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AN OFFSPRING OF MULTIVARIATE EXTREME VALUE THEORY: THE MAX-CHARACTERISTIC FUNCTION

MICHAEL FALK[(1)] & GILLES STUPFLER[(2)]

Abstract. This paper introduces max-characteristic functions (max-CFs), which are an offspring of multivariate extreme-value theory. A max-CF characterizes the distribution of a random vector in $\mathbb{R}^d$, whose components are nonnegative and have finite expectation. Pointwise convergence of max-CFs is shown to be equivalent to convergence with respect to the Wasserstein metric. The space of max-CFs is not closed in the sense of pointwise convergence. An inversion formula for max-CFs is established.

1. Introduction

Multivariate extreme-value theory (MEVT) is the proper toolbox for analyzing several extremal events simultaneously. Its practical relevance in particular for risk assessment is, consequently, obvious. But on the other hand MEVT is by no means easy to access; its key results are formulated in a measure theoretic setup; a common thread is not visible.

Writing the 'angular measure' in MEVT in terms of a random vector, however, provides the missing common thread: Every result in MEVT, every relevant probability distribution, be it a max-stable one or a generalized Pareto distribution, every relevant copula, every tail dependence coefficient etc. can be formulated using a particular kind of norm on multivariate Euclidean space, called D-norm; see below. For a summary of MEVT and D-norms we refer to Falk et al. [10], Aulbach et al. [1, 2, 3, 4, 5], Falk [9]. For a review of copulas in the context of extreme-value theory, see, e.g., Genest and Nešlehová [11].

A norm $\|\cdot\|_D$ on $\mathbb{R}^d$ is a *D-norm*, if there exists a random vector (rv) $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ with $Z_i \ge 0$, $\mathrm{E}(Z_i) = 1$, $1 \le i \le d$, such that

$$\|\boldsymbol{x}\|_D = \mathrm{E}\left\{\max_{1\le i\le d}(|x_i|\, Z_i)\right\}, \qquad \boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

(1) Universität Würzburg, 97074 Würzburg, Germany

(2) School of Mathematical Sciences, The University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom

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In this case the rv $\boldsymbol{Z}$ is called *generator* of $\|\cdot\|_D$. Here is a list of D-norms and their generators:

- $\|\boldsymbol{x}\|_\infty = \max_{1\le i\le d}|x_i|$ is generated by $\boldsymbol{Z} = (1,\dots,1)$,
- $\|\boldsymbol{x}\|_1 = \sum_{i=1}^d |x_i|$ is generated by $\boldsymbol{Z}$ = random permutation of $(d,0,\dots,0)\in\mathbb{R}^d$ with equal probability $1/d$,
- $\|\boldsymbol{x}\|_\lambda = \left(\sum_{i=1}^d |x_i|^\lambda\right)^{1/\lambda}$, $1<\lambda<\infty$. Let $X_1,\dots,X_d$ be independent and identically Fréchet-distributed random variables, i.e., $\Pr(X_i\le x) = \exp(-x^{-\lambda})$, $x>0$, $\lambda>1$. Then $\boldsymbol{Z}=(Z_1,\dots,Z_d)$ with
$$Z_i = \frac{X_i}{\Gamma(1-1/\lambda)},\quad i=1,\dots,d,$$
generates $\|\cdot\|_\lambda$. By $\Gamma(p)=\int_0^\infty x^{p-1}e^{-x}\,dx$, $p>0$, we denote the usual Gamma function.

D-norms are a powerful tool when analyzing dependence in MEVT. The first letter of the word "dependence" is, therefore, the reason for the index D.

The generator of a D-norm is not uniquely determined, even its distribution is not. Let, for example, $X\ge 0$ be a random variable with $\mathrm{E}(X)=1$ and put $\boldsymbol{Z}=(X,\dots,X)$. Then $\boldsymbol{Z}$ generates $\|\cdot\|_\infty$ as well. However, we can, given a generator $\boldsymbol{Z}$ of a D-norm, design a D-norm in a simple fashion so that it characterizes the distribution of $\boldsymbol{Z}$: consider the D-norm on $\mathbb{R}^{d+1}$
$$(t,\boldsymbol{x})\mapsto \mathrm{E}\left\{\max(|t|,|x_1|Z_1,\dots,|x_d|Z_d)\right\}.$$
Then it turns out that the knowledge of this D-norm fully identifies the distribution of $\boldsymbol{Z}$; it is actually enough to know this D-norm when $t=1$, as Lemma 1.1 below shows, and this shall be the basis for our definition of a max-characteristic function.

Lemma 1.1. *Let* $\boldsymbol{X}=(X_1,\dots,X_d)\ge\boldsymbol{0}$, $\boldsymbol{Y}=(Y_1,\dots,Y_d)\ge\boldsymbol{0}$ *be random vectors with* $\mathrm{E}(X_i),\mathrm{E}(Y_i)<\infty$ *for all* $i\in\{1,\dots,d\}$. *If we have for each* $\boldsymbol{x}>\boldsymbol{0}\in\mathbb{R}^d$
$$\mathrm{E}\left\{\max(1,x_1X_1,\dots,x_dX_d)\right\} = \mathrm{E}\left\{\max(1,x_1Y_1,\dots,x_dY_d)\right\},$$
then $\boldsymbol{X}=_d\boldsymbol{Y}$, *where "$=_d$" denotes equality in distribution.*

Proof. Fubini's theorem implies $\mathrm{E}(X)=\int_0^\infty \Pr(X>t)\,dt$ for any random variable $X\ge 0$. consequently, we have for $\boldsymbol{x}>\boldsymbol{0}$ and $c>0$
$$\begin{aligned}\mathrm{E}\left\{\max\left(1,\frac{X_1}{cx_1},\dots,\frac{X_d}{cx_d}\right)\right\} &= \int_0^\infty 1-\Pr\left\{\max\left(1,\frac{X_1}{cx_1},\dots,\frac{X_d}{cx_d}\right)\le t\right\}dt\\ &= \int_0^\infty 1-\Pr(1\le t, X_i\le tcx_i,\ 1\le i\le d)\,dt\\ &= 1+\int_1^\infty 1-\Pr\left(X_i\le tcx_i,\ 1\le i\le d\right)\,dt.\end{aligned}$$

The substitution $t \mapsto t/c$ yields that the right-hand side above equals

$$1 + \frac{1}{c} \int_c^\infty 1 - \Pr(X_i \le tx_i, 1 \le i \le d)\, dt.$$

Repeating the preceding arguments with Y_i in place of X_i, we obtain for all $c > 0$ from the assumption the equality

$$\int_c^\infty 1 - \Pr(X_i \le tx_i, 1 \le i \le d)\, dt = \int_c^\infty 1 - \Pr(Y_i \le tx_i, 1 \le i \le d)\, dt.$$

Taking right derivatives with respect to c we obtain for $c > 0$

$$1 - \Pr(X_i \le cx_i,\, 1 \le i \le d) = 1 - \Pr(Y_i \le cx_i,\, 1 \le i \le d),$$

and, thus, the assertion. □

Let $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ be a random vector, whose components are nonnegative and integrable. Then we call

$$\varphi_{\boldsymbol{Z}}(\boldsymbol{x}) = \mathrm{E}\left\{\max\left(1, x_1 Z_1, \dots, x_d Z_d\right)\right\}, \quad \boldsymbol{x} = (x_1, \dots, x_d) \ge \boldsymbol{0} \in \mathbb{R}^d,$$

the *max-characteristic function* (max-CF) pertaining to $\boldsymbol{Z}$. Lemma 1.1 shows that the distribution of a nonnegative and integrable random vector $\boldsymbol{Z}$ is uniquely determined by its max-CF.

Some obvious properties of $\varphi_{\boldsymbol{Z}}$ are $\varphi_{\boldsymbol{Z}}(\boldsymbol{0}) = 1$, $\varphi_{\boldsymbol{Z}}(\boldsymbol{x}) \ge 1$ for all $\boldsymbol{x}$ and

$$\varphi_{\boldsymbol{Z}}(r\boldsymbol{x}) \begin{cases} \le r\varphi_{\boldsymbol{Z}}(\boldsymbol{x}) & \text{if } r \ge 1, \\ \ge r\varphi_{\boldsymbol{Z}}(\boldsymbol{x}) & \text{if } 0 \le r \le 1. \end{cases}$$

It is straightforward to show that any max-CF is a convex function and, thus, it is continuous and almost everywhere differentiable; besides, its derivative from the right exists everywhere. This fact will be used in Section 2.2, where we will establish an inversion formula for max-CFs.

When $\boldsymbol{Z}$ has bounded components, we have $\varphi_{\boldsymbol{Z}}(\boldsymbol{x}) = 1$ in a neighborhood of the origin. Finally, the max-CF of $\max(\boldsymbol{Z}_1, \boldsymbol{Z}_2)$ (where the max is taken componentwise) evaluated at $\boldsymbol{x}$ is equal to the max-CF of the vector $(\boldsymbol{Z}_1, \boldsymbol{Z}_2)$ evaluated at the point $(\boldsymbol{x}, \boldsymbol{x})$.

Remark 1.2. When $d = 1$, the max-CF of a nonnegative and integrable random variable Z is

$$\begin{aligned} \varphi_Z(x) = \mathrm{E}\left\{\max\left(1, xZ\right)\right\} &= 1 + \int_1^\infty \Pr(xZ > t)\, dt \\ &= 1 + x \int_{1/x}^\infty \Pr\left(Z > z\right)\, dz \\ &= 1 + x\mathrm{E}\{(Z - 1/x)\mathbf{1}_{\{Z > 1/x\}}\}. \end{aligned}$$

The latter expression is connected to the expected shortfall of Z; see Embrechts et al. [8]. Indeed, if q_Z is the quantile function of Z then the expected shortfall of Z is defined, for all $\alpha \in (0,1)$, by

$$\mathrm{ES}_Z(\alpha) = \frac{1}{1-\alpha}\int_\alpha^1 q_Z(\beta)\, d\beta.$$

When the distribution function (df) of Z is continuous, defining

$$g(\beta) = \min\left(\frac{\beta}{1-\alpha}, 1\right) = \begin{cases} \dfrac{\beta}{1-\alpha} & \text{if} \quad \beta \le 1-\alpha, \\ 1 & \text{otherwise}, \end{cases}$$

for all $\beta \in (0,1)$, then

$$\mathrm{ES}_Z(\alpha) = \frac{1}{1-\alpha}\int_0^{1-\alpha} q_Z(1-\beta)\, d\beta = \int_0^1 q_Z(1-\beta)\, dg(\beta).$$

An integration by parts and the change of variables $\beta = \Pr(Z > z)$ give

$$\begin{aligned} \mathrm{ES}_Z(\alpha) = \int_0^1 g(\beta) dq_Z(1-\beta) &= \int_0^\infty g\{\Pr(Z>z)\} dz \\ &= q_Z(\alpha) + \frac{1}{1-\alpha}\int_{q_Z(\alpha)}^\infty \Pr\left(Z > z\right)\, dz. \end{aligned}$$

A similar argument in the more general context of Wang distortion risk measures is given in El Methni and Stupfler [7]. Letting $x = x_\alpha = 1/q_Z(\alpha)$, $\alpha \in (0,1)$, we obtain

$$\varphi_Z(x_\alpha) = 1 + x_\alpha(1-\alpha)\{\mathrm{ES}_Z(\alpha) - q_Z(\alpha)\}.$$

If the stop-loss premium risk measure of Z is defined as

$$\mathrm{SP}_Z(\alpha) = (1-\alpha)\{\mathrm{ES}_Z(\alpha) - q_Z(\alpha)\} = \int_{q_Z(\alpha)}^\infty \Pr\left(Z>z\right)\, dz,$$

see Embrechts et al. [8], then

$$\varphi_Z(x_\alpha) = 1 + x_\alpha \mathrm{SP}_Z(\alpha).$$

This remark suggests that max-CFs are closely connected to well-known elementary objects such as conditional expectations and risk measures; a particular consequence of it is that computing a max-CF is, in certain cases, much easier than computing a standard CF, i.e., a *Fourier transform*. The following example illustrates this idea.

Example 1.3. Let Z be a random variable having the generalized Pareto distribution with location parameter $\mu \ge 0$, scale parameter $\sigma > 0$ and shape parameter $\xi \in (0,1)$, whose distribution function is

$$\Pr(Z \le z) = 1 - \left(1 + \xi\frac{z-\mu}{\sigma}\right)^{-1/\xi}, \qquad z \ge \mu.$$

The expression of the characteristic function of this distribution is a fairly involved one which depends on the Gamma function evaluated in the complex plane. However, it is straightforward that, for all $x > 0$,

$$\int_x^\infty \Pr(Z > z)\, dz = \begin{cases} \mathrm{E}(Z) - x = \mu - x + \dfrac{\sigma}{1-\xi} & \text{if } x < \mu, \\ \dfrac{\sigma}{1-\xi}\left(1 + \xi \dfrac{x-\mu}{\sigma}\right)^{1-1/\xi} & \text{if } x \geq \mu. \end{cases}$$

Hence the max-CF of Z is

$$\varphi_Z(x) = \begin{cases} x\mathrm{E}(Z) = x\left(\mu + \dfrac{\sigma}{1-\xi}\right) & \text{if } x > \dfrac{1}{\mu}, \\ 1 + \dfrac{\sigma x}{1-\xi}\left(1 + \xi \dfrac{1-\mu x}{\sigma x}\right)^{1-1/\xi} & \text{if } x \leq \dfrac{1}{\mu}. \end{cases}$$

The following example is a consequence of the Pickands–de Haan–Resnick representation of a max-stable distribution function; see, e.g., Falk et al. [10, Theorems 4.2.5, 4.3.1]. In this paper, all operations on vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ such as $\boldsymbol{x} + \boldsymbol{y}$, $\boldsymbol{x}/\boldsymbol{y}$, $\boldsymbol{x} \leq \boldsymbol{y}$, $\max(\boldsymbol{x}, \boldsymbol{y})$ etc. are always meant componentwise.

Example 1.4. Let G be a d-dimensional max-stable distribution function with identical univariate Fréchet-margins $G_i(x) = \exp(-x^{-\alpha})$, $x > 0$, $\alpha > 1$. Then there exists a D-norm $\|\cdot\|_D$ on $\mathbb{R}^d$ such that $G(\boldsymbol{x}) = \exp\left(-\|1/\boldsymbol{x}^\alpha\|_D\right)$, $\boldsymbol{x} > \boldsymbol{0} \in \mathbb{R}^d$. Let the random vector $\boldsymbol{\xi}$ have distribution function G. Its max-CF is

$$\begin{aligned} \varphi_{\boldsymbol{\xi}}(\boldsymbol{x}) &= 1 + \int_1^\infty 1 - \exp\left(-\frac{\|\boldsymbol{x}^\alpha\|_D}{y^\alpha}\right) dy \\ &= 1 + \|\boldsymbol{x}^\alpha\|_D^{1/\alpha} \int_{1/\|\boldsymbol{x}^\alpha\|_D^{1/\alpha}} 1 - \exp(-y^{-\alpha})\, dy, \qquad \boldsymbol{x} \geq \boldsymbol{0} \in \mathbb{R}^d. \end{aligned}$$

This paper is organized as follows. In Section 2 we establish among others the fact that pointwise convergence of max-CFs is equivalent to convergence with respect to the Wasserstein distance. In Section 2.1 we list some general remarks on max-CFs. In particular, it is shown that the space of max-CFs is not closed in the sense of pointwise convergence. An inversion formula for max-CF, by which the distribution function of a nonnegative and integrable random variable can be restored by knowing its max-CF, is established in Section 2.2.

2. Convergence of max-characteristic functions

Denote by $d_W(P, Q)$ the Wasserstein metric between two probability distributions on $\mathbb{R}^d$ with finite first moments, i.e.,

$$d_W(P, Q) = \inf\left\{\mathrm{E}\left(\|\boldsymbol{X} - \boldsymbol{Y}\|_1\right) : \boldsymbol{X} \text{ has distribution } P,\ \boldsymbol{Y} \text{ has distribution } Q\right\}.$$

It is well known that convergence of probability measures P_n to P_0 with respect to the Wasserstein metric is equivalent to weak convergence together with convergence of the sequence of moments

$$\int_{\mathbb{R}^d} \|\boldsymbol{x}\|_1 \ P_n(d\boldsymbol{x}) \to_{n\to\infty} \int_{\mathbb{R}^d} \|\boldsymbol{x}\|_1 \ P_0(d\boldsymbol{x});$$

see, for example, Definition 6.8 of Villani [12].

Let $\boldsymbol{X}, \boldsymbol{Y}$ be integrable random vectors in $\mathbb{R}^d$ with distributions P and Q. By $d_W(\boldsymbol{X}, \boldsymbol{Y}) = d_W(P, Q)$ we denote the Wasserstein distance between $\boldsymbol{X}$ and $\boldsymbol{Y}$. The next result states that pointwise convergence of max-CFs is equivalent to convergence with respect to the Wasserstein metric.

Theorem 2.1. *Let $\boldsymbol{Z}$, $\boldsymbol{Z}^{(n)}$, $n \in \mathbb{N}$, be nonnegative and integrable random vectors in $\mathbb{R}^d$ with corresponding max-CF $\varphi_{\boldsymbol{Z}}$, $\varphi_{\boldsymbol{Z}^{(n)}}$, $n \in \mathbb{N}$. Then $\varphi_{\boldsymbol{Z}^{(n)}} \to_{n\to\infty} \varphi_{\boldsymbol{Z}}$ pointwise $\Leftrightarrow d_W\left(\boldsymbol{Z}^{(n)}, \boldsymbol{Z}\right) \to_{n\to\infty} 0$.*

Proof. Suppose that $d_W(\boldsymbol{Z}^{(n)}, \boldsymbol{Z}) \to_{n\to\infty} 0$. Then we can find versions $\boldsymbol{Z}^{(n)}$, $\boldsymbol{Z}$ such that $\mathrm{E}\left(\left\|\boldsymbol{Z}^{(n)} - \boldsymbol{Z}\right\|_1\right) \to_{n\to\infty} 0$. This implies, for $\boldsymbol{x} = (x_1, \dots, x_d) \geq 0$,

$$\begin{aligned}
\varphi_{\boldsymbol{Z}^{(n)}}(\boldsymbol{x}) &= \mathrm{E}\left(\max[1, x_1\{Z_1 + (Z_1^{(n)} - Z_1)\}, \dots, x_d\{Z_d + (Z_d^{(n)} - Z_d)\}]\right) \\
&\begin{cases} \leq \mathrm{E}\{\max(1, x_1 Z_1, \dots, x_d Z_d)\} + \|\boldsymbol{x}\|_\infty \mathrm{E}\left(\|\boldsymbol{Z}^n - \boldsymbol{Z}\|_1\right) \\ \geq \mathrm{E}\left\{\max(1, x_1 Z_1, \dots, x_d Z_d)\right\} - \|\boldsymbol{x}\|_\infty \mathrm{E}\left(\|\boldsymbol{Z}^n - \boldsymbol{Z}\|_1\right) \end{cases} \\
&= \varphi_{\boldsymbol{Z}}(\boldsymbol{x}) + o(1).
\end{aligned}$$

Suppose next that $\varphi_{\boldsymbol{Z}^{(n)}} \to_{n\to\infty} \varphi_{\boldsymbol{Z}}$ pointwise. We have for $t > 0$ and $\boldsymbol{x} = (x_1, \dots, x_d) \geq \boldsymbol{0}$

$$t\varphi_{\boldsymbol{Z}^{(n)}}\left(\frac{\boldsymbol{x}}{t}\right) = \mathrm{E}\{\max(t, x_1 Z_1^{(n)}, \dots, x_d Z_d^{(n)})\}.$$

This gives

$$\begin{aligned}
t\varphi_{\boldsymbol{Z}^{(n)}}\left(\frac{\boldsymbol{x}}{t}\right) &= \int_0^{+\infty} \Pr\{\max(t, x_1 Z_1^{(n)}, \dots, x_d Z_d^{(n)}) > y\} dy \\
&= t + \int_t^{+\infty} \Pr\{\max(x_1 Z_1^{(n)}, \dots, x_d Z_d^{(n)}) > y\} dy
\end{aligned}$$

so that

$$t\varphi_{\boldsymbol{Z}^{(n)}}\left(\frac{\boldsymbol{x}}{t}\right) = t + \int_t^{\infty} 1 - \Pr(x_i Z_i^{(n)} \leq y, 1 \leq i \leq d)\, dy. \tag{1}$$

Now, for $\varepsilon > 0$ and $1 \leq i \leq d$

$$\begin{aligned}
\mathrm{E}(Z_i^{(n)}) - \mathrm{E}\,(Z_i) &= \int_0^{\infty} 1 - \Pr(Z_i^{(n)} \leq y)\, dy - \int_0^{\infty} 1 - \Pr\,(Z_i \leq y)\ dy \\
&= \int_{\varepsilon/2}^{\infty} 1 - \Pr(Z_i^{(n)} \leq y)\, dy - \int_{\varepsilon/2}^{\infty} 1 - \Pr\,(Z_i \leq y)\ dy + R_{n,i}(\varepsilon)
\end{aligned}$$

where

$$|R_{n,i}(\varepsilon)| = \left| \int_0^{\varepsilon/2} \Pr(Z_i^{(n)} \le y) - \Pr\left(Z_i \le y\right) \, dy \right| \le \varepsilon/2.$$

Equation (1) then gives

$$\left| \mathrm{E}(Z_i^{(n)}) - \mathrm{E}\left(Z_i\right) \right| \le \varepsilon \quad \text{for large enough } n,$$

which entails convergence of $\mathrm{E}(Z_i^{(n)})$ to $\mathrm{E}\left(Z_i\right)$. Consequently, we have to establish weak convergence of $\boldsymbol{Z}^{(n)}$ to $\boldsymbol{Z}$. From Equation (1) we obtain for $0 < s < t$ and $\boldsymbol{x} = (x_1, \dots, x_d) \ge 0$

$$\begin{aligned} t\varphi_{\boldsymbol{Z}^{(n)}}\left(\frac{\boldsymbol{x}}{t}\right) - s\varphi_{\boldsymbol{Z}^{(n)}}\left(\frac{\boldsymbol{x}}{s}\right) &= \int_s^t \Pr(x_i Z_i^{(n)} \le y, 1 \le i \le d) \, dy \\ &\to_{n\to\infty} t\varphi_{\boldsymbol{Z}}\left(\frac{\boldsymbol{x}}{t}\right) - s\varphi_{\boldsymbol{Z}}\left(\frac{\boldsymbol{x}}{s}\right) \\ &= \int_s^t \Pr\left(x_i Z_i \le y, 1 \le i \le d\right) \, dy. \end{aligned} \tag{2}$$

Let $\boldsymbol{x} = (x_1, \dots, x_d) \ge 0$ be a point of continuity of the distribution function of $\boldsymbol{Z}$. Suppose first that $\boldsymbol{x} > 0$. Then we have

$$\Pr(\boldsymbol{Z}^{(n)} \le \boldsymbol{x}) = \Pr\left(\frac{1}{x_i} Z_i^{(n)} \le 1, 1 \le i \le d\right).$$

If

$$\limsup_{n\to\infty} \Pr\left(\frac{1}{x_i} Z_i^{(n)} \le 1, 1 \le i \le d\right) > \Pr\left(\frac{1}{x_i} Z_i \le 1, 1 \le i \le d\right)$$

or

$$\liminf_{n\to\infty} \Pr\left(\frac{1}{x_i} Z_i^{(n)} \le 1, 1 \le i \le d\right) < \Pr\left(\frac{1}{x_i} Z_i \le 1, 1 \le i \le d\right),$$

then Equation (2) readily produces a contradiction by putting $s = 1$ and $t = 1 + \varepsilon$ or $t = 1$ and $s = 1 - \varepsilon$ with a small $\varepsilon > 0$. We, thus, have

$$\Pr(\boldsymbol{Z}^{(n)} \le \boldsymbol{x}) \to_{n\to\infty} \Pr\left(\boldsymbol{Z} \le \boldsymbol{x}\right) \tag{3}$$

for each point of continuity $\boldsymbol{x} = (x_1, \dots, x_d)$ of the distribution function of $\boldsymbol{Z}$ with strictly positive components.

Suppose next that $x_j = 0$ for $j \in T \subset \{1, \dots, d\}$, $x_i > 0$ for $i \notin T$, $T \ne \emptyset$. In this case we have

$$\Pr(\boldsymbol{Z} \le \boldsymbol{x}) = \Pr(Z_i \le x_i, i \notin T, Z_j \le 0, j \in T) = 0$$

by the continuity from the left of the distribution function of $\boldsymbol{Z}$ at $\boldsymbol{x}$. We thus have to establish

$$\limsup_{n\to\infty} \Pr(\boldsymbol{Z}^{(n)} \le \boldsymbol{x}) = \limsup_{n\to\infty} \Pr\left(Z_i^{(n)} \le x_i, i \notin T, Z_j^{(n)} \le 0, j \in T\right) = 0.$$

Suppose that

$$\limsup_{n\to\infty} \Pr\left(Z_i^{(n)} \le x_i, i \notin T, Z_j^{(n)} \le 0, j \in T\right) = c > 0.$$

Choose a point of continuity $\boldsymbol{y} > \boldsymbol{x}$. Then we obtain

$$0 < c \le \limsup_{n\to\infty} \Pr(\boldsymbol{Z}^{(n)} \le \boldsymbol{y}) = \Pr(\boldsymbol{Z} \le \boldsymbol{y})$$

by Equation (3). Letting $\boldsymbol{y}$ converge to $\boldsymbol{x}$ we obtain $\Pr(\boldsymbol{Z} \le \boldsymbol{x}) \ge c > 0$ and, thus, a contradiction. This completes the proof of Theorem 2.1. □

Convergence of a sequence of max-CFs is therefore stronger than the convergence of standard CFs: the example of a sequence of real-valued random variables (Z_n) such that

$$\Pr(Z_n = e^n) = \frac{1}{n} \quad \text{and} \quad \Pr(Z_n = 0) = 1 - \frac{1}{n}$$

is such that $Z_n \to 0$ in distribution, as can be seen from computing the related sequence of CFs, but $\mathrm{E}(Z_n) = e^n/n \to \infty \neq 0$.

Corollary 2.2 below, which is obtained by simply rewriting Theorem 2.1, is tailored to applications to MEVT.

Corollary 2.2. *Let $\boldsymbol{X}^{(n)}$, $n \in \mathbb{N}$, be independent copies of a random vector $\boldsymbol{X}$ in $\mathbb{R}^d$ that is nonnegative and integrable in each component. Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)$ be a max-stable random vector with Fréchet margins $\Pr(\xi_i \le x) = \exp(-1/x^{\alpha_i})$, $x > 0$, $\alpha_i > 1$, $1 \le i \le d$. Then we obtain from Theorem 2.1 the equivalence*

$$d_W\left(\frac{\max_{1\le i\le n} \boldsymbol{X}^{(i)}}{\boldsymbol{a}^{(n)}}, \boldsymbol{\xi}\right) \to_{n\to\infty} 0$$

for some norming sequence $\boldsymbol{0} < \boldsymbol{a}^{(n)} \in \mathbb{R}^d$ if and only if

$$\varphi_n \to_{n\to\infty} \varphi_{\boldsymbol{\xi}} \qquad \textit{pointwise},$$

where φ_n denotes the max-CF of $\max_{1\le i\le n} \boldsymbol{X}^{(i)}/\boldsymbol{a}^{(n)}$, $n \in \mathbb{N}$.

The following example shows a nice application of the use of max-CFs to the convergence of the componentwise maxima of independent generalized Pareto random vector in the total variation distance.

Example 2.3. Let U be a random variable that is uniformly distributed on $(0,1)$ and let $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ be the generator of a D-norm $\|\cdot\|_D$ with the additional property that each Z_i is bounded, i.e., $Z_i \le c$, $1 \le i \le d$, for some constant $c \ge 1$. We require that U and $\boldsymbol{Z}$ are independent.

Then the random vector

$$\boldsymbol{V} = (V_1, \dots, V_d) = \frac{1}{U^{1/\alpha}}(Z_1^{1/\alpha}, \dots, Z_d^{1/\alpha})$$

with $\alpha > 0$ follows a *multivariate generalized Pareto distribution*; see, e.g., Buishand et al. [6] or Falk et al. [10, Chapter 5]. Precisely, we have for $\boldsymbol{x} \geq (c^{1/\alpha}, \ldots, c^{1/\alpha}) \in \mathbb{R}^d$

$$\Pr(\boldsymbol{V} \leq \boldsymbol{x}) = \Pr\left(U \geq \max_{1 \leq i \leq d} \frac{Z_i}{x_i^\alpha}\right) = 1 - \mathrm{E}\left(\max_{1 \leq i \leq d} \frac{Z_i}{x_i^\alpha}\right) = 1 - \left\| \frac{1}{\boldsymbol{x}^\alpha} \right\|_D .$$

Let now $\boldsymbol{V}^{(1)}, \boldsymbol{V}^{(2)}, \ldots$ be independent copies of $\boldsymbol{V}$ and put

$$\boldsymbol{Y}^{(n)} = \frac{\max_{1 \leq i \leq n} \boldsymbol{V}^{(i)}}{n^{1/\alpha}}.$$

Then we have for $\boldsymbol{x} > \boldsymbol{0} \in \mathbb{R}^d$ and n large

$$\Pr(\boldsymbol{Y}^{(n)} \leq \boldsymbol{x}) = \left(1 - \left\| \frac{1}{n\boldsymbol{x}^\alpha} \right\|_D \right)^n \to_{n\to\infty} \exp\left(- \left\| \frac{1}{\boldsymbol{x}^\alpha} \right\|_D \right) = \Pr(\boldsymbol{\xi} \leq \boldsymbol{x}), \tag{4}$$

where $\boldsymbol{\xi}$ is a max-stable random vector with identical Fréchet margins $\Pr(\xi_i \leq x) = \exp(-1/x^\alpha)$, $x > 0$. Choose $\alpha > 1$; in this case the components of $\boldsymbol{V}$ and $\boldsymbol{\xi}$ have finite expectations. By writing

$$\varphi_{\boldsymbol{Y}^{(n)}}(\boldsymbol{x}) = 1 + \int_1^\infty 1 - \Pr(\boldsymbol{Y}^{(n)} \leq t/\boldsymbol{x})\, dt$$

and using Equation (4), elementary arguments such as a Taylor expansion make it possible to show that the sequence of max-CF $\varphi_{\boldsymbol{Y}^{(n)}}$ converges pointwise to the max-CF $\varphi_{\boldsymbol{\xi}}$ of $\boldsymbol{\xi}$. Since convergence with respect to the Wasserstein metric is equivalent to convergence in distribution, denoted by $\to_d$, together with convergence of the moments, we obtain from Theorem 2.1 that in this example we actually have both $\boldsymbol{Y}^{(n)} \to_d \boldsymbol{\xi}$ and $\mathrm{E}(Y_i^{(n)}) \to_{n\to\infty} \mathrm{E}(\xi_i) = \Gamma(1 - 1/\alpha)$ for $1 \leq i \leq d$.

Example 2.4. Let $\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}, \ldots$ be independent copies of the random vector $\boldsymbol{U} = (U_1, \ldots, U_d)$, which follows a copula C on $\mathbb{R}^d$, i.e., each U_i is uniformly distributed on $(0, 1)$. It is well-known (see, e.g., Falk et al. [10, Section 5.2]) that there exists a non-degenerate random vector $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d)$ on $(-\infty, 0]^d$ such that

$$\boldsymbol{V}^{(n)} = n\left(\max_{1 \leq j \leq n} \boldsymbol{U}^{(j)} - \boldsymbol{1}\right) \to_d \boldsymbol{\eta}$$

if and only if there exists a D-norm $\|\cdot\|_D$ on $\mathbb{R}^d$ such that, for all $\boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$,

$$\Pr(\boldsymbol{V}^{(n)} \leq \boldsymbol{x}) \to_{n\to\infty} \exp\left(- \|\boldsymbol{x}\|_D\right) = G(\boldsymbol{x}),$$

or if and only if there exists a D-norm $\|\cdot\|_D$ on $\mathbb{R}^d$ such that

$$C(\boldsymbol{u}) = 1 - \|\boldsymbol{1} - \boldsymbol{u}\|_D + o\left(\|\boldsymbol{1} - \boldsymbol{u}\|\right)$$

as $\boldsymbol{u} \to \boldsymbol{1}$, uniformly for $\boldsymbol{u} \in [0, 1]^d$.

We have for $1 \leq i \leq d$

$$\mathrm{E}\left\{n\left(1 - \max_{1 \leq j \leq n} U_i^{(j)}\right)\right\} = \frac{n}{n+1} \to_{n\to\infty} 1$$

and, thus, we obtain from Theorem 2.1 the characterization

$$\boldsymbol{V}^{(n)} \to_d \boldsymbol{\eta} \Leftrightarrow d_W\left(\boldsymbol{V}^{(n)}, \boldsymbol{\eta}\right) \to_{n\to\infty} 0 \Leftrightarrow \varphi_{-\boldsymbol{V}^{(n)}} \to_{n\to\infty} \varphi_{-\boldsymbol{\eta}} \text{ pointwise.}$$

For instance, when $d = 2$, straightforward computations yield that $-\boldsymbol{\eta}$ arises as a weak limit above if and only if it has a max-CF of the form

$$\varphi_{-\boldsymbol{\eta}}(\boldsymbol{x}) = 1 + x_1 \exp(-1/x_1) + x_2 \exp(-1/x_2) - \frac{1}{\|1/\boldsymbol{x}\|_D} \exp(-\|1/\boldsymbol{x}\|_D).$$

Corollary 2.5. *Let $\boldsymbol{Z}$, $\boldsymbol{Z}^{(n)}$, $n \in \mathbb{N}$, be generators of D-norms on $\mathbb{R}^d$. Then $\varphi_{\boldsymbol{Z}^{(n)}} \to_{n\to\infty} \varphi_{\boldsymbol{Z}}$ pointwise $\Leftrightarrow \boldsymbol{Z}^{(n)} \to_d \boldsymbol{Z}$.*

Interestingly, the convergence of a sequence of max-CFs of generators of D-norms also implies pointwise convergence of the related D-norms. We denote by $\|\cdot\|_{D,\boldsymbol{Z}}$ that D-norm, which is generated by $\boldsymbol{Z}$.

Corollary 2.6. *Let $\boldsymbol{Z}$, $\boldsymbol{Z}^{(n)}$, $n \in \mathbb{N}$, be generators of D-norms in $\mathbb{R}^d$ with respective max-CF $\varphi_{\boldsymbol{Z}}$, $\varphi_{\boldsymbol{Z}^{(n)}}$, $n \in \mathbb{N}$. Then the pointwise convergence $\varphi_{\boldsymbol{Z}^{(n)}} \to_{n\to\infty} \varphi_{\boldsymbol{Z}}$ implies $\|\cdot\|_{D,\boldsymbol{Z}^{(n)}} \to_{n\to\infty} \|\cdot\|_{D,\boldsymbol{Z}}$ pointwise.*

Proof. We have for $\boldsymbol{x} = (x_1, \dots, x_d) \geq \boldsymbol{0}$

$$\begin{aligned}
\|\boldsymbol{x}\|_{D^{(n)}} &= \mathrm{E}\left\{\max_{1\le i\le d}\left(x_i Z_i^{(n)}\right)\right\} = \mathrm{E}\left[\max_{1\le i\le d}\left\{x_i Z_i + x_i\left(Z_i^{(n)} - Z_i\right)\right\}\right] \\
&= \mathrm{E}\left\{\max_{1\le i\le d}(x_i Z_i)\right\} + O\left\{\mathrm{E}\left(\left\|\boldsymbol{Z}^{(n)} - \boldsymbol{Z}\right\|_1\right)\right\} \\
&\to_{n\to\infty} \mathrm{E}\left\{\max_{1\le i\le d}(x_i Z_i)\right\} = \|\boldsymbol{x}\|_D
\end{aligned}$$

with proper versions of $\boldsymbol{Z}^{(n)}$ and $\boldsymbol{Z}$. □

2.1. **Some general remarks on max-characteristic functions.** The goal of this section is to give a few elements about the structure of the set of max-characteristic functions. This is done by constructing a particular functional mapping between max-CFs for generators of D-norms, and then iterating this mapping to draw our conclusions. Specifically, in what follows we let, for any $p \in (0, 1]$, T_p be the functional mapping which sends any function $f : \mathbb{R}^d \to \mathbb{R}$ to

$$T_p(f) = 1 - p + pf\left(\frac{\cdot}{p}\right).$$

Lemma 2.7. *If φ is the max-CF of a generator of a D-norm then, for any $p \in (0, 1]$, so is the function $T_p(\varphi)$.*

Proof. Let $\boldsymbol{Z}$ be a generator of the max-CF φ. Pick a Bernoulli random variable U having expectation p and independent of $\boldsymbol{Z}$, and set

$$\psi(\boldsymbol{x}) = \mathrm{E}\left\{\max\left(1, x_1\frac{U}{p}Z_1, \dots, x_d\frac{U}{p}Z_d\right)\right\}.$$

Then clearly ψ is the max-CF of the generator of a D-norm, and

$$\begin{aligned}\psi(\boldsymbol{x}) &= \mathrm{E}\left\{\mathbf{1}_{\{U=0\}} + \max\left(1, \frac{x_1}{p}Z_1, \dots, \frac{x_d}{p}Z_d\right)\mathbf{1}_{\{U=1\}}\right\}\\ &= \Pr(U=0) + \Pr(U=1)\mathrm{E}\left\{\max\left(1, \frac{x_1}{p}Z_1, \dots, \frac{x_d}{p}Z_d\right)\right\}\end{aligned}$$

by the independence of U and Z. The result follows because of the right-hand side being exactly $1-p+p\varphi(\boldsymbol{x}/p)$. □

Lemma 2.8. *For any integer $k \geq 1$, the kth iterate of the functional T_p is*

$$f \mapsto T_p^{(k)}(f) = \Pr(X \leq k) + p^k f\left(\frac{\cdot}{p^k}\right),$$

where X is a geometric random variable having parameter $1-p$.

Proof. The result is clearly true for $k=1$. That the conclusion holds for every integer k follows by straightforward induction because

$$(1-p)+p\Pr(X \leq k) = (1-p)+p\sum_{j=1}^{k} p^{j-1}(1-p) = \sum_{j=1}^{k+1} p^{j-1}(1-p) = \Pr(X \leq k+1)$$

whenever X has a geometric distribution with parameter $1-p$. □

In the following lemma, the phrase "$\boldsymbol{x} \to \infty$ in $\mathbb{R}_+^d$" means $\|\boldsymbol{x}\|_\infty \to \infty$ and $\boldsymbol{x} \in \mathbb{R}_+^d$.

Lemma 2.9. *If $\varphi_{\boldsymbol{Z}}$ is the max-CF of a generator $\boldsymbol{Z}$ of a D-norm, then*

$$\max(1, \|\boldsymbol{x}\|_{D,Z}) \leq \varphi_{\boldsymbol{Z}}(\boldsymbol{x}) \leq 1 + \|\boldsymbol{x}\|_{D,\boldsymbol{Z}} \quad \textit{for all} \quad \boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}_+^d.$$

Especially, if $\mathcal{G}$ denotes the set of all generators of D-norms,

$$\sup_{\boldsymbol{Z}\in\mathcal{G}}\left|\frac{\varphi_{\boldsymbol{Z}}(\boldsymbol{x})}{\|\boldsymbol{x}\|_{D,\boldsymbol{Z}}} - 1\right| \to 0 \quad \textit{as} \quad \boldsymbol{x} \to \infty \ \textit{in}\ \mathbb{R}_+^d.$$

Proof. The lower bound is obtained by noting that

$$(5)\quad 1 \leq \max(1, x_1Z_1, \dots, x_dZ_d), \quad \max(x_1Z_1, \dots, x_dZ_d) \leq \max(1, x_1Z_1, \dots, x_dZ_d)$$

and taking expectations. The upper bound is a consequence of the inequality $\max(a,b) \leq a+b$, valid when $a, b \geq 0$. Finally, the uniform convergence result is obtained by writing

$$1 \leq \frac{\varphi_{\boldsymbol{Z}}(\boldsymbol{x})}{\|\boldsymbol{x}\|_{D,\boldsymbol{Z}}} \leq 1 + \frac{1}{\|\boldsymbol{x}\|_{D,\boldsymbol{Z}}} \quad \text{for all} \quad \boldsymbol{Z} \in \mathcal{G} \text{ and } \boldsymbol{x} \in \mathbb{R}_+^d \setminus \{\mathbf{0}\}.$$

Because $\|\cdot\|_{D,\boldsymbol{Z}} \geq \|\cdot\|_\infty$, this entails

$$\sup_{\boldsymbol{Z}\in\mathcal{G}} \left| \frac{\varphi_{\boldsymbol{Z}}(\boldsymbol{x})}{\|\boldsymbol{x}\|_{D,\boldsymbol{Z}}} - 1 \right| \leq \frac{1}{\|\boldsymbol{x}\|_\infty} \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}_+^d \setminus \{\boldsymbol{0}\}$$

from which the conclusion follows. □

It is noteworthy that the inequalities of Lemma 2.9 are sharp, in the sense that for $\boldsymbol{Z} = (1, \ldots, 1)$, $\varphi_{\boldsymbol{Z}}(\boldsymbol{x}) = \max(1, \|\boldsymbol{x}\|_\infty) = \max(1, \|\boldsymbol{x}\|_{D,Z})$ and therefore the leftmost inequality is in fact an equality in this case, while the rightmost inequality $\varphi_{\boldsymbol{Z}}(\boldsymbol{x}) \leq a + b\|\boldsymbol{x}\|_{D,\boldsymbol{Z}}$ can only be true if $a, b \geq 1$ because of the leftmost inequality again.

Lemma 2.9 has the following corollary, which can also be obtained as a consequence of the monotone convergence theorem.

Corollary 2.10. *No constant function can be the max-CF of a generator of a D-norm.*

Such a result is of course not true for standard CFs, since the CF of the constant random variable 0 is the constant function 1.

The next result looks at what can be said when examining the pointwise limit of iterates of the functional T_p on the set of max-CFs.

Proposition 2.11. *If $\varphi_{\boldsymbol{Z}}$ is the max-CF of a generator $\boldsymbol{Z}$ of a D-norm, then for any $p \in (0, 1)$, the sequence of mappings $\{T_p^{(k)}(\varphi_{\boldsymbol{Z}})\}$ has a pointwise limit which is independent of p and equal to*

$$T(\varphi_{\boldsymbol{Z}}) = 1 + \|\cdot\|_{D,\boldsymbol{Z}}.$$

Proof. By Lemma 2.8, we have for any $\boldsymbol{x} \in \mathbb{R}_+^d$, $\boldsymbol{x} \neq \boldsymbol{0} \in \mathbb{R}^d$, and any $k \geq 1$ that

$$T_p^{(k)}(\varphi)(\boldsymbol{x}) = \Pr(X \leq k) + p^k \varphi_{\boldsymbol{Z}}\left(\frac{\boldsymbol{x}}{p^k}\right).$$

On one hand, when $k \to \infty$, the first term on the right-hand side converges to 1; on the other hand, because $p \in (0, 1)$, we have $\boldsymbol{x}/p^k \to \infty$ in $\mathbb{R}_+^d$ and therefore

$$\lim_{k\to\infty} p^k \varphi_{\boldsymbol{Z}}\left(\frac{\boldsymbol{x}}{p^k}\right) = \|\boldsymbol{x}\|_{D,\boldsymbol{Z}} \lim_{k\to\infty} \frac{\varphi_{\boldsymbol{Z}}(\boldsymbol{x}/p^k)}{\|\boldsymbol{x}/p^k\|_{D,\boldsymbol{Z}}} = \|\boldsymbol{x}\|_{D,\boldsymbol{Z}}$$

by Lemma 2.9. The conclusion follows by adding these limits. □

Corollary 2.12. *If $\|\cdot\|_{D,\boldsymbol{Z}}$ is any $D-$norm then there is an explicit, iterative way to realize the function $1 + \|\cdot\|_{D,\boldsymbol{Z}}$ as a limit of max-CFs. In particular, the expression of a $D-$norm is explicitly determined by the knowledge of the max-CF of any of its generators.*

Note that this result certainly cannot be true the other way around, since a single $D-$norm can in general be generated by different generators.

The next result looks a bit further into the range of the map $\boldsymbol{Z} \mapsto \varphi_{\boldsymbol{Z}}$. By considering the generator $(1, \dots, 1) \in \mathbb{R}^d$ that generates the D-norm $\|\cdot\|_\infty$, it is obvious that $\max(1, \|\cdot\|_\infty)$ is actually the max-CF of a generator of a D-norm. Looking at Lemma 2.9, one may wonder if this remains true if $\|\cdot\|_\infty$ is replaced by some other $D-$norm, or, in other words, if the lower bound $\max(1, \|\cdot\|_{D,Z})$ in Lemma 2.9 can be achieved as a $D-$norm, and similarly for the upper bound $1 + \|\cdot\|_{D,Z}$. The next result says that this is not the case.

Proposition 2.13. *Let $\boldsymbol{Z}$ be a generator of a D-norm.*

(i) The mapping $1+\|\cdot\|_{D,\boldsymbol{Z}}$ cannot be the max-CF of a generator of a D-norm.

(ii) If moreover $\|\cdot\|_{D,\boldsymbol{Z}} \neq \|\cdot\|_\infty$, then $\max(1, \|\cdot\|_{D,\boldsymbol{Z}})$ cannot be the max-CF of a generator of a D-norm.

Proof. We start by proving (i). Suppose there is a generator of a D-norm $\boldsymbol{Y}$ such that $\varphi_{\boldsymbol{Y}} = 1 + \|\cdot\|_{D,\boldsymbol{Z}}$. By Proposition 2.11, the sequence of mappings $T_p^{(k)}(\varphi_{\boldsymbol{Y}})$, $k \geq 1$, has the pointwise limit

$$T(\varphi_{\boldsymbol{Y}}) = 1 + \|\cdot\|_{D,\boldsymbol{Y}}.$$

Besides, if X is a geometric random variable with parameter $1-p$, then for all $\boldsymbol{x} \in \mathbb{R}_+^d$

$$T_p^{(k)}(\varphi_{\boldsymbol{Y}})(\boldsymbol{x}) = T_p^{(k)}(1+\|\cdot\|_{D,\boldsymbol{Z}})(\boldsymbol{x}) = \Pr(X \leq k) + p^k(1+\|\boldsymbol{x}/p^k\|_{D,\boldsymbol{Z}}) \to 1+\|\boldsymbol{x}\|_{D,\boldsymbol{Z}}$$

as $k \to \infty$, so that $\|\cdot\|_{D,\boldsymbol{Y}} = \|\cdot\|_{D,\boldsymbol{Z}}$.

We now conclude by using Theorem 2.1: the random vector

$$\boldsymbol{Y}^{(n)} = \frac{U_1 \cdots U_n}{p^n} \mathbf{Y}$$

where $U_1, \dots, U_n$ are independent Bernoulli random variables with mean p which are independent of $\mathbf{Y}$, is the generator of a D-norm, with max-CF

$$\varphi_{\boldsymbol{Y}^{(n)}} = T_p^{(n)}(\varphi_{\boldsymbol{Y}});$$

see the proof of Lemma 2.7 and Lemma 2.8. By Proposition 2.11, $\varphi_{\boldsymbol{Y}^{(n)}} \to_{n\to\infty} 1 + \|\cdot\|_{D,\boldsymbol{Y}} = \varphi_{\boldsymbol{Y}}$ pointwise, and thus Theorem 2.1 yields $d_W\left(\boldsymbol{Y}^{(n)}, \boldsymbol{Y}\right) \to_{n\to\infty} 0$. But

$$\Pr(\boldsymbol{Y}^{(n)} \neq \mathbf{0}) = p^n \to_{n\to\infty} 0$$

which shows that $\boldsymbol{Y}^{(n)}$ converges in distribution to $\mathbf{0}$. This is a contradiction and (i) is proven.

We turn to the proof of (ii). Again, suppose there is a generator of a D-norm $\boldsymbol{Y}$ such that $\varphi_{\boldsymbol{Y}} = \max(1, \|\cdot\|_{D,\boldsymbol{Z}})$. We shall prove that $\|\cdot\|_{D,\boldsymbol{Z}} = \|\cdot\|_\infty$. The sequence of mappings $T_p^{(k)}(\varphi_{\boldsymbol{Y}})$, $k \geq 1$, has the pointwise limit

$$T(\varphi_{\boldsymbol{Y}}) = 1 + \|\cdot\|_{D,\boldsymbol{Y}},$$

and if X is a geometric random variable with parameter $1-p$ then for all $\boldsymbol{x} \in \mathbb{R}_+^d$,

$$\begin{aligned} T_p^{(k)}(\varphi_{\boldsymbol{Y}})(\boldsymbol{x}) &= T_p^{(k)}\{\max(1, \|\cdot\|_{D,\boldsymbol{Z}})\}(\boldsymbol{x}) \\ &= \Pr(X \leq k) + p^k \max(1, \|\boldsymbol{x}/p^k\|_{D,\boldsymbol{Z}}) \\ &= \Pr(X \leq k) + \max(p^k, \|\boldsymbol{x}\|_{D,\boldsymbol{Z}}) \\ &\to 1 + \|\boldsymbol{x}\|_{D,\boldsymbol{Z}} \end{aligned}$$

as $k \to \infty$, so that $\|\cdot\|_{D,\boldsymbol{Y}} = \|\cdot\|_{D,\boldsymbol{Z}}$. Consequently:

$$\varphi_{\boldsymbol{Y}}(\boldsymbol{x}) = \mathrm{E}\{\max(1, x_1 Y_1, \ldots, x_d Y_d)\} = \max[1, \mathrm{E}\{\max(x_1 Y_1, \ldots, x_d Y_d)\}]$$

for all $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}_+^d$. For all $i \in \{1, \ldots, d\}$, specializing $x_j = 0$ for $j \neq i$ and $x_i = 1$ gives

$$\mathrm{E}\{\max(1, Y_i)\} = \max\{1, \mathrm{E}(Y_i)\} = 1 \quad \text{for all} \quad i \in \{1, \ldots, d\}.$$

Because Y_i has expectation 1, this implies that the random variables $\max(1, Y_i) - 1$ and $\max(1, Y_i) - Y_i$, being nonnegative and having expectation zero, must be almost surely zero. In other words, $Y_i \leq 1$ and $Y_i \geq 1$ almost surely, and thus $Y_i = 1$ almost surely for all i. But then $Y = (1, \ldots, 1)$ is a generator of the norm $\|\cdot\|_\infty$, so that $\|\cdot\|_{D,\boldsymbol{Z}} = \|\cdot\|_{D,\boldsymbol{Y}} = \|\cdot\|_\infty$. The proof is complete. □

Combining Propositions 2.11 and 2.13(i), we get the following corollary.

Corollary 2.14. *The set of max-CFs of generators of D-norms is not closed in the sense of pointwise convergence.*

It should be noted that Corollary 2.14 is also true for usual characteristic functions, as we can see with the example of a sequence of random variables (X_n) such that for every n, X_n is normally distributed, centered, and has variance n^2. Then

$$\varphi_n(t) = \mathrm{E}\left(e^{itX_n}\right) = e^{-n^2 t^2/2} \quad \text{for all} \quad t \in \mathbb{R}$$

so that the sequence (φ_n) converges pointwise to the indicator function of $\{0\}$, which is not a characteristic function because it is not continuous.

2.2. **An inversion formula for max-characteristic functions.** As mentioned in the Introduction, any max-CF is a convex function and thus it is continuous and almost everywhere differentiable; furthermore, its derivative from the right exists everywhere.

Recall that for a vector $\boldsymbol{x} \in \mathbb{R}^d$, the notation $\boldsymbol{x} > \mathbf{0}$ means that $\boldsymbol{x}$ has strictly positive components. The next result contains both an inversion formula for max-CFs and a criterion for a function to be a max-CF.

Proposition 2.15. *Let $\boldsymbol{Z}$ be a nonnegative and integrable random vector with max-CF $\varphi_{\boldsymbol{Z}}$.*

(i) We have, for all $\boldsymbol{x} = (x_1, \ldots, x_d) > \mathbf{0}$,

$$\Pr(Z_j \le x_j,\ 1 \le j \le d) = \frac{\partial_+}{\partial t}\left\{ t\varphi_{\boldsymbol{Z}}\left(\frac{1}{t\boldsymbol{x}}\right)\right\}\Big|_{t=1}$$

where $\partial_+/\partial t$ denotes the right derivative with respect to the univariate variable t.

(ii) If ψ is a continuously differentiable function such that

$$\begin{aligned} \frac{\partial}{\partial t}\left\{ t\psi\left(\frac{1}{t\boldsymbol{x}}\right)\right\}\Big|_{t=1} &= \Pr(Z_j \le x_j,\ 1 \le j \le d) \\ \text{and} \quad \lim_{t\to\infty} t\left\{\psi\left(\frac{1}{t\boldsymbol{x}}\right) - 1\right\} &= 0 \end{aligned}$$

for all $\boldsymbol{x} = (x_1, \ldots, x_d) > \mathbf{0}$, then $\psi = \varphi_{\boldsymbol{Z}}$ on $(0,\infty)^d$.

Proof. Notice first that, similarly to equation (1), we have

$$t\varphi_{\boldsymbol{Z}}\left(\frac{1}{t\boldsymbol{x}}\right) = t + \int_t^{+\infty} 1 - \Pr(Z_j \le yx_j,\ 1 \le j \le d)\,dy.$$

Note that the above representation yields $\lim_{t\to\infty} t[\varphi_{\boldsymbol{Z}}\{1/(t\boldsymbol{x})\} - 1] = 0$.

To show (i), notice that taking right derivatives with respect to t yields

$$\frac{\partial_+}{\partial t}\left\{ t\varphi_{\boldsymbol{Z}}\left(\frac{1}{t\boldsymbol{x}}\right)\right\} = \Pr(Z_j \le tx_j,\ 1 \le j \le d).$$

Setting $t = 1$ concludes the proof of (i). To prove (ii), remark that

$$\frac{\partial}{\partial t}\left\{ t\psi\left(\frac{1}{t\boldsymbol{x}}\right)\right\} = \psi\left(\frac{1}{t\boldsymbol{x}}\right) - \frac{1}{t}\sum_{i=1}^d \frac{1}{x_j}\partial_j\psi\left(\frac{1}{t\boldsymbol{x}}\right) \quad \text{for all} \quad t > 0,$$

where $\partial_j\psi$ denotes the partial derivative of ψ with respect to its jth component. In particular, because

$$\Pr(Z_j \le x_j,\ 1 \le j \le d) = \frac{\partial}{\partial t}\left\{ t\psi\left(\frac{1}{t\boldsymbol{x}}\right)\right\}\Big|_{t=1} = \psi\left(\frac{1}{\boldsymbol{x}}\right) - \sum_{i=1}^d \frac{1}{x_j}\partial_j\psi\left(\frac{1}{\boldsymbol{x}}\right)$$

we obtain by replacing $\boldsymbol{x}$ with $t\boldsymbol{x}$ that for all $t > 0$,

$$\frac{\partial}{\partial t}\left\{t\psi\left(\frac{1}{t\boldsymbol{x}}\right)\right\} = \Pr(Z_j \le tx_j,\ 1 \le j \le d).$$

Write now

$$\begin{aligned} t\psi\left(\frac{1}{t\boldsymbol{x}}\right) &= t - \int_t^\infty \frac{\partial}{\partial y}\left[y\left\{\psi\left(\frac{1}{y\boldsymbol{x}}\right) - 1\right\}\right] dy \\ &= t + \int_t^\infty 1 - \Pr(Z_j \le yx_j,\ 1 \le j \le d)\, dy \\ &= t\varphi_{\boldsymbol{Z}}\left(\frac{1}{t\boldsymbol{x}}\right) \end{aligned}$$

to conclude the proof of (ii). □

Remark 2.16. This result makes it possible to improve upon the result of Proposition 2.13(i). Assume that $\varphi_{\boldsymbol{Z}}$ is the max-CF of a nonnegative and integrable random vector such that

$$\varphi_{\boldsymbol{Z}}(\boldsymbol{x}) = \psi(1, \|\boldsymbol{x}\|),$$

where $\psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a 1-homogeneous function and $\|\cdot\|$ is a norm on $\mathbb{R}^d$. Then informally,

$$\frac{\partial_+}{\partial t}\left\{t\varphi_{\boldsymbol{Z}}\left(\frac{1}{t\boldsymbol{x}}\right)\right\}\Big|_{t=1} = \frac{\partial_+}{\partial t}\left\{\psi(t, \|1/\boldsymbol{x}\|)\right\}\Big|_{t=1} = \partial_{1,+}\psi(1, \|1/\boldsymbol{x}\|)$$

if $\partial_{1,+}$ denotes the right derivative with respect to the first component. In particular,

$$\frac{\partial_+}{\partial t}\left\{t\varphi_{\boldsymbol{Z}}\left(\frac{1}{t\boldsymbol{x}}\right)\right\}\Big|_{t=1} \to \begin{cases} \partial_{1,+}\psi(1,0) & \text{if } \boldsymbol{x} \to \boldsymbol{\infty}, \\ \partial_{1,+}\psi(1,\infty) & \text{if } \boldsymbol{x} \to \boldsymbol{0}. \end{cases}$$

In other words, by Proposition 2.15, unless $\partial_{1,+}\psi(1, y)$ both converges to 1 as $y \to 0$ and to 0 as $y \to \infty$, the function $\psi(1, \|\cdot\|)$ cannot be a max-CF. Applying this to the example $\psi(x, y) = x + y$, we find the result of Proposition 2.13(i) again.

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E-mail address: michael.falk@uni-wuerzburg.de, Gilles.Stupfler@nottingham.ac.uk