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On The Asphericity of a Family of Positive Relative Group Presentations

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Abstract

Excluding four exceptional cases, the asphericity of the relative presentation $\mathcal{P} = \langle G, x | x^m g x h \rangle$ for $m \geq 2$ is determined. If $H = \langle g, h \rangle \leq G$, then the exceptional cases occur when H is isomorphic to C_5 or C_6 .

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1 Introduction

A *relative group presentation* is a presentation of the form $\mathcal{P} = \langle G, \mathbf{x} | \mathbf{r} \rangle$, where G is a group and \mathbf{x} is a set disjoint from G . Denoting the free group on \mathbf{x} by $\langle \mathbf{x} \rangle$, \mathbf{r} is a set of cyclically reduced words in the free product $G * \langle \mathbf{x} \rangle$. The group defined by \mathcal{P} is $\hat{G} = G * \langle \mathbf{x} \rangle / N$, where N is the normal closure in $G * \langle \mathbf{x} \rangle$ of \mathbf{r} . A relative presentation is said to be *aspherical* if every spherical picture over it contains a dipole. These notions were defined and studied in [3] where it is shown that if \mathcal{P} is aspherical then group theoretic information about \hat{G} can be deduced.

There has been much interest in determining asphericity of \mathcal{P} particularly when $\mathbf{x} = \{x\}$ and $\mathbf{r} = \{r\}$ both consist of a single element. Indeed, if $r = x^{\varepsilon_1} g_1 \dots x^{\varepsilon_k} g_k$ where $g_i \in G$, $\varepsilon_i = \pm 1$ and $g_i = 1$ implies $\varepsilon_i + \varepsilon_{i+1} \neq 0$ ($1 \leq i \leq k$, subscripts mod k), then the asphericity of \mathcal{P} has been determined (modulo some exceptional cases) when $k \leq 3$ or $r \in \{xg_1xg_2xg_3xg_4, xg_1xg_2xg_3x^{-1}g_4, xg_1xg_2xg_3xg_4xg_5, (xg_1)^{l_1}(xg_2)^{l_2}(xg_3)^{l_3} (l_i > 1, 1 \leq i \leq 3)\}$ [1-3] [7-9]. This list includes $x^m g x^{-1} h$ ($g, h \in G \setminus \{1\}$) for $m \leq 3$, and when $m \geq 4$ asphericity (modulo exceptional cases) has been determined in [6].

In this paper we consider $x^m g x h$ ($g, h \in G \setminus \{1\}$). If $m = 2$ then a complete classification of when \mathcal{P} is aspherical has been obtained in [3]. Modulo some exceptions the

cases $m = 3$ and $m = 4$ were determined in [2] and [8] respectively. Before stating our main result observe that $x^m g x h = 1$ if and only if $x^{-m} h^{-1} x^{-1} g^{-1} = 1$, and it follows that we can work modulo $g \leftrightarrow h^{-1}$.

We list the following exceptional cases.

(E1) $g = h^2$, $|h| = 5$ and $m \geq 5$.

(E2) $g \in \{h^2, h^3, h^4\}$, $|h| = 6$ and $m \geq 3$.

Theorem 1.1. *Let \mathcal{P} be the relative presentation $\mathcal{P} = \langle G, x | x^m g x h \rangle$, where $m \geq 2$, $x \notin G$, $g, h \in G \setminus \{1\}$. Suppose that none of the conditions in (E1) or (E2) holds. Then \mathcal{P} is aspherical if and only if (modulo $g \leftrightarrow h^{-1}$) none of the following holds:*

1. $g = h^{\pm 1}$ has finite order.
2. $g = h^2$ has finite order and $m = 2$.
3. $g = h^2$, $|h| = 4$ and $m \geq 3$.
4. $g = h^2$, $|h| = 5$ and $3 \leq m \leq 4$.
5. $g \in \{h^3, h^4\}$, $|h| = 6$ and $m = 2$.
6. $|g| = 2$, $|h| = 3$ and $[g, h] = 1$.
7. $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, where $\frac{1}{\infty} := 0$.

If $m = 2, 3, 4$ (respectively) then the proof of Theorem 1.1 can be deduced from results in [3], [2], [8] (respectively) apart from two exceptional cases for $m = 3$ (E4 and E5 of [2]) which are dealt with here together with the case $m \geq 5$. In Section 2 we discuss the method of the proof where the concept of pictures is needed. In Section 3 some preliminaries results are stated. The proof of Theorem 1.1 is given in Section 4.

2 Method of Proof

2.1 Pictures and Curvature

The definitions of this subsection are taken from [3]. The reader is referred to [3] and [2] for more details.

A *picture* \mathbb{P} is a finite collection of pairwise disjoint discs $\{D_1, \dots, D_m\}$ in the interior of a disc D^2 , together with a finite collection of pairwise disjoint simple arcs

$\{\alpha_1, \dots, \alpha_n\}$ embedded in the closure of $D^2 - \bigcup_{i=1}^m D_i$ in such a way that each arc meets $\partial D^2 \cup \bigcup_{i=1}^m D_i$ transversely in its end points. The *boundary* of \mathbb{P} is the circle ∂D^2 , denoted by $\partial\mathbb{P}$. For $1 \leq i \leq m$, the *corners* of D_i are the closures of the connected components of $\partial D_i - \bigcup_{j=1}^n \alpha_j$, where ∂D_i is the boundary of D_i . The *regions* Δ of \mathbb{P} are the closures of the connected components of $D^2 - (\bigcup_{i=1}^m D_i \cup \bigcup_{j=1}^n \alpha_j)$. An *inner region* of \mathbb{P} is a simply connected region of \mathbb{P} that does not meet $\partial\mathbb{P}$. The picture \mathbb{P} is *non-trivial* if $m \geq 1$, is *connected* if $\bigcup_{i=1}^m D_i \cup \bigcup_{j=1}^n \alpha_j$ is connected, and is *spherical* if it is non-trivial and if none of the arcs meets the boundary of D^2 . The number of edges in $\partial\Delta$ is called the *degree* of the region Δ and is denoted by $d(\Delta)$. A region of degree n will be called an *n-region*. If \mathbb{P} is a spherical picture, the number of different discs to which a disc D_i is connected is called the *degree* of D_i , denoted by $d(D_i)$. The discs of a spherical picture \mathbb{P} are also called *vertices* of \mathbb{P} .

Suppose that the picture \mathbb{P} is labelled in the following sense: each arc α_j is equipped with a normal orientation, indicated by a short arrow meeting the arc transversely, and labelled by an element of $\mathbf{x} \cup \mathbf{x}^{-1}$. Each corner of \mathbb{P} is oriented *clockwise* (with respect to D^2) and labelled by an element of G . If κ is a corner of a disc D_i of \mathbb{P} , then $W(\kappa)$ will be the word obtained by reading in a clockwise order the labels on the arcs and corners meeting ∂D_i beginning with the label on the first arc we meet as we read the clockwise corner κ . If we cross an arc labelled x in the direction of its normal orientation, we read x , else we read x^{-1} .

A *picture over \mathcal{P}* is a picture \mathbb{P} labelled in such a way the following are satisfied:

1. For each corner κ of \mathbb{P} , $W(\kappa) \in \mathbf{r}^*$, the set of all cyclic permutations of $\mathbf{r} \cup \mathbf{r}^{-1}$ which begin with a member of \mathbf{x} .
2. If g_1, \dots, g_l is the sequence of corner labels encountered in *anticlockwise* traversal of the boundary of an inner region Δ of \mathbb{P} , then the product $g_1 g_2 \dots g_l = 1$ in G . We say that $g_1 g_2 \dots g_l$ is the label of Δ , denoted by $l(\Delta) = g_1 g_2 \dots g_l$.

A *dipole* in a labelled picture \mathbb{P} over \mathcal{P} consists of corners κ and κ' of \mathbb{P} together with an arc joining the two corners such that κ and κ' belong to the same region and such that if $W(\kappa) = Sg$ where $g \in G$ and S begins and ends with a member of $\mathbf{x} \cup \mathbf{x}^{-1}$, then $W(\kappa') = S^{-1}g^{-1}$. The picture \mathbb{P} is *reduced* if it does not contain a dipole. A relative presentation \mathcal{P} is called *aspherical* if every connected spherical picture over \mathcal{P} contains a dipole. If \mathcal{P} is not aspherical then there is a reduced spherical picture over \mathcal{P} .

The *star graph* \mathcal{P}^{st} of a relative presentation \mathcal{P} is a graph whose vertex set is $\mathbf{x} \cup \mathbf{x}^{-1}$ and edge set is \mathbf{r}^* . For $R \in \mathbf{r}^*$, write $R = Sg$ where $g \in G$ and S begins and ends with a member of $\mathbf{x} \cup \mathbf{x}^{-1}$. The initial and terminal functions are given as follows: $\iota(R)$ is the first symbol of S , and $\tau(R)$ is the inverse of the last symbol of S . The labelling function on the edges is defined by $\lambda(R) = g^{-1}$ and is extended to paths in the usual way. A non-empty cyclically reduced cycle (closed path) in \mathcal{P}^{st} will be called *admissible* if it has trivial label in G . Each inner region of a reduced picture over \mathcal{P} supports an admissible cycle in \mathcal{P}^{st} .

A *weight function* θ is a real-valued function on the set of edges of \mathcal{P}^{st} which satisfies $\theta(Sg) = \theta(S^{-1}g^{-1})$ where $Sg = R \in \mathbf{r}^*$. The weight of a closed cycle is the sum of the weights of the constituent edges. A weight function is *weakly aspherical* if the following conditions are satisfied:

1. Let $R \in \mathbf{r}^*$, with $R = x_1^{\varepsilon_1} g_1 \dots x_n^{\varepsilon_n} g_n$. Then

$$\sum_{i=1}^n (1 - \theta(x_i^{\varepsilon_i} g_i \dots x_n^{\varepsilon_n} g_n x_1^{\varepsilon_1} g_1 \dots x_{i-1}^{\varepsilon_{i-1}} g_{i-1})) \geq 2.$$

2. The weight of each admissible cycle in \mathcal{P}^{st} is at least 2.

If \mathcal{P}^{st} admits a weakly aspherical weight function, then \mathcal{P} is aspherical [3] and this method will be used in the proofs.

Another method is *curvature distribution* (see, for example [7]). Let \mathbb{P} be a reduced spherical picture over \mathcal{P} . We proceed as follows. An *angle function* on \mathbb{P} is a real-valued function on the set of corners of \mathbb{P} . Given this, the *curvature* of a vertex of \mathbb{P} is defined to be 2π less the sum of the angles at that vertex. The *curvature* $c(\Delta)$ of a k -gonal region Δ of \mathbb{P} is the sum of all the angles of the corners of Δ less $(k-2)\pi$. Our method of associating angles ensures that vertices have zero curvature and it follows from this that $\sum c(\Delta) = 4\pi$ where the sum is taken over all the regions Δ of \mathbb{P} . Assuming that none of conditions 1-7 holds, our strategy will be to show that the positive curvature that exists in \mathbb{P} can be sufficiently compensated by the negative curvature. To this end we locate each Δ satisfying $c(\Delta) > 0$ and distribute $c(\Delta)$ to near regions $\hat{\Delta}$ of Δ . For such regions $\hat{\Delta}$ define $c^*(\hat{\Delta})$ to equal $c(\hat{\Delta})$ plus all the positive curvature $\hat{\Delta}$ receives during this *distribution procedure*. We prove that $c^*(\hat{\Delta}) \leq 0$ and, since the total curvature of \mathbb{P} is at most $\sum c^*(\hat{\Delta})$, this yields a contradiction which shows that \mathcal{P} is aspherical.

2.2 Construction of pictures and Defined angle functions

For this subsection we assume $g \neq h^{\pm 1}$. Let \mathbb{P} be a reduced spherical picture over $\mathcal{P} = \langle G, x | x^m g x h \rangle$. Then each vertex (disc) in \mathbb{P} has one of the forms given by Figure 2.2.1(i) and (ii); and the star graph \mathcal{P}^{st} of \mathcal{P} is given by Figure 2.2.1(iii). Note that when drawing figures the edge arrows shown in Figure 2.2.1 will often be omitted.

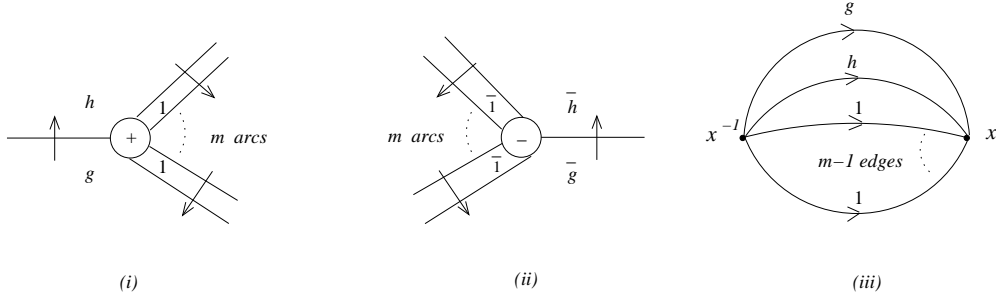


Figure 2.2.1: + disc, - disc and \mathcal{P}^{st} .

If there are $k - 1$ consecutive regions of degree 2, then the k arcs in the boundary of these regions constitute a k -bond. We will refer to a 1-bond as a single bond. Given that $g \neq h^{\pm 1}$ there are (up to inversion) only two types of $(m - 1)$ -bonds in a reduced picture \mathbb{P} (see Figure 2.2.2). For simplicity, in our figures $(m - 1)$ -bonds will be drawn as bold 2-bonds (see Figure 2.2.2). Note that there are no m -bonds or $(m + 1)$ -bonds in \mathbb{P} , indeed a vertex of degree 2 can only occur in a reduced picture if $g = h$ or $g = 1$ or $h = 1$. Also, for simplicity, the vertex of degree 3 of the form shown in Figure 2.2.3 (i) will be drawn as shown in Figure 2.2.3 (ii), where $m_1 \geq 2$, $m_2 \geq 2$ and $m_1 + m_2 = m$.

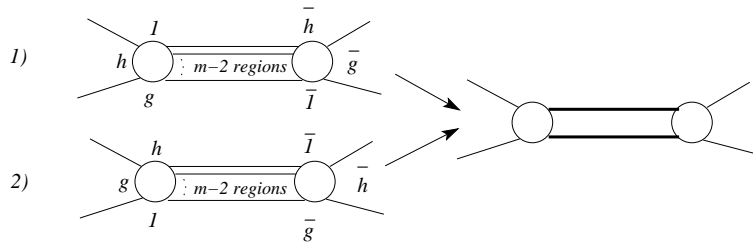


Figure 2.2.2: $(m - 1)$ -bond.

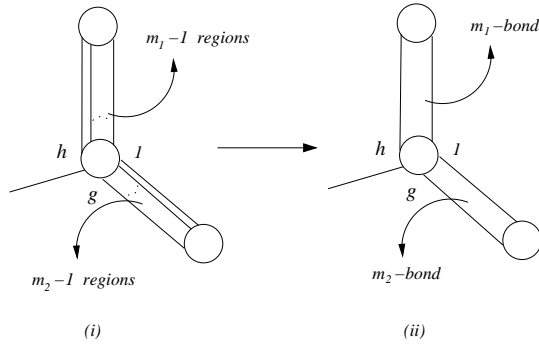


Figure 2.2.3.

Remarks 2.1.

1. Each arc connects a + disc to a - disc, and so each region has even degree.
2. A word w obtained from reading the labels on the edges of a cyclically reduced cycle in \mathcal{P}^{st} does not contain (up to cyclic permutation and inversion) gg^{-1} or hh^{-1} although it can contain 11^{-1} provided different edges in \mathcal{P}^{st} are used. We will call such words w cyclically reduced.
3. Each region in a reduced spherical picture \mathbb{P} over \mathcal{P} supports a cyclically reduced word in $\{g, h, 1\}$.

There are (up to inversion) three types of vertices of degree 3 and these are shown in Figure 2.2.4.

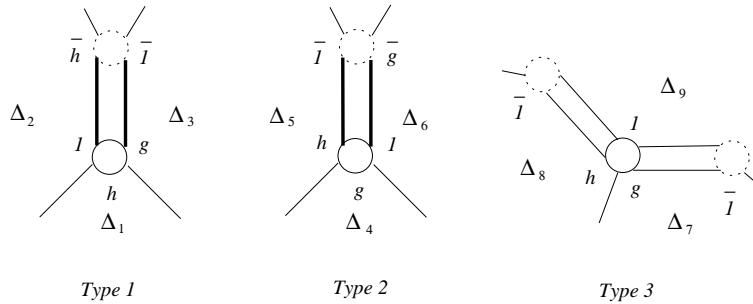


Figure 2.2.4.

For the proofs, we define the following angle functions on the vertices v of \mathbb{P} . The angle function α is defined as follows. Each corner within a 2-bond has angle zero, while each of the other corners has angle $\frac{2\pi}{d(v)}$. We will refer to α as the *standard angle function*.

The *angle function* α_1 is defined as follows. Again, corners within 2-bonds have angle zero. For vertices of degree 3 of Type 1-3, α_1 is given by Figure 2.2.5. If $d(v) > 3$, then each corner in v has angle $\frac{2\pi}{d(v)}$.

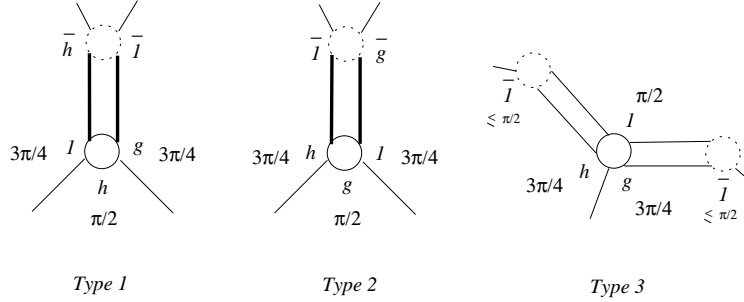


Figure 2.2.5: Angle function α_1 for vertices of degree 3.

Define an *angle function* α_2 on \mathbb{P} as follows. Corners within 2-bonds have angle zero. In vertices of degree 3, corners labelled by $h^{\pm 1}$ have angle π , each of the other two corners has angle $\frac{\pi}{2}$ (see Figure 2.2.6). Corners in vertices of degree > 3 have angle $\frac{2\pi}{d(v)}$.

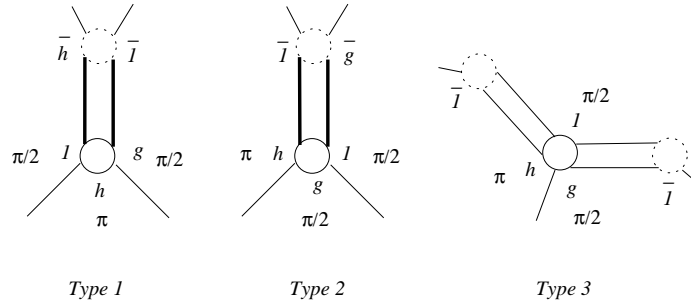


Figure 2.2.6: Angle function α_2 for vertices of degree 3.

Finally, the *angle function* α_3 on \mathbb{P} is given as follows. Corners within 2-bonds have angle zero. For vertices of degree 3, corners labelled by $1^{\pm 1}$ have angle π , each of the other two corners has angle $\frac{\pi}{2}$ (see Figure 2.2.7). Corners in vertices of degree > 3 have angle $\frac{2\pi}{d(v)}$.

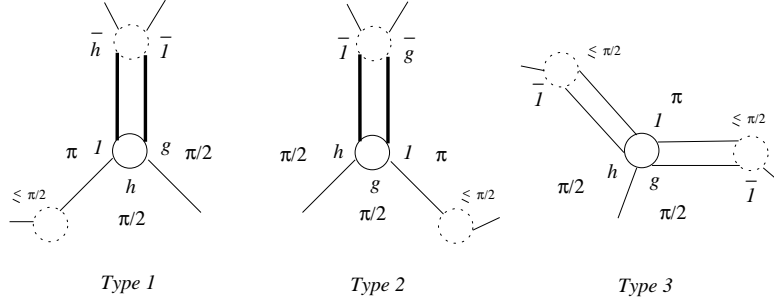


Figure 2.2.7: Angle function α_3 for vertices of degree 3.

Remarks 2.2.

1. The corners in each 2-bond have angle 0 in each of the above angle functions. It follows that the curvature of regions of degree 2 is 0, and so we can treat each k -bond as a single bond.
2. By assigning the angle function α_1 to the corners of \mathbb{P} , the following are satisfied:
 - (i) Since $(2 - 8)\pi + 8 \cdot \frac{3\pi}{4} = 0$, positive regions can only have degree 4 or 6.
 - (ii) Both corners adjacent to the $(m - 1)$ -bond in a boundary of a region have angle $\frac{3\pi}{4}$; while the two corners adjacent to the m_1 -bond or m_2 -bond in a boundary of a region cannot both have angle $\frac{3\pi}{4}$ (see Figure 2.2.5).
3. By assigning the angle function α_2 to the corners of \mathbb{P} , the following are satisfied:
 - (i) In any region Δ of \mathbb{P} , there are no consecutive corners with angle π , else \mathbb{P} is not reduced. Hence, $c(\Delta) \leq (2 - n)\pi + \frac{n}{2} \cdot \pi + \frac{n}{2} \cdot \frac{\pi}{2} = \pi(\frac{8-n}{4})$ and so positively curved regions can only be 4-regions or 6-regions.
 - (ii) If Δ is a positive 4-region, then it has at least one corner labelled by $h^{\pm 1}$ with angle π (otherwise $c(\Delta) \leq -2\pi + 4 \cdot \frac{\pi}{2} = 0$).
 - (iii) If Δ is a positive 6-region, then it contains at least three $h^{\pm 1}$ -corners each with angle π (else $c(\Delta) \leq -4\pi + 2\pi + 4 \cdot \frac{\pi}{2} = 0$).
4. By assigning the angle function α_3 to the corners of \mathbb{P} , the following are satisfied:
 - (i) There are no consecutive corners with angle π in the boundary of a region Δ of \mathbb{P} (otherwise \mathbb{P} is not reduced). Thus, $c(\Delta) \leq (2 - n)\pi + \frac{n}{2} \cdot \pi + \frac{n}{2} \cdot \frac{\pi}{2} = \pi(\frac{8-n}{4})$ and so positive regions can only be 4-regions or 6-regions.
 - (ii) If Δ is a positive 4-region, then it contains at least one corner labelled by $1^{\pm 1}$ with angle π (otherwise $c(\Delta) \leq -2\pi + 4 \cdot \frac{\pi}{2} = 0$).

(iii) If Δ is a positive 6-region, then it contains three occurrences of $1^{\pm 1}$ -corners each with angle π (else $c(\Delta) \leq -4\pi + 2\pi + 4 \cdot \frac{\pi}{2} = 0$).

3 Preliminary Lemmas

Assume that $m \geq 5$. We first state a series of lemmas followed by their proofs. Recall that we assume $g, h \in G \setminus \{1\}$.

3.1 Statement of Lemmas

Lemma 3.1. *If \mathcal{P} is not aspherical, then at least one of the following conditions holds:*

1. $g = h^{\pm 1}$;
2. $g = h^2$ or $h = g^2$;
3. $2 \in \{|g|, |h|\}$;
4. $|gh^{-1}| = 2$ and $3 \in \{|g|, |h|\}$.

Lemma 3.2. *If $g = h^{\pm 1}$, then \mathcal{P} is aspherical if and only if g has infinite order.*

Lemma 3.3. *Let $g = h^2$. If $|h| = 4$, then \mathcal{P} is not aspherical, while if $|h| > 6$, then \mathcal{P} is aspherical.*

Lemma 3.4. *If $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$, then \mathcal{P} is not aspherical.*

Lemma 3.5. *If $|gh^{-1}|$ is infinite, then \mathcal{P} is aspherical.*

Lemma 3.6. *Suppose that $|g| = 2$.*

1. *If $|gh^{-1}| = 2$ and $|h| = \infty$, then \mathcal{P} is aspherical.*
2. *If $|gh^{-1}| = 3$, $|h| \geq 6$ and \mathcal{P} is not aspherical, then $g = h^3$, in particular $|h| = 6$.*
3. *If $|gh^{-1}| \geq 4$, $|h| \geq 4$ and $g \neq h^2$, then \mathcal{P} is aspherical.*
4. *If $|gh^{-1}| \geq 6$ and $|h| = 3$, then \mathcal{P} is not aspherical if and only if $[g, h] = 1$.*

Lemma 3.7. *If $|g| = 3$, $|gh^{-1}| = 2$, $|h| \geq 6$ and \mathcal{P} is not aspherical, then $g = h^4$ and $|h| = 6$.*

3.2 Proof of Lemma 3.1.

Let \mathbb{P} be a reduced spherical picture over \mathcal{P} . It can be assumed without any loss of generality **(A)** that the number of regions of degree 4 cannot be decreased by bridge moves [5]. Suppose that none of the Conditions 1, 2 or 3 holds.

First assign the standard angle function α to the vertices of \mathbb{P} . Since for any n -region Δ in \mathbb{P} , $c(\Delta) \leq \pi(\frac{6-n}{3})$, $c(\Delta) > 0$ only if $n = 4$. A positively curved 4-region Δ has at least one vertex of degree 3. If $\Delta \in \{\Delta_i : 1 \leq i \leq 8\}$ which are shown in Figure 2.2.4, then at least one corner of Δ is not labelled by $1^{\pm 1}$. By considering all cyclically reduced words of length at most 4 in $\{g^{\pm 1}, h^{\pm 1}\}$ (which are compatible with our hypotheses on g and h), we obtain $l(\Delta) = (gh^{-1})^{\pm 2}$. If $\Delta = \Delta_9$ then $l(\Delta)$ gives a contradiction or Δ is the positive 4-region shown in Figure 3.2.1. Since $m_1 > B$ a sequence of bridge moves transforms Δ into a region of degree > 4 without creating a new region of degree 4. This contradicts assumption **(A)** and so by assigning α we obtain $|gh^{-1}| = 2$.

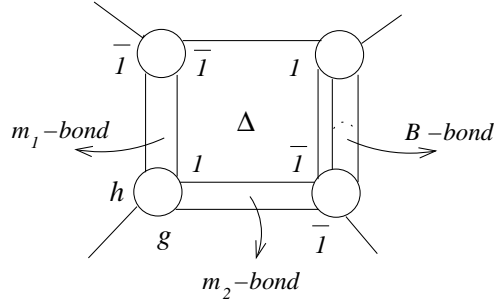


Figure 3.2.1.

Now apply the angle function α_1 . By Remark 2.2.(2)(i), positively curved regions can only be 4-regions or 6-regions. A positively curved 4-region Δ has at least one corner with angle $\frac{3\pi}{4}$ in its boundary and so $\Delta = \Delta_i$ for some $i \in \{2, 3, 5, 6, 7, 8\}$. This implies that Δ has at least one corner not labelled by $1^{\pm 1}$. Also, it implies that $l(\Delta) \neq (gh^{-1})^{\pm 2}$. All other choices contradict our assumptions on g and h and so there are no positive 4-regions. It follows that Δ is a 6-region which contains at least five corners with angle $\frac{3\pi}{4}$ in its boundary (else, $c(\Delta) \leq (2-6)\pi + 4 \cdot \frac{3\pi}{4} + 2 \cdot \frac{\pi}{2} = 0$). By Remark 2.2(2)(ii) Δ contains at least two $(m-1)$ -bonds in its boundary and a third bond which is either an $(m-1)$ -bond, an m_1 -bond or m_2 -bond. If the $(m-1)$ -bonds in the boundary of Δ are inwardly oriented, then $l(\Delta) = (g1^{-1})^{\pm 3}$, while if the $(m-1)$ -bonds are oriented outward Δ , then $l(\Delta) = (h1^{-1})^{\pm 3}$. It follows that $|gh^{-1}| = 2$ and $3 \in \{|g|, |h|\}$ which is Condition 4, as required.

3.3 Proof of Lemma 3.2.

If $g = h$ then $x^m g x h = 1$ if and only if $x^{m-1}(xg)^2 = 1$. By Lemma 1 in [9], \mathcal{P} is aspherical if and only if $|g| = \infty$.

If $g = h^{-1}$ and g has infinite order, then Lemma 3 in [2] applies to show that \mathcal{P} is aspherical. But $x^m g x g^{-1} = 1$ and $|g| < \infty$ implies $|x| < \infty$ and by Theorem 1 in [2] \mathcal{P} is not aspherical.

3.4 Proof of Lemma 3.3.

Let $g = h^2$. For $|h| = 4$ there is the sphere shown in Figure 3.4.1. On the other hand if $|h| = k > 6$, then the ordinary presentation $\langle x, h | x^m h^2 x h = 1 = h^k \rangle$ is a C(4)-T(4) presentation, hence \mathcal{P} is aspherical (for more details see [2]).

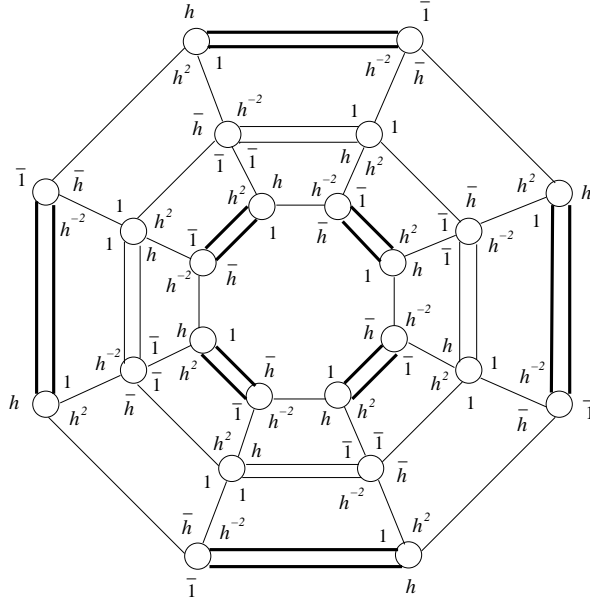


Figure 3.4.1: $g = h^2$ and $|h| = 4$.

3.5 Proof of Lemma 3.4.

If $\frac{1}{|g|} + \frac{1}{|gh^{-1}|} + \frac{1}{|h|} > 1$ then there are spherical pictures \mathbb{P} over \mathcal{P} . For example if $(|g|, |gh^{-1}|, |h|) = (2, 3, 4)$ then \mathbb{P} is given by Figure 3.5.1. The other spheres are constructed in a similar way, we omit the details.

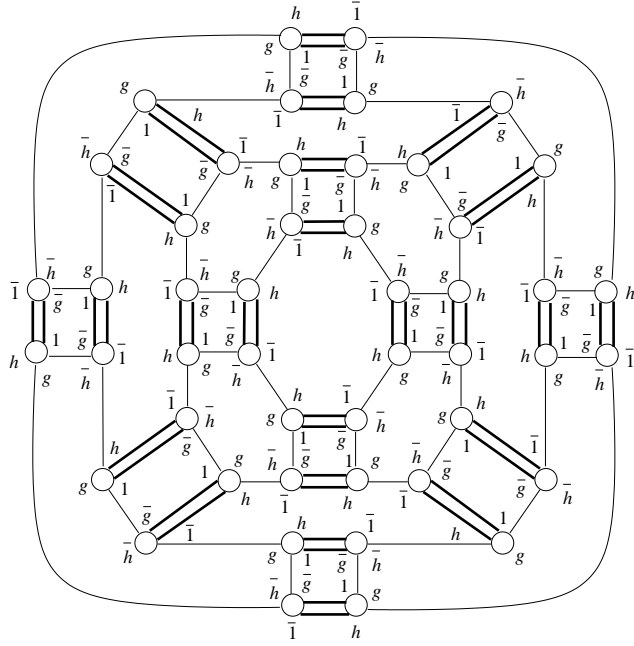


Figure 3.5.1: $(|g|, |gh^{-1}|, |h|) = (2, 3, 4)$.

3.6 Proof of Lemma 3.5.

Suppose that $|gh^{-1}|$ is infinite. If we have a relation of the form $(gh^{-1})^k g = 1$ or $h^{-1}(gh^{-1})^k = 1$ in G then $H = \langle g, h \rangle$ is infinite cyclic generated by gh^{-1} , and so \mathcal{P} is aspherical by Lemma 3 in [2]. So assume otherwise.

Define the following weight function θ on \mathcal{P}^{st} (see Figure 2.2.1(iii)): $\theta(e_g) = 0 = \theta(e_h)$ and $\theta(s_i) = 1$ for $(1 \leq i \leq m-1)$, where e_g, e_h, s_i ($1 \leq i \leq m-1$) are the edges of \mathcal{P}^{st} labelled $g, h, 1$ (respectively). Clearly Condition 1 of weakly aspherical weight function is satisfied. The assumptions on g and h imply that each admissible cycle in \mathcal{P}^{st} must involve at least 2 edges labelled by the identity, and so has weight at least 2. Therefore θ is an aspherical weight function which proves that \mathcal{P} is aspherical.

3.7 Proof of Lemma 3.6(1): Case(2, 2, ∞)

In this case, $|g| = 2, |gh^{-1}| = 2$ and $|h| = \infty$. Let \mathbb{P} be a reduced spherical picture over \mathcal{P} and assign the angle function α_2 to \mathbb{P} . By Remark 2.2 (3)(i), positive regions can only be 4-regions or 6-regions. By Remark 2.2 (3)(iii), positive 6-regions involve three occurrences of $h^{\pm 1}$ -corners and each possible label yields a contradiction. By Remark 2.2 (3)(ii) a positive 4-region must contain $h^{\pm 1}$ forcing the label $(gh^{-1})^{\pm 2}$. Hence, there are (up to inversion) two types of positive regions as shown in Figure 3.7.1. (Note that

the maximum possible curvature is always indicated.)

We adopt the notation of [2] and define the following distribution scheme (distributing positive curvature from Δ to $\hat{\Delta}$) which is given in Figure 3.7.1:

$$\Gamma(\Delta, \hat{\Delta}) = \begin{cases} c(\Delta) & \text{if } 0 < c(\Delta) \leq \frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single bond } S \\ & \text{that is oriented from } \Delta \text{ to } \hat{\Delta} \text{ such that } S \text{ is adjacent to an} \\ & h^{\pm 1}\text{-corner in } \Delta \text{ with angle } \pi \\ c(\Delta)/2 & \text{if } \frac{\pi}{2} < c(\Delta) \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single bond} \\ & \text{that is oriented from } \Delta \text{ to } \hat{\Delta} \\ 0 & \text{otherwise} \end{cases}$$

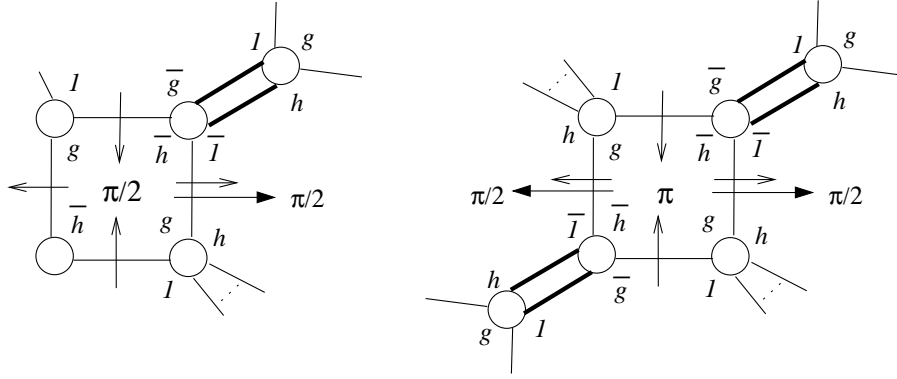


Figure 3.7.1: Positive regions and distribution scheme in Case(2, 2, ∞).

Let $\Gamma(\Delta, \hat{\Delta}) > 0$ and let r be the number of corners of angle π in $\hat{\Delta}$. By Remark 2.2(3)(i), $r \leq \frac{n}{2}$ where $n = d(\hat{\Delta})$. Set $\Gamma_2 = \Gamma_2(\hat{\Delta}) = |\{\Delta : \Gamma(\Delta, \hat{\Delta}) = \frac{\pi}{2}\}| \leq \frac{n}{2}$ (since $\hat{\Delta}$ receives $\pi/2$ only across edges that are oriented inwards - see Figure 3.7.1). Then $c^*(\hat{\Delta}) \leq (2 - n)\pi + r\pi + (n - r)\frac{\pi}{2} + \Gamma_2 \cdot \frac{\pi}{2} = 2\pi - \frac{\pi}{2}(n - r - \Gamma_2) \leq 2\pi$. It follows that if $\Gamma_2 \leq \frac{n}{2} - 4$ then $c^*(\hat{\Delta}) \leq 0$, so assume otherwise.

If $\Gamma_2 = \frac{n}{2}$ or $\frac{n}{2} - 1$, then (see Figure 3.7.1) the labelling of $\hat{\Delta}$ implies that either $h^{\pm \frac{n}{2}} = 1$ or $g = h^{\pm \frac{n}{2}}$, contradicting $|h| = \infty$. This leaves $\Gamma_2 = \frac{n}{2} - 3$ and $r = \frac{n}{2}$; or $\Gamma_2 = \frac{n}{2} - 2$ and $r \geq \frac{n}{2} - 1$ (otherwise $c^*(\hat{\Delta}) \leq 0$). First assume that $r = \frac{n}{2} - 1$. Then $\Gamma_2 = \frac{n}{2} - 2$ and $c^*(\hat{\Delta}) \leq \frac{\pi}{2}$. The fact that $\Gamma_2 = \frac{n}{2} - 2$ means that there are two inwardly oriented edges in $\partial\hat{\Delta}$ across which $\hat{\Delta}$ does not receive $\frac{\pi}{2}$. Figure 3.7.2 (i) shows the first case (consecutive), which forces $l(\hat{\Delta}) = h^{\frac{n}{2}-1}w_1w_2w_3$, where $w_1, w_3 \in \{1^{-1}, g^{-1}\}$ and $w_2 \in \{1, g, h\}$; and it follows that $|h| < \infty$, a contradiction. The second case is given by Figure 3.7.2 (ii) and $l(\hat{\Delta}) = z_1h^{\alpha_1}z_2h^{\alpha_2}$, where $z_1, z_2 \in \{1^{-1}, g^{-1}\}$. If $z_1 = 1^{-1}$ or

$z_2 = 1^{-1}$ then $|h| = \infty$, a contradiction, so assume otherwise. But if $z_1 = g^{-1}$ in Figure 3.7.2 (ii) then either the h -corner in the vertex v_1 has angle $\leq \frac{\pi}{2}$, or Δ_1 contains an m -bond in its boundary and so it cannot be either of the positive regions shown in Figure 3.7.1. (i.e. $\hat{\Delta}$ does not receive $\frac{\pi}{2}$ from Δ_1). Either way, $c^*(\hat{\Delta})$ will be decreased by $\frac{\pi}{2}$ and so $c^*(\hat{\Delta}) \leq 0$.

Now let $r = \frac{n}{2}$ in which case $\Gamma_2 = \frac{n}{2} - 2$ or $\frac{n}{2} - 3$ and $c^*(\hat{\Delta}) \leq \pi$. Since $g^2 = (gh^{-1})^2 = 1$ it follows that any word in g and h can be rewritten in the form $g^{\alpha_1} h^{\alpha_2}$. If $g^{\pm 1}$ appears an odd number of times in $l(\hat{\Delta})$ then $|h| < \infty$. Also, if $g^{\pm 1}$ occurs at least four times in $l(\hat{\Delta})$ then $\Gamma_2 \leq \frac{n}{2} - 4$, a contradiction, and so $g^{\pm 1}$ appears exactly twice in $l(\hat{\Delta})$. Since $r = \frac{n}{2}$, each of these two g^{-1} -corners is adjacent to two h -corners in $\partial\hat{\Delta}$. Thus, arguing as in the case $z_1 = g^{-1}$ above it follows that $c^*(\hat{\Delta}) \leq \pi - 2 \cdot \frac{\pi}{2} = 0$.

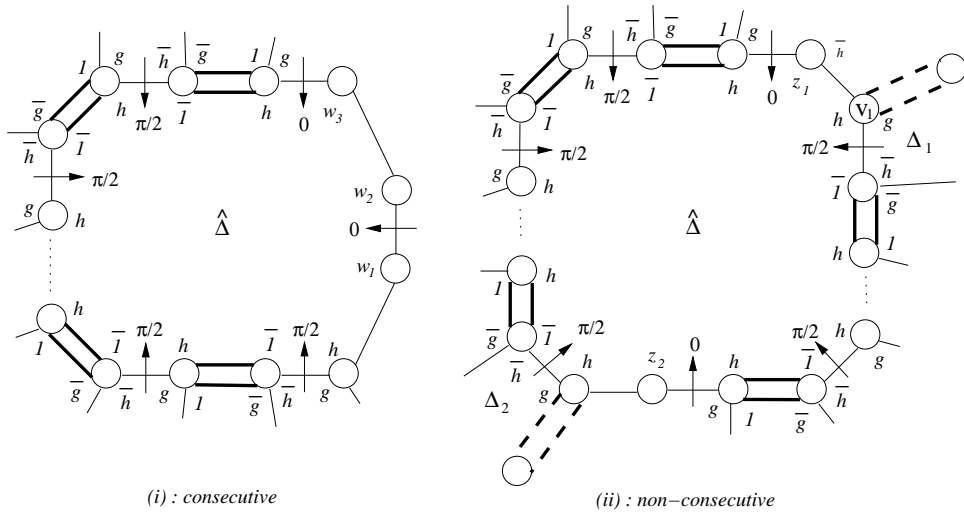


Figure 3.7.2.

3.8 Proof of Lemma 3.6(2): Case $(2, \bar{3}, \bar{6})$

Here we assume that $|g| = 2$, $|gh^{-1}| \geq 3$ and $|h| \geq 6$. Suppose that $\mathcal{P} = \langle G, x | x^m g x h \rangle$ is not aspherical. We show that $H = gp\{g, h\}$ is cyclic of order 6 generated by h and $g = h^3$. Let \mathbb{P} be a reduced spherical picture over \mathcal{P} to which we assign the angle function α_2 . All possible labels for a positive 4-region give a contradiction since, by Remark 2.2 (3)(ii), each must involve $h^{\pm 1}$. For positive 6-regions, by Remark 2.2 (3)(iii), there are three occurrences of $h^{\pm 1}$ and the only possible labels not yielding a contradiction imply $(gh^{-1})^{\pm 3} = 1$ or $g = h^3$ (and we are done). Therefore there is (up to inversion) only one positive region which is shown in Figure 3.8.1.

Apply the following distribution scheme:

$$\Gamma(\Delta, \hat{\Delta}) = \begin{cases} c(\Delta)/3 & \text{if } c(\Delta) > 0 \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single bond} \\ & \text{that is oriented from } \Delta \text{ to } \hat{\Delta} \\ 0 & \text{otherwise} \end{cases}$$

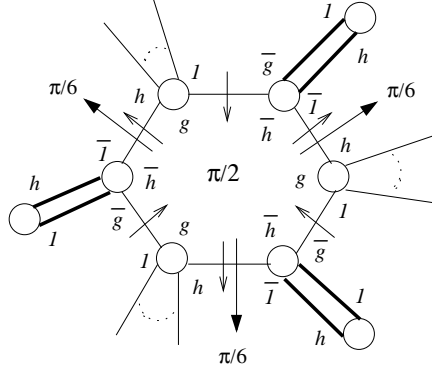


Figure 3.8.1: Positive regions and distribution scheme in Case(2, $\bar{3}$, $\bar{6}$).

As shown in Figure 3.8.1, if $\Gamma(\Delta, \hat{\Delta}) > 0$, then $(h1^{-1}h)^{\pm 1}$ is a sublabel of $\hat{\Delta}$. For a fixed region $\hat{\Delta}$ set $\Gamma_6(\hat{\Delta}) = |\{\Delta : \Gamma(\Delta, \hat{\Delta}) = \frac{\pi}{6}\}|$.

Remarks 3.8.

1. Since $\hat{\Delta}$ receives $\pi/6$ only through edges that are oriented towards $\hat{\Delta}$, $\Gamma_6 \leq \frac{n}{2}$.
2. For each $\pi/6$ that $\hat{\Delta}$ receives, there is an $(m-1)$ -bond in the boundary of $\hat{\Delta}$ which gives $(h1^{-1})^{\pm 1}$ as a sublabel of $\hat{\Delta}$.
3. $l(\hat{\Delta}) = h1^{-1}hw$ and so $d(\hat{\Delta}) > 6$; since if $d(\hat{\Delta})=6$ then $l(\hat{\Delta})$ yields a contradiction or $g = h^3$.

Observe that by Remarks 2.2 (3)(i) and 3.8.1, $c^*(\hat{\Delta}) \leq (2-n)\pi + \frac{n}{2} \cdot \pi + \frac{n}{2} \cdot \frac{\pi}{2} + \frac{n}{2} \cdot \frac{\pi}{6}$ and so $c^*(\hat{\Delta}) > 0$ implies $n < 12$.

Let $\hat{\Delta} = (n, r)$ denote a region of degree n with $\Gamma_6 = r$. We need to check $c^*(\hat{\Delta})$ for $\hat{\Delta} = (n, r) = (10, 5), (10, 4), (10, 3), (10, 2), (10, 1), (8, 4), (8, 3), (8, 2)$ and $(8, 1)$. The region $(n, r) \neq (10, 5)$ or $(8, 4)$ else it gives $h^{\pm 5} = 1$ or $h^{\pm 4} = 1$ (respectively) contradicting $|h| \geq 6$. All possible labels for $\hat{\Delta} = (n, r) = (10, 4)$ or $(8, 3)$ yields a contradiction.

For example, $\hat{\Delta} = (8, 3)$ gives either $h^{\pm 4} = 1$ or $g = h^4$: the first contradicts $|h| \geq 6$ and the second implies $h = 1$. For $\hat{\Delta} = (10, r \leq 3)$, $c^*(\hat{\Delta}) \leq (2 - 10)\pi + 5\pi + 5 \cdot \frac{\pi}{2} + 3 \cdot \frac{\pi}{6} = 0$. Finally, since $(2 - 8)\pi + 3\pi + 5 \cdot \frac{\pi}{2} + 2 \cdot \frac{\pi}{6} = -\frac{\pi}{6} < 0$, $c^*(\hat{\Delta}) > 0$ for $\hat{\Delta} = (8, r \leq 2)$ only if it contains 4 corners with angle π (up to inversion $\hat{\Delta}$ is shown in Figure 3.8.2), and each possible $l(\hat{\Delta})$ yields a contradiction.

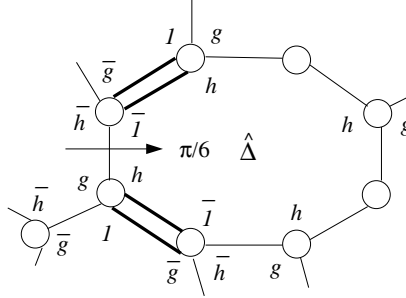


Figure 3.8.2: $\hat{\Delta} = (8, r \leq 2)$ with $c^*(\hat{\Delta}) > 0$.

3.9 Proof of Lemma 3.6(3): Case $(2, \bar{4}, \bar{4})$

Here $|g| = 2$, $|gh^{-1}| \geq 4$ and $|h| \geq 4$. Let \mathbb{P} be a reduced spherical picture over \mathcal{P} and assign the angle function α_2 to \mathbb{P} . By Remark 2.2(3)(i) a positive region $\hat{\Delta}$ can only have degree 4 or 6. It follows from Remarks 2.2(3)(ii) and (iii) that $l(\hat{\Delta})$ will yield a contradiction. Therefore, in this case \mathcal{P} is aspherical.

3.10 Proof of Lemma 3.6(4): Case $(2, \bar{6}, 3)$

If $[g, h] = 1$ then $|gh^{-1}| = 6$, $(gh^{-1})^3 = g$ and $(gh^{-1})^2 = h$. It follows that $\mathcal{P} = \langle G, x | x^m b^3 x b^2, b^6 \rangle$ and this presentation has been shown to be not aspherical by Bogley and Williams [4] (indeed it can be shown that b is conjugate to x^{m+1}). So it can be assumed that $[g, h] \neq 1$. We prove that $\mathcal{P} = \langle G, x | x^m g x h \rangle$ is aspherical. Let \mathbb{P} be a reduced spherical picture over \mathcal{P} with the assumption **(A)** stated in the proof of Lemma 3.1 and assign the angle function α_3 . By Remark 2.2(4)(i) the degree of a positive region Δ can only be 4 or 6. If Δ is a positive 4-region with an $h^{\pm 1}$ -corner then $l(\Delta)$ yields a contradiction, so assume otherwise. If now Δ has a $g^{\pm 1}$ corner then $l(\Delta)$ yields the 4-regions shown in Figure 3.10.1. This leaves $l(\Delta) = 11^{-1}11^{-1}$ which contradicts **(A)** as in the proof of Lemma 3.1.

If Δ is a 6-region, then either there is a contradiction or $l(\Delta) \in \{1^{-1}11^{-1}11^{-1}1, 1^{-1}11^{-1}g1^{-1}g, 1^{-1}h1^{-1}h1^{-1}h\}$. The first two cannot be positive, while the last gives the positive 6-region shown in Figure 3.10.1.

Define the following distribution scheme which is given in Figure 3.10.1:

$$\Gamma(\Delta, \hat{\Delta}) = \begin{cases} c(\Delta)/2 & \text{if } c(\Delta) = \pi \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single bond} \\ & \text{that is oriented from } \Delta \text{ to } \hat{\Delta} \\ c(\Delta) & \text{if } 0 < c(\Delta) \leq \frac{\pi}{2}, \Delta \text{ is separated from } \hat{\Delta} \text{ by a single bond } S \\ & \text{that is oriented from } \Delta \text{ to } \hat{\Delta} \text{ and } S \text{ is adjacent to a 1-corner} \\ & \text{in } \Delta \text{ with angle } \pi \\ \pi/6 & \text{if } c(\Delta) = \frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by a single bond} \\ & \text{that is oriented from } \hat{\Delta} \text{ to } \Delta \\ 0 & \text{otherwise} \end{cases}$$

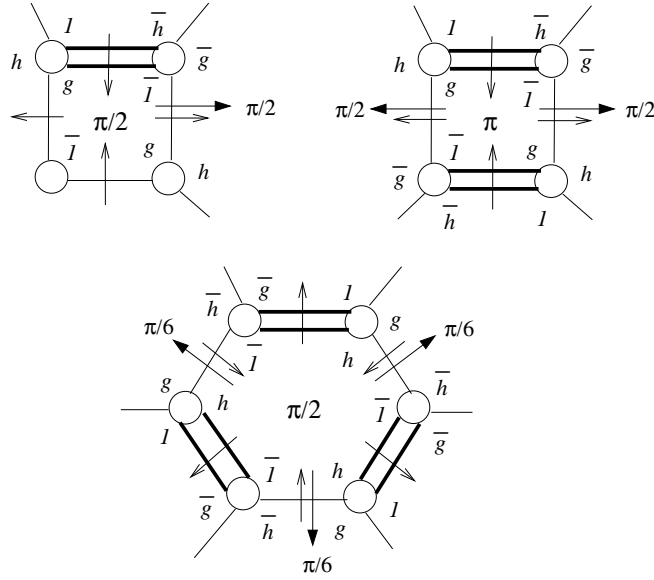


Figure 3.10.1: Positive regions and distribution scheme in Case(2, $\bar{6}$, 3).

Let r be the number of corners of angle π in Δ . Then $r \leq \frac{n}{2}$ (by Remark 2.2(4)(i)). Let s denote the number of pairs $(\frac{\pi}{2}, \frac{\pi}{6})$ or $(\frac{\pi}{6}, \frac{\pi}{2})$ such that $\hat{\Delta}$ receives $\frac{\pi}{2}$ and $\frac{\pi}{6}$ across adjacent edges in $\partial\hat{\Delta}$, with the understanding that each $\frac{\pi}{2}$ and $\frac{\pi}{6}$ that $\hat{\Delta}$ receives appears at most once in these pairs. Denote the remaining number of $\frac{\pi}{2}$ that $\hat{\Delta}$ receives by s_1 . Also, let s_2 denote the remaining number of $\frac{\pi}{6}$ that $\hat{\Delta}$ receives. As an example to show how to get the values s, s_1 and s_2 see Figure 3.10.2.

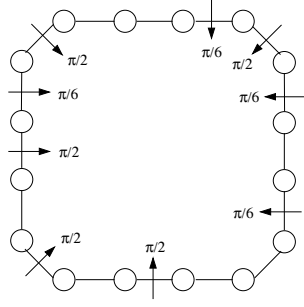


Figure 3.10.2: $n = 16, s = 2, s_1 = 3, s_2 = 2$.

Remarks 3.9.

1. As shown in Figure 3.10.1, $l(\hat{\Delta}) \in \{hg^{-1}w, h^{-1}gw\} \Rightarrow d(\hat{\Delta}) > 6$ for otherwise $l(\hat{\Delta})$ yields a contradiction.
2. $r \leq \frac{n}{2} - (s + s_1 + s_2)$.
3. $s + s_2 \leq \frac{n}{2}$.

Let $\hat{\Delta}$ be a region such that $c^*(\hat{\Delta}) > 0$. Then $c^*(\hat{\Delta}) \leq (2 - n)\pi + [\frac{n}{2} - (s + s_1 + s_2)]\pi + (\frac{n}{2} + s + s_1 + s_2)\frac{\pi}{2} + s(\frac{1}{2} + \frac{1}{6})\pi + s_1 \cdot \frac{\pi}{2} + s_2 \cdot \frac{\pi}{6} = \frac{\pi}{12}(24 - 3n + 2s - 4s_2)$, and so $c^*(\hat{\Delta}) > 0$ implies $24 - 3n + 2s - 4s_2 > 0 \Rightarrow 3n < 24 - 4s_2 + 2s \leq 24 - 4s_2 + 2(\frac{n}{2} - s_2) = 24 - 6s_2 + n \Rightarrow n < 12$.

Let $n = 10$. Then $c^*(\hat{\Delta}) > 0 \Rightarrow 24 - 3(10) + 2s > 4s_2 \geq 0 \Rightarrow s > 3$. If $s=4$ or 5 , then either $l(\hat{\Delta}) = (g^{-1}h)^4g^{-1}1$ which contradicts $[g, h] \neq 1$ or $l(\hat{\Delta}) = (g^{-1}h)^5$ which contradicts $|gh^{-1}| \geq 6$. This leaves $n = 8$. But checking the possible labels shows that $l(\hat{\Delta}) = hg^{-1}1g^{-1}h1^{-1}h1^{-1} \Rightarrow c^*(\hat{\Delta}) \leq -\frac{\pi}{3}$ (see Figure 3.10.3).

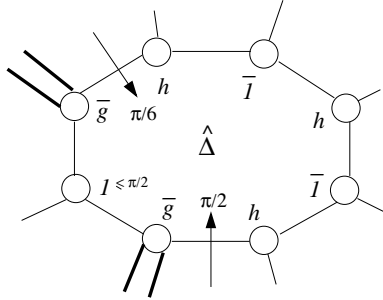


Figure 3.10.3: $n = 8$.

3.11 Proof of Lemma 3.7: Case(3, 2, $\bar{6}$)

Here, we assume that $|g| = 3, |gh^{-1}| = 2$ and $|h| \geq 6$. Let \mathbb{P} be a reduced spherical picture over \mathcal{P} and assign the angle function α_1 to \mathbb{P} . Observe that if $c(\hat{\Delta}) > 0$, then

$l(\hat{\Delta}) \in \{1^{-1}gw, h1^{-1}w\}$ (see Figure 2.2.6). It follows that all positively curved regions are shown in Figure 3.11.1.

Define the following distribution scheme which is given in Figure 3.11.1:

$$\Gamma(\Delta, \hat{\Delta}) = \begin{cases} \pi/6 & \text{if } c(\Delta) = \frac{\pi}{2} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by an } (m-1)\text{-bond} \\ c(\Delta)/2 & \text{if } 0 < c(\Delta) \leq \frac{\pi}{4} \text{ and } \Delta \text{ is separated from } \hat{\Delta} \text{ by an } (m-1)\text{-bond} \\ 0 & \text{otherwise} \end{cases}$$

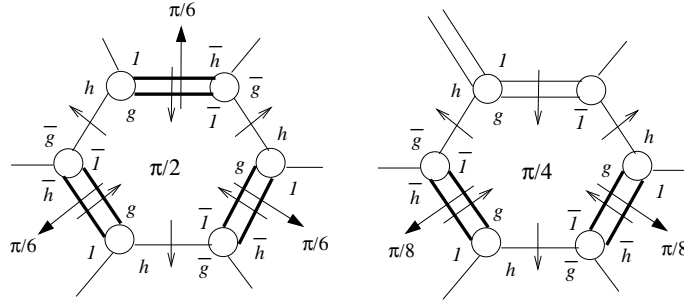


Figure 3.11.1: Positive regions and distribution scheme in Case(3, 2, $\bar{6}$).

For a fixed region $\hat{\Delta}$ again set $\Gamma_6(\hat{\Delta}) = |\{\Delta : \Gamma(\Delta, \hat{\Delta}) = \frac{\pi}{6}\}|$.

Remarks 3.10.

1. The region $\hat{\Delta}$ receives each $\pi/6$ through an $(m-1)$ -bond in its boundary which gives $(1h^{-1})^{\pm 1}$ as a sublabel of $\hat{\Delta}$.
2. $\hat{\Delta}$ receives $\pi/6$ only through edges that are oriented outwards $\hat{\Delta}$, and so $\hat{\Delta}$ does not receive $\pi/6$ through consecutive edges in its boundary ($\Gamma_6 \leq \frac{n}{2}$). Also, for each $\pi/6$ that $\hat{\Delta}$ receives, there are two corners in $\hat{\Delta}$ with angle $\frac{3\pi}{4}$. Therefore, $\Gamma_6 \leq \frac{r}{2}$, where r is the number of corners with angle $\frac{3\pi}{4}$ in the boundary of $\hat{\Delta}$.
3. As shown in Figure 3.11.1, $l(\hat{\Delta}) = 1h^{-1}w$, which implies that $d(\hat{\Delta}) > 4$ for otherwise $l(\hat{\Delta})$ yields a contradiction.

By using $\Gamma_6 \leq \frac{r}{2}$, $c^*(\hat{\Delta}) \leq (2-n)\pi + r \cdot \frac{3\pi}{4} + (n-r) \cdot \frac{\pi}{2} + \frac{r}{2} \cdot \frac{\pi}{6}$, and so $c^*(\hat{\Delta}) > 0 \Rightarrow 2r > 3n - 12$. Since $r \leq n$, this implies that $n < 12$.

Let $\hat{\Delta} = (n, r)$ denote a region of degree n with r corners of angle $\frac{3\pi}{4}$ and assume that $c^*(\hat{\Delta}) > 0$. Since $2r > 3n - 12$ it follows that if $n = 10$ then $r = 10$; if $n = 8$ then $r = 7$ or 8 ; and if $n = 6$ then $r = 4, 5$ or 6 . If $(n, r) = (10, 10)$ or $(8, 8)$ then $l(\hat{\Delta})$ implies that $h^5 = 1$ or $h^4 = 1$ contradicting $|h| \geq 6$. If $(n, r) = (8, 7)$ then $\hat{\Delta}$ is given by Figure 3.11.2 (i) and $l(\hat{\Delta})$ implies either $h^4 = 1$ or $g = h^4$ which is **(E2)**. This leaves $d(\hat{\Delta}) = 6$ and checking shows that $l(\hat{\Delta}) = 1h^{-1}gh^{-1}g1^{-1}$ as in Figure 3.11.2 (ii), otherwise there is a contradiction or condition **(E2)** occurs. But observe that if $r > 3$ in Figure 3.11.2 (ii) then $r = 4$ and since the $\hat{\Delta}$ corners of vertices u and v cannot have angle $\frac{3\pi}{4}$, this forces $x = h^{-1}$, a contradiction which completes the proof.

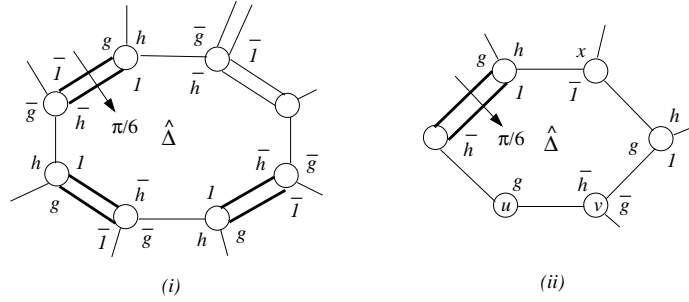


Figure 3.11.2.

4 Proof of Theorem 1.1

As mentioned in the introduction, the proof of Theorem 1.1 has been done previously for $m = 2, 3$ and 4 except for the exceptional cases **E4** and **E5** of [2]. But by following the proof of Lemma 3.6(3), Lemma 8(3) in [2] can be amended as follows: if $|g| = 2$, $|gh^{-1}| \geq 4$, $|h| \geq 4$ and $g \neq h^2$, then \mathcal{P} is aspherical even if $[g, h] = 1$; and so in these two cases \mathcal{P} is also aspherical. So it can be assumed that $m \geq 5$. The ‘only if’ direction of Theorem 1.1 follows from Lemmas 3.2, 3.3, 3.4 and 3.6(4). For the rest of the proof we assume that none of the Conditions (1)-(7) of Theorem 1.1 is satisfied. We show that either \mathcal{P} is aspherical or exceptional.

If none of the conditions of Lemma 3.1 holds, then \mathcal{P} is aspherical. Assume that Condition 1 of Lemma 3.1 holds. Then $|g| = \infty$ (since Condition 1 of Theorem 1.1 does not hold), and so \mathcal{P} is aspherical by Lemma 3.2. So assume from now on that $g \neq h^{\pm 1}$.

If Condition 2 of Lemma 3.1 holds, then it can be assumed without any loss that $g = h^2$. Then $|h| \geq 5$ (by the negation of Condition 3 of Theorem 1.1). If $|h| \in \{5, 6\}$ then \mathcal{P} is exceptional of type **(E1)** or **(E2, $g = h^2$)**; and if $|h| \geq 7$, then \mathcal{P} is aspherical by Lemma 3.3. So assume from now on that $g \neq h^2$.

If Condition 3 of Lemma 3.1 holds, then it can be assumed without any loss that $|g| = 2$. Since $g \neq h$, $|gh^{-1}| \geq 2$. If $|gh^{-1}| = 2$ then $|h| = \infty$ (Condition 7 of Theorem 1.1) and it follows that \mathcal{P} is aspherical by Lemma 3.6(1). If $|gh^{-1}| = 3$, then $|h| \geq 6$ (Condition 7 of Theorem 1.1). By Lemma 3.6(2), \mathcal{P} is aspherical if $g \neq h^3$, while if $g = h^3$ then \mathcal{P} is exceptional of type **(E2, $g = h^3$)**. If $|gh^{-1}| = 4$ or 5 then $|h| \geq 4$ (Condition 7 of Theorem 1.1), and so \mathcal{P} is aspherical by Lemma 3.6(3). Now suppose that $|gh^{-1}| \geq 6$. By Lemma 3.5, if $|gh^{-1}| = \infty$ then \mathcal{P} is aspherical, so assume otherwise. Then $|h| \geq 3$ (Condition 7 of Theorem 1.1). If $|h| = 3$ then $[g, h] \neq 1$, otherwise Condition 6 of Theorem 1.1 holds, and so \mathcal{P} is aspherical by Lemma 3.6(4). If $|h| \geq 4$, then \mathcal{P} is aspherical by Lemma 3.6(3).

Finally, if Condition 4 of Lemma 3.1 is satisfied then it can be assumed without loss that $|g| = 3$ and $|gh^{-1}| = 2$. Hence $|h| \geq 6$ (else, Condition 7 of Theorem 1.1 applies). If $g = h^4$ then \mathcal{P} is exceptional of type **(E2, $g = h^4$)**; otherwise \mathcal{P} is aspherical by Lemma 3.7.

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