted through the following:

Proposition 37 *Let α be a sequence, and the following relations on the terms of α:*

- *R: A relation where $(a_i, a_j) \in R$ if and only if the term a_i lies strictly below and to the right of a_j.*

- *L: A relation where* $(a_i, a_j) \in L$ *if and only if the term* a_i *lies strictly above and to the left of* a_j.

- *B: a relation where* $(a_i, a_j) \in B$ *if and only if* a_i *and* a_j *lie in the same sum component of* α.

Then both L and R are transitive. Moreover B is the smallest equivalence class containing both R and L.

PROOF: R and L are transitive, that is a trivial result. B is an equivalence relation, being symmetric, reflexive and transitive. Terms related by R or L do lie in the same sum component, therefore B does contain L and R. By Proposition 34 any two distinct terms in one sum component are contained in an increasing oscillating subsequence of that sum component. Logically consecutive terms in an increasing oscillating sequence are related by either L or R, thus B is the smallest equivalence class to contain both L and R. ∎

Corollary 38 *Two terms* a_i *and* a_j *of a sequence* α *are related by* B*, defined as above, if and only if there exists a sequence* $a_{f(0)} \dots a_{f(n)}$ *of terms such that* $a_{f(0)} = a_i$ *and* $a_{f(n)} = a_j$ *and each* $a_{f(k)}$ *is related to its successor* $a_{f(k+1)}$ *by either L or R. (The terms* $a_{f(0)} \dots a_{f(n)}$ *are in logical order, they are not necessarily listed in the order in which they appear in* α*.)*

Moreover there exists a subsequence $a_{g(0)} \dots a_{g(m)}$ *of* $a_{f(0)} \dots a_{f(n)}$ *in which:*

- $a_{g(0)} = a_{f(0)}$ *and* $a_{g(m)} = a_{f(n)}$.

- *For each non-terminal term of the chain* $a_{g(0)} \dots a_{g(m)}$*, that is for every* $a_{g(i)}$ *where* $g(0) < g(i) < g(m)$*, either* $(a_{g(i-1)}, a_{g(i)}) \in R$ *and* $(a_{g(i)}, a_{g(i+1)}) \in L$ *or vice versa.*

- *Each term is related to its predecessor (if it exists) and its successor (if it exists) and to no other term.*

PROOF: The former follows from the fact that B is the smallest equivalence containing both L and R, and that $L = R^{-1}$, which is a sufficient condition to establish primeval chains. The latter can be deduced from the former and the fact that L and R are both transitive. ■

The second half of the corollary is consistent with the oscillating of minimal sum indecomposable sequences.

2.2.2 The form of Basis Elements of $Y \oplus Z$

Let Y and Z be closed classes. Let δ be a basis element of $Y \oplus Z$. We will examine δ.

Let $\delta = \delta_1 \oplus \delta_2 \oplus \ldots \oplus \delta_n$ where the δ_i are the sum components of δ. δ must involve a basis element of Z. Let k be the largest number such that $\delta_k \oplus \delta_{k+1} \oplus \ldots \oplus \delta_n$ involves a basis element of Z.

$\delta_1 \oplus \ldots \oplus \delta_k$ must involve a basis element of Y, or else $\delta_1 \oplus \ldots \oplus \delta_k \in Y$, $\delta_{k+1} \oplus \ldots \oplus \delta_n \in Z$ and $\delta \in Y \oplus Z$. Consider an embedding of a basis element of Y in $\delta_1 \oplus \ldots \oplus \delta_k$. Note that that embedding must involve every term of $\delta_1 \oplus \ldots \oplus \delta_{k-1}$, for otherwise δ is not a minimal permutation outwith $Y \oplus Z$. Similarly if we regard any embedding of a basis element of Z in $\delta_k \oplus \ldots \oplus \delta_n$, that embedding must involve every term of $\delta_{k+1} \oplus \ldots \oplus \delta_n$.

Denote by Υ the terms of δ involved in some embedding of a basis element of Y in $\delta_1 \oplus \ldots \oplus \delta_k$. Similarly denote by Ξ the terms of δ involved in some embedding of a basis element of Z in $\delta_k \oplus \ldots \oplus \delta_n$. Denote by Υ_f the terms of the last sum component of Υ and denote by Ξ_f the terms of the first sum component of Ξ.

If every term of Ξ lies above and to the right of every term of Υ then we may deduce that δ consists of nothing but Υ and Ξ because no sum such as $\Upsilon \oplus \Xi$ of a basis element of Y and a basis element of Z is in $Y \oplus Z$.

If not then all the terms of Υ_f and Ξ_f lie in δ_k which is, be reminded, a sum component. Thus there exists a minimal set of terms, Λ, such that the subsequence of δ consisting of the elements of Υ_f, Λ and Ξ_f is sum indecomposable. The subsequence of δ consisting of the terms of Υ, Λ and Ξ is not order isomorphic to a permutation in $Y \oplus Z$. Thus these terms constitute δ.

Finally note that the terms of Λ, of which there may be none, form an increasing oscillating sequence. (To see this take any term from Υ_f and any from Ξ_f. They lie in the same sum component of $\Upsilon_f \cup \Lambda \cup \Xi_f$ therefore there exists a minimal set of terms containing both terms that is sum indecomposable; and by Lemma 34 this is an increasing oscillating sequence. That increasing oscillating sequence, by minimality of Λ, contains every term of Λ, making Λ a subsequence of an increasing oscillating sequence. Now use the relations R and L of Proposition 37 to show that Λ is, by minimality, a sum indecomposable subsequence of the increasing oscillating sequence, and we are done.)

2.2.3 Corollaries

Proposition 39 *If Y and Z are finitely based and either Y or Z has a basis element that is a subpermutation of an increasing oscillating sequence then*

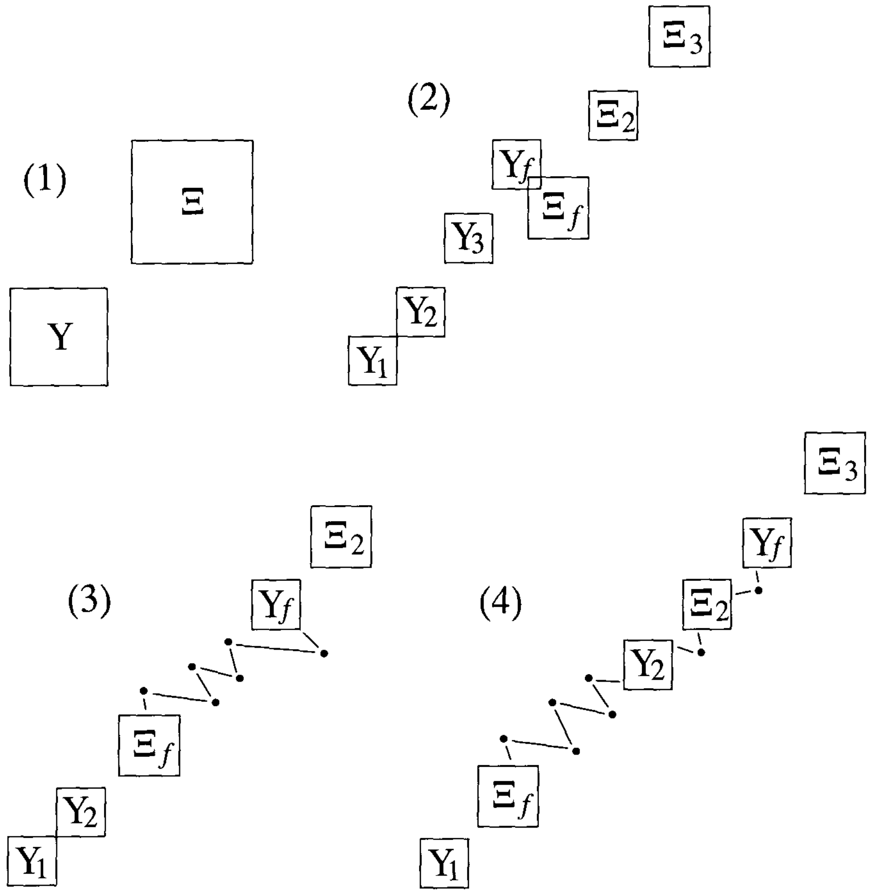


Figure 2.3: Basis elements of Y and Z in increasing stages of complexity. All contain subsequences Υ and Ξ, order isomorphic to basis elements of Y and Z respectively. In (1) the terms of Ξ lie above and to the right of those of Υ, and that is sufficient to ensure that the permutation does not lie in $Y \oplus Z$. In all other cases the last sum component of Υ and the first of Ξ must lie in the same sum component of the final permutation. In (2) the respective positions of Υ and Ξ are sufficient to ensure this, in (3) and (4) more terms, in the form of an increasing oscillating sequence are needed. In (4) some of the terms of the increasing oscillating sequence coincide with terms of Υ and Ξ.

$Y \oplus Z$ is finitely based.

PROOF: Briefly: Suppose that Y has a basis element that is involved in some increasing oscillating sequence. As the length of basis elements in Y and Z is limited the only way in which $Y \oplus Z$ can be infinitely based is that it has basis elements that consist of a basis element of Y and a basis element of Z bound together with an arbitrarily long increasing oscillating sequence.

That however cannot occur because we can take such a hypothetical "long" basis element of $Y \oplus Z$, call it δ, and within the basis element of Z and the increasing oscillating sequence find a shorter sequence also not in $Y \oplus Z$, which is a contradiction.

To be precise let us call Ξ the subsequence order isomorpic to a basis element of Z and Λ the terms of the binding increasing oscillating sequence, in accordance with our earlier notation.

If we take the sequence consisting of Λ and Ξ and remove the rightmost term of Λ not in Ξ then we have a sequence definitely shorter than δ. If $|\delta|$ was chosen to be sufficently long we will find that the first sum component of this sequence will contain an increasing oscillating sequence long enough to involve a basis element of Y, so this sequence is not in $Y \oplus Z$ and we are done. ■

Corollary 40 *If Y is finite and Z is finitely based then $Y \oplus Z$ is finitely based. By symmetry, if Y is finitely based and Z finite then $Y \oplus Z$ is finitely based.*

PROOF: If Y is finite then $I_m = 12\ldots m$ is a basis element of Y for some m. ■

Proposition 41 *There exist closed classes Y, Z such that $Y \oplus Z$ is finitely based but where neither Y nor Z has a basis element that is a subpermutation of an increasing oscillating sequence.*

This indicates that Proposition 39 cannot be strengthened to an if and only if condition. Even barring the obvious counterexamples where $Y \subseteq Z$ there is the following:

Example 42 Let $Y = \mathcal{A}(4321)$ and $Z = \mathcal{A}(4132, 4312, 3421, 2431)$. As all the basis elements of Y and Z are sum indecomposable $Y \oplus Z$ can only be infinitely based if $\mathcal{B}(Y \oplus Z)$ contains a permutation involving a copy of 4321 that is strictly above and to the right of every embedding of any basis element of Z.

If however an attempt is made to "bind" that sequence to the rest of the basis element with an oscillating sequence it will instantly be found that this is not possible.

Example 43 Let A be any infinite antichain of sum indecomposable permutations, none of which involves 321 and all of which have length at least three. The the elements of sufficient length in the antichain ${}^{I_2}U_{I_2}$, listed in the Bibliothek, form an example of such an antichain. Let $X = Sub(4321) \oplus (Sub(A) \setminus A)$ and let $Y = \mathcal{A}(321)$. Then:

No element of A is an element of X. No element of A is involved in 4321, no element is contained in $Sub(A) \setminus A$, and as the elements of A are sum indecomposable no element of A can belong to X unless it is an element of at least one of these two. Furthermore, every permutation properly involved in an element of A is by necessity an element of $Sub(A) \setminus A$ and therefore an element of X. Thus elements of A are also basis elements of X and as A is infinite, so is the basis of X.

The the sum of X and Y is finitely based, indeed as $Sub(A) \setminus A$ is a subset of $\mathcal{A}(321)$ and as $\mathcal{A}(321) \oplus \mathcal{A}(321) = \mathcal{A}(321)$ we have that:

$$X \oplus Y = Sub(4321) \oplus (Sub(A) \setminus A) \oplus \mathcal{A}(321) = Sub(4321) \oplus \mathcal{A}(321)$$

which, by Corollary 40 is finitely based.

2.2.4 $X \oplus Y$ Infinitely Based

Proposition 44 *There exist finitely based closed classes X and Y such that $X \oplus Y$ is not finitely based.*

This is due to Atkinson in [2]. As proof we may use the following example:

Proposition 45 *Let A be the infinite antichain, listed in the Bibliothek under the title ${}^{I_2}U_{I_2}$, the first few elements of which are:*

$A_1 =$ 3 4 1 2

$A_2 =$ 2 3 6 1 4 5

$A_3 =$ 2 3 5 1 8 4 6 7

$A_4 =$ 2 3 5 1 7 4 10 6 8 9

Let X be the set of all permutations order isomorphic to sequences of the form $\gamma_1\gamma_2\ldots\gamma_{n-1}$ where $\gamma_1\gamma_2\ldots\gamma_n$ is an element of $Sub(A)$. That is to say, let X be the set of all permutations order isomorphic to elements of $Sub(A)$ with the last term removed.

Similarly let $Y = \partial(Sub(A))$, i.e. the set of all permutations order isomorphic to elements of $Sub(A)$ with the first term removed.

Then both X and Y are finitely based but $X \oplus Y = Sub(A) \setminus A$, which is infinitely based.

PROOF:

First we show that every permutation in $X \oplus Y$ is in $Sub(A) \setminus A$: By examining the elements of A note that if A_{i+j+2} is expressed as $a_1 \ldots a_{2(i+j+3)}$ then the sequence obtained from A_{i+j+2} by removing all the terms $\{a_{2i+2}, a_{2i+3}, a_{2i+4}, a_{2i+5}\}$, denoted $A_{i+j+2} \setminus \{a_{2i+2}, a_{2i+3}, a_{2i+4}, a_{2i+5}\}$ is sum decomposable. The sum components consist of the first $2i+1$ terms and the last $2j+1$ terms of A_{i+j+2}. The first $2i+1$ terms of A_{i+j+2} form a sequence order isomorphic to A_i with its last term removed, and similarly the last $2j+1$ terms of A_{i+j+2} are order isomorphic to A_j with its first term removed. We conclude that every element of $X \oplus Y$ is also an element of $Sub(A) \setminus A$.

To show that $Sub(A) \setminus A$ is a subset of $X \oplus Y$ it is sufficient to show that if any one term, a_i, is removed from an element $A_n = a_1 \ldots a_{2n+2}$ of A then the resulting sequence is order isomorphic to an element of $X \oplus Y$. We are obliged to consider four generic cases:

- If $i = 1$ then A_n with a_i removed is order isomorphic to the last $2n+1$ terms of A_n and is therefore an element of Y.

- If i is odd but not one then the permutation A_n with a_i removed is order isomorphic to $(a_1 a_2 \ldots a_{i-1} a_{i+1}) \oplus (a_{i+2} a_{i+3} \ldots a_{2n+2})$. The first part of this sum composition is order isomorphic to an element of X,

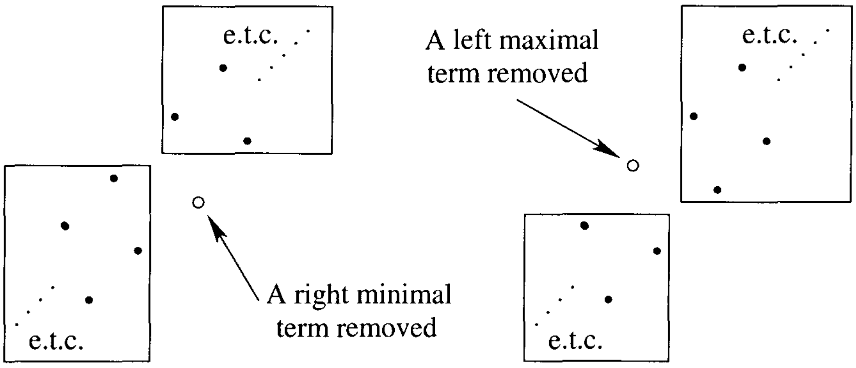


Figure 2.4: An increasing oscillating sequence $a_1 \ldots a_n$ with the k^{th} term removed becomes sum decomposable into two parts. The components are the first $k-2$ terms and the terms $a_{k-1}a_{k+1}a_{k+2}\ldots a_n$ (the largest $n-k+1$) if a_k is right minimal, or $a_1 \ldots a_{k-1}a_{k+1}$ (the smallest k) and the last $n-1-k$ terms if a_k is left maximal.

even when $i = 2n+1$, and the second part is either empty or order isomorphic to an element of Y.

- If $i = 2n+2$, in which case a_i is the last term of A_n, then A_n with a_i removed is order isomorphic to the first $2n+1$ terms of A_n and is therefore an element of X.

- If n is even but not $2n+2$ then the permutation A_n with a_i removed is order isomorphic to $a_1a_2\ldots a_{i-2} \oplus a_{i-1}a_{i+1}a_{i+2}\ldots a_{2n+2}$. The first part of this sum decomposition is order isomorphic to an element of X and the second part is either empty or order isomorphic to an element of Y.

In every case we have our desired result. Finally, to show that $Sub(A) \setminus A$ is infinitely based note that as A is an antichain, every element of A is a basis element of that set. ∎

Another example of this is given implicitly in [2]. There, to prove that a certain class (the sum completion of $\mathcal{A}(321654)$, as defined later) is infinitely based, an infinite list of basis elements is given and infinitely many of those are also basis elements of $\mathcal{A}(321)$ summed with itself: $\mathcal{A}(321) \oplus \mathcal{A}(321)$.

2.2.5 Sum, Skew and Strong Completion

Definition 46 Let X be a closed class. The *sum completion* of X is the set of permutations of the form $a_1 \oplus a_2 \oplus \ldots \oplus a_n$ for any n where each a_i is an element of X. If X is equal to its own sum completion then X is said to be *sum complete.*

Example 47 $R = \mathcal{A}(12)$ is the set of all decreasing sequences. The sum completion of R has basis $\{312, 231\}$ and consists of the elements of R, the elements $R \oplus R$, of $R \oplus R \oplus R$ and so on.

It is clear that basis elements of the completion must be sum indecomposable. It is then easy to observe that the basis elements of the completion are precisely the minimal sum indecomposable permutations involving basis elements of X. From the last section we know that these are merges of basis elements of X with increasing oscillating sequences, that merge being further specified by some technical details. We will not retrace our steps.

We will however comment on the relation between the size of the basis of a closed class and the size of the basis of its sum completion.

- It is possible for the sum completion of a finitely based class to be infinitely based. An example of such a class is given in [2]. Equally there exist finitely based classes that are sum complete, such as $\mathcal{A}(321)$.

- The finitely based classes whose sum completion is finitely based have not been characterised, however partial characterisations exist. Corollaries 39 and 40 and have parallels, and specifically in the latter case we have that the sum completion of a finite class is finitely based.

- There exist infinitely based classes whose sum completion is infinitely based, and others whose sum completion is finitely based. Examples of both are easy to generate.

- For finitely based classes whose sum completion is finitely based, it is possible for the number of basis elements of the class to be either greater or less than the number of basis elements of the sum completion. Examples of each are given by $\mathcal{A}(12)$ and $\mathcal{A}(213, 231, \{R_i \oplus R_{n-i} \mid i \in \mathbb{Z}_n^+\})$, for some integer n greater than one.

- For every finitely based class there is however a computable upper bound for the length that basis elements of the sum completion can have, if it is to be finitely based. Furthermore there exists a decision mechanism that can determine whether the sum completion of a finitely based class is finitely based.

Many are also interested in the enumerative question for the class; however, we do not consider it here. For information, the question is: Given the number of permutations of each length in some closed class (i.e. given the enumeration for some given class), what is the number of permutations of each length in its sum completion? There are equinumerous classes one of which is sum complete, the other not, hence any solution to this question would have to consider a variety of factors.

The skew sum is defined as one might expect:

Definition 48 Let X be a closed class. The *skew completion* of X is the set of permutations of the form $a_1 \ominus a_2 \ominus \ldots \ominus a_n$ for any n where each a_i is an element of X.

The strong completion of X is defined recursively:

Definition 49 Let X be a set of permutations. Then the *strong completion*, $X^{\oplus\ominus}$, of X is the set of permutations generated by the following recursion:

- If $\alpha \in X$ then $\alpha \in X^{\oplus\ominus}$.
- If $\alpha, \beta \in X^{\oplus\ominus}$ then $\alpha \oplus \beta \in X^{\oplus\ominus}$ and $\alpha \ominus \beta \in X^{\oplus\ominus}$.

Example 50 The strong completion of $\mathcal{A}(I_5, R_7, 132)$ is $\mathcal{A}(2413, 3142)$. The sum completion of $\mathcal{A}(I_5, R_7, 132)$ is $\mathcal{A}(2413, 3142, 234561, 612345, R_7)$

Example 51 The strong completion of $\mathcal{A}(123)$ is $\mathcal{A}(23514, 24513, 35124, 25134, 34152, 41253, 31452, 41523, 31524, 24153)$. The sum completion avoids all the basis elements of the strong completion but in addition also 2341 and 4123.

The strong completion has the potential to produce more interesting basis elements than we have seen so far, however only to a limited extent: The basis elements of the strong completion of a closed class need to be minimal subject to a) involving a basis element of that class, b) being sum indecomposable and c) being skew indecomposable. Thus it is possible to start with a skew decomposable permutation and add a minimal set of terms so as to produce a permutation that is skew indecomposable, but sum decomposable. To that permutation one might add terms to produce another that was sum indecomposable, but skew decomposable, and so on. However in permutations this chain cannot continue indefinitely. The following hold, assuming that X is a closed class with basis B:

- If β is a basis element of the strong completion of X and if there exists some γ, a basis element of X involved in β and having length strictly less than $|\beta| - 4$ then β is a basis element of either the sum or the skew completion of X.

- If β is a basis element of the sum (or skew) completion of X and if there exists a permutation γ in the basis of X, involved in β and having length strictly less than $|\beta| - 1$ then β is strongly indecomposable and a basis element of the strong completion of X. (This is an easy observation given that every increasing oscillating sequence of length greater than or equal to four is strongly indecomposable.)

We prove neither point, although neither is technically difficult. We do not regard it as being one of the crucial points of this thesis, although to prove them would be to prove the following useful corollary.

Corollary 52 *The strong completion of a finitely based class is finitely based if and only if both its sum and its skew completion are finitely based.*

This is useful because it is decidable whether the sum and skew completions of a finitely based class are finitely based.

We do however note that sum and the skew completions being finitely based are thoroughly independent matters:

Proposition 53 *There exists a finitely based class whose sum completion is infinitely based but whose skew completion is finitely based.*

PROOF: Consider $\mathcal{A}(321654)$. Its basis element is skew indecomposable, hence it is skew complete. However it is shown in [2] that this class is infinitely based. Indeed the infinite antichain $U_{R_2}^{R_2}$ (in the Bibliothek) is an infinite subset of its basis. ■

We also note that the unproved Corollary 52 cannot be extended to infinitely based classes:

Proposition 54 *There exists an infinitely based class X whose sum completion is infinitely based, but whose strong completion is finitely based.*

PROOF: Let A be the infinite antichain listed in the Bibliothek as ${}_{I_2}^{I_2}U$ whose first few terms are:

$A_1 = 2\ 3\ 5\ 1\ 6\ 7\ 4$

$A_2 = 2\ 3\ 5\ 1\ 7\ 4\ 8\ 9\ 6$

$A_3 = 2\ 3\ 5\ 1\ 7\ 4\ 9\ 6\ 10\ 11\ 8$

$\ldots$

$A_n = 2\ 3\ 5\ 1\ 7\ 4\ \ldots 2n+1\ 2n-2\ 2n+3\ 2n\ 2n+4\ 2n+5\ 2n+2$

...

Let X be the class whose basis consists of the following:

All permutations of the form $A_i \ominus A_j$ where $i, j \in \mathbb{Z}^+$,

31524, 41523, 41532,

31524, 35124, 35214,

25314, 41352,

31(54)2, (43)152, (32)514, 2(54)13.

(These brackets are inserted only to highlight the structure of these permutations. The reader already familiar with intervals and P-frames will notice that the bracketed terms are the only non-trivial proper intervals in these permutations, which have top P-frame 2413 or 3142.)

Every basis element of X is sum indecomposable and therefore X is sum complete. However any strongly indecomposable permutation involving a permutation of the form $A_i \ominus A_j$ involves at least one of the strongly indecomposable basis elements of X, of which there are finitely many. Thus the strong completion of X is finitely based.

■

From that example we also have that:

Proposition 55 *There exists an infinitely based class whose sum completion is infinitely based but whose skew completion is finitely based.*

Questions that we have not yet answered are as follows:

Question 56 If the strong completion of an infinitely based class is infinitely based does it follow that at least one of the sum and the skew completion of that class must be infinitely based?

Conjecture 57 *There exists an infinitely based class whose sum completion is infinitely based, whose skew completion is infinitely based but whose strong completion is finitely based.*

The Separable Permutations

Before abandoning the strong completion altogether we introduce the following set of permutations, as its regular appearance deserves. We use the name given in [2].

Definition 58 The set of *separable* permutations is the strong completion of 1.

Thus separable permutations can be decomposed by repeated applications of sum and skew decomposition into single terms.

We hesitate to attach a letter to it, although Hebrew does offer two, both little used in this field and both with a fitting pronunciation. The paper [2] gives various properties of separable permutations, with repect to the wreath product, completion and expansion. In [4] it is shown, amongst other things, that the class of separable permutations is partially well ordered.

2.2.6 Sum, Skew and Strong Decidability

Basis elements of the sum completion $X^{\oplus}$ of some class X consist of basis elements of X in which the sum components of X are bound together with increasing oscillating sequences. We will show that if one of these binding sequences has sufficient length then we may construct longer and shorter basis elements of X by *extending* or *contracting* that increasing oscillating sequence (an operation described in detail below). This permits us to determine whether or not the sum completion of X is finitely based:

If the basis of $X^{\oplus}$ has an element of some sufficient length then we may deduce that that permutation contains a "long" binding increasing oscillating subsequence. In that case we may gradually extend that subsequence to produce arbitrarily long basis elements, and thereby demonstrate that $X^{\oplus}$ is infinitely based.

By the ability to reduce the length of increasing oscillating sequences we also determine whether $X^{\oplus}$ is infinitely based. If it is, we can choose a "long" basis element and reduce the length of the increasing oscillating sequences that bind it together to produce a basis element whose length lies in some specified range. By checking that $X^{\oplus}$ has no basis elements whose length lies in that range we can demonstrate that $X^{\oplus}$ has no longer basis elements and is therefore finitely based. That range can be chosen to be sufficiently great that only an infinitely based $X^{\oplus}$ can have a basis element with length in that range. That completes the test.

By an extension of the same argument we will show that it is decidable whether the sum or skew sum of two finitely based class is finitely based. The size of the basis of the strong completion of a closed class is related to those of the sum and skew completion as related in the previous section.

Extending an Increasing Oscillating Sequence

It was shown in Proposition 34 that if a_i and a_j are terms of a sequence α and if $\alpha(A)$ is a minimal sum indecomposable sequence of α containing both those terms then $\alpha(A)$ is an increasing oscillating sequence. Using the relations L and R as used in the later Proposition 37 and the subsequent corollary we may deduce that if β and γ are sum indecomposable subsequences of α, a sequence, and if $\alpha(B)$ is a subsequence of α minimal subject to $\beta \cup \gamma \cup \alpha(B)$ being sum indecomposable then $\alpha(B)$ is increasing oscillating. Indeed if we assume that the terms of β lie to the left of and below those of γ and if for neatness sake we assume that $\beta \cup \gamma \cup \alpha(B)$ constitutes the entirety of α then we may go further and describe α as consisting of the following:

- Three disjoint subsequences β, δ and γ, where every term of β lies below and to the left of every term of δ and where every term of δ is below and to the left of every term of γ; and where δ, like β and γ, is sum indecomposable.

- Two terms, possibly identical, denoted d_1 and d_2 and having the property that d_1 lies either below and to the right or above and to the left of some term of β, and that d_2 satisfies the same with respect to γ, and that $\delta \cup \{d_1, d_2\}$ is an increasing oscillating sequence.

From now on let us suppose that α consists of nothing other than the terms of β, γ and $\alpha(D) = \delta \cup \{d_1, d_2\}$. This will permit us to concentrate on $\beta \cup \gamma \cup \alpha(D)$ and will give us a shorter notation for that sequence. We

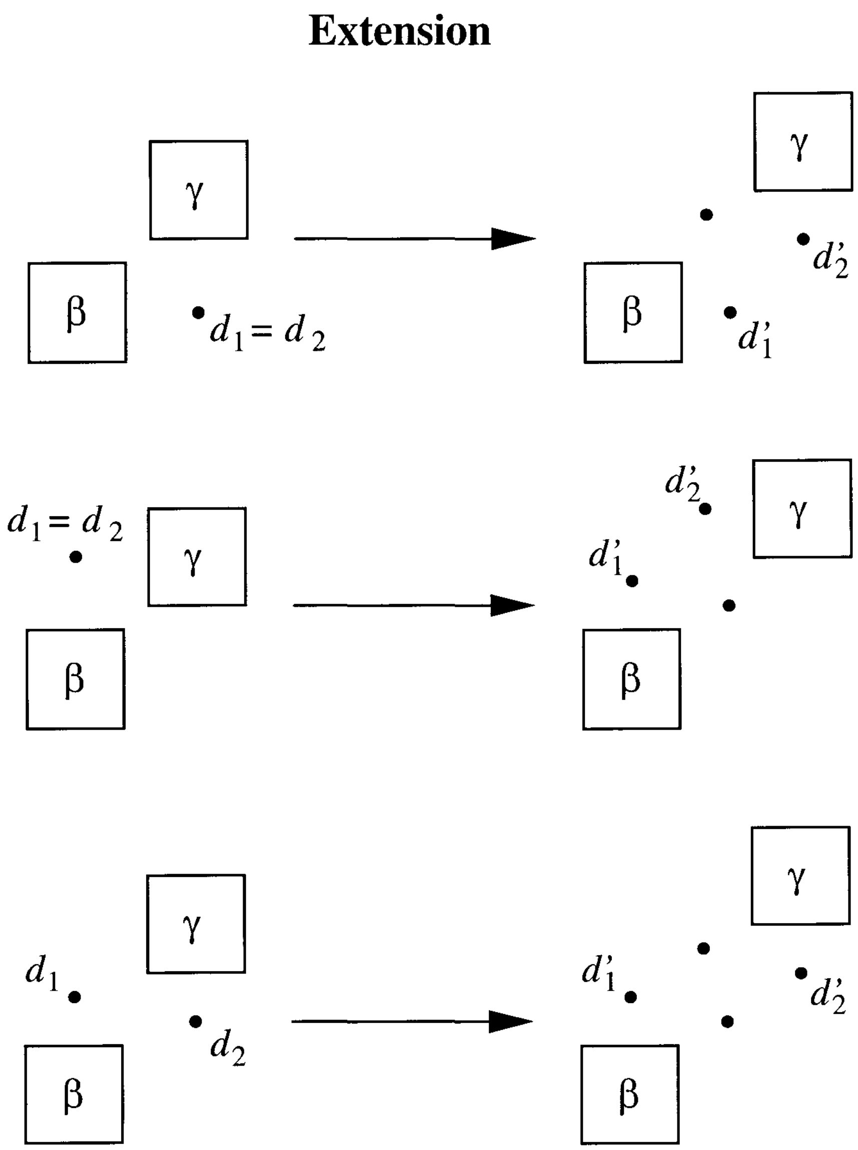


Figure 2.5:

will define the modification where α is *extended by two* as that permutation consisting of:

- Three disjoint subsequences β', δ' and γ', each of which lies to the left of and below the next. Thus $\beta' \cup \gamma' \cup \delta' = \beta'\delta'\gamma' \cong \beta' \oplus \delta' \oplus \gamma'$.

- Two terms denoted d_1' and d_2' and having the properties that $\beta' \cup \{d_1'\}$ is order isomorphic to $\beta \cup \{d_1\}$, with d_1' corresponding to d_1, an added restriction; and that $\gamma' \cup \{d_2'\}$ is similarly order isomorphic to $\gamma \cup \{d_2\}$ with d_2' corresponding to d_2; and that $\delta' \cup \{d_1', d_2'\}$ is an increasing oscillating sequence of length $|\delta \cup \{d_1, d_2\}| + 2$.

A few moments' consideration will convince the reader that this extension by two is well defined. Examples of extension can be found in Figures 2.5 and 2.6. We define *contraction by two* to be the inverse of extension by two.

Proposition 59 *Let α, β, γ, δ, d_1, d_2 be defined as above. Let σ be a subsequence of α. If at least one of the following conditions is satisfied then σ is also involved in the sequence obtained from α by extending the increasing oscillating sequence $\delta \cup \{d_1, d_2\}$ by two:*

1. *σ does not contain every element of $\delta \cup \{d_1, d_2\}$.*

2. *σ contains no element of β.*

3. *σ contains no element of γ.*

PROOF: A diagram, it is said, is worth more than a hundred words, and one is given in Figure 2.6, however we also provide the following:

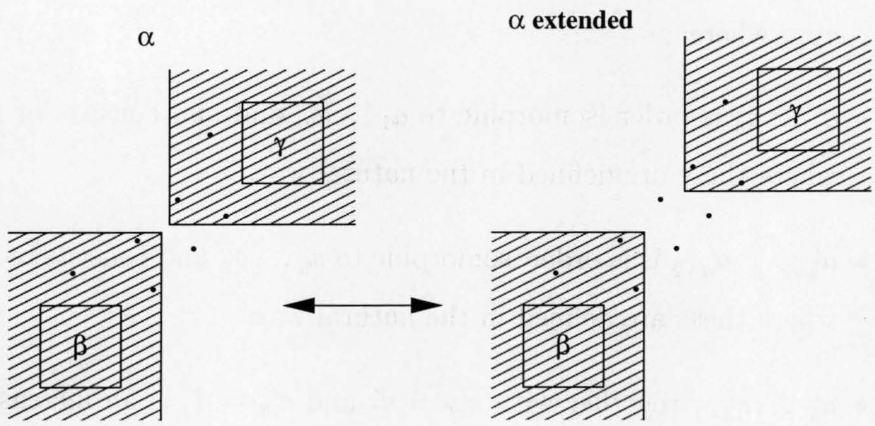


Figure 2.6: The shaded subsequences are order isomorphic. Thus if a subsequence of α does not contain some term of $\delta \cup \{d_1, d_2\}$ then that subsequence is also involved in α extended. Similarly if a subsequence of α extended does not contain some three terms of $\delta \cup \{d_1, d_2\}$ order isomorphic to either 312 or 231 then that subsequence is also involved in α.

A general sequential notation for α is difficult to establish because its expression in any standard system will change depending on whether d_1 and d_2 lie above and to the left or below and to the right of some term of β or γ respectively. Instead we will prove only the case where d_1 is above and to the left of some term of β and d_2 is below and to the right of some term of γ; the other cases are entirely analogous.

Let $\alpha = a_1 a_2 \dots a_n$. Let a_{df} and a_{dl} denote the terms d_1 and d_2. Let a_b denote the last term of β, and a_g the first term of γ. Note that as a_{df} $(= d_1)$ is the only term not in β preceding a_b we have that $\beta = (a_1 \dots a_b) \setminus \{a_{df}\}$, and similarly we have that $\gamma = (a_g \dots a_n) \setminus \{a_{dl}\}$. The terms of δ, if any, are $a_{b+1} \dots a_{g-1}$.

Extended by two, we will denote α by α'. The terms of α' are then

$a'_1 \dots a'_{n+2}$ where:

- $a'_1 \dots a'_b$ is order isomorphic to $a_1 \dots a_b$, and and consists of β' and d'_1 where these are defined in the natural way.
- $a'_{g+2} \dots a'_{n+2}$ is is order isomorphic to $a_g \dots a_n$ and consists of γ' and d'_2 where these are defined in the natural way.
- $a'_{b_1} \dots a'_{g+1}$ together with $a'_{df} = d'_1$ and $a'_{dl} = d'_2$ is an increasing oscillating sequence.

$$\begin{aligned}\alpha' &= \langle \underbrace{a_1, \dots, a_b,}_{\beta' \text{ and } d'_1} \underbrace{(b+3), b, (a_{b+1}+2), (a_{b+2}+2), \dots, (a_{g-1}+2),}_{\delta'} \\ &\quad \underbrace{(a_g+2), \dots, (a_n+2)}_{\gamma' \text{ and } d'_2} \rangle\end{aligned}$$

Now let σ be given, a subsequence of α, and suppose that σ contains no element of γ. As $\beta \cup \delta \cup \{d_1, d_2\}$, the smallest g terms of α, form a sequence identical to the sequence formed by the smallest g terms of α' we have that σ is also involved in α', as required. Similarly if σ contains no term in β then we have our desired result. Now suppose that σ does not contain some term a_{lost} of $\delta \cup \{d_1, d_2\}$, which is $a_{df} a_{b+1} a_{b+2} \dots a_{g-2} a_{g-1} a_{dl}$ in our new notation. Then:

Recall that if the k^{th} term is removed from an increasing oscillating sequence then that sequence becomes sum decomposable, and that the nature of that sum decomposition depends on whether the removed term is left maximal or right minimal, see Figure 2.4. Similarly here $\alpha \setminus \{a_{lost}\}$ is sum decomposable into the first $|a_{lost}|$ terms of α and the largest $n - 1 - a_{lost}$

terms, if α is right minimal. Note that the first $|a_{lost}|$ terms of α are order isomorphic and indeed equal to the first $|a_{lost}|$ terms of α', as demonstrated in Figure 2.4. The largest $n-1-a_{lost}$ terms of α are similarly order isomorphic to those largest of α', and we conclude that σ is therefore also involved in α'.

If a_{lost} is not right minimal then it must be left maximal, with similar results: $\alpha \setminus \{a_{lost}\}$ is sum decomposable into the smallest $|a_{lost}| - 2$ and the rightmost $n + 1 - |a_{lost}|$ terms of α, and the sequences that consist of these terms are order isomorphic to those consisting of the smallest $|a_{lost}| - 2$ and the rightmost $n+1-|a_{lost}|$ terms of α'. We are therefore done. In every case σ is also involved in α'. ■

Corollary 60 *If σ is a permutation involved in α and having length no greater than $|\delta \cup \{d_1, d_2\}|+1$ then σ is involved in α with $\delta \cup \{d_1, d_2\}$ extended by two.*

Proposition 61 *Let α, β, γ, δ, d_1, d_2 be defined as above. Let τ be a subsequence of α and let there be three logically consecutive terms d_7, d_8, d_9 of the increasing oscillating sequence $\delta \cup \{d_1, d_2\}$ not in τ. Then τ is involved in α contracted by two.*

PROOF: The length of α is n. As d_7, d_8, d_9 are logically consecutive then they either appear in α as the subsequence $d_8d_7d_9$ which is order isomorphic to 312, or as the subsequence $d_7d_9d_8$ which is order isomorphic to 231.

If d_7 is the $k + 2^{th}$ term of α then in the first case $\alpha \setminus \{d_7, d_8, d_9\}$ is sum decomposable into the first k and the greatest $n - k - 3$ terms of α, which form sequences order isomorphic to the first k and the greatest $n - k - 3$ terms of α'. In the second, $\alpha \setminus \{d_7, d_8, d_9\}$ is sum decomposable into the smallest k and the rightmost $n-k-3$ terms of α, order isomorphic to the smallest k and rightmost $n-k-3$ terms of α'.

In either case we have that τ is involved in α'.

■

Corollary 62 *Let τ be a permutation involved in α having length n and having m sum components. If $\delta \cup \{d_1, d_2\} \leq n + 2m + 3$ then τ is also involved in α contracted by two.*

Proposition 63 *Let X be a closed class having no basis element with length greater than n or with more than m sum components. Then if $X^{\oplus}$ has a basis element θ of length k where $k \geq (2(m-1)+n)(m-1)+n+1$ then $X^{\oplus}$ also has a basis element of length $k+2$.*

PROOF: If all basis elements of X are sum indecomposable then X is sum complete, therefore its sum completion does not contain a basis element of length $n+1$ or more. Thus this case is trivial, as is the case when X has no basis elements. Let us therefore assume that n is no less than one, and that m is no less than two.

Now, let θ be given. θ contains a subsequence β order isomorphic to a basis element of X and with sum components that we will denote $\beta_1 \ldots \beta_z$. If any term of θ not in β is removed then the result will be a sequence that is an element of $X^{\oplus}$; but as it still involves the basis element β of X it must be sum decomposable and not every β_i can lie in the same sum component. Thus we may choose a function f from the terms of $\theta \setminus \beta$ to the integers $\{1, 2, \ldots, (z-1)\}$ such that if $h_j \in \theta \setminus \beta$ then $\beta_{f(h_j)}$ and $\beta_{f(h_j)+1}$ lie in different sum components of $\theta \setminus \{h_j\}$.

As θ has at least $(2(m-1)+n)(m-1)+n+1$ terms of which at most n belong to β, and as $m-1$ is at least as great as $z-1$ it follows that there exists a number $i \in \{1,\ldots,(z-1)\}$ such that $f^{-1}(i)$ is at least $2(m-1)+n+1$. Let such i be given and let η denote all the terms $h_j \in \theta \setminus \beta$, including those in $f^{-1}(i)$, such that β_i and β_{i+1} lie in different sum components of $\theta \setminus \{h_j\}$.

η is a minimal set of terms such that $\beta_i \cup \beta_{i+1} \cup \eta$ is sum indecomposable, hence η is an increasing oscillating sequence. Furthermore η contains precisely one term that lies either above and to the left of some term of β_i or below and to the right of it, and one term that satisfies the same conditions with respect to β_{i+1}. We denote these two terms by d_1 and d_2 respectively. All remaining terms in η lie above and to the right of β_i, below and to the left of β_{i+1}. We denote these terms by δ. We will no longer use η, preferring to refer to d_1, d_2 and δ.

We will also condense our notation for the terms of θ not in either δ or $\{d_1, d_2\}$. Those terms form a sequence that is sum decomposable into two or more parts, with b_i and b_{i+1} lying in consecutive components. Let θ_{beg} denote those terms in the sum component of b_i and in preceding sum components, and let θ_{end} denote those other terms in the sum component of b_{i+1} and succeeding sum components. θ may be thought of accurately as the sequences θ_{beg} and θ_{end} bound together by an increasing oscillating sequence $\delta \cup \{d_1, d_2\}$, indeed the following description of θ will be familiar:

θ consists of:

- Three disjoint subsequences θ_{beg} δ and θ_{end}, where the terms of each lie above and to the right of those of its predecessor.

- Two terms, possibly identical, denoted d_1 and d_2 and having the prop-

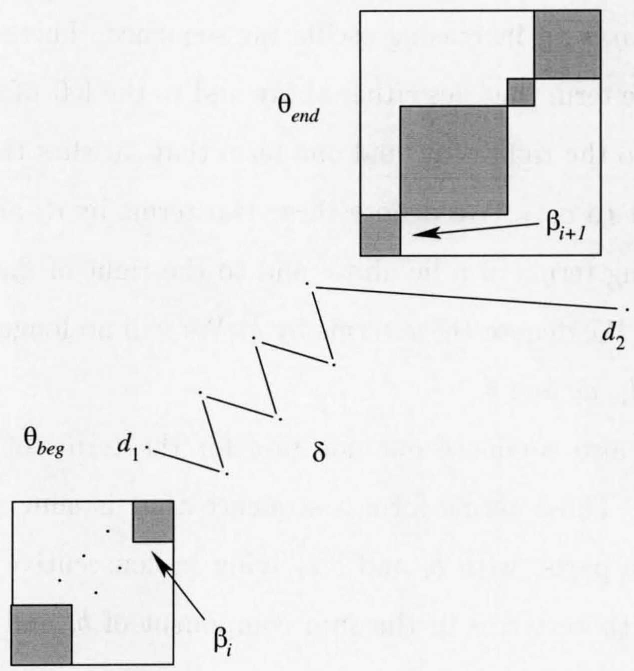


Figure 2.7: The permutation θ.

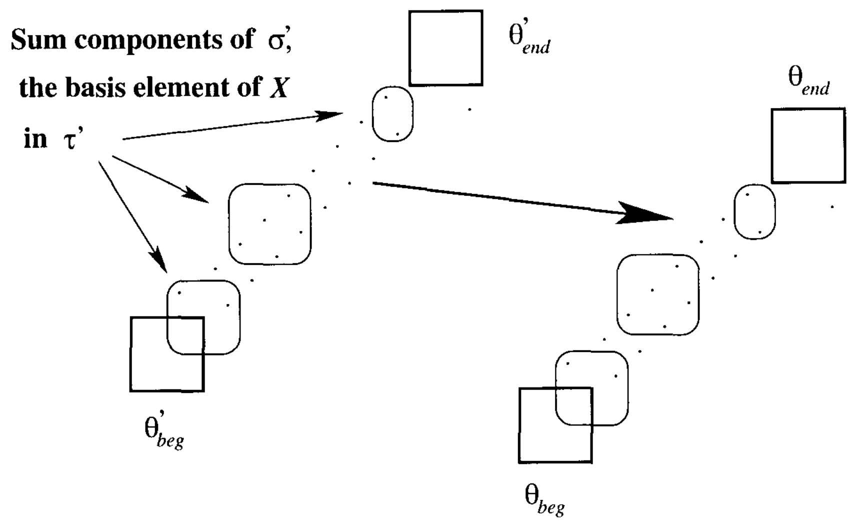


Figure 2.8: The situation where θ' has a subsequence τ' order isomorphic to a basis element of $X^{\oplus}$, and where τ' contains every term of $\delta \cup \{d'_1, d'_2\}$. We have that τ' consists of a basis element (with a limited number of terms) bound together with a limited number of increasing oscillating sequences. By the length of $\delta \cup \{d'_1, d'_2\}$ we have that one of these binding sequences has at least three terms in $\delta \cup \{d'_1, d'_2\}$, which permits us to find a subsequence of length $|\tau'| - 2$ in θ, but not in $X^{\oplus}$.

erty that d_1 lies either below and to the right or above and to the left of some term of θ_{beg}, and that d_2 satisfies the same with respect to θ_{end}, and that $\delta \cup \{d_1, d_2\}$ is an increasing oscillating sequence.

Let θ' be θ with δ extended by two. We claim that θ' is a basis element of the sum completion of X. For otherwise suppose that τ' is a basis element of $X^{\oplus}$ properly involved in θ'. We will obtain a contradiction.

If τ' does not contain every term of $\delta' \cup \{d_1', d_2'\}$ then τ' does not include both d_1' and d_2'. Suppose therefore that d_2' is not contained in τ' and let us suppose that d_2' lies below and to the right of some term of θ'_{end}. Then d_2 similarly lies below and to the right of some term of θ_{end} and if we denote that term by d_3 then we have that the subsequence $\theta' \setminus (\theta'_{end} \cup \{d_2'\})$ of θ', identical to $\theta'_{beg} \cup d_1' \cup \delta'$, is order isomorphic to $(\theta \setminus \theta_{end}) \cup \{d_3\} = \theta_{beg} \cup \delta \cup \{d_1, d_2, d_3\}$. Now if we claim that d_3 is not the only term of θ_{end} then we can obtain a contradiction as we would have that θ is properly involved in θ, and we do claim this: If θ_{end} consists of a single term then θ_{end} is order isomorphic to d_2, which also lies above and to the right of θ_{beg}. Thus $\theta \setminus \theta_{end} = \theta_{beg} \cup \delta \cup \{d_1, d_2\}$ involves a basis element of X and as it is also sum indecomposable it is not in the sum completion of X. This contradicts the notion that θ is a basis element of $X^{\oplus}$ and so our claim stands.

Thus, by various symmetries, we have that τ' contains all $\delta' \cup \{d_1', d_2'\}$. But now we dissect τ': It consists of a basis element σ of X bound together with at most $m-1$ increasing oscillating sequences. Specifically if we examine $\delta' \cup \{d_1', d_2'\}$ then at most n of the terms in here are contained in τ. The rest, which number at least $2(m+1)+1$, are split amongst the binding sequences. This implies that there is one binding sequence that has at least three terms in $\delta' \cup \{d_1', d_2'\}$. Thus we can contract τ' by two at that binding sequence to obtain a permutation τ that has the following properties:

- τ is sum indecomposable and involves a basis element of X, thus it is not in $X^{\oplus}$.
- τ is involved in θ.
- The length of τ is two less than the length of τ'. Thus τ is properly involved in θ if and only if τ' is properly involved in θ'.

This completes the proof. ■

Proposition 64 *Let X be a closed class having no basis element with length greater than $n \geq 2$ or with more than m sum components. Then if $X^{\oplus}$ has a basis element θ of length k where $k \geq n + (m-1)n + 1 = mn + 1$ then $X^{\oplus}$ also has a basis element of length $k - 2$.*

PROOF:

Let θ_{beg}, θ_{end}, d_1, d_2 and δ be defined as in the previous proof, except that here we require that the binding increasing oscillating sequence $\delta \cup \{d_1, d_2\}$ must have length at least $n + 1$ instead of $2(m-1) + n + 1$. Let θ' be the permutation obtained by *contracting* θ at the oscillating sequence $\delta \cup \{d_1, d_2\}$. Let θ'_{beg}, θ'_{end}, d_1', d_2' and δ' be parts of θ' corresponding to the parts of θ. Thus θ' consists of θ'_{beg} and θ'_{end} bound together by the increasing oscillating sequence $\delta \cup \{d_1, d_2\}$, which has length at least $n - 1$.

If β is defined as in the previous proof then β is a basis element of X involved in θ, indeed it is involved in $\theta_{beg} \cup \theta_{end}$ which is order isomorphic to $\theta'_{beg} \cup \theta'_{end}$. Thus θ' is not in X and as θ' is sum indecomposable neither is it in the sum completion $X^{\oplus}$. Let τ' be a subsequence of θ' order isomorphic to a basis element of $X^{\oplus}$.

τ' consists of a subsequence σ' order isomorphic to a basis element of X and sufficient other terms that τ' is sum indecomposable. Let $\sigma'_1 \dots \sigma'_y$ denote the sum components of σ', the sum components that are bound together by the remaining terms of τ' into a single sum indecomposable sequence. We claim that σ'_1 and θ'_{beg} have at least one term in common, and that the same holds for σ'_y and θ'_{end}:

Note that $\theta' \backslash \theta'_{end}$ is involved in $\theta \backslash \theta_{end}$, a proper and sum indecomposable subsequence of θ, which yields a contradiction if $\theta' \setminus \theta'_{end}$ involves the entirety of a basis element of X. To see that $\theta' \setminus \theta'_{end}$ is involved in $\theta \setminus \theta_{end}$ we must consider the two cases where d'_2 lies above and to the left of some term of θ'_{end}, or below and to the right. If it lies above and to the left then $\theta' \setminus \theta'_{end}$ is a prefix of θ'. That prefix is order isomorphic to the prefix of θ that has the same length. The other case is similar, which demonstrates that σ'_y and θ'_{end} share a term, and by symmetry we have the result for σ'_1 and θ'_{beg}.

We can conclude from this two things: That, as τ' is sum indecomposable, τ' contains every term of $\delta' \cup \{d'_1, d'_2\}$, and that as σ' has no more than $n-2$ terms that are also in the no less than $n-1$ terms of $\delta' \cup \{d'_1, d'_2\}$, there exists a term of τ' in $\delta' \cup \{d'_1, d'_2\}$ but not in σ'. This implies that we may take τ' and extend it by two at the binding sequence $\delta' \cup \{d'_1, d'_2\}$ to obtain a permutation θ that has the properties that:

- τ involves σ', a basis element of X. Furthermore τ is sum indecomposable which implies that τ is not in $X^{\oplus}$.

- τ is involved in θ, and as the latter is a basis element of $X^{\oplus}$ we have that τ is equal to θ.

- τ has length $|\tau'| + 2$. As the length of θ is two greater than that of θ' this implies that τ' is equal to θ'.

Recall that τ' is a basis element of $X^{\oplus}$. The conclusion that θ' is equal to τ' completes this proof. ■

Theorem 65 *It is decidable whether the sum completion of a given finitely based closed class is finitely based.*

PROOF: This is a corollary of the last two Propositions. ■

Proposition 66 *Let X and Y be closed classes. Let X and Y have no basis elements of length greater than $n \geq 2$ or $s \geq 2$ respectively, and none with more than $m \geq 1$ or $r \geq 1$ sum components. If $X \oplus Y$ has a basis element of length k where $k \geq (2(m+r-1)+n+s)(m+r-1)+n+s+1$ then it has a basis element of length $k+2$. If it has a basis element of length l where $l \geq (m+r)(n+s)+1$ then it also has a basis element of length $l-2$.*

PROOF: Recall the description of a basis element of the sum of two classes. Such a basis element consists of the merge of a basis element of X and a basis element of Y, and a minimal other set of terms subject to certain terms lying in the same sum component of the eventual permutation. Therefore to perform arguments essentially identical to those of Propositions 63 and 64 we need only count the maximum number of terms and the maximum number of sum components that can be in a merge of a basis element of X and of Y. These are easily seen to be $n+s$ and $m-r$ respectively. This gives our desired result.

It will be noted that requiring terms to make only part of a permutation sum indecomposable is rather an advantage than a disadvantage. Q.E.D. ■

Theorem 67 *Let X and Y be finitely based closed classes. It is decidable whether the sum $X \oplus Y$ is finitely based.*

PROOF: This follows from Proposition 66. ■

2.3 Expansion and the Wreath Product

Definition 68 Let X be a set of permutations. Then let the *expansion* of X be the set Y defined as follows:

- If $\alpha \in X$ then $\alpha \in Y$.
- If $\alpha \in Y$ and $\beta = b_1, b_2, \ldots, b_n$ is a permutation such that

  $$b_1, \ldots, b_i, b_{i+2}, b_{i+3}, \ldots, b_n \cong \alpha$$

  for some i and $b_{i+1} = b_i + 1$ then $\beta \in Y$.

  β can be regarded as the permutation obtained from α by replacing the ith term of α with an increasing pair of terms. Repeated applications of this operation are equivalent to replacing the ith term with an arbitrarily long increasing sequence.

Example 69 The expansion of $R = \mathcal{A}(12) = \{1, 21, 321, \ldots\}$ is $\mathcal{A}(132, 213) = Sub(I \ominus I \ominus I \ominus \ldots)$, the skew completion of I.

The *skew expansion* of a class X is defined in an entirely analogous way. Whereas in the direct expansion terms may be replaced with increasing sequences, in the skew expansion terms may be replaced with decreasing sequences. In the *strong expansion* terms of X may be replaced with any separable permutation, as these are the permutations that can be generated by replacing given terms by increasing or decreasing sequences.

Definition 70 Let X be a set of permutations. Then let the *skew expansion* of X be the set Y defined as follows:

- If $\alpha \in X$ then $\alpha \in Y$.
- If $\alpha \in Y$ and $\beta = b_1, b_2, \ldots, b_n$ is a permutation such that $b_1, \ldots, b_i, b_{i+2}, b_{i+3}, \ldots, b_n \cong \alpha$ for some i and $b_{i+1} = b_i - 1$ then $\beta \in Y$.

Definition 71 Let X be a set of permutations. Then let the *strong expansion* of X be the set Y defined as follows:

- If $\alpha \in X$ then $\alpha \in Y$.
- If $\alpha \in Y$ and $\beta = b_1, b_2, \ldots, b_n$ is a permutation such that $b_1, \ldots, b_i, b_{i+2}, b_{i+3}, \ldots, b_n \cong \alpha$ and $b_1, \ldots, b_{i-1}, b_{i+1}, b_{i+2}, \ldots, b_n \cong \alpha$ for some i then $\beta \in Y$.

  Note: If β satisfies the above then $b_i = b_{i+1} \pm 1$.

The *wreath product* is a generalisation of expansion. Instead of replacing terms with increasing or decreasing pairs we permit them to be replaced with a wider range of possible *intervals*:

Definition 72 Let $\alpha = a_1, \ldots, a_n \in S$ be any permutation. An *interval* of α is a contiguous subsequence of α. That is, a subsequence of the form $a_i a_{i+1} \ldots a_{j-1} a_j$ where there does not exist a term a_k with $k \notin [i, j]$ such that a_k is greater than the least term of, and less than the greatest term of, $a_i, \ldots a_j$ is called an *interval.* The interval is said to be proper if it does not constitute the whole of α, and is said to be trivial if it consists of a single term.

Definition 73 Let X and Y be sets of permutations. The *wreath product* $X \wr Y$ is the set of all permutations $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$ where:

- Each subsequence α_i is an interval.
- Each α_i is order isomorphic to an element of Y.
- If a_i is a term of α_i then $a_1 a_2 \ldots a_n$ is order isomorphic to an element of X. Note that, because of the first condition, the order isomorphism of $a_1 a_2 \ldots a_n$ is independent of the choice of each $a_i \in \alpha_i$.

If we wish to describe an element of $X \wr Y$ precisely then we may give a permutation γ in X and a list of permutations in Y that are to replace the terms of γ. For instance $2413 \wr (1, 12, 123, 321)$ may be used to denote 489123765.

Direct, skew and strong expansion, and the wreath product are introduced in [2]. We will see much more of the wreath product. It is one of the two most important and powerful constructions that exist, on a par with sum (and skew sum).

A special case of the wreath product, namely the profile class, appears in [1] and we will find it useful in describing some atomic classes listed in the Bibliothek.

Definition 74 Let γ be any permutation. Then the *profile class* $Prof(\gamma)$ is the wreath product $Sub(\gamma) \wr I$, also known as the expansion of $Sub(\gamma)$.

Given a set of permutations we can construct another by allowing any term to be replaced with an arbitrarily long increasing sequence by taking the expansion of the set or by writing the new set as a profile class. However, sometimes we may wish to allow only some terms to be replaced by arbitrarily long increasing sequences whilst restricting the length of increasing sequence that other terms may be replaced with. In this case we use the notation of profile classes but superscript terms by the length of sequence they are limited to. For instance the profile class $Prof(2^1\ 4^1\ 1\ 3)$ describes the set of all permutations of the form $i, n, 1, 2, 3, \ldots, i-1, i+1, i+2, \ldots, n-1$ and their subpermutations. This is a noteworthy example because $Prof(2^1\ 4^1\ 1\ 3)$ cannot be written as the sum or skew sum of lesser classes, our more usual deconstruction method. For comparison the permutation 4213 is separable and therefore $Prof(4\ 2^3\ 1\ 3^1)$ is expressible as $Sub(I \ominus ((123 \ominus I) \oplus 1))$. It may be seen why even in simpler cases the profile notation may be clearer.

There are also cases that may argue for a notation in which superscripts may indicate that some particular term may be replaced with any element of some particular class. For instance if we denote $\mathcal{A}(132, 123)$ by D then $\mathcal{A}(132, 4123)$, which is represented as a diagram in the Bibliothek, might be expressed as $Prof(1^I\ -1^D\ 2^I\ -2^D\ 3^I\ -3^D\ \ldots)$. As I merely stands for the set of increasing sequences it could be omitted in this example.

2.3.1 Wreath Structure and P

We develop a decomposition of permutations sufficiently fundamental and unifying almost to warrant the name of "Fundamental theorem of decomposition". We divide up the terms of any permutation into disjoint intervals and by giving the relative positions of the intervals can define the permutation exactly. We choose this decomposition to make it unique for every permutation and our choice has discrete properties.

First note that sum components and skew components are special cases of intervals, and so:

- If a permutation α is sum decomposable then it is uniquely expressible in the form:

  $$\alpha = I_n \wr (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

  where $\alpha_1 \ldots \alpha_n$ are sum indecomposable permutations (isomorpic to sum components). Respectively:

- If α is skew decomposable then it is uniquely expressible as:

  $$\alpha = R_n \wr (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

  where $\alpha_1 \ldots \alpha_n$ are skew indecomposable.

In the case of permutations that are non-trivial and strongly indecomposable we find that maximal proper intervals, that is maximal intervals that are not equal to the entire permutation, are disjoint and partition the permutation.

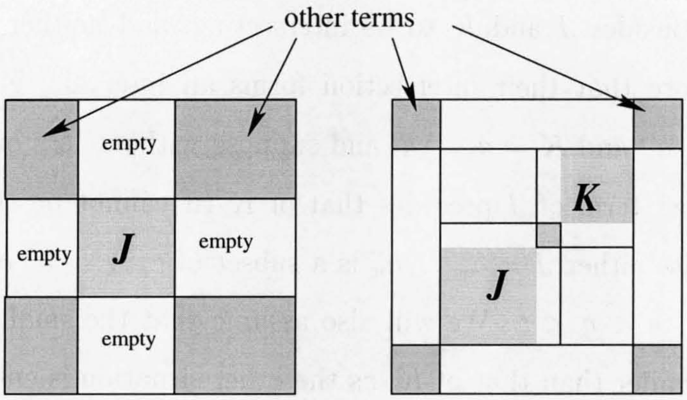


Figure 2.9: On the left, the definition of an interval, on the right the pattern formed by two intervals partially intersecting.

Lemma 75 *Let α be a permutation of length greater than one that is neither sum nor skew decomposable. Let J and K be distinct maximal proper intervals of α. Then:*

1. *If J and K are not disjoint then their union is an interval and therefore constitutes the entirety of α.*

2. *If J and K are not disjoint then their union is either sum or skew decomposable into three sum or skew components.*

3. *Therefore J and K are disjoint.*

PROOF: See Figure 2.9. It follows straight from the definition that if two intervals intersect then their union is also an interval. Here neither J nor K is a subset of the other (they are distinct and maximal) therefore their union is strictly larger than either J or K. By maximality the union, being an iterval, cannot be proper, therefore the union is equal to all α.

Now consider J and K to be intersecting, and neither a subset of the other. Note that their intersection forms an interval. For suppose that $J = a_m \ldots a_n$ and $K = a_s \ldots a_t$ and suppose without loss of generality that the leftmost term of J precedes that of K (it cannot be the same as that of K or else either $J = a_m \ldots a_n$ is a subset of $a_s \ldots a_t = K$ or vice versa). Thus $m < s < n < t$. We will also assume that the smallest term of J is strictly smaller than that of K, as the other situation is entirely analogous. It should now be evident that the intersection $a_s \ldots a_n$ contains all the terms no less than the smallest of K and no greater than the largest of J, making it an interval.

This also demonstrates that in this case $J \cup K$ is equal to $(J \setminus K) \oplus (J \cap K) \oplus (K \setminus J)$, and in the analogous case we have that $J \cup K = (J \setminus K) \ominus (J \cap K) \ominus (K \setminus J)$. This makes α, which is equal to this union, either sum or skew decomposable, a contradiction. Q.E.D. ■

Now we need permutations that can describe in lieu of I or R how these disjoint intervals are arranged. Specifically let us write $\alpha = \beta \wr (\alpha_1, \ldots, \alpha_n)$ where $\alpha_1 \ldots \alpha_n$ now correspond to the maximal proper intervals of α. We wish to define the set of possible values for β (β is fixed for any α but different α may yield different β). Clearly β cannot itself have a proper non-trivial interval or else this would contradict the maximality of at least two elements of $\alpha_1 \ldots \alpha_n$. In fact it turns out that the best locus that we can choose for β consists of the following set:

Definition 76 Let P be the set of all permutations having no proper non-trivial interval, and let P not include the permutations 1, $1\,2$, $2\,1$.

Example 77 The elements of P of length four and five are:

- 2413, 3142
- 25314, 35142, 31524, 42513, 24153, 41352

P has no elements of length three.

This set P yields:

Theorem 78 *Let $\alpha \in S$ be any permutation. Then α is precisely one of the following:*

1. *Trivial or empty.*
2. *Sum decomposable and expressible as $I_n \wr (a_1, a_2, \ldots, a_n)$, where $a_1 \ldots a_n$ are the sum components of α.*
3. *Skew decomposable and expressible as $R_n \wr (a_1, a_2, \ldots, a_n)$, where $a_1 \ldots a_n$ are the sum components of α.*
4. *Uniquely expressible as $\nu \wr (a_1, a_2, \ldots, a_n)$, where ν is an element of P and where $a_1, \ldots, a_n$ are non-empty maximal proper intervals of α.*

It is evident that every permutation falls into at least one of the above four categories. It remains only to show that no permutation is in more than one of those categories, and the only interesting part is to show that the states of (2) and (4) or (3) and (4) cannot coexist. That is proved by Lemma 75.

Definition 79 Let α be any permutation. The *top RIP frame* of α is a permutation equal to:

- α itself if α is either trivial or empty.
- $I_n = 123 \ldots n$ if α is sum decomposable with n sum components.
- $R_n = n\ n-1 \ldots 3\ 2\ 1$ if α is skew decomposable with n skew components.
- The element ν of P such that $\alpha = \nu \wr (a_1, a_2, \ldots, a_n)$ for some non-empty permutations $a_1, \ldots, a_n$, if α is non-trivial, non-empty and strongly indecomposable.

In the case of a strongly indecomposable permutation the *second level RIP-frames* are the top RIP-frames of the permutation's maximal proper intervals. In the case of a sum or skew decomposable permutation the *second level RIP-frames* are the top RIP-frames of the sum or skew components respectively. A trivial or empty permutation does not have a second or lower level RIP-frame. The third level and consecutive RIP frames are defined in a similar manner. The *bottom RIP-frames* are the lowest non-trivial entries of this tree. Bottom level frames need not all lie on the same level, as some branches may become extinct before others. These bottom frames correspond precisely to the intervals of α maximal subject to being elements of either I, R or P.

We call those permutations whose top RIP frames are elements of P *P-framed permutations*, and *pure P-framed permutations* if *all* RIP frames are elements of P. These permutations are relevant when examining partially well ordered classes because of the following results:

Proposition 80 *Let α and β be P-framed permutations. Then α is involved in β if and only if either:*

1. *α is involved in a maximal proper interval of β, or:*

2. *The top P-frame of α is involved in the top P-frame of β and the top intervals of α are involved in corresponding top intervals of β. That is to say:*

   If the top P-frames of α and β are $P(\alpha) = g_1 \dots g_m$ and $P(\beta) = h_1 \dots h_n$ respectively and if $\alpha = P(\alpha) \wr (\alpha_1, \alpha_2, \dots, \alpha_m)$ and $\beta = Q(\beta) \wr (\beta_1, \beta_2, \dots, \beta_n)$ then there exists an increasing injective map $j : \mathbb{Z}_m^+ \to \mathbb{Z}_n^+$ such that the subsequence $h_{j(1)} h_{j(2)} \dots h_{j(m)}$ of $P(\beta)$ is order isomorphic to $P(\alpha)$ and such that for each $j(k)$ the interval $\beta_{j(k)}$ involves α_k.

In the case of pure P-frame permutations this has a pleasing inductive quality. The above result also holds if β is not required to be P-framed.

2.3.2 Wreath Product Bases

Proposition 81 *Let X be a closed class with basis B. Then the basis elements of the completion of X under wreath product are precisely those permutations minimal subject to: 1) involving an element of B and: 2) having no proper non-trivial interval. Thus every basis element of the wreath closure of X is either 1, 12, 21 or an element of P.*

This follows directly from the definitions of wreath and basis and is sufficiently evident that we omit the proof.

A full description of the basis of one closed class wreathed with another is not given here, as it would be long and not very productive. However two special cases are examined in detail and we give a result that illustrates the essence of the general case.

Proposition 82 *Let B be a set of permutations every one of which is either 12, 21 or an element of P. Let D be any set of permutations. Then the basis of $\mathcal{A}(B) \wr \mathcal{A}(D)$ is the set of permutations minimal subject to involving an element of D and to having a top RIP-frame that involves an element of B.*

PROOF: By its basis $\mathcal{A}(B)$ is a wreath complete class. Thus if the top RIP-frame of some given permutation does not involve an element of B, then the permutation lies in $\mathcal{A}(B) \wr \mathcal{A}(D)$ if and only if the maximal proper intervals of the permutation do. ∎

(This last result is especially interesting if both $\mathcal{A}(B)$ and $\mathcal{A}(D)$ are wreath complete.)

Proposition 83 *Let ρ and μ be elements of P and let α be the permutation obtained by replacing some term j of ρ with the permutation μ. Let Y be any closed set. Then the basis of $\mathcal{A}(\alpha) \wr Y$ is the set of permutations λ minimal subject to satisfying at least one of the following conditions:*

- *λ involves a basis element of Y, is a P-framed permutation and has a top P-frame that involves α.*
- *λ is a permutation that may be obtained by replacing the term j of ρ with some permutation ν that is a basis element of $\mathcal{A}(\mu) \wr Y$.*

PROOF: Let λ be any permutation.

If λ is described by the first condition in the Proposition then λ is not in $\mathcal{A}(\alpha) \wr Y$.

Let us therefore assume that the top P-frame of λ does not involve the entirety of α. If the top P-frame does not involve ρ then λ is in $\mathcal{A}(\alpha) \wr Y$ if and only if every maximal proper interval of λ is. Thus if λ is to be a basis element its top P-frame must involve ρ.

Now suppose that for every subsequence of the top P-frame of λ order isomorphic to ρ, the λ interval corresponding to the term j of ρ is contained in $\mathcal{A}(\mu) \wr Y$. Then λ is in $\mathcal{A}(\alpha) \wr Y$ if and only if the remaining maximal intervals of λ are all order isomorphic to elements of $\mathcal{A}(\alpha) \wr Y$. If λ is to be a basis element this implies that λ must involve a permutation satisfying the second condition of the Proposition. Since every permutation satisfying that second condition is not an element of the wreath, we have that λ itself satisfies that condition. Q.E.D. ∎

Proposition 84 *Let α be any permutation in S^+, the set of all non-empty permutations, and let Y be any set of permutations, closed or otherwise, containing the trivial permutation 1. Then there exists a unique minimal permutation $f(\alpha)$ such that $\alpha \in \{f(\alpha)\} \wr Y$. That permutation is given by the following inductive mechanism:*

- *If $\alpha \in Y$ then $f(\alpha) = 1$.*
- *If $\alpha \notin Y$ and α has a top P-frame then let $\alpha = \rho \wr (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where ρ is an element of P of length n and where $\alpha_1, \ldots, \alpha_n$ are the maximal proper intervals of α.*

  Then $f(\alpha) = \rho \wr (f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n))$.

- *If $\alpha \notin Y$ and α is sum decomposable then let $\alpha = \alpha_1 \oplus \alpha_2 \oplus \ldots \oplus \alpha_n$ where $\alpha_1 \ldots \alpha_n$ are the sum components of α.*

  Let $\alpha_{j(1)}, \alpha_{j(2)}, \ldots, \alpha_{j(z)}$ be the sum components of α not in Y. The components $\alpha_{j(k)}$ partition the remaining sum components of α into $z+1$ categories, so that we may write $\alpha = \eta_1 \oplus \alpha_{j(1)} \oplus \eta_2 \oplus \alpha_{j(2)} \oplus \ldots \oplus \alpha_{j(z)} \oplus \eta_{z+1}$. For each η_k let $g(k)$ be the smallest non-negative integer such that $\eta_k \in I_{g(k)} \wr Y$. If η_k is the empty permutation then $g(k)$ may equal zero.

  Then $f(\alpha) = I_{g(1)} \oplus f(\alpha_{j(1)}) \oplus I_{g(2)} \oplus \ldots \oplus f(\alpha_{j(z)}) \oplus I_{g(z+1)}$.

- *If $\alpha \notin Y$ and α is skew decomposable then define $f(\alpha)$ in a manner analogous to that where α is sum decomposable.*

We claim that the method of generating $f(\alpha)$ given in this proposition is the best proof that may be given and therefore we do not add another. We only note that if X and Y are closed classes and the function f is defined as above then a permutation β is in $X \wr Y$ if and only if $f(\beta)$ is in X.

Conjecture 85 *Let X be a closed class having finite basis $\mathcal{B}(X)$. Let m be the length of the longest increasing interval of any element of $\mathcal{B}(X)$ and let n be the length of the longest decreasing interval of any element of $\mathcal{B}(X)$.*

Let Y be a finitely based closed class such that $Y\oplus^i = \underbrace{Y \oplus Y \oplus \ldots \oplus Y}_{i \text{ times}}$ is finitely based for all $i \leq m$ and such that $Y\ominus^j$ is finitely based for all $j \leq n$.

Then $X \wr Y$ is finitely based.

We believe that this conjecture is readily provable but that the proof will be long, not highly original and at present not necessary, therefore we have omitted it. We further believe that an upper bound for the length of basis elements of $X \wr Y$ can readily be calculated, based on the lengths of basis elements of X and Y.

We complete this section with a supplement to [2]. We give an example of a class defined by a single basis element whose wreath completion is infinitely based:

Proposition 86 *There exists a finitely based closed class whose completion under wreath is infinitely based.*

PROOF: Let $\alpha = 41352$. Let $X = \mathcal{A}(\alpha \oplus \alpha)$. Then the following are all basis elements of $X^{\wr}$:

$$\delta_1 = \underbrace{4\ 1\ 3\ 6\ 2}_{\cong\alpha}\ 8\ 5\ \underbrace{11\ 7\ 10\ 12\ 9}_{\cong\alpha}$$

$$\delta_2 = \underbrace{4\ 1\ 3\ 6\ 2}_{\cong\alpha}\ 8\ 5\ 10\ 7\ \underbrace{13\ 9\ 12\ 14\ 11}_{\cong\alpha}$$

$$\delta_3 = \underbrace{4\ 1\ 3\ 6\ 2}_{\cong\alpha}\ 8\ 5\ 10\ 7\ 12\ 9\ \underbrace{15\ 11\ 14\ 16\ 13}_{\cong\alpha}$$

... *et cetera.*

The above are also all elements of the basis of the sum completion of X. A sample of an infinite set of basis elements of the wreath completion but not the sum completion of X are shown in Figure 2.10 (This infinite set is generated in a way analogous to that of the antichain W.)

■

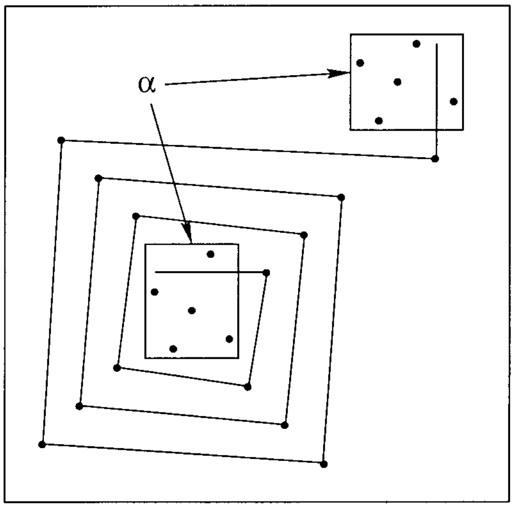


Figure 2.10: Note that the only subsequences of this permutation order isomorphic to 41352 are those in the two copies of α shown. Note also that this permutation is an element of P, having no proper non-trivial intervals, however if any term is removed, barring perhaps one ocurring in a copy of α, then the resulting sequence is not an element of P. By increasing the number of loops in the spiral we can construct arbitrarily long permutations of this form.

2.4 Juxtaposition

Definition 87 Let α and β be two permutations of length m and n respectively. Then a permutation γ is said to be a *juxtaposition* of α and β if γ has $m+n$ terms, the first m being order isomorphic to α and the last n order isomorphic to β.

Example 88 The permutation 5164237 is a juxtaposition of 3142 and 123. Indeed $5164 \cong 3142$ and $237 \cong 123$.

Note: There are $\binom{m+n}{n}$ distinct juxtapositions of any two permutations of length m and n respectively. With juxtapositions being so numerous there are abundant examples of hypotheses disproved by counterexamples.

The definition of juxtaposition extends to sets in the natural way:

Definition 89 If α and β are permutations then $\alpha\, juxt\, \beta$ is the set of all juxtapositions of α and β. Similarly if U and V are sets of permutations then $U\, juxt\, V$ is the set of all juxtapositions of permutations of U and V.

2.4.1 Basis of Juxtaposition

Mike Atkinson showed in *Restricted Permutations* [1] that if U and V are finitely based classes then their juxtaposition is finitely based. Indeed the basis elements of U *juxt* V are the minimal permutations of the form $\gamma\delta$, where γ is a sequence order isomorphic to an element of the basis of U and δ likewise for V, or of the form $\gamma g \delta$ where g is a single term, γg is a sequence order isomorphic to a basis element of U and where $g\delta$ is order isomorphic to a basis element of V. Thus the basis elements have length no greater than the sum of the lengths of the longest basis element of U and the longest basis element of V. The basis elements can be generated easily and furthermore since all permutations of the latter form are basis elements of U *juxt* V we have that if either of U, V or both is infinitely based then so is U *juxt* V.

2.4.2 A Conjecture

In this section we give counter examples to a structure hypothesis involving juxtaposition of atomic classes.

Conjecture 90 *An atomic class is expressible as a juxtaposition of two closed classes if and only if it is expressible as $\mathcal{B}(A, B, \pi)$ where the order types of A and B are 2ω and ω respectively.*[1]

It seems that the conjecture is not true in either direction.

Proposition 91 *Let* $A = \{1, 2, 3, \ldots, \omega+1, \omega+2, \ldots\}$, $B = \{1, 2, 3, \ldots\}$,

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \ldots & \omega+1 & \omega+2 & \omega+3 & \omega+4 & \omega+5 & \ldots \\ 2 & 4 & 6 & \ldots & 3 & 1 & 5 & 7 & 9 & \ldots \end{pmatrix}.$$

Then $X = \mathcal{B}(A, B, \pi)$ is not expressible as the juxtaposition of two non-empty closed classes.

Proof: Suppose that X is expressible as the juxtaposition of two non-empty closed classes C and D. Note that $321 \in X$ and that therefore either $21 \in C$ or $21 \in D$ or both. We consider the following four cases:

[1] ω is a notation used to describe the ordering of the natural numbers by size. The ordering $-\omega\omega$ is that of all integers, also by size. Similarly $2\omega = \omega\omega$ describes the ordering of, for instance, all the elements of the set $(\{-1 - 1/p | p \in \mathbb{N}\} \cup \{-1/q | q \in \mathbb{N}\})$.

1. Suppose that $21 \in C$ and D contains no permutation of length 2. Then as $2134 \in X$ we have that $213 \in C$. Thus $2143 \in C\, juxt\, D$ but $2143 \notin X$. Reductio ad absurdum.

2. Suppose that $21 \in C$ and D contains a permutation of length 2. Then either $4312 \in CD$ or $4321 \in CD$ or both, but neither is an element of X. Reductio ad absurdum.

3. Suppose that C is finite, all the elements of C are increasing, and $21 \in D$. Then as $(3, 4, 5, \ldots, n, 2, 1) \in X$ for all $n > 2$, and specifically for all $n > 2 + |C|$, we have $321 \in D$. Thus $4321 \in CD$ but $4321 \notin X$. Reductio ad absurdum.

4. Suppose that C is infinite, all the elements of C are increasing and $21 \in D$. Then, $12 \in C$ and $1432 \in CD$ but $1432 \notin X$. Reductio ad absurdum. ∎

Remark 92 Note that the above proof holds in the more general situation where π satisfies:

- π restricted to $\{1, 2, 3, \ldots\}$ is increasing.
- π restricted to $\{\omega + 3, \omega + 4, \ldots\}$ is increasing.
- $\pi(\omega + 2) < \pi(\omega + 1) < \pi(\omega + 3)$

The other direction of the conjecture does not seem plausible unless C and D are representable as $\mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$. Slightly more surprisingly it is not true even with this added restriction.

Proposition 93 *Not every juxtaposition has the join property. Not even the juxtaposition of atomic classes is necessarily atomic.*

PROOF: Let $C, D = \{\emptyset, (1)\}$. Then $12 \in CD$ and $21 \in CD$ but no element of CD contains both of these as subpermutations.

■

Remark 94 If D is the set of all permutations involved in 2 1 3 4 5 ... then D juxtaposed with itself provides a counterexample where the juxtaposed classes are infinite.

Proposition 95 *Let C be the set of all finite increasing permutations and D be the set of all finite permutations involved in* $2\ 1\ 3\ 4\ 5\ \ldots\ n-1\ \ldots$. *Then C juxtaposed with D has the join property.*

This last proposition is sufficiently self evident that we leave readers to convince themselves of its truth.

Proposition 96 *Let C and D be defined as in the previous proposition. Then the atomic class X that is the juxtaposition of C and D is not expressible in the form $\mathcal{B}(A, B, \pi)$ where A, B have order types 2ω and ω respectively.*

PROOF: Note that $C\,\mathit{juxt}\,D$ does not contain a permutation of the form $abcd$ where ab and cd are decreasing pairs.

Suppose that $C\,\mathit{juxt}\,D$ is expressible in the form $\mathcal{B}(A, B, \pi)$ where $A = \{1, 2, 3, \ldots, \omega + 1, \omega + 2, \ldots\}$ and $B = \{1, 2, 3, \ldots\}$. Let $A_1 = \{1, 2, 3, \ldots\}$ and $A_2 = \{\omega + 1, \omega + 2, \ldots\}$.

Claim 1: π restricted to A_1 is increasing. For otherwise let a, b be a pair in A_1 such that $a < b$ but $\pi(a) > \pi(b)$ and let d be an element of A_2. There exists c in A_1 that appears after b such that $\pi(c) > \pi(d)$. Then $\pi(a)\pi(b)\pi(c)\pi(d)$ is order isomorphic to a permutation in $\mathcal{B}(A, B, \pi)$ but both $\pi(a)\pi(b)$ and $\pi(c)\pi(d)$ are decreasing pairs. Reductio ad absurdum.

Claim 2: π restricted to $\{\omega + 2, \omega + 3, \ldots\}$ is increasing. Indeed, assume that we have $c, d \in A \backslash \{\omega+1\}$ such that $c < d$ but $\pi(c) > \pi(d)$. Let $b = \omega+1$, and let $a \in A_1$ be such that $\pi(a) > \pi(b)$. Then $\pi(a)\pi(b)\pi(c)\pi(d)$ is order isomorphic to a permutation in $\mathcal{B}(A, B, \pi)$ but both $\pi(a)\pi(b)$ and $\pi(c)\pi(d)$ are decreasing pairs. Reductio ad absurdum.

Claim 3: $\pi(\omega + 1) > \pi(\omega + 2)$. This is because $321 \in C\, juxt\, D$.

Claim 4: $\pi(\omega + 1) < \pi(\omega + 3)$. Because $4312 \notin C\, juxt\, D$.

However, by Remark 92, we know that an atomic class represented by $\mathcal{B}(A, B, \pi)$ where A, B, π have this form is not the juxtaposition of any two closed nonempty classes. Reductio ad absurdum. ■

Finally, we make a few more observations:

Proposition 97 *The juxtaposition of any two sum complete closed classes is expressible in the form $\mathcal{B}(A, B, \pi)$ where A and B have order types 2ω and ω respectively.*

PROOF: Let X and Y be sum complete closed classes of permutations. Both X and Y have a countably infinite number of elements, and therefore so does $X\, juxt\, Y$. List all the elements of $X\, juxt\, Y$ as $\alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3, \ldots$ where each $\alpha_i\beta_i$ is a permutation whose subsequences α_i and β_i are order isomorphic to elements of X and Y respectively. Let $A = \mathbb{N} \cup (\omega + \mathbb{N})$, let $B = \mathbb{N}$ and define $\pi : A \to B$ to be the function where:

- For every $n \in \mathbb{N}$ the preimage under π of:

  $$\left(\sum_{i=1}^{n-1} |\alpha_i b_i|\right) + \{1, 2, 3, \ldots, |\alpha_n \beta_n|\}$$

  is:

  $$\left[\left(\sum_{i=1}^{n-1} |\alpha_i|\right) + \{1, 2, 3, \ldots, |\alpha_n|\}\right] \cup \left[\omega + \left(\sum_{i=1}^{n-1} |\beta_i|\right) + \{1, 2, 3, \ldots, |\beta_n|\}\right]$$

- The terms of π equal to

  $$\left(\sum_{i=1}^{n-1} |a_i b_i|\right) + \{1, 2, 3, \ldots, |\alpha_n \beta_n|\}$$

  form a sequence order isomorphic to $\alpha_n \beta_n$.

Then $\mathcal{B}(A, B, \pi)$ is an atomic class equal to $X \, juxt \, Y$. (See Figure 2.11)

■

Proposition 98 *If $X = \mathcal{B}(A, B, \pi)$ where A, B have order types 2ω and ω respectively and where π restricted to either the first or the last ω elements of A is sum complete, then X is not necessarily the juxtaposition of two non-empty closed classes.*

PROOF: A counterexample is provided by $A = \{1, 2, 3, \ldots, \omega{+}1, \omega{+}2, \ldots\}$, $B = \{1, 2, 3, \ldots\}$, $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \ldots & \omega+1 & \omega+2 & \omega+3 & \ldots \\ 2 & 1 & 5 & 4 & 8 & 7 & \ldots & 3 & 6 & 9 & \ldots \end{pmatrix}$.
Here, if $\mathcal{B}(A, B, \pi) = X \, juxt \, Y$ for some non-empty sets X and Y then it will be found that $21 \in X$ and $1 \in Y$, but $312 \notin \mathcal{B}(A, B, \pi)$. Reductio ad absurdum. ■

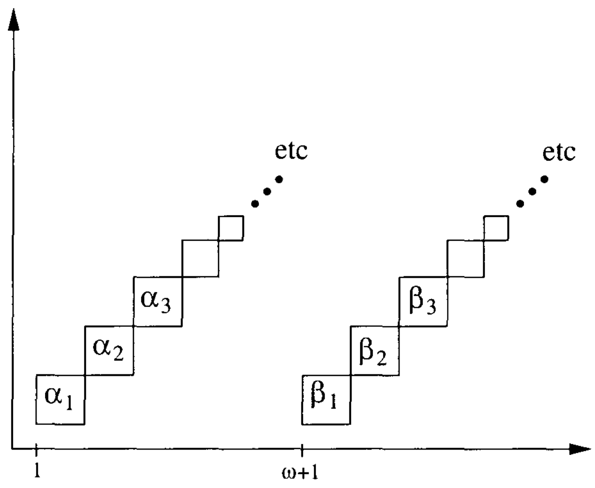


Figure 2.11: Sum complete classes juxtaposed.

2.5 Differentiation

Definition 99 Let X be a set of permutations. Then the *derivative* ∂X of X is the set of permutations order isomorphic to elements of X with the first term removed.

Example 100 If 2413 is an element of X then 312, which is order isomorphic to 413, is an element of ∂X.

The most interesting results involving differentiation in this text are those relating to natural classes, which are atomic classes of the form $\mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$. We refer the reader to the chapter on natural classes.

2.5.1 Differentiation and Atomic Classes

The following statements are all elementary and unproved. They are given to provide a little feel for the behaviour of differentiation.

If $X = \mathcal{B}(A, B, \pi)$ and the domain A has a first element which we may call a then $\partial X = \mathcal{B}(A \setminus \{a\}, B, \pi)$, with π restricted to its new domain. If A does not have a first element then $\partial X = X$.

If X is the union of a number of sets of permutations, each denoted by Y_i for some index i, then the derivative of X is equal to the union of the derivatives of all Y_i.

2.5.2 Differentiation and the Basis

Theorem 101 *There exists a finitely based class X such that ∂X is infinitely based.*

Proof is provided by the following lemma:

Lemma 102 *Let X be defined by the basis:*

2 3 5 1 4

3 2 5 1 4

4 2 5 1 3

6 2 5 1 3 4

1 4 6 2 3 5

Then the following are basis elements of ∂X:

3 5 1 2 8 4 6 7

3 5 1 2 7 4 10 6 8 9

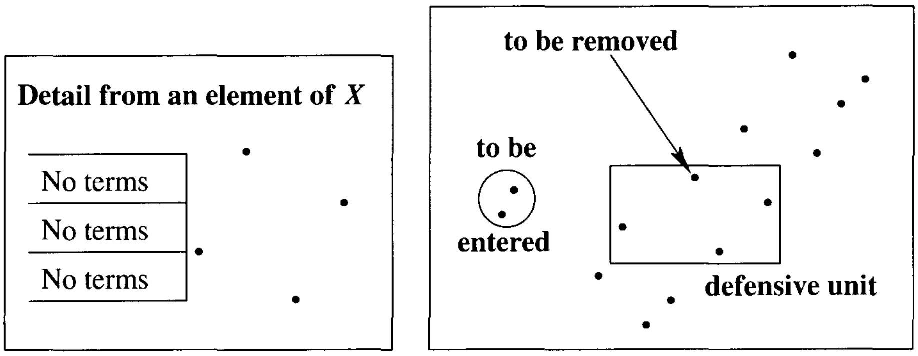


Figure 2.12: On the left we have the basic defensive unit: A sequence order isomorphic to 2413. Wherever such a sequence is found in an element of X the first three basis elements of X assure us that no terms will be found in any of the indicated regions. However this defence is fragile as illustrated on the right. Shown is a basis element of ∂X. If a single term, such as that indicated, is removed then further terms may be added. This ensures that every permutation properly involved in the illustrated element is an element of ∂X, as required. However this only explains the need for the first three basis elements of X. The last two basis elements defend against terms lying not directly to the left, but above or below and to the left of embedded basis elements of $\partial(X)$.

3 5 1 2 7 4 9 6 12 8 10 11

3 5 1 2 7 4 9 6 11 8 14 10 12 13

ad infinitum.

PROOF: In this preamble we explain the rationale behind this example, before giving the full proof. The basis of X is designed to exclude all elements of $1\, juxt\, A$, where A is the infinite antichain chosen to be in the basis of ∂X and which is listed above. The basis of X is also designed so that every term of the antichain A is essential to the maintenance of this "shield", by ensuring that if any one term is removed from an element A_i of A then $1\, juxt\, A_i$ contains an element of X. This fragile nature ensures that every element of A is not only not an element of ∂X, but is in fact a basis element of ∂X. The proof:

To demonstrate that no element of $1\ juxt\ A$ is an element of X, consider an embedding of an arbitrary element A_i of A in an element γ of X.

No term of γ preceding all the terms of the embedding may be both greater than the smallest term of the embedding and smaller than the greatest term of the embedding. For any such term would precede a subsequence order isomorphic to 2 4 1 3 in a manner forbidden by the basis elements 2 3 5 1 4, 3 2 5 1 4 and 4 2 5 1 3 of X.

No term may precede and be greater than all the terms of the embedding of A_i, for that term, combined with the largest five terms of the embedding would form a subsequence order isomorphic to 6 2 5 1 3 4.

No term may precede and be less than all the terms of the embedding of A_i, for that term combined with the smallest five terms of the embedding would form a subsequence order isomorphic to 1 4 6 2 3 5.

Thus we may conclude that no term precedes an embedding of A_i in any element of X. Thus $1\ juxt\ A_i$ has no element in X, hence A_i is not in ∂X and by generalisation neither is any other element of A in ∂X.

To demonstrate that A is a subset of the basis of ∂X we must demonstrate that every permutation properly involved in an element of A is an element of ∂X. As ∂X is closed it suffices to show that if any single term is removed from any element A_i of A then the resulting sequence is order isomorphic to an element of ∂X. This we do. Consider any term in A_i.

If any one of the smallest two or the rightmost two terms of A_i are removed to produce a sequence order isomorphic to a permutation β then either $1 \oplus \beta \in X$ or $1 \ominus \beta \in X$. Thus $\beta \in \partial X$.

To consider the case when any other term is removed write $A_i = a_1 a_2 \ldots a_m$ and let a_j be the term to be removed. Note that the permutation $a_j a_1 a_2 \ldots$ $\ldots a_{j-1} a_{j+1} a_{j+2} \ldots a_m$ is an element of X. Thus the permutation order isomorphic to A_i with a_j removed is contained in ∂X.

Thus we have our desired result. Q.E.D.

■

Proposition 103 *There exists an infinitely based class X such that ∂X is finitely based.*

Example 104 Let X be the class:

$$Sub(3\ 2\ 5\ 1\ 7\ 8\ 4\ 10\ 11\ 12\ 6\ 14\ 15\ 16\ 17\ 9\ \ldots).$$

Then X is infinitely based. Indeed the following are all basis elements of X:

$$3\ 2\ 4\ 5\ 1,\quad 3\ 2\ 5\ 1\ 6\ 7\ 8\ 4,\quad 3\ 2\ 5\ 1\ 7\ 4\ 8\ 9\ 10\ 11\ 6,\ \ldots$$

$$\underbrace{3\ 2}\ 5\ 1\ 7\ 4\ 9\ 6\ 11\ 8\ \ldots\ 2n+1\ \ 2n-2\ \ \underbrace{2n+2\ \ 2n+3\ \ \ldots\ 3n+1}\ \ 2n$$

etc ad infinitum

(Essentially these are increasing oscillating sequences with certain terms replaced by longer sequences, which are marked.)

However ∂X is finitely based. Its basis is $\{321, 4123, 314625\}$. See Figure 2.13

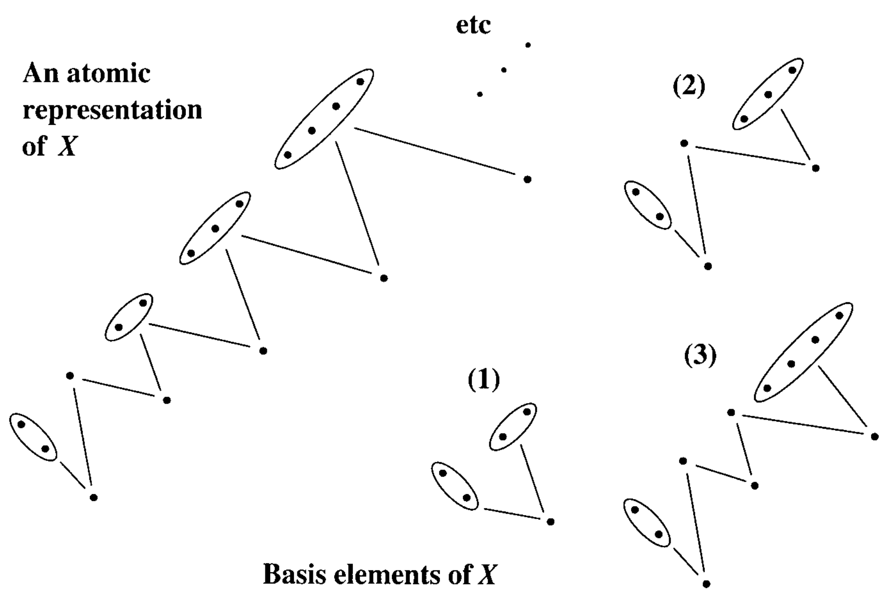


Figure 2.13: An infinitely based natural class X that becomes finitely based when differentiated. On the left is an infinite increasing oscillating sequence, modified, whose Sub defines X. On the right are three of an infinite sequence of basis elements of X. Note that the atomic representation on the left has a unique decreasing subsequence of length three. Thus were it to involve (1) its second left maximal interval would have to contain two terms instead of one; were it to involve (2) its third left maximal interval would have to contain three terms instead of two, and so on. Finally note that any sequence obtained by removing a term from (1), (2) or (3) is involved in the atomic representation, either because the demands it makes of left maximal intervals is less than that of its parent, or else because it brakes by sum decomposition into two parts that are then easily accommodated. Differentiating X destroys the "handle" or "irregularity" that is the decreasing triple. Without that there is no uniquely emeddable sequence that can be used to define the relative locations of the other irregulatities.

This example proves Proposition 103.

Question 105 Is it possible to conceive an efficient mechanism that constructs the basis of ∂X for an arbitrary closed class X?

Conjecture 106 *There does not exist a finitely based natural class X such that ∂X is infinitely based.*

Theorem 107 *There exists a finitely based atomic class X such that ∂X is infinitely based.*

As proof we provide the following lemma.

Lemma 108 *Let U, V and W be defined as follows:*

$$U = Sub(\ 4\ 5\ 8\ 1\ 2\ 11\ 6\ 7\ 14\ 9\ 10\ 17\ 12\ 13\ \ldots)$$

$$V = Sub(2\ 3\ 5\ 1\ 7\ 4\ 9\ 6\ 11\ 8\ \ldots)$$

$$W = Sub(\ldots\ -8\ -11\ -6\ -9\ -4\ -7\ -1\ -5\ -3\ -2)$$

Let $X = (U \ominus V) \oplus W$. Let A be the infinite set consisting of all the following permutations:

$A_1 = 2\ 3\ 6\ 1\ 4\ 5$

$A_2 = 2\ 3\ 5\ 1\ 8\ 4\ 6\ 7$

$A_3 = 2\ 3\ 5\ 1\ 7\ 4\ 10\ 6\ 8\ 9$

$A_4 = 2\ 3\ 5\ 1\ 7\ 4\ 9\ 6\ 12\ 8\ 10\ 11$

et cetera

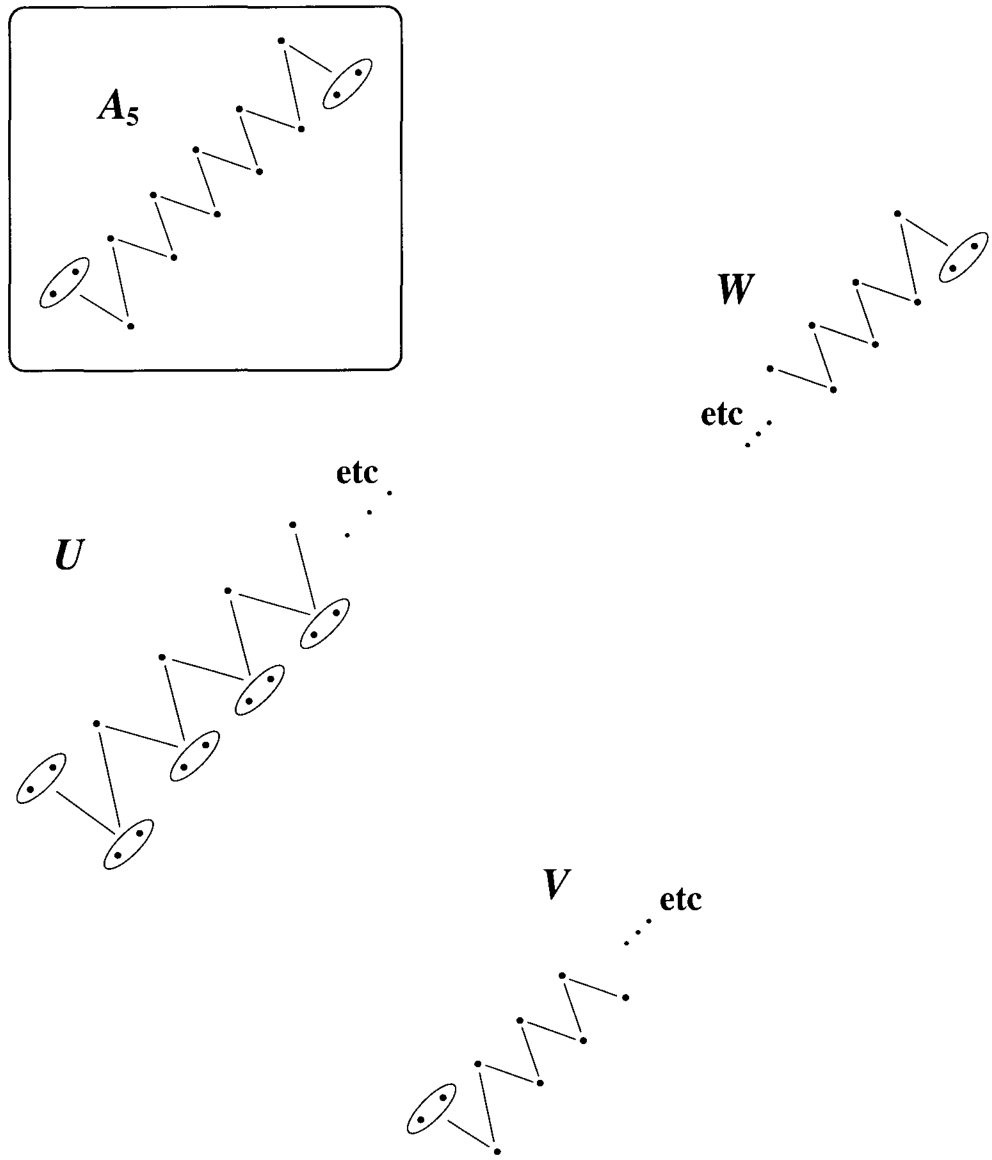


Figure 2.14: From Lemma 108, a finitely based atomic class that is infinitely based when differentiated. Note that A_5, a sample element of A, is involved in U but not in U differentiated, and that every permutation properly involved in A_5 is involved in the sum of V and W.

Then X has the finite basis given below, but A is a subset of the basis of ∂X.

$$\{13452, 31452, 52431, 53241, 54321, 4567321,\ 6723451,\ 6734521,\ 7234561,\ 7623451\}$$

PROOF: First we prove that every element of A is a basis element of ∂X. Note that U involves every element of A but that U differentiated once involves none. Notice also that $Sub(A) \setminus A = V \oplus W$. Thus we have that every permutation properly involved in an element of A is an element of $V \oplus W$, a subset of ∂X, but that no element of A is itself contained in ∂X. Thus A is contained within the basis of ∂X.

The basis of X was obtained by computational means, as follows:

The basis of U is:

$$\begin{aligned} B(U) &= \{13452,\ 23451,\ 24513,\ 31452,\ 34152, \\ &\quad 45123,\ 251364,\ 261345,\ 512364,\ 612345\} \end{aligned}$$

The basis of V is:

$$B(V) = \{321,\ 3412,\ 4123,\ 13452,\ 23451,\ 31452\}$$

The basis of $U \ominus V$ was obtained by solving a problem, equivalent by symmetry, with a terminating program that calculates the basis of the sum of two classes. Termination is safeguarded by an upper bound on the size of basis elements of the sum, if the sum is indeed finitely based. The program can also be used to determine whether the class is finitely based, although in this case it may be noted that as 321 is a basis element of V, the skew sum $U \ominus V$ will be finitely based.

The basis of W can be obtained by symmetry from the basis of V:

$$B(W) = \{321,\ 3412,\ 2341,\ 41235,\ 51234,\ 41253\}$$

Thus, again by computational methods, we obtained the basis of $(U \ominus V) \oplus W = X$. ■

It may be stated that due to the inefficiency of programs currently available the above results required considerable computer time to obtain; one week on a 7600/132 Power Macintosh.

2.5.3 Recent Development

Within the last few days of August 2002 we have found finitely based classes, both atomic and non-atomic, that never stabilize under differentiation. Unfortunately there is insufficient time to include either example in this thesis. The author has also obtained a result, a corollary of which is that the following holds:

Theorem 109 *Let X be any finitely based atomic class expressible as $\mathcal{B}(A, B, \pi)$ where A has order type ω. Then there exists an integer N such that for all $n \geq N$ we have that $\partial^N(X) = \partial^n(X)$.*

2.6 Merges

Definition 110 Let α, β be permutations. Then a permutation γ is a *merge* of α and β if it consists of two subsequences order isomorphic to α and β.

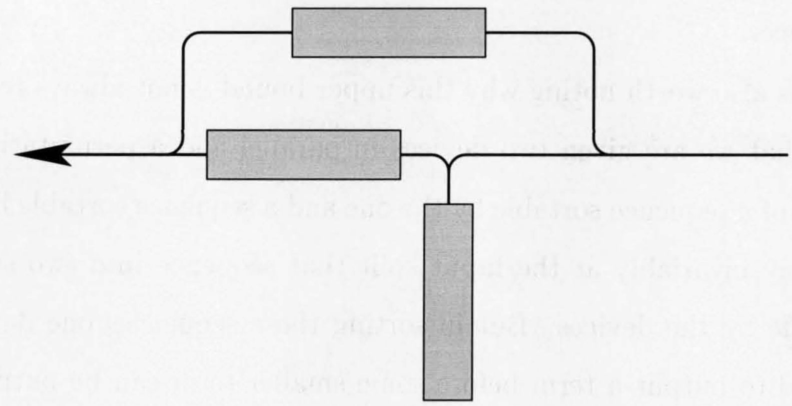


Figure 2.15: In parallel, a queue and a series arrangement of a stack and a queue. The class this sorts is the merge of $\mathcal{A}(21)$ and $\mathcal{A}(231)$. The class $\mathcal{A}(21)$ is the set of permutations sortable by the top branch, $\mathcal{A}(231)$ is the set sortable by the bottom branch. Were we to omit the bottom queue we could not sort $632541 \in \mathcal{A}(21)$ *merge* $\mathcal{A}(231)$.

Example 111 1 3 6 5 4 2 is a merge of 1 2 3 and 3 2 1. It may be noticed immediately that there are three ways of dividing this merge into subsequences order isomorphic to 1 2 3 and 3 2 1. This is not at all unusual.

Merges appear naturally in networks of sorting devices: Subject to the choices of device, two sorting devices placed in parallel are capable of sorting precisely those permutations that are merges of permutations sortable by the one device and permutations sortable by the other device, see Figure 2.15.

The merge is invariably an upper bound for the sorting capability of two devices placed in parallel, and it is worth noting why this is. At the input to the combined device, the input sequence is split into two subsequences that are fed term by term into the two constituent machines for sorting. At the output the two now sorted subsequences are reassembled into a single sorted sequence.

It is also worth noting why this upper bound is not always reached. Suppose that we are given two devices in parallel and a permutation that is a merge of a sequence sortable by the one and a sequence sortable by the other. We may invariably at the input split that sequence into two subsequences sortable by the devices. But in sorting these sequences one device may be obliged to output a term before some smaller term can be output from the other device. In essence the timing of inputs of the two machines is fixed, but it may not be possible to synchronise their outputs suitably.

For an example let us consider:

Example 112 The sorting machine Q_3 is a queue with maximum capacity three. It cannot rearrange any terms input and so it 'sorts' only the increasing sequences I. Consider two such devices in parallel. It is clear that they cannot sort 23451 although this is in $I merge I$.

Alternatively consider the single stack S as a sorting device. It is well known and readily seen that the permutations that S can sort are precisely those in $\mathcal{A}(231)$. Now consider two stacks in parallel. They cannot sort $24531 \in 245$ *merge* 31 even though this is in $\mathcal{A}(231)$ *merge* $\mathcal{A}(231)$. (Again this is evident. For more details on stacks in parallel see [33].)

It is worth noting that if both of two devices each have the ability to sort by inputting all terms and then later outputting all terms then the permutations sortable by their parallel combination are precisely the merges of permutations sortable by the individual devices. This is because outputs from each device can be made at leisure, each term waiting until all smaller terms have been output. It can be seen that an unlimited queue has this potential for waiting.

The condition that both machines have waiting potential is too strong to characterize all cases when two machines in parallel reach the merge limit. It can be sufficient for only one of the two devices to have the potential for waiting, as in the case when an unlimited queue and a finite capacity queue are placed in parallel. This example is the only one known at present, to the best of the author's knowledge.

The waiting property can be added to any sorting device by placing the device in series with a queue.

Question 113 What is the basis of the merge of two classes?

In the case of merging increasing and decreasing sequences Mike Atkinson showed in [10] that the basis (of I *merge* R) consists of the wreath products of $1\,2$ and $2\,1$, namely $\{3\,4\,1\,2, 2\,1\,4\,3\}$. This is not true in general, but the truth would appear to be not far off, in the few cases that we consider here.

Remark 114 $\mathcal{A}(12)$ *merge* $\mathcal{A}(12) = \mathcal{A}(123)$, hence $12 \wr 12$ is not a basis element of $\mathcal{A}(12)$ *merge* $\mathcal{A}(12)$.

Conjecture 115 *The basis of* $\mathcal{A}(132)$ *merged with* $\mathcal{A}(213)$ *is* $\{2154376, 132 \wr 213, 213 \wr 132\}$

Chapter 3

Antichains

Definition and Example, Antichain Classes, Maximal, Trim, Fundamental, Finitely and Infinitely Based Antichain Classes, Partially Well Ordered, Strongly Finitely Based, Higman's Theorem, Small Basis Case Studies

3.1 Definition

We briefly remind the reader of the definition of an antichain: Involvement is a partial order on the set of all permutations. A set of permutations where no one element is involved in any other, barring itself, is said to be an antichain. In this chapter we discuss structures of antichains, especially infinite antichains.

3.2 Infinite Antichains

We have in this thesis already used infinite antichains for various purposes but we have not yet demonstrated what it is that makes these sets of permutations into antichains. We will take a sample infinite antichain of the simplest kind and show that it is indeed what it claims to be:

Let W be the set of permutations shown in Figure 3.1:

$$W_1 = \ 3\ 7\ 6\ 2\ 5\ 1\ 4, \quad W_2 = \ 3\ 8\ 2\ 5\ 1\ 4\ 6\ 7$$

$$W_3 = \ 5\ 9\ 4\ 7\ 3\ 6\ 2\ 1\ 8, \quad W_4 = \ 2\ 3\ 6\ 10\ 5\ 8\ 4\ 7\ 1\ 9$$

et cetera

We will show that W is an antichain.

1. To begin, we will note that every single element of W contains a unique subsequence order isomorphic to 32514. In Figure 3.1 these subsequences have been circled, or in the above lists it will be found that 32514 is this unique subsequence in W_1, in W_2 it is again 32514, in W_3 it is 54736, in W_4 it is 65847 and so on. The antichain was constructed specifically so that each element would have this unique subsequence.

   The unique subsequences order isomorphic to 32514 permit us to divide the remaining terms of each sequence into 6^2 parts, according to where the terms lie horizontally and vertically compared with those of the subsequence, and again this is shown in the figure.

2. To our original sequence 32514 we add a single term to produce $3\bar{6}2514$. The 6 corresponds to the added term and is therefore marked. This

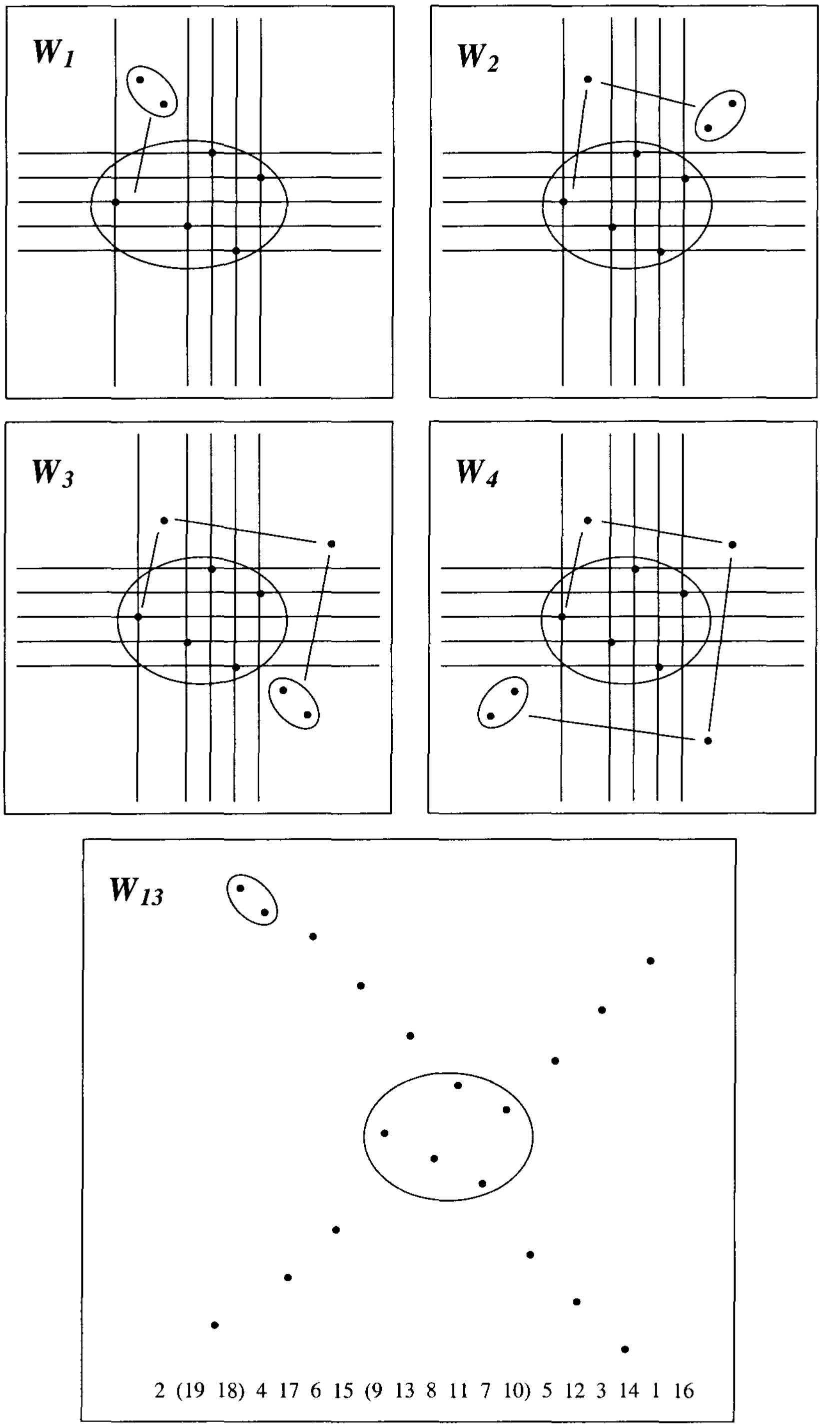


Figure 3.1: Constructing an antichain. The permutations can be thought of as regular spirals where the ends have been made distinct from the rest of the spiral. Whether the circled pairs are increasing or decreasing does not affect whether W is an antichain. Less obvious is that of the central marked sets, the top two terms may be removed from each permutation and we will still have an antichain.

new sequence is still involved in every element of W but we lose the property of uniqueness, but only in one case: Every element of W contains a unique subsequence order isomorphic to 362514, barring W_1 alone, which has two.

We therefore have that W_1 is pairwise incomparable to every other element of W. Every other element of W is longer than W_1 and is therefore not involved in W_1, but any permutation that involves W_1 has at least two subsequences order isomorphic to 362514. We discard W_1 from our future concerns.

3. Every element of $W \setminus W_1$ contains a unique subsequence order isomorphic to 362514. We will however add another term to this sequence to obtain 372514$\bar{6}$. Note that barring W_2 which has two, every element of $W \setminus W_1$ has a unique subsequence order isomorphic to 3725146. As previously, W_2 is the shortest element of $W \setminus W_1$, and therefore we may ignore W_2 in our future concerns.

4. We continue our argument by adding terms one at a time to produce the sequences 483625$\bar{1}$7, $\bar{2}$59473618 and so on, thereby disposing of the elements of W one by one. This completes our demonstration that W is an antichain.

In Figure 3.1 we have shown construction lines that indicate the unique subsequences order isomorphic to 23514, 362514 and so on. This particular antichain was designed to show that there exists an infinite antichain every element of which avoids 3142. An examination of the diagram may convince the reader that the antichain satisfies that requirement. A nicer version of this antichain is given in the Bibliothek, however it is constructed in essentially the same manner.[1]

More complex antichains can be constructed from existing ones. For instance if A and B are antichains then the following are also antichains:

1. $A \oplus B = \{\alpha \oplus \beta \mid \alpha \in A, \beta \in B\}$ and $A \ominus B$.
2. $1 \oplus B$ or indeed $\gamma \oplus B$ for any permutation γ. This is a special case of (1) as any single permutation forms an antichain of one element.
3. A *Juxt* B, the set of all juxtapositions of elements of A and B.
4. $A \wr B$, the wreath product of A and B.
5. The merge of A and B need not be an antichain but the minimal merge of any two sets is by definition an antichain.

Of these examples the only one that may be difficult[2] to verify is 4. Consider the manner in which two intervals may overlap. With a little observation you may note that if B is an antichain and $\alpha \in S \wr B$, where S is the set of all permutations, then there exists a unique permutation γ such that $\alpha \in \gamma \wr B$. The result follows.

It is doubtless not worth classifying the many different variations that can be produced on an antichain, as these variations show progressively less tangible structure. It is the antichains that are in some sense most restricted

[1] The letter W stands for *Widderschin*, to indicate the spiral nature of this antichain.

[2] For the readers familiar with wreath products this should take three minutes, for the unfamiliar, fifteen. Think simple.

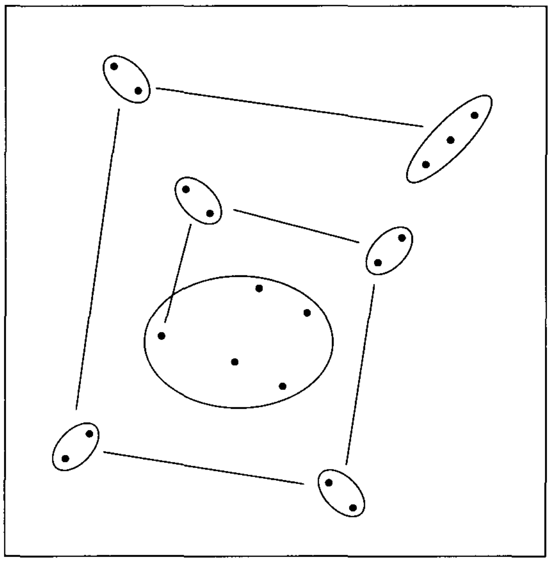


Figure 3.2: A sample element of an infinite antichain constructed in a manner similar to W.

that show the strongest patterns and therefore we will later expend our efforts in trying to classify those.

However first we will consider antichains in their natural context of bases and sets of maximal elements in a closed class.

3.3 Antichain Classes

It was noted in the introduction that if X is any set of permutations, and specifically if X is a closed class, then the set M of maximal elements of X is an antichain, as is the basis $\mathcal{B}$. In order to exercise our intuition of antichains in these two roles we will examine the following specific question: "If X is infinite is there any strong connection between the size of M and that of $\mathcal{B}$?"

It will be shown that there is none, at least not if we permit ourselves to only recognise whether a cardinality is infinite or finite. The solution to our question when M is finite is well known:

If $X = I$, the set of increasing permutations then $\mathcal{B}$ is the single permutation $\{21\}$ and X has no maximal elements. Thus there exists a class with finite basis and with finitely many maximal elements.

The infinite antichain ${}^{I_2}U^{I_2}$, the first in the library, is well known. It is also fairly well known that every permutation properly involved in an element of ${}^{I_2}U^{I_2}$ is also involved in every subsequent element of ${}^{I_2}U^{I_2}$. This implies that the set of permutations properly involved in an element of ${}^{I_2}U^{I_2}$ has no maximal elements: If α is properly involved in one element of ${}^{I_2}U^{I_2}$ then we may pluck from a subsequent element of ${}^{I_2}U^{I_2}$ a longer proper subsequence that involves α. The set of permutations properly involved in

$${}^{I_2}U^{I_2}$$

is denoted $PropSub({}^{I_2}U^{I_2})$ and as every element of ${}^{I_2}U^{I_2}$ is a basis element of $PropSub({}^{I_2}U^{I_2})$ this gives us an example of an infinitely based class with no maximal elements.

We now treat the cases that we have not yet considered, where the set of maximal elements is infinite. Note that if A is an antichain then every element of A is a maximal element of the closure of A. As examples of infinite antichains are readily available this gives us an easy method of generating the classes we need.

In the following section we show that the closure of an antichain, very much like ${}^{I_2}U_{I_2}$, is finitely based. This gives us a finitely based class with infinitely many maximal elements.

Intuitively the next obvious step is to take the antichain ${}^{I_2}U_{I_2}$ and divide it into its odd terms U_o and its even U_e. The closure of U_e has infinitely many maximal elements. Since every permutation properly involved in an element of U_o is involved in the subsequent element of ${}^{I_2}U_{I_2}$, in U_e, it follows that $PropSub(U_e)$ is infinitely based and has infinitely many maximal permutations; however that example is too easy, too nice:

The antichain U_e is not maximal as an antichain: It is possible to add another permutation to U_e and obtain a bigger set that is still an antichain. We are exposed to the following conjecture: "If X is a class with infinitely many maximal elements and whose maximal elements form a maximal antichain then X is finitely based". In Section 3.3.2 we counter that conjecture by expending a little effort finding an example of a maximal antichain whose closure is infinitely based.

Before entering fully into this chapter we mention that we have nothing to say about finite classes. They have, invariably, basis elements and maximal elements, and finitely many of both.

3.3.1 Finitely Based Infinite Antichain Classes

Theorem 116 *Let $A = \{A_i | i \in \mathbb{Z}^+\}$ be the infinite set of permutations given by*

$$A_1 = \ 3\ 2\ 5\ 1\ 9\ 4\ 8\ 6\ 7, \quad A_2 = \ 3\ 2\ 5\ 1\ 7\ 4\ 11\ 6\ 10\ 8\ 9,$$

$$A_3 = \ 3\ 2\ 5\ 1\ 7\ 4\ 9\ 6\ 13\ 8\ 12\ 10\ 11, \quad A_4 = \ 3\ 2\ 5\ 1\ 7\ 4\ 9\ 6\ 11\ 8\ 15\ 10\ 14\ 12\ 13,$$

$$\ldots$$

$A_n = \langle 3, 2, 5, 1, 7, 4, 9, 6 \ldots 2n+3, 2n, 2n+7, 2n+2, 2n+6, 2n+4, 2n+5 \rangle.$

$$\ldots$$

Then A is an antichain. Furthermore if $X = Sub(A)$ then every basis element of X has length less than 24. Thus X is finitely based.

In this section we will prove[3] this theorem. For this purpose we will, throughout this section, refer to A, X and A_n as in the above theorem.

Lemma 117 *Let $A_m = a_1, a_2, \ldots, a_{2m+7}$ be an element of the set A. Then the subsequence of A_m obtained by removing any of $a_3, a_4, \ldots, a_{2m+4}$ is sum decomposable. In fact if the removed a_i is left maximal then*

$$\langle a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{2m+7} \rangle \cong \langle a_1, a_2, \ldots, a_{i-1}, a_{i+1} \rangle \oplus \langle a_{i+2}, \ldots, a_{2m+7} \rangle.$$

If a_i is right minimal then

$$\langle a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{2m+7} \rangle \cong \langle a_1, a_2, \ldots, a_{i-2} \rangle \oplus \langle a_{i-1}, a_{i+1}, \ldots, a_{2m+7} \rangle.$$

By left maximal we mean greater than all the preceding terms. In this case the left maximal terms in the available range are those with odd suffixes, namely $a_3, a_5, a_7, \ldots, a_{2m+3}$. By right minimal we mean smaller than all succeeding terms, in this case the terms $a_4, a_6, a_8, \ldots, a_{2m+4}$.

This result is easily observed. It may be worth noting that the result holds for every element of X with sufficient length, although the decomposition may differ from the above.

[3] Graecum est: non legitur. That is: The proof is complex.

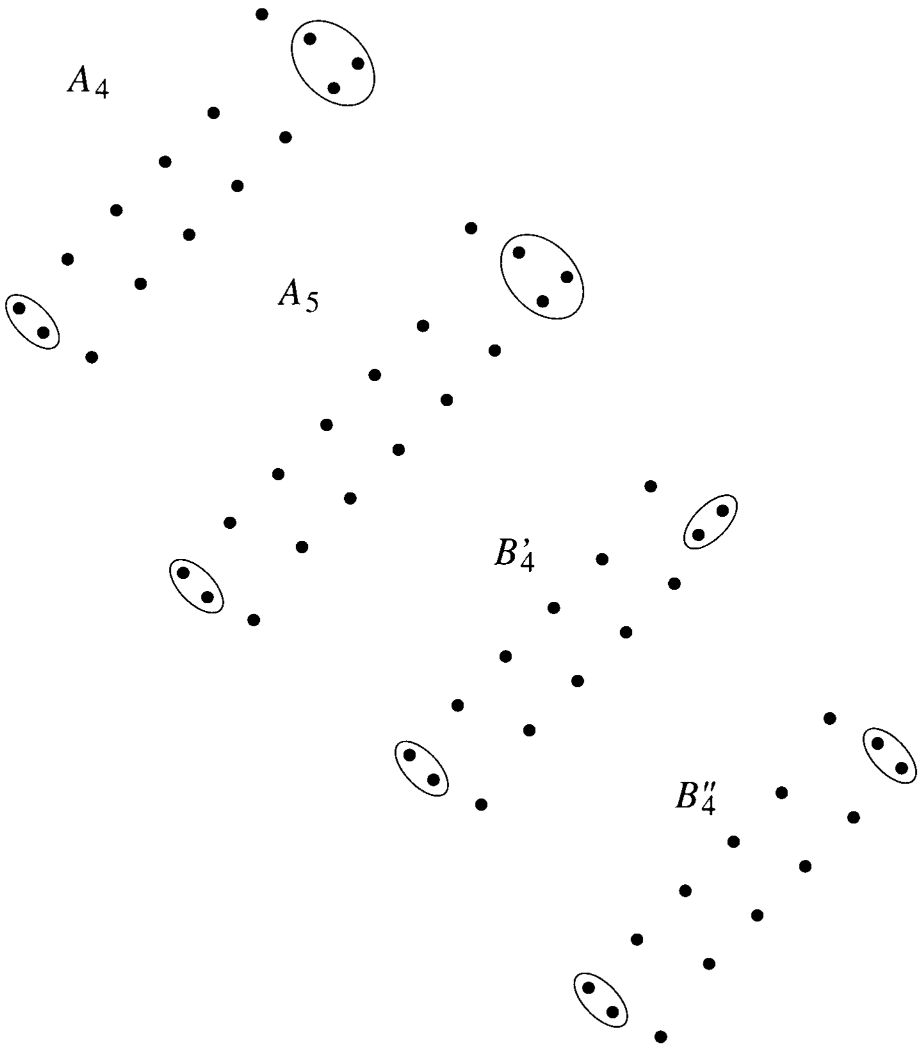


Figure 3.3: Every permutation properly involved in A_4 is involved either in A_5 or in one of B'_4, B''_4.

Proposition 118 *The permutations in A are pairwise incomparable and therefore A is an antichain.*

PROOF: Let A_i and A_j be distinct elements of A, and suppose that A_i is involved in A_j.

Note that both A_i and A_j have unique subsequences order isomorphic to 2341, these subsequences consisting in each case of the first four terms of A_i and A_j.

Note also that both A_i and A_j have unique subsequences order isomorphic to 51423, these subsequences consisting of the last five terms of A_i and A_j.

Thus the first four and the last five terms of A_j must be involved in any embedding of A_i into A_j. Further note that A_i is sum indecomposable and so, by Lemma 117, all other terms of A_j are also involved in any embedding. Thus A_i and A_j must be equal. Reductio ad absurdum. ■

Lemma 119 *Let $\tau = t_1 t_2 \ldots t_n$ be a basis element of X with $n \geq 7$. Then for all terms t_i there are at most three terms t_j to the left of and greater than t_i (i.e. such that $j < i$ and $t_j > t_i$), and at most four elements to the right of and less than t_i (i.e. such that $i < j$ and $t_i > t_j$).* [4]

[4]This lemma and its proof generalise quite easily to all 'finite drop' closed classes and their basis elements. 'Finite drop' is an intuitive notion having a number of different and non-equivalent definitions including: A class is finite drop if there is a constant limit for the number of terms that may be found below and to the right of any given term of an element of the class. The most popular definition is the stronger: A class X has the finite drop property if there is a constant integer $k \geq 0$ such that for any two terms a_i and a_j with $i < j$ of any element of X, we have that $a_i - k < a_j$. The lemma generalises for all of these definitions.

PROOF:

Consider any element $A_m = a_1a_2\ldots a_{2m+7}$ of the antichain A. For every a_i there are at most three a_j to the left of and greater than a_i, and at most four a_j to the right of and smaller than a_i. These restrictions must hold equally for any element of $X = Sub(A)$.

Now consider a basis element $\tau = t_1t_2\ldots t_n$ with $n \geq 7$, and choose a t_i. If $t_i = 1$ and $i = n$ then $t_2t_3\ldots t_n$ is order isomorphic to a permutation in X, for it is a proper subpermutation of a basis element. But then $t_2, t_3, \ldots, t_{n-1}$, which number at least five, are all to the left of and greater than t_n, and this cannot be.

Thus, having excluded this extreme situation, there must be a term t_j that is either to the right of or below, t_i, or both. But $t_1t_2\ldots t_{j-1}t_{j+1}\ldots t_n$ is order isomorphic to a permutation in X. Thus we have that there can be at most three terms in τ to the left of and greater than t_i.

By a similar argument we can demonstrate that there can be at most four elements of τ to the right of and less than t_i. Q.E.D. ∎

Lemma 120 *If* $\rho = r_1r_2\ldots r_m \in X$ *and if* $\sigma = s_1s_2\ldots s_n \in X$ *then* $r_1r_2\ldots r_{m-2} \oplus s_2s_3\ldots s_n \in X$.

PROOF:

Suppose that $\rho \preceq A_i = \langle 3, 2, 5, 1, 7, 4, \ldots, 2i+7, 2i+2, 2i+6, 2i+4, 2i+5\rangle$ and $\mu \preceq A_j = \langle 3, 2, 5, 1, 7, 4, \ldots, 2j+7, 2j+2, 2j+6, 2j+4, 2j+5\rangle$.

Note that the first $2i+5$ terms of A_i, the last of which is $2i+6$, form a sequence order isomorphic to the first $2i+4$ terms and the $2i+6^{\text{th}}$ term of A_{i+j+3}.

Similarly the last $2j+6$ terms of A_j are order isomorphic to the $2i+7^{\text{th}}$ and last $2j+5$ terms of A_{i+j+3}. (The first of the last $2j+6$ terms of A_j has value 3. The $2i+7^{\text{th}}$ term of A_{i+j+3} has value $2i+9$. The first of the last $2j+5$ terms of A_{i+j+3} is the $2i+9^{\text{th}}$ term of A_{i+j+3} and has value $2i+11$.)

Finally note that if the $2i+5^{\text{th}}$ and $2i+8^{\text{th}}$ terms of A_{i+j+3} are removed the resulting sequence is sum decomposable into two components, the first component having length $2i+7$, the second having length $2j+8$.

■

Corollary 121 *Let τ be a basis element of X. Then τ cannot be written in the form $\rho \oplus \sigma$ where $|\rho| \geq 2$ and $|\sigma| \geq 3$.*

PROOF: Suppose that a basis element $\tau = \rho \oplus \sigma$ where $|\rho| \geq 2$ and $|\sigma| \geq 3$. Let $\rho = r_1 \dots r_i$ and $\sigma = s_1 \dots s_j$. Then $r \oplus (s_1 s_2) \in X$. Similarly $r_i \oplus \sigma \in X$ Thus, by the above Lemma 120 we have that $\tau = \rho \oplus \sigma \in X$, quid non est. This completes the proof.

■

Lemma 122 *Let $\tau = t_1 t_2 \dots t_n$ be a basis element of X with length $n \geq 5$. Then there exists a proper subsequence $t_u t_{u+1} \dots t_{v-1} t_v$ of τ which has length at least $n-3$ and which is sum indecomposable. (This does* not *imply that $t_u \dots t_v$ is a sum component of τ.)*

PROOF:

Suppose that τ is sum decomposable. Then by Corollary 121 we can express τ as $\rho \oplus \lambda \oplus \mu$ where $0 \leq |\rho| \leq 1$ and $0 \leq |\mu| \leq 2$, where ρ and μ are not both empty, and where λ is sum indecomposable. Finally, λ has length at least $n-3$.

If τ is sum indecomposable then let $t_1t_2\dots t_{n-1} \cong \alpha \oplus \beta$ where α is the first sum component of $t_1t_2\dots t_{n-1}$ and where β is the remainder of $t_1t_2\dots t_{n-1}$. The latter sequence, β, may be empty but then we have that α is sum indecomposable and is of length $n-1$. Therefore we will consider the case when β is not empty. To preserve sum indecomposability of τ we have that t_n must be smaller than every term of β and smaller than at least one term of α. By Lemma 119 there are at most three terms to the left of and greater than t_n and so we have that $0 \leq |\beta| \leq 2$. Again α has length at least $n-3$.

■

Proposition 123 *The class X is finitely based and has no basis element of length 24 or more.*

PROOF: Suppose that $\tau = t_1t_2\dots t_n$ is a basis element of X with length 24 or more.

The function f

By Lemma 122 there exist integers u, v such that $t_ut_{u+1}\dots t_v$ is sum indecomposable, has length at least $n-3$ and is a proper subsequence of τ. Let such u, v be given, and let the sequence $t_u\dots t_v$ be denoted by λ.

$t_u\dots t_v$ is order isomorphic to an element of X. Let A_m be the shortest element of A involving λ. Let f be an order preserving function (i.e. if $i > j$ then $f(i) > f(j)$) such that $f(t_u)f(t_{u+1}),\dots f(t_v)$ is a subsequence of

$$A_m = \langle 3, 2, 5, 1, 7, 4, \dots, 2m+3, 2m, 2m+7, 2m+2, 2m+6, 2m+4, 2m+5\rangle.$$

First note that both 1 and $2m+7$ must be in the image of f. For suppose that 1 is not in the image of f. If either of 2 and 3 is in the image then λ is sum decomposable (see Lemma 117). If neither 2 nor 3 is in the image then λ may be embedded into A_{m-1}, contradicting the choice of A_m.

The argument demonstrating the presence of $2m+7$ in the image is similar: Were $2m+7$ *not* in the image then either one of $2m+6$, $2m+4$, $2m+5$ would be in the image and λ would be sum decomposable, or else no such term would be in the image and m would not be minimal.

Now note that, by Lemma 117, all of $5, 1, 7, 4, \ldots 2m{+}3, 2m, 2m{+}7, 2m{+}2$ must be in the image of f, for otherwise λ is sum decomposable with $f^{-1}(1)$ and $f^{-1}(2m+7)$ in different sum components.

Regarding the length of this subsequence, note that:

$$\begin{aligned} |5,1,7,4,\ldots 2m+7, 2m+2| &= |A_m| - 5 \\ &\geq |t_u \ldots t_v| - 5 \\ &\geq (n-3) - 5 \\ &= n - 8 \end{aligned}$$

Let $f^{-1}(5) = t_p$. As $|\tau| \geq 24$ we have that:

$$t_p \ldots t_{p+15} \cong 3\ 1\ 5\ 2\ 7\ 4\ 9\ 6\ 11\ 8\ 13\ 10\ 15\ 12\ 16\ 14$$

See Figure 3.4.

The function g

$t_1 \ldots t_{p+13}$ is a proper subsequence of τ and is therefore an element of X. Let A_k now be defined as the shortest element of the antichain A involving $t_1 \ldots t_{p+13}$. Let g be an order preserving function such that $g(t_1) \ldots g(t_{p+13})$ is a subsequence of $A_k = \langle 3, 2, 5, 1, 7, 4, \ldots, 2k+3, 2k, 2k+7, 2k+2, 2k+6, 2k+4, 2k+5 \rangle$.

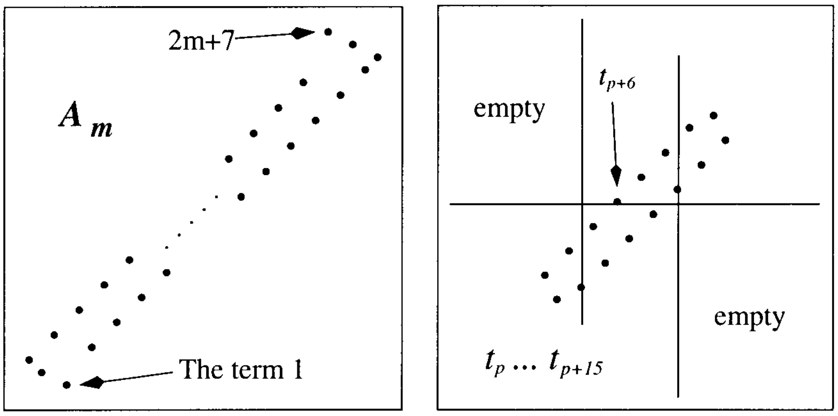


Figure 3.4: All but the first two and last three terms of A_m must be contained in $f(\lambda)$, which is order isomorphic to $\lambda = t_u \dots t_v$. This permits us to plot $t_p \dots t_{p+15}$. Later we show that all terms preceding t_p are less than t_{p+6} and all the terms succeeding t_{p+15} are greater than t_{p+6}.

Note that $2k+6, 2k+4, 2k+5$, the last three terms of A_k, are not in the image of g. Otherwise $g(t_{p+13})$, the last term of the image of g would have to be one of them, a situation which engenders the following contradiction: There is an increasing pair of terms, namely t_{p+10} and t_{p+12}, that precede and are greater than t_{p+13} but there is no increasing pair preceding and greater than any of $2k+6, 2k+4, 2k+5$ in A_m.

Similarly note that $2, 3 \notin \{g(t_p), \dots, g(t_{p+13})\}$ due to the fact that t_p is followed by an increasing pair of terms in τ, both smaller than t_p.

Also note that $2k+2$ is in the image of g, for otherwise A_k is not the shortest antichain element involving $g(t_1) \dots g(t_{p+13})$. (The sequence $g(t_1), \dots g(t_{p+13})$ is of length at least 10, therefore $k > 1$ and A_{k-1} exists. If our statement is not true then it can be shown that $g(t_1) \dots g(t_{p+13})$ is involved in A_{k-1}.)

Thus we may conclude that $g(t_{p+13}) = 2k + 2$.

Now, as t_{p+13} is preceded by t_{p+10} and t_{p+12}, both of which are greater than t_{p+13}, we may deduce that $g(t_{p+12}) = 2k+7$ and that $g(t_{p+10}) = 2k+3$. Thus $g(t_{p+11}) = 2k$.

By continuing this argument we may show that if $A_k = \langle a_1 \dots a_{2k+9} \rangle$ then $g(t_p) \dots g(t_{p+13}) = a_{2k-9} \dots a_{2k+4}$. It is especially worth noting that:

$$g(t_{p+4}) \dots g(t_{p+11}) = \langle 2k-3, 2k-6, 2k-1, 2k-4, 2k+1, 2k-2, 2k+3, 2k \rangle$$

The function h

$t_{p+2} \dots t_n$ is a proper subsequence of τ and is therefore an element of X. Let A_l now be defined as the shortest element of the antichain A involving $t_{p+2} \dots t_n$. Let h be an order preserving function such that $h(t_{p+2}) \dots h(t_n)$ is a subsequence of $A_l = \langle 3, 2, 5, 1, 7, 4, \dots, 2l+3, 2l, 2l+7, 2l+2, 2l+6, 2l+4, 2l+5 \rangle$.

By an argument very similar to that in the last section we may show that:

$$h(t_{p+4}) \dots h(\tau_{p+11}) = 7\ 4\ 9\ 6\ 11\ 8\ 13\ 10$$

Note that for $p+4 \leq i \leq p+11$ we have that $g(t_i) = h(t_i) + 2k - 10$.

Let F be defined as follows:

$$F(\tau_i) = \begin{cases} g(t_i) & \text{if } i \leq p+11 \\ h(t_i) + 2k - 10 & \text{if } i > p+11 \end{cases}$$

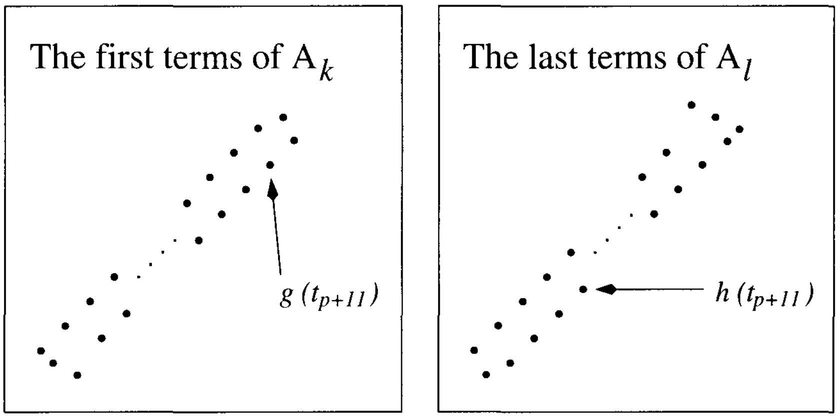


Figure 3.5: The functions g and h permit us to plot permutations that involve the beginning and end of τ. Information gleaned from the function f permits us to marry the two plots together into a permutation that is an element of A. This shows that τ is in the closure of A and provides the necessary contradiction.

We claim that $F(t_1)\dots F(t_n)$ is a subsequence of A_{k+l-2}, and order isomorphic to τ.

To see that F is an increasing function let us consider any i,j with $i < j$.

- If $j \leq p+11$ then $F(t_i) = g(t_i)$ and $F(t_j) = g(t_j)$. As g is an increasing function it follows that $F(t_i) < F(t_j)$ if and only if $t_i < t_j$.

- Similarly if $i > p+11$ then $F(t_i) = h(t_i)$ and $F(t_j) = h(t_j)$ and as h is an increasing function we obtain the same result: $F(t_i) < F(t_j)$ if and only if $t_i < t_j$.

- The previous point may be strengthened as follows: Note that if $i \geq p+4$ then $F(t_i) = g(t_i) = h(t_i) + 2k - 10$. Thus if $i \geq p+4$ then as h is an increasing function we have that $F(t_i) < F(t_j)$ if and only if $t_i < t_j$.

- Now we consider the case when $i < p+4$ and $j > p+11$:

  Consider τ. We have that $t_i < t_{p+6}$, because otherwise $t_{p+4}, t_{p+5}, t_{p+6}, t_{p+7}, t_{p+9}$ all follow t_i, and are all be smaller than t_i which, by Lemma 119, is impossible.

  Similarly $t_j > t_{p+6}$, due to the terms $t_{p+6}, t_{p+8}, t_{p+10}, t_{p+11}$ and Lemma 119.

  Thus if $i < p+4$ and $j > p+11$ then $t_i < t_j$. Now, as g and h are increasing we have that $F(t_i) < F(t_j)$ because:

  $$F(t_i) = g(t_i) < g(t_{p+6}) = h(t_{p+6}) + 2i - 10 < h(t_j) + 2i - 10 = F(t_j)$$

Thus we may conclude that $F(t_1)\dots F(t_n)$ is order isomorphic to τ.

Now we demonstrate that $F(t_1)\dots F(t_n)$ is a subsequence of A_{k+l-5}.

Note that the first $2k+2$ terms of A_k are equal to the first $2k+2$ terms of A_{k+l-5} and recall that the sequence $g(t_1)\dots g(t_{p+11}) = F(t_1)\dots F(t_{p+11})$ is a subsequence of the first $2k+2$ terms of A_k. Thus the first $p+11$ terms of the image of F are a subsequence of the first $2k+2$ terms of A_{k+l-5}.

Similarly note that the last $2l-5$ terms of A_l are order isomorphic to the last $2l-5$ terms of A_{k+l-5}, indeed the terms of the former differ from those of A_{k+l-5} by $2k-10$. Thus the sequence $F(\tau_{p+12})\dots F(\tau_n)$ is a subsequence of the last $2l-5$ terms of A_{k+l-5}.

We conclude that $F(\tau_1)\dots F(\tau_n)$ is a subsequence of A_{k+l-5}. This demonstrates that τ is an element of X and completes the proof.

■

Corollary 124 *There exists a finitely based closed class that cannot be written as the union of a finite set of atomic classes.*

PROOF: The class X of this section provides an example. It is finitely based and is equal to $Sub(A)$ where A is an infinite antichain.

■

Remark 125 Subsequent to the above result being proved the basis of $Sub(A)$ was calculated by computer. The result was as follows:

$$\begin{aligned} B(Sub(A)) &= \{2341, 2431, 3412, 3421, 4213, 4231, 4321, \\ &\quad 14325, 14352, 41235, 41253, 41325, 41352, 43125, \\ &\quad 43152, 51234, 51243, 54123, 326145, 326154\} \end{aligned}$$

3.3.2 Infinitely Based Antichain Classes

In the last section we have a class, $X = Sub(A)$ where A is an infinite antichain and where X is finitely based. We can therefore construct an infinite maximal antichain whose closure is finitely based by letting C be the set of all basis elements of X that do not involve an element of A and taking the union $A \cup C$ of A and C. The argument showing that this union is an antichain and is maximal is given in Proposition 136. To see that $Sub(A \cup C)$ is finitely based note that $Sub(A)$ is finitely based, C is finite and therefore finitely based, and the union of two finitely based classes is also finitely based. We now construct a further antichain that is maximal but whose closure is infinitely based:

Let B be the antichain produced by replacing every element

$$A_{2m} = \langle 3, 2, 5, 1, 7, 4, \ldots 4m+3, 4m, 4m+7, 4m+2, 4m+6, 4m+4, 4m+5 \rangle.$$

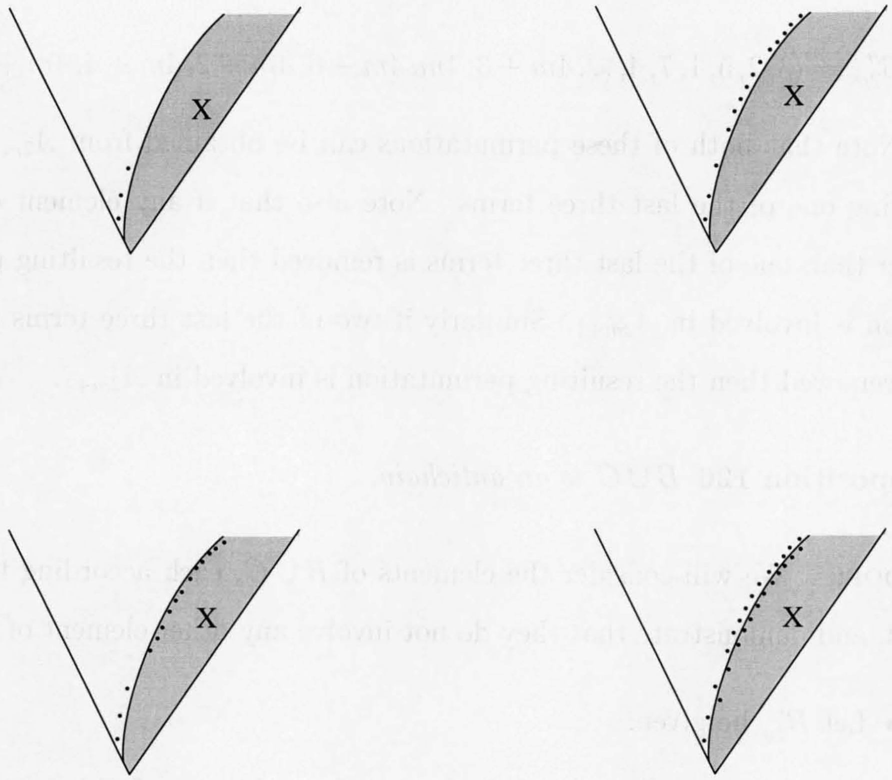


Figure 3.6: If M is the set of maximals and B is the basis of some class then M and B can both be finite (top left), M can be finite whilst B is infinite (top right), M can be infinite whilst B is finite (bottom left) and also when B is infinite (bottom right).

of A where A_{2m} is at least two longer[5] than the longest element of C with the two permutations

$$B'_{2m} = \langle 3,2,5,1,7,4,\ldots 4m+3, 4m, 4m+6, 4m+2, 4m+5, 4m+4\rangle$$

$$B''_{2m} = \langle 3,2,5,1,7,4,\ldots 4m+3, 4m, 4m+6, 4m+2, 4m+4, 4m+5\rangle.$$

Note that both of these permutations can be obtained from A_{2m} by removing one of the last three terms. Note also that if any element of A_{2m} other than one of the last three terms is removed then the resulting permutation is involved in A_{2m+1}. Similarly if two of the last three terms of A_{2m} are removed then the resulting permutation is involved in A_{2m+1}.

Proposition 126 *$B \cup C$ is an antichain.*

PROOF: We will consider the elements of $B \cup C$, each according to their kind, and demonstrate that they do not involve any other element of $B \cup C$.

- Let B'_{2n} be given.

  As $B'_{2n} \prec A_{2n}$ we have that no A_i and no element of C is involved in B'_{2n}.

  Every B''_{2i} involves $\langle 4,1,2,3\rangle$ but B'_{2n} does not, thus no B''_{2i} is involved in B'_{2n}.

  Suppose that some B'_{2i} with $2i \neq 2n$ is involved in B'_{2n}. Note that both B'_{2n} and B'_{2i} have unique subsequences order isomorphic to $\langle 3,2,4,1\rangle$,

[5] Assuming that the basis of $Sub(A)$ given in the last section is correct, no element of C has length greater than six and every permutation in A having an even suffix is replaced. The length restriction ensures that no B' or B'' is involved in any element of C.

these being the first four terms of B'_{2n} and B'_{2i}. Similarly note that both have unique subsequences order isomorphic to $\langle 4,1,3,2\rangle$, these being the last four terms of those permutations.

Thus both the first four and the last four elements of B'_{2n} are involved in any embedding of B'_{2i} in B'_{2n}. Furthermore note that as B'_{2i} is sum indecomposable every other element of B'_{2n} must be involved in the embedding. Thus $B'_i = B'_{2n}$. Reductio ad absurdum. Thus no B'_{2i} with $2i \neq 2n$ is involved in B'_{2n}.

- Similarly B''_{2n} does not involve any element of $B \cup C = \{B'_{2i}, B''_{2i}, A_{2i+1} | \text{all } i\}$ barring itself.

- No A_{2n+1} properly involves any element of $A \cup C$.

  By noting that A_{2n+1} and every B'_{2i} have unique subsequences order isomorphic to $\langle 3,2,4,1\rangle$ and $\langle 4,1,3,2\rangle$ we may demonstrate, by the argument given above, that if some B'_{2i} is involved in A_{2n+1} then B'_{2i} has length divisible by four, which is not the case (B'_i would have to have length $|A_{2n+1}| - 1$.). Similarly we may demonstrate that no B''_{2i} is involved in any A_{2n+1}.

- No element $C_i \in C$ properly involves any element of $A \cup C$. Every element of B' and of B'' is longer than every element of C and therefore no element of C properly involves an element of $B \cup C$.

■

Proposition 127 *The antichain $B \cup C$ is maximal.*

PROOF:

The antichain $A \cup C$ is maximal, and therefore every permutation either involves or is involved in an element of $A \cup C$. In order to classify permutations we may restate this as: Every permutation either involves or is properly involved in an element of $A \cup C$.

Let any permutation γ be given. If γ involves an element of $A \cup C$ then, as every element of $A \cup C$ involves an element of $B \cup C$, we have that γ involves an element of $B \cup C$.

If γ is properly involved in an element of $A \cup C$ then we have several cases to consider:

- γ is properly involved in an element of C or of an A_{2m+1}.
- γ is either a B'_{2m} or an B''_{2m}.
- γ is properly involved in an A_{2m} but is not of the form B'_{2m} or B''_{2m}. In this case γ is involved in A_{2m+1}.

In each of the above cases, which exhaust all possibilities, γ is involved in an element of $B \cup C$. Q.E.D.

■

Proposition 128 *The class $Sub(B \cup C)$ is infinitely based.*

PROOF: Every A_{2m} is a basis element of $Sub(B \cup C)$. ■

Antichain Classes

Cardinality of the basis of $Sub(A)$	Cardinality of antichain A finite	infinite
finite	possible	possible
infinite	not possible	possible

Figure 3.7:

3.4 Maximal, Trim, Fundamental

We will now consider a set of permutations later called *fundamental.* The reason for our curiosity can be seen in the list of antichains given in the Bibliothek, not repeated here for lack of space: The antichains are all of a type that we call fundamental and they all exhibit some very regular patterns.

Intuitively it is as though in an antichain element there are invariably two things of interest: distinct sets of "marker terms" that form "irregularities", and regular sets of terms that measure out the "distance" between the irregularities, or marker terms. In different antichain elements it is the differences in distance between irregularities that makes the antichain elements incomparable. In fundamental antichains it is as though there is only one distance that is measured out, between only two irregularities. (In graphs under deletion of edges and vertices, and in posets under inclusion, fundamental antchains can also be defined. There, the two irregularities typical of fundamental antichains do not always appear. Instead the single measure of distance measures a circular path, which needs neither beginning nor end.) We strongly encourage the reader to examine the Bibliothek antichains in this intuitive light.

Further support for our general aim can be found if we look at structures other than permutations under involvement. If we consider graphs having no repeated edges and no trivial loops under the partial order generated by removing an edge, or a vertex and its adjacent edges, then we will find that there are precisely two maximal infinite fundamental antichains. (Of these four, one consists of the set of all loops, plus one further graph, and the elements of the other three consist largely of chains with modified endings.) Another structure that shows this is the set of all finite partial orders under inclusion: One partial order is contained in the other if there exists an injective homomorphism f from the elements of the one to the other, a homomorphism in the sense that for any two a, b in the domain of f we have that $a < b$ if and only if $f(a) < f(b)$. In this case there are again finitely many maximal fundamental antichains, this time two. See Figure 3.8.

It may indeed be that the best approach that can now be made is to classify infinite fundamental antichains in a variety of other contexts, such as graphs under contraction, graphs with coloured edges under deletion of vertices, and other such systems. This may provide general results or insight sufficient to classify the antichains in this much more complex context.

In this section we compare four restrictions that we can apply to antichains. We are unable, in doing so, to explain or characterize the regular patterns that appear in fundamental antichains, but we do encounter a nice connection between infinite fundamental antichains and atomic classes. Nice because it is in general difficult to determine whether a given basis yields an atomic class whereas fundamental antichains show promise of regularity and of being classifiable. We proceed:

3.4.1 Definitions

Definition 129 An antichain A is *maximal* if there does not exist $\gamma \notin A$ such that $A \cup \{\gamma\}$ is an antichain.

Definition 130 An antichain A is *trim* if there does not exist $\alpha \in A$ and $\beta \prec \alpha$ such that $(A \setminus \{\alpha\}) \cup \{\beta\}$ is an antichain.

Lemma 131 *Let A be an antichain of size greater than one. Then A is trim if and only if every permutation β that is properly involved in one element of A, is also involved in another element of A.*

The exception arises from the lack of an 'other' element in any antichain of size one, which might indicate that all one permutation antichains are trim, which is not the case.

PROOF: Let A be given.

If A is not trim then there exist permutations $\alpha \in A$ and $\beta \prec \alpha$ such that $(A \setminus \alpha) \cup \{\beta\}$ is an antichain. The permutation β cannot be an element of A because $\alpha \in A$. Therefore β is properly involved in α but is not involved in any other element of A. This proves one direction of the lemma.

Suppose that there exist permutations $\alpha \in A$ and $\beta \prec \alpha$ such that β is not involved in any other element of A. No element of A is involved in β, as $\beta \prec \alpha \in A$ and A is an antichain. Therefore $(A \setminus \alpha) \cup \{\beta\}$ is an antichain, and A is not trim. QED ■

Definition 132 An antichain A is *strongly trim* if there does not exist a permutation β properly involved in an element of A and a subset B of A such that $\{\beta\} \cup B$ is an antichain having the same size as A.

Lemma 133 *An antichain having size at least two is strongly trim if and only if every permutation β properly involved in an element of A is involved in at least two elements of A, and indeed is involved in all but finitely many elements of A.*

An antichain is strongly trim if and only if every subset of the same size is trim.

The proof is almost identical to the proof of the last lemma, and we omit it.

It is easy to see that for a finite antichain the conditions of being trim and strongly trim are equivalent.

Definition 134 An antichain A is said to be *fundamental* if for every antichain $B \not\subseteq A$ with $|B| = |A|$ there does not exist a map $f : B \to A$ with the property that for all $\beta \in B$ $\beta \preceq f(\beta)$.

Lemma 135 *An antichain A is fundamental if and only if its closure contains no antichains of the same size as A, except those that are subsets of A.*

It will be shown that 'fundamental' is a strictly stronger quality than 'strongly trim', which in turn is strictly stronger than mere 'trimness'. In contrast maximal appears to be a quality independent of all of these. Although we will continue for two sections with proofs regarding this, the proofs will only emphasise the separation.

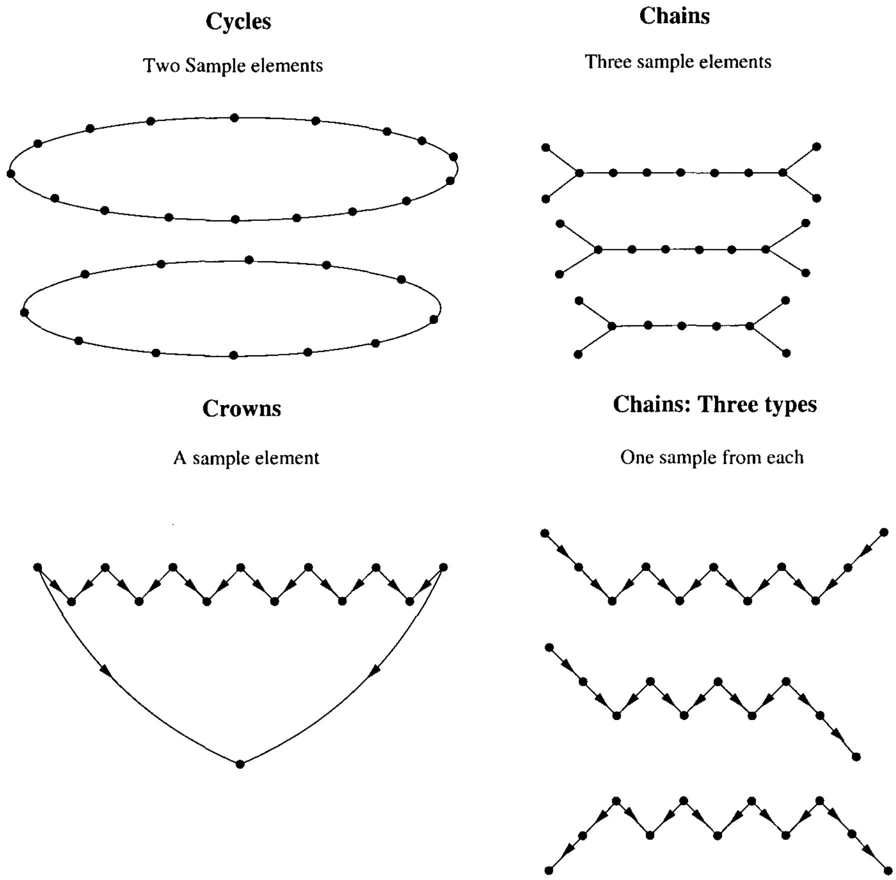


Figure 3.8: Infinite fundamental antichains in other contexts: In graphs under deletion of edges, or vertices and adjacent edges (above) and in finite partial orders under inclusion (below).

3.4.2 Some Technical Results

Proposition 136 *Let A be any antichain. Let C be the set of basis elements of $Sub(A)$ that do not involve any element of A. Then $A \cup C$ is a maximal antichain and the closed class with basis $A \cup C$ is precisely $PropSub(A) = Sub(A) \setminus A$. Furthermore, if A is trim then so is $A \cup C$.*

PROOF:

To show that $A \cup C$ is maximal let γ be any permutation. If $\gamma \in Sub(A)$ then γ is involved in an element of A. If $\gamma \notin Sub(A)$ then γ involves a basis element β of $Sub(A)$. Either $\beta \in C$, in which case we are done, or else β involves an element of A, in which case we are also done.

To show that $\mathcal{A}(A \cup C) = Sub(A) \setminus (A)$ is similar: As $A \cup C$ is maximal every permutation, γ, either involves or is involved in an element of $A \cup C$. If γ is properly involved in an element of C then γ is properly involved in an element of A and is therefore an element of $Sub(A) \setminus (A)$. If γ is properly involved in an element of A then γ is certainly an element of $Sub(A) \setminus (A)$. If γ is equal to or properly involves an element of A or of C then γ is not in $Sub(A) \setminus (A)$, as required.

To show that if A is trim then so is $A \cup C$ we use Lemma 131 and note that every permutation properly involved in an element of C is also involved in an element of A.

■

Corollary 137 *If there exists an infinite antichain A such that $Sub(A)$ is finitely based then there exists an infinite maximal antichain $A' \supseteq A$ such that $Sub(A')$ is finitely based.*

PROOF: If $Sub(A)$ is finitely based, then let C be defined as in the above proof. $Sub(C)$ is finite and therefore finitely based. $Sub(A \cup C) = Sub(A) \cup Sub(C)$, and the union of two finitely based classes is finitely based. ∎

Conjecture 138 *We may add to the end of Proposition 136 the following two clauses:*

1. *If A is strongly trim then so is $A \cup C$.*
2. *If A is fundamental then so is $A \cup C$.*

The conjecture certainly holds when C is finite, but C may be infinite.

Proposition 139 *If B is an infinite subset of a fundamental antichain A then B is a fundamental antichain. The same need not be true if B is finite, independent of whether A is infinite or finite.*

PROOF: If B is infinite and not fundamental then there exists an injective map $f : C \to B$ from C, an infinite antichain, to B with the property that for all $\gamma \in C$, $\gamma \preceq f(\gamma)$. f is also a map to A. This completes the proof for the case when B is infinite.

Fundamental antichains of size at least three exist. Therefore there exists such a fundamental antichain containing two elements γ_1, γ_2 such that $12 \preceq \gamma_1$ and $21 \preceq \gamma_2$. However $\{\gamma_1, \gamma_2\}$ is not a fundamental antichain, for it involves the antichain $\{1\,2, 2\,1\}$ without containing it. Q.E.D.

∎

3.4.3 Comparison of Maximal, Trim and Fundamental

Proposition 140 *The following hold:*

1. *Every fundamental antichain is strongly trim, but the converse does not hold.*
2. *Every strongly trim antichain is trim, but the converse does not hold.*
3. *A finite antichain is trim if and only if it is strongly trim.*

Thus for infinite antichains fundamental is a strictly stronger quality than strongly trim, which in turn is a strictly stronger quality than trim. For finite antichains fundamental is still a stronger quality than strongly trim, but strongly trim and trim are equivalent. Maximal, however, is a quality that will appear to be independent of trim, strongly trim and fundamental.

PROOF: It is sufficiently evident from the definitions that every fundamental antichain is strongly trim and every strongly trim antichain is trim that we omit that part of the proof. However we exhibit examples that demonstrate the converse parts of 1 and 2:

1. The finite strongly trim antichain $\{132, 312\}$ is not fundamental because it involves the antichain $\{12, 21\}$.

   For an infinite example of the same we direct the reader to Proposition 153.

2. We will now construct an example of an infinite trim but not strongly trim antichain: There exists an infinite fundamental antichain no element of which involves 321. The antichain ${}^{I_2}U_{I_2}$ from the library is

one such example, and we will use that. For reference, the first few elements of ${}^{I_2}U_{I_2}$ are:

$${}^{I_2}U_{I_2}(1) = 3\ 4\ 1\ 2 \qquad {}^{I_2}U_{I_2}(2) = \ 2\ 3\ 6\ 1\ 4\ 5$$

$${}^{I_2}U_{I_2}(3) = \ 2\ 3\ 5\ 1\ 8\ 4\ 6\ 7 \quad {}^{I_2}U_{I_2}(4) = \ 2\ 3\ 5\ 1\ 7\ 4\ 10\ 6\ 8\ 9$$

Let U be the set of all elements of ${}^{I_2}U_{I_2}$ that involve 123. The set U is infinite and trim.

Let U^R be the set of all elements of U written in reverse, so that a permutation $\alpha = a_1 a_2 \ldots a_m$ is in U^R if and only if the reversed permutation $a_m a_{m-1} \ldots a_2 a_1$ is an element of U. This set U^R is trim. Indeed every permutation properly involved in an element of U is involved in every longer element of U and by symmetry the same holds for U^R.

The union of U and U^R is an antichain, as every element of U involves 123 but avoids 321 and the reverse is true of U^R, whose every element involves 321 and avoids 123. Furthermore $U \cup U^R$ is trim. However it is not strongly so because $\{321\} \cup U$ is also an infinite antichain, hence equinumerous with $U \cup U^R$, and because one element, 321, is properly involved in an element of $U \cup U^R$ whilst the rest, U, form a subset of $U \cup U^R$. This completes the proof.

3. Part (3) of the proposition is simply a corollary of Lemma 133.

■

Proposition 141 *There exist the following:*

1. *A finite fundamental antichain that is not maximal.*

2. *An infinite fundamental antichain that is not maximal.*

3. *A finite maximal antichain that is not trim.*

4. *An infinite maximal antichain that is not trim.*

PROOF:

1. Let F be the set of all permutations of length three barring the increasing and decreasing sequences 123 and 321. Thus $F = \{132, 213, 231, 312\}$. First we claim that F is fundamental. Were F not fundamental there would be another antichain, G, with four elements every one of which is involved in an element of F. As G must be distinct from F this implies that G contains a permutation of length two or less. However it is easily seen that no antichain containing a permutation of length two can have more than two elements, and no antichain having an element of length one can have size greater than one. Thus G cannot exist and F is fundamental. However F is not maximal, it is a subset of S_3, the set of all permutations of length three, which is also an antichain.

2. To give an infinite example we use Proposition 139, which states that every infinite subset of a fundamental antichain is also fundamental. Constructing an example is now trivial.

3. The antichain $\{12, 321\}$ is maximal but not trim.

4. The antichain A of Theorem 116 is not trim. Indeed every element A_j of A involves a permutation denoted B'_j of length $|A_j|-1$ and which has the property that $A \cup \{B'_j\} \setminus \{A_j\}$ is an antichain. Furthermore $Sub(A)$

is finitely based. Let us denote by C the set of all basis elements of $Sub(A)$ that do not involve an element of A. Then $A \cup C$ is a maximal antichain, and by choosing a sufficiently long element of A we may demonstrate that $A \cup C$ is not trim.

■

3.4.4 Further Comparison

We have a heirachy by which trim is weaker than strongly trim, which is weaker than fundamental. Here we show that being maximal will not make a trim antichain strongly trim, or a strongly trim antichain maximal.

Proposition 142 *There exists a finite maximal trim antichain that is not fundamental.*

PROOF: Let M be the set consisting of 123, 321 and all permutations of length 4 avoiding both these triples:

$$M = \{123, 321, 3142, 2413, 2143, 3412\}$$

We will show that M is maximal. Note that every non-monotonic permutation of length three is involved in at least two of the elements of M. (A little more observation will show that this antichain is in fact trim!) Every permutation of length four either involves an increasing triple or a decreasing triple, or else is an element of M, and every permutation of length five or more involves either an increasing or a deceasing triple. Thus we have our desired result: M is maximal and trim. However it is not fundamental because it involves S_3, the set of all permutations of length three, a set that is an antichain with six elements[6]. ■

Proposition 143 *There exists an infinite maximal trim antichain that is neither fundamental nor strongly trim.*

PROOF: Let U and U^R be defined as in the proof of Proposition 140. (We merely need to modify the proof of that theorem a little for our new purpose.) U and U^R are both infinite and trim antichains. Every element of U avoids 321 and involves 123, every element of U^R involves 321 and avoids 123. Let C be the set of all basis elements of $Sub(U \cup U^R)$ that do not involve elements of either U or U^R. By Proposition 136 we have that $U \cup U^R \cup C$ is a maximal trim antichain. However it is not strongly trim because $321 \cup U$ is an antichain. ■

Proposition 144 *No strongly trim antichain T contains as a subset an infinite fundamental antichain F unless T is itself fundamental.*

PROOF: Let T be given. Suppose for the sake of argument that S is an infinite antichain, suppose that S is involved in T and suppose that S is not

[6]The equality of the numbers 3! and the number of permutations of length four avoiding both 123 and 321, plus two (from the permutations 123 and 321) seems to be coincidental. The equivalent calculation for increasing and decreasing sequenses of length four is as follows: The number of permutations of length 4 is 24. All permutations avoiding both 1234 and 4321 have length no greater than nine. There are 1764 permutations of length nine that avoid both 1234 and 4321. Even before adding two, this is greater than 24. However, concidentally perhaps, there are also 1764 permutations of length eight avoiding both 1234 and 4321. The quest for consistent numerical coincidences continues.

a subset of T. This implies that T is not fundamental. However, let F be given, an infinite fundamental subset of T. The set S contains an element σ properly involved in an element of T, and therefore in an element of F. Therefore, due to the fact that F is fundamental, only finitely many elements of S lie in $Sub(F)$. The remainder must all lie in $T \setminus F$, and let S' be that remainder. When σ is added to S' we obtain an infinite antichain, every element barring one of which is in T. This contradicts the assumtion that T is strongly trim. ■

3.4.5 A Chain Structure for Strongly Trim Antichains?

Here we struggle to find stronger statements that may help us to classify fundamental antichains.

Conjecture 145 *If A is a fundamental antichain then there exists a sequence $A_1, A_2, A_3, \ldots$ containing all but finitely many elements of A, with the property that any permutation properly involved in A_i, for some fixed i, is also involved in all subsequent elements of the sequence.*

It is readily proved that if A is an infinite fundamental antichain then there exists an infinite sequence $A_1 A_2 A_3 \ldots$ of elements of A having the property that every permutation properly involved in A_i, for any fixed i, is also properly involved in A_{i+1} and therefore by induction is involved in all consequent elements of the sequence.

We conjecture that for every fundamental antichain A, all but finitely many elements of A may be arranged into a single such sequence.

The conjecture is readily proved for infinite fundamental antichains, if instead of demanding that $A_1, A_2, A_3 \dots$ contains all but finitely many elements of A, we merely require the sequence to be infinite. However the strengthened conjecture, if true, has much greater value.

Conjecture 146 *If A is a fundamental antichain then there exist at most finitely many lengths n such that A has two or more permutations of length n.*

3.4.6 Antichains as Bases: Atomic classes

Proposition 147 *Let G be an infinite strongly trim antichain. Then $Sub(G) \setminus G$ is an atomic class.*

PROOF: By Lemma 133 we have that every permutation properly involved in an element of $Sub(G)$ is involved in all but finitely many elements of G. Thus if α and β are both permutations involved in elements of G then there exist only finitely many elements of G that do not properly involve both of them. Thus $Sub(G) \setminus G$ has the join property and therefore is atomic. ■

Proposition 148 *Every closed class having an infinite, maximal and strongly trim antichain B as its basis is atomic. Indeed such a class is equal to $Sub(B) \setminus B$.*

PROOF: As B is a maximal antichain we have that $\mathcal{A}(B) = Sub(B) \setminus (B)$. The result now follows from Proposition 147. ■

This is a pleasant result as it is in general difficult to determine whether a given closed class is atomic. We will demonstrate that the conditions of B being maximal, infinite and strongly trim cannot easily be weakened:

Proposition 149 *There exist the following:*

1. *A finite maximal fundamental and therefore also strongly trim antichain whose avoidance set $\mathcal{A}(G)$ is not atomic.*

2. *An infinite fundamental non-maximal antichain G whose avoidance class $\mathcal{A}(G)$ is not atomic.*

3. *An infinite, maximal, trim, but not strongly trim antichain G whose avoidance class $\mathcal{A}(G)$ is not atomic.*

This shows that every one of the conditions in Proposition 147 is necessary.

PROOF: (1) An example of a non-atomic class with a finite maximal and fundamental antichain as its basis is provided by $\mathcal{A}(S_3)$, where S_3 is the set of all permutations of length three. This makes $\mathcal{A}(S_3)$ the set of all permutations of length no greater than two, a class in which 12 and 21 do not join. It remains only to note that S_3 is indeed fundamental.

(2) Let A be an infinite fundamental antichain whose closure $Sub(A)$ is finitely based with basis B. Let C be the set of all elements of B that do not involve any element of A. The set $A \cup C$ is an infinite fundamental antichain. Moreover every permutation not in the closure of A involves an element of B, therefore an element of C or an element of A shorter than the longest element of B. In fact it may also be shown that $A \cup C$ is an infinte fundamental and maximal antichain. Let α_1 and α_2 be any two distinct elements of A at least as long as the longest element of B.

The set $(A \cup C) \setminus \{\alpha_1, \alpha_2\}$ is a fundamental antichain, an infinite subset of a fundamental antichain. The class with this set as a basis is however not atomic because α_1, α_2 do not join in it: A permutation that involves both α_1 and α_2 does not lie in the closure of A, therefore it involves an element of B, therefore it involves an element of C or an element of A shorter than the longest element of B. The construction is complete, of an infinitely and fundamentally based non-atomic closed class.

(3) To construct a nonatomic class with an infinite maximal and trim antichain for a basis we use the antichain $U \cup U^R$ of Proposition 140, which is infinite and trim. Every element of U properly involves 123, every element of U^R properly involves 321, but no element of either set involves both. Let C be the set of all basis elements of $Sub(U \cup U^R)$ that do not involve elements of $U \cup U^R$. By Lemma 136 we have that $U \cup U^R \cup C$ is a maximal and trim antichain whose avoidance class is $Sub(U \cup U^R) \setminus (U \cup U^R)$. However as noted this class contains both 123 and 321 but does not contain any permutation that involves both 123 and 321. Thus 123 and 321 do not join and this class is not atomic. This completes the proof.

■

Question 150 The antichain S_2 consisting of the two permutations of length two is both maximal and fundamental. The class that avoids it consists of the trivial permutation 1 and is therefore, trivially, atomic. Does there exist a non-trivial finite atomic class having a maximal fundamental antichain as its basis?

Proposition 151 *There exists an atomic class with a finite non-fundamental antichain as a basis.*

PROOF: The class $\mathcal{A}(123, 231)$, plucked from the Bibliothek, is atomic. An atomic representation for that class is also given there. ■

Question 152 Does there exist an infinitely based atomic class the basis of which is not fundamental?

3.4.7 The Delayed Example

Here we give an example of an infinite strongly trim but non-fundamental antichain.

Proposition 153 *There exists an infinite non-fundamental antichain Q having the property that every permutation properly involved in an element of Q is involved in all but finitely many elements of Q. Therefore there exists an infinite non-fundamental strongly trim antichain.*

PROOF:

We begin by constructing an antichain that satisfies the first statement in the proposition. By Lemma 133 the second statement follows directly from the first.

We will need for our construction an auxiliary infinite antichain every element of which has no non-trivial intervals. For this we use an antichain chosen from the library and easily verified to be an antichain, and expand it a little to end up with three elements of length eight and two of every longer even length.

Let A be the set of all elements in the antichain listed under 'Both ends tied by one' in the Bibliothek with length at least eight (the shorter elements of that antichain have non-trivial intervals). The first few elements of A are:

$$A_1 = \ 4\ 1\ 6\ 3\ 8\ 5\ 2\ 7 \quad A_2 = \ 4\ 1\ 6\ 3\ 8\ 5\ 10\ 7\ 2\ 9$$

$$A_3 = \ 4\ 1\ 6\ 3\ 8\ 5\ 10\ 7\ 12\ 9\ 2\ 11 \quad A_4 = \ 4\ 1\ 6\ 3\ 8\ 5\ 10\ 7\ 12\ 9\ 14\ 11\ 2\ 13$$

and in general:

$$A_n = 4\ 1\ 6\ 3\ 8\ 5\ \ldots\ 2n+4\ \ 2n+1\ \ 2n+6\ \ 2n+3\ \ 2\ \ 2n+5$$

Note that no element of A contains a proper non-trivial interval. That is, the elements of A are elements of P and are equal to their own P-frames. Also note that every permutation properly involved in an element A_i of A is also involved in all subsequent elements of A.

Let B be the set of permutations consisting of the following:

- The permutation A_1 rotated in an anticlockwise direction by 90 degrees, which we will denote B_0 and which is:

$$B_0 = \ 5\ 8\ 3\ 6\ 1\ 4\ 7\ 2$$

- The elements of A reversed. We will denote the reversal of A_i by B_i. Thus, for instance:

$$B_3 = \ 11\ 2\ 9\ 12\ 7\ 10\ 5\ 8\ 3\ 6\ 1\ 4$$

$$B_4 = \ 13\ 2\ 11\ 14\ 9\ 12\ 7\ 10\ 5\ 8\ 3\ 6\ 1\ 4$$

We claim that the elements of A and of B are incomparable under involvement: No element of A is involved in any other element of A because A is an antichain. Every element of A involves 1234 and avoids 4321 whilst the reverse is true of the elements of B, from which we infer that no element

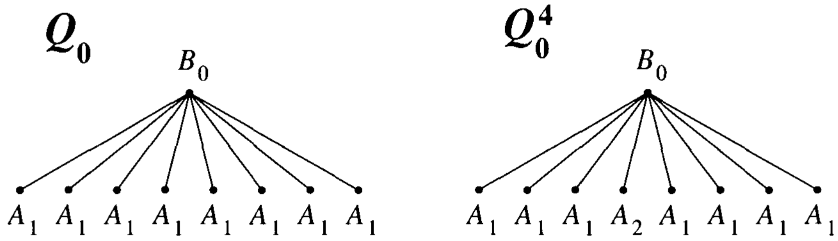


Figure 3.9: The P-frame structure of Q_0^4 is derived from that of Q_0.

of A is involved in an element of B and vice versa. All the elements of B barring B_0 are obtained from A by the same symmetry, hence $B \setminus B_0$ is an antichain. No element of $B \setminus B_0$ is involved in B_0 and by noting that B_0, unlike every other element of B, involves 3214 we have that no element of B is involved in any other. Thus $A \cup B$ is an antichain as required.

Note also that the elements of B, being obtained by symmetries from those in A, have no proper non-trivial intervals. We now construct our antichain Q using the wreath product:

- Let Q_0 be the permutation obtained by replacing every term of B_0 with the permutation A_1. That is, let $Q_0 = B_0 \wr \langle A_1, A_1, A_1, A_1, A_1, A_1, A_1, A_1 \rangle$. Thus Q_0 has length $8^2 = 64$ and some of the terms of Q_0 are given:

$$\begin{aligned} Q_0 &= \underbrace{36\ 33\ 38\ 35\ 40\ 37\ 34\ 39}_{\cong A_1}\ \underbrace{60\ 57\ 62\ 59\ 64\ 61\ 58\ 63}_{\cong A_1} \ldots \\ &\quad \ldots \underbrace{12\ 9\ 14\ 11\ 16\ 13\ 10\ 15}_{\cong A_1} \end{aligned}$$

- We construct Q_1:

  We denote by Q_0^1 the permutation $B_0 \wr \langle A_2, A_1, A_1, A_1, A_1, A_1, A_1, A_1 \rangle$. This can equally be constructed by replacing the first minimal non-trivial interval of Q_0 by the permutation A_2. Similarly we will let Q_0^2 be defined by $Q_0^2 = B_0 \wr \langle A_1, A_2, A_1, A_1, A_1, A_1, A_1, A_1 \rangle$ and in general we will let Q_0^i denote the permutation obtained by replacing the i^{th} minimal non-trivial interval of Q_0 with A_2.

Note now, for later reference, that Q_0 is not involved in any Q_0^i. Similarly note that, as every permutation properly involved in A_1 is also involved in A_2, every permutation properly involved in Q_0 is also involved in some Q_0^i. Indeed we can state that if a single term from the i^{th} minimal non-trivial interval of Q_0 is removed then the remains of that interval are involved in A_2 and therefore Q_0 with that term removed is involved in Q_0^i. To summarise, we have defined eight Q_0^i.

B_1 has length eight. Let Q_1 be the permutation obtained by replacing the first term of B_1 with Q_0^1, replacing the second term of B_1 with Q_0^2 and in general replacing the i^{th} term of B_1 with Q_0^i, where $i \in \{1, 2, \ldots, 8\}$.

- Q_1 does not involve Q_0 because:

  Q_0 is not involved in any of the maximal proper intervals of Q_1, as these are of the form Q_0^i. Therefore Q_0 can only be involved in Q_1 if the top RIP frame of Q_0 is involved in that of Q_1. However the top RIP frame of Q_0 is B_0, that of Q_1 is B_1, two elements of an antichain. Thus Q_0 is not involved in Q_1.

- Q_1 involves every permutation properly involved in Q_0: Trivially, because Q_1 involves every Q_0^i.

- Summary: Q_1 has top P-frame B_1, second level P-frames B_0 and the first, or lowest level P-frames are elements of A. The lowest level P-frames are the minimal non-trivial intervals of Q_1. Of these there are eight that are order isomorphic to A_2 and the remaining $8^2 - 8$ are order isomorphic to A_1. This, incidentally, permits us to calculate the length of Q_1 as $10 * 8 + 8 * (8^2 - 8) = 528$.

- We construct Q_2:

  Q_1 has 64 minimal non-trivial intervals, all order isomorphic to either A_1 or A_2. Moreover these intervals contain every term of Q_1. As with Q_0 define Q_1^i to be the permutation obtained by replacing the i^{th} minimal non-trivial interval of Q_1 with A_3. As previously Q_1 is not involved in any Q_1^i but all permutations properly involved in Q_1 are involved in at least one of $Q_1^1 \dots Q_1^{64}$.

  B_{29} has length 64. Let Q_2 be the permutation obtained by replacing each i^{th} term of B_{29} with Q_1^i. As previously Q_2 does not involve Q_1 or Q_0 but Q_2 does involve every permutation properly involved in Q_1 and by induction every permutation properly involved in Q_0.

- Summary: The top P-frame of Q_2 is B_{29}. The second level frames are shorter and are all equal to B_1. The third level frames are all B_0. The bottom frames, corresponding to minimal non-trivial intervals, are A_1, A_2 or A_3. Of the last, there are $8^2 = 64$ equal to A_3, which has length 12. There are $(8^2 - 1) * 8$ equal to A_2 which has length 10, and $(8^2 - 1)(8^2 - 8)$ equal to A_1 which has length 8. Thus the length of Q_2 is 34032.

- Now in general assume that Q_j ($j > 0$) has the following properties:

    1. Q_j has n minimal non-trivial intervals, where $n = 8^{2^j}$. Each of these is order isomorphic to one of $A_1 \dots A_{j+1}$.

    2. The top P-frame of Q_j is B_k where $k = \frac{\sqrt{n}-6}{2}$. (E.g. when $j = 2$ $n = 64^2 = 5096$ and $k = 29$.)

    3. All second level P-frames of Q_j are elements of B shorter than its top P-frame, and are identical to the top frame of Q_{j-1}. Similarly the third level P-frames are shorter than the second and are identical to the top frame of Q_{j-2}, and so on. The bottom level but one P-frames are identical to the top frame of Q_0. The bottom level P-frames are elements of A and coincide with the minimal non-trivial intervals.

    4. No previous element $Q_p \in Q$ ($p < j$) is involved in Q_j.

    5. Every permutation properly involved in a predecessor of Q_j is involved in Q_j.

Now to construct Q_{j+1}.

As previously for each $i \in \{1, \dots, n\}$ define Q_j^i to be the permutation obtained by replacing the ith minimal non-trivial interval of Q_j with A_{j+2}. It is important not to forget that each Q_j^i is identical to Q_j except for this replaced interval; and that, for instance, the number of minimal non-trivial intervals of Q_j^i is the same as that of Q_j. Also recall that none of the n classes $Sub(Q_j^i)$ involves Q_j but that between them they involve every element of $Propsub(Q_j)$.

The element B_l of B where $l = \frac{n-6}{2}$ has n terms. Define Q_{j+1} to be the permutation obtained by replacing the first term of B_l with Q_j^1 and in general replacing the ith term of B_l with Q_j^i, where $i \in \{1, \ldots, n\}$. Now:

1. It is clear that Q_{j+1} has n^2 minimal non-trivial intervals, which makes the number of these intervals $8^{2^{j+1}}$.
2. The top P-frame of Q_{j+1} has been chosen to be B_l where $l = \frac{n-6}{2} = \frac{\sqrt{n^2}-6}{2}$.
3. The maximal intervals of Q_{j+1} are order isomorphic to permutations of the form Q_j^i which are identical to Q_j except for the substitution of one minimal non-trivial interval for a longer element of A.

   Thus, in order of decreasing length: The second level P-frames of Q_{j+1} are the top level P-frames of Q_j, the third level P-frames are the top P-frames of Q_{j-1}, and so on down to the bottom level but one and bottom P-frames that are B_0 or elements of $\{A_1 \ldots A_{j+2}\}$ respectively.
4. (3) gives rise to the pyramidal structure illustrated in Figure 3.10. Note that Q_{j+1} does not involve any of $Q_1, \ldots, Q_j$. Indeed consider Q_p where $Q_p \in \{Q_1, \ldots, Q_j\}$:

   Every interval of Q_{j+1} has a top P-frame that is an element of the antichain $A \cup B$, and the top P-frame of Q_p is also an element of $A \cup B$. Thus if Q_p is involved in Q_{j+1} then Q_p must be involved in one of the intervals that has top P-frame the same as that of

Q_p. However those intervals are identical to Q_p except that they are mutilated by substituting minimal non-trivial intervals, that are order isomorphic to elements of A, by longer elements of A. This means that Q_p is not involved in Q_{j+1}.

These results 1-4 correspond to assumptions 1-4 and by induction the assumptions hold for all elements of Q, after Q_0.

- It can be seen that the length of Q_{j+1} can be calculated quite easily from that of Q_j by:

$$|Q_{j+1}| = 8^{2^j} * 2(j+5) + \left(8^{2^j} - 1\right)|Q_j|.$$

This indicates that the elements of Q grow rapidly in length! Q_3 has length 139418384, Q_4 has length just under $2.34 * 10^{15}$.

Thus we have an infinite antichain Q, not fundamental, in which every permutation properly involved in an element of Q is involved in every subsequent and therefore all but finitely many elements of Q. Q.E.D. ∎

3.5 Partially Well Ordered

Given a set under a partial order it is always interesting, in its own right, to know whether the set contains an infinite antichain. Here we consider closed classes and call them *partially well ordered* if they do not contain such an antichain. Normally the definition of partially well ordered also forbids infinite descending chains, however in the context of permutations, which have finite positive integer length, that condition is always satisfied for them.

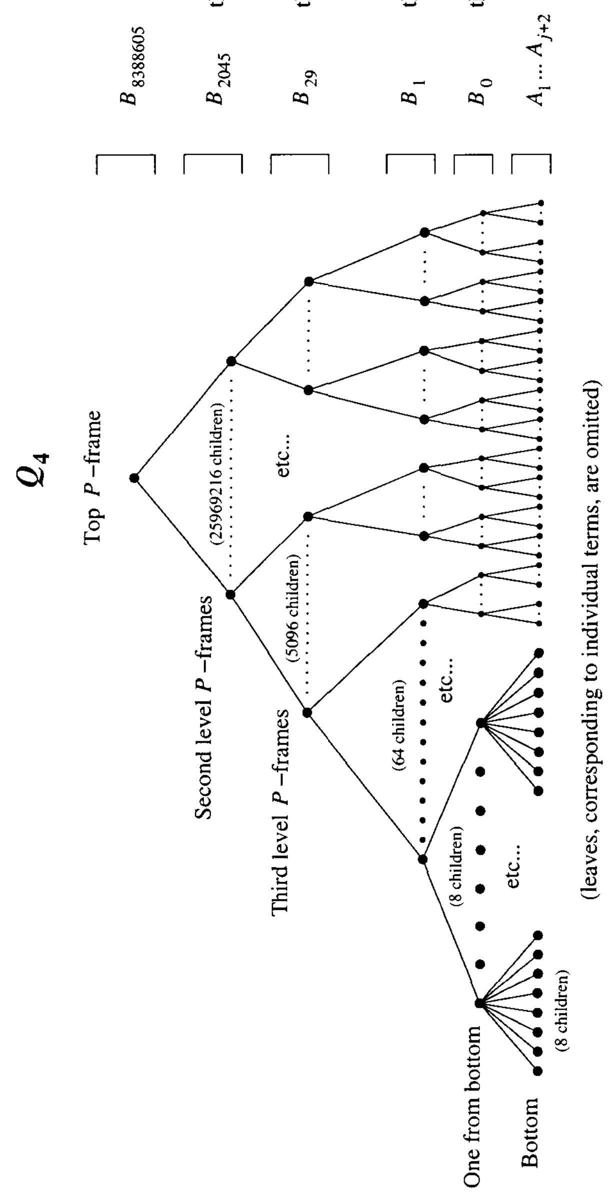


Figure 3.10: The P-frame structure of an element of Q.

Specific rewards that we expect are in classifying all fundamental antichains and in determining whether a class is atomic or not. The first depends on the further understanding of infinite antichains gained in this exercise, the latter has more tangible effects which are given in the next chapter.

We will apply Higman's Theorem [12] to the sum, skew sum and wreath constructions and with that prove that certain classes are strongly finitely based. We then show that there exists an algorithm that will terminate if a given finitely based class has finitely many interval free permutations. This, combined with our results from Higman's Theorem give us a mechanism for producing the beginning of a list of strongly finitely based classes. We proceed:

3.5.1 Definition

Definition 154 Let X be a closed class. X is said to be *partially well ordered* if it does not contain an infinite antichain. X is said to be *strongly finitely based* if its basis is finite and it is partially well ordered.

Example 155 The set $\mathcal{A}(12) = I$ of all strictly increasing sequences is partially well ordered, indeed every two elements of I are comparable which means that not only does I contain no infinite antichain, it contains no antichain of size greater than one. This is, up to equivalence, the only class with one basis element of length two.

A crude initial example of a non-partially well ordered class is $\mathcal{A}(\emptyset) = S$, the set of all permutations. Since an infinite antichain of permutations exists, it is contained in this class. A more interesting example can be found among the classes with two basis elements of length three:

In [8] D. Spielman and M. Bóna constructed an infinite antichain that lies in $\mathcal{A}(123)$. This is remarkable because up to symmetry the only other class with one basis element of length three is $\mathcal{A}(132)$ which is partially well ordered, as proved in unpublished work by M. Atkinson and N. Ruškuc. It would appear that although $\mathcal{A}(123)$ and $\mathcal{A}(132)$ have the rare property of equal enumeration they differ in almost every other respect.

The partial well orderedness of $\mathcal{A}(132)$ can also be derived on results in [4], repeated in the following section, that imply that $\mathcal{A}(2413, 3142)$ is partially well ordered. That $\mathcal{A}(132)$ is partially well ordered follows easily because it is a subset of $\mathcal{A}(2413, 3142)$.

3.5.2 Higman's Theorem and Strong Completion

Proposition 156 *If X is a closed class not containing an infinite antichain then the strong completion of X, which is the completion under $\oplus$ and $\ominus$, does not contain an infinite antichain.*

PROOF: Let f be a 1-1 correspondence from a set A of symbols to the strongly indecomposable elements of X. Define the set X' inductively as follows:

- If $a \in A$ then $a \in X'$.
- If $a, b \in X'$ then $(a + b) \in X'$.
- If $a, b \in X'$ then $(a - b) \in X'$.

Note that + and − are not associative on X', indeed X' represents not the elements of X or its strong completion but the *decompositions* of elements of the *strong completion of* X. To illustrate this note that 123 has two distinct decompositions: $(1 \oplus (1 \oplus 1))$ and $((1 \oplus 1) \oplus 1)$. We will therefore be strict in maintaining brackets in expressions in X'.

We define an ordering on X' by the following means. If $a, b \in X'$ then we define $a \leq b$ if one of the following holds:

- $a, b \in A$ and $f(a) \preceq f(b)$.
- $b = (x + y)$ and $a \leq x$ or $a \leq y$.
- $b = (x - y)$ and $a \leq x$ or $a \leq y$.
- $a = (w + x)$ and $b = (y + z)$ and $w \leq y$, $x \leq z$.
- $a = (w - x)$ and $b = (y - z)$ and $w \leq y$, $x \leq z$.

Note that $\leq$ is a partial ordering. Restricted to elements of A, $b \leq c$ is equivalent to $f(b) \preceq f(c)$. The qualities of being reflexive, antisymmetric and transitive follow inductively from this base.

Now, Higman's theorem on Ordering by Divisibility [12] states that providing the following conditions hold, the set X' has no infinite antichain. We list the conditions almost word for word from Higman's paper, and it will be found that in this, our case, some of the requirements are trivially satisfied.

1. A has no infinite antichain under $\leq$.
2. $\leq$ is a quasi ordering, i.e. it is transitive.

3. If $a \leq b$ then $(a+c) \leq (b+c)$ for all $a, b, c \in X'$.

4. If $a \leq b$ then $(a-c) \leq (b-c)$ for all $a, b, c \in X'$.

5. If $a \leq b$ then $(c+a) \leq (c+b)$ for all $a, b, c \in X'$.

6. If $a \leq b$ then $(c-a) \leq (c-b)$ for all $a, b, c \in X'$.

   (Requirements (3)-(6) represent the constraints of X' being an ordered algebra.)

7. $a \leq a+b$ and $a \leq b+a$ for all $a, b \in X'$.

8. $a \leq a-b$ and $a \leq b-a$ for all $a, b \in X'$.

   (Requirements (7) and (8) are the constraints of divisibility.)

(1) is satisfied by definition. X contains no antichain under involvement of permutations. Thus the set of $\oplus$ and $\ominus$ indecomposable permutations of X contains no antichain under involvement. Thus A contains no infinite antichain under $\leq$. (2) is satisfied because $\leq$ is a partial ordering. Constraints (3)-(6) are satisfied because of the last two points in the definition of $\leq$ and the reflexivity of $\leq$. Constraints (7) and (8) are satisfied because of the second and third points in the definition of $\leq$ and the reflexivity of $\leq$.

Thus we conclude that X' has no infinite antichain under $\leq$.

Now we demonstrate that the strong completion of X contains no infinite antichain under involvement. First we must extend the function f so that f maps from X' to the strong completion of X. As X' represents all the decompositions of elements of the strong completion of X this is easily done by induction:

- We already have that f is a bijection from A to the indecomposable permutations of X.
- Define $f(a+b) = f(a) \oplus f(b)$ for all $a, b \in X'$. (As $\oplus$, like $\ominus$, is an associative operation no brackets are required.)
- Define $f(a-b) = f(a) \ominus f(b)$ for all $a, b \in X'$.

Note that f is surjective but that f need not be and indeed is not injective. Further note that f is order preserving in that $a \leq b$ implies that $f(a) \preceq f(b)$.

Now suppose that D is an infinite antichain of permutations in the strong completion of X. Then let D' be a minimal set of elements of X' such that $f(D') = D$. As f is a surjective function D' exists, and furthermore as $|D'| = |D|$ we have that D' is infinite. As X' contains no infinite antichain there exist two distinct terms $a, b \in D'$ such that $a \leq b$. But then $f(a) \neq f(b)$ and $f(a) \preceq f(b)$. Reductio ad absurdum.

■

Corollary 157 *The set of decomposable permutations, which is the $\oplus$ and $\ominus$ closure of 1, contains no infinite antichain.*

Note: By [2] the set of $\oplus - \ominus$ decomposable permutations are precisely those avoiding 2413 and 3142.

Corollary 158 *A closed class X has an infinite antichain if and only if the set of indecomposable elements of X contains an infinite antichain under involvement.*

Conjecture 159 *If X is strongly finitely based then the completion of X under $\oplus$ and $\ominus$ is partially well ordered but need not be finitely based.*

3.5.3 Some Classes with Small Bases

Up to symmetry there are only two permutations of length three. Example 155 states that $\mathcal{A}(132)$ is partially well ordered and that $\mathcal{A}(123)$ is not.

The only permutations of length four that avoid both 123 and 321 are, up to symmetry, 2413 and 2143. The infinite antichain W listed in the Bibliothek and in [2] avoids both these permutations. Thus we may conclude that all classes with a single basis element of length four contain an infinite antichain and are therefore not partially well ordered.

All permutations of length five or more involve either 123 or 321 and therefore all classes with a single basis element of length five are therefore not partially well ordered.

All the classes with two basis elements of length three contain 132 or a permutation symmetric to it, barring the class $\mathcal{A}(123, 321)$, which is finite. Thus all classes that have two basis elements of length three are partially well ordered.

Of the classes with one basis element of length three and one basis element of length four, some such as $\mathcal{A}(321, 2413)$ are partially well ordered, others such as $\mathcal{A}(321, 3412)$ are not. The Bibliothek gives a listing of all such pairs up to symmetry and indicates which classes are partially well ordered. A similar but incomplete listing exists for classes with two basis elements of length four.

3.5.4 Higman's Theorem and the Wreath Product

The operation of sum ($\alpha \oplus \beta$) is a special case of the wreath product, being equal to replacing the first and second terms of 12 with a pair of permutations. (In symbols, $\alpha \oplus \beta = 12 \wr (\alpha, \beta)$.). If we replace 12 with any other permutation η then we will equally obtain an operation: If η has length n then the operation will be on the set of all n-tuples of permutations, for instance $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and will yield $\eta \wr (\alpha_1, \ldots, \alpha_n)$.

We can extend the results on strong completion and partial well orderedness to any set of permutations obtained from 1 by a set of wreath operations. As each wreath operation, like sum and skew sum, can act as operations with divisibility in an ordered algebra, this is easily done providing that we consider only finitely many operations. The special case of Higman's Theorem that we use specifies that there are only finitely many operations.

Now, every permutation can be deconstructed by RIP frames, or conversely constructed from the trivial permutation by means of sum, skew sum, or a wreath operation of the sort that we have mentioned, with the added restriction that now η must be an element of P (the set of all interval free permutations of length strictly greater than two). This, by closure, implies that if a closed class contains only finitely many interval free permutations then it is partially well ordered.

Here we prove a structure result for permutations in P, and that structure result gives a decision mechanism that halts if a given closed class has only finitely many elements of P. All we require from the class is a method of determining whether a given permutation is in the class or not. This is a much weaker condition than finite basedness, which gives our result very wide applicability!

The structure result also permits us to find the elements of P in a class quite quickly as it permits us to generate them instead of having to sift through all $n!$ permutations of each length n.

The Demi-Closed nature of P

The main result of this section is Theorem 164, which states that for all $\alpha \in P$ with $\alpha \notin G$ (a straightforward set of permutations to be defined below), there exists a $\beta \in P$ such that $\beta \prec \alpha$ and $|\beta| = |\alpha| - 1$.

We call a permutation α in P *basic* if there is no term $a_i \in \alpha$ such that $\alpha \setminus \{a_i\}$ is order isomorphic to an element of P. This is equivalent to stating that α is a basic element of P if there does not exist a permutation $\beta \in P$ such that $\beta \prec \alpha$ and $|\beta| = |\alpha| - 1$. We claim that the basic elements of P are those of the set G, which consists of permutations of the form:

$$2\ 4\ 6\ \ldots\ 2m-4\ 2m-2,\ 2m\ 1\ 3\ 5\ \ldots\ 2m-3\ 2m-1$$

and symmetries of it. Thus the permutations in G of length up to eight are:

Length four:	2413	3142
Length six:	246135	531642
	415263	362514
Length eight:	24681357	75318642
	51627384	48372615

It is readily seen that the elements of G are in fact basic elements of P, we have only the small task of demonstrating that these are all the elements of P. (That converse result has been verified by computer for all permutations of length up to ten, which is irrelevant to our proof, but is consistent with it.)

Rules Concerning Adjacency

The hardest part of our proof consists of the following observations. Suppose that $\alpha = a_1 \ldots a_n$ is a permutation in P, that a_i is a term of P and that M is a set of terms that form a proper non-trivial interval of $\alpha \setminus \{a_i\}$ (α with the term a_i removed).

Firstly, if we choose M to be **minimal**, in the sense that no subset of M has the same qualities, then:

- Either M is of size two.
- Or the terms of M form a sequence order isomorphic to an element of P.

Concerning the position of M compared with that of a_i we have that:

- Either M contains both a_{i+1} and a_{i-1} (we call this *horizontal separation*),
- Or M contains both $a_i + 1$ and $a_i - 1$ (*vertical separation*).

If neither of the above holds then all the terms of M must all lie above and to the right of a_i, or all lie below and to the right, or all lie above or below and to the left. In any of these four cases M is a proper interval not only of $\alpha \setminus \{a_i\}$ but also of α itself, a contradiction.

There are other positions that a_i cannot have with respect to M. If J is any set of terms then we say that a_i is *horizontally adjacent* to J if J

contains either a_{i+1} or a_{i-1}, and we say that a_i is *vertically adjacent* if J contains either $a_i + 1$ or $a_i - 1$. Now suppose that a_i is both horizontally and vertically adjacent to J (we simply say *adjacent*) and suppose that J is an interval of α or of $\alpha \setminus \{a_i\}$, it does not matter which. Then a_i and J together form an interval of α. If J is M then this interval is both proper and non-trivial and therefore we have a contradiction. Therefore we have that:

- a_i cannot be adjacent to M.

That is all. The coming lemmata are targets that we prove by these principles. The reader looking for intuition can disregard the proofs. All the intuition is contained within the basic principles by which the proofs are generated, and those are given by the observations. We merely consider one term, either prove our lemma by the above or else deduce the existence of one or two other terms to which we can apply our considerations. This gives rise to a tree of possibilities, but always a finite one. The fact that it was easy to guess the structure of P-permutations demonstrates, as is never sufficiently often demonstrated, that intuition probably has a rather better system of logic than that which this proof relies on!

Reduction to Pairs

If M is a minimal non-trivial interval of α with some term removed then M either has size two, or else is order isomorphic to an element of P. These two options describe precisely the set of all sequences that have no proper non-trivial interval. We here show that if α is a basic element of P then there exists a term a_k of α such that $\alpha \setminus \{a_k\}$ has an interval of size two.

Lemma 160 *Let $\alpha = a_1 \dots a_n$ be an element of P. Then there exists a term a_i such that $\alpha \setminus \{a_i\}$ is either (i) order isomorphic to an element of P or (ii) has an interval of size two.*

PROOF: Given any term a_i of α we have that at least one of the following must hold:

1. $\alpha \setminus \{a_i\}$ is order isomorphic to an element of P.

2. $\alpha \setminus \{a_i\}$ has a proper interval of size two.

3. $\alpha \setminus \{a_i\}$ has a proper interval M order isomorphic to an element of P.

(1 excludes the possibilities 2 and 3, but 2 and 3 can occur together.)

If there exists a term a_i such that either 1 or 2 holds then we are done. We will show, by a technical argument, that if 3 holds and 1 does not, then 2 must hold. For this purpose let a_i and M be chosen such that M has the smallest possible size; that is, let there not exist a subsequence β of α and a term a_j of α such that (i) the sequence β is order isomorphic to an element of P, and (ii) the sequence β is a proper interval of $\alpha \setminus \{a_j\}$, and (iii) the size, that is the number of terms, of β is strictly less than the size of M. Clearly a_i and M can be chosen to satisfy this.

Notationally it will also greatly assist us if we assume, without loss of generality, that a_i precedes all the terms of M. We can assume this by the eight symmetries that can be performed on a given permutation. We denote the terms of M by $a_{m+1} \dots a_{n+1}$ and we also can assume that i is strictly less than m; because if $i = m$ then a_i is adjacent to M making M and a_i together a proper interval of α, a contradiction. Now:

Consider M as a sequence in itself, and let a_k be some term of M. Either $M \setminus \{a_k\}$ has an interval of size two or else it does not. In the latter case we claim that $\alpha \setminus \{a_k\}$ is order isomorphic to an element of P, in which case case (1) is satisfied. Indeed suppose first that $M \setminus \{a_k\}$ is order isomorphic to an element of P. Then:

- Assume that $\alpha \setminus \{a_k\}$ has some proper non-trivial interval J.

- If J contains two or more terms of $M \setminus \{a_k\}$ (an element of P) then J contains all terms in $M \setminus \{a_k\}$, which implies that a_k is adjacent to J and that a_k and J together form a proper interval of α, a contradiction.

- J must contain either both $a_k + 1$ and $a_k - 1$ or both a_{k-1} and a_{k+1}. Therefore J contains at least one term of M, therefore J contains precisely one term of M. (It cannot be that a_k is the largest term of M and that a_i is one smaller than a_k or else $M \setminus \{a_k\}$ is an interval of $\alpha \setminus \{a_k\}$, contradicting the minimality of M. Similarly a_k cannot be the smallest term of M and a_i be a term one greater than a_k.)

- The terms of $\alpha \setminus \{a_k\}$ can be divided into those of $M \setminus \{a_k\}$, into a_i, and into the remaining terms which are distributed either above and to the left of all terms of $M \setminus \{a_k\}$, above and to the right of all terms of $M \setminus \{a_k\}$, below and to the left or below and to the right of all terms of $M \setminus \{a_k\}$. As $M \setminus \{a_k\}$ is order isomorphic to an element of P, its largest and leftmost cannot be the same, therefore if J contains a term that lies above and to the left of $M \setminus \{a_k\}$ then J contains at least two terms of $M \setminus \{a_k\}$, a contradiction.

Similarly J does not contain a term that lies above and to the right of all $M \setminus \{a_k\}$, nor one that lies in either of the other two classifications of this form. Nor can J contain a_i because i is strictly less than m and the leftmost term of $M \setminus \{a_k\}$ is either a_{m+1} or a_{m+2}, and so a_m would have to be a term of J.

Thus J consists of only terms in $M \setminus \{a_k\}$, and only one of those. Reductio ad absurdum.

The only other case that we need consider is that where $M \setminus \{a_k\}$ has a proper interval N order isomorphic to an element of P. In that case:

- N cannot be an interval of $\alpha \setminus \{a_k\}$ because this would contradict the minimal size of M. However N is an interval of $\alpha \setminus \{a_i, a_k\}$, which implies that the terms of $\alpha \setminus \{a_k\}$ can be divided into those of N, the term a_k, and the remaining terms that must lie either above and to the left of all terms in N, above and to the right of all terms in N, below and to the left or below and to the right of all terms in N.

  We can therefore argue as before: Assume that J is a proper non-trivial interval of $\alpha \setminus \{a_k\}$ and we will find that J contains only a single term, and that one in N.

That completes the proof of our claim. So either case (1) holds, or else $M \setminus \{a_k\}$ has an interval of size two.

If $M \setminus \{a_k\}$ has an interval of size two then either that interval is also an interval of $\alpha \setminus \{a_k\}$ or else it is not. If it is not then that must be because a_i vertically separates the terms of the interval, in which ase we have simply chosen a_k badly, in some sense. Up to symmetry there are six cases in which this may happen, and these are illustrated in Figure 3.5.4.

There, in cases (i)-(iv) we should choose a_i+2 as a new choice for a_k, and our argument will now work. This is because if $M \setminus \{a_i+2\}$ has an interval of size two then that interval, it will be found, must be either $a_i+1\ a_i+2$ or else $a_i-1\ a_i-2$, neither of which is separated by a_i. In the remaining cases note that our original choice for a_k must be strictly greater than a_i+2 in value, because otherwise a_i+1 and a_i+2 form an interval of α. It will again be found that a_{i+2} will work as our new choice of a_k. Whatever pair a_i+2 may separate in M, it will not be $a_i+1\ a_i-1$.

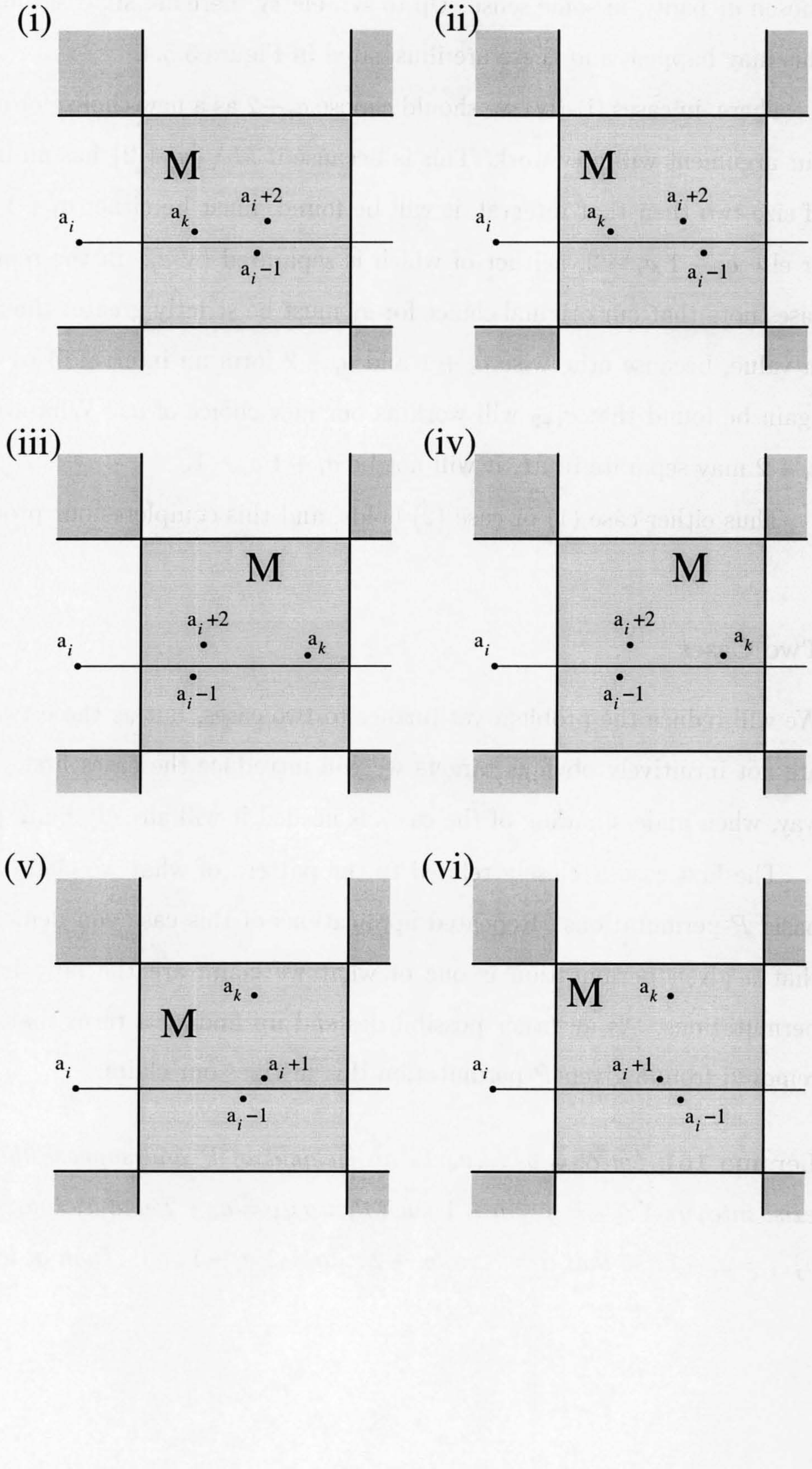
Thus either case (1) or case (2) holds, and this completes our proof. ■

Two Cases

We will reduce the problem yet further to two cases, but as these two cases are not intuitively obvious targets we will introduce the cases first. In this way, when understanding of the cases is needed it will already be in place.

The first case is closely related to the pattern of what we claim are the basic P-permutations. Repeated applications of this case can demonstrate that a given permutation is one of what we claim are the only basic P-permutations. As all other possibilities end up finding a term that can be removed from a given P-permutation this justifies our claim.

Lemma 161 *Let $\alpha = a_1 \dots a_n$ be an element of P and suppose that there exist integers $1 \leq i < j \leq n-1$ such that $a_{i+1} = a_i + 2$ and $a_j = a_i - 1$ and $a_{j+1} = a_i + 1$ (so that $\alpha = \dots a_i\ a_i+2 \dots a_i-1\ a_i+1 \dots$). Then at least one*


(i)
M
a_i+2
a_k
a_i
a_i-1
(ii)
M
a_i+2
a_k
a_i
a_i-1
(iii)
M
a_i+2
a_i
a_k
a_i-1
(iv)
M
a_i+2
a_i
a_k
a_i-1
(v)
a_k
M
a_i
a_i+1
a_i-1
(vi)
M
a_k
a_i
a_i+1
a_i-1

of the following hold: (i) There exists a term a_k such that $\alpha \setminus \{a_k\}$ is order isomorphic to an element of P. (ii) $a_{i+1} = a_i + 2$ is the largest term of α. (iii) The terms a_{i+2} and a_{j+2} have values $a_i + 4$ and $a_i + 3$ respectively, so that $\alpha = \ldots\ a_i\ a_i + 2\ a_i + 4\ \ldots a_i - 1\ a_i + 1\ a_i + 3\ \ldots$.

It will be noticed that if possibility (iii) holds then we may re-apply the lemma to the subsequence $a_{i+1}a_{i+2}a_{j+1}a_{j+2}$. Also note that by symmetry this lemma states that either (i) holds, or else a_j is the smallest term of α, or else a_{i-1} and a_{j-1} have values $a_i - 2$ and $a_i - 3$, making $\alpha = \ldots\ a_i - 2\ a_i\ a_i + 2\ \ldots\ a_i - 3\ a_i - 1\ a_i + 1\ \ldots$. The symmetry involved is rotation by 180°; perhaps it is easiest to imagine the permutation α rotated rather than the Lemma rotated.

In any case, this shows that if only cases (ii) and (iii) apply then α must have the form:

$$2\ 4\ 6\ \ldots\ 2m-4\ 2m-2\ 2m\ 1\ 3\ 5\ \ldots\ 2m-3\ 2m-1$$

where $2m = n$ (n must be even n this case). This is one of our claimed basic elements of P.

Now for the proof, which is technical and perhaps not of general interest.

PROOF: We have that $\alpha = a_1 \ldots a_n = \ldots a_i\ a_i + 2 \ldots a_i - 1\ a_i + 1 \ldots$. Let us consider the term $a_i + 2 = a_{i+1}$. There are, as ever, a number of possibilities:

1. $\alpha \setminus \{a_i + 2\}$ is order isomorphic to an element of P.

2. $\alpha \setminus \{a_i + 2\}$ has a proper interval J containing the terms a_i and a_{i+2} ($a_i + 2$horizontally separates J.)

3. $\alpha \setminus \{a_i + 2\}$ has a proper interval J containing the terms $a_i + 1$ and $a_i + 3$ ($a_i + 2$vertically separates J.)

If 1 holds then we are done, as this corresponds to (i) in the lemma.

If 2 holds then we have that $a_i + 2$ is the largest term in α (iii). Indeed:

- a_j is an element of J, because either a_{i+2} (in J) is equal to a_j or else a_{i+2} is smaller than a_j in which case $J \ni a_i > a_j > a_{i+2} \in J$. a_{i+2} cannot be larger than $a_j = a_i - 1$ because then a_{i+2} is at least a_{i+1}, making J and $a_i + 2$ together a proper interval of α, a contradiction.

- Therefore J, a_i+2 and a_{j+1} together form an interval of α and therefore constitute all α. To see this note that the terms a_j and a_i are in J, and $a_{j+1} = a_i + 1$ and $a_{i+1} = a_i + 2$.

- J cannot contain terms larger than $a_i + 2$, or else J and $a_i + 2$ form a proper interval of α. Therefore $a_i + 2$ is the largest term of α, as required.

If 3 holds then we have that $a_{j+2} = a_i + 3$. Indeed:

- J may be assumed to be minimal, therefore either of size two or else order isomophic to an element of P.

- J contains $a_{j+1} = a_i + 1$ but no term either smaller or to the left of a_{j+1}. If J contains $a_{j+1} - 1 = a_i$ then J contains terms preceding a_{i+1} (e.g. a_i), terms succeeding a_{i+1} (e.g. a_{j+1}), terms smaller than and terms larger than a_{i+1}, which means that J and a_{i+1} together form a proper interval of α.

- Therefore J is sum decomposable, therefore J consists of an increasing pair and therefore $a_{j+2} = a_{i+3}$, as required.

We still need to close the case for 3, and we aim to show (iii) in the lemma but (i) may also occur. To this end let us consider what may happen if $a_{j+2} = a_i + 3$ is removed from α. At least one of the following must hold:

3.1. $\alpha \setminus \{a_i + 3\}$ is order isomorphic to an element of P.

3.2. $\alpha \setminus \{a_i + 3\}$ has a proper interval J containing both a_{j+1} and a_{j+3} (horizontal separation).

3.3. $\alpha \setminus \{a_i + 3\}$ has a proper interval J containing both $a_i + 4$ and $a_i + 2$ (vertical separation).

In case 3.1 corresponds to (i) in the lemma, and so in this case we are done.

We claim that case 3.2 cannot occur because if it does then a_{j+2} is adjacent to J, making $\{a_{j+2}\} \cup J$ a proper interval of α. We argue as follows. The terms a_{j+1} and a_{j+3} must be in J. There is no difficulty with $a_{j+1} = a_i + 1$ but there is a difficulty with a_{j+3}. The terms $a_i - 1, a_i, a_i + 1, a_i + 2$ all precede a_{j+3}, therefore the latter must be either greater than $a_i + 2$, or less than $a_i - 1$. In either case $a_i + 2$ must be in J, in the former case because $J \ni a_i + 1 < a_i + 2 < a_{j+3} \in J$, in the latter case because $J \ni a_{j+3} < a_i < a_i + 1 \in J$ which implies that $a_i \in J$ and $a_i + 2$ succeeds $a_i \in J$ and precedes $a_i + 1 \in J$. This makes $a_{j+2} = a_i + 3$ adjacent to J, as required.

In case 3.3 we claim that $a_{i+2} = a_i + 4$ which means that we are done as this corresponds to (iii) in the lemma. To see this consider J.

- We can choose J to be minimal subject to being a proper interval and containing $a_i + 2$ and $a_i + 4$. In that case J either has size two or is order isomorphic to an element of P.

- $a_i + 2 = a_{i+1}$ is the smallest and leftmost term of J. Indeed the term immediately below $a_i + 2$ is $a_i + 1 = a_{j+1}$, the immediate predecessor of a_{j+2}, therefore if this term is in J then a_{j+2} and J together form a proper interval of α, a contradiction. And if a_i, the immediate predecessor of $a_{i+1} = a_i + 2$ is in J then this contradicts $a_i + 2$ being the smallest term in J.

- Therefore J is sum decomposable, $a_i + 2$ forming the first sum component. Therefore J is not an element of P but has size two and is an increasing pair.

- Therefore $a_{i+3} = a_i + 4$, as required.

This completes or proof. ■

The other case is a little more elementary, because there is only one possible outcome.

Lemma 162 *Let $\alpha = a_1 \dots a_n$ be an element of P and suppose that there exist two integers $2 \leq i < j \leq n-1$ such that $a_{i-1} = a_i + 2$ and $a_j = a_i + 1$ and $a_{j+1} = a_i + 3$. Then either there exists a term a_k such that $\alpha \setminus \{a_k\}$ is order isomorphic to an element of P (i), or else $i < j - 2$ and $a_{i+1} = a_i - 2$ and $a_{j-1} = a_i - 1$.*

As with the last case of the previous lemma, if (ii) holds then we can apply the lemma again and again until we have that (i) holds.

We omit the proof because it is mechanical and in the same style as the last. The proof is comparatively speaking very short. Should the reader (for some unimaginable reason!) wish to become familiar with the technique then this is a good starting exercise.

Reduction to Cases

Lemma 163 *Let $\alpha = a_1 \dots a_n$ be an element of P. Then there exists a symmetry σ of α (one of the eight possible for a permutation, including the identity symmetry) for which one of the following holds: (i) There exists a term of σ that when removed yields a sequence order isomorphic to an element of P. (ii) If $\sigma = s_1 \dots s_n$ then there exist integers $1 \le i < j \le n-1$ such that $s_{i+1} = s_i + 2$ and $s_j = s_i - 1$ and $s_{j+1} = s_i + 1$ (As in the first of the two last lemmata, Lemma refP.reduction.case.one.) (iii) If $\sigma = s_1 \dots s_n$ then there exist integers $1 \le i < j \le n-1$ such that $s_{i-1} = s_i + 2$ and $s_j = s_i + 1$ and $s_{j+1} = s_i + 3$ (As in the second of the two preceding lemmata.)*

Again we omit the proof. If a computer may be used to prove anything, then it should be permitted to prove this. The proof is the longest of the four but it requires no more intuition than the simplest of them.

Theorem 164 *Let $\alpha \in P$. If $\alpha \notin G$ then there exists a term $a_p \neq a_1$ of α such that $(\alpha \setminus a_p) \in P$.*

PROOF: This follows from Lemmata 160–163. ∎

Theorem 165 *Let X be a closed class. If X contains no elements of P of length n and $n+1$ for some integer $n \ge 4$ then X has no elements of P of length $n+2$ or greater.*

PROOF: This is based on Theorem 164 and on the observation that every basic element of P having length greater than four involves another basic element of P two terms shorter. Q.E.D. ∎

3.5.5 Results

- The results obtained from looking at finitely based classes in the light of P are included in the library entries. The results are not outstanding. The classes most under scrutiny at the time of writing were those with two basis elements of length four. There are 57 such pairs. Of these 33 contained known fundamental antichains, 24 did not. Two of the twenty four were already known to be strongly finitely based (one is finite, the other is the set of Separable permutations which contains no elements of P). Five more of the twenty four have finitely many elements of P or have sufficiently simple sets of elements in P that it is easy to demonstrate that the classes are strongly finitely based. The remaining cases are undecided.

- The best strategy for finding a decision mechanism for determining whether a class is partially well ordered almost certainly lies in classifying all infinite fundamental antichains, which we believe to be feasible and interesting in itself.

- Doubtless examining the elements of P in a class does still, for the time being, have a place in attempting to determine whether a class is strongly finitely based. It is a cheap check to make, to see whether a class contains a finite or very simple infinite set of elements in P.

Generating the elements requires five short lines of computer code, plus another twenty for an efficient method of disposing the generated elements that are not in the class under consideration.

- There is no obvious connection between the number of permutations in a class and the number of elements of P in the class. In a brief survey of the results for classes with two basis elements of length four (4-4 classes), there are 25 classes that potentially have the same overall enumeration as some other 4-4 class. Of these only two classes potentially have the same number of elements of P of each length, the same *P-enumeration*, as we will call it in future.

- Many 4-4 classes do however have an obvious and simple pattern in the number of P elements of each length (a brief glance revealed 16 out of 57). This is astronomically better than the situation with the general enumeration problem, where the pattern in each case is teased out with difficulty and ingenuity. A link between the two would therefore be clearly advantageous from the point of view of solving the general enumeration problem.

Exercise 166 Let α be an element of P with length n. Show that there are $(n-1)^2 - 4 = (n+1)(n-3)$ distinct elements of P of length $n+1$ that involve α.

Exercise 167 Let α be an element of P and let β be an arbitrary permutation of length $|\alpha| + 1$ that involves α. Show that β satisfies one of the following: (i) $\beta \in P$, or (iii) β has an interval of size two, or (iii) β is either sum or skew decomposable.

This simple observation is essential for quick generation of elements of P.

The class of separable permutations, given by $\mathcal{A}(2413, 3142)$, and which we have already shown to be strongly finitely based, contains no elements of P. The following are some other classes that have only finitely many elements of P. Under 'P enumeration' the number of permutations in P and in the class is given for each length, starting with length four.

Brief Notes on Structure

The natural ways of arranging a list of classes that have an infinite intersection with P and a list of classes that have a finite P intersect are different:

In the first case it is sensible to list the most restrictive, smallest classes that contain an infinite P intersect, larger classes containing these classes will inherit the infinite intersect with P. A small restricted class implies a numerous basis, therefore in the absence of known minima it almost makes sense to start with large bases.

In the second case, of classes that have a finite intersect with P it makes sense to list large classes, maximal classes with few basis elements.

We will follow the second pattern, listing classes that have small numbers of basis elements first, and gradually increasing the numbers. We will in addition list a few sample classes that have been found to contain infinite numbers of elements of P, especially classes that we consider to be "highly restrictive".

Discussion and Conjectures

There is no guarantee that classes minimal subject to containing infinitely many elements of P exist. It is possible to imagine, with concepts such as the infinite fundamental antichain, that such sets do not exist at all or that they appear only in certain limited contexts.[7]

Never the less, and in spite of the tendency towards large bases as we choose ever smaller such classes, we believe that the following conjecture is highly likely to hold:

Conjecture 168 *Let X be a class minimal subject to having an infinite intersection with P. Then X is finitely based.*

I believe that infinite sets of elements of P that are "minimal" are regular in some sense, like fundamental antichains. Certain regular classes are are also very likely to have a finitely based closure. I believe that these probabilities are in fact true.

It is likely that there exist only finitely many infinite subsets of P that are minimal in the same way that infinite maximal fundamental antichains are minimal: That the proper closure of these infinite sets contain only finitely many elements of P [8].

This brings us on to the other question. I also believe, perhaps to a slightly weaker extent, that the following holds:

[7] It is known whether such minimal classes exist.

[8] Proper closure is not in fact quite the right phrase, but it nearly is. As soon as some research is done on these matters the "right" description will soon become apparent, I think!

Conjecture 169 *Let X be a class maximal subject to having a finite intersection with P. Then X is finitely based. Furthermore every class that contains only finitely many elements of P is contained in a class such as X.*

There is more to be said about the number of such maximal classes (e.g. either there are infinitely many, or else there is only one, *therefore there is only one*) but this margin is too small, and mathematics of this sort too easily generated! The reason for our conjecture is that we believe that if we tweak the definition of a "minimal infinite set of elements of P" in the right way then we will find that the elements of such a set can be arranged in a sequence, every element being contained in the next. If that belief is proved then we will be well on our way to proving the conjecture.

Classes with One Basis Element

Classes with one basis element of length two or less clearly cannot contain any elements of P whatsoever.

It was proved in [4] that in the cases of the only two permutations of length three (up to symmetry) that $\mathcal{A}(123)$ contains an infinite antichain, and that $\mathcal{A}(132)$ does not. Indeed the latter is a subset of the separable permutations (see *Higman's Theorem and Strong completion*).

Every permutation of length five involves either 123 or 321, therefore by the above does not forbid an infinite antichain.

This leaves only the length four, and that same paper [4] contains examples of antichains, also contained in the Bibliothek, that show that all classes with one basis element of length four contain an infinite antichain.

Classes with Two Basis Elements - of Length Three

From the results on classes with one basis element we have that the only class with two basis elements of length three that has a hope of containing an infinite antichain is that where all basis elements are symmetric to 123, namely $\mathcal{A}(123, 321)$. This class is finite, therefore certainly partially well ordered.

Classes with Two Basis Elements - of Length Three and Four

Up to symmetry the only such classes that might be non-partially well ordered are those that have 123 as a basis element. We list the classes that have finitely many elements of P:

1. $\mathcal{A}(123, 2431)$. P enumeration is:

$$(1, 1, 0, 0)$$

   Thus this class has no elements in P of length 6 or greater.

2. $\mathcal{A}(123, 3421)$. P enumeration is:

$$(2, 2, 1, 0, 0)$$

   Thus this class has no elements in P of length 7 or greater.

3. $\mathcal{A}(123, 4, 2, 3, 1)$. P enumeration is:

$$(2, 2, 1, 0, 0)$$

   Thus this class has no elements in P of length 7 or greater.

All other classes have infinitely many elements of P and do in fact contain at least one infinite antichain. The number of elements of P in each is constant with length, or oscillates between 0 and 1, depending on whether an odd or even length is chosen.

Classes with Two Basis Elements - Length Three and One Other Length

We expect that every such situation is also fully decidable, like the above.

Classes with Two Basis Elements - Length Four

Again discarding the cases where we already know the solution (finite classes, classes that contain known infinite antichains, classes that contain a class that contains an infinite antichain) we are left with:

Two Basis Elements of Length Four

1. $\mathcal{A}(2413, 3142)$. This is the set of separable permutations, contains only sum and skew decomposable permutations, no elements of P and is partially well ordered.

2. $\mathcal{A}(4321, 3124)$. P enumeration is:

$$(2, 10, 33, 74, 120, 155, 177, 167, 105, 36, 5, 0, 0)$$

   Thus this class has no elements in P of length 15 or greater.

3. $\mathcal{A}(4321, 3124)$. P enumeration is:

$$(2, 10, 32, 75, 130, 153, 107, 38, 5, 0, 0)$$

Thus this class has no elements in P of length 13 or greater.

4. $\mathcal{A}(4312, 1234)$. P enumeration is:

$$(2, 12, 48, 130, 241, 295, 221, 89, 14, 0, 0)$$

Thus this class has no elements in P of length 13 or greater.

The equivalent statistics for some classes for which partial well orderedness is still undecided are as follows. It may well be that a more extensive search of this type will uncover more partially well ordered classes. Not all growth rates are as regular as those exhibited here.

1. $\mathcal{A}(4231, 2143)$. P enumeration is:

$$(2, 8, 20, 42, 86, 178, 362, 726, ???)$$

2. $\mathcal{A}(4231, 3142)$. P enumeration is:

$$(1, 3, 6, 12, 24, 48, 96, 192, 384, 768, ???)$$

3. $\mathcal{A}(4231, 3124)$. P enumeration is:

$$(2, 6, 12, 24, 48, 96, 192, 384, 768, ???)$$

Three Basis Elements of Length Four

1. $\mathcal{A}(1234, 2143, 4312)$. P enumeration is:

$$(2, 10, 28, 38, 22, 4, 0, 0)$$

Thus this class has no elements in P of length 10 or greater.

2. $\mathcal{A}(1234, 2143, 4231)$. P enumeration is:

$$(2, 8, 14, 10, 6, 2, 0, 0)$$

Thus this class has no elements in P of length 10 or greater.

3. $\mathcal{A}(1234, 2143, 4213)$. P enumeration is:

$$(2, 8, 19, 26, 26, 28, 30, 24, 11, 2, 0, 0)$$

Thus this class has no elements in P of length 14 or greater.

4. $\mathcal{A}(1234, 3142, 4312)$. P enumeration is:

$$(1, 4, 8, 7, 3, 1, 0, 0)$$

Thus this class has no elements in P of length 10 or greater.

5. $\mathcal{A}(1234, 3142, 4231)$. P enumeration is:

$$(1, 3, 3, 1, 0, 0)$$

Thus this class has no elements in P of length 8 or greater.

6. $\mathcal{A}(1234, 3142, 4213)$. P enumeration is:

$$(1, 3, 5, 5, 4, 2, 0, 0)$$

Thus this class has no elements in P of length 10 or greater.

7. $\mathcal{A}(1234, 2413, 4213)$. P enumeration is:

$$(1, 4, 9, 11, 9, 7, 6, 4, 1, 0, 0)$$

Thus this class has no elements in P of length 13 or greater.

The equivalent statistics for a class for which partial well orderedness is still undecided is as follows:

1. $\mathcal{A}(1234, 2143, 4312)$. P enumeration is:

$$(2, 12, 48, 136, 302, 572, 986, 1616, ???)$$

Chapter 4

Atomic Classes

Closed Classes as the Union of Maximal Atomic Subclasses, Closed Classes as an Independent Union of Atomic Classes, Atomic Classes and their Bases

4.1 Introduction

To be atomic is to have a strong and intuitive property. In the first part of this chapter we present some thoughts relevant to deciding whether a finitely based class is atomic. We recap from Theorem 15 some of the essential features of atomic classes:

- A class is *atomic* if it can be written in the form $\mathcal{B}(A, B, \pi)$ or, equivalently, as $Sub(\pi)$, where $\pi : A \to B$ is an injective function and A, B are linearly ordered sets. A class is atomic if and only it has the join property.

- A class X has the *join property* if for every two elements β and γ of

X, there exists an element η of X that involves both β and γ. Thus a class has the join property if and only if it cannot be written as a union of two strictly smaller closed classes.

- If A is an atomic class and a subset of a closed class X, and if $X = Y \cup Z$ where Y and Z are closed classes, then A is a subset of at least one of Y and Z. That is to say, atomic classes are indivisible by union. (Indivisible by a finite number of unions, at least. When infinite unions are permitted the situation changes.) This is a corollary of Theorem 15, not a direct quotation.

- The sum and skew sum of two atomic classes is atomic, as is the sum and skew completion, the wreath product of two atomic classes and the wreath completion of an atomic class. This again is unproved but elementary.

  The union or intersection of two atomic classes need not be atomic, neither need be the juxtaposition or merge of two atomic classes.

Of the constructions that do not preserve atomicity it is union that most interests us. We compare two 'minimal' ways of decomposing a closed class into a union of atomic classes: maximal and independent decompositions. We show that the maximal decomposition always exists, but is not necessarily independent. An independent decomposition may not exist. We demonstrate that there exists an atomic class which can be written as an infinite union of pairwise incomparable atomic classes.

The second part of the chapter contains some early thoughts and partial work on the relation between atomic classes and their bases. We outline the decision problem for whether or not a finitely based closed class is atomic. We show that if we add or remove a basis element from an atomic class the result need not be atomic. We show that there are certain sets of permutations that, if contained within the basis of a closed class, ensure that that class is not atomic.

4.2 Closed Classes as Unions of Atomic Classes

In this section we investigate how one may write classes as unions of atomic classes.

A note of caution:

Proposition 170 *There exists an atomic class that may be written as an infinite union of pairwise incomparable atomic classes.*

PROOF: Let $A = A_1, A_2, A_3, \ldots$ be any infinite antichain the elements of which are sum indecomposable. (For example take the antichain W from the Bibliothek.) Let $X = Sub(A_1 \oplus A_2 \oplus A_3 \oplus \ldots)$. For every positive integer i let $X_i = Sub(A_1 \oplus \ldots \oplus A_{i-1} \oplus A_{i+1} \oplus A_{i+2} \oplus \ldots \textit{ad inf.})$. Then every pair of X_i are incomparable, for if $i \neq j$ then $A_i \in X_j \setminus X_i$ and $A_j \in X_i \setminus X_j$. Furthermore $X = \bigcup_{i \in \mathbb{Z}^+} X_i$. Q.E.D.

■

Now we can start considering uniqueness of representations:

Proposition 171 *Every closed class can be written as a union of maximal atomic classes.*

PROOF: Let X be a closed class. Let A be the set of all atomic classes that are subsets of X. A partial ordering may be constructed on the elements of A by $A_1 < A_2$ if and only if $A_1 \subset A_2$.

Consider any chain $A_1 < A_2 < A_3 < \ldots$ of elements of A and let $B = A_1 \cup A_2 \cup \ldots$. Then B is closed, for if $\gamma \in B$ and $\delta \preceq \gamma$ then there exists an element A_i of the chain such that $\gamma \in A_i$ and as every atomic class is closed we have that $\delta \in A_i \subseteq B$. Furthermore B has the join property, for suppose that $\gamma, \delta \in B$ with $\gamma \in A_i$ and $\delta \in A_j$ and $i \leq j$. Then γ, δ are both elements of A_j. As A_j is atomic, γ and δ join in A_j and therefore also in B. Thus B is atomic.

By Zorn's Lemma we may conclude that every atomic class in A is contained in a maximal class of A. For every permutation $\gamma \in X$ we have that $Sub(\gamma)$ is an atomic subclass of X and so we have that every permutation of X is contained in a maximal element of A. Thus X is the union of the maximal elements of A. By the nature of the partial ordering these maximal classes are incomparable. Q.E.D.

■

Note: A representation of a class as a union of maximal atomic subclasses is not necessarily unique.

Definition 172 Let L be a family of closed classes with members denoted by L_x, x being an element of some index set Ξ. We denote the union of all members of L by $\bigcup_{x\in\Xi} L_x$, and this contains precisely those permutations that are contained in at least one member of L. We say that the members of L are *independent* in union if for every member L_y of L we have that $\bigcup_{y\neq x\in\Xi} L_x$ is distinct from, and therefore strictly smaller than, $\bigcup_{x\in\Xi} L_x$.

Proposition 173 *Let L be a family of atomic classes. Then L is independent under union if and only if every $X_i \in L$ contains a permutation, γ, not contained in any other element of L.*

PROOF: Trivial. ∎

Proposition 174 *The set of all atomic classes is uncountable.*

PROOF: Let $A_1, A_2, \ldots$ be the elements of an infinite antichain of sum indecomposable permutations arranged in a sequence. For every subsequence $A_{j_1}, A_{j_2}, A_{j_3}, \ldots$ of $A_1, A_2, A_3, \ldots$ define the class B_j to be $B_j = Sub(A_{j_1} \oplus A_{j_2} \oplus A_{j_3} \oplus \ldots)$. There are uncountably many such classes and they are all distinct. Q.E.D.

∎

Proposition 175 *A representation of a closed class as a union of independent atomic classes is unique. (Not every closed class is representable as a union of independent atomic classes.)*

PROOF: Let X be a closed class and suppose that X can be written in two distinct ways as a union of independent atomic classes: $X = A_1 \cup A_2 \cup \ldots$ and $X = B_1 \cup B_2 \cup \ldots$.

Consider any B_i. As $B_1 \cup B_2 \cup \ldots$ are independent there exists some γ such that $\gamma \in B_i$ and $\gamma \notin B_j$ for any $j \neq i$. $\gamma \in X$ and therefore $\gamma \in A_k$ for some k. It follows that $A_k \subset B_i$. For otherwise there exists some permutation $\delta \in A_k \setminus B_i$. As A_k is atomic γ and δ join in some $\eta \in A_k$. $\eta \in X$, and therefore $\eta \in B_j$ for some j. As η involves γ we have that $\eta \in B_i$. But then by closure $\delta \in B_i$, a contradiction.

By symmetry $B_i \subseteq A_l$ for some l. But if $k \neq l$ then $A_k \not\subseteq A_l$. Thus $A_k = B_i$. Thus every B_i is equal to some A_k and we conclude that X cannot be written distinctly as two independent unions of atomic classes.

■

Proposition 176 *Let X be a closed class and let $X = Y_1 \cup Y_2 \cup \ldots$ where $Y_1, Y_2, \ldots$ are independent atomic classes. Then every Y_i is a maximal atomic class in X.*

PROOF: Suppose that some Y_i is not maximal in X. Let $\gamma_i \in Y_i$ with $\gamma_i \notin Y_j$ if $i \neq j$. As Y_i is not maximal in X there exists some greater atomic class Y in X. Let $\delta \in Y \setminus Y_i$. As Y is atomic there exists some join η of γ_i and δ in Y. $\eta \in X$, and as it involves γ_i, $\eta \in Y_i$. By closure $\delta \in Y_i$. Reductio ad absurdum. ■

Proposition 177 *Let X be a closed class and let L be a set of maximal atomic classes in X. Then the elements of L are pairwise incomparable.*

PROOF: Let $Y, Z \in L$. As $Y \not\subseteq Z$ there exists $\gamma \in Y \setminus Z$. ■

Proposition 178 *There exists a closed class that cannot be written as an independent union of atomic classes and that has an uncountable set of maximal atomic subclasses.*

PROOF:

Let A be an antichain whose elements, for later simplicity, are denoted $A_{(1,0)}, A_{(1,1)}, A_{(2,0)}, A_{(2,1)}, \ldots$, having the following properties:

- For every $A_{(i,j)}$ in A, every permutation properly involved in $A_{(i,j)}$ is involved in every $A_{(k,0)}$ and every $A_{(k,1)}$ with $k > i$.
- Every element of A is sum indecomposable.

We consider the second entry in the antichain indices to be taken modulo 2, so that $A_{(m,1+1)} = A_{(m,0)}$. The library antichain ${}^{R_2}U_{R_2}$ is suitable, it gives:

$$\begin{array}{llll} A_{(1,0)} = & 3\,2\,6\,1\,5\,4 & A_{(1,1)} = & 3\,2\,5\,1\,8\,4\,7\,6 \\ A_{(2,0)} = & 3\,2\,5\,1\,7\,4\,10\,6\,9\,8 & A_{(2,1)} = & 3\,2\,5\,1\,7\,4\,9\,6\,12\,8\,11\,10 \\ A_{(3,0)} = & 3\,2\,5\,1\,7\,4\,9\,6\,11\,8\,14\,10\,13\,12 & A_{(3,1)} = & 3\,2\,5\,1\,7\,\ldots\,16\,12\,15\,14 \\ & \vdots & & \vdots \end{array}$$

Let X be the set of all permutations that may be written in the form $\gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \ldots$ where every γ_i is involved in either $A_{(i,0)}$ or $A_{(i,1)}$, and may be the empty permutation. In other words let:

$$\begin{aligned} X &= Sub(\{A_{(1,0)}, A_{(1,1)}\} \oplus \{A_{(2,0)}, A_{(2,1)}\} \oplus \{A_{(3,0)}, A_{(3,1)}\} \oplus \ldots) \\ &= (Sub(A_{(1,0)}) \cup Sub(A_{(1,1)})) \oplus (Sub(A_{(2,0)}) \cup Sub(A_{(2,1)})) \oplus \ldots \end{aligned}$$

Let $\mathcal{E}$ be the set of all classes of the form $Sub(\gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \ldots)$ where every γ_i is equal to either $A_{(i,0)}$ or $A_{(i,1)}$. Note that $\mathcal{E}$ is uncountably infinite.

Claim: The elements of $\mathcal{E}$ are maximal atomic subclasses of X.

Proof: Suppose that some $E = Sub(A_{(1,j_1)} \oplus A_{(2,j_2)} \oplus A_{(3,j_3)} \oplus \ldots) \in \mathcal{E}$ is not maximal. Then there exists a superior atomic class C in X properly containing E. Let γ be a minimal element of $C \setminus E$ in the sense that every subsequence of γ is an element of E.

Let $\gamma = \eta \oplus \theta$ where θ is sum indecomposable and non-empty. As γ is an element of X, θ must be involved in an element of the antichain A. Indeed it must be an antichain element for otherwise $\theta \in A_{(m,j_m)}$ for sufficiently large m (a property of the antichain) and as, by minimality, $\eta \in E$ we have that $\gamma = \eta \oplus \theta \in E$, a contradiction. Thus we may let $\theta = A_{(m,n)}$ for some m, n. Note that $n = j_m$ because both $A_{(m,n)}$ and $A_{(m,j_m)}$ are elements of C, which is atomic, but $A_{(m,0)}$ and $A_{(m,1)}$ do not join in X.

Next note that $A_{(1,j_1)} \oplus A_{(2,j_2)} \oplus \ldots \oplus A_{(m,j_m)} \in E \subset C$. Therefore there exists a minimal join, δ, of $A_{(1,j_1)} \oplus \ldots \oplus A_{(m,j_m)}$ and γ in C. By minimality and as there exists at most one embedding of $A_{(m,j_m)}$ in any element of X we have that the last sum component of δ is $A_{(m,j_m)}$. Thus, by considering the definition of X we have that:

$$\delta \in D = Sub((A_{(1,0)} \cup A_{(1,1)}) \oplus (A_{(2,0)} \cup A_{(2,1)}) \oplus \ldots \oplus (A_{(m-1,0)} \cup A_{(m-1,1)}) \oplus A_{(m,i_m)}))$$

However note that $\beta = A_{(1,j_1)} \oplus A_{(2,j_2)} \oplus \ldots \oplus A_{(m,j_m)}$ is a maximal element of D. Hence $\gamma \preceq \delta = \beta = A_{(1,j_1)} \oplus A_{(2,j_2)} \oplus \ldots \oplus A_{(m,j_m)} \in E$. Reductio ad absurdum.

■

CLAIM: Every maximal atomic subclass of X is an element of $\mathcal{E}$.

PROOF:

Let M be a maximal atomic subclass of X. To show that M is an element of E it suffices to show that for every m there exists an antichain element of the form $A_{(m,j_m)}$ in M, where $j_m = 0$ or 1.

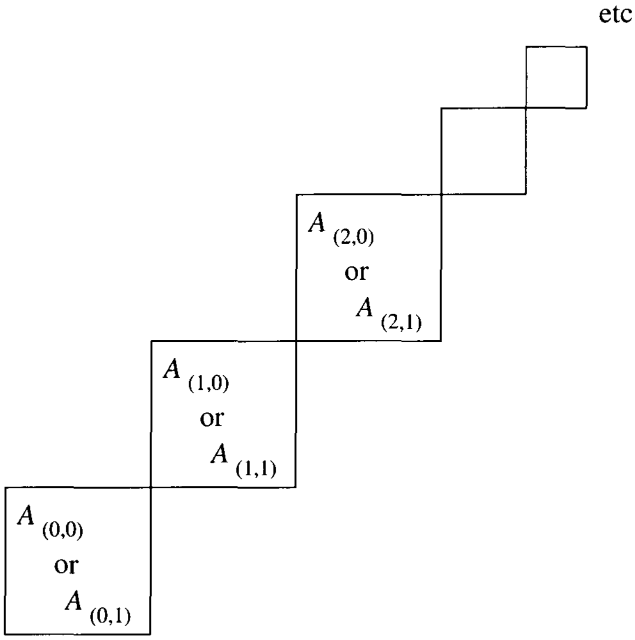


Figure 4.1: A typical class in E is made by taking the closure or Sub of an infinite sequence of the above type. Clearly there are uncountably many such classes in E. However their union, X, forms a closed class (containing only countably many permutations), the maximal atomic subclasses of which are precisely the classes in E. Thus X is not an independent union of all its maximal atomic subclasses. The situation is in fact far worse: X cannot be expressed as an independent union of atomic subclasses.

First note that M contains an antichain element, for otherwise $Sub(A_{(1,0)}) \oplus M$ is an atomic class in X properly containing M, which contradicts the maximality of M. To argue a little more precisely, suppose that M does not contain an element of A and let $\gamma \in M$. As $\gamma \in X$, γ is expressible in the form $\gamma = \gamma_1 \oplus \gamma_2 \oplus \ldots \oplus \gamma_m$ where each γ_i is a possibly empty permutation properly involved in either $A_{(i,0)}$ or $A_{(i,0)}$. By recalling the properties of the antichain we have that each γ_i is also involved in $A_{(i+1,0)}$. Thus $A_{(1,0)} \oplus \gamma$ is involved in $A_{(1,0)} \oplus A_{(2,0)} \oplus \ldots \oplus A_{(m+1,0)}$, and is therefore an element of X, which yields the required contradiction.

Furthermore if M contains an antichain element $A_{(m,j_m)}$ then M must contain an element of the form $A_{(n,j_n)}$ for every $n \leq m$, where each j_n is either 0 or 1. We demonstrate:

Consider any antichain element $A_{(m,j_m)}$ in M. All permutations of X involving $A_{(m,j_m)}$ are of the form $\alpha \oplus A_{(m,j_m)} \oplus \beta$ where α and β are possibly empty permutations of X. X, and therefore M, has but finitely many permutations of the form $\alpha \oplus A_{(m,j_m)}$. Thus, as M is atomic there exists a $\gamma \in M$ such that $\gamma \oplus A_{(m,j_m)}$ is an element of M involving every permutation in M of the form $\alpha \oplus A_{(m,j_m)}$. As $\gamma \oplus A_{(m,j_m)} \in X$ we may write $\gamma \oplus A_{(m,j_m)} = \gamma_1 \oplus \gamma_2 \oplus \ldots \oplus \gamma_{m-1} \oplus A_{(m,j_m)}$ where each γ_i is involved in some antichain element $A_{(i,j_i)}$. Thus $\gamma \oplus A_{(m,j_m)}$ is involved in $A_{(1,j_1)} \oplus A_{(2,j_2)} \oplus \ldots \oplus A_{(m,m_j)}$. Let $\Delta = \{\gamma | A_{(m,j_m)} \oplus \gamma \in M\}$. $Sub(A_{(1,j_1)} \oplus A_{(2,j_2)} \oplus \ldots \oplus A_{(m,j_m)}) \oplus \Delta)$ is an atomic subclass of X containing M and by the maximality of M equal to M. Thus M contains an element of the form $A_{(n,j_n)}$ for every $n \leq m$, where each j_n is either 0 or 1.

Finally, M contains an infinite number of antichain elements of A. For otherwise consider the antichain element $A_{(m,j)}$ in M where m is maximal. Let $\Gamma = \{\gamma | \gamma \oplus A_{(m,j)} \in M\}$ and let $\Delta = \{\gamma | A_{(m,j)} \oplus \gamma \in M\}$. Then $Sub(\Gamma \oplus A_{(m,j)} \oplus A_{(m+1,j)} \oplus \Delta)$ is an atomic class in X properly containing M.

■

CLAIM: X is not expressible as an independent union of atomic classes.

PROOF: Suppose that $\mathcal{D}$ is a set of independent atomic classes whose union is X. Then every element of $\mathcal{D}$ is maximal and therefore an element of B. Consider any $D \in \mathcal{D}$. By independence D contains a permutation γ not contained in any other. There must be an antichain element $A_{(m,j)}$ such that $\gamma \oplus A_{(m,j)} \in D$. Now, $\gamma \oplus A_{(m,j+1)} \in X$ and $\gamma \oplus A_{(m,j+1)}$ cannot be contained in any element of $\mathcal{D}$ other than D. (Else, by closure, that other element would contain γ.). However if $\gamma \oplus A_{(m,j+1)} \in D$ then by closure both $A_{(m,j)}$ and $A_{(m,j+1)}$ are elements of D and must therefore join in D. However $A_{(m,j)}$ and $A_{(m,j+1)}$ do not join in X. Reductio ad absurdum.

■

■

Note: The antichain used in the proof has the "finite drop" property: There are at most three terms below and to the right of any term of any antichain element. Thus by considering the number of ways in which a new largest element can be added to any permutation in X we have that X has at most 4^{n-1} permutations of length n. A similar exponential bound would apply if any finite drop or finite rise antichain was used in the construction of X.

4.3 The Decision Problem

It is not known if it is decidable from its basis whether or not an arbitrary finitely based closed class X is atomic or not. What we will find out shortly is the following: If X is not atomic then it is expressible as the union of two strictly smaller classes Y and Z. Furthermore we can arrange for Y and Z each to have precisely one basis element that is not a basis element of X, these two basis elements being any two permutations in X that do not join in X.

Thus the problem is that a program can be written that takes as an input a finite basis and terminates if the class defined by that basis is non-atomic. It merely looks for a pair of permutations that do not join in that class and then utilises it to express the class as the union of two strictly smaller closed classes. However it is not known whether there exists a program that will definitely terminate if the basis of an atomic class is given as an input. All that is known is that there are certain special classes of permutations, such as the finite ones, for which atomicity is fully decidable.

Now let us consider the subclasses Y and Z. One might hope that these subclasses are in some way simpler than the superclass X. Such a simplification, if it exists, is far from obvious. The bases of Y and Z can have more elements than that of X and it is possible for basis elements of Y and Z to be longer than those of X. Let us clarify a point. There is a concept of an *elementary non-joining pair* which is as follows: If α and β are non-joining elements of X and if α' and β' are permutations that involve α and β respectively then it follows that α' and β' similarly do not join in X. Thus it makes sense to define a minimal or *elementary* non-joining pair to be a pair of permutations $\alpha, \beta \in X$ that do not join, but where every permutation properly involved in α joins with β and likewise every permutation properly involved in β joins with α. Even so, there exist finitely based closed classes that contain arbitrarily long elementary non-joining pairs. At best we can hope to obtain an upper bound for the minimum combined length of any such pair.

Again let us consider the possibilities that a mechanism of expressing a non-atomic class as the union of two lesser classes might give us: Not only may the bases of the two subclasses be more complex than that of the original, neither of the two subclasses need be atomic. However by again splitting these subclasses into yet smaller classes we may eventually obtain atomic classes. This is the front of present research, although this front is not very active. The following points are pivotal in this consideration:

- There exists a finitely based non-atomic class that is the union of infinitely many (finitely based) maximal atomic subclasses (See Theorem 116 in the section titled "Antichain Classes" of Chapter 3.). Thus a process of splitting a non-atomic class into two lesser closed classes and splitting the subclasses into two yet smaller subclasses need never terminate.

- Further to the above, we remark that unless care is taken it may be possible eternally to split a given closed class into unions of lesser classes and never obtain an atomic class.

- If a class is partially well ordered (i.e. it contains no infinite antichain) then the class is the union of finitely many atomic classes and furthermore a procedure of splitting such a class into two subclasses and subclasses into subclasses and so on will invariably terminate[1].

- It is not known whether there exists a finitely based class that is the union of infinitely based atomic classes. It is not even known whether there is a finitely based class that is not expressible as a union of finitely based maximal atomic subclasses. There is a finitely based class that contains an infinitely based atomic subclass (See Proposition 186) but in our known example the closure of the class, less the subclass, is the class itself. This implies that this maximal atomic subclass will never be essential when expressing the class as the union of lesser classes.

- If a closed class is finite, something which is easily determined, then it is determinable whether or not the class is atomic. There also exist certain sets of permutations that, if contained in the basis of a class, guarantee that the class is non-atomic. Beyond that, no general solutions are known.

Active questions are as follows:

Question 179 Is it determinable whether a given closed class is non-atomic?

Question 180 Given two closed classes X and Y, defined by their bases, what is the basis of the closure of $X \setminus Y$?

We discuss the above points individually.

[1]At least, such a procedure will come to a point where it can no longer split subclasses into lesser classes but the procedure may not be able to determine whether or not it has reached that stage.

Finitely Based Classes as the Union of Finitely Based Classes

From Theorem 24, which we repeat here:

Theorem 24 *If X and Y are closed classes then $X \cup Y$ is closed. Furthermore if $X = \mathcal{A}(B_1)$ and $Y = \mathcal{A}(B_2)$ then the basis of $X \cup Y$ is the minimal merge of B_1 and B_2.*

We have that:

Lemma 181 *Let X be a closed class equal to the union of A and B, two closed classes neither of which is a subset of the other. If there exists a basis element γ of A not involved in any basis element of X then X is equal to the union of B and $\mathcal{A}(\mathcal{B}(A) \setminus \{\gamma\})$.*

This is a direct corollary of Theorem 24. It gives us a mechanism to prove the first part of the following:

Theorem 182 *Let X be a finitely based non-atomic closed class. Then:*

- *X is the union of two proper subclasses, and these may be chosen to be finitely based.*
- *X may be expressed as the union of two proper subclasses Y and Z each of which has at most one basis element that is not a basis element of X.*
- *It is possible that if X is written as the union of two proper subclasses Y and Z then both Y and Z must have at least $|\mathcal{B}(X)|+1$ basis elements.*

PROOF: **Finitely based decomposition:** Suppose that $X = Y \cup Z$ where Y and Z are incomparable closed classes. By Lemma 181 we may discard any basis elements of Y and Z that are not involved in basis elements of X to obtain two new classes Y' and Z' whose union is still the whole of X. As X is finitely based, so will Y' and Z' be, but we do need to ensure that we do not end up with the degenerate case where either Y' or Z' is equal to X.

As Y and Z are incomparable, Y has a basis element α that is an element of Z and Z has a basis element β that lies in Y. Let C be the set of basis elements of Y that are involved in basis elements of X, and let D be the equivalent set for Z. Both C and D are finite. Let $Y' = \mathcal{A}(C \cup \{\alpha\})$ and let $Z' = \mathcal{A}(D \cup \{\beta\})$. Then $X = Y' \cup Z'$, by Lemma 181, and furthermore Y' and Z' are incomparable. Q.E.D.

Limitations on the basis of Y and Z: Let α and β be permutations in X that do not join in X. Since every element of X avoids at least one of α and β we have that X is the union of $X \cap \mathcal{A}(\alpha)$ and $X \cap \mathcal{A}(\beta)$. The former is equal to $Y = \mathcal{A}(\{\alpha\} \cup \mathcal{B}(X))$ and the latter to $Z = \mathcal{A}(\{\beta\} \cup \mathcal{B}(X))$. Both are proper subclasses of X. Q.E.D.

The basis elements of A and B may be numerous: $Sub\{213, 231\}$ is non-atomic and can be written as a union of incomparable closed classes in only one way, namely as $Sub(213) \cup Sub(231)$. The basis of the first component is $\mathcal{B}(Sub(213)) = \{123, 132, 231, 312, 321\}$, that of the second is $\mathcal{B}(Sub(231)) = \{123, 132, 213, 312, 321\}$, and that of the whole is $\mathcal{B}(Sub(213, 231)) = \{123, 132, 312, 321\}$. Q.E.D.

■

Arbitrarily Long Non-Joining Pairs

Proposition 183 *There exists a closed class X such that for every natural number N there exists an elementary non-joining pair α, β of elements in X such that the lengths of α and β are both greater than N.*

PROOF: Let A be an infinite fundamental antichain, and let X be the closure $Sub(A)$. Recall from Proposition 147 in the Antichains chapter that as A is infinite and fundamental, $Sub(A) \setminus A$ is an atomic class. Hence any pair of elements that do not join in X must contain at least one element of A. Let A_i be an element of A. If β is an element of $Sub(A) \setminus A$ and β has length at least as great as that of A_i then β does not join with A_i in X, else we have a contradiction with the fact that A is an antichain and $X = Sub(A)$. Let β therefore be a permutation minimal under inclusion, in X, subject to not joining with A_i. Then β, A_i is an elementary non-joining pair. If a term from either is removed then the result is two sequences that join in X.

Furthermore if the antichain ${}^{I_2}U_{I_2}$ is used for A then β must have length at least $|A_i|/2$. If that antichain is examined then the reader will be persuaded of this. This completes the proof.

■

Splitting a Class into Constituent Parts

Question 184 Given an elementary non-joining pair α, β in any closed class X, we can express X as the union of $Y = \mathcal{A}(\mathcal{B}(X) \cup \{\alpha\})$ and $Z = \mathcal{A}(\mathcal{B}(X) \cup \{\beta\})$. We will call this an *elementary split* of X. Our question is as follows: If X is the union of finitely many atomic classes, will a process of performing an elementary split on X, another on the subclasses obtained and so on invariably yield, after a finite number of steps, the constituent atomic classes of X?

Let us suppose that X is expressible as the union of atomic subclasses denoted by $A_1, \ldots, A_n$. If the classes obtained by an elementary split are unions of some of these atomic classes then the answer is, trivially, yes: Each subclass is strictly smaller than its ancestor, therefore consists of the union of a strictly smaller list of maximal atomic classes. Therefore after a finite number of steps none of the suclasses obtained in this way can consist of more than one atomic subclass. On the other hand, if the split is not as well behaved we might have the following scenario:

- $X = A_1 \cup A_2$.
- $X = Y \cup Z$ by elementary split and: $Y = A_1 \cup B_2$, $Z = A_2 \cup B_1$ where $B_1 \subset A_1$ and $B_2 \subset A_2$.
- By cruel chance we might have the same problem when attempting to split A_1 from B_2 and A_2 from B_1. In that case we might have:

$$\begin{aligned} X &= Y \cup Z = (A_1 \cup B_2) \cup (A_2 \cup B_1) \\ &= [(A_1 \cup C_2) \cup (B_2 \cup C_1)] \cup [(A_2 \cup D_1) \cup (B_1 \cup D_2)] \end{aligned}$$

  where C_1 and C_2 are subsets of A_1 and B_2 respectively, and similarly for D_1, D_2 in A_2 and B_1. There is no apparent reason why this might not continue indefinitely.

These hypothetical objections may however be disprovable.

More general questions are as follows:

Question 185 Let X be any non-atomic closed class. Then:

1. Will a finite number of elementary splits invariably yield at least one maximal atomic subclass of X?

2. Is it possible with sufficient knowledge to choose an atomic split of X into two subclasses, choose an atomic split of one of those subclasses and repeat this until a maximal atomic subclass of X is obtained?

   (The answer to the above is affirmative if X contains a finitely based maximal atomic subclass, therefore the above is equivalent to (3) which follows.)

3. Is it possible for X to be the union of finitely many infinitely based atomic classes?

Were X a closed class over an arbitrary partial order then (3) in the above would be perfectly possible; it would even be possible to use in the demonstration a partial order in many respects very similar to the partial order on permutations. We note the following:

Proposition 186 *There exists a finitely based closed class that contains an infinitely based maximal atomic subclass.*

PROOF: Let X be the closure of the infinite antichain ${}^{I_2}U_{I_2}$, which is fundamental and is listed in the Bibliothek. The closure X is finitely based, it is proved for a similar class at the beginning of the chapter on antichains. As ${}^{I_2}U_{I_2}$ is an infinite fundamental antichain the set of permutations properly involved in its elements is an atomic class, and furthermore this class is infinitely based. Q.E.D. ■

Note that in the above proof the set X, less its infinitely based atomic subclass consists of the antichain ${}^{I_2}U_{I_2}$. Were ever X to be expressed as the union of lesser classes then the antichain elements would have to be contained in some of those subclasses. The closure of the antichain constitutes the whole of X, therefore our infinitely based maximal atomic subclass would never have a role to play in such a situation.

The Core

Definition 187 The *core*, K, of a closed class X is the set of elements of X that join with every element of X. Thus the *core* of X is the set of permutations $K \subseteq X$ such that if $\alpha \in X$ and $\xi \in K$ then there exists an element $\beta \in X$ that involves both X and ξ.

It was hoped that the core of a closed class might provide a key for determining whether or not a closed class is atomic. Specifically, in the case of a finitely based class it was hoped that it might provide an upper bound for the minimal combined length of two elements of the class that did not join, if the class was non-atomic. This would leave us with having to check whether some finite set of permutations joined in X, if so then the class would be atomic, if not then not.

However the following points outline the character of the core, and its limitations. Assume that X is a closed class and that K is its core.

1. The core K is a subset of every maximal atomic subclass of X, indeed it is the largest such set.

2. K is a closed class.

3. All those basis elements of K that lie in X, if any, do not join with some element of X.

4. It is possible for the core K to be infinitely based whilst X is finite. (If U is an infinite fundamental antichain whose closure $Sub(U)$ is finitely based then the core of $Sub(U)$ contains every element of U as a basis element.)

5. It is possible for all basis elements of the core to join in X. (If $X = Sub(231) \cup I \cup R$) then the core of X consists of the trivial permutation, 1. The permutation 21 does not join with 1234, the permutation 12 not with 4321. However the two basis elements of K join in X.)

The most useful observation that the core has led us to is the following:

4.3.1 Partially Well Ordered Classes

Recall that a set of permutations is *partially well ordered* if it does not contain an infinite antichain. (This does not mean that they cannot be either infinite or infinitely based.) Partially well ordered classes have the following nice properties:

Proposition 188 *Let X be a partially well ordered closed class. Then:*

- *X has only finitely many maximal atomic subclasses.*
- *X is expressible as an independent union of finitely many atomic classes.*
- *If X is expressed as an infinite union of closed classes then these closed classes will not be pairwise incomparable: At least one will be a subset of another.*

If X is finitely based then it is the union of finitely many finitely based atomic classes.

- *Given a binary tree having the following properties:*
  - *The root of the tree is labelled by X; every other node is labelled by a subclass of X.*
  - *A node of the tree is a leaf if and only if the node label is an atomic class.*
  - *If a node label Y is non-atomic then the node has precisely two children, children that are labelled by proper subclasses of Y and whose union is equal to Y. (There may be several nodes with the same label.)*

  The tree must be finite.

PROOF: **Maximal subclasses:** Let $M_1, M_2, M_3, \ldots$ be an infinite list of distinct maximal atomic subclasses of X. Given a maximal class M_i, those of its basis elements that are not basis elements of X are elements of X itself, and are therefore finite in number. We will denote them by E_i. Every subclass of any closed class is always precisely defined by those basis elements that are not basis elements of the superclass, and so this approach is entirely standard. Now one M_i is contained in another, M_j, if and only if every element of E_j involves an element of E_i (else there is a permutation in M_i that is not in M_j). This concept can be used to define a partial order on finite sets of permutations: Given two sets U and V, we will write $U \leq^* V$ if and only if every element of V involves at least one element of U.

Now the classes $M_1, M_2, \ldots$ are maximal in X, therefore the sets $E_1, E_2, \ldots$ are pairwise incomparable under $\leq^*$, in other words the sets $E_1, E_2, \ldots$ form an (infinite) antichain under $\leq^*$. However by Higman's Theorem (see Chapter 3, Proposition 156, or [12]) it is not possible for an infinite antichain to arise out of a partially well ordered set in this way. This is therefore a contradiction and completes the proof.

An independent union: This follows from the previous: X has only finitely many maximal atomic subclasses, therefore each maximal atomic subclass contains a permutation not contained in any other (from which the desired result follows directly):

For suppose that $M_1, \ldots, M_n$ are the maximal atomic subclasses of X. The class M_1 is not a subset of M_2 therefore M_1 contains an element β_2 not in M_2. Similarly M_1 contains an element β_3 not contained in M_3 and so on. Since M_1 is atomic and the list $\beta_2, \ldots, \beta_n$ is finite it follows that M_1 contains some permutation that involves all these $\beta_2, \ldots, \beta_n$. Such a permutation is contained in M_1 but in no other maximal subclass.

Thus an element unique to M_1 is established, and by symmetry elements unique to every other one of the maximal atomic subclasses. Therefore not only is X equal to the union of all its maximal atomic subclasses, the union is also independent in that if any of these subclasses is omitted then the union of the remaining does not equal X.

An infinite union: This again follows directly from Higman's Theorem.

Binary Tree: Suppose that a tree exists as described except that it is infinite. Then there exists an infinite descending chain, starting at the root, of nodes each below the other. This implies that there exists an infinite sequence of subclasses of X, each properly contained in its predecessor, which is impossible. X is partially well ordered and if such a sequence, which we will denote by $Y_1, Y_2, Y_3, \ldots$ exists then we can choose an infinite antichain in X from their bases. Each class has only finitely many basis elements that are in X but each class also properly contains its successor, which means that the basis of its successor must be more restrictive, either by having shorter basis elements or by having more basis elements. Since permutations cannot be made arbitrarily short this gives us all we need:

Select any element of $\mathcal{B}(Y_1) \cap X$, that is any basis element of Y_1 that is not a basis element of X. That basis element involves, properly or improperly, a basis element of Y_2 because Y_2 is a subset of Y_1. Similarly the chosen basis element of Y_2 involves a basis element of Y_3, and so we can choose an infinite sequence of basis elements, each involving its successor. Now since permutations cannot be arbitrarily short there must be some element of the sequence that is equal to all its successors. Denote this permutation by β_1 and let Y_{N_1} be a class having β_1 as a basis element. The permutation β_1 will be the first of an infinite antichain in X.

Y_{N_1+1} is a proper subset of Y_{N_1}, therefore there exists a basis element of Y_{N_1+1} that is not a basis element of X and that is not equal to β_1. Let such an element be given.

Again we can construct an infinite sequence of basis elements: Our given basis element of Y_{N_1+1} involves a basis element of Y_{N_1+2}, that involves a basis element of Y_{N_1+3} and so on, and as with β_1 this sequence must stabilize. There is an element of the sequence equal to all its successors. Call this permutation β_2 and let β_2 be a basis element of some Y_{N_2} where $N_2 > N_1$.

Note that since the first element of this sequence is independent from and does not involve β_1, we have that β_2 also is not equal to β_1. Indeed since β_1 and β_2 are distinct basis elements of Y_{N_2} they must be pairwise incomparable.

Now choose a basis element of Y_{N_2+1} that is not a basis element of X and that is not equal to either β_1 or β_2. Use this basis element by our established method to choose a basis element β_3 of some Y_{N_3} $(N_3 > N_2)$.

The permutations β_1, β_2, β_3 are all elements of X and they form an antichain. By repeating our method we can extend this into an infinite antichain, every element of which is an element of X, which completes our proof.

■

4.4 Modifying the Basis

Given a closed class with given basis, another class can be obtained by either adding or removing a basis element of X. We show that neither operation preserves atomicity.

Proposition 189 *Let X be an atomic class and let γ be a basis element of X. Then $\mathcal{A}(B(X) \setminus \{\gamma\})$ need not be atomic.*

PROOF: We give two examples of an atomic class where the removal of a single basis element results in a non-atomic class. One example is infinitely based, the other finitely based.

Let $U = U_1, U_2, \ldots$ be the infinite antichain ${}^{I_2}U_{I_2}$ described in the Bibliothek, the first few elements of which are:

$$3\,4\,1\,2,\quad 2\,3\,6\,1\,4\,5,\quad 2\,3\,5\,1\,8\,4\,6\,7,$$
$$2\,3\,5\,1\,7\,4\,10\,6\,8\,9,\quad \ldots$$

It can be shown that $Sub(U)$ is finitely based, indeed U is very similar to the antichain in Theorem 116 and the closure of that antichain is shown there to be finitely based. A similar argument can be applied here.

Let C be the set of all basis elements of $Sub(U)$ that do not involve an element of U. We will consider the class $\mathcal{A}(U \cup C)$ which is equal to $PropSub(U)$. This can be seen directly; no supporting lemma is required.

$\mathcal{A}(U \cup C) = PropSub(U)$ is an atomic class. This can be seen either by noting that U is a fundamental antichain and using Proposition 147 which states that the set of permutations properly involoved in an infinite strongly trim (a weaker condition than fundamental) antichain is atomic. Alternatively it can be seen directly that:

$$PropSub(U) = Sub(2\,3\,5\,1\,7\,4\,9\,6\,11\,8\,\ldots) \oplus Sub(\ldots 3\,0\,5\,2\,7\,4\,10\,6\,8\,9)$$

Now, let m be the length of the longest basis element of $Sub(U)$, and note that U_m is an example of an element of U with length greater than m. If we delete U_m from the basis of $PropSub(U)$ then we obtain the class:

$$\mathcal{A}(U \cup C \setminus \{U_m\}) = PropSub(U) \cup \{U_m\}$$

In this class U_m does not join with any permutation longer than itself. Thus we have taken an atomic class, deleted a basis element and obtained a non-atomic class, as required.

In addition a finitely based counterexample is afforded by the following.

Let $X = Sub\{231\}$. Then the basis of X consists of all permutations of length three barring 231. X is atomic. Now consider what occurs if we remove 213 from the basis of X. The resulting class, which is not atomic, is $Sub\{231, 213\}$. (This is easily seen once it has been noted that there are no permutations of length 4 in the resulting class.) ■

Proposition 190 *Let X be a non-atomic closed class and let γ be a basis element of X. It does not necessarily follow that if γ is removed from the basis of X that the resulting class is still non-atomic.*

PROOF: Suppose that the proposition did not hold for finitely based X. Then the basis elements of X could be removed one by one until the set of all permutations was obtained, which is certainly atomic.

We could also cease removing basis elements when there was but one left: That basis element would have to be either sum or skew indecomposable, and therefore its avoidance class would be either skew or sum complete, and therefore atomic.

For an infinitely based example consider the following:

Let U be any infinite antichain of skew indecomposable permutations. Such antichains exist, and examples are available in the Bibliothek. Let U_1 be any element of U and let U' be the set of elements of U with length strictly greater than three times that of U_1. Note that the class with basis U' is skew complete, atomic and infinitely based.

We will create a finite set K of permutations such that $K \cup U'$ is an antichain and such that the class with $K \cup U'$ as its basis is non-atomic. We proceed:

Let K be the set of all permutations minimal subject to containing at least three distinct but not necessarily termwise disjoint subsequences order isomorphic to U_1. This set K exists, every one of its elements has length no more than $3|U_1|$, and K together with U' forms an antichain, as predicted.

In the next six paragraphs we construct two permutations in $K \cup U'$ that do not join.

That there exist permutations that have precisely two subsequences order isomorphic to U_1 is a permutation specific technical result, but an easy one: Let $U_1 = u_1 u_2 \dots u_m$ and let u_i be any term of U_1. If the term preceding u_i is equal to $u_i - 1$ or if the succeeding term is $u_i + 1$ then u_i is part of an increasing contiguous subsequence, of length greater than one. In this case we replace u_i by wreath with a decreasing pair of terms.

For example if $U_1 = 123$ and $u_i = 2$ then we obtain 1324.

Alternatively we replace u_i with an increasing pair of terms. This defends us against the alernative possibility of u_i belonging to a decreasing contiguous subsequence of U_1.

A little thought should now persuade the reader that the permutation thus obtained has precisely two subsequences order isomorphic to U_1, and that these subsequences together constitute the entirety of the constructed permutation. We denote this permutation $f(u_i)$.

The fact that U is an infinite antichain implies that U_1 has at least two, indeed at least three terms. Thus we can construct two distinct permutations $f(u_i)$ and $f(u_j)$, each of which has precisely two subsequences order isomorphic to U_1. That $f(u_i)$ and $f(u_j)$ are distinct for distinct terms u_i and u_j is another small permutation specific technical result.

Both the permutations $f(u_i)$ and $f(u_j)$ are elements of $\mathcal{A}(K \cup U')$, but any merge of the two must have at least three subsequences order isomorphic to U_1, thus these two permutations form a non-joining pair in $\mathcal{A}(K \cup U')$.

To finish the proof, we now have two classes, $\mathcal{A}(U')$ and $\mathcal{A}(K \cup U')$. The former is atomic, the latter we have shown is not. We can remove basis elements of the latter one at a time until we obtain a class identical to the former. All the classes obtained in this gradual reduction are infinitely based. Thus as the transition from non-atomic to atomic must be made at least once, there exists an infinitely based non-atomic class with a basis element that, when removed, yields an atomic class. ■

The reader may have noted, or may now note that the technical machinery used in the last proof is exactly that which is needed to do Exercise 16, which states that there are precisely n^2+1 permutations of length $n+1$ that involve any given permutation of length n.

4.5 Further Modification of the Basis

We consider results that hold for arbitrary partial orders with regard to the concepts of 'basis' and 'join property', and apply them to involvement. We occasionally use technical knowledge about involvement to further constrain the obtained results. We deduce the possibility and demonstrate the existence of certain antichains that cannot be contained as a subset of the basis of an atomic class.

Let X be a closed class with basis B. We first consider the case when X is atomic, and under what circumstances it is possible to add permutations to the basis of X and thereby make in non-atomic.

1. If every element of X is contained in $Sub(B)$ then B is a maximal antichain and no further elements can be added to the basis of X. Therefore we can disregard this case. (We do note however, trivially, that in this case B is infinite if and only if X is infinite.)

2. If not every element of X is in the closure $Sub(B)$ then, except in the case where $X \setminus Sub(B)$ is linearly ordered under inclusion, it is certainly possible to add finitely many elements to the basis of X in a manner that produces a non-atomic class. Choose any two non-comparable elements α, β of $X \setminus Sub(B)$ and append $X \cap (\alpha \ minmerge \ \beta)$ to the basis of X.

3. If $X \setminus Sub(B)$ is linearly ordered then it is always possible to add a single element to the basis of X and thereby produce a non-atomic class, except possibly in the unusual circumstance when the smallest element of $X \setminus Sub(B)$ involves every element of $Sub(B)$. This indicates that B is finite. In that case either:

   (a) $X \setminus Sub(B)$ has only on element. In this case X is finite and it is easily decidable whether it is possible to add elements to B and thereby produce a non-atomic class.

   (b) $X \setminus Sub(B)$ has at least two elements, one greater than the other. Let us consider any two such elements, α and β, and let us choose them in such a way that the length of α is one greater than the length of β. Note that closure implies that if we delete any term from α then the result is order isomorphic to β. Examination of permutations quickly permits us to deduce that the only permutations that involve only one permutation of length one shorter than themselves, are the elements of I and R. From this we deduce that X is equal to the entirety of precisely one of these classes.

Thus either X is finite, and it is easily decidable whether it is possible to add permutations to the basis of X to manufacture a non-atomic class, or else X is equal to either I or R.

That completes the analysis of whether it is possible to restrict an atomic class and thereby make it non-atomic. We now consider the case when X is non-atomic, and when it is possible to add permutations to the basis of B and thereby make X atomic.

1. When restricting X there is only one set of permutations that we are obliged to keep in the new class. These are the elements of $PropSub(B)$. Any class that contains this and does not contain any element of B can be created by adding permutations to the basis B of X. (For finite B it is trivially decidable whether such a class can be atomic.)

Thus there may be certain antichains that will never appear in the basis of an atomic class. Trojan horse like, their presence brings non-atomicity:

Theorem 191 *Let H be an antichain. There exists an atomic class with H as a subset of its basis if and only if there does not exist an antichain C with the property that i) every element of C is properly involved in an element of H, and ii) there is no atomic subclass of $\mathcal{A}(H)$ that contains C.*

The proof we omit; it is elementary. The theorem yields a fast decision algorithm of whether a given finite set of permutations may be found in the basis of an atomic class. Fast because the only antichain C that we need consider in this case is the set of maximal elements of $PropSub(H)$. To determine whether C is contained in an atomic subclass of $\mathcal{A}(H)$ we search for a permutation in $\mathcal{A}(H)$ that involves every element of C.

We give only one example of a 'Trojan' set, as it is clear how to generate them and as they tend to be quite large sets.

Example 192 Every closed class whose basis set contains $132, 213, 231, 312$ is non-atomic. Every such class contains both 12 and 21 but these permutations cannot join in the class. We note that $\mathcal{A}(132, 213, 231, 312) = I \cup R$.

We see no reason why minimal 'Trojan' antichains should not be infinite.

4.5.1 A List of Non-Atomic Classes

These are but a few:

1. $\mathcal{A}(321, 1234)$ Finite and non-atomic. A short pair of elements that do not join is: (214365,456123). Since atomicity is decidable for finite classes, and since finitude of a class is decidable from the basis, we give no further finite non-atomic examples.

2. $\mathcal{A}(321, 1324)$, equal to $Prof(21354) \cup Prof(351624)$ where $Prof(21354) = \mathcal{A}(321, 1324, 2413, 3142)$ and $Prof(351624) = \mathcal{A}(321, 1324, 21354)$. A non-joining pair of minimal combined length is (2413,21354).

3. $\mathcal{A}(321, 2143)$, equal to $\mathcal{A}(321, 2143, 3142) \cup \mathcal{A}(321, 2143, 2413)$.

4. $\mathcal{A}(132, 4321)$, equal to $Prof(42135) \cup Prof(32415)$ where $Prof(42135) = \mathcal{A}(132, 4321, 3241)$ and $Prof(32415) = \mathcal{A}(132, 4321, 4213)$. A non-joining pair of minimal combined length is (3241,4213).

5. $\mathcal{A}(123, 231, 2143)$, equal to $\mathcal{A}(123, 231, 132) \cup \mathcal{A}(123, 231, 213)$.

6. $\mathcal{A}(123, 231, 4132)$, equal to $\mathcal{A}(123, 231, 132) \cup \mathcal{A}(123, 231, 312)$.

7. $\mathcal{A}(123, 312, 2143)$, equal to $\mathcal{A}(123, 312, 132) \cup \mathcal{A}(123, 312, 213)$.

8. $\mathcal{A}(123, 312, 2431)$, equal to $\mathcal{A}(123, 312, 132) \cup \mathcal{A}(123, 312, 231)$.

Chapter 5

Natural Classes

Definition and Stability under Differentiation, Two Types of Natural Class, Bounds on the number of Differentiations required to produce Stability, Naturality is Decidable, Infinitely Based Natural Classes

5.1 Introduction

A natural class is a particular type of atomic class. We show here that natural classes are very close to being sum complete. Furthermore we are able to show that it is decidable whether a class defined by a finite basis is natural, a result that has not been forthcoming with arbitrary atomic classes.

5.2 Natural and Sum Complete Classes

Definition 193 A class of permutations X is *natural* if $X = \mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$ for some permutation π of the natural numbers.

Let us remind the reader that differentiating a sequence is removing the first term of the sequence. In the following Results 194 to 196 we show that a finitely based natural class stabilizes under differentiation into a sum complete class, the basis of which is easily obtainable.[1]

Theorem 194 *Let $X = \mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$ be a natural class with finite basis B. Let C be the set of final sum components of elements of B. Then there exists $N \in \mathbb{N}$ such that $\partial^N X = \mathcal{A}(C)$.*

Corollary 195 *As all the elements of C are sum indecomposable $\mathcal{A}(C)$ is sum complete. Thus every finitely based natural class becomes sum complete when differentiated sufficiently many times.*

Proof: [of Theorem 194]

First we will construct a suitable N. If $C \setminus B = \emptyset$ then $C = B$, X is sum complete and $N = 0$ satisfies the theorem.

Otherwise for every $\gamma \in C \setminus B$ choose $\beta \in B$ such that $\beta = \mu \oplus \gamma$ for some permutation μ. We have that $\mu \in X$ and therefore there exists a subsequence

[1] If an atomic class is $X = \mathcal{B}(A, B, \pi)$ then, retrospectively, it seems reasonable that if A has a "first or smallest element" then this, and any initial regularities at the beginning of elements of X could be removed by differentiation. And if A does not have a first element then X is invariant under differentiation. What is special about natural classes is that by differentiating one also removes the smallest terms, thus obtaining sum completion.

S of the natural numbers $\mathbb{N}$ such that $\pi(S) \cong \mu$. Let $\pi(i)$ be the largest term in $\pi(S)$. There exist precisely $\pi(i)$ numbers j such that $\pi(j) \leq \pi(i)$. Let m_γ be the largest such number.

Then $\partial^{m_\gamma} X \subseteq \mathcal{A}(\gamma)$. For otherwise there exists a subsequence T of $\mathbb{N}$ such that all elements of T are greater than m_γ and $\pi(T) \cong \gamma$. But then as every term of $\pi(T)$ is greater than every term of $\pi(S)$ we have that $\pi(S,T) \in X$ and $\pi(S,T) \cong \mu \oplus \gamma = \beta$, a contradiction.

Let N be equal to the greatest m_γ. We may conclude that $\partial^N X \subseteq \mathcal{A}(C)$. To prove that $\mathcal{A}(C) \subseteq \partial^N X$ note that for all $\alpha \in \mathcal{A}(C)$, $\underbrace{\alpha \oplus \alpha \oplus \ldots \oplus \alpha}_{N+1 \text{ times}} \in X$ and so $\alpha \in \partial^N X$. Q.E.D.

■

Theorem 196 *A natural class X is sum complete if and only if $X = \partial X$.*

PROOF: Suppose that X is sum complete. Let $\sigma \in X$. As $1 \in X$ we have that $1 \oplus \sigma \in X$ and so $\partial(1 \oplus \sigma) = \sigma \in \partial X$. By closure $\partial X \subseteq X$. Thus $X = \partial X$.

To prove the converse suppose that $X = \mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$ is not sum complete and therefore has a decomposable basis element $\beta \oplus \gamma$. Now β, $\gamma \in X$ but by Theorem 194 there exist only a finite number of subsequences S of $\mathbb{N}$ such that $\pi(S)$ is order isomorphic to γ. Of all the elements of such sets, let $m \in \mathbb{N}$ be the largest. Then $\pi(1, 2, \ldots m) \in X$ but $\pi(1, 2, \ldots, m) \notin \partial X$. Thus $X \neq \partial X$.

■

Theorem 197 *If X is a non-empty sum complete class of permutations then X is natural.*

PROOF: There exist only countably many finite permutations. Thus the elements of X can be listed as a sequence $\xi_1, \xi_2, \ldots$. Let π be the infinite permutation $\xi_1 \oplus \xi_2 \oplus \ldots$. π is a permutation of the natural numbers and furthermore $X = Sub(\pi)$, which means that $X = \mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$.

■

5.3 The Structure of π

We have learnt that barring some finite discrepancies, every finitely based natural class X is sum complete. The nature of that initial discrepancy can have a profound effect on how X is expressible in the form $\mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$. Essentially we divide finitely based natural classes into two types. If C is the set of final sum components of basis elements of X then either $X = Sub(\gamma) \oplus \mathcal{A}(C)$ for some finite permutation γ, or it is not. If X is expressible in this form then γ, which is unique only if a minimal γ is chosen, represents the initial discrepancy. It may as well be noted that unless $\mathcal{A}(C)$ is the class I of increasing permutations, $\mathcal{A}(C)$ can be represented in many different ways as a natural class, and therefore X can also be represented as $\mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$ for many different π, but that is an aside. If X is not expressible as $Sub(\gamma) \oplus \mathcal{A}(C)$ then π is severely restrained, is unique and has a very specific structure.

Theorem 198 *If $X = \mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$ is a finitely based natural class not expressible in the form $Sub(\gamma) \oplus \mathcal{A}(C)$, where C is the set of final sum components of basis elements of X, then there exist $m, n \in \mathbb{N}$ such that $m \neq n$ and $\pi_m\ \pi_{m+1}\ \pi_{m+2} \ldots \cong \pi_n\ \pi_{n+1}\ \pi_{n+2} \ldots$.*

This implies that π becomes cyclic with a cycle length equal to or dividing $m - n$.

PROOF:

π is a permutation of $\mathbb{N}$, and as such can be considered to be either sum decomposable or sum indecomposable. There exists a number k, that we can choose to be minimal, such that $\partial^k(X) = A(C)$. Now, unless X is expressible in the form $Sub(\gamma) \oplus A(C)$, we may conclude that π has only finitely many sum components and that the last sum component, which must be infinite, involves an element of C.

The consequence of this for k is that $\pi(k)$ is in the last sum component of π. We utilise this:

Select some terms of π, as follows: Firstly, let $\pi(p_1)$ be the rightmost term of π not bigger than $\pi(k)$. Then, for each $j \geq 1$ let $\pi(q_j)$ be the largest term that either precedes or is equal to $\pi(p_j)$, and for each $i \geq 2$ let $\pi(p_i)$ be the rightmost term either less than or equal to $\pi(q_i)$. As we might expect, all these terms are distinct:

CLAIM: The relative positions and sizes of the terms $\pi(p_i)$ and $\pi(q_j)$ satisfy the following inequalities:

$$q_1 < p_1 < q_2 < p_2 < q_3 < p_3 < q_4 < p_4 < \dots$$
$$\pi(p_1) < \pi(p_2) < \pi(q_1) < \pi(p_3) < \pi(q_2) < \pi(p_4) < \pi(q_3) < \dots$$

PROOF: For each positive integer i, both $\pi(q_i)$ and $\pi(q_{i+1})$ precede or are equal to $\pi(p_{i+1})$. As $\pi(q_{i+1})$ is the greatest term preceding or equal to $\pi(p_{i+1})$ and $\pi(q_i)$ is greater than all the terms it succeeds we have that

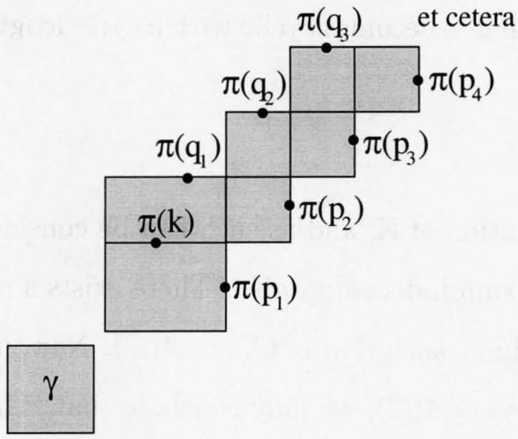


Figure 5.1: The terms of π as mapped out by $\pi(p_i)$ and $\pi(q_i)$. In our notation Λ_i denotes all the terms lying (inclusively) to the left of $\pi(p_i)$ and Υ_i denotes all the terms (inclusively) below or underneath $\pi(q_i)$.

$q_i \leq q_{i+1}$ and $\pi(q_i) \leq \pi(q_{i+1})$. Similarly for each i we have that $p_i \leq p_{i+1}$ and that $\pi(p_i) \leq \pi(p_{i+1})$.

Now suppose that $p_i = p_{i+1}$ for some integer i. All the terms preceding or equal to $\pi(p_i)$ are less than or equal to $\pi(q_i)$. Also, as $\pi(p_{i+1})$ is the rightmost term smaller than or equal to $\pi(q_i)$ we have that all the terms that succeed $\pi(p_i)$ are strictly greater than $\pi(q_i)$. This is not possible as $\pi(p_i)$ is contained in the last sum component of π.

Moreover if $q_j = q_{j+1}$ for some j then $p_{j+1} = p_{j+2}$ and so we reach the same contradiction. This proves the above inequalities.

∎

Thus the terms $\pi(p_i)$ and $\pi(q_j)$ can be used to demark π, and eventually to break it up into chunks. To this end for all integers $i > 0$ define $\Lambda(i) =$

$\{\pi(j)|j \leq p_i\}$ and $\Upsilon(i) = \{\pi(j)|\pi(j) \leq \pi(q_i)\}$. We will use our convention that permits Λ_i and Υ_i to represent either a set of terms or the sequence formed by that set of terms. Note that, due to our first claim, $\Upsilon(i) \subset \Lambda(i) \subset \Upsilon(i+1)$ for all integers $i \geq 0$; see Figure 5.1.

Let c be the length of the longest element of $A(C)$. If c is even then let $c' = c/2$, if odd let $c' = (c+1)/2$.

CLAIM: No embedding of an element of C in π contains a term succeeding $\pi(p_{c'+1})$.

PROOF: Suppose that α contains a term to the right of $\pi(p_{c'+1})$. Recall from the definition of k that α must contain a term preceding or equal to $\pi(k)$ and note that such a term must precede or be equal to $\pi(p_1)$. Elements of C are sum indecomposable and therefore we conclude that α must contain the following:

- At least one term in Λ_1 (preceding or equal to $\pi(k)$).
- At least one term in $\pi \setminus \Lambda_{c'+1}$ (by assumption).
- For each integer i such that $1 \leq i \leq c'$ at least one of the terms in $\Upsilon(i) \setminus \Lambda(i)$ and at least one in $\Lambda(i+1) \setminus \Upsilon(i)$ (to preserve sum indecomposability).

However the length of α is no greater than c, unlike the minimum number of terms required by the above, which is $2c' + 2$.

■

It follows directly from the preceding claim that of all terms contained in at least one embedding of an element of C in π, there exists a rightmost, which we denote $\pi(l)$. The term $\pi(l)$ marks a change in π that is manifest in a number of ways. Firstly:

CLAIM: For every p_i such that $p_i > l$ there exists unique subsequences of π order isomorphic to $\Lambda(p_i)$ and $\Upsilon(p_i)$. This is due to the fact that there is a unique subsequence of π order isomorphic to $\pi(1)\ldots\pi(l)$.

PROOF: The last term of $\pi(1)\ldots\pi(l)$ is the last term of a subsequence of π order isomorphic to an element of C. The last term of any alternative subsequence of π order isomorphic to $\pi(1)\ldots\pi(l)$ would have to succeed $\pi(l)$. Thus π would contain a subsequence order isomorphic to an element of C and whose last term lay to the right of $\pi(l)$. Thus we are maintained by contradiction.

By extension if f is a non-identity, order preserving map from $\Lambda(i)$ to π for some $p_i > l$ then $f(\pi(p_i))$ lies strictly to the right of $\pi(p_i)$. Since $\pi(p_i)$ is the rightmost term smaller than $\pi(q_{i-1})$ this implies that $f(\pi(q_{i-1}))$ is greater than $\pi(q_{i-1})$. This in turn implies that $f(\pi(p_{i-1}))$ lies to the right of $\pi(p_{i-1})$ and so on until we have that $f(\pi(p_0))$ lies to the right of $\pi(p_0)$. This, by order preservation implies that $f(\pi(k))$ is strictly greater than $\pi(k)$. However $k < l < p_i$ and we have above proved that f must be the identity map on $\pi(1)\ldots\pi(l)$, another contradiction.

The proof for $\Upsilon(i)$ is nearly identical to that for $\Lambda(i)$, and we omit it. This completes the proof of the claim. ■

Secondly, and consequent to this, we have that:

CLAIM: For every positive integer i such that $p_i \geq l$ the sequence $\pi(p_{i+1} +$

$1)\dots\pi(p_{i+2})$ has length no greater than $2(b-1)^2$, where b is the maximum number of terms in any basis element of X.[2]

PROOF: The sequence $\pi(p_{i+1}+1)\dots\pi(p_{i+2})$ is equal to $\Lambda(p_{i+2}) \setminus \Lambda(p_{i+1})$, which can be divided into the terms greater than $\pi(q_{i+1})$ and those less than $\pi(q_{i+1})$, namely $\Lambda(i+2) \setminus \Upsilon(i+1)$ and $\Upsilon(i+1) \setminus \Lambda(i+1)$. We show that neither of these sets contains an increasing or a decreasing sequence of length b. By [15] this implies our result.

Given any permutation μ we can test whether it is involved in $\Upsilon(i+1) \setminus \Lambda(i+1)$ by the following means: Consider the sequence consisting of $\Upsilon(i)$ and $\pi(q_{i+1})$. It can be expressed in the form:

$$\sigma\ \pi(q_{i+1})\ \tau$$

where $\pi(q_{i+1})$ is the largest term and where $\sigma\tau$ together is $\Upsilon(i)$, the rightmost term of which is $\pi(p_{i+1})$. We introduce μ in this manner: Denote by $(\mu+q_i)$ the sequence obtained by increasing the value of each term in μ by q_i, which has the effect of making all the terms of $(\mu+q_i)$ greater than those of $\Upsilon(i)$. Then the sequence

$$\sigma\ (\pi(q_{i+1})+|\mu|)\ \tau\ (\mu+q_i)$$

is order isomorphic to a permutation in X if and only if μ is in $\Upsilon(i+1) \setminus \Lambda(i+1)$. To see this consider any embedding of the above sequence into π. There exists a unique embedding of $\Upsilon(i)$ in π, therefore $\sigma\tau$ must be mapped onto that and the rightmost term of $\sigma\tau$ must be mapped onto $\pi(p_{i+1})$. This means that $(\mu+q_i)$ must be mapped onto the part of π strictly to the right of $\pi(p_{i+1})$, namely $\pi\setminus\Lambda(i+1)$. Furthermore in the above sequence $(\pi(q_{i+1})+|\mu|)$

[2] A little thought can reduce this further to $2(b-2)^2$.

is still the largest term, and so this must be mapped to some term in π no greater than $\pi(q_{i+1})$. From which we have that all the terms in $(\mu + q_i)$ are mapped to terms in π smaller than $\pi(q_{i+1})$, which puts them in $\Upsilon(i+1)$. That completes the requirement.

We denote the testing sequence above by $T(\mu)$, where μ is the sequence being tested. Now, let the longest increasing sequence in $\Upsilon(i+1) \setminus \Lambda(i+1)$ have length z. Then $T(I_z)$ is in X but $T(I_{z+1})$ is not. From this we have that $T(I_{z+1})$ involves a basis element of X, and that every one of the last $z+1$ terms of $T(I_{z+1})$ is involved in every embedding of a basis element of X in $T(I_{z+1})$. Thus $z + 1 \leq b$, as required. Similarly the maximum length of any decreasing sequence in $\Upsilon(i+1) \setminus \Lambda(i+1)$ is bounded by $b-1$. The combination of these two results demonstrates that $\Upsilon(i+1) \setminus \Lambda(i+1)$ has no more than $(b-1)^2$ terms.

The result for $\Lambda(i+2) \setminus \Upsilon(i+1)$ is obtained in the same way. This completes the proof of the claim. ■

We intend to demonstrate that for each $i \in \mathbb{N}$ there exists data of finite and uniformly bounded size from which we can calculate firstly the permutation order isomorphic to $\Lambda(1+i+c') \setminus \Lambda(i)$; and secondly, data, within the bound, sufficient to calculate $\Lambda(2+i+c') \setminus \Lambda(1+i)$. As the data is bounded, the sequence of these permutations must eventually cycle.

Claim: For each positive integer i such that $p_i > l$ we have that: The permutation order isomorphic to $\pi(1+p_i) \ldots \pi(p_{1+i+c'})$ is the longest permutation $\alpha = \alpha_1 \ldots \alpha_n$ satisfying all of the following:

- $\alpha_1 \ldots \alpha_{p_{i+c'}-p_i}$ is order isomorphic to $\pi(p_i + 1) \ldots \pi(p_{i+c'})$.

- All the terms of α succeeding $\alpha(p_{i+c'} - p_i)$ are greater than $\alpha(q_{i+c'-1} - p_i)$.
- $\alpha(n)$ is less than $\alpha(q_{i+c'} - p_i)$.
- α is an element of $A(C)$.

PROOF: Note first that the permutation order isomorphic to $\pi(1 + p_i) \ldots \pi(p_{1+i+c'})$ does satisfy these conditions, excepting perhaps that of maximality. Now let α be a permutation satisfying all the conditions of the lemma. We show that α is involved in $\pi(1 + p_i) \ldots \pi(p_{1+i+c'})$ (which proves that a maximal α exists and is order isomorphic to $\pi(1 + p_i) \ldots \pi(p_{1+i+c'})$), as follows:

Given $\alpha = \alpha(1) \ldots \alpha(w)$ we construct another permutation $\beta = \beta(1) \ldots \beta(p_i + w)$ as follows:

- Let $\beta(1) \ldots \beta(p_{i+c'})$ be order isomorphic to $\pi(1) \ldots \pi(p_{i+c'})$.
- Let $\beta(p_i + 1) \ldots \beta(p_i + w)$ be order isomorphic to α.

Note that β is well defined. Its length is prescribed. The sizes of the terms up to $\beta(p_{i+c'})$ relative one to another is given. By this we mean that given any two of the first $p_{i+c'}$ terms we know from the definition of β which term is greater. Similarly the sizes of the terms succeeding $\beta(p_i)$ relative one to one another is given. Finally note that c' is at least one, which may be verified by noting c must be at least three; we may deduce that the terms of β succeeding $\beta(p_{i+c'})$ are all greater than those preceding or equal to $\beta(p_i)$.

We claim that β is an element of X. Once we have proved that all the rest follows in this way: The permutation β must be embedded in π. The subsequence order isomorphic to $\pi(1)\ldots\pi(p_{i+c'})$ in π is unique, hence we know the first $p_{i+c'}$ terms of any such subsequence. It will then follow that the last $m + p_i - p_{i+c'}$ terms of such a subsequence must lie in the range $\pi(p_{i+c'} + 1)\ldots\pi(p_{i+c'+1})$.

Our claim that β is an element of X holds, for the alternative is that β contains a subsequence order isomorphic to an element of the basis of X, which cannot be for the following reasons: The first $p_{i+c'}$ elements of β are order isomorphic to an element of X, hence a subsequence $\beta(B)$ order isomorphic to a basis element of X must contain at least one term not in $\beta(1)\ldots\beta(p_{i+c'})$, but succeeding these terms. Especially the last sum component of $\beta(B)$ is order isomorphic to an element of C and must contain a term succeeding $\beta(p_{i+c'})$. Now α avoids all elements of C and the last n terms of β are order isomorphic to α, hence the first term of the last sum component of $\beta(B)$ must lie in $\beta(1)\ldots\beta(p_i)$. This is enough, for no element of C has more than $c \leq 2c' + 1$ terms, but the sum indecomposability of elements of C requires that the last sum component of $\beta(B)$ has at least $2c' + 2$ terms, as follows:

- At least one term succeeding $\beta(p_{i+c'})$.
- At least one term preceding or equal to $\beta(p_i)$.
- For each j such that $i \leq j < i + c'$ at least one term in the range $\beta(p_{i+1})\ldots\beta(p_{i+1})$ greater than $\beta(q_i)$.
- For each j such that $i \leq j < i + c'$ at least one term in the range $\beta(p_{i+1})\ldots\beta(p_{i+1})$ less than $\beta(q_i)$.

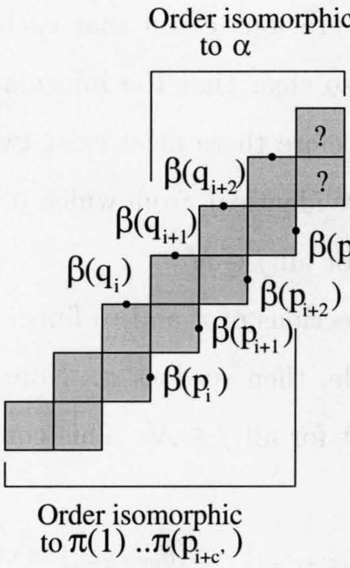


Figure 5.2: The permutation β.

Figure 5.2 shows β and it will be evident from there why all these terms are required to maintain sum indecomposability.

■

We define the *cell* $\Theta(i)$ to be an ordered collection of data consisting of:

- The permutation α_i order isomorphic to $\Lambda(i + c' + 1) \setminus \Lambda(i)$.
- The numbers $q_{i+c'-1} - p_i$ and $q_{i+c'} - p_i$.
- The numbers $p_{i+j} - p_i$ for $j = 1, 2, \ldots, c'$.

Note that none of the permutations contained in any cell $\Theta(i)$ with i such that $p_i > l$ has length greater than $2(c' + 1)(b - 1)^2$, and that none of the integers contained in any of these cells lies outwith the range $[1, 2(c' + 1)(b - 1)^2]$.

It is clear from the previous claim that each cell can be derived from its predecessor. It is also clear that the information carried in each cell is bounded above and therefore there must exist two numbers m' and n' such that $\Theta(m')$ and $\Theta(n')$ are identical, from which it follows that $\Theta(m'+j)$ and $\Theta(n'+j)$ are identical for all $j \in \mathcal{N}$.

The cells represent sections of π and to finish the proof we need to show that when the cells cycle, then so does π. Note that due to the cell cycle $p_{n'+j} - p_{m'+j}$ is constant for all $j \in \mathcal{N}$. This constant we call the *period* P of *pi*.

To note that also $\pi(q_{n'+c'+j}) - \pi(q_{m'+c'+j})$ is constant we need utilize the fact that c, the longest length of any final sum component of a basis element of X, must be at least three, giving $c' \geq 2$. This implies that not only is the permutation $alpha_i$ in each cell order isomorphic to $\Lambda(i+c'+1) \setminus \Lambda(i)$, but the difference $\pi(j) - \alpha(j-p_i)$ is constant for all j such that $\pi(j) \in \Upsilon(i+c') \setminus \Upsilon(i)$. Since this range includes the terms $\pi(q_{i+c'-1})$ and $\pi(q_{i+c'})$ this means that each cell $\Theta(i)$ specifies the difference $\pi(q_{i+c'}) - \pi(q_{i+c'-1})$, and our result follows.

Now let some term $\pi(j)$ be given with $j > p_{m'+c'-1}$. Then:

$$\pi(j) \in \Lambda(i+c') \setminus \Lambda(i+c'-1) \subset \Upsilon(i+c') \setminus \Upsilon(i)$$

for some $i \geq m'$. Therefore:

$$\pi(j) = \pi(q_{i+c'}) + \alpha_i(j-p_i) - \alpha_i(q_{i+c'} - p_i)$$

Now we will examine the term one period on from $\pi(j)$, which is the term:

$$\pi(j+P)$$

Due to periodicity of cells we have that:

$$\pi(j+P) \in \Lambda(i+c'+n'-m') \setminus \Lambda(i+c'+n'-m'-1)$$

which means that:

$$\begin{aligned}\pi(j+P) &= \pi(q_{i+c'+n'-m'}) + \alpha_{i+n'-m'}(j-p_i) - \alpha_{i+n'-m'}(q_{i+c'}-p_i) \\ &= \pi(q_{i+c'+n'-m'}) + \alpha_i(j-p_i) - \alpha_i(q_{i+c'}-p_i) \\ &= \pi(q_{i+c'+n'-m'}) - \pi(q_{i+c'}) + \pi(j)\end{aligned}$$

and since $\pi(q_{i+c'+n'-m'}) - \pi(q_{i+c'})$ is constant for $i - c' \geq m'$ it follows that:

$$\begin{aligned}&\pi(p_{m'+c'-1}+1)\pi(p_{m'+c'-1}+2)\pi(p_{m'+c'-1}+3)\ldots \\ &\quad \cong \pi(p_{n'+c'-1}+1)\pi(p_{n'+c'-1}+2)\pi(p_{n'+c'-1}+3)\ldots\end{aligned}$$

This completes the proof of Theorem 198.

■

This result has been followed up in [11] where cyclic and eventually cyclic classes amongst others are termed *regular classes* and a method is given for obtaining the basis of a cyclic class if only we can generate its elements by a certain grammar, a grammar that suits our purposes very well. All this carries the important consequence that it is decidable whether a class defined by a basis is a natural class of this cyclic form, but more of this later. First we need some numerical constraints and a result.

5.4 Bounds on Differentiation

5.4.1 Bases Remain Finite

Apart from being a nice result in its own right, we will need to know for a shortly to appear decidability result that a finitely based natural class is still finitely based when differentiated. This we prove:

Definition 199 Let G be a set of permutations. Then G is said to have the *finite drop* property if there exists $m \in \mathbb{N}$ such that for every $\gamma = \gamma_1 \dots \gamma_n \in G$ and for every $i, j \in \mathbb{N}$ such that $1 \leq i \leq j \leq n$, $\gamma_i \leq \gamma_j - m$.

Proposition 200 *Let X be finitely based and have the finite drop property. Then ∂X is finitely based.*

This result does not hold for all finitely based classes, as was shown in Sections 2.5.2 and 2.5.3.

PROOF: As X is finitely based there exists a number n such that X has no basis element of length greater than n. As X has the finite drop property there exists a number m such that no term of any permutation in X is succeeded by m or more terms less than that term.

Suppose now that there exists a basis element $\delta = \delta_1 \dots \delta_k$ of ∂X such that $|\delta| > (n+1)m$. Define R to be the set consisting of the smallest m terms of δ. For each integer i satisfying $1 \leq i \leq m$ define $\alpha^i = a_1^i \dots a_{k+1}^i$ to be the permutation satisfying $a_1^i = i$ and $a_2^i \dots a_{k+1}^i \cong \delta$. Define $f_i : \delta \to \alpha^i$ by $f_i(\delta_j) = a_{j+1}^i$, an injective map. Each α^i involves a basis element of X, for otherwise $\delta \in \partial X$, and therefore we may define for each i, γ^i to be a subsequence of α^i order isomorphic to a basis element of X.

For each i, $|f_i^{-1}(\gamma^i)| \leq n$, and $|R| = m$. Thus there exists a term δ_p of δ such that δ_p is not in R or any $f_i^{-1}(\gamma^i)$. But then $\delta \setminus \delta_p$ is not an element of ∂X. For suppose that $\mu = \mu_1 \dots \mu_k$ is a permutation in X such that $\mu_2 \dots \mu_k \cong \delta \setminus \delta_p$. Let $g : (\delta \setminus \delta_p) \to (\mu \setminus \mu_1)$ be an order preserving map.

$\mu_1 \leq m$, because of the finite drop property of X. But then $\mu_1, g(f_{\mu_1}^{-1}(\gamma^{\mu_1}))$ is a subsequence of μ order isomorphic to γ_{μ_1}, which is a basis element of X.

Thus δ is not a minimal element not contained in ∂X and is therefore not a basis element of ∂X. ■

Corollary 201 *If X is a finitely based natural class then ∂X is also finitely based.*

PROOF: If X is expressible in the form $Sub(\gamma) \oplus \mathcal{A}(C)$ where γ is some permutation and C is the set of final sum components of basis elements of X then ∂X is also expressible in that form. If Y is a finite class and Z is a finitely based class then $Y \oplus Z$ is finitely based. Thus ∂X is finitely based.

If X is not expressible in the form $\gamma \oplus \mathcal{A}(C)$ then X has the finite drop property. ■

5.4.2 A Bound on Minimal n such that $\partial^n X = \partial^{n+1} X$

Let X be a finitely based closed class. Let C be the set of all final sum components of basis elements of X.

If X is natural, in which case we may express X as $\mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$, then there exists a number k such that $\partial^k X = \mathcal{A}(C)$. $\mathcal{A}(C)$ is a sum complete class. If we let k be the largest number such that $\pi(k), \pi(k+1), \ldots$ involves an element of C then k is also the smallest number such that $\partial^k X = \mathcal{A}(C)$. In this section we establish a coarse upper bound on k. This provides us with a terminating point for tests that establish whether or not X is natural.

Lemma 202 *Every infinite closed class contains as a subclass either I or R or both. (See [14].)*

For a proof see [14]. From this it is an easy corollary that:

Corollary 203 *A closed class is finite if and only if it does not contain I_m and R_n for some positive integers m and n.*

Thus we can define a function:

Definition 204 Let $F(m, n)$ be the maximum length of the permutations avoiding both I_m and R_n. ($F(m, n)$ is simply $(m - 1)(n - 1)$. See Theorem 20.)

Proposition 205 *Let X be a finitely based natural class. Let C be the set of all final sum components of basis elements of X. Then if X has no basis element of length greater than b and if $\mathcal{A}(C)$ has no basis elements of length greater than c then $\partial^{(c+1)*F(b,b)+1}X = \mathcal{A}(C)$.*

In practice a considerably smaller bound can usually be found. The following proof gives a method of establishing a bound no greater than the one given above.

PROOF:

Let k be the smallest number such that $\partial^k X = \mathcal{A}(C)$. It follows that $\pi(k)$ is the leftmost element of some embedding of a basis element, ν, of $\mathcal{A}(C)$ in π.

We wish to classify the terms preceding $\pi(k)$ according to their size and therefore we denote the terms involved in this embedding of ν by $\pi(\nu_1), \ldots \pi(\nu_n)$ where $\pi(\nu_i) < \pi(\nu_j)$ if $i < j$. We claim that of the terms preceding $\pi(k)$ there are at most $F(b, b)$ that are smaller than $\pi(\nu_1)$, at most $F(b, b)$ that are greater than $\pi(\nu_n)$, and for each $0 < i < n$ at most $F(b,b)$ terms that are greater than $\pi(\nu_i)$ and less than $\pi(\nu_{i+1})$. To demonstrate this we show that there is no increasing or decreasing sequence of length b in any of these categories. Finally, as $n \leq c$ we can deduce our bound for the number of terms preceding $\pi(k)$.

Let $\theta = i, i+1, i+2, \ldots i+k-1, \mu_1, \mu_2, \ldots \mu_n$ be a permutation where $\mu_1 \ldots \mu_n$ is order isomorphic to the permutation ν. At most $k-1$ elements may precede any embedding of a basis element of $\mathcal{A}(C)$ and so θ is not contained in X and involves a basis element of X. If $k \leq b$ then we may conclude that in π there is no increasing subsequence of length b preceding $\pi(k)$, greater than $\pi(\nu_{i-1})$ and less than $\pi(\nu_i)$ (if $i = 1$ or $i = n+1$ then we may conclude that there is no increasing sequence of length b preceding $\pi(k)$ and less than $\pi(\nu_1)$ or greater than $\pi(\nu_n)$ respectively).

But if $k > b$ then we may draw the same conclusion because an embedding of a basis element of X in θ will involve at most b terms of the initial increasing sequence and so that same basis element of X can also be embedded in the permutation order isomorphic to $i, i+1, i+2, \ldots i+b-1, \mu_1, \mu_2, \ldots \mu_n$.

Similarly by considering permutations of the form $i+k-1, i+k-2, i+k-3, \ldots i+1, i, \mu_1, \mu_2, \ldots \mu_n$ it may be demonstrated that there is no decreasing sequence of length b in any of the aforenamed categories. Thus we have our required result. Q.E.D.

■

5.4.3 X as $Sub(\gamma) \oplus \mathcal{A}(C)$

We have that finitely based natural classes when differentiated eventually stabilize as sum complete classes. We used the way in which this can occur to divide finitely based natural classes into two types: those of the form $Sub(\gamma) \oplus \mathcal{A}(C)$ and the rest, which must be eventually cyclic. For the first type we here now find a bound for the length of γ, for each class, obtained from its basis.

Theorem 206 *Suppose that X is a finitely based natural class and that X is expressible in the form $Sub(\gamma) \oplus \mathcal{A}(C)$, where γ is some finite permutation and C is the set of all final sum components of basis elements of X. Let b be the length of the longest basis element of X and let c be the length of the longest basis element of $\mathcal{A}(C)$. Then X can be written in the form $Sub(\gamma) \oplus \mathcal{A}(C)$, where γ is a permutation having length no greater than $((c+1)^2 - 1) * F(b,b) + c + 2b * F(b,b)$.*

PROOF: Without loss of generality suppose that X is not sum complete, so that X has at least one sum decomposable basis element. Then there is at least one element of C in X, and therefore γ is non-empty. Without loss of generality we may assume that the last sum component of γ involves an element of C.

We may express X as $\mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$ where the first h terms of π are order isomorphic to γ and where the remaining terms of π are all greater than these initial h terms. In future γ will be used to refer to the first h terms of π. Let b be the maximum length of any basis element of X and let c be the maximum basis element of any element of $\mathcal{A}(C)$.

Let $\pi(\nu_1), \pi(\nu_2), \ldots, \pi(\nu_n)$ be a subsequence of π order isomorphic to a basis element ν of $\mathcal{A}(C)$ and let it be a rightmost such sequence, in the sense that there does not exist another such subsequence of π whose rightmost element is a successor of $\pi(\nu_n)$. $\pi(\nu_1), \ldots \pi(\nu_n)$ is contained in the last sum component of γ. All terms of γ either precede the rightmost term of $\pi(\nu_1), \ldots \pi(\nu_n)$, are smaller than the greatest term of $\pi(\nu_1), \ldots, \pi(\nu_n)$, or else are above and to the right of every term of $\pi(\nu_1), \ldots, \pi(\nu_n)$ but are contained in the same sum component of π that also contains $\pi(\nu_1), \ldots \pi(\nu_n)$.

We will consider the terms in each of these categories.

Consider any set of terms Θ of π that is undivided by any terms of $\pi(\nu_1) \ldots \pi(\nu_n)$. That is, there does not exist a pair of elements $\pi(i), \pi(j) \in \Theta$ and a term $\pi(\nu_l)$ such that either $\pi(i) \leq \pi(\nu_l) \leq \pi(j)$ or $i \leq \nu_l \leq j$ (or both). If all the terms of Θ precede the rightmost term of $\pi(\nu_1), \ldots, \pi(\nu_n)$ or are less than the largest term of $\pi(\nu_1), \ldots, \pi(\nu_n)$ then by an argument almost identical to the one employed in the main proof of the last section, the subsequence of π represented by the terms of Θ avoids both I_b and R_b and therefore $|\Theta| \leq F(b, b)$. Thus there are at most $((c+1)^2 - 1) * F(b, b) + c$ terms of π that are not both greater than the greatest term of $\pi(\nu_1) \ldots \pi(\nu_n)$ and further to the right than the rightmost term of $\pi(\nu_1) \ldots \pi(\nu_n)$. (This includes all terms of sets such as Θ as well as the terms of $\pi(\nu_1) \ldots \pi(\nu_n)$, of which there are at most c.)

Now we will consider the terms of γ succeeding and greater than $\pi(\nu_1) \ldots \pi(\nu_n)$. We must consider two possible cases separately, but the argument for one case is symmetric to that of the other.

Let $\pi(q_0)$ be the greatest term of $\pi(\nu_1) \ldots \pi(\nu_n)$. Let $\pi(p_0)$ so that the rightmost term of $\pi(\nu_1)\dots\pi(\nu_n)$.

Now we must consider two different cases. Let $\pi(p)$ be the rightmost term of π less than $\pi(q_0)$. If the greatest term preceding $\pi(p)$ lies to the right of $\pi(p_0)$ then we make the following definitions:

For every $i \geq 1$ let $\pi(p_i)$ be the rightmost term of π smaller than $\pi(q_{i-1})$, and let $\pi(q_i)$ be the greatest term of π preceding $\pi(p_i)$.

Otherwise we define:

For every $i \geq 1$ let $\pi(q_i)$ be the greatest term of π preceding $\pi(p_{i-1})$, and let $\pi(p_i)$ be the rightmost term of π smaller than $\pi(q_i)$.

The two situations are entirely analogous. The former situation is assumed and examined in detail. Thereafter the reader should be quite sufficiently well equipped to deal with the latter case.

We note the following:

- Every $\pi(p_i)$ and every $\pi(q_i)$ is in the same sum component of π as $\pi(\nu_1)\dots\pi(\nu_n)$.

- As γ is finite there exist only finitely many distinct terms of the form $\pi(p_i)$ or $\pi(q_i)$. Indeed there are at most $2b+2$ such terms. (Consider the infinite set of permutations $\pi(\nu_1)\dots\pi(\nu_n), \pi(q_0)+2, \pi(p_1), \pi(q_0)+4, \pi(q_0)+1, \pi(q_0)+6, \pi(q_0)+3, \dots, \pi(q_0)+2d+1$. Recall that $\pi(q_0)$ is the largest term of $\pi(\nu_1)\dots\pi(\nu_n)$. There are infinitely many of these sequences, they are sum indecomposable, involve an element of C and therefore at least one of them must involve a basis element of X. No basis element of X has length greater than b. Every subsequence of length b or less of any of the above sequences is involved in

$\pi(\nu_1)\ldots\pi(\nu_n), \pi(q_0)+2, \pi(p_1), \pi(q_0)+4, \pi(q_0)+1, \pi(q_0)+6, \pi(q_0)+3, \ldots, \pi(q_0)+2b+1$. Thus we have our bound.)

Another bound for the number of such terms is $c+4$. This is because for every basis element δ of X, if $\delta = \delta_1 \oplus \delta_2$ where δ_2 is sum indecomposable then δ_1 is involved in $\pi(1), \ldots, \pi(p_0)$. For otherwise $\pi(1), \ldots, \pi(p_0) \oplus \delta_2 \in X$. But any embedding of this permutation into π would place δ_2 to the right of $\pi(\nu_1)\ldots\pi(\nu_n)$, contradicting the rightmost choice of $\pi(\nu_1)\ldots\pi(\nu_n)$. Thus the terms q_i, $i \geq 2$, p_j, $j \geq 3$ must avoid the elements of C.

- Unless $q_i = q_{i+1}$, in which case $q_i = q_{i+2} = q_{i+3} = \ldots$ and $p_i = p_{i+1} = p_{i+2} = \ldots$ we have that for every $i \geq 1$:

  $$q_i < p_i < q_{i+1} < p_{i+1}$$

  and:

  $$\pi(p_i) < \pi(q_{i-1}) < \pi(p_{i+1}) < \pi(q_{i+1}).$$

  This result is obtained easily by induction.

- If $p_i = p_{i+1}$ then $\pi(p_i)$ is the rightmost term of γ. For in this case every term preceding $\pi(p_i)$ is less than or equal to $\pi(q_i)$ and every term succeeding $\pi(p_i)$ is greater than $\pi(q_i)$. (Were there a term of π preceding $\pi(p_i)$ greater than $\pi(q_i)$, this would contradict the choice of $\pi(q_i)$. Were there a term of π less than $\pi(q_i)$ succeeding $\pi(p_i)$ this would contradict the choice of $\pi(p_{i+1})$. Thus $\pi(p_i)$ represents the end of the last sum component of γ.)

- Every term $\pi(j)$ of γ above and to the right of $\pi(\nu_1), \ldots, \pi(\nu_n)$ satisfies one of the following for some $i \geq 0$:

$$\pi(j) > \pi(q_i) \text{ and } p_i < j < p_{i+1}$$

or:

$$j > p_i \text{ and } \pi(q_i) < j < \pi(q_{i+1})$$

We may further note that there are at most $F(b,b)$ terms in each of these categories. Thus we may conclude that there are at most $2b * F(b,b)$ terms above and to the right of $\pi(\nu_1), \ldots, \pi(\nu_n)$. This bound may be improved, for as $\pi(1), \ldots, \pi(p_0)$ involves every basis element of X barring its final sum component we have that there are at most $F(c,c)$ terms of π in each of the following categories:

$$\pi(j) > \pi(q_i) \text{ and } p_i < j < p_{i+1} \text{ where } i \geq 2$$

$$j > p_i \text{ and } \pi(q_i) < j < \pi(q_{i+1}) \text{ where } i \geq 2$$

Thus we may conclude that there are at most $2b * F(b,b)$ terms of π in γ above and to the right of $\pi(\nu_1), \ldots, \pi(\nu_n)$.

Thus we have our bound of $((c+1)^2 - 1) * F(b,b) + c + 2b * F(b,b)$ for the number of terms in γ. $((c+1)^2 - 1) * F(b,b)$ is an upper bound on the number of terms of γ not above and to the right of $\pi(\nu_1), \ldots, \pi(\nu_n)$, excluding the terms $\pi(\nu_1), \ldots, \pi(\nu_n)$. There are at most c terms in $\pi(\nu_1), \ldots, \pi(\nu_n)$. Finally there are at most $2b * F(b,b)$ terms of γ above and to the right of $\pi(\nu_1), \ldots, \pi(\nu_n)$.

■

It is worth noting that if X is expressible as $\gamma \oplus \mathcal{A}(C)$ then a minimal γ is unique. The representation of $\mathcal{A}(C)$ need not be. Indeed unless $\mathcal{A}(C) = I$ it may never be.

5.5 The Decision Problem

Given a basis we may wish to determine whether the closed class defined by it is natural. The results and machinery of this chapter give us the following theorem:

Theorem 207 *Given a finite basis, it is decidable whether the closed class defined by it is natural.*

This is remarkable because in general it is very difficult to determine whether a given basis defines an atomic class, and here we are able to determine for an entire spectrum of classes whether the class is atomic of a certain type. We justify the theorem with the following notes, from which, it will be seen, a full and detailed decision mechanism can be derived. The first two notes are sufficient justification, but the others may be helpful when realizing the decision mechanism.

- Given an arbitrary finitely based closed class X it is decidable whether or not X is a natural class of the form $Sub(\gamma) \oplus \mathcal{A}(C)$. If X is of this form then we have an upper bound for the length of minimal γ. We can generate maximal permutations in X, no longer than that upper bound and containing within their final sum components a final sum component of a basis element of X. If X is of the desired form then this maximal permutation exists, is unique and may be called γ. If C is the set of final sum components of basis elements of X then we can find the basis of $Sub(\gamma) \oplus \mathcal{A}(C)$, (see Section 2.2.6) and compare this with the basis of X.

The paper [11] gives a mechanism for obtaining the basis of classes such as $Sub(\pi)$ when π is eventually cyclic.

- It is also decidable whether or not a finitely based class is natural but not of the form $Sub(\gamma) \oplus \mathcal{A}(C)$. If X is indeed cyclic then $X = \mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$ for some *unique* π. The cycles of π have a length with a computable bound. Thus the first few terms of π can be generated, until we are guaranteed that at least two entire cycle has been completed, at which point the cyclic nature becomes evident. The cyclic pattern of π can be used to translate $\mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$ into a form acceptable to the results in the paper [11] on so-called *regular classes*. At that point the actual basis of $\mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$ can be calculated using the results of that paper, and compared with the basis of X. If the match is good then our assumption that X is of this form is correct.

- Neither of the above two decision processes suggests itself as being computationally easy. But it can be said that in neither of the two processes must all permutations of some given length be investigated for some property (the task that makes most algorithms almost ridiculously difficult). All the permutations we use can be generated by high polynomial time algorithms. With the first process, when generating γ, it is not necessary to find all maximal permutations in X involving an element of C in their final sum components. We only need to find one maximal element for each basis element of $\mathcal{A}(C)$. Of the resulting permutations we select the largest and continue with the second part of the decision process. Thus the algorithm it suggests is not entirely infeasible.

When attempting to determine whether or not a class is natural, and if so what form it takes, it is worth remembering the following:

- If every basis element of X is sum indecomposable then X is sum complete and, trivially, expressible as $\gamma \oplus \mathcal{A}(C)$, where C is the set of final sum components of basis elements of X. (In this case $C = X$ and any permutation of X will form a suitable γ.)

- If X is natural, C is the set of final sum components of basis elements of X and C contains an increasing oscillating sequence (i.e. a subpermutation of $\ldots 0, -3, 2, -1, 4, 1, 6, 3, \ldots$) then X is of the form $\gamma \oplus \mathcal{A}(C)$

- If X is natural and not expressible as $Sub(\gamma) \oplus \mathcal{A}(C)$ then X has the finite drop property. This can be used as a fast filter that tests whether a given basis might be natural and of this type, because it is both quick and easy to determine from a basis whether the class defined by it has the finite drop property. (If it is finite drop then there is a bound on the number of terms that can lie below and to the right, or above and to the left of any given term. It is easy to determine whether it is possible to bound the number of terms in any given place because a class is infinite if and only if it contains at least one of I and R as subclasses.)

5.6 Miscellaneous

We note that an infinitely based natural class may be finitely based when differentiated.

Proposition 208 *If X is natural and ∂X is finitely based then X need not be finitely based.*

PROOF: For an example of an infinitely based natural class X such that ∂X is finitely based see Example 103. ■

Proposition 209 *There exists an infinitely based cyclic natural class X. That is to say, there exists an infinitely based class $X = \mathcal{B}(\mathbb{N}, \mathbb{N}, \pi)$ and a number m such that for all subsequences $\pi(i_1), \ldots, \pi(i_n)$ of π, $\pi(i_1), \ldots, \pi(i_n) \cong \pi(i_1 + m), \ldots, \pi(i_n + m)$.*

PROOF: Let $X = Sub(2\ 3\ 5\ 1\ 7\ 8\ 4\ 10\ 6\ 12\ 13\ 9\ 15\ 11 \ldots)$. Then all of the following are basis elements of X:

2 3 4 5 1

2 3 5 1 7 4 8 9 6

2 3 5 1 7 4 9 6 11 8 12 13 10

...

$2\ 3\ 5\ 1\ 7\ 4\ 9\ 6 \ldots 4n-3\ 4n-6\ 4n-1\ 4n-4\ 4n\ 4n+1\ 4n-2$

Essentially, π is an increasing oscillating sequence with every other left maximal term replaced with an increasing pair. The listed basis elements are increasing oscillating sequences where two left maximal terms, separated by an *even* number of left maximal terms, are replaced by an increasing pair. ■

Chapter 6

Bibliothek

Infinite Fundamental Antichains, Finitely Based Classes

We list various infinite fundamental antichains, which are the antichains on which, to some extent, all infinite antichains are based. We then systematically list various closed classes with basis elements of length up to four, and for each give various pieces of information, if known, such as whether the class is atomic, how many permutations of each length it contains, and whether it is partially well ordered.

The list of infinite fundamental antichains can never be complete as there are infinitely many of them, although we do not prove this. (The interested reader may wish to do so, a proof can be obtained as follows: Examine the permutation listed under "Medley ..." in Figure 6.1 and confirm that it is indeed an element of an infinite fundamental antichain of similar permutations. Now note that the antichain elements are largely formed by a regular pattern which consists in part of increasing oscillating sequences, all of the same length, and that that length has been arbitrarily fixed. By varying the oscillating length more antichains of a similar nature can be formed, all fundamental.)

However it may be possible to characterise or classify all fundamental antichains. All the infinite fundamental antichains known to the author have great similarity to at least one of those lited in this chapter. Unfortunately we are unable to state briefly what we mean by 'similarity', but we attempt to illustrate by giving a spectrum of antichains 'similar' to our first antichain, ${}^{I_2}U^{I_2}$, when it appears.

We hope that the reader will be able to recognise heuristically the following three concepts that we use to think, if not to write:

1. The regular chain-like pattern that is to be found in (almost) every element of an infinite fundamental antichain.

2. The distinctions that mark the two ends of the regular pattern in each antichain element.

3. The possibility in some cases of melding together several of the regular patterns exhibited in this thesis so as to generate more complex regular patterns and more complex infinite fundamental antichains. (This possibility of melding is shown in only one example, in Figure 6.1, however it should not be difficult for the reader to see how other, much more extensive medleys can be formed.)

What we are writing is not rigorous, but we feel that the situation is, like many things, aptly described by Bagehot in "The English Constitution" when he states that it is not wrong to yearn for something before it is possible to achieve it, indeed that often the yearning is an essential prerequisite to the achieving. This was in discussing the early, crushed, uprisings that eventually led to the liberation and unification of Italy in 1852. We believe that an attack on fundamental antichains will lead to classification in terms of the three points mentioned above.

6.1 Antichains

The Antichain $^{I_2}U^{I_2}$

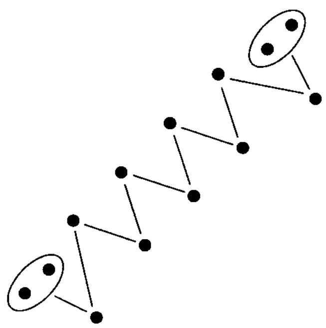
(2 3) (4 5) 1

(2 3) 5 1 (6 7) 4

(2 3) 5 1 7 4 (8 9) 6

(2 3) 5 1 7 4 9 6 (10 11) 8

(2 3) 5 1 7 4 9 6 11 8 (12 13) 10



et cetera

(2 3) 5 1 7 4 9 6 ... $[2n-4]$ $[2n+1]$ $[2n-2]$ $([2n+2]$ $[2n+3])$ $[2n]$

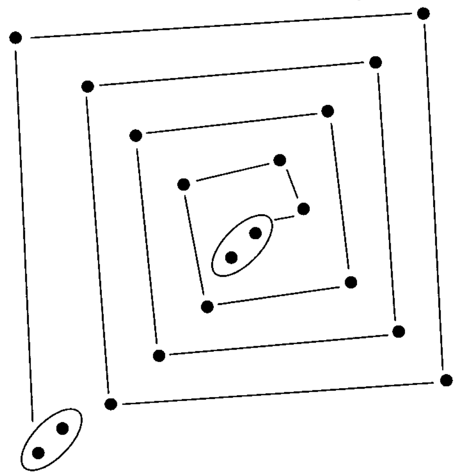
An antichain essentially identical to this was introduced by Spielman and Bóna in [8].

There is a family of antichains like this one, where each antichain element is constructed by replacing some two terms of an increasing oscillating sequence by pairs of terms. The notation $^{I_2}U^{I_2}$ is meant to reflect which terms are replaced. Our heuristic is as follows:

The terms of an increasing oscillating sequence can be partitioned uniquely into two monotonic increasing subsequences, an upper and a lower. To create an antichain element of type U we first replace a term at the left end of the increasing oscillating sequence with an increasing pair of terms. The replaced term may be the leftmost of either the upper or of the lower subsequence, and we denote this by ${}^{I_2}U$ and ${}_{I_2}U$ respectively. Similarly at the right end of the oscillating sequence we replace a rightmost term from either the upper or the lower subsequence, and this is denoted U^{I_2} or U_{I_2} respectively. Combining these two systems yields, for instance, ${}^{I_2}U^{I_2}$.

Terms may also be replaced by decreasing pairs, indeed Figure 6.1 lists as Variation 2 an element of antichain that we would denote as ${}^{R_2}U_{I_2}$ in which a term at the left end of an increasing oscillating sequence is replaced by a decreasing pair.

All U-type antichains have finitely based closure. For instance $Sub({}^{I_2}U^{I_2})$ has basis: {321, 3412, 4123, 23451, 134526, 134625, 314526, 314625}.



The Antichain W (Widderschin)

The elements of W are as follows:

6 (1 2) (3 4) 7 5

10 (1 2) 8 3 (5 6) 11 13 4

14 (1 2) 12 3 10 5 (7 8) 11 9 13 6 15 4

18 (1 2) 16 3 14 5 12 7 (9 10) 13 11 15 8 17 6 19 4

22 (1 2) 20 3 18 5 16 7 14 9 (11 12) 15 13 17 10 19 8 21 6 23 4

et cetera

$[4n+2]$ (1 2) $[4n]$ 3 $[4n-2]$ 5 $[4n-4]$ $\ldots$ $[2n+4]$ $[2n-1]$ ($[2n+1]$ $[2n+2]$) $[2n+5]$ $[2n+3]$ $[2n+7]$ $[2n]$ $\ldots$ $[4n-1]$ 8 $[4n+1]$ 6 $[4n+3]$ 4

The closure of this antichain has the following as basis elements. It is thought, strongly but modulo computational error, by the author that these are all the basis elements:

Type 0 basis elements (3): $\{2143, 2413, 3412\}$ (These permit only permutations that are a merge of an increasing and a decreasing sequence, and that avoid 2413.)

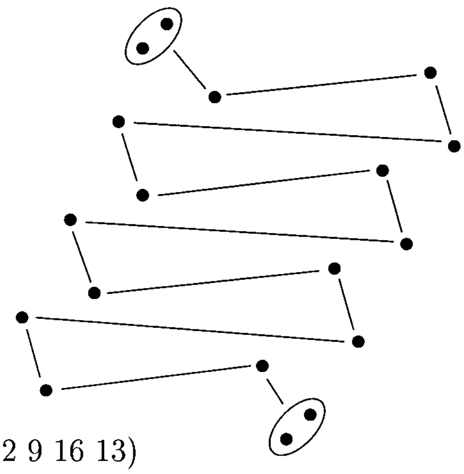
Type 1 basis elements (6): $\{412563, 512643, 415632, 431562, 541263, 541632\}$ (Given 3142 it is possible to replace any term with one of 12 or 21 (or 123 when 1 is replaced, as 1 lies in the third quadrant) and still be in the closure of Widderschin. However it is not possible to replace two such terms with the same effect. There are six ways of choosing two terms from four, hence this logic leads to six basis elements.)

Type 2 basis elements (4): $\{314562, 516432, 543162, 6123475\}$ (Given 3142 it is possible to replace the term 1 with an increasing triple and still be in the closure of W, but not an increasing four. Each of the terms 3, 4 and 2 can be replaced with an either increasing or decreasing pair of terms, but with no more than that. The four basis elements of type 2 indicate these limitations.)

Type 3 basis elements (4): $\{17234685, 61235748, 86123574, 72346851\}$

Type 4 basis elements (4): $\{17236485, 61253748, 86125374, 72364851\}$ (The permutations 6123574 and 6125374 appear only in the outside ring. The types 3 and 4 prevent them from lying anywhere but in that ring.)

Type 5 basis elements (2): $\{71346852, 71364852\}$ (These are inside ring basis elements.)



Parallel

((5 6) 3) (4 (1 2))

(6 3 (9 10) 7) (4 (1 2) 8 5)

(6 3 10 7 (13 14) 11) (4 (1 2) 8 5 12 9)

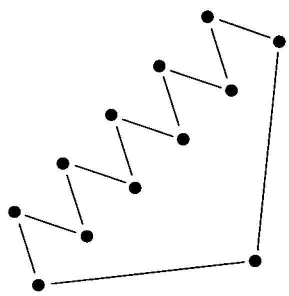
(6 3 10 7 14 11 (17 18) 15) (4 (1 2) 8 5 12 9 16 13)

(6 3 10 7 14 11 18 15 (21 22) 19) (4 (1 2) 8 5 12 9 16 13 20 17)

et cetera

(6 3 10 7 14 11 ... [$4n-6$] [$4n-9$] [$4n-2$] [$4n-5$] ([$4n+1$] [$4n+2$]) [$4n+1$]) (4 (1 2) 8 5 12 9 ...[$4n-4$] [$4n-7$] [$4n$] [$4n-3$])

Beginning and End Tied by One



4 1 2 3

4 1 6 3 2 5

4 1 6 3 8 5 2 7

4 1 6 3 8 5 10 7 2 9

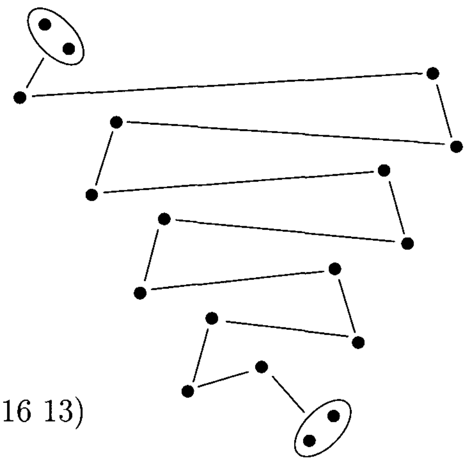
4 1 6 3 8 5 10 7 12 9 2 11

et cetera

4 1 6 3 8 5 ... [$2n$] [$2n-3$] [$2n+2$] [$2n-1$] 2 [$2n+1$]

The basis of the closure of this antichain has three elements of length four, six of length five, eight of length six, none of length seven, eight or nine. The known basis elements are: 3412, 4312, 4321, 12543, 15243, 21543, 25143, 34521, 52143, 124563, 142563, 214563, 234156, 234165, 235146, 235164, 241562.

The notion of "Beginning and End Tied by One" can also be used with the other types of antichain: Spiral, parallel, etc, as in Figure 6.1 (1).



The antichain V

(3 6 5) (4 1 2)

(7 10 9 3 6) (4 1 2 8 5)

(11 14 13 7 10 3 6) (4 1 2 8 5 12 9)

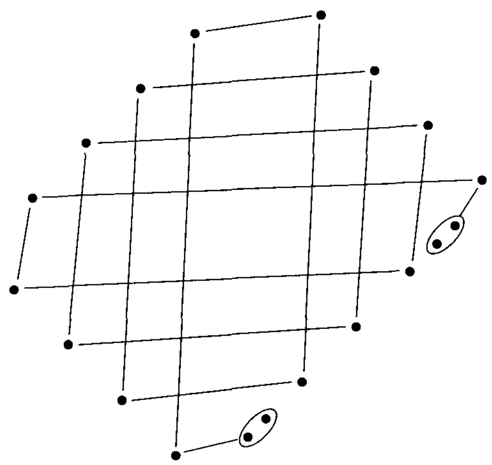
(15 18 17 11 14 7 10 3 6) (4 1 2 8 5 12 9 16 13)

(19 22 21 15 18 11 14 7 10 3 6) (4 1 2 8 5 12 9 16 13 20 17)

et cetera

$([4n-1]\ ([4n+2]\ [4n+1])\ [4n-5]\ [4n-2]\ [4n-9]\ [4n-6]\ \ldots\ 7\ 10\ 3\ 6)$
$(4\ (1\ 2)\ 8\ 5\ 12\ 9\ \ldots [4n-4]\ [4n-7]\ [4n]\ [4n-3])$

It may be noted that the elements of this set may be obtained from those of the antichain Parallel by reversing the order of appearance of the first half of the terms.



Quasi-square

1 6 (2 3) (4 5) 7

4 8 1 10 (2 3) 5 11 (6 7) 9

6 10 4 12 1 14 (2 3) 5 15 7 13 (8 9) 11

8 12 6 14 4 16 1 18 (2 3) 5 19 7 17 9 15 (10 11) 13

10 14 8 16 6 18 4 20 1 22 (2 3) 5 23 7 21 9 19 11 17 (12 13) 15

et cetera

$[2n]$ $[2n+4]$ $[2n-2]$ $[2n+6]$ $[2n-4]$ $[2n+8]$ $\ldots$ $[4n]$ 1 $[4n+2]$ (2 3) 5 $[4n+3]$ 7 $[4n+1]$ 9 $\ldots$ $[2n-1]$ $[2n+9]$ $[2n+1]$ $[2n+7]$ $([2n+2]$ $[2n+3])$ $[2n+5]$

The pattern does not form a square. The closure of a "tilted square" does not contain an infinite antichain, no matter what the angle of the tilt.

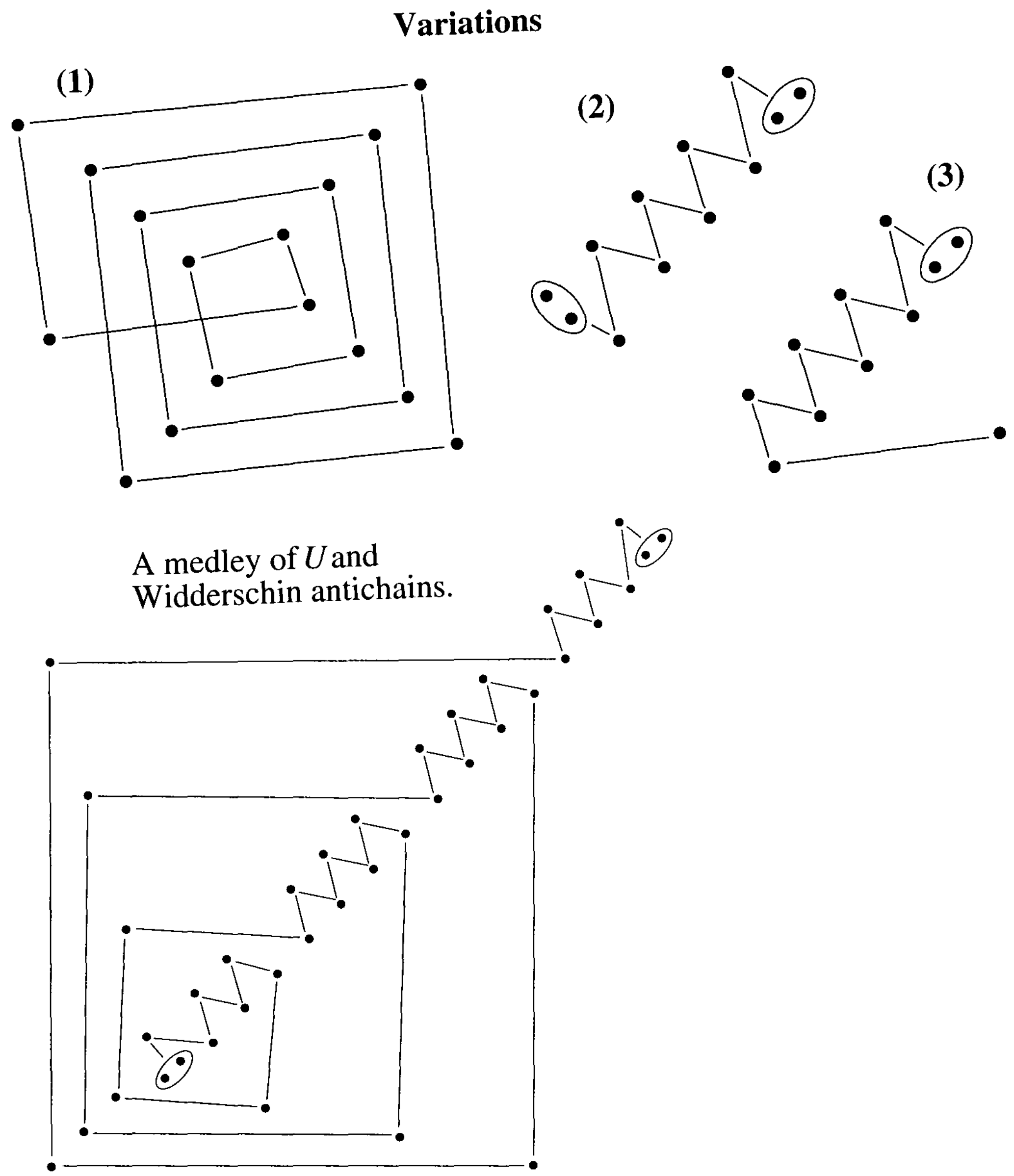


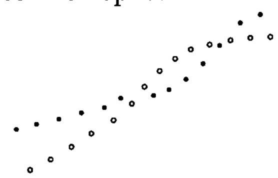
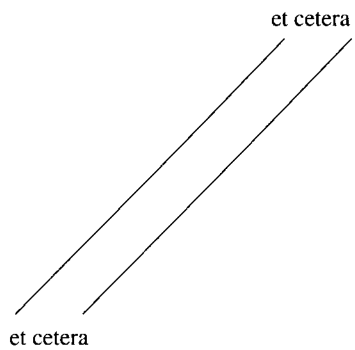
Figure 6.1: A multitude of antichains

6.2 Short Bases

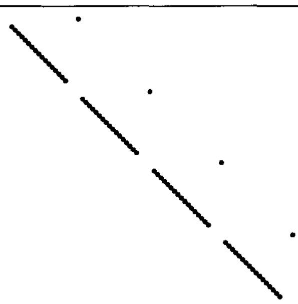
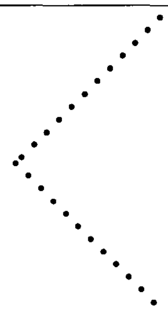
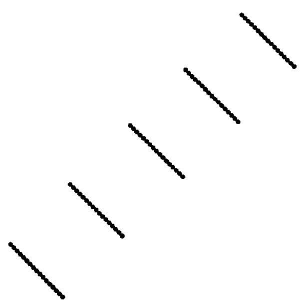
We begin a systematic compilation of classes with short basis elements.

6.2.1 One Basis Element of Length Three

All permutations of length three are symmetric to either 321 or 132.

1	321	Atomic but not partially well ordered. The elements consist of an interleaving of two increasing subsequences [17]. The number of permutations of length n in this class, i.e. the enumeration, is $\frac{2n!}{n!(n+1)!}$ (the nth Catalan number).	(A typical example, not an atomic representation:)  Atomic representation: 
2	132	Atomic and partially well ordered [4]. The elements are those of the class X generated as follows: • $1 \in X$ • If $\alpha, \beta \in X$ then $\alpha \ominus \beta \in X$ • If $\alpha \in X$ then $\alpha \oplus 1 \in X$ Enumeration: $\frac{2n!}{n!(n+1)!}$ (the nth Catalan number).	An atomic representation. 

6.2.2 Two Basis Elements of Length Three

1	$123, 321$	Finite and non-atomic. (3412 and 2413 do not join.) Longest permutation has length 4.	
2	$123, 231$	Neither sum nor skew complete. $Sub(R \ominus (R \oplus R))$ Atomic. Partially well ordered. Enumeration: $1 + \frac{n(n-1)}{2}$.	
3	$123, 132$	Not sum complete but skew complete. $Sub(\pi \ominus \pi \ominus \ldots$ *ad inf.*) where $\pi = R \oplus 1$. Atomic. Partially well ordered. Enumeration: 2^{n-1}.	 e.t.c.
4	$132, 312$	Neither sum nor skew complete. $Sub(0, 1, -1, 2, -2, 3, -3, \ldots)$ Atomic. Partially well ordered. Enumeration: 2^{n-1}.	
5	$231, 312$	Sum complete, not skew complete. $Sub(R \oplus R \oplus \ldots$ *ad inf.*) Atomic. Partially well ordered. Enumeration: 2^{n-1}.	e.t.c. 

6.2.3 One Basis Element of Length Four

Atomicity: Every class with only a single basis element is either sum or skew complete and therefore atomic.

Enumeration: All permutations of length four are symmetric to one of the following:

1234, 1243, 2143 1432, 1324, 1342, 2413

Theorem 210 (West) *For every permutation γ and every number n the number of permutations of each length in $\mathcal{A}(I_n \oplus \gamma)$ and $\mathcal{A}(R_n \oplus \gamma)$ is equal.*

This theorem has been proved for $n = 2$ in [29] and for all n in an as yet unpublished article. This theorem can be used to demonstrate that $\mathcal{A}(1234), \mathcal{A}(1243), \mathcal{A}(2143), \mathcal{A}(1432)$ are equinumerous classes. It is also known that $\mathcal{A}(1342)$ and $\mathcal{A}(2413)$ are equinumerous, see [30].

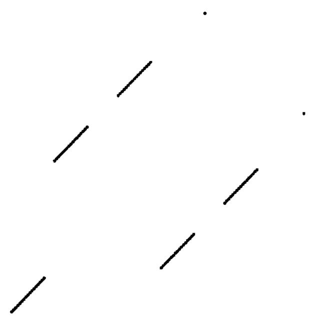
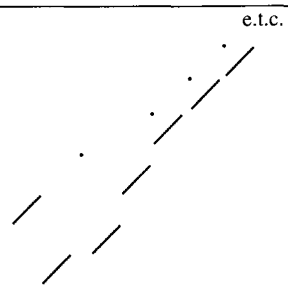
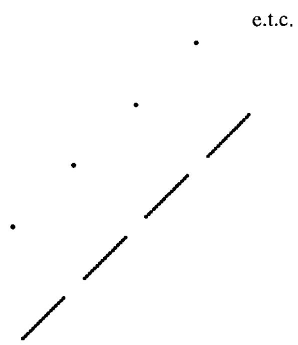
6.2.4 One Basis Element of Length Three and One of Length Four

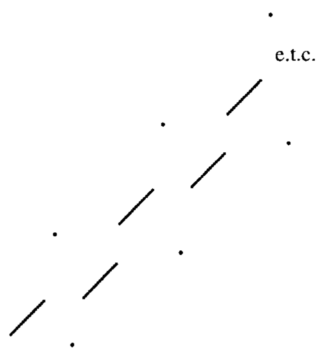
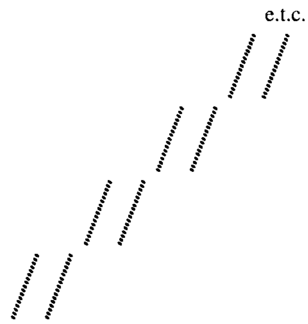
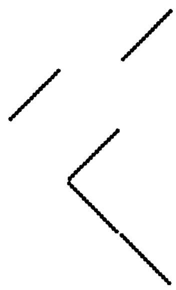
The entries of order type in the following tables are a partial answer to the following question: If X is an atomic class then what are in some sense the simplest cardinalities that two sets A and B can have where X is expressible in the form $\mathcal{B}(A, B, \pi)$.

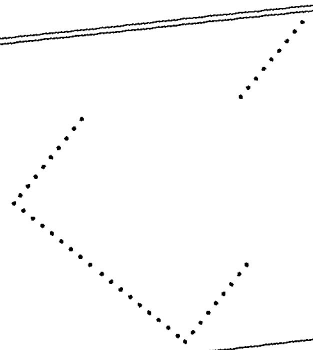
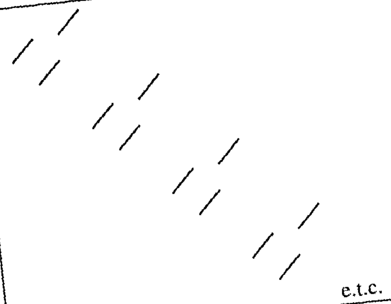
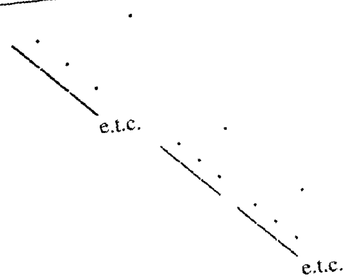
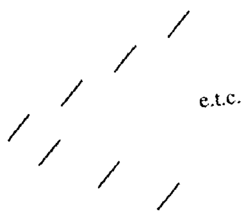
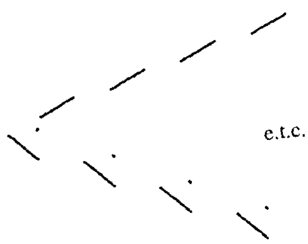
All other entries should be self explanatory.

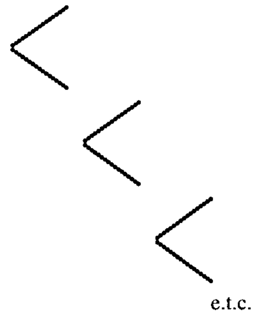
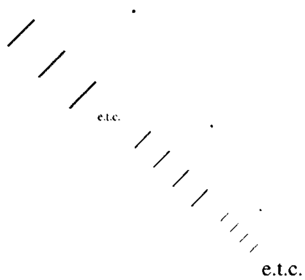
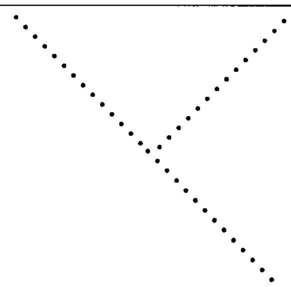
Enumeration: A table giving formulae for the number of permutations of given length in each of the following classes can be found in [1].

Partially Well Ordered: Of the following only $\mathcal{A}(321, 2341)$ and $\mathcal{A}(321, 3412)$ are not partially well ordered.

1	321, 1234	Neither sum nor skew complete. Not Atomic All perms: *length* ≤ 7. 214365 and 456123 do not join. $X = \mathcal{A}(321, 1234, 214365) \cup \mathcal{A}(321, 1234, 456123)$	
2	321, 2134	Neither sum nor skew complete. Atomic $Sub(1, 2, 3 \ldots, 3\omega, 3\omega + 1, 3\omega + 2, \ldots, 5\omega, 5\omega + 1, 5\omega + 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, 6\omega, 2\omega, 2\omega + 1, 2\omega + 2, \ldots 4\omega)$ Profile Class $Prof(1\ 4\ 6\ 2\ 7^1\ 3\ 5^1)$. Order Type: Expressible in the form $\mathcal{B}(\omega\omega\omega\omega 1\omega 1, \omega\omega\omega\omega 1\omega 1, \pi)$	
3	321, 1324	Neither sum nor skew complete. Not Atomic $Prof(2\ 1\ 3\ 5\ 4) \cup$ $Prof(3\ 5\ 1\ 6\ 2\ 4)$	
4	321, 1342	Neither sum nor skew complete. Atomic $Prof(3\ 1\ 5^1\ 2\ 4) \oplus \mathcal{A}(321, 231)$ Order Type: Expressible in the form $\mathcal{B}(\omega\omega 1\omega\omega\omega, \omega\omega\omega\omega 1\omega, \pi)$.	
5	321, 2341	Sum complete, not skew complete. Atomic. Not partially well ordered. Defined by the infinite profile class $Prof(3^1, 1, 5^1, 2, 7^1, 4, 9^1, 6, \ldots)$. The pattern is related to that of increasing oscillating sequences. Order Type: The class is sum complete and therefore expressible as a natural class, in the form $\mathcal{B}(\omega, \omega, \pi)$.	

6	$321, 3412$	Sum complete, not skew complete. Atomic. Not partially well ordered. Expressible as the infinite profile class $Prof(2^1\ 3\ 6^1\ 1^1\ 4\ 7\ 10^1\ 5^1\ 8\ 11\ \ 14^1\ 9^1\ 12\ 15\ 18^1\ \ldots)$ Order Type: $\mathcal{A}(321, 3412)$ is sum complete and is therefore natural.	
7	$321, 3142$	Sum complete, not skew complete. Atomic The sum completion of $I\ juxt\ I$. Order Type: The class is sum complete and is therefore natural.	
8	$321, 2143$	Neither sum nor skew complete. Not Atomic $\mathcal{A}(321, 2143, 3142) \cup \mathcal{A}(321, 2143, 2413)$	
9	$132, 4321$	Neither sum nor skew complete. Not Atomic $Prof(4\ 2\ 1\ 3\ 5) \cup Prof(3\ 2\ 4\ 1\ 5)$	
10	$132, 4312$	Neither sum nor skew complete. Atomic $Sub(2\omega, 2\omega + 1, 2\omega + 2, \ldots, \omega, \omega + 1, \omega - 1, \omega + 2, \omega - 2, \omega + 3, \omega - 3, \ldots, 3\omega, -1, 3\omega + 1, -2, 3\omega + 2, -3, 3\omega + 3, -4, \ldots)$ Order Type: Expressible in the form $\mathcal{B}(\omega\omega, \omega^R\omega^R\omega\omega\omega, \pi)$.	

11	132, 4231	Neither sum nor skew complete. Atomic $(\mathcal{A}(132,312) \ominus \mathcal{A}(231,132)) \oplus I$ Order Type: Expressible in the form $\mathcal{B}(\omega\omega^R\omega\omega, \omega\omega^R\omega\omega, \pi)$.	
12	132, 3214	Not sum complete but skew complete. Atomic The skew completion of $\mathcal{A}(132,321)$. Order Type: The class is skew complete, therefore expressible in the form $\mathcal{B}(\omega,\omega^R,\pi)$.	
13	132, 1234	Not sum complete but skew complete. Atomic The skew completion of $\mathcal{A}(132,123) \oplus 1$. Order Type: The class is skew complete and therefore expressible in the form $\mathcal{B}(\omega,\omega^R,\pi)$.	
14	132, 4213	Neither sum nor skew complete. Atomic This is the infinite profile class $Prof(0\ 1\ -1\ 2\ -2\ 3\ -3\ \ldots)$. Order Type: Expressible in the form $\mathcal{B}(\omega,\omega^R\omega,\pi)$.	
15	132, 4123	Neither sum nor skew complete. Atomic (For the derivation consider $\mathcal{A}(123,132)$, then $\mathcal{A}(132,4123,3124)$ containing 123, then $\mathcal{A}(132,4123)$ containing 3124.) Order Type: Expressible in the form $\mathcal{B}(\omega,\omega^R\omega,\pi)$.	

16	$132, 3124$	Not sum complete but skew complete. Atomic The skew completion of $\mathcal{A}(132, 312)$. Order Type: The class is skew complete and therefore expressible in the form $\mathcal{B}(\omega, \omega^R, \pi)$.	
17	$132, 2134$	Not sum complete but skew complete. Atomic The skew completion of $\mathcal{A}(132, 2134, 3241, 4213)$, where $\mathcal{A}(132, 2134, 3241, 4213) = (I \ominus I \ominus I \ominus I \ldots) \oplus 1$ Order Type: The class is skew complete and therefore expressible in the form $\mathcal{B}(\omega, \omega^R, \pi)$.	
18	$132, 3412$	Neither sum nor skew complete. Atomic $Sub(\ldots, 6, 4, 2, 0, 1, -1, 3, -2, 5, -3, \ldots)$ Order Type: Expressible in the form $\mathcal{B}(\omega^R\omega, \omega^R\omega, \pi)$.	

6.3 Two Basis Elements of Length Four

The following lists all the pairs of basis elements of length four, distinct up to isomorphism. There are 56 such pairs. The total number of permutations of each length is given for some small lengths and general results are given where known. The same is done for the number of interval-free permutations P_n, starting with length four.

Basis		Partially		Permutations of Length				
		Well Ordered?		5	6	7	8	
4321	4312	no	U	90	394	1806		Kremer I (1 of 9)
		high exp? $P_n = 2\ 6\ 30\ 118\ 488\ \ldots$						
4321	4231	no	U	90	396	1837		Unique
		high exp? $P_n = 2\ 6\ 28\ 109\ 459\ \ldots$						
4321	4213	no	U	89	380	1678		Unique
		high exp? $P_n = 2\ 4\ 19\ 62\ 217\ 742\ \ldots$						
4321	4123	no	U	86	342	1366		Type B? (1 of 4)
		high exp? $P_n = 2\ 6\ 30\ 91\ 263\ 749\ \ldots$						
4321	3412	no	U	86	342	1366	5462	Type B?
		$P_n = 2\ 4\ 22\ 72\ 226\ 690\ \ldots$						
4321	3214	no	U	89	376	1611		Unique
		$P_n = 2\ 6\ 30\ 107\ 371\ 1213\ \ldots$						
4321	3142	???		86	338	1314		Unique
		$P_n = 1\ 1\ 6\ 10\ 20\ 30\ 47\ 65\ \ldots \frac{57-4n-18n^2+4n^3+39(-1)^n}{48}$						
4321	3124	yes	P finite	86	330	1198		Unique
		P finite: $P_n = 2\ 4\ 19\ 39\ 49\ 43\ 33\ 24\ 11\ 2\ 0\ 0$						
4321	2143	???		86	333	1235		Unique
		$P_n = 2\ 4\ 22\ 52\ 112\ 200\ 346\ 564\ \ldots$						
4321	1324	yes	P finite	86	332	1217		Unique
		P finite: $P_n = 2\ 6\ 28\ 73\ 130\ 153\ 107\ 38\ 5\ 0\ 0$						
4321	1234	yes	class finite	86	306	882	1764	Finite

Basis		Partially Well Ordered?		5	6	7	8	
4312	4231	no	U	90	394	1806		Kremer VII
		high exp? $P_n = 2\ 6\ 24\ 88\ 340\ 1327\ \ldots$						
4312	4213	no	U	90	394	1806		Kremer IX
		high exp? $P_n = 2\ 4\ 17\ 58\ 218\ 822\ \ldots$						
4312	4123	no	U	89	382	1711		Unique
		high exp? $P_n = 2\ 6\ 18\ 60\ 208\ 748\ \ldots$						
4312	3421	no	U	87	354	1459	6056	Type C? (1 of 2)
		$P_n = 2\ 6\ 20\ 55\ 163\ 475\ \ldots$						
4312	3412	no	U	90	394	1806		Kremer VI
		$P_n = 2\ 4\ 14\ 46\ 158\ 552\ \ldots$						
4312	3214	no	U	88	365	1540		Unique
		$P_n = 2\ 6\ 21\ 65\ 217\ 727\ \ldots$						
4312	3142	???		88	367	1568		Unique
		$P_n = 1\ 3\ 5\ 11\ 21\ 43\ 85\ 171\ 341\ 683\ \ldots$						
		$P_{n+1} = 2 * P_n + (-1)^{n+1}$						
4312	3124	???		88	363	1507		Unique
		apprx. exp. growth: $P_n = 2\ 4\ 10\ 21\ 44\ 89\ 178\ 352\ 692\ \ldots$						
4312	2341	no	U	86	338	1318		Type D? (1 of 2)
		$P_n = 2\ 6\ 18\ 43\ 94\ 216\ 510\ \ldots$						
4312	2143	???		86	337	1295		Unique
		$P_n = 2\ 4\ 12\ 26\ 62\ 136\ 302\ 654\ \ldots$						
4312	2134	???		86	330	1206	4174	Unique
		observed polynomial growth: $P_n = 3n^2 - 23n + 46$						
4312	1324	???		86	335	1266		Unique
		poly $\bar{c}$ osc.? $P_n = 2\ 6\ 14\ 29\ 53\ 88\ 137\ 204\ 295\ 419\ 590\ \ldots$						
4312	1234	???		86	321	1085	3266	Unique
		finite: $P_n = 2\ 6\ 30\ 82\ 139\ 140\ 73\ 14\ 0\ 0$						

Basis		Partially Well Ordered?		5	6	7	8	
4231	4213	no	U	90	394	1806		Kremer VIII
		high exp? $P_n = 2\ 4\ 16\ 52\ 186\ 664\ \ldots$						
4231	4123	no	U	89	380	1677		Unique
		$P_n = 2\ 6\ 16\ 46\ 133\ 393\ 1174\ \ldots$						
4231	3412	no	U	88	366	1552	6652	Bona 1
		$P_n = 2\ 4\ 12\ 30\ 80\ 208\ 546\ \ldots$						
4231	3214	no	U	87	352	1428		Unique
		$P_n = 2\ 6\ 19\ 51\ 149\ 427\ \ldots$						
4231	3142	???		88	366	1552	6652	Bona 5
		$P_n = 1\ 3\ 5\ 11\ 21\ 43\ 85\ 171\ 341\ 683\ \ldots$						
		$P_{n+1} = 2 * P_n + (-1)^{n+1}$						
4231	3124	y??	by P	88	363	1508		Unique
		observed exp. growth: $P_n = 2^{n-3}$						
4231	2143	???		86	335	1271		Unique
		$P_n = 2\ 4\ 10\ 18\ 40\ 80\ 162\ 322\ 646\ \ldots$						
4231	1324	???		86	336	1282		Unique
		$P_n = 2\ 6\ 10\ 24\ 58\ 140\ 338\ 816\ \ldots$						

Basis		**Partially Well Ordered?**	**5**	**6**	**7**	**8**	
4213	4132	???	90	394	1806		Kremer VIII
		exp? $P_n = 2\ 2\ 7\ 14\ 37\ 90\ 233\ 602\ \ldots$					
4213	4123	no U	90	394	1806		Kremer IV
		$P_n = 2\ 4\ 10\ 25\ 65\ 173\ 470\ \ldots$					
4213	3421	no U	88	367	1571		Unique
		P finite: $P_n = 2\ 4\ 14\ \ 39\ 121\ 371\ 1167\ \ldots$					
4213	3412	no U	88	368	1584		Type F? 2
		$P_n = 2\ 3\ 10\ 27\ 78\ 225\ 664\ \ldots$					
4213	3241	no U	88	366	1552	6652	Bona 4
		$P_n = 2\ 3\ 11\ 28\ 82\ 230\ 660\ \ldots$					
4213	3142	yes	89	379	1664		Unique
		$P_n = 1\ 0\ 1\ 0\ 1\ 0\ 1\ \ldots$					
		P frames cannot support an infinite antichain.					
4213	3124	yes	88	366	1552	6652	Bona 2
		Constant : $P_n = 2$					
		P frames cannot support infinite antichain.					
4213	2431	no U	87	354	1459		Type C?
		If $n + 1$ is divisible by 3 then $P_n = 2\ 3\ 11\ 28\ 82\ 230\ 660\ \ldots$					
4213	2413	???	90	394	1806		Kremer V
		$P_n = 1\ 1\ 4\ 9\ 27\ 71\ 204\ 590\ \ldots$					
4213	2341	no U^{-1}	86	336	1290		Unique
		$P_n = 2\ 4\ 11\ 24\ 48\ 96\ 192\ 384\ 768\ \ldots$					
		After the first few terms P_n simply doubles each time.					

Basis		Partially Well Ordered?	5	6	7	8	
4213	3214	???	90	394	1806		Kremer III
		$P_n = 2\ 4\ 17\ 57\ 211\ 779\ \ldots$					
4213	2314	???	88	366	1552	6652	Bona 3
		$P_n = 2\ 3\ 7\ 13\ 25\ 46\ 84\ 151\ 269\ 475\ \ldots$					
4213	2143	???	88	366	1556		Unique
		$P_n = 2\ 7\ 14\ 39\ 102\ 288\ 820\ \ldots$					
4213	1432	no U	87	352	1434		Unique
		$P_n = 2\ 4\ 11\ 22\ 54\ 128\ 322\ 818\ \ldots$					
4213	1342	yes	86	338	1318	5106	Type D?
		P constant: $P_n = 2$					
		P frames cannot support infinite antichain.					
4213	1243	???	86	337	1299		Unique
		observed linear growth ($n \leq 20$): $P_n = 3n - 10$					

Basis		Partially Well Ordered?		5	6	7	8	
4123	3412	no	U^R	89	381	1696		Unique
		$P_n = 2\ 4\ 17\ 52\ 180\ 615\ \ldots$						
4123	3214	no	U	86	342	1366	5462	Type B?
		$P_n = 2\ 6\ 18\ 44\ 106\ 248\ 572\ \ldots$						
4123	3142	???		88	368	1584		Type F?
		$P_n = 1\ 1\ 3\ 4\ 8\ 12\ 21\ 33\ 55\ 88\ 144\ 232\ 377\ 609\ \ldots$ If $n+1$ is divisible by 3 then $P_{n+2} = P_n + P_{n+1} + 1$ otherwise $P_{n+2} = P_n + P_{n+1}$. (Observed only, not proved.)						
4123	2341	no	U^R	87	348	1374	5335	Unique
		$P_n = 2\ 6\ 17\ 37\ 81\ 176\ 397\ 912\ \ldots$						
4123	2143	no	U^R	86	342	1366	5462	Type B?
		$P_n = 2\ 6\ 17\ 37\ 81\ 176\ 397\ 912\ \ldots$						

3412	2413	no	W	90	395	1823		Unique
		$P_{n+2} = 1\ 1\ 5\ 13\ 45\ 149\ 522\ \ldots$						
3412	2143	no	W	86	340	1340		Unique
		$P_n = 2\ 8\ 16\ 44\ 108\ 284\ 740\ \ldots$ Merge of I and R. Enumeration: See [10].						

3142	2413	yes		90	394	1806		Kremer X
		Class of Separable permutations; has no elements of P.						

There would appear from the above to be at most five types of enumeration that appear more than once, namely those listed as having enumeration of type A-F.

Type A is now listed as "Kremer" as these classes were proved to be equinumerous by Darla Kremer in [9], enumeration being given by the Large Schröder numbers.

Type E, now listed as "Bona" have been shown in [7] to be equinumerous to what is known as the "Schubert Class" or "Smooth class" which is $\mathcal{A}(3412, 4231)$.

The four classes labelled as possibly being of Type B are consistent in that they would all appear to have enumeration given by $(4^{n+1} + 2)/3$. However this has not been verified. I am not aware of any work on the enumeration of possible types C, D and F.

There does not appear to be an obvious connection between the number of permutations in a class and the number of elements of P in the class.

6.4 Three Basis Elements

Extensive computational results are available for classes with three basis elements, but these classes are too numerous to list here. A program is available for Apple Macintosh computers that will generate these results for anyone who is interested.

Chapter 7

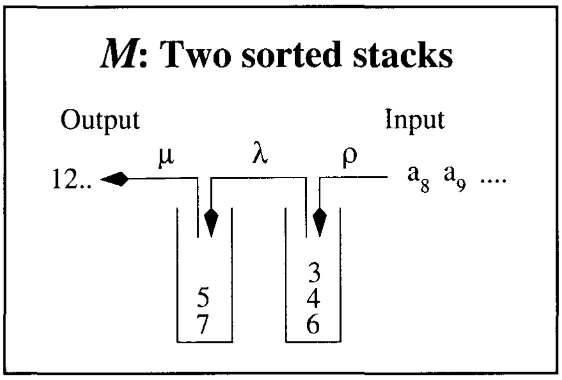
A Practical Example: The M Machine

In this chapter we will consider the set of permutations M that can be sorted over two stacks in series, with no restriction whatsoever except that the contents of both stacks remain ordered throughout the sorting process. By ordered we mean that if one term lies on top of another in a stack then, Hanoi like, the value of the upper must be strictly less than that of the lower. A notation for the moves performed on S^2, an arbitrary pair of stacks in series, was presented in Chapter 1, and we will use that notation again. Terms may be moved from the input to the output only by means of:

ρ: An operation that takes a single term from the input and places it on the right stack.

λ: An operation that takes a term from the right stack and places it on the left.

μ: An operation that takes a term from the left stack and puts it in the final output.



A sequence is sorted by two ordered stacks in series.

We let M refer to both this machine and to the set of all permutations that can be sorted by it

This chapter is essentially contained the paper by M.D. Atkinson, N. Ruškuc and myself, due to appear in Theoretical Computer Science under the title "Sorting with two ordered stacks in series" [3], which, in turn, benefited from an anonymous referee's comments. The machine M was introduced in a restricted form by Julian West in [27]. He considered an algorithm for this machine by which terms were obliged to be input into the machine as soon as possible: Thus if a ρ operation could be performed then it had to be, and it follows from the ordering of the stacks that if no ρ operation was possible then a λ operation had to be performed, and failing that a μ. It can be seen that this is not a universal algorithm for M by observing that 3241 is a permutation that cannot be sorted by that algorithm: We would be obliged to place the terms 3 and 2 immediately in the right stack which leads the algorithm to fail but if the obligation is ignored then it is possible to sort 3241 over M. We also note that if W is the set of permutations sortable by the algorithm then W is not a closed class: 3241 cannot be sorted but 35241 can be.

However M is closed and so we will find the basis of M and, as Zeilberger enumerated the number of permutations of each length in W, see [28], so we will enumerate the number of permutations of each length in M.

Theorem 211 *The basis of M is the infinite set*

$$B = \{(2, 2m-1, 4, 1, 6, 3, 8, 5, \ldots, 2m, 2m-3) \mid m = 2, 3, 4, \ldots\}.$$

Theorem 212 *Let z_n be the number of permutations of length n in M. Then*

$$\sum_{n=0}^{\infty} z_n x^n = \frac{32x}{-8x^2 + 20x + 1 - (1-8x)^{3/2}}$$

We will in fact prove these by imitating Julian West's algorithm, but reversing the order in which we favour moves. Where the algorithm for W is *right greedy*, having the order of preference for moves $\rho\ \lambda\ \mu$, we will consider the *left greedy* algorithm where the order of preference is $\mu\ \lambda\ \rho$. We will furthermore find that the left greedy algorithm does not impose a restriction at all: Every permutation in M can be sorted by the left greedy algorithm, and this is our first assistant proposition:

Proposition 213 *Every permutation in M can be sorted by the greedy algorithm.*

Corollary 214 *There is an algorithm which decides whether a permutation σ belongs to M, and which has linear time complexity in the length of σ.*

We will use this left greedy algorithm, that we will simply call the *greedy algorithm*, to prove Theorem 211. We will then represent every greedy algorithm that sorts an element of M by a word over ρ, λ and μ and by associating the latter to a labelled plane tree, enumerate M. In doing so we will obtain the following surprising result: The number of permutations of given length that lie in M is identical to the number of permutations of that length that avoid 1342. Indeed the plane trees that we use to enumerate M are the same as those used in [6] to enumerate $\mathcal{A}(1342)$. This is the first known case of two equinumerous closed classes that have different sizes of basis, and the difference is somewhat extreme.

We will first prove in Section 7.1 that the basis of M is given accurately by Theorem 211, providing that every element of M can indeed be sorted by the greedy algorithm. In Section 7.2 we describe sortings over two stacks in series as words and prove in passing that every permutation in M is greedily sortable. Then we bind the words to plane trees and perform our enumeration.

7.1 The basis of M

To prove Theorem 211 we first note that none of the permutations in B can be sorted by the greedy algorithm. Furthermore, we can readily check that, if any term of one of these permutations is deleted, the resulting sequence can be sorted. This proves that B is a subset of the basis of M.

To prove that B is the whole of the basis we shall consider an arbitrary basis permutation β of length n, examine how the greedy algorithm must fail when applied to β, and thereby identify enough properties of β to demonstrate that $\beta \in B$. We denote by $\beta \setminus \{i\}$ the sequence obtained by removing the term i from β.

Lemma 215 *Before the greedy algorithm applied to β fails, the term $n-1$ has been processed by a ρ-operation but not by a λ-operation.*

PROOF: If the greedy algorithm fails before attempting to apply a ρ-operation to the term n then it would also fail on $\beta \setminus \{n\}$ and this contradicts the minimality of β. On the other hand if it fails after a ρ-operation has been successfully applied to n then, from the fact that $\beta \setminus \{n\}$ can be sorted by the greedy algorithm, we easily see that β itself can be sorted. It follows that the greedy algorithm fails exactly at the point where it attempts to carry out a ρ-operation on n (failing because the right stack is non-empty).

We compare the action of the greedy algorithm on each of β and $\beta \setminus \{n-1\}$. It is clear that, up to the point of failure in β, these algorithms must be performing identically except that, for $\beta \setminus \{n-1\}$, all operations involving $n-1$ are absent. However, the greedy algorithm on $\beta \setminus \{n-1\}$ would not fail on the ρ-operation to insert n into the right stack (by minimality) and so it follows that, when applied to β, $n-1$ must be present in the right stack at the point of failure. In other words $n-1$ has been processed by a ρ-operation but not by a λ-operation. ∎

To complete the proof of Theorem 211 we shall construct a subsequence $a_{i_1} a_{i_2} \ldots$ of β order isomorphic to a permutation of B. By minimality this will be the whole of β. We shall label the subscripts i_j so that they suggest the relative values of the a_{i_j}.

Consider the point, guaranteed by Lemma 215, at which the greedy algorithm inserts $n-1$ into the right stack by a ρ-operation. The left stack is not empty (otherwise a λ-operation could be applied to $n-1$, contradicting

Lemma 215) and contains a largest element a_2. Thus

$$\beta = \ldots\ a_2\ \ \theta\ \ n-1\ \ \phi.$$

Now, it cannot be possible to empty the left stack by μ-operations (for that would permit a λ-operation on $n-1$) and so there must exist some a_1 within ϕ with $a_1 < a_2$ and we choose the rightmost such a_1. So we have

$$\beta = \ldots\ a_2\ \ \theta\ \ n-1\ \ \phi_1\ \ a_1\ \ldots$$

Within θ there are no terms larger than a_2. Indeed, since the right stack must be empty in order to insert $n-1$ into it, each term of θ must either be output or on the left stack; in either case it is smaller than a_2. However, within ϕ_1 there must be a term larger than a_2 (else when a_1 is processed by a ρ-operation the left stack and all of the right stack except for $n-1$ could be moved to the output and that would allow $n-1$ to move to the left stack). If ϕ_1 contains n then β contains the subsequence $a_2, n-1, n, a_1$ which is order isomorphic to $2341 \in B$ and we are finished. Otherwise the terms in ϕ_1 that are larger than a_2 cannot be a set of contiguous terms contiguous with a_2 (for the same reason as before, that they could all be output once a_1 was processed by a ρ-operation). Hence ϕ_1 contains some largest a_4 and there is a smaller a_3 to the right of a_1 also larger than a_2; we choose the rightmost such a_3. Now we have

$$\beta = \ldots a_2 \ldots n-1 \ldots a_4 \ldots\ a_1\ \phi_3\ a_3\ \ldots$$

Essentially we now repeat the argument of the last paragraph until we run out of terms. We do it explicitly once more for clarity. Within ϕ_3 there are some terms larger than a_4 (else when a_3 is processed by a ρ-operation the contents of both stacks, except for $n-1$, could be moved to the output and $n-1$ could move to the left stack). If ϕ_3 contains n, then β contains $a_2, n-1, a_4, a_1, n, a_3$ which is order isomorphic to $254163 \in B$. Otherwise, the set of terms in ϕ_3 that are larger than a_4 cannot be a contiguous set contiguous with a_4 (or again all the terms in both stacks, except for $n-1$, could be output). Hence ϕ_3 has a largest term a_6 and there is a rightmost smaller term a_5 greater than a_4 but smaller than a_6 and to the right of a_3. The situation now is

$$\beta = \ldots a_2 \ldots n-1 \ldots a_4 \ldots a_1 \ldots a_6 \ldots \; a_3 \; \phi_5 \; a_5 \; \ldots$$

In this way we define more and more terms of β:

$$a_2 \;\; n-1 \;\; a_4 \;\; a_1 \;\; a_6 \;\; a_3 \;\; \ldots \;\; a_{2k} \;\; a_{2k-3} \;\; \phi_{2k-1} \;\; a_{2k-1}$$

and we do this until ϕ_{2k-1} contains n, in which case we obtain a sequence order isomorphic to $\langle 2, 2k-1, 4, 1, 6, 3, \ldots, 2k, 2k-3 \rangle \in B$ as required. This completes the proof of Theorem 211.

7.2 Algorithms as Words

This section begins the proof of Theorem 212. In that theorem the generating function has constant term 1 corresponding to the empty permutation. However, for technical reasons, we shall from now on consider only non-empty permutations.

An algorithm for sorting a permutation of length n through two stacks in series is a sequence of appropriate stack operations, and so can be described as a word of length $3n$ over the alphabet ρ, λ, μ. We call these S^2-*words*. For a word W over $\{\rho, \lambda, \mu\}$ and $x \in \{\rho, \lambda, \mu\}$, we denote by $\sharp_x(W)$ the number of occurrences of x in W.

It is clear that a word W over $\{\rho, \lambda, \mu\}$ is an S^2-word if and only if it describes how to take a permutation through two stacks in series (without necessarily sorting it). Indeed, if W transforms a permutation $\sigma = (i_1, \ldots, i_n)$ into the permutation $\tau = (j_1, \ldots, j_n)$, then W sorts the permutation $\tau^{-1}\sigma$. From this it now easily follows that W is an S^2-word if and only if the following two conditions are satisfied:

(S1) $\sharp_\rho(W) = \sharp_\lambda(W) = \sharp_\mu(W)$;

(S2) for any initial subword (prefix) U of W we have $\sharp_\rho(U) \geq \sharp_\lambda(U) \geq \sharp_\mu(U)$.

It is also true that for every S^2-word there is a unique permutation σ which it sorts (σ can be found by applying the S^2-word in reverse so that it defines an algorithm for transforming an output sequence $1, 2, \ldots, n$, via the two stacks, to produce σ in the input). The converse, however, is not necessarily true: it may be possible to sort a given permutation in several different ways.

In what follows we will find it useful to label the letters of an S^2-word W as follows. If $\pi = (a_1, \ldots, a_n)$ is the permutation sorted by W, then we denote by ρ_i $(1 \leq i \leq n)$ the occurrence of ρ in W which corresponds to moving a_i from the input onto the right stack. Similarly, λ_i moves a_i from the right stack to the left stack, and μ_i outputs a_i from the left stack.

Those S^2-words which represent sortings of permutations whilst respecting the characteristic sorted stack property of M are called *M-words*; and those *M-words* that also represent greedy sortings are called *Greedy M-words*, or *GM-words* for short. We characterise M-words and GM-words in Propositions 218 and 219 but first we point out our reason for studying GM-words.

Lemma 216 *The number of GM-words of length $3n$ is equal to the number of permutations of length n in the set M.*

PROOF: There is a natural one-to-one correspondence between GM-words and permutations of M. Every permutation of M can be sorted by the greedy algorithm and so determines a GM-word. On the other hand, as already observed, each GM-word sorts a unique permutation which necessarily belongs to M. ■

Lemma 217 *Let W be an S^2-word, and let $\pi = (a_1, \ldots, a_n)$ be the permutation it sorts. Then W is not an M-word if and only if in applying W to π there is a pair of elements a_i and a_j that are adjacent in both stacks.*

PROOF: The 'if' part is obvious. For the 'only if' part let a_i, a_j $(i < j)$ be a pair that violates the stack ordering (necessarily on the right stack). Thus we have $a_i < a_j$, and at some stage a_j lies above a_i in the right stack, while at some later stage a_i lies above a_j in the left stack. In addition, assume that a_i and a_j are chosen so that the length of the subword $\lambda_j \ldots \lambda_i$ of W is minimal possible. We claim that a_i and a_j are actually adjacent in both stacks.

Assume first that a_i and a_j are not adjacent on the right stack, and let a_k be an entry which lies between them. Since $a_j > a_i$, we must have $a_j > a_k$ or $a_k > a_i$. In the former case, the pair a_k, a_j violates the stack ordering and the sequence $\lambda_j \ldots \lambda_k$ is a proper subword of $\lambda_j \ldots \lambda_i$, while in the latter case the pair a_i, a_k violates the stack ordering and $\lambda_k \ldots \lambda_i$ is a proper subword of $\lambda_j \ldots \lambda_i$. In both cases we obtain a contradiction with the choice of a_i and a_j, and so they must be adjacent in the right stack.

Assume now that a_i and a_j are not adjacent on the left stack, and let a_l be an entry which lies between them. In particular, we have $a_l > a_i$. Since a_i and a_j were adjacent in the right stack, a_l must have been moved onto the right stack after a_j had left it, but before a_i had done so. Therefore, the pair a_i, a_l violates the stack ordering, and $\lambda_l \ldots \lambda_i$ is a proper subword of $\lambda_j \ldots \lambda_i$, which is again in contradiction with the choice of a_i and a_j. ■

Proposition 218 *An S^2-word W is an M-word if and only if it contains no subword of the form $\lambda U \lambda$, where U is empty or an S^2-word.*

PROOF: Let $\pi = (a_1, \ldots, a_n)$ be the permutation sorted by W. If W contains a subword $\lambda_j \lambda_i$ then obviously a_i and a_j are adjacent in both stacks, and W is not an M-word by Lemma 217. If W contains a subword of the form $\lambda_j U \lambda_i$, where U is an S^2-word, then after a_j has been moved to the left stack, U transfers a collection of elements from the input, via the two stacks, into the output, and then a_i is moved onto the left stack. We see that again a_i and a_j are adjacent in both stacks, and so W is not an M-word.

Conversely, if W is not an M-word, then, by Lemma 217, there is a pair a_i, a_j, such that they are adjacent in both stacks. Consider the subword $\lambda_j U \lambda_i$ of W, and assume that U is non-empty. We see that, after a_j has been moved onto the left stack, no element already on either of the stacks must be moved before a_i is moved on top of a_j. Therefore, U must transfer a group of terms from the input, via the two stacks, to the output; in other words U must be an S^2-word. ■

Proposition 219 *An S^2-word W is a GM-word if and only if the following are satisfied:*

(GM1) *W does not contain a subword $\lambda\lambda$;*

(GM2) *W does not contain a subword $\rho\mu$;*

(GM3) *W does not contain a subword $U\lambda$, where U is an S^2-word.*

PROOF: ($\Rightarrow$) If W contains a subword $\lambda\lambda$ then it is not an M-word by Proposition 218. If W contains a subword $\rho\mu$, say $W = V\rho\mu\ldots$, W is not greedy, because a μ can follow V. Similarly, if $W = VU\lambda\ldots$, where U is an S^2-word, then W is not greedy, because a λ can follow V, while the first letter of U is ρ.

($\Leftarrow$) Assume now that W is not a GM-word. If W is not even an M-word then, by Proposition 218, it either contains a subword $\lambda\lambda$, or a subword $\lambda U\lambda$, where U is an S^2-word, and the proof is finished. So, let us now consider the case where W is an M-word, but is not greedy. Let π be the permutation sorted by M, and let V be the shortest initial segment of W after which the greedy algorithm condition fails. Thus, if we write $W = VxV_1$, where $x \in \{\rho, \lambda, \mu\}$, there exists another M-word sorting π of the form VyV_2, where x precedes y in the list ρ, λ, μ. So we can distinguish the following three cases.

Case 1: $x = \rho$, $y = \lambda$. Let a_j be the top element in the right stack after V has been applied to π, and write $W = V\rho V_3\lambda_j V_4$. Now note that ρV_3 must not move any of the elements which are already on either of the stacks. Indeed, the elements on the right stack cannot be moved before a_j (because a_j is on the top of the stack), and the elements on the left stack cannot be output before a_j (because a_j is smaller than any of them). Also, since any element input by ρV_3 is smaller than a_j, it must also be output by ρV_3 (i.e. before λ_j). Therefore, ρV_3 is an S^2-subword of W preceding a λ.

Case 2: $x = \rho$, $y = \mu$. Let a_j be the top element of the left stack after V has been applied to π, and write $W = V\rho V_3 \mu_j V_4$. Notice that a_j is the least element that has not yet been output. Therefore, V_3 cannot contain any occurrences of either λ or μ, and hence W contains a subword $\rho\mu$.

Case 3: $x = \lambda$, $y = \mu$. This case cannot occur, for if a_j is the top element on the left stack after V has been applied to π, then again it is the least element that has not yet been output, and so applying a λ move would violate the left stack ordering.

This completes the proof of the proposition. ■

We can now prove en passant:

Proposition 220 *Every permutation in M can be sorted by the greedy algorithm.*

PROOF: Let W be an M word but not a GM word. We will construct a GM word that sorts the same sequence as W as follows:

By Proposition 219 we have that W must contain a subword of the form $\rho\mu$ or $U\lambda$ where U is a non-empty S^2 word. By Proposition 218 we can discard the possibility that W contains $\lambda\lambda$ as a subword.

Suppose first that W has a subword of the form $\rho\mu$ and that we may therefore express W as $V_1\rho\mu V_2$ where V_1 and V_2 are words over ρ, λ, μ. Note that the effect of performing a ρ immediately followed by a μ at any point in a sorting is exactly the same as that of performing a μ followed by a ρ instead at that same point. The one violates stack ordering if and only if the other does. Thus $V_1\mu\rho V_2$ also is an M word and this word sorts the same sequence as W. Now let us attach to each letter in W the position of the letter, attaching the number 1 to the first letter, 2 to the second and so on. If we sum the numbers attached to μ terms in W and perform a similar calculation for $V_1\mu\rho V_2$ then the number obtained for the latter sequence will be one less than for the former. Thus we may a finite number of times substitute a subword of the form $\rho\mu$ in W for a subword $\mu\rho$ and obtain a new word W' that contains no subwords $\rho\mu$, that is an M word and that sorts the same sequence as W.

Now suppose that W' has a subword of the form $V_1U\lambda V_2$ where V_1 and V_2 are words over ρ, λ, μ. Note that since U merely represents a set of small terms moving from the input directly to the output without disturbing either stack, we have that $V_1\lambda UV_2$ is an M word that sorts the same sequence as W'. Note also that since the first symbol of U is a ρ and the last a μ we have that $V_1\lambda UV_2$ contains no subword $\rho\mu$. If we attach numbers to the symbols of W' and $V_1\lambda UV_2$ as before then the μ sum of $V_1\lambda UV_2$ will be at least one less than that of W' because U contains at least one μ. Thus we can perform a finite number of substitutions of subwords of the form λU for equivalent subwords $U\lambda$ and obtan a word W'' that is a GM-word and that sorts the same sequence as W. ∎

7.3 Algorithms and Plane Trees

A GM-word W is *reducible* if $W = W_1W_2$, where both W_1 and W_2 are GM-words, and is *irreducible* (or IGM for short) otherwise. Reducibility of

GM-words corresponds precisely to sum decomposability of the permutation whose sorting by the greedy algorithm the GM-word represents: If a permutation is sum decomposable and greedily sortable then the GM-word that represents its sorting will be reducible, and a reducible GM-word represents the greedy sorting of some decomposable permutation.

In this section, we are going to show how to associate a rooted plane tree with labelled edges to every IGM-word, and then we are going to establish a recurrence formula for the number of IGM-words corresponding to a rooted plane tree without labels.

Lemma 221 *The number of IGM-words of length $3n$ is equal to the number of sum indecomposable permutations of length n in the set M.*

PROOF: Restrict the one-to-one correspondence given in the proof of Lemma 216 to IGM-words. ■

Let W be an IGM-word. Since it represents a greedy algorithm, W must begin with $\rho\lambda$, and it must end with μ; in other words W can be written as $W = \rho\lambda W'\mu$.

We define the *derived* word $\partial(W)$ of a GM-word W to be the word obtained from W by removing the λ symbols. The properties of $\partial(W)$ inherited from conditions (S1) and (S2) are those of well-balanced strings of parentheses and allow a well-known description by a plane tree. In this description $\partial(W)$ is obtained by walking around the tree beginning at the root traversing each edge twice, first downwards for a ρ symbol (opening parenthesis) and later upwards for the corresponding μ symbol (closing parenthesis). We shall use this only in the case of an IGM-word. For such words the tree corresponding to $\partial(W)$ has root of degree 1 and it is convenient to remove the root and its incident edge. The resulting rooted plane tree will be denoted by $T(W)$. By construction, if W has length $3n$, then $T(W)$ has n vertices.

Although $T(W)$ uniquely determines $\partial(W)$ it certainly does not determine W itself. To capture the more detailed information present in W we attach labels to the edges of $T(W)$.

Each edge of $T(W)$ corresponds, as described above, to a ρ-μ pair of W. To each such edge e we attach a label from the set

$$\mathcal{L} = \{(\rho, \mu), (\rho, \mu\lambda), (\rho\lambda, \mu), (\rho\lambda, \mu\lambda)\}$$

depending on whether ρ and μ corresponding to e are followed by λ in W. Trees arising in this way are called *IGM-trees*. We also call the unlabelled tree $T(W)$ the *shape* of the IGM-word W.

Example 222 The IGM-word

$$\rho\lambda\ \rho\lambda\rho\lambda\rho\lambda\mu\mu\rho\rho\lambda\mu\mu\mu\lambda\rho\rho\lambda\mu\mu\lambda\ \mu$$

gives rise, after derivation, to the IGM-tree shown in Figure 2.

The IGM-word W can be reconstructed from its IGM-tree $T(W)$ in the intuitively obvious way, which can be formalised as follows. Let T be any plane tree with edges labelled by elements of the set $\mathcal{L}$. To each vertex V of T we associate two words $\sigma(V)$ and $\tau(V)$ defined recursively. If V is a leaf, and if the edge leading to it is labelled by (x, y) then

$$\sigma(V) = x\mu,\ \tau(V) = xy.$$

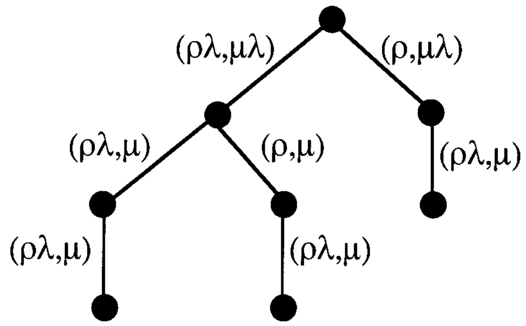


Figure 7.1: An IGM-tree

(Note that if T is an IGM-tree then $\sigma(V) = \tau(V) = \rho\lambda\mu$, because W does not contain a subword $\rho\mu$ or $\rho\lambda\mu\lambda$ by Proposition 219.) If V is neither a leaf nor the root, with children $C_1, \ldots, C_p$, and with the edge from V to its parent labelled by (x, y), then define

$$\sigma(V) = x\tau(C_1)\ldots\tau(C_p)\mu,$$
$$\tau(V) = x\tau(C_1)\ldots\tau(C_p)y.$$

Note that either $\sigma(V) = \tau(V)$ or else $\tau(V) = \sigma(V)\lambda$. Finally, if V is the root, and if its children are $C_1, \ldots, C_p$, then define

$$\sigma(V) = \tau(V) = \rho\lambda\tau(C_1)\ldots\tau(C_p)\mu.$$

Clearly, if $T = T(W)$ is an IGM-tree, and if R is its root, then $\sigma(R) = \tau(R) = W$.

We now give some properties of the words $\sigma(V)$ and $\tau(V)$.

Lemma 223 *Let $T = T(W)$ be an IGM-tree, and let V be any vertex of T.*

(i) *For every initial segment Z of $\sigma(V)$ we have $\sharp_\rho(Z) \geq \sharp_\lambda(Z)$.*

(ii) *For every terminal segment* U *of* $\sigma(V)$ *we have* $\sharp_\mu(U) \geq \sharp_\lambda(U)$.

(iii) *If* $\sharp_\rho(\sigma(V)) = \sharp_\lambda(\sigma(V))$ *then* $\sigma(V)$ *is an* S^2*-word, and hence* $\sigma(V) = \tau(V)$.

(iv) $\sharp_\rho(\tau(V)) \geq \sharp_\lambda(\tau(V))$.

PROOF: If V is a leaf then all the statements hold. Consider now the case where V is not a leaf, and assume inductively that all the statements (i)–(iv) hold for all of its children $C_1, \ldots, C_p$. By the definition we have

$$\sigma(V) = x\tau(C_1)\ldots\tau(C_p)\mu,$$
$$\tau(V) = x\tau(C_1)\ldots\tau(C_p)y.$$

(i) Let Z be an initial segment of $\sigma(V)$. If the length of Z is 1 or 2 the statement is obvious. Otherwise, if Z is a proper initial segment of length greater than 2, we can write

$$Z = x\tau(C_1)\ldots\tau(C_k)Z',$$

where Z' is either empty or else it is a proper prefix of $\tau(C_{k+1})$ (and hence a prefix of $\sigma(C_{k+1})$). We have

$$\sharp_\rho(Z) = 1 + \sum_{i=1}^{k} \sharp_\rho(\tau(C_i)) + \sharp_\rho(Z'),$$
$$\sharp_\lambda(Z) = \sharp_\lambda(x) + \sum_{i=1}^{k} \sharp_\lambda(\tau(C_i)) + \sharp_\lambda(Z').$$

Now note that $1 \geq \sharp_\lambda(x)$, as x is either ρ or $\rho\lambda$. Next note that, by induction, we have $\sharp_\rho(\tau(C_i)) \geq \sharp_\lambda(\tau(C_i))$ (property (iv)) and $\sharp_\rho(Z') \geq \sharp_\lambda(Z')$ (property(i)). We conclude that $\sharp_\rho(Z) \geq \sharp_\lambda(Z)$ in this case. Finally, the case where $Z = \sigma(V)$ follows also by noting that the last μ of $\sigma(V)$ does not contribute anything to either of $\sharp_\rho(\tau(Z))$ or $\sharp_\lambda(\tau(Z))$.

(ii) Let U be a terminal segment of $\sigma(V)$. If U has length 1, the statement is obvious. Otherwise, if U is a proper terminal segment of $\sigma(V)$ of length greater than 1 we can write

$$U = U'\tau(C_k)\ldots\tau(C_p)\mu,$$

where either (1) $U' = \lambda$, or (2) $\sigma(C_{k-1}) = \tau(C_{k-1})$ and U' is a terminal segment of $\sigma(C_{k-1})$, or (3) $U' = U''\lambda$ and U'' is a terminal segment of $\sigma(C_{k-1})$. We now have that

$$\sharp_\mu(U) = \sharp_\mu(U') + \sum_{i=k}^{p} \sharp_\mu(\tau(C_i)) + 1, \tag{7.1}$$

$$\sharp_\lambda(U) = \sharp_\lambda(U') + \sum_{i=k}^{p} \sharp_\lambda(\tau(C_i)). \tag{7.2}$$

By induction (property (iv)) we have

$$\sharp_\mu(\tau(C_i)) = \sharp_\rho(\tau(C_i)) \geq \sharp_\lambda(\tau(C_i)) \ (i = k, \ldots, p). \tag{7.3}$$

Also, in each of the three possibilities for U' we have

$$\sharp_\mu(U') + 1 \geq \sharp_\lambda(U'). \tag{7.4}$$

Indeed in the case (1) this is obvious, while in the cases (2) and (3) it follows from the inductive hypothesis (property (ii)). Combining (7.1)–(7.4) we conclude that $\sharp_\mu(U) \geq \sharp_\lambda(U)$, as required.

(iii) We are going to show that conditions (S1) and (S2) are satisfied for $\sigma(V)$. Indeed, (S1) is satisfied by assumption. Also, if Z is an initial segment of $\sigma(V)$ then $\sharp_\rho(Z) \geq \sharp_\lambda(Z)$ by (i). Write now $\sigma(V)$ as $\sigma(V) = ZU$. By (ii) we have $\sharp_\mu(U) \geq \sharp_\lambda(U)$, and hence we have

$$\sharp_\lambda(Z) = \sharp_\lambda(\sigma(V)) - \sharp_\lambda(U) \geq \sharp_\mu(\sigma(V)) - \sharp_\mu(U) = \sharp_\mu(Z),$$

thus proving (S2) as well. The final statement follows from the assumption that W is an IGM-word and condition (GM3) for such words.

(iv) If $\tau(V) = \sigma(V)$ this follows from (i). Otherwise we have $\tau(V) = \sigma(V)\lambda$. We know that $\sharp_\rho(\sigma(V)) \geq \sharp_\lambda(\sigma(V))$. In fact, we must have $\sharp_\rho(\sigma(V)) > \sharp_\lambda(\sigma(V))$ by (iii). Since $\sharp_\lambda(\tau(V)) = \sharp_\lambda(\sigma(V)) + 1$, the statement follows. ■

Definition 224 Let T be a plane tree with edges labelled by elements of the set $\mathcal{L}$, and let V be a vertex of T. The λ-*deficit* at V is the number

$$d(V) = \sharp_\rho(\tau(V)) - \sharp_\lambda(\tau(V)).$$

In the next result we give a characterisation of IGM-trees.

Proposition 225 *A plane tree T with edges labelled by elements of the set $\mathcal{L}$ is an IGM-tree if and only if the following conditions are satisfied:*

(T1) *every leaf edge is labelled by $(\rho\lambda, \mu)$;*

(T2) $d(V) \geq 0$ *for every vertex V;*

(T3) $d(R) = 0$, *where R is the root.*

PROOF: ($\Rightarrow$) If $T = T(W)$ is an IGM-tree, then (T1) follows from (GM2) and (GM3), (T2) is Lemma 223 (iv), and (T3) follows from (S1).

($\Leftarrow$) Assume now that T satisfies (T1)–(T3). We are going to check that the word $W = \sigma(R)$, where R is the root of T, satisfies all conditions (S1), (S2), (GM1)–(GM3), and that it is irreducible. Indeed, (S1) follows from (T3), (GM1) follows from the definition of the words $\sigma(V)$, (GM2) follows from (T1), and (GM3) follows from (T2).

Next we prove that for every initial segment Z of W we have $\sharp_\rho(Z) \geq \sharp_\lambda(Z)$, (the first inequality in (S2)). We do this by induction on the length of Z. If Z has length 1, or, more generally, if Z contains no occurrence of μ, the assertion is obvious. Otherwise Z has one of the forms $Z_1\sigma(V)$ or $Z_1\tau(V)Z_2$, where V is a vertex and Z_2 contains no occurrences of μ. (This is obtained by 'reading' Z until its last μ, and then finding the corresponding ρ in front of it.) Then we have

$$\sharp_\rho(Z_1) \geq \sharp_\lambda(Z_1)$$

by induction,

$$\sharp_\rho(\sigma(V)) \geq \sharp_\lambda(\sigma(V)), \ \sharp_\rho(\tau(V)) \geq \sharp_\lambda(\tau(V))$$

by (T2), and

$$\sharp_\rho(Z_2) \geq \sharp_\lambda(Z_2)$$

since Z_2 contains no μ. Combining the above inequalities as appropriate we conclude that $\sharp_\rho(Z) \geq \sharp_\lambda(Z)$.

A similar induction shows that $\sharp_\mu(U) \geq \sharp_\lambda(U)$ for any terminal segment U of W. This, together with (T3) implies that $\sharp_\lambda(Z) \geq \sharp_\mu(Z)$ for any initial segment Z of W, thus completing the proof of property (S2). Finally, again by induction, one easily proves that $\sharp_\rho(Z) > \sharp_\mu(Z)$ for a proper initial segment Z of W, and this implies that W is irreducible. For, if $W = W_1W_2$, where W_1 and W_2 are GM-words, then W_1 is a proper initial segment of W with $\sharp_\rho(W_1) = \sharp_\mu(W_1)$. ■

Definition 226 Let T be a rooted plane tree, and let V be a non-root vertex of T. We define the *branch* of V to be the tree $H(V)$ obtained by taking the subtree of T rooted in V and adding to it the parent P of V and the edge linking P and V.

Definition 227 Let T be a rooted plane tree. To each non-root vertex V we associate a sequence

$$\Delta(V) = (\delta_0, \delta_1, \delta_2, \ldots),$$

where δ_r is the number of different labellings of edges of $H(V)$ by elements of the set $\mathcal{L}$ which satisfy conditions (T1) and (T2) (but not necessarily (T3)) and for which $d(V) = r$.

Proposition 228 *Let T be a rooted plane tree, let R be its root, let $C_1, \ldots, C_p$ be the children of R, and let*

$$\Delta(C_i) = (\delta_{i0}, \delta_{i1}, \delta_{i2}, \ldots) \ (i = 1, \ldots, p).$$

The number of IGM-trees with shape T is equal to $\delta_{10}\delta_{20} \ldots \delta_{p0}$.

PROOF: For a given labelling of edges of T satisfying (T1) and (T2) from Proposition 225, we have that $\sigma(R)$ is an IGM-word if and only if (T3) is satisfied, i.e. if and only if $d(R) = \sharp_\rho(\tau(R)) - \sharp_\lambda(\tau(R)) = 0$. Since

$$\tau(R) = \rho\lambda\tau(C_1) \ldots \tau(C_p)\mu,$$

and since $d(C_i) \geq 0$ $(i = 1, \ldots, p)$, we have that $d(R) = 0$ if and only if $d(C_i) = 0$ for all $i = 1, \ldots, p$. The number of different labellings of any $H(C_i)$ satisfying $d(C_i) = 0$ is precisely δ_{i0} and the result follows. ∎

We now give a recurrence for the sequences $\Delta(V)$.

Proposition 229 *Let T be a rooted plane tree, let V be a non-root vertex of T, and let*

$$\Delta(V) = (\delta_0, \delta_1, \delta_2, \ldots).$$

If V is a leaf then

$$\delta_0 = 1,\ \delta_i = 0\ (i > 0).$$

Otherwise, if $C_1, \ldots, C_p$ are the children of V with

$$\Delta(C_i) = (\delta_{i0}, \delta_{i1}, \delta_{i2}, \ldots),$$

then

$$\begin{aligned} \delta_r &= \sum_{j_1+\ldots+j_p=r-1} \delta_{1j_1}\delta_{2j_2}\ldots\delta_{pj_p} + 2\left(\sum_{j_1+\ldots+j_p=r} \delta_{1j_1}\delta_{2j_2}\ldots\delta_{pj_p}\right) \\ &\quad + \sum_{j_1+\ldots+j_p=r+1} \delta_{1j_1}\delta_{2j_2}\ldots\delta_{pj_p}. \end{aligned}$$

PROOF: Note that

$$d(V) = \sharp_\rho(xy) - \sharp_\lambda(xy) + \sum_{i=1}^{p} d(C_i),$$

where (x, y) is the label of the edge connecting V to its parent. Clearly

$$\sharp_\rho(xy) = 1,\ \sharp_\lambda(xy) = \begin{cases} 0 & \text{if } x = \rho,\ y = \mu, \\ 1 & \text{if } x = \rho\lambda,\ y = \mu \text{ or } x = \rho,\ y = \mu\lambda, \\ 2 & \text{if } x = \rho\lambda,\ y = \mu\lambda. \end{cases}$$

Hence, to be able to label $H(V)$ so that $d(V) = r$, the trees $H(C_i)$ ($i = 1, \ldots, p$) must be labelled so that

$$\sum_{i=1}^{p} d(C_i) \in \{r-1, r, r+1\}.$$

A labelling of $H(C_i)$ $(i = 1, \ldots, p)$ with $\sum_{i=1}^{p} d(C_i) = r \pm 1$ can be extended in a unique way to a labelling of $H(V)$ with $d(V) = r$ by setting $x = \rho\lambda$, $y = \mu\lambda$ or $x = \rho$, $y = \mu$ respectively. Similarly, a labelling of $H(C_i)$ $(i = 1, \ldots, p)$ with $\sum_{i=1}^{p} d(C_i) = r$ can be extended in two ways to a labelling of $H(V)$ with $d(V) = r$ by setting $x = \rho\lambda$, $y = \mu$ and $x = \rho$, $y = \mu\lambda$. Finally note that the number of labellings of $H(C_i)$ $(i = 1, \ldots, p)$ with $\sum_{i=1}^{p} d(C_i) = k \in \{r-1, r+1, r\}$ is precisely

$$\sum_{j_1+\ldots+j_p=k} \delta_{1j_1}\delta_{2j_2}\ldots\delta_{pj_p},$$

proving the formula. We remark that the argument remains valid for $r = 0$, when the term

$$\sum_{j_1+\ldots+j_p=-1} \delta_{1j_1}\delta_{2j_2}\ldots\delta_{pj_p}$$

is zero, reflecting the fact that the λ-deficit of every C_i is non-negative (Proposition 225 (T2)). ■

Remark 230 Although $\Delta(V)$ is an infinite sequence, only finitely many of its entries are non-zero. In other words the generating function $\Delta(V, x) = \sum \delta_i x^i$ of $\Delta(V)$ is a polynomial. From the recurrence of Proposition 229 we easily derive the polynomial equations:

$$\Delta(V, x) = 1 \tag{7.5}$$

if V is a leaf, and

$$\Delta(V, x) = \frac{1}{x}((1+x)^2 \prod_{i=1}^{p} \Delta(C_i, x) - \prod_{i=1}^{p} \Delta(C_i, 0)) \tag{7.6}$$

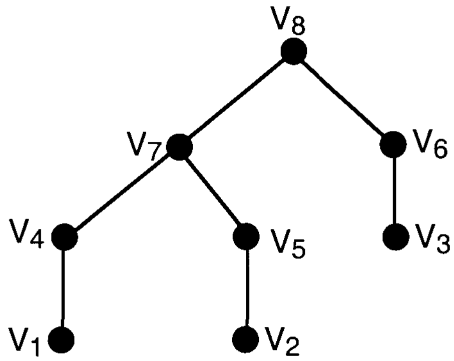


Figure 7.2:

if V is a non-leaf, non root vertex with children $C_1, \ldots, C_p$. By Proposition 228 the number of IGM-trees of shape T is

$$\prod_{i=1}^{p} \Delta(C_i, 0) \tag{7.7}$$

where $C_1, \ldots, C_p$ are the children of the root.

Example 231 Consider the plane tree of Figure 3. For each vertex V_i ($i = 1, \ldots, 7$) we calculate the corresponding polynomial $\Delta(V_i, x)$. First

$$\Delta(V_1, x) = \Delta(V_2, x) = \Delta(V_3, x) = 1,$$

because they are the leaves. Now, V_4 has one child, V_1. Applying (7.6) we obtain

$$\Delta(V_4, x) = \frac{1}{x}((1+x)^2 - 1) = 2 + x.$$

Similarly,

$$\Delta(V_5, x) = \Delta(V_6, x) = 2 + x.$$

The vertex V_7 has two children, V_4 and V_5, so

$$\Delta(V_7, x) = \frac{1}{x}((2+x)^2(1+x)^2 - 4) = 12 + 13x + 6x^2 + x^3.$$

By Proposition 228 we can conclude that there are $\Delta(V_6, 0) \cdot \Delta(V_7, 0) = 2 \cdot 12 = 24$ IGM-trees with this given shape.

7.4 IGM-Trees and $\beta(0,1)$-Trees: enumeration

In this section we introduce the concept of $\beta(0,1)$-trees, as rooted plane trees with labelled vertices. Then we establish a recurrence formula, giving the number of $\beta(0,1)$-trees over a given rooted plane tree (with no labels), and establish connections between this recurrence relation and the one from the last section. From here it then follows that the numbers of IGM-trees and $\beta(0,1)$-trees over a given rooted plane tree are equal. Finally, we use this fact to give a proof of Theorem 212.

Definition 232 A $\beta(0,1)$-tree is a rooted plane tree with non-negative integer labels $l(V)$ on its vertices, satisfying the following conditions:

(B1) if V is a leaf then $l(V) = 0$;

(B2) if V is an internal vertex, and if $C_1, \ldots, C_p$ are its children then $l(V) \leq l(C_1) + l(C_2) + \ldots + l(C_p) + 1$;

(B3) if V is the root then $l(V) = 0$.

We note that this differs from the definition of a $\beta(0,1)$-tree given in [6] (and the more general definition in [19]) where one requires that $l(V) = l(C_1) + l(C_2) + \ldots + l(C_p)$ when V is the root with children $C_1, \ldots, C_p$.

However, this difference will not affect the number of $\beta(0,1)$-trees, as both give no freedom of choice for the label of the root.

The following result was given in [6]. Proofs may be found in [21] and [22].

Proposition 233 *The number t_n of $\beta(0,1)$-trees on n vertices is equal to the number of rooted bicubic maps on $n-1$ vertices.*

From the enumeration of rooted bicubic maps given in [26] (see also [8] for a combinatorial proof) we have $t_1 = 1$ and, for $n > 1$, $t_n = 3 \cdot 2^{n-2} \cdot (2n-2)!/(n+1)!(n-1)!$.

Definition 234 Let T be a rooted plane tree. To each vertex V we associate a sequence

$$B(V) = (\beta_0, \beta_1, \beta_2, \ldots),$$

where β_r is the number of different labellings of the subtree of T rooted in V which satisfy conditions (B1) and (B2) of Definition 232 (but not necessarily condition (B3)), and in which V is labelled by r.

Remark 235 If R is the root of T, and if $B(R) = (\beta_0, \beta_1, \beta_2, \ldots)$, then the number of $\beta(0,1)$-trees with shape T is equal to β_0.

Remark 236 As with $\Delta(V)$, we see that $B(V)$ is an infinite sequence with only finitely many non-zero entries.

In the following proposition we give a recurrence for computing the sequences $B(V)$ in an arbitrary rooted plane tree.

Proposition 237 *Let T be a rooted plane tree, let V be any vertex in it, and let $B(V) = (\beta_0, \beta_1, \beta_2, \ldots)$. If V is a leaf, then*

$$\beta_0 = 1, \ \beta_i = 0 \ (i \geq 1).$$

If V is not a leaf, and if $C_1, \ldots, C_p$ are its children, with

$$B(C_i) = (\beta_{i0}, \beta_{i1}, \beta_{i2}, \ldots),$$

then

$$\beta_r = \sum_{j_1+\ldots+j_p \geq r-1} \beta_{1j_1}\beta_{2j_2}\ldots\beta_{pj_p}.$$

PROOF: A leaf must be labelled by 0, hence the first assertion. For the second, note that one is allowed to label V by r if and only if the sum of its children's labels is at least $r-1$. ■

Remark 238 The recurrence of Proposition 237 also gives relations between the polynomials $B(V,x) = \sum \beta_i x^i$. Indeed we have:

$$B(V,x) = 1 \tag{7.8}$$

if V is a leaf, and

$$B(V,x) = \frac{1}{x-1}(x^2 \prod_{i=1}^{p} B(C_i,x) - \prod_{i=1}^{p} B(C_i,1)) \tag{7.9}$$

if V is a non-leaf vertex with children $C_1, \ldots, C_p$. An easy consequence of (7.9) is

$$\prod_{i=1}^{p} B(C_i,1) = B(V,0). \tag{7.10}$$

Also, from the definition $B(V,0) = \beta_0$. So if R is the root node and $C_1, \ldots, C_p$ are the children of the root then the number of $\beta(0,1)$-trees of shape T is

$$B(R,0) = \prod_{i=1}^{p} B(C_i, 1). \tag{7.11}$$

Example 239 Consider the plane tree shown in Figure 3. For each vertex V_i we calculate the corresponding polynomial $B(V_i, x)$. First

$$B(V_1, x) = B(V_2, x) = B(V_3, x) = 1.$$

because they are the leaves. The vertex V_4 has but one child, V_1, giving:

$$B(V_4, x) = \frac{x^2 - 1}{x - 1} = 1 + x.$$

Similarly,

$$B(V_5, x) = B(V_6, x) = 1 + x.$$

Now, V_7 has two children, V_4 and V_5. From their polynomials we obtain:

$$B(V_7, x) = \frac{(1+x)^2 x^2 - 4}{x - 1} = 4 + 4x + 3x^2 + x^3.$$

Finally, a similar calculation for V_8 involving its children V_6 and V_7, gives:

$$B(V_8, x) = \frac{(4 + 4x + 3x^2 + x^3)(1 + x)x^2 - 24}{x - 1} = 24+24x+20x^2+12x^3+5x^4+x^5.$$

We conclude that there are $B(V_8, 0) = 24$ $\beta(0,1)$-trees with this given shape.

If we compare the equations of Remarks 230 and 238 we obtain:

Proposition 240 *Let T be a plane tree and let V be a non root vertex of T. Then the following polynomial equality holds:*

$$\Delta(V, x - 1) = B(V, x)$$

PROOF: We prove the proposition by induction. If V is a leaf then

$$\Delta(V, x-1) = 1 = B(V, x)$$

Otherwise, if V has children $C_1, \ldots, C_p$ and if we assume that the proposition holds for $C_1, \ldots, C_p$, then

$$\begin{aligned}
\Delta(V, x-1) &= \frac{1}{x-1}(x^2 \prod_{i=1}^{p} \Delta(C_i, x-1) - \prod_{i=1}^{p} \Delta(C_i, 0)) && \text{(by (7.6))} \\
&= \frac{1}{x-1}(x^2 \prod_{i=1}^{p} B(C_i, x) - \prod_{i=1}^{p} B(C_i, 1)) && \text{(induction)} \\
&= B(V, x) && \text{(by (7.9))}
\end{aligned}$$

as required, thus completing the proof.

■

Theorem 241 *Let T be a rooted plane tree. The number of IGM-trees with shape T is equal to the number of $\beta(0,1)$-trees with shape T.*

PROOF: Let R be the root of T and let $C_1, \ldots, C_p$ be its children. Then the number of IGM-trees with shape T is $\prod_{i=1}^{p} \Delta(C_i, 0)$, by Remark 230. On the other hand, by Remark 238, the number of $\beta(0,1)$-trees with shape T is $\prod_{i=1}^{p} B(C_i, 1)$. By Proposition 240 these are equal. ■

PROOF OF THEOREM 212: First we observe that the number of IGM-words of length $3n$ is equal to the number, t_n, of $\beta(0,1)$-trees on n vertices. This follows from Theorem 241 by summation over all tree shapes, and the fact that IGM-words and IGM-trees are in one-to-one correspondence. By Lemma 221 this number is also equal to the number of indecomposable permutations of length n in M; so this number is t_n.

We complete the proof by following an argument similar to that used in [6]. Every permutation σ of M has a unique factorisation (as a word) $\sigma = \tau_1\tau_2\ldots\tau_m$ with $a < b$ whenever $a \in \tau_i, b \in \tau_{i+1}$. The subwords τ_i are order isomorphic to indecomposable permutations of M. Conversely, every sequence of indecomposable permutations of M determines a permutation of M in this way. It follows from this that the generating function for the numbers z_n of permutations of length n in the set M, including the empty permutation, is

$$\sum_{k=0}^{\infty} F(x)^k = \frac{1}{1 - F(x)}$$

where $F(x)$ is the generating function for the (non-empty) indecomposable permutations of M. However, Tutte [26] has proved that

$$F(x) = \sum_{n=1}^{\infty} t_n x^n = \frac{8x^2 + 12x - 1 + (1 - 8x)^{3/2}}{32x}$$

and our theorem now follows. ∎

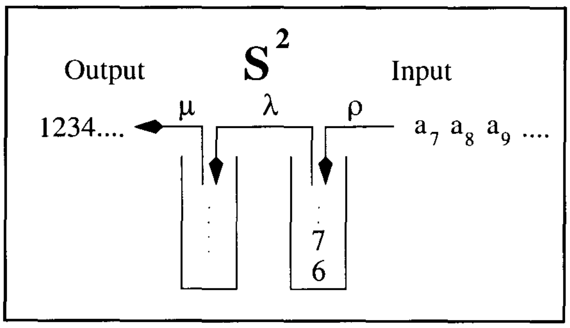
Chapter 8

A Harder Example: S^2

8.1 Introduction

By now the sorting machine S^2, which first appeared in Chapter 1, needs no introduction. It is the free form of the machine M that consists of two stacks in series, unordered instead of ordered.

Neither do the moves ρ, λ and μ that move terms through S^2 need another introduction, and S^2 words are fully defined and characterised in the last chapter. S^2 words are sequences of moves that sort an input through the stacks.



What we do need is an overview of the current problems concerning the sorting machine S^2 and the closed class sorted by this machine, also denoted S^2. We give an overview of the basis, enumeration problems and any interesting decision problems related to this set.

We show that the basis is certainly infinite. The basis is large: It has 219 elements of length nine or less. It also appears unruly and has not been characterised. The table in Figure 8.1 lists some of the first basis elements, and a full list of all basis elements of length up to nine is extant.

The number of permutations of each length in S^2 forms a list. Any conjecture for the number of entries in that list does require some direct computation to obtain raw data. As the shortest length at which not all permutations lie in the class is seven enumerative results will be needed at least for length seven and preferably a reasonable number of greater lengths. The sheer number of permutations that need to be checked for such a computation is tremendous and no serious attempt has been made as yet. For what it is worth the number of permutations of each length in S^2 is, for the known cases, given in Figure 8.2.

There is another reason why enumerative statistics are lacking: Given an arbitrary single permutation there are many different ways in which the permutation might be sorted through S^2, so not only must many permutations be checked, the calculation per permutation can take significant time. It is this problem that principally we address in this chapter.

To expand a little: The decision problem for whether a given permutation is in S^2 can be expressed as a 3SAT graph theoretic problem, as explained later. The reader may be reminded that the general 3SAT problem is NP-

Short S^2 basis elements

length 7 (all 22)	length 8 (all 51) and length 9 (15 of 146)		
2 4 3 5 7 6 1	6 3 2 8 5 4 7 1	4 7 2 5 8 1 6 3	2 6 8 4 3 5 7 1
2 5 3 6 4 7 1	5 3 2 8 6 4 7 1	4 6 2 5 8 1 7 3	2 5 7 4 6 1 8 3
2 5 3 7 4 6 1	6 2 4 3 8 5 7 1	4 6 2 8 5 1 7 3	2 5 7 3 6 1 8 4
2 4 6 3 5 7 1	5 2 4 8 3 7 6 1	3 7 2 8 4 6 5 1	2 5 8 4 6 3 7 1
2 5 4 7 3 6 1	6 2 4 7 5 3 8 1	3 5 2 6 8 7 4 1	3 5 7 6 2 4 8 1
2 6 4 7 3 5 1	5 2 4 8 6 3 7 1	5 7 6 3 2 4 8 1	2 4 7 6 8 3 5 1
3 2 4 6 5 7 1	5 3 6 2 8 7 4 1	3 7 4 2 6 5 8 1	2 4 6 5 7 3 8 1
3 2 5 4 7 6 1	4 3 6 2 8 7 5 1	3 7 4 2 6 8 5 1	7 4 2 5 8 1 6 9 3
3 2 6 4 7 5 1	5 2 8 4 3 7 6 1	4 6 5 2 8 3 7 1	7 3 2 5 8 1 6 9 4
3 2 5 7 4 6 1	6 2 7 3 5 4 8 1	4 6 5 2 7 3 8 1	6 4 2 5 8 1 7 9 3
4 2 5 3 7 6 1	5 2 7 4 6 1 8 3	2 7 4 3 6 5 8 1	6 3 2 5 8 1 7 9 4
4 2 6 3 5 7 1	4 2 8 3 6 5 7 1	2 7 5 3 6 8 4 1	7 4 2 5 8 6 1 9 3
4 2 6 3 7 5 1	5 2 7 4 8 1 6 3	2 7 4 3 6 8 5 1	7 3 2 5 8 6 1 9 4
5 2 6 3 7 4 1	4 2 7 5 8 1 6 3	3 7 4 6 2 5 8 1	7 3 2 4 8 6 1 9 5
4 2 5 7 3 6 1	4 2 6 5 8 1 7 3	2 7 3 5 4 6 8 1	6 4 2 5 8 7 1 9 3
4 2 6 5 3 7 1	4 2 7 6 8 3 5 1	2 6 4 7 1 5 8 3	6 3 2 5 8 7 1 9 4
3 5 2 4 7 6 1	5 3 6 8 2 7 4 1	2 7 6 8 3 5 4 1	5 7 2 8 3 6 1 9 4
3 6 2 4 7 5 1	4 3 6 8 2 7 5 1	2 6 5 7 3 8 4 1	7 4 2 8 3 6 1 9 5
3 5 2 7 4 6 1	4 3 5 7 2 8 6 1	2 6 4 8 5 1 7 3	6 5 2 8 3 7 1 9 4
4 6 2 5 3 7 1	5 2 6 8 4 1 7 3	2 6 4 7 5 1 8 3	7 4 3 9 6 2 8 5 1
4 6 2 7 3 5 1	5 7 3 2 4 6 8 1	2 7 3 8 4 6 5 1	6 4 2 9 7 1 5 8 3
3 5 7 2 4 6 1	5 7 3 2 4 8 6 1	2 5 4 7 6 3 8 1	6 4 2 9 7 5 1 8 3

Figure 8.1: Short basis elements of S^2.

S^2

Length	Sortable	Unsortable	Basis Elements
5	120	0	0
6	720	0	0
7	5018	22	22
8	39374	946	51
9	336870	26010	146
10	3066695	562105	

Figure 8.2: S^2 statistics are scarce and hard to compute.

complete, which indicates that decision algorithms may be slow. We present an extensively tested but unproved conjecture that this problem can be simplified to a 2SAT problem, soluble in polynomial time.

That is the principal content of this chapter, but we do also briefly examine S^3 (three stacks in series) and S^n. A plausible conjecture appears to be that S^n can sort all permutations of length up to and including $(n+1)!$.

Relations H and V

Definition 242 Let w be an S^2 word sorting the permutation $\alpha = a_1 a_2 \ldots a_n \in S^2$. For any terms a_i, a_j of α define $a_i \leq^H a_j$ if either $i = j$ or if at some stage in the sorting process represented by w, the term a_j is placed above a_i in the right stack.

Similarly define $a_i \leq^V a_j$ if either $i = j$ or if at some stage in the sorting process represented by $w\ a_j$ is placed above a_i in the left stack [1].

H and V are called the *cohabitation relations* generated by w.

Example 243 $\rho\rho\rho\lambda\mu\lambda\lambda\mu\lambda\mu$ is an S^2 word that represents a sorting of 231. For this sorting, in addition to the reflexive properties of H and V the following hold: $2 \leq^H 3$, $3 \leq^H 1$, $2 \leq^H 1$, $3 \leq^V 2$.

H and V are partial orders, however for most of our arguments it suffices to know whether two terms are comparable by H and V. If so we say that they are H or V related.

The Order Property:

Lemma 244 *Let a_i and a_j with $i < j$ be distinct terms of $\alpha \in S^2$. Then for any algorithm that sorts α and thereby generates cohabitation relations H and V precisely one of the following holds:*

- *$a_i < a_j$ and $a_i \leq^H a_j$ and $a_j \leq^V a_i$.*
- *$a_i < a_j$ and a_i, a_j are incomparable by both H and V.*
- *$a_i > a_j$ and $a_i \leq^H a_j$ but a_i, a_j are incomparable by V.*
- *$a_i > a_j$ and $a_i \leq^V a_j$ but a_i, a_j are incomparable by H.*

[1]The letters V and H refer to the properties described in Lemmata 246 and 245 which may be regarded as dual vertical and horizontal properties. V and H also refer to the stacks of their definition, vensten and höge being Norwegian for left and right.

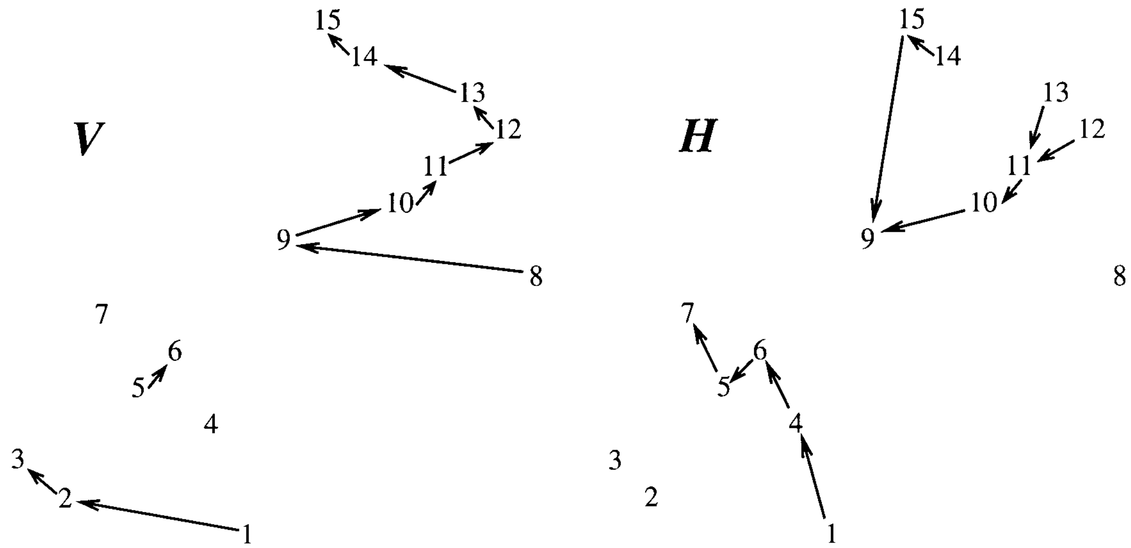


Figure 8.3: The Hasse diagram of partial orders V and H generated by a sorting of a permutation of length 16. Note that these are in fact forests with arrows pointing towards the root. In the case of H, on the right, all arrows point to the left and roots are leftmost terms in components. Note that with H components never overlap horizontally. They are arranged sequentially from left to right. In the case of V arrows point up, roots are highest terms and components are arranged bottom to top. In the sorting that produced these relations roots correspond to terms that lie at the very bottom of the left (V) or right (H) stacks. Arrows indicate which terms lie above which others. Given particular cohabitation relations it is always posible to deduce the set of sortings that yield the relations. The (unique) sorting that can produce the illustrated relations is: $\rho_3\lambda_3\rho_2\lambda_2\rho_7\rho_5\rho_6\rho_4\rho_1\lambda_1\mu_1\mu_2\mu_3\lambda_4\mu_4\lambda_6\lambda_5\mu_5\mu_6\lambda_7\lambda_7\ldots$

PROOF:

If a_i and a_j are comparable by H then although a_j enters the right hand stack after a_i it passes over it and leaves before it. Alternatively if a_i and a_j are not comparable by H then no change takes place and they leave the right hand stack in the order in which they entered.

Similarly a_i and a_j being V related is equivalent to a_i and a_j leaving the left hand stack in the order opposite to that in which they entered.

As the smaller of a_i and a_j must leave the left hand stack first we may conclude that if a_i, a_j is a decreasing pair, that is $a_i > a_j$, then a_j must coexist in passing with a_i in either the left or the right hand stack, but not both. In either case whilst coexisting in a stack a_i must lie below a_j. These two possibilities correspond to the first two listed in the lemma.

However if $a_i < a_j$, and a_i, a_j is an increasing pair, they must be related either by both V and H or by neither. By again noting which term lies above the other in each stack, if at all, we obtain the third and fourth possibilities listed in the lemma. As a_i and a_j must be either increasing or decreasing this exhausts all possibilities. Q.E.D.

■

The Direct Implications of V and H :

Lemma 245 *Let $\alpha \in S^2$, let w be an S^2 word that sorts α and generates relations of cohabitation V and H on the terms of α. Let a_i, a_j, a_k be terms of α such that $i < j < k$. If a_i and a_k are H related then also a_i and a_j are H related.*

PROOF: The terms a_i, a_j, a_k enter the right hand stack in the order a_i, a_j, a_k.

If a_i and a_k are H related then i must be in the right hand stack when a_k is entered and therefore must still have been in the right hand stack when a_j was entered. Thus a_i and a_j are H related.

■

Lemma 246 *Let $\alpha \in S^2$, let w be an S^2 word that sorts α and generates relations of cohabitation V and H on the terms of α. Let a_i, a_j, a_k be terms of α such that $a_i < a_j < a_k$. If a_i and a_k are V related then also a_j and a_k are V related.*

PROOF: The terms must leave the left hand stack in order a_i, a_j, a_k. Any term placed on the left hand stack must remain there until it is removed. If a_i and a_k are V related then a_k must be on the left hand stack at some stage prior to a_j being removed from the left stack. Thus as the removal of a_j is immediately preceded by a_j being on the left stack, a_j and a_k must cohabit the left stack and are therefore V related.

■

8.1.1 Algorithms versus Relations.

In sorting some permutation α the order in which adjacent ρ and μ operations are performed does not matter; the order can be changed without affecting the outcome of the sorting. To give an example, $\rho\lambda\rho\mu\lambda\mu$ and $\rho\lambda\mu\rho\lambda\mu$ both sort 1 2. Note that if both sortings are run in parallel for comparison then the stacks, inputs and outputs of both systems will be in the same state except after the third operation. Permuting adjacent ρ and μ symbols gives rise to an equivalence relation on S^2 words:

Definition 247 Let w_1 and w_2 be S^2 words. We say that w_1 and w_2 can be obtained one from another by $\rho\mu$ exchange if w_1 is equal to the word obtained from w_2 by exchanging two consecutive ρ and μ symbols. The ρ, μ *equivalence* on S^2 words is the equivalence relation generated by $\rho\mu$ exchange.

Lemma 248 *$\rho\mu$ equivalent S^2 words represent sortings of the same input sequence.*

The proof is easy and we omit it.

Any algorithm sorting a sequence α by two stacks in series generates cohabitation relations H and V on the terms of α. H and V are partial orders satisfying the properties of ORDER and IMPLICATION. Conversely any two partial orders I_H and I_V on the terms of α satisfying the properties of Order and Implication are generated as cohabitation relations in some sorting of α. Thus ORDER and IMPLICATION characterise the partial orders H and V; and furthermore there is a surjective map from the set of S^2 algorithms to the set of paired cohabitation relations.

Cohabitation relations are specific not to S^2 words but rather to $\rho\mu$ equivalence classes of S^2 words. If the input is fixed then these equivalence classes are in turn in one to one correspondence with what we call *midsequences* of algorithms, which are the orders in which algorithms move terms from the right to the left stack. This order is of course unaffected by $\rho\mu$ exchanges in S^2 words. Midsequences are precisely the permutations of $\mathcal{A}(231)$.

Proposition 249 *Let I_H and I_V, partial orders on the terms of some permutation α, satisfy the conditions given below. Then I_H and I_V are cohabitation relations generated by some sorting of α.*

ORDER:

For any two distinct terms a_i, a_j with $i < j$, precisely one of the following hold:

- $a_i < a_j$ *and* $a_i I_H a_j$ *and* $a_j I_V a_i$.
- $a_i < a_j$ *and* a_i, a_j *are unrelated by* I_V *and* I_H.
- $a_i > a_j$ *and* $a_i I_H a_j$ *but* a_i, a_j *are unrelated by* V.
- $a_i > a_j$ *and* $a_i I_V a_j$ *but* a_i, a_j *are unrelated by* H.

IMPLICATION:

- *If* $i < j < k$ *and* $a_i\ I_H\ a_k$ *then* $a_i\ I_H\ a_j$
- *If* $a_i < a_j < a_k$ *and* $a_i\ I_V\ a_k$ *then* $a_j\ I_V\ a_k$.

PROOF: Briefly: The proof we give centers around midsequences. Note that a permutation is S^2 sortable if and only if its terms can be permuted through one stack into an 'intermediate' sequence, and then permuted again through another stack to emerge fully sorted. The fact that in S^2 a pop operation from the first stack must coincide with a push operation on the second is irrelevant. Whenever two stacks are placed in series it is always possible to arrange the timing of the operations in the first and the second so that pop operations from the first stack and push operations on the second coincide. Now:

There is a method of passing the permutation α through a single stack so that the induced one stack cohabitation relation is I_H. This is assured by the IMPLICATION property for I_H. The passing of α through a stack need not be a sorting, it merely corresponds to passing α through the right stack of S^2. We denote the order in which terms emerge from this single stack operation by M_1; a permutation of length n.

It is possible to rearrange the terms of α to form another permutation M_2 that can be sorted over a single stack, and furthermore we can choose M_2 and the sorting in such a way that the induced cohabitation relations are precisely I_V. Similarly to the case for I_H, this follows from the IMPLICATION property of I_V.

Note that it is an easy matter, given any two terms of α, to determine which precedes the other in M_1 or in M_2. If the two terms are I_H related then they apper in M_1 in the order opposite to the order in which they appear in α, else they appear in the same order. Similarly if two terms are I_V related then they form a decreasing pair in M_2, else they form an increasing pair.

It follows from the ORDER properties that the permutations M_1 and M_2 are identical, and thence that α can be sorted over two stacks in series. Q.E.D. ■

Proposition 250 *If w_1 and w_2 are S^2 words representing sortings of a permutation α then the following are equivalent:*

- *w_1 and w_2 generate the same midsequences.*
- *w_1 and w_2 are $\rho\mu$ equivalent.*
- *w_1 and w_2 generate the same cohabitation relations.*

This is fairly elementary, and as we do not use it, we omit the proof.

The Word Problem

S^2 words are called *fully equivalent* or simply *equivalent* if they represent sortings of the same sequence. The $\rho\mu$ equivalence is contained within the full equivalence. The $\rho\mu$ equivalence can be defined by the substitution law $\rho\mu = \mu\rho$, but no similar set of substitution laws are known that define the full equivalence on S^2 words. Some substitution laws that hold in the full equivalence are given below.

- If W is itself an S^2 word then $W\lambda = \lambda W$.
- $\rho\mu = \mu\rho$
- If W_1 is a balanced word over ρ, λ and if W_2 is a balanced word over λ, μ then $W_1 W_2 = W_2 W_1$.
- If W is an S^2 word then $(\lambda\rho)^n \rho^{n-1} W \mu^{n-1} (\mu\lambda)^n = \rho^{n-1}(\rho\lambda)^n W (\lambda\mu)^n \mu^{n-1}$. In particular $\rho\lambda W \lambda\mu = \lambda\rho W \mu\lambda$.

8.1.2 Duality

Definition 251 Let α be a permutation of length n. Then α^D is the permutation defined by $\alpha^D(a) = n + 1 - \alpha^{-1}(n + 1 - a)$. That is, $\alpha^D = R_n \alpha^{-1} R_n$.

Example 252 If $\alpha = 1\,4\,2\,3$ then $\alpha^D = 3\,1\,2\,4$.

Lemma 253 *Let α be a permutation of length n. Then the following hold:*

- $\alpha \in S^2$ *if and only if* $\alpha^D \in S^2$.

- *If w is an S^2 word that sorts α, then the word obtained from w by reading w back to front and replacing every occurrence of ρ in w by μ and vice versa is an S^2 word that sorts α^D.*

- *Let w be an S^2 word that sorts α. Denote by w' the word obtained from w by replacing every occurrence of ρ in w by μ and vice versa. Let V and H be the cohabitation relations on the terms of α generated by w. Let V' and H' be the cohabitation relations on the terms of α^D generated by w'. Then the ith and jth terms of α are H or respectively V related if and only if the terms i and j of α^D are V' or respectively H' related.*

- $(\alpha^D)^D = \alpha$.

PROOF: Suppose that $\alpha \in S^2$ and that w is an S^2 word that represents an algorithm that sorts α. If w', defined as in the lemma, is allowed to operate on an input of $R_n = n\ n-1\ \ldots\ 2\ 1$ it will output terms in the order $R_n\alpha = a_n\ a_{n-1}\ \ldots\ a_2\ a_1$. Indeed such a performance is equivalent to running the algorithm represented by w in reverse, taking terms from the output and moving them back to the input. Thus by applying the permutation α^{-1} to the input and output sequences R_n and $R_n\alpha$ we conclude that w' represents the algorithm that sorts $R_n\alpha^{-1}R_n = \alpha^D$. By this analogy we also conclude that the cohabitation relations generated by w and w' are dual in the manner described in the lemma.

Further note that $(\alpha^D)^D = R_n(R_n\alpha^{-1}R_n)^{-1}R_n = R_nR_n\alpha R_nR_n = \alpha$. This completes the proof. Q.E.D.

■

Example 254 $(32451)^D = 23541$. Thus $32451 \in S^2$ if and only if $23541 \in S^2$. It is later proved that for any cohabitation relations generated by a sorting of 23541, 2 and 3 must be both H and V related. Thus a similar statement holds stating that the terms 4 and 5 of 32451 must be both H and V related.

8.1.3 Some Consequences involving V and H

Lemma 255 *Let $\alpha = a_1, a_2, \ldots, a_n$ be a sequence in S^2 containing a subsequence a_i, a_j, a_k, a_l order isomorphic to 1 3 2 4. Then in any sorting algorithm that sorts α, a_i and a_l will be neither V nor H related.*

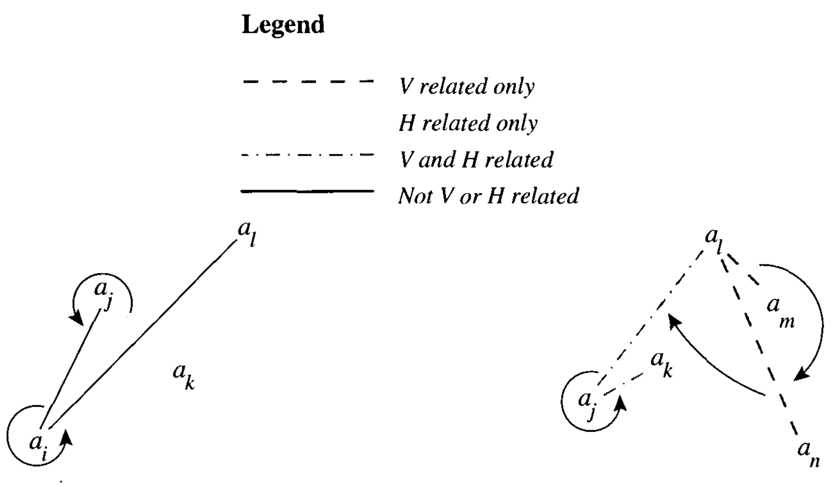
PROOF: The decreasing pair a_j, a_k must be either V related or H related. Suppose that it is H related.

As a_j and a_k are not V related, neither, by IMPLICATION, are a_i and a_j. So by ORDER a_i and a_j are neither V nor H related, and specifically not H related. By IMPLICATION a_i and a_l are not H related, and therefore by ORDER not H related either, as required.

The proof is dual for the case where a_j and a_k are H related. The terms a_i and a_j cannot be V related and therefore neither are they H related. Thus a_i and a_l can be neither V nor H related. Q.E.D.

■

Lemma 256 *Let $\alpha = a_1, a_2, \ldots, a_n$ be a sequence in S^2 containing a subsequence a_j, a_k, a_l, a_m, a_n order isomorphic to 2 3 5 4 1. Then in any sorting algorithm that sorts α, a_j and a_k will be both V and H related.*



The case where a_j, a_k are H related. The case when a_l, a_m are V related.

Figure 8.4: Basic constants: On the left, a_i and a_l are invariably related by neither V nor H. The opposite is true for a_j and a_k on the right, that must always be related by both V and H.

PROOF: a_l, a_m is a decreasing pair and must by ORDER be either V or H related, but not both. Suppose that it is V related. By IMPLICATION a_l and a_n are not H related but, by ORDER, V related. IMPLICATION then gives that a_j and a_l are V related and being an increasing pair ORDER adds that they are also H related. Thus a_j and a_k are H related and therefore also V related, as required.

If a_l and a_m are H related then they are not V related and neither are a_k and a_l. Thus a_k and a_l are not H related, and neither are a_k and a_n. Therefore a_k and a_n are V related and therefore so are a_j and a_k, which must also be H related, as required.

■

Incidentally, note that in the above proofs no use whatsoever has been made of the fact that as partial orders, H and V are transitive. This has growing significance.

8.2 Some Infinite Sets of Basis Elements

Proposition 257 *The following is an infinite set of basis elements of* S^2.

$D_0 =$ 2 4 3 5 7 6 1

$D_1 =$ 6 8 7 2 9 3 5 4 1

$D_2 =$ 8 10 9 6 11 2 7 3 5 4 1

$D_3 =$ 10 12 11 8 13 6 9 2 7 3 5 4 1

$D_4 =$ 12 14 13 10 15 8 11 6 9 2 7 3 5 4 1

$\vdots$

$D_i = 2i+4,\ 2i+6,\ 2i+5,\ 2i+2,\ 2i+7,\ 2i,\ 2i+3,\ 2i-2,\ 2i+1,\ 2i-4,\ 2i-1,\ \ldots,\ 8,\ 11,\ 6,\ 9,\ 2,\ 7,\ 3$

PROOF:

It is a routine matter to demonstrate that if any term is removed from one of the above permutations then the resulting permutation is in S^2. We will therefore omit that part of the proof and merely prove that the above permutations are not S^2 sortable.

In the case of D_0 note that the terms 2 4 3 5 are order isomorphic to 1 3 2 4 and therefore, by Lemma 255 in any algorithm that sorts D_0, 2 and 5 will be neither H nor V related. However notice also that the terms 2 5 7 6 1 are order isomorphic to 2 3 5 4 1, and therefore in any algorithm

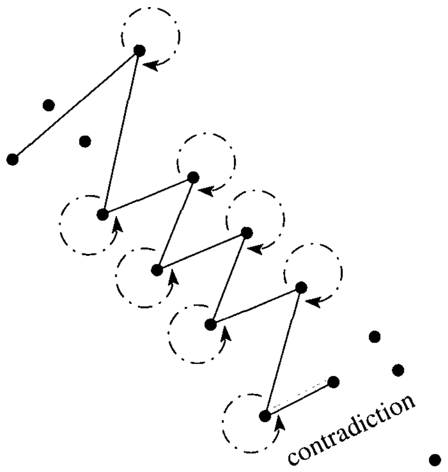


Figure 8.5: D_4 is a basis element of S^2.

that sorts D_0, 2 and 5 will be both V and H related, a contradiction. Thus no algorithm can sort D_0.

Now let us consider any other D_i. Suppose that some algorithm exists that can sort D_i, thereby generating cohabitation relations H and V. The first $4+2i$ terms of D_i are:

$$2i+4, 2i+6, 2i+5, 2i+2, 2i+7, 2i, 2i+3, 2i-2, 2i+1, 2i-4, 2i-1,$$

$$\ldots, 8, 11, 6, 9$$

The terms $2i+4, 2i+6, 2i+5, 2i+7$ are order isomorphic to 1324 and therefore by Lemma 255, in the algorithm $2i+4$ and $2i+7$ can be neither H nor V related. As this pair is not V related, $2i+2$ and $2i+7$ cannot be V related and, being an increasing pair, they cannot be H related either. Thus

$2i+2$ and $2i+3$ cannot be H related, and are not V related. Therefore $2i$ and $2i+3$ cannot be V related, and so the argument continues until we conclude that 2 and 3 are neither H nor V related.

However if we now consider the remaining terms of D_i note that 23541 is a subsequence of D_i and therefore by Lemma 256 we have that 2 and 3 must be both V and H related, a contradiction. Thus we have that D_i is not S^2 sortable. Q.E.D.

■

The basis elements in the last lemma are essentially decreasing. Terms start large near the beginning of the sequence and become with occasional deviations gradually smaller and the smallest terms are at the end. We might well expect such a permutation to be hard to sort as all the early large terms must be accommodated in the stacks, suitably, until the smallest term is output, a difficult task. We here have another set of basis elements that is overall increasing, for contrast. This second infinite set can also be used to show that there is no version of the greedy algorithm that will work for S^2. The reason for this is clarified in the following Corollary 259

Lemma 258 *The following is an infinite set of basis elements of X.*

$E_1 = 3\ 2\ 4\ 7\ 6\ 1\ \ 8\ 10\ 9\ 5$

$E_2 = 3\ 2\ 4\ 7\ 6\ 1\ \ 10\ 9\ 5\ \ 13\ 12\ 8\ \ 14\ 16\ 15\ 11$

$E_3 = 3\ 2\ 4\ 7\ 6\ 1\ \ 10\ 9\ 5\ \ 13\ 12\ 8\ \ 16\ 15\ 11\ \ 19\ 18\ 15\ \ 20\ 22\ 21\ 17$

$E_4 = 3\ 2\ 4\ 7\ 6\ 1\ \ 10\ 9\ 5\ \ 13\ 12\ 8\ \ 16\ 15\ 11\ \ 19\ 18\ 15\ \ 22\ 21\ 17$
$\quad 25\ 24\ 20\ \ 26\ 28\ 27\ 23$

$\ldots$

$E_n = 3\ 2\ 4\ 7\ 6\ 1\ \ 10\ 9\ 5\ \ 13\ 12\ 8\ \ 16\ 15\ 11\ \ 19\ 18\ 14\ \ldots$
$\quad \ldots 6n+1\ 6n\ 6n-4\ \ 6n+2\ 6n+4\ 6n+3\ 6n-1$

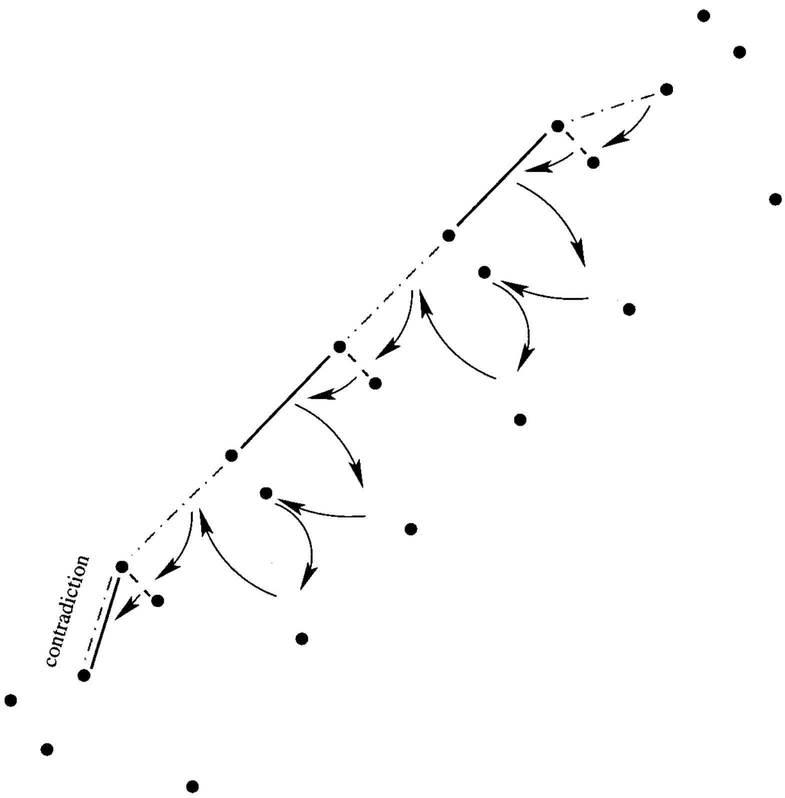


Figure 8.6: E_4 is a basis element of S^2

...

PROOF: Let us consider any of the proffered permutations. It is an elementary and routine matter to demonstrate that if any term is removed from the permutation then the resulting sequence is S^2 sortable. However let us suppose that the entire permutation is sortable by some algorithm that generates cohabitation relations V and H. We will reconstruct something of V and H.

The last few terms of the permutation are:

$$\ldots\ 6n+1\ \ldots\ 6n+2\ 6n+4\ 6n+3\ 6n-1$$

As $6n+1\ 6n+2\ 6n+4\ 6n+3\ 6n-1$ is order isomorphic to 23541 we can note from Lemma 256 that $6n+1$ and $6n+2$ must be related by both V and H. Casting our eye further back we see that we have the following terms:

$$\ldots\ 6n-2\ \ldots\ 6n+1\ 6n\ \ldots\ 6n+2\ \ldots$$

As $6n+1$ and $6n+2$ are H related, so too must $6n+1$ and $6n$ be, by Lemma 245. As $6n+1\ 6n$ is a decreasing pair it cannot be both V and H related, and so as it is not V related, we have by Lemma 246 that $6n+1$ and $6n-2$ are also not V related. As it is an increasing pair $6n-2$ and $6n+1$ are neither V nor H related. We consider another subsequence:

$$\ldots\ 6n-5\ \ldots\ 6n-2\ 6n-3\ 6n-7\ 6n+1\ \ldots 6n-4\ \ldots$$

$6n-2$ and $6n-1$ are not H related, and so neither are $6n-2$ and $6n-4$ which, as a decreasing pair, must therefore be V related. As $6n-2$ and $6n-4$

are V related, so too are $6n-2$ and $6n-3$, and as $6n-2$ and $6n-3$ form a decreasing pair they are not H related. Thus $6n-2$ and $6n-7$ are not H related, and so, as a decreasing pair, they must be V related. Therefore $6n-5$ and $6n-2$ are V related and, as an increasing pair, they must be both V and H related.

We continue examining subsequences of the form $6i-2, 6i+1, 6i, 6i+2$ and $6i-5, 6i-2, 6i-3, 6i-7, 6i+1$ until we conclude that every increasing pair $6i-2, 6i+1$ is neither V nor H related and that every increasing pair of the form $6i-5, 6i-2$ is both V and H related. By this rule 4 and 7 are neither V nor H related. However the subsequence 3 2 4 7 1 is order isomorphic to 3 2 4 5 1 and so by duality and Lemma 256 we have that 4 and 7 are both V and H related, a contradiction.

■

Corollary 259 *All the following permutations R_i are in S^2. However if i is odd then the first two letters of any S^2 word representing an algorithm that sorts R_i are $\rho\rho$, whereas if i is even then the first two letters are $\rho\lambda$.*

$R_1 = 4\ 3\ 1\ \ 5\ 7\ 6\ 2$

$R_2 = 4\ 3\ 1\ \ 7\ 6\ 2\ \ 8\ 10\ 9\ 5$

$R_3 = 4\ 3\ 1\ \ 7\ 6\ 2\ \ 10\ 9\ 5\ \ 11\ 13\ 12\ 8$

$R_4 = 4\ 3\ 1\ \ 7\ 6\ 2\ \ 10\ 9\ 5\ \ 13\ 12\ 8\ \ 14\ 16\ 15\ 11$

$R_5 = 4\ 3\ 1\ \ 7\ 6\ 2\ \ 10\ 9\ 5\ \ 13\ 12\ 8\ 16\ 15\ 11\ \ 17\ 19\ 18\ 14$

...

$R_n = 4\ 3\ 1\ 7\ 6\ 2\ \ 10\ 9\ 5\ \ 13\ 12\ 8\ \ 6n+1\ 6n\ 6n-4\ \ 6n+2\ 6n+4\ 6n+3\ 6n-1$

...

Thus it is not possible to determine from the first n terms of an arbitrary S^2 permutation α of length at least n, n being a fixed number, whether some two letters can be the first two letters of an S^2 word w representing a sorting of α.

PROOF:

The permutations R_i are order isomorphic to basis elements of S^2 listed in Lemma 258 with either the first three or the first six terms removed. Thus every R_i is in S^2 and an algorithm exists that sorts it. Consider any S^2 word w that represents a sorting of R_i and suppose that the sorting generates cohabitation relations V and H. From the proof of Lemma 258 we gather that if i is odd then the first increasing pair involving 4, which is 4 5 in the case of R_1 and 4 7 in all other cases, must be both V and H related wheras if i is even that increasing pair is neither V nor H related. If i is odd we may conclude that the first two terms of R_i are H related, in which case the first two letters of w must be $\rho\rho$. If i is even then as the terms 4 and 7 of R_i are not R related the decreasing pair 4 2 must be V related, wherefore the decreasing pair 4 3 must be V but not H related and the first two letters of w must be $\rho\lambda$. Q.E.D.

■

8.3 Conjecture

It has been conjectured that the problem of S^2 membership is NP complete, see[3]. However we would like to present another conjecture:

Conjecture 260 *If, for a permutation $\alpha = a_1, \ldots, a_n$, there exist reflexive and antisymmetric relations J_V and J_H on the terms of α and satisfying the conditions of ORDER and IMPLICATION (defined in Proposition 249) then $\alpha \in S^2$.*

Note that J_V and J_H have all the qualities that characterise cohabitation relations in Proposition 249, except for transitivity. However J_V and J_H are not required to be cohabitation relations for some sorting of α. Indeed if $\alpha = 2\ 3\ 1$ then it is sufficient for J_V and J_H to be reflexive, antisymmetric, and contain the relations $2J_V1$ and $3J_H1$ to satisfy the conditions of the conjecture. These are not cohabitation relations for any sorting of 2 3 1. The conjecture is unproved but the following are worth noting:

- The set of permutations for which relations J_H and J_V, as defined in the conjecture, exist is closed and contains S^2.

- To prove the conjecture it suffices to prove that it holds for all basis elements of S^2. (The set of permutations for which relations J_V and J_H exist, is a superset of S^2. If containment is proper then there is an S^2 basis element for which relations J_V and J_H exist.)

- The conjecture has been tested by computer for all permutations of length 9 or less, and especially on all basis elements of S^2 with length 9 or less. Complete lists of longer basis elements are not known.

- The conjecture has been tested by hand for permutations of length 18 and less. Sadly the record of this work was discarded when it did not suggest a method of either proving or disproving the conjecture.

There are many permutations of length 18 but with a little effort a highly refined search can vastly reduce the number of cases that need to be tested. Voluminous work done by hand is liable to minor errors but apart from such possibilities the author of this text is reasonably convinced that the search was both accurate and exhaustive. This result is given only because the weight of this evidence may encourage further research.

- In [27] J. West considers sorting sequences through two consecutive stacks. However the reader should be cautious because the set of permutations that is regarded in that paper as being two stack sortable is strictly less than the set S^2 of this.

If the conjecture holds it can be readily shown by even a crude argument that an algorithm exists that can determine whether a given permutation of length a is in S^2 in time at worst proportional to a^5. One such argument is outlined here:

- Let τ, a permutation of length a be given. If τ has length a then there are $a(a-1)/2$ unordered pairs of distinct terms of τ. The largest component of this expression is a^2 and so a list of these pairs can be compiled in $O(a^2)$ time.

- We will seek to find relations J_H and J_V on the terms of τ that satisfy the requirements of Conjecture 260. In such relations an increasing pair of terms must lie in either both or neither of the relations, a decreasing pair in either one or the other but not both. We will therefore shortly attempt to colour each pair of terms: Red or black if the pair is increasing, corresponding to the pair lying in both or neither relations respectively, blue or green if the pair is decreasing, corresponding to the pair being in J_V or J_H.

We can, in linear time with respect to the length of the list of pairs (quadratic in the number of terms), add to each pair of terms the information of whether or not it is increasing or decreasing and two text fields, one of which can contain a permanent label, the other a temporary label for experimental purposes. If the permanent text field is empty then we call the corresponding pair of terms unlabelled.

- Choose an unlabelled edge e and temporarily label it with a colour, either red or black if increasing, blue or green if decreasing.

- Temporarily mark out the consequences of our labelling. This involves the following: For each edge that shares a vertex with e, IMPLICATION rules may or may not indicate what that edge should be labelled. If there is indication then label temporarily. Now for the set of edges temporarily labelled but whose consequences have not yet been found (a list of these should be kept), find the consequences. And so on until either all consequences have been found, or an edge is given two different temporary labels, thereby yielding a contradiction. (An extra temporary label can prevent the consequences of one particular label ever having to be found more than once.) There are $a(a-1)/2$ edges and for each there are $2(a-1)$ consequent edges that may have to be labelled. Thus this operation takes $O(a^3)$ time. At the end of this we will either have a contradiction, with two different labels on one edge, or our labelling will be consistent.

- If the labelling of e was consistent then make the label on e permanent. We also know that if the e labelling was consistent then all other temporary labels that we inserted were consistent, so we can make them permanent. Note that no labelling of an as yet unlabelled edge d can contradict a permanent label, else a consequence of the permanent label would be that d can have only one colour, in which case d would by now also be permanently labelled.

- If the labelling of e was not consistent then try the other possible labelling. We will either get a consistent labelling, or else no consistent labelling exists and we can terminate this operation.

- Continue in this way, establishing permanent and consistent labels. This must be done at most $a(a-1)/2$ times. This gives the overall mechanism a time complexity of at most $O(a^5)$, a loose upper bound.

The difference between the complexity of finding a non-contradictory assignment to a logical problem where each rule has two variables, known as 2SAT, and the problem where rules can have three variables (3SAT) is well known. The former is definitely polynomial time soluble, the latter is in general NP complete.

ORDER and IMPLICATION both have only two variables, transitivity has three. Thus by disposing of transitivity we are taking a problem that is potentially NP complete and making it definitely polynomial time soluble. If it transpires that transitivity cannot be disposed of as in our conjecture, then all may not be lost. There do exist classes of 3SAT problems where satisfiability is soluble in polynomial time.

- At least one program that follows this description is available from the author, albeit in a format that can be run only on Macintosh computers. (It is still to be seen whether it will run on the latest Apple computers.)

8.4 Onwards and Upwards: S^n

Sortings of permutations of length a over n stacks in series can be expressed uniquely by balanced words of length $a(n+1)$ over an alphabet of size $n+1$, as was done in the case of S^2. The number of such words is equal to the n-dimensional Catalan number

$$\frac{0! * 1! * \ldots * (n-1)! * (n*a)!}{a! * (a+1)! * \ldots * (a+n-1)!},$$

see [31], and certainly less than $(n+1)^{a(n+1)}$ whereas there are $a!$ permutations of length a. Thus no n is great enough for S^n to contain all finite permutations. However bounds on maximal a such that S^n contains all permutations of length a or less have been sought.

The permutations that can be sorted by a single stack are known to be those of $\mathcal{A}(231)$. Thus we have that a single stack in series can sort all permutations of length 2. Additionally it is known that the number of permutations of length m sortable over a single stack is given by the mth Catalan number $(2m)!/(m!(m+1))$. No such result is known for any other number of sequential stacks.

The shortest basis element of S^2 has length 7, thus we have that all permutations of length 6 or less are S^2 sortable.

From this brief glimpse it seems possible that maximal a is given by $(n+1)!$. Bounds on a for arbitrary n are found in the following results. They do not affirm this possibility but at least they do not contradict it.

First an upper bound on a:

Proposition 261 *For every positive integer n there exists a permutation of length S_n, the nth term of the Sylvester sequence, that is not sortable by n stacks in series.*

The first few terms of the Sylvester sequence are: $3, 7, 43, 1807, \ldots$. *The terms are given by the recurrence $S_{n+1} = (S_n)^2 - S_n + 1$.*

This follows from the following:

Proposition 262 *If there exists a permutation not of S^n and having length k then there exists a permutation of length $k^2 - k + 1$ not in S^{n+1}*

PROOF: Let $\alpha = a_1, \ldots, a_m$ be a permutation not sortable by n stacks in series. Construct β as follows: Let $\beta = \beta_1\beta_2 \ldots \beta_m$ where β_m is a single term and every other β_i is a contiguous interval order isomorphic to $R\alpha = a_m, a_{m-1}, \ldots, a_1$. Furthermore let β be such that if $b_1, \ldots, b_m$ be a subsequence of β with each b_i being a term of β_i, then $b_1, \ldots, b_m$ is order isomorphic to α. The fact that each b_i is a contiguous interval ensures that this last requirement holds for all such sequences if it holds for any one.

β has length $a(a-1)+1 = a^2 - a + 1$ and it is not in S^{n+1}. For suppose that an algorithm exists that can sort β over $n+1$ stacks. During the sorting process it is never possible for all terms of some β_i, $i < m$ to be placed in the first stack, for otherwise these terms would not be sortable over the remaining n stacks. Thus some term b_1 of β_1 enters the second stack before some term b_2 of β_2, which in turn enters before some term of β_3 and so on. But this sequence $b_1, \ldots, b_m$ is order isomorphic to α and is therefore unsortable over the remaining n stacks. Thus we have a contradiction, and β is not in S^{n+1}. Q.E.D.

■

If we apply the construction of this proof to 2 3 1, the basis element of S^1, we obtain the sequence 2 4 3 5 7 6 1, which is a basis element of S^2. Repeating the construction to get a permutaton not in S^3 we obtain:

2 7 8 6 4 5 3	16 21 22 20 18 19 17	9 14 15 13 11 12 10	
23 28 29 27 25 26 24	37 42 43 41 39 40 38	30 35 36 34 32 33 31	1

If the 1 is removed then the resulting sequence is S^3 sortable. But if the 2 is removed then the sequence obtained is not S^3 sortable, hence the above is not a basis element. It would further appear that if four of the first seven basis elements are removed then the resulting permutation is still unsortable:

7 8 6	16 21 22 20 18 19 17	9 14 15 13 11 12 10	
23 28 29 27 25 26 24	37 42 43 41 39 40 38	30 35 36 34 32 33 31	1

It is not known whether this sequence is order isomorphic to a basis element of S^3.

Now we give a lower bound for a.

Proposition 263 *Every basis element of S^n is sortable over $n+1$ stacks.*

This assures us that $a > n$. The argument used to prove this is a similar to and simpler than the proof of the following and is therefore not given.

Proposition 264 *If every permutation of length k is in S^n then every permutation of length $2k$ is in S^{n+1}.*

PROOF: Suppose that all permutations of length a are sortable over n stacks in series and let α be a permutation of length no greater than $2a$. Then given a sorting mechanism with $n+1$ stacks in series we may sort the terms of α no greater than a by utilising only the last n stacks that lie nearest the output. These terms can be passed straight through te first stack, which is nearest the input, as they are required whilst terms greater than a can be left idle on the first stack where they will not interfere with the sorting process. Once all the lesser terms have been output the terms greater than a remaining in the first stack and the input can be sorted as they number no more than a. Q.E.D.

■

Still stands the conjecture:

Conjecture 265 *S^n contains all permutations of length $(n+1)!$ or less.*

Chapter 9

Gessel's enumerative conjecture

Uncountably Many Closed Classes With and Without Identical Enumeration

This comment concerns itself with Ira Gessel's conjecture in [32] that all closed classes with finite basis have enumeration that satisfies a recursive formula with polynomial coefficients. At this time of writing the conjecture is open, with all known finitely based (and infinitely based) closed class enumerations satisfying rational recursive formulae, as required.

It is our purpose to prove that the conjecture cannot be extended to all infinitely based closed classes. We prove that there are too many closed classes for them to be enumerable in this manner. The argument is plucked from [2] in which it is argued that since there are only countably many rational recursive formulae but uncountably many closed classes, the suggestion that all closed classes have rational recursive enumeration implies that there must be uncountably many classes all having identical enumeration. This was there considered unlikely.

We take the direct approach of generating an uncountable family of closed classes all having distinct enumeration. We furthermore show that the original indirect approach cannot work because there do in fact exist uncountable families of closed classes, all having identical enumeration.

Theorem 266 *There exists a family $\mathcal{X}$ with cardinality $2^{\aleph_0}$, the members of which are closed classes and where no two distinct members of the family have the equal enumeration.*

It follows that there exist closed classes that do not satisfy a recursive enumeration formula with polynomial coefficients, as the number of such formulae is only $\aleph_0$.

PROOF: The second statement in the theorem is evidently justifiable once the first statement has been proved, therefore we omit its proof.

We now construct a family $\mathcal{X}$ having the properties claimed above. Let A be any infinite antichain, with elements denoted by $A_1, A_2, A_3, \ldots$, no two of which have the same length.

By definition *Propsub*(A) is the set of all permutations properly involved in some element of A. Since A is an antichain we have that *Propsub*(A) and A are disjoint.

Now, *Propsub*$(A) \cup \{A_1\}$, which, importantly, is a closed class, has identical enumeration to *Propsub*(A) except that the number of permutations with length equal to that of A_1 is one greater in *Propsub*$(A) \cup \{A_1\}$.

Similarly if A_i is any element of A and if $f(n)$ is is a function giving the number of permutations of length n in *Propsub*(A) then the number of permutations of each length in $Propsub(A) \cup \{A_i\}$ is given by:

$$g(n) = \begin{cases} f(n) & \text{if } n \neq |A_i| \\ f(n) + 1 & \text{otherwise} \end{cases}$$

We can extend this method of increasing the size of $Propsub(A)$ at one point only to increasing the size at any number of points, in an entirely independent manner. Given any subset O of A we have that the number of permutations of each length in $Propsub(A) \cup O$ is given by:

$$h(n) = \begin{cases} f(n) & \text{if } O \text{ has no element of length } n. \\ f(n) + 1 & \text{otherwise} \end{cases}$$

Now there are uncountably many infinite subsets of A and we claim that for any two distinct subsets O' and O'' the enumeration of $Propsub(A) \cup O'$ and $Propsub(A) \cup O''$ is distinct. We claim that this is evident, and indeed if A_j is an element of O' but not of O'' then the number of permutations in $Propsub(A) \cup O'$ of length $|A_j|$ is one greater than that in $Propsub(A) \cup O''$. This completes the proof. ■

Theorem 267 *There exists a family $\mathcal{Y}$ with cardinality $2^{\aleph_0}$ whose members are closed classes, all of which have the same enumeration.*

PROOF: Let $A = A_1, A_2, \ldots$ be any infinite antichain of sum indecomposable permutations. For every bijection $\rho : \mathbb{N} \to \mathbb{N}$ define the Y_ρ to be the closed class:

$$\begin{aligned} Y_\rho \;=\; & Propsub(A)^{\oplus} \oplus Sub(A_{\rho(1)}) \oplus Propsub(A)^{\oplus} \oplus Sub(A_{\rho(2)}) \oplus \\ & Propsub(A)^{\oplus} \oplus Sub(A_{\rho(3)}) \oplus \ldots \textit{etc ad inf} \end{aligned}$$

(Recall that $\oplus$ as a superscript indicates sum completion.)

Thus any element of A can appear at most once in any element of Y_ρ, and then only as a sum component. Two or more elements of A may conceivably appear in an element of Y_ρ but if so then they appear in a specific order. The $Propsub(A)^\oplus$ sections are spacers that absorb excess terms that appear later. Note that any sequence in Y_ρ that does not involve an element of A is involved in $Propsub(A)^\oplus$. That is the property that makes these spacers useful.

The number of bijections, such as ρ, on $\mathbb{N}$ is uncountable and in fact has cardinality $2^{\aleph_0}$. Thus this method can be used to generate uncountably many closed classes. We let these classes be the members of $\mathcal{Y}$.

We claim that these closed classes are all distinct. Let ρ and σ be any two distinct bijections on $\mathbb{N}$. Then there must be two numbers $i, j \in \mathbb{N}$ such that Y_ρ contains $A_i \oplus A_j$ whereas Y_σ only contains $A_j \oplus A_i$, because $\rho^{-1}(i) < \rho^{-1}(j)$ but $\sigma^{-1}(i) > \sigma^{-1}(j)$. That demonstrates the claim.

We also claim that all these classes all have the same enumeration. We will prove this by defining a bijection f from Y_ρ to Y_σ. This map will be length preserving, and will therefore demonstrate the equal enumeration. To define the map note that every element of Y_ρ can be expressed as:

$$\alpha = \alpha_1 \oplus A_{k_1} \oplus \alpha_2 \oplus A_{k_2} \oplus \alpha_3 \oplus A_{k_3} \oplus \ldots \alpha_n \oplus A_{k_n} \oplus \alpha_{n+1}$$

where $A_{k_1}, A_{k_2}, \ldots$ are elements of A and where $\alpha_1, \alpha_2, \ldots$ are elements of $Propsub(A)$. The only reason why this permutation may conceivably not be an element of Y_σ is that the elements of A may appear in the wrong order. So we rearrange them. Let:

$$f(\alpha) = \alpha_1 \oplus A_{\sigma(\rho^{-1}(k_1))} \oplus \alpha_2 \oplus A_{\sigma(\rho^{-1}(k_2))} \oplus \ldots \alpha_n \oplus A_{\sigma(\rho^{-1}(k_n))} \oplus \alpha_{n+1}.$$

That completes the proof. ■

Appendix A

Markov Chain selection

In the following we present some thoughts on the problem of selecting a random permutation of given length from a given closed class X. The reasons for wishing to solve this problem is this: If we know what a random element of X looks like, then we will have a good idea of what an arbitrary element of X looks like, and that insight is almost as valuable as having an atomic representation for X. Furthermore it may well lead to a conjectured atomic representation.

I acknowledge that I am aware that Michael Albert and Mike Atkinson have put some thought into producing random elements of a closed class, doubtless they know much that can be added to this text. The following are merely thoughts had since the discussion with them, and since having read much of Rajeev Motwani and Prabhakar Raghavan's "Randomized Algorithms".

An important recognition from material relating to the latter is that it is possible in $n\log(n)$ time, and using of the order of $n\log(n)$ random bits to generate random elements of the set of all permutations of length n (by what is known as the quickselect algorithm). This is both fast and cheap. The only way in which our demands here differ is that we need random elements not of all permutations, but of some specified closed class X. This does suggest that unless the elements of length n in our class are scarce compared with the equivalent in the set of all permutations, we should simply use the fast and cheap $n\log(n)$ algorithm repeatedly and keep those generated permutations that lie in the target class. However there are still classes whose membership grows much more slowly with length than the $n!$ of S_n, therefore there can still be the need to continue. Indeed if the Stanley-Wilf conjecture (which states that for every closed class, the number of elements of each length is bounded above by an exponential bound) holds, then all closed classes with at least one basis element 'grow' much more slowly than $n!$.

Here are three conceivable ways of manufacturing random permutations in X:

1. Starting with the trivial permutation, 1, and adding terms at random to produce arbitrary permutations of length $2, 3, 4, \ldots, n$ in X.

2. Starting with a permutation of length n in the class and walking by random transpositions to other permutations of length n in X. By transpositions we mean that we select two terms and swap them. If the resulting permutation is in the class then fine, we repeat choosing pairs of terms and swapping them until we are satisfied that we must have a suitably random element in the class. If the result of the swap is however not in the class then we may have chosen our pair of terms badly and we should look for a different pair of terms to swap.

3. Starting with a permutation of length n in the class and by random deletion-insertions taking a walk to other permutations of length n in the class. This can be thought of in a graph theoretical context as follows: Take the simple graph whose vertices are the permutations in the class with length $n-1$ and n and where two vertices are adjacent if and only if they are comparable by involvement. This graph is bipartite. Now start at any vertex that is a permutation of length n and take a random walk of even length. The walk is equivalent to arbitrarily many times deleting a term from the permutation and then adding another randomly, but judicially so that we do not step outside the class. It is worth noting that we do not need to generate this graph to take a walk on it. It is enough to remember the basis of X and our present location.

The first of these is the method that I myself use. It is quick and it is clear that, given a closed class, any permutation of length n in the class can be obtained by this mechanism, subject only to the random choices of where to add terms.

Considering uniformity of distribution however, we have none except in exceptional circumstances, which is not a very good promise. In fact if we specify our algorithm a little more precisely and insist that given a permutation of length i we add a term with value $i+1$, i.e. we add a new largest term, then the potential moves of the algorithm form a tree like structure. The root, which we will regard as the bottom element of our tree, is labelled by 1, and every node labeled by $\alpha = a_1 \dots a_n$ has higher children that are labelled by those permutations of the form $a_1 \dots a_j \ (n+1) \ a_{j+1} \dots a_n$ that are in the class. Now at a given node with our algorithm we choose a child uniformly at random. However we have no guarantee in general that one child can potentially lead to as many permutations of length n as any other. Thus our statement that distribution may not be uniform.

Another fact that anyone considering use of this first method should bear in mind is that not every ascending chain of permutations in a class need have an element of length n. We may find ourselves in the circumstance of having a permutation with length less than n and to which we cannot add another term without leaving the class. What we have done here is to generate a permutation in a finite maximal atomic subclass. There is nothing we can do about this except start again, or remove a few terms from our permutation and hope thereby to get out of the dead end.

The second method is under investigation. It suffers from the difficulty that not every permutation of length n in the class can be reached from every other. The adjacency graph is undirected but split up into components in no very natural way.

The third method seems the most interesting. Although in the graph there described not all permutations of length n in the class lie in the same component, the following facts hold:

- If α and β lie in a common atomic subclass Y of X then there is a walk by deletion-insertions within Y from α to β and vice versa.

- If two atomic subclasses Y and Z of X have at least one element of length n in common then it is possible to walk by deletion insertions from any element of Y to any element of Z.

- Specifically all infinite atomic subclasses of X, and all sufficiently large atomic subclasses of X contain at least one of I_n and R_n. Thus permutations in infinite atomic subclasses of X can be found in at most two components of the deletion-insertion graph.

The following demonstrates the final niceties of the deletion-insertion method:

Given a permutation $\alpha = a_1 \dots a_n \in X$ there may exist two distinct terms a_i and a_j $(i < j)$ such that removing either from α has the same effect, in that:

$$\alpha \setminus \{a_i\} \cong \alpha \setminus \{a_j\}.$$

However it transpires that this can occur only if $a_i \dots a_j$ is an increasing or decreasing contiguous subsequence of α, that is that:

$$a_i \dots a_j = a_i \ (a_i + 1) \ (a_i + 2) \ (a_i + 3) \ \dots \ (a_j - 1) \ a_j$$

or:

$$a_i \dots a_j = a_i \ (a_i - 1) \ (a_i - 2) \ (a_i - 3) \ \dots \ (a_j + 1) \ a_j$$

which means that we can very easily prevent an algorithm from considering separately two different deletions that have the same effect: We do not permit it to delete a term a_j if $a_{j-1} = a_j \pm 1$. After all in those circumstances deleting a_{j-1} would have the same effect and it is clear that if we abide by this rule then we can never make two different deletions that have the same end result. This means that if we look at the deletion insertion graph, and specifically at a node that is a permutation of length n then it is very easy when taking a random walk to ensure that we choose each of the outgoing paths with equal probability.

Adding terms to permutations of length $n-1$, which corresponds to the other type of step in the deletion-insertion graph, is equally easy to make uniform. Whereas with deletion of a term from α, two deletions are equivalent if and only if the two deleted terms belong to the same contiguous increasing or decreasing subsequence, two possible insertions of a term into a permutation β are equivalent if and only if they insert into the same increasing or decreasing contiguous subsequence. It is a logically equivalent statement. Thus it is easy, given a permutation of length $n-1$ in the deletion-insertion graph, to ensure that all outgoing edges are chosen with equal probability. The only care that we must take is with respect to ensuring that we are still in the class X after having added the new term, which is not guaranteed in any way.

Our summary of random selection of permutations in X by deletion-insertion is that it is highly amenable to standard arguments involving Markov chains on an undirected graph. Should we ever have to make a serious analysis of random selection in closed classes we should start here.

Appendix B

Notes for Programmers

We give some comments that may be useful to anyone proposing to write programs relating to closed classes.

Involvement

Suppose that we have a permutation α of length n and a basis element β of length r and we wish to know whether the basis element is involved in the permutation. The worst case scenario is that when the permutation does not involve the basis element because there we need to keep looking until we have checked every possible way in which the basis element might be involved in α. The crude way of doing this is to look at all subsequences of length r in α and see whether any of them is order isomorphic to β. There are $\binom{n}{r}$ such subsequences, an r degree polynomial in n, so this operation can potentially be quite expensive. It is worth attempting to speed up this crude algorithm, and this can be done by a factor of $r!$ (approximately) as follows:

When choosing a subsequence of length r from α:

1. Choose the first term (from the first $n+1-r$ terms of α).

2. Choose the second term (succeeding the first but still among the first $n+2-r$ terms). Now if the subsequence we are about to generate is conceivably to be order isomorphic to β then these first two terms that we have chosen must be order isomorphic to the first two terms of β. We have just disposed of the need to continue with about half of our searches.

3. Similarly when plucking the third term (after the second, but amongst the first $n+3-r$ in α), these first three terms must as a sequence be order isomorphic to the first three terms of β. This means that of all possible choices of third term in the α subsequence, only about one third are interesting and all other cases can be discarded.

4. And so on.

Although this analysis does not take into account such things as the time taken for all the discarded searches, and the time taken to check order isomorphism of partial sequences, it can be seen approximately how this strategy leads to an $r!$ improvement.

This strategy can be expanded to the case where our basis has more than one element, as follows: Let r be the length of the longest basis element. (We do not propose checking avoidance of an infinite basis using this search on a computer!) Then:

1. Choose a first term in α (some constraints on this first term can still be made.)

2. Choose a second term from α, succeeding the first. From our entire basis discard those whose first two terms are not order isomorphic to the two terms we have selected so far.

3. Select a third term, and again discard uninteresting basis elements.

4. And so on. Stop this particular set of choices either when there are no more permutations that our subsequence might conceivably be order isomorphic to, or when our subsequence is order isomorphic to a basis element.

5. Keep testing sets of choices until either every set of choices has been tested and passed (in which case α avoids the basis), or until it is found that α involves a basis element.

This is still a substantial saving on the "crude approach".

A further variant on this checking involvement algorithm is one in which α is represented as an $n \times n$ matrix with entries of 0 or 1, instead of as a sequence of integers. In the matrix the jth column of the ith row would be one if and only if the ith term of α was j, the entry would be zero otherwise. This in principle might make it quick and easy to select terms satisfying both position and size constraints. The improvement does however not seem to be obviously substantial and so I have never implemented this.

I have involvement program implementations in S-Algol. S-Algol compilers for Macintosh are freely available, however it would be worth while converting the program into more accessible C or Java format.

Sum junction, Sum completion

A program exists that, given a class basis, calculate the basis of the sum completion of the class, providing that the sum basis is finite. The program indicates if the sum completion is not finitely based. Another program exists that calculates the sum of two classes, given their bases, failing only if the sum is infinitely based, a state which the program duly reports. These are straight implementations of the analysis in Sections 2.2.2 and 2.2.5. They are fast implementations, I doubt that they can be improved by any substantial amount. They are both long and complex. It is perhaps therefore much better for a new user simply to copy this existing program and use that, than to give a detailed analysis. The only progress that should be seriously considered is converting the program to a more accessible language.

Strong completion

I know of no implementations of programs that return the strong completion of a class, using basis as a representation. This, given the existence of the sum completion program, is purely due to lack of need. It would not require a great deal of work to extend that to deal with strong completion.

Merge, Join

Programs exist that, given two permutations, find the list of all their merges (permutations that consist of two subsequences, not necessarily disjoint, one isomorphic to each of the input permutations), and a program that returns a list of all minimal merges (merges minimal under involvement). These are useful for, amongst other things, looking for pairs of permutations that do not join in a given class. That is essentially a direct test of atomicity, but not usually a terminating test.

As usual this program is in S-Algol. A conversion to C or Java might be useful.

Generating elements of P

This is very quick and easy to program, no programmer should hesitate to at least try to enumerate the number of elements of P in a given class.

Closure generation

There exists a program that takes a permutation of arbitrary length (usually about 24) and returns the set of all permutations involved in it, sorted by length. This is a simple but somewhat expensive way of finding the basis of the closure of a given permutation or set of permutations. The nicest thing about the program is the waisted shape of the output. Here is the number of permutations of each length in one permutation of length 17:

17	1
16	11
15	61
14	218
13	549
12	1005
11	1348
10	1315
9	927
8	486
7	210
6	87
5	36
4	14
3	5
2	2
1	1

A more effective method of finding the basis of the closure of some given finite set is to generate all permutations of length n in the class and add single new largest terms to these. This generates some basis elements, some elements not in the class but also not basis elements, and all the permutations of length $n+1$ in the class. Which of these applies is done by simple comparison with the permutations of which we are taking closure and with shorter known basis elements. In spite of the large numbers involved even in this, this strategy has been more than quick enough for my uses, that usually involve finding e.g. all basis elements of length up to 9 in the closure of some permutation of length 30 or 60, for hypothesis and result testing.

S^2: Hypothetical and Real solutions.

Programs exist that test whether a given sequence is S^2 (two unsorted stacks in series) sortable, by direct attempts to sort. Programs also exist that test whether a permutation has relations H and V, described in Chapter 8. The set of permutations that pass the $V-H$ test form a superset of the S^2 sortable permutations. It is conjectured that it is an identical set, but direct computer tests cannot deal with permutations of length greater than 10. The $V-H$ testing program is (naturally) much quicker than the S^2 testing program.

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Index

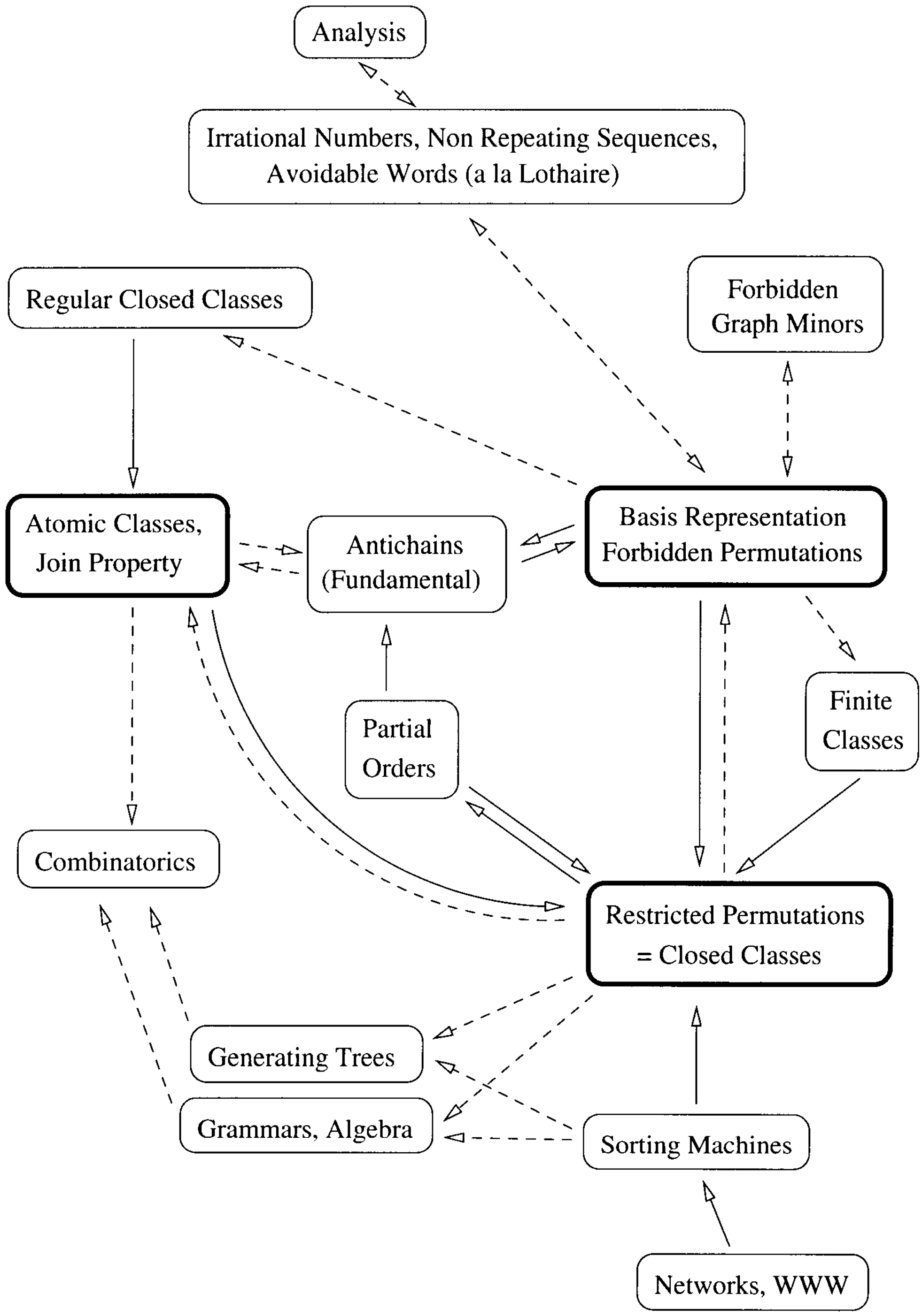

Figure B.1: Closed Classes: Neighbouring Subjects.