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Valid Inequalities for a Single Constrained 0-1 MIP Set Intersected with a Conflict Graph

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Abstract

In this paper a mixed integer set resulting from the intersection of a single constrained mixed 0-1 set with the vertex packing set is investigated. This set arises as a subproblem of more general mixed integer problems such as inventory routing and facility location problems. Families of strong valid inequalities that take into account the structure of the simple mixed integer set and that of the vertex packing set simultaneously are introduced. In particular, the well-known mixed integer rounding inequality is generalized to the case where incompatibilities between binary variables are present. Exact and heuristic algorithms are designed to solve the separation problems associated to the proposed valid inequalities. Preliminary computational experiments show that these inequalities can be useful to reduce the integrality gaps and to solve integer programming problems.

Keywords: mixed integer programming; valid inequality; separation; vertex packing set; conflict graph; independent set;

1. Introduction

It is well-known that the use of strong valid inequalities as cuts can be very effective in solving mixed integer problems. One classical approach to generate these valid inequalities is to study the polyhedral structure of simple sets which occur as relaxations of the feasible sets of those general problems. Two such successful examples are the use of Mixed Integer Rounding (MIR) inequalities, derived from a basic mixed integer set [14, 19], and the use of valid inequalities for conflict graphs, resulting from logical relations between binary variables, for solving mixed integer programs [5].

The goal of this paper is to investigate the polyhedral structure of a mixed integer set that results from the intersection of two well-known sets: a simple mixed integer set and the vertex packing set associated with a conflict graph.

Let X be the set of points $(s, x) \in \mathbb{R} \times \mathbb{Z}^n$ satisfying

$$s + c \sum_{i \in N_1} x_i \geq d, \tag{1}$$

$$x_i + x_j \leq 1, \ \{i, j\} \in E, \tag{2}$$

$$x_i \in \{0, 1\}, \ i \in N, \tag{3}$$

$$s \geq 0, \tag{4}$$

where $N = \{1, \ldots, n\}$ is the index set of binary variables, and E is the set of pairs of indices of incompatible nodes, $N_1 \subseteq N$, and $c > 0$, $d > 0$. The graph $G = (N, E)$ is known as the conflict graph of pairwise conflicts between binary variables (see [1, 5]).

Let $N_0 = N \setminus N_1$. Although the general results and the validity of the inequalities presented in the paper hold for the case where N_0 is empty, some facet-defining conditions need to be adjusted. Therefore, to ease the reading of the paper, N_0 is assumed to be nonempty. When $c > d$, the inequality $s + c \sum_{i \in N_1} x_i \geq d$ can be replaced by the stronger inequality $s + d \sum_{i \in N_1} x_i \geq d$. Thus, henceforward, it is also assumed that $c \leq d$.

Set X is the intersection of two sets: $X = X_{VP} \bigcap X_{SMI}$, where X_{VP} is the vertex packing set defined by (2)–(3), that results by considering the conflict graph $G = (N, E)$, and X_{SMI} is a simple mixed integer set defined by $\{(s, x) \in \mathbb{R} \times \mathbb{B}^{|N_1|} | \text{ satisfying (1) and (4)}\}$. The convex hulls of X, X_{VP}, and X_{SMI}, are denoted by P, P_{VP}, and P_{SMI}, respectively.

The set X_{SMI} has been intensively used as a relaxation of several mixed integer sets, see [19] for examples. It is well-known that in order to describe P_{SMI}, when $|N_1| \geq \lceil \frac{d}{c} \rceil$, it suffices to add to the defining inequalities (1), (4), $x_i \geq 0$, and $x_i \leq 1, i \in N_1$, the following MIR inequality

$$s + r \sum_{i \in N_1} x_i \geq r \left\lceil \frac{d}{c} \right\rceil, \tag{5}$$

where $r = d - c(\lceil \frac{d}{c} \rceil - 1)$.

On the contrary, a complete description of P_{VP} is not known and since optimizing a linear function over X_{VP} is a NP-hard problem, there is not much hope in finding such a description. Nevertheless, families of valid inequalities are known, see [9, 10, 16, 17]. The derivation of inequalities for integer programs based on conflict graphs have also been considered in the past (see [5] for further details).

Although the two sets X_{SMI} and X_{VP} have been intensively considered in the past, to the best of our knowledge, set X has only been considered in a preliminary version of this paper [4]. The most related mixed integer sets considered before are the mixed vertex packing set studied by Atamtürk et al. [6] and the flow set with partial order studied by Atamtürk and Zang [7].

Cuts from valid inequalities for X_{SMI} and X_{VP} are commonly used by researchers using MIP solvers, by identifying these sets as relaxations of the original feasible set. This work aims at deriving new inequalities that can be used when those structures are present simultaneously. Such structures can be found in various mixed integer problems, such as inventory routing, production planning, facility locations, network design, etc. The practical examples that motivated this research stemmed from maritime Inventory Routing Problems (IRPs), see [2, 3]. Constraint (1) results from the relaxation of inventory constraints, where s is the stock level at a given location, d is the aggregated demand at that location during a set of periods, c is the vehicle capacity (when several vehicles are considered one may assume this capacity to be constant for all vehicles, otherwise one can take c as the maximum of these capacities) and x_i represents an arc traveled by a vehicle. N_1 is the index set of arcs entering to that particular node. Constraints (2) represent incompatible arcs, that is, arcs that cannot belong to the same route, for instance, due to time constraints. The two sets X_{SMI} (e.g. in [2]) and X_{VP} (e.g. in [3]) were considered as relaxations of the set of feasible solutions previously in such problems. However they have never been considered simultaneously.

From the theoretical point of view, valid inequalities for X_{VP} and valid inequalities for X_{SMI} are valid for X. As, in general, P is strictly included in $P_{VP} \bigcap P_{SMI}$, there are fractional solutions that cannot be cut off by valid inequalities derived either for P_{VP} or P_{SMI}. Hence, in this paper, the focus is on valid inequalities derived for P that take into account properties from the two sets simultaneously. In particular, valid inequalities are proposed that extend the well-known MIR inequalities to the case where incompatibility constraints are imposed on pairs of binary variables. This leads to new inequalities, some of them resembling MIR inequalities, that incorporate variables in N_0 that do not appear in the set X_{SMI}. Notice however that, similar to what happens to P_{VP}, the complete linear description of P remains unknown.

The outline of this paper is as follows. In Section 2, basic properties of P are discussed and related with P_{SMI} and P_{VP}. Furthermore, conditions for the MIR inequality, the defining inequality $s \geq 0$, and other known inequalities for X_{VP} to define facets of P are established. In Section 3, several families of valid inequalities for X are derived and, in particular, a new family of inequalities, called conflict MIR inequalities, is introduced that strengthens the well-known MIR inequalities for set X by incorporating conflicts between the variables into the inequality. In addition, conditions for some of those inequalities to be facet-defining

are provided. In Section 4, exact and heuristic procedures are discussed to solve the separation problems associated to those valid inequalities. In Section 5, computational experiments on randomly generated instances of a single node fixed-charge set with conflicts on arcs are reported. Finally, in Section 6, the main conclusions and future lines of research are presented.

2. Basic polyhedral results

In this section some basic results on set X are provided.

Proposition 2.1. *Polyhedron P is full-dimensional.*

Proof. It suffices to consider the following $n + 2$ affinely independent points belonging to X :

(*i*) for all $j \in N_1, x_j = 1; x_i = 0, i \in N \setminus \{j\}; s = d - c$;

(*ii*) for all $j \in N_0, x_j = 1; x_i = 0, i \in N \setminus \{j\}; s = d$;

(*iii*) $x_i = 0, \forall i \in N; s = d$;

(*iv*) $x_i = 0, \forall i \in N; s = 2d$.

$\square$

Proposition 2.2. *Polyhedron P is unbounded, with one extreme ray $v = (1, \mathbf{0})$, where $\mathbf{0}$ is the null vector of dimension n.*

Proof. The characteristic cone of polyhedron P is the following.

$$char.cone(P) = \left\{ (s, x) \mid s + c \sum_{i \in N_1} x_i \geq 0, x_i + x_j \leq 0, \{i, j\} \in E, s \geq 0, x_i = 0, i \in N \right\}$$

$$= \left\{ (s, x) \mid s \geq 0, x_i = 0, i \in N \right\}.$$

Hence, P has an extreme ray $(1, \mathbf{0})$.

$\square$

Proposition 2.3. *Inequality* (1) *defines a facet of P.*

Proof. It suffices to consider the first $n + 1$ points given in the proof of Proposition 2.1.

$\square$

It is easy to check that the projection of X onto the space of x variables, $Proj_x(X)$, coincides with X_{VP}, which is stated in the following proposition.

Proposition 2.4. $Proj_x(X) = X_{VP}$.

The following result establishes a relation between facet-defining inequalities for P_{VP} and some facet-defining inequalities for P.

Proposition 2.5. *Every facet-defining inequality $\sum_{i \in N} \alpha_i x_i \geq \delta$, for P_{VP} is a facet-defining inequality for P. Conversely, every facet-defining inequality $\sum_{i \in N} \alpha_i x_i + \beta s \geq \delta$, for P with $\beta = 0$, is a facet-defining inequality of P_{VP}.*

Proof. Assume $\sum_{i \in N} \alpha_i x_i \geq \delta$ is valid for X_{VP}, and defines a facet of P_{VP}. Since X includes all the constraints defining X_{VP}, and $\sum_{i \in N} \alpha_i x_i \geq \delta$ is valid for X_{VP}, then it is also valid for X. As $(1, \mathbf{0})$ is a ray of P, then each facet-defining inequality of P_{VP} defines also a facet of P.

Next, assume $\sum_{i \in N} \alpha_i x_i + \beta s \geq \delta$ defines a facet of P with $\beta = 0$. As $Proj_x(X) = X_{VP}$, and since $\sum_{i \in N} \alpha_i x_i + \beta s \geq \delta$ is valid for X with $\beta = 0$, then it is also valid for X_{VP}. Suppose $\sum_{i \in N} \alpha_i x_i \geq \delta$ does not define a facet of P_{VP}. This assumption implies that all the points in P_{VP} satisfying $\sum_{i \in N} \alpha_i x_i = \delta$ also satisfy the inequality $\pi x \geq \pi_0$ as equation. Then, all the points in the corresponding facet of P would also satisfy $\pi x = \pi_0$, which is a contradiction.

$\square$

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As a consequence of Proposition 2.5, one can conclude that the interesting inequalities (those that combine the structure of the vertex packing set with the simple mixed integer set) must include the continuous variable.

The following notation is used throughout this paper. Consider graph $G = (N, E)$. For $j \in N$, $N(j) = \{i \in N \mid \{i,j\} \in E\}$ is set of vertices in N which are in conflict with node j, $N_1(j) = \{i \in N_1 \mid \{i,j\} \in E\}$, and $N_0(j) = \{i \in N_0 \mid \{i,j\} \in E\}$. In addition, for $S \subseteq N$, $N_1(S) = \bigcup_{j \in S} N_1(j)$, $\widetilde{N}_1(S) = \bigcap_{j \in S} N_1(j)$, and $N_0(S) = \bigcup_{j \in S} N_0(j)$. Notice that if S is a singleton then $\widetilde{N}_1(S) = N_1(S)$. Moreover, $G[S]$ denotes the subgraph induced by set S and $\alpha(G[S])$ represents the independence number of the corresponding graph. For $C \subseteq N$ and $b \in \mathbb{Z}_+$, $\mathcal{I}(C)$ denotes the set of all independent sets of $G[C]$ which includes the empty set, and $\mathcal{I}_b(C)$ denotes the set of all independent sets of $G[C]$ with cardinality equal to b.

A class of well-known clique inequalities (see [16, 17]) for set X_{VP} is given next.

Theorem 2.1. *An inequality $\sum_{i \in K} x_i \leq 1$, where $K \subseteq N$, is a facet of P_{VP} if and only if K is a maximal clique in the conflict graph G.*

Theorem 2.1 and Proposition 2.5 ensure that inequality $\sum_{i \in K} x_i \leq 1$, where $K \subseteq N$ is a maximal clique in G, defines a facet of P. In particular, they give conditions for trivial inequalities to define facets of P, see (ii) and (iii) in the following proposition. A single node (case (ii)) defines a maximum clique if it has no neighbors, and a pair of adjacent nodes (case (iii)) defines a maximum clique if they do not have any common neighbor.

Proposition 2.6. *(i) $x_i \geq 0, i \in N$ is facet-defining for P.*
(ii) $x_i \leq 1, i \in N$ defines a facet of P if and only if $N(i) = \emptyset$.
(iii) $x_i + x_j \leq 1$ defines a facet of P if and only if $N(i) \cap N(j) = \emptyset$.

Next, sufficient conditions for inequalities $s \geq 0$ and MIR to be facet-defining for P are established. Furthermore, the idea of constructing an auxiliary graph presented in [13], to prove that the rank inequalities define facets, is implemented to achieve the following result.

Define the graph $G'_a = (N', E'), a \in \mathbb{Z}_+$, having N' as node set and whose edges are defined as follows: two nodes i and j are adjacent in G'_a if and only if there exists an independent set $I \in \mathcal{I}_a(N')$ such that $i \in I, j \notin I$, and $(I \setminus \{i\}) \cup \{j\} \in \mathcal{I}_a(N')$.

Proposition 2.7. *Inequality $s \geq 0$ defines a facet of P if the following conditions hold.*

(i) $\alpha(G[N_1]) \geq \left\lceil \frac{d}{c} \right\rceil + 1$.

(ii) $G'_{\lceil \frac{d}{c} \rceil}$ with $N' = N_1$ is connected.

(iii) $\alpha(G[N_1 \setminus N_1(j)]) \geq \left\lceil \frac{d}{c} \right\rceil, \forall j \in N_0$.

Proof. Define $K = P \cap \{(s, x) \mid s = 0\}$ and show that inequality $s \geq 0$ is facet-defining by showing that whenever the inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$, is valid for P and satisfies the condition $\gamma s + \sum_{i \in N} \beta_i x_i = \gamma_0, \forall (s, x) \in K$, then $\gamma s + \sum_{i \in N} \beta_i x_i$ and s are identical linear forms up to positive multiple.

For each $j \in N_0$, condition (iii) ensures that there exists $T_j \subseteq N_1 \setminus N_1(j)$ such that $|T_j| = \lceil \frac{d}{c} \rceil$. Hence, for each $j \in N_0$, consider the following points belonging to K.

(P1) $s = 0; x_i = 1, i \in T_j; x_i = 0, i \in N \setminus T_j;$

(P2) $s = 0; x_i = 1, i \in T_j; x_i = 0, i \in N \setminus (T_j \cup \{j\}); x_j = 1.$

Points (P1) and (P2) imply $\beta_j = 0, \forall j \in N_0$. Now take $i, j \in N_1$ and assume that they are adjacent in graph $G'_{\lceil \frac{d}{c} \rceil}$. From the definition of $G'_{\lceil \frac{d}{c} \rceil}$, there exists an independent set I such that $I \subseteq N_1, i \in I, j \notin I, I' = (I \setminus \{i\}) \cup \{j\}$ is an independent set and $|I| = |I'| = \lceil \frac{d}{c} \rceil$. Consider the points $s = 0; x_t = 1, t \in I; x_t = 0, t \in N \setminus I$ and $s = 0; x_t = 1, t \in I'; x_t = 0, t \in N \setminus I'$ in X that belong to K. Substituting these two

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points in (8) and subtracting the resultant equations gives $\beta_i = \beta_j$. It now follows from the connectivity of graph $G'_{\lceil \frac{d}{c} \rceil}$ (condition (ii)) that $\beta_i = \beta, \forall i \in N_1$.

Finally, from (i), there exists $T \subseteq N_1$, $|T| = \lceil \frac{d}{c} \rceil + 1$ such that the point (P3) $s = 0, x_i = 1, i \in T, x_i = 0, i \in N \setminus T$ belongs to K. Now, considering the point (P4) $s = 0, x_i = 1, i \in T \setminus \{\ell\}, x_i = 0, i \in (N \setminus T) \cup \{\ell\}$ also in K, it follows that $\beta = 0$ and therefore $\gamma_0 = 0$. □

The facet-defining conditions for the MIR inequality are established and presented as follows.

Proposition 2.8. *The MIR inequality* (5) *defines a facet of P if the following conditions hold.*

(i) $\alpha\big(G[N_1]\big) \geq \lceil \frac{d}{c} \rceil$.

(ii) $G'_{\lfloor \frac{d}{c} \rfloor} = (N_1, E')$ *is connected.*

(iii) $\alpha\big(G[N_1 \setminus N_1(j)]\big) \geq \lfloor \frac{d}{c} \rfloor, \forall j \in N_0$.

Proof. Consider the equation

$$s + r \sum_{i \in N_1} x_i = r \left\lceil \frac{d}{c} \right\rceil. \tag{6}$$

Define $K = P \cap \{(s, x) \mid (s, x) \text{ satisfies } (6)\}$. One can prove that inequality (5) is facet-defining by showing that whenever the inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ is valid for P and satisfies the condition

$$\gamma s + \sum_{i \in N} \beta_i x_i = \gamma_0, \forall (s, x) \in K, \tag{7}$$

then equality (7) is a multiple of (6).

For each $j \in N_0$, condition (iii) ensures that there exists a set $T_j \in \mathcal{I}_{\lfloor \frac{d}{c} \rfloor}(N_1 \setminus N_1(j))$, such that the following feasible points belong to K.

(P1) $s = r; x_i = 1, i \in T_j; x_i = 0, i \in N \setminus T_j$;

(P2) $s = r; x_i = 1, i \in T_j; x_j = 1; x_i = 0, i \in N \setminus (T_j \cup \{j\})$.

By substituting the points of type (P1) and (P2) in equation (7) and subtracting the resultant equations it follows that $\beta_j = 0, \forall j \in N_0$. Thus, equality (7) can be rewritten as

$$\gamma s + \sum_{i \in N_1} \beta_i x_i = \gamma_0. \tag{8}$$

Now take $i, j \in N_1$ and assume that they are adjacent in graph $G'_{\lfloor \frac{d}{c} \rfloor}$. From the definition of $G'_{\lfloor \frac{d}{c} \rfloor}$, there exists an independent set I such that $I \subseteq N_1, i \in I, j \notin I, I' = (I \setminus \{i\}) \cup \{j\}$ is an independent set and $|I| = |I'| = \lfloor \frac{d}{c} \rfloor$. Consider the points $s = r; x_i = 1, i \in I; x_i = 0, i \in N \setminus I$ and $s = r; x_i = 1, i \in I'; x_i = 0, i \in N \setminus I'$ in X that belong to K. Substituting the two points in (8) and subtracting the resulting equations gives $\beta_i = \beta_j$. It now follows from the connectivity of graph $G'_{\lfloor \frac{d}{c} \rfloor}$ that $\beta_i = \beta, \forall i \in N_1$.

Condition (i) ensures the existence of the points of the following form, which are in K,

$$\forall \, T \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(N_1), s = 0; x_i = 1, i \in T; x_i = 0, i \in N \setminus T.$$

Replacing these points in equation (8), it follows that $\beta \lceil \frac{d}{c} \rceil = \gamma_0$. Now, using points of type (P1) gives $\gamma r + \beta \lfloor \frac{d}{c} \rfloor = \gamma_0$. These two equalities imply $\beta = \gamma r$ and $\gamma_0 = \gamma r \lceil \frac{d}{c} \rceil$ and so (7) is a multiple of (6). □

Conditions (i) and (iii) of Proposition 2.8 are necessary conditions for (6) to define a facet. The following example shows that condition (ii) is not a necessary condition.

Example 2.1. *Consider the set X with $d = 20, c = 9, N = \{1, \ldots, 8\}, N_1 = \{1, \ldots, 6\}$, and the conflict graph depicted in Figure 1. It can be seen that the MIR inequality (5), $s + 2 \sum_{j \in N_1} x_j \geq 6$, defines a facet of P. In addition to the points of type (P1) and (P2) the following (s, x) points $e_{6,7}, e_{5,7}, e_{5,6}, e_{1,2}, e_{1,3}, e_{2,3}$, are tight, where e_{ij} is the vector with 1 in positions i and j, and zero elsewhere. Using these points, graph $G'_{\lfloor \frac{d}{c} \rfloor}$ depicted in Figure 2 is obtained. Although this graph is not connected, following the proof of Proposition 2.8, such points are enough to enforce $\beta_i = \beta, i \in N_1$ in equation (8).*
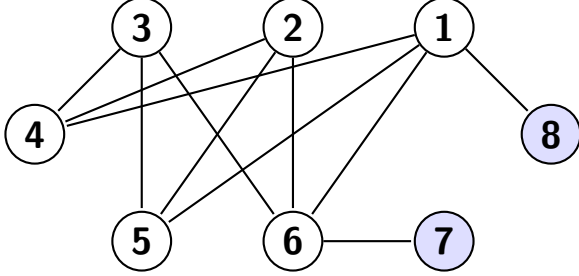

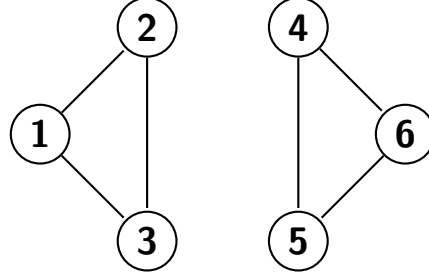Figure 1: Conflict graph corresponding to Example 2.1.

Figure 2: Graph $G'_{\lfloor \frac{d}{c} \rfloor}$ corresponding to Example 2.1.

The following proposition shows that if $\alpha(G[N_1]) \leq \lfloor \frac{d}{c} \rfloor$, then all non-trivial facet-defining inequalities for P are those from the vertex packing polytope.

Proposition 2.9. *Let $\alpha(G[N_1]) \leq \lfloor \frac{d}{c} \rfloor$. If inequality*

$$\sum_{i \in N} \alpha_i x_i + \beta s \geq \gamma, \tag{9}$$

with $\beta \neq 0$, defines a facet of P, then inequality (9) is a multiple of inequality (1).

Proof. First, note that since $(1, \mathbf{0})$ is an extreme ray, then $\beta \geq 0$. As $\beta \neq 0$, assume that $\beta > 0$. Then every point of X satisfying inequality (9) as equation also satisfies $s + c \sum_{i \in N_1} x_i = d$. Otherwise, if there exists a point $(s^*, x^*) \in X$ such that $s^* + c \sum_{i \in N_1} x_i^* > d$ and $\sum_{i \in N} \alpha_i x_i^* + \beta s^* = \gamma$, then condition $\alpha(G[N_1]) \leq \lfloor \frac{d}{c} \rfloor$ implies $s^* > 0$. Thus, the feasible point $(s^* - \epsilon, x^*) \in X$ with $0 < \epsilon \leq s^* + \sum_{i \in N_1} x_i^* - d$ violates inequality (9), which is a contradiction. $\square$

Henceforward, assume $\alpha(G[N_1]) \geq \lceil \frac{d}{c} \rceil$.

2.1. Application to single node fixed-charge set with conflicts on arcs

Set X discussed in this paper can occur as a relaxation of several more complex feasible sets of general mixed integer programs. Here a set Y is introduced that can be seen as an intermediate set between those general mixed integer sets and the set X. This set is a variant of the single node fixed-charge set where incompatibilities between arcs are considered, and it is defined as follows.

$$Y = \left\{ (s, y, x) \in \mathbb{R} \times \mathbb{R}^{|N_1|} \times \mathbb{B}^{|N|} \mid s + \sum_{i \in N_1} y_i \geq d, y_i \leq c x_i, i \in N_1, \right.$$

$$\left. x_i + x_j \leq 1, \{i, j\} \in E, s \geq 0, y_i \geq 0, i \in N_1 \right\},$$

where $N_1 \subset N$, and E is the edge set.

Set X is a restriction of Y by setting $y_i = c x_i, \forall i \in N_1$. Obviously, valid inequalities for X_{VP} are valid for Y. Furthermore, the following proposition establishes the relation between valid inequalities for X and Y.

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Proposition 2.10. *Any valid inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ for X is also valid for Y.*

Proof. Suppose not. That is, there exists $(s^*, y^*, x^*) \in Y$ such that $\gamma s^* + \sum_{i \in N} \beta_i x_i^* < \gamma_0$. Then the inequality is also violated by $(s^*, y', x^*) \in Y$ where $y_i' = c x_i^*$. Thus $(s^*, x^*) \in X$ and inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ is violated by this point, which is a contradiction. $\qquad \square$

The computational tests discussed in Section 5 are conducted for set Y.

3. Valid inequalities

This section presents new families of valid inequalities for X. These inequalities will be grouped into three large families: the lifted $s \geq 0$ inequalities, the residual inequalities and the conflict MIR inequalities.

3.1. Lifted $s \geq 0$ inequalities

To derive the first family of inequalities, notice that if $x_j = 1$ for some $j \in N$, then $x_i = 0, \forall i \in N_1(j)$. Hence, it follows

$$s \geq l_j x_j, \tag{lift 0}$$

is valid for X, where $l_j = (d - \alpha (G[N_1 \setminus N_1(j)]) c)^+$ and $(x)^+ = \max\{0, x\}$. This inequality can be regarded as the lifting of inequality $s \geq 0$ when this inequality does not define a facet. Inequality (lift 0) can be extended in two directions. One is to extend the right-hand side of the inequality for each clique. The other direction is to consider a subset of N_1 in the left-hand side. The following proposition gives the valid inequality for the general case.

Proposition 3.1. *Let $S \subseteq N$ be a clique in G and $T \subseteq N_1 \setminus S$. Then the following inequality is valid for X.*

$$s + c \sum_{i \in T} x_i \geq \sum_{i \in S} (d - p_i c)^+ x_i, \tag{lift 1}$$

where $p_i = \alpha (G[N_1 \setminus (N_1(i) \cup T)])$.

Proof. Let $(s, x) \in X$. Notice that since S is a clique then $\sum_{i \in S} x_i \leq 1$. If $\sum_{i \in S} x_i = 0$ then inequality (lift 1) is implied by nonnegativity of $x_i, i \in T$ and s.

Assume $x_i = 1$ for some $i \in S$. This implies $x_j = 0, j \in N_1(i)$. If $(d - p_i c)^+ = 0$, then the inequality trivially holds. Hence, assume $d - p_i c > 0$. Then from (1) it follows

$$s + c \sum_{i \in N_1} x_i = s + c \sum_{i \in T} x_i + c \sum_{i \in N_1(i) \setminus T} x_i + c \sum_{i \in N_1 \setminus (N_1(i) \cup T)} x_i \geq d,$$

which implies

$$s + c \sum_{i \in T} x_i \geq d - c \sum_{i \in N_1 \setminus (N_1(i) \cup T)} x_i \geq d - c p_i = (d - c p_i)^+ x_i = \sum_{i \in S} (d - p_i c)^+ x_i.$$

$\qquad \square$

Proposition 3.2. *If the following conditions hold, then inequality (lift 1) defines a facet of P.*

(i) *For each $i \in N_1 \setminus (T \cup S), \alpha(G[N_1 \setminus (T \cup S \cup N_1(i) \cup \{i\})]) \geq \lceil \frac{d}{c} \rceil$.*

(ii) *For each $i \in N_0 \setminus S, \alpha(G[N_1 \setminus (T \cup S \cup N_1(i))]) \geq \lceil \frac{d}{c} \rceil$.*

(iii) *For each $i \in T$, there exists at least one $j \in S$ with $\{i, j\} \notin E$, and $p_j < \lfloor \frac{d}{c} \rfloor$ such that*

$$\alpha(G[N_1 \setminus (N_1(j) \cup T \setminus \{i\})]) \geq p_j + 1.$$
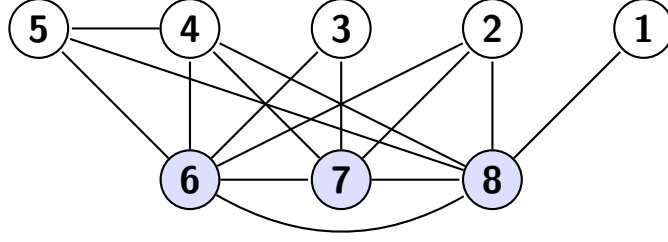
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Figure 3: Conflict graph considered in Example 3.1.

Proof. Without loss of generality assume that $d - p_i c > 0, i \in S$. Consider the equality

$$s + c \sum_{i \in T} x_i = \sum_{i \in S} (d - p_i c) x_i, \tag{10}$$

and let $K = P \cap \{(s, x) \mid (s, x) \text{ satisfies (10)}\}$. Now assume inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ is valid for X and satisfies the condition

$$\gamma s + \sum_{i \in N} \beta_i x_i = \gamma_0, \forall (s, x) \in K. \tag{11}$$

So one can show that equality (11) is a multiple of (10) by generating the following points belonging to K. Conditions (i) and (ii) ensure the existence of the following points.

(P1) $\forall \overline{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(N_1 \setminus (T \cup S)), s = 0; x_i = 1, i \in \overline{T}; x_i = 0, i \in N \setminus \overline{T};$

(P2) $\forall j \in N_1 \setminus (T \cup S), \forall \overline{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(N_1 \setminus (T \cup S \cup N_1(j) \cup \{j\})), s = 0; x_i = 1, i \in \overline{T}; x_j = 1; x_i = 0, i \in N \setminus (\overline{T} \cup \{j\});$

(P3) $\forall j \in N_0 \setminus S, \forall \overline{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(N_1 \setminus (T \cup S \cup N_1(j))), s = 0; x_i = 1, i \in \overline{T}; x_j = 1; x_i = 0, i \in N \setminus (\overline{T} \cup \{j\}).$

Points (P1), (P2) and (P3) imply $\beta_i = 0, i \in N_1 \setminus (T \cup S), \beta_i = 0, i \in N_0 \setminus S$ and $\gamma_0 = 0$. For each $j \in S$, from the definition of p_j, there exits $\overline{T} \in \mathcal{I}_{p_j}(N_1 \setminus (T \cup N_1(j)))$. Considering the point $s = d - p_j c; x_i = 1, i \in \overline{T}; x_j = 1; x_i = 0, i \in N \setminus (\overline{T} \cup \{j\})$ and substituting it in equation (11) gives $\beta_i = -\gamma(d - p_i c), i \in S$. Finally, for each $i \in T$, and each $j \in S$ such that condition (iii) is satisfied, consider the point $s = d - (p_j + 1)c; x_k = 1, k \in \overline{T} \in \mathcal{I}_{p_j}(N_1 \setminus (T \cup N_1(j))); x_j = x_i = 1; x_k = 0, k \in N \setminus (\overline{T} \cup \{j, i\})$. Replacing these points in equation (11) implies $\beta_i = \gamma c, i \in T$. Hence, (11) is a multiple of (10). $\square$

Facet-defining inequalities of type (lift 1) are illustrated in the following example.

Example 3.1. *Let $d = 20, c = 9, N = \{1, \ldots, 8\}, N_1 = \{1, \ldots, 5\}$ and the conflict graph G shown in Figure 3. One can check that the following inequalities*

$$s + 9x_5 \geq 11x_6 + 11x_7 + 11x_8,$$
$$s \geq 11x_6 + 2x_7 + 11x_8,$$

define facets of P with $S = \{6, 7, 8\}, T = \{5\}$, and $S = \{6, 7, 8\}, T = \emptyset$, respectively.

Remark 3.1. *Consider valid inequality* (lift 1) *by setting $T = N_1 \setminus \widetilde{N}_1(S)$. Then, one can check that $p_i = 0, \forall i \in S$. Thus, the following inequality is valid for X.*

$$s + c \sum_{i \in N_1 \setminus \widetilde{N}_1(S)} x_i \geq d \sum_{i \in S} x_i. \tag{12}$$

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For the particular case of $d - p_i c = r$, the following class of valid inequalities can be derived where S is not restricted to be a clique.

Proposition 3.3. *Let $S \subseteq N_0$, and $T \subseteq N_1$ such that*

$$\alpha\big(G[S]\big) \leq \left\lceil \frac{d}{c} \right\rceil,$$

and

$$\alpha\big(G[T \setminus N_1(\bar{S})]\big) \leq \left\lceil \frac{d}{c} \right\rceil - |\bar{S}|, \forall \bar{S} \in \mathcal{I}(S). \tag{13}$$

Then the following inequality is valid for X.

$$s + r \sum_{i \in N_1 \setminus T} x_i \geq r \sum_{i \in S} x_i. \tag{lift 2}$$

Proof. If $\sum_{i \in S} x_i = 0$, then validity of (lift 2) follows from nonnegativity of s and $x_i, i \in N_1 \setminus T$. Assume $\sum_{i \in S} x_i \geq 1$. Let $\bar{S} = \{i \in S : x_i = 1\}$. Thus $\sum_{i \in S} x_i = |\bar{S}|$ where $\bar{S}$ is an independent set. Then

$$s + r \sum_{i \in N_1 \setminus T} x_i \geq r\Big(\left\lceil \frac{d}{c} \right\rceil - \sum_{i \in T} x_i\Big) = r\Big(\left\lceil \frac{d}{c} \right\rceil - \sum_{i \in T \setminus N_1(\bar{S})} x_i\Big) \geq r\Big(\left\lceil \frac{d}{c} \right\rceil - \alpha\big(G[T \setminus N_1(\bar{S})]\big)\Big)$$

$$\geq r\Big(\left\lceil \frac{d}{c} \right\rceil - \left\lceil \frac{d}{c} \right\rceil + |\bar{S}|\Big) = r \sum_{i \in S} x_i,$$

where the first inequality follows from the validity of the MIR inequality, the second inequality follows from the definition of independent set, and the third inequality follows from (13). $\square$

Proposition 3.4. *Consider sets S and T as defined in the statement of Proposition 3.3. Suppose*

$$\mathcal{S} = \Big\{ \bar{S} \in \mathcal{I}(S) \mid \alpha\big(G[T \setminus N_1(\bar{S})]\big) = \left\lceil \frac{d}{c} \right\rceil - |\bar{S}| \Big\} \neq \emptyset,$$

and consider the following two graphs:

$G' = (N_1 \setminus T, E')$, *where $\{i, j\} \in E'$ if there exist $\bar{S} \in \mathcal{S}, \bar{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil - |\bar{S}|}(T \setminus N_1(\bar{S}))$, and an independent set $I \subseteq N_1 \setminus (T \cup N_1(\bar{S}) \cup N_1(\bar{T}))$ such that $|I| \in \{|\bar{S}| - 1, |\bar{S}|\}$, $i \in I$, $j \notin I$, and $I' \cup \bar{S} \cup \bar{T}$ is an independent set where $I' = (I \setminus \{i\}) \cup \{j\}$;*

$G'' = (S, E'')$, *where $\{i, j\} \in E''$ if there exist $\bar{S} \in \mathcal{S}, \bar{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil - |\bar{S}|}(T \setminus N_1(\bar{S}))$, and an independent set $I \subseteq N_1 \setminus (T \cup N_1(\bar{S}) \cup N_1(\bar{T}))$ such that $|I| \in \{|\bar{S}| - 1, |\bar{S}|\}$, $i \in \bar{S}$, $j \notin \bar{S}$, $\bar{S}' = (\bar{S} \setminus \{i\}) \cup \{j\} \in \mathcal{S}$ and sets $\bar{S} \cup \bar{T} \cup I$ and $\bar{S}' \cup \bar{T} \cup I$ are independent.*

Then inequality (lift 2) defines a facet of P if the following conditions hold.

(i) *For each $i \in T, \alpha\big(G[T \setminus (N_1(i) \cup \{i\})]\big) \geq \lceil \frac{d}{c} \rceil$.*

(ii) *For each $i \in N_0 \setminus S, \alpha\big(G[T \setminus N_1(i)]\big) \geq \lceil \frac{d}{c} \rceil$.*

(iii) *For each $\bar{S} \in \mathcal{S}$ there exists $\bar{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil - |\bar{S}|}(T \setminus N_1(\bar{S}))$ such that*

$$\alpha\big(G[N_1 \setminus (T \cup N_1(\bar{S}) \cup N_1(\bar{T}))]\big) \geq |\bar{S}|.$$

(iv) *Graph $G' = (N_1 \setminus T, E')$ is connected.*

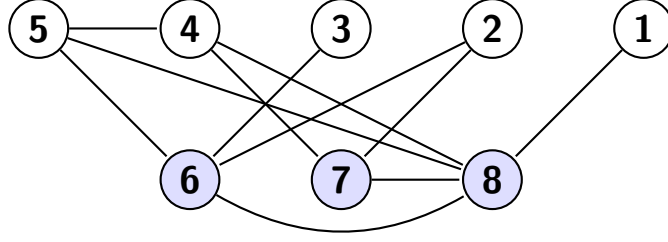(v) *Graph $G'' = (S, E'')$ is connected.*

Figure 4: Conflict graph corresponding with Example 3.2.

The proof is left to the Appendix.

Example 3.2. *Consider the data given in Example 3.1 and the conflict graph G shown in Figure 4. By setting $S = \{6, 7, 8\}$ and $T = \{1, 2, 3, 4\}$, one can check that the validity and the facet-defining conditions established in Proposition 3.3 and Proposition 3.4 hold. Hence, the following inequality of type (lift 2) defines a facet of P.*

$$s + 2x_5 \geq 2x_6 + 2x_7 + 2x_8.$$

3.2. Residual inequalities

Next, a new family of valid inequalities is introduced where the residuum $c - r = c\lceil \frac{d}{c} \rceil - d$ occurs as the independent term.

Proposition 3.5. *Let $S \subseteq N_0$ such that $\alpha(G[S]) \leq \lceil \frac{d}{c} \rceil$ and*

$$\alpha\big(G[N_1 \setminus N_1(\bar{S})]\big) \leq \left\lceil \frac{d}{c} \right\rceil - |\bar{S}|, \forall \bar{S} \in \mathcal{I}(S). \tag{14}$$

Then the following inequality is valid for X.

$$s + (c - r) \geq c \sum_{i \in S} x_i. \tag{residual 1}$$

The proof of Proposition 3.5 is omitted since a proof of a more general class will be given later. Next, it is shown that, if $\widetilde{N}_1(S) \neq \emptyset$, then (residual 1) does not define a facet. Let

$$\mathcal{F} = \left\{ (s, x) \in X \mid s = c \sum_{i \in S} x_i - (c - r) \right\}.$$

As $-(c - r) < 0$ and $s \geq 0$ then $\sum_{i \in S} x_i > 0, \forall (s, x) \in \mathcal{F}$. This implies that if $i \in \widetilde{N}_1(S)$, then $x_i = 0, \forall (s, x) \in \mathcal{F}$. Thus, (residual 1) does not define a facet when $\widetilde{N}_1(S) \neq \emptyset$. In order to obtain a stronger inequality, $x_i, i \in \widetilde{N}_1(S)$ are lifted as follows. Consider $R \subseteq \widetilde{N}_1(S)$ such that R is a clique in $G[\widetilde{N}_1(S)]$. Hence, it suffices to find coefficients $l_i, i \in R$ such that inequality

$$s + (c - r) \geq c \sum_{i \in S} x_i + \sum_{i \in R} l_i x_i, \tag{15}$$

remains valid for X. If $x_i = 0, \forall i \in R$, then inequality (15) is trivially valid. So assume $x_j = 1$, for some $j \in R$. Notice that since R is a clique, then $x_j = 1$ implies $x_i = 0, \forall i \in R \setminus \{j\}$. Thus, in order for inequality

$$s + (c - r) \geq c \sum_{i \in S} x_i + l_j, \forall (s, x) \in X|_{x_j = 1},$$
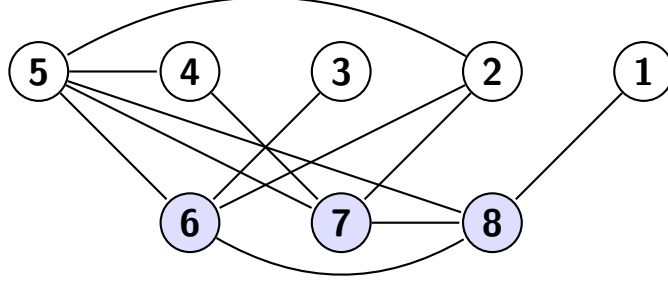
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Figure 5: Conflict graph corresponding with Example 3.3.

to be valid, l_j must satisfy $l_j \leq s + (c - r) - c \sum_{i \in S} x_i, \forall (s, x) \in X|_{x_j=1}$. Since $j \in \widetilde{N}_1(S)$, so $x_j = 1$ implies $x_i = 0, \forall i \in S$. Hence

$$l_j \leq s + (c - r), \forall (s, x) \in X|_{x_j=1} \implies l_j \leq min_{(s,x) \in X|_{x_j=1}} \{s\} + (c - r).$$

The minimum value which s attains can be obtained by maximizing the number of binary variables in N_1 set to one.

$$l_j = (c - r) + \left[d - \left(\alpha \big(G[N_1 \setminus (j \cup N_1(j))] \big) + 1 \right) c \right]^+. \tag{16}$$

Therefore, since R is a clique, inequality (15) is valid for X where $l_i, i \in R$, is defined by (16). Moreover, if condition $\alpha(G[N_1 \setminus (i \cup N_1(i))]) \geq \lfloor \frac{d}{c} \rfloor$ holds, then $s = 0$ implies $l_i = c - r, i \in R$.

Example 3.3. *Consider the data given in Example 3.1 and the conflict graph G shown in Figure 5. Taking $S = \{6, 7\}$ implies that the inequality $s + 7 \geq 9x_6 + 9x_7$ of type* (residual 1) *is valid for X. Since $\widetilde{N}_1(S) = \{2, 5\}$ is a clique, then let $R = \widetilde{N}_1(S)$. One can check that the following lifted inequality, with lifting coefficients $l_2 = l_5 = c - r = 7$, defines a facet of P.*

$$s + 7 \geq 7x_2 + 7x_5 + 9x_6 + 9x_7.$$

Now, inequality (residual 1) is generalized as follows.

Proposition 3.6. *Let $S \subseteq N_0$ with $\alpha(G[S]) \leq \lceil \frac{d}{c} \rceil$, and $T \subseteq N_1$ such that*

$$\alpha\big(G[T \setminus N_1(\bar{S})]\big) \leq \left\lceil \frac{d}{c} \right\rceil - |\bar{S}|, \forall \bar{S} \in \mathcal{I}(S).$$

Then the following inequality is valid for X.

$$s + c \sum_{i \in N_1 \setminus T} x_i + (c - r) \geq c \sum_{i \in S} x_i. \tag{residual 2}$$

Proof. Consider $(s, x) \in X$. If $\sum_{i \in S} x_i = 0$, then validity of (residual 2) is implied by the nonnegativity of variables x_i and s. Assume $x_i = 1, i \in \bar{S} \subseteq S$ and $x_i = 0, i \in S \setminus \bar{S}$. From (1) it follows that $s + c \sum_{i \in N_1 \setminus T} x_i \geq d - c \sum_{i \in T} x_i$. Thus

$$s + c \sum_{i \in N_1 \setminus T} x_i + (c - r) \geq d - c \sum_{i \in T} x_i + (c - r) \geq d - \alpha\big(G[T \setminus N_1(\bar{S})]\big)c + (c - r)$$

$$\geq d - c\left(\left\lceil \frac{d}{c} \right\rceil - |\bar{S}| \right) + (c - r) = c \left\lfloor \frac{d}{c} \right\rfloor + r - c\left(\left\lceil \frac{d}{c} \right\rceil - |\bar{S}| \right) + (c - r) = c |\bar{S}| = c \sum_{i \in S} x_i.$$

$\square$

Similarly to inequalities (residual 1), inequalities (residual 2) can be strengthened by lifting variables in $\widetilde{N}_1(S)$. These variables are lifted by taking $R \subseteq \widetilde{N}_1(S)$ such that R is a clique. It suffices to find lifting coefficients $l_i, i \in R$ such that inequality $s + (c - r) \geq c \sum_{i \in S} x_i - c \sum_{i \in N_1 \setminus T} x_i + \sum_{i \in R} l_i x_i$, remains valid for X. Following the same steps used to lift inequality (residual 1), the following general family of valid inequalities can be derived.

Proposition 3.7. *Consider the sets $S \subseteq N_0$, $T \subseteq N_1$, and $R \subseteq \widetilde{N}_1(S)$ such that $\alpha(G[S]) \leq \lceil \frac{d}{c} \rceil$,*

$$\alpha\big(G[T \setminus N_1(\bar{S})]\big) \leq \left\lceil \frac{d}{c} \right\rceil - |\bar{S}|, \forall \bar{S} \in \mathcal{I}(S),$$

and R is a clique. Then following inequality is valid for X.

$$s + c \sum_{i \in N_1 \setminus T} x_i + (c - r) \geq c \sum_{i \in S} x_i + \sum_{i \in R} l_i x_i, \qquad \text{(residual 3)}$$

where

$$l_i = (c - r) + \left[d - \Big(\alpha\big(G[N_1(S) \setminus (i \cup N_1(i))]\big) + 1 \Big) c \right]^+, i \in T.$$

If $\alpha(G[N_1(S) \setminus (i \cup N_1(i))]) \geq \lfloor \frac{d}{c} \rfloor$, then $l_i = c - r, i \in T$.

3.3. Conflict MIR inequalities

Next, families of valid inequalities, called conflict MIR inequalities, are introduced that can be regarded as an extension of MIR inequalities to the case where a conflict graph representing incompatibilities between pairs of variables is present. To do so, initially consider the following weaker MIR inequality obtained from a restriction of set X. For each $T \subset N_1$, let $s' = s + c \sum_{i \in N_1 \setminus T} x_i$. Then the MIR inequality

$$s' + r \sum_{i \in T} x_i \geq r \left\lceil \frac{d}{c} \right\rceil,$$

is valid for X. When this inequality does not define a facet (see Proposition 2.1), it could be lifted as follows.

Proposition 3.8. *Consider $S \subseteq N_0$ with $\alpha(G[S]) \leq \lfloor \frac{d}{c} \rfloor$ and $T \subseteq N_1$ such that*

$$\alpha\big(G[T \setminus N_1(\bar{S})]\big) \leq \left\lfloor \frac{d}{c} \right\rfloor - |\bar{S}|, \ \forall \bar{S} \in \mathcal{I}(S). \qquad (17)$$

Then the following inequality is valid for X.

$$s + c \sum_{i \in N_1 \setminus T} x_i + r \sum_{i \in T} x_i \geq r \left\lceil \frac{d}{c} \right\rceil + (c - r) \sum_{i \in S} x_i. \qquad \text{(cMIR 1)}$$

Proof. Let $(s, x) \in X$. If $\sum_{i \in S} x_i = 0$, then the validity is implied by the MIR inequality (5) as follows.

$$s + c \sum_{i \in N_1 \setminus T} x_i + r \sum_{i \in T} x_i \geq s + r \sum_{i \in N_1 \setminus T} x_i + r \sum_{i \in T} x_i \geq r \left\lceil \frac{d}{c} \right\rceil.$$

Assume $\sum_{i \in S} x_i \geq 1$. So $\sum_{i \in S} x_i = |\bar{S}|$ where $\bar{S} \subseteq S$ is an independent set. Now let $\sum_{i \in T \setminus N_1(\bar{S})} x_i = \lfloor \frac{d}{c} \rfloor - |\bar{S}| - k$ where $0 \leq k \leq \lfloor \frac{d}{c} \rfloor - |\bar{S}|$. As

$$\sum_{i \in N_1} x_i = \sum_{i \in N_1 \setminus T} x_i + \sum_{i \in T \setminus N_1(\bar{S})} x_i = \sum_{i \in N_1 \setminus T} x_i + \left\lfloor \frac{d}{c} \right\rfloor - |\bar{S}| - k,$$

then, using inequality (1) gives

$$s + c \sum_{i \in N_1} x_i \geq d \iff s + c \sum_{i \in N_1 \setminus T} x_i + c \sum_{i \in T \setminus N_1(\bar{S})} x_i \geq d.$$

Thus

$$s + c \sum_{i \in N_1 \setminus T} x_i \geq d - c \left(\left\lfloor \frac{d}{c} \right\rfloor - |\bar{S}| - k \right) \geq |\bar{S}| c + kc + r$$

$$\geq |\bar{S}| c + (k+1) r \geq r \left\lceil \frac{d}{c} \right\rceil + (c - r) |\bar{S}| - r \left(\left\lfloor \frac{d}{c} \right\rfloor - |\bar{S}| - k \right).$$

Hence

$$s + c \sum_{i \in N_1 \setminus T} x_i + r \left(\left\lfloor \frac{d}{c} \right\rfloor - |\bar{S}| - k \right) = s + c \sum_{i \in N_1 \setminus T} x_i + r \sum_{i \in T \setminus N_1(\bar{S})} x_i \geq r \left\lceil \frac{d}{c} \right\rceil + (c - r) |\bar{S}|.$$

$\square$

Definition 3.1. *For $S \subseteq N_0$ and $T \subseteq N$, $\bar{\alpha}(G[T \cup S])$ denotes the independence number of the subgraph induced by $T \cup S$ such that at least one node from set S appears in the corresponding independent set.*

In the following proposition sufficient conditions for inequality (cMIR 1) to be facet-defining are presented.

Proposition 3.9. *Consider S and T as defined in the statement of Proposition 3.8. Suppose*

$$\mathcal{S}_1 = \left\{ \bar{S} \in \mathcal{I}(S) \mid \alpha\big(G[T \setminus N_1(\bar{S})]\big) = \left\lfloor \frac{d}{c} \right\rfloor - |\bar{S}| \right\} \neq \emptyset,$$

and consider the following graph:
   $G'' = (S, E'')$, *where $\{i, j\} \in E''$ if there exists $J \in \mathcal{S}_1$ such that $i \in J$, $j \notin J$, and $J' = (J \setminus \{i\}) \cup \{j\} \in \mathcal{S}_1$.*
   Then inequality (cMIR 1) is facet-defining for P if the following conditions hold.

   (i) $\alpha\big(G[T]\big) \geq \left\lceil \frac{d}{c} \right\rceil$.

   (ii) For each $i \in N_1 \setminus T$, $\bar{\alpha}\big(G[(T \cup S) \setminus N(i)]\big) \geq \left\lfloor \frac{d}{c} \right\rfloor$.

   (iii) For each $i \in N_0 \setminus S$, $\bar{\alpha}\big(G[(T \cup S) \setminus N(i)]\big) \geq \left\lfloor \frac{d}{c} \right\rfloor$.

   (iv) Graph $G'_{\lfloor \frac{d}{c} \rfloor} = (T, E')$ is connected.

   (v) Graph $G'' = (S, E'')$ is connected.

The proof is left to the Appendix.
When $S \subseteq N_0$ is a clique, inequalities (cMIR 1) can be strengthened as follows.

Proposition 3.10. *Let S be a clique in $G[N_0]$, and $T \subseteq N_1$ such that*

$$\alpha\big(G[T \setminus N_1(i)]\big) \leq \left\lfloor \frac{d}{c} \right\rfloor - p_i, \forall i \in S,$$

where $p_i \in \left\{ 1, \ldots, \lfloor \frac{d}{c} \rfloor \right\}, i \in S$. Then the following inequality is valid for X.

$$s + c \sum_{i \in N_1 \setminus T} x_i + r \sum_{i \in T} x_i \geq r \left\lceil \frac{d}{c} \right\rceil + \sum_{i \in S} p_i (c - r) x_i. \tag{cMIR 2}$$

Proof. Let $(s, x) \in X$. Assume $\sum_{i \in S} x_i = 0$. Then validity is implied by the MIR inequality (5) similarly to the proof of the same case given in Proposition 3.8.

Let $\sum_{i \in S} x_i = 1$. So assume $x_j = 1$, for some $j \in S$. Then $\sum_{i \in T \setminus N_1(j)} x_i = \lfloor \frac{d}{c} \rfloor - k_j$ where $p_j \le k_j \le \lfloor \frac{d}{c} \rfloor$. From (1), using $\sum_{i \in N_1(j)} x_i = 0$ and $\sum_{i \in T \setminus N_1(j)} x_i = \lfloor \frac{d}{c} \rfloor - k_j$, then

$$s + c \sum_{i \in N_1 \setminus T} x_i + c \sum_{i \in T \setminus N_1(j)} x_i + c \sum_{i \in N_1(j)} x_i \ge d \Leftrightarrow s + c \sum_{i \in N_1 \setminus T} x_i + r \sum_{i \in T \setminus N_1(j)} x_i \ge d - (c - r) \sum_{i \in T \setminus N_1(j)} x_i$$

$$\ge d - (c-r)\left(\left\lfloor \frac{d}{c} \right\rfloor - k_j\right) = r + c\left\lfloor \frac{d}{c} \right\rfloor - (c-r)\left(\left\lfloor \frac{d}{c} \right\rfloor - k_j\right) = r\left\lceil \frac{d}{c} \right\rceil + (c-r)k_j \ge r\left\lceil \frac{d}{c} \right\rceil + (c-r)p_j. \qquad \square$$

Inequalities (cMIR 2) can be lifted as follows.

Proposition 3.11. *Let $S \subseteq N_0$ define a clique in G, $k \in N_0 \setminus S$ such that $S \cup \{k\}$ does not define a clique, and $T \subseteq N_1$ such that*

$$\alpha\big(G[T \setminus N_1(i)]\big) \le \left\lfloor \frac{d}{c} \right\rfloor - p_i, \forall i \in S \cup k,$$

$$\alpha\big(G[T \setminus N_1(\{k,j\})]\big) \le \left\lfloor \frac{d}{c} \right\rfloor - p_j - p_k, \forall j \in S_1,$$

where $p_i \in \left\{1, \ldots, \lfloor \frac{d}{c} \rfloor\right\}, i \in S \cup \{k\}, 1 \le p_j + p_k \le \lfloor \frac{d}{c} \rfloor, j \in S_1 = \{j \in S : \{j, k\} \notin E\}$. Then the following inequality is valid.

$$s + c \sum_{i \in N_1 \setminus T} x_i + r \sum_{i \in T} x_i \ge r\left\lceil \frac{d}{c} \right\rceil + \sum_{i \in S} p_i(c - r)x_i + p_k(c - r)x_k. \qquad \text{(cMIR 3)}$$

Proof. If $x_k = 0$ or $x_k = 1$ and $\sum_{i \in S} x_i = 0$, then validity of (cMIR 3) follows from validity of (cMIR 2). The proof of case $x_k = 1$ and $\sum_{i \in S} x_i = 1$ is similar to the proof of validity of (cMIR 2). $\qquad \square$

The following example presents facet-defining inequalities of types (cMIR 1), (cMIR 2), and (cMIR 3).

Example 3.4. *Assume $d = 20, c = 9, N = \{1, \ldots, 8\}, N_1 = \{1, \ldots, 5\}$ and consider the conflict graph G depicted in Figure 6. Then it can be checked that condition (17) is satisfied for $S = \{6, 7, 8\}$ and $T = \{2, 3, 4, 5\}$. So the following inequality of type (cMIR 1) is valid for X.*

$$s + 9x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 \ge 6 + 7x_6 + 7x_7 + 7x_8.$$

One can check that the previous inequality as well as the following inequalities of type (cMIR 1) define facets of P.

$$s + 2x_1 + 2x_2 + 2x_3 + 2x_4 + 9x_5 \ge 6 + 7x_6 + 7x_7 + 7x_8,$$
$$s + 2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 \ge 6 + 7x_6 + 7x_7.$$

The following inequalities of type (cMIR 2) are facet-defining for P.

$$s + 9x_1 + 2x_2 + 2x_3 + 2x_4 + 9x_5 \ge 6 + 14x_6 + 14x_7,$$
$$s + 2x_1 + 2x_2 + 9x_3 + 2x_4 + 9x_5 \ge 6 + 7x_7 + 14x_8,$$
$$s + 9x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 \ge 6 + 14x_6 + 7x_7,$$
$$s + 2x_1 + 2x_2 + 9x_3 + 2x_4 + 2x_5 \ge 6 + 14x_8.$$

The unique facet-defining inequality of type (cMIR 3) is obtained with $S = \{6, 7\}, k = \{8\}$ and $T = \{2, 3, 4\}$, and is given by

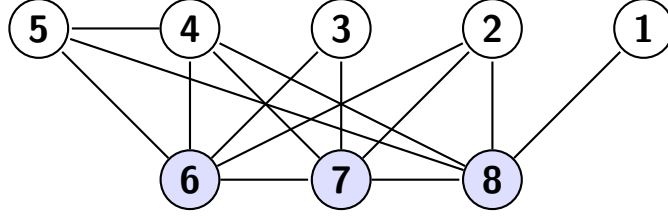$$s + 9x_1 + 2x_2 + 2x_3 + 2x_4 + 9x_5 \ge 6 + 7x_6 + 14x_7 + 7x_8.$$

Figure 6: Conflict graph considered in Example 3.4.

The following proposition generalizes inequalities (cMIR 1).

Proposition 3.12. *Let* $S \subseteq N_0$, $T \subseteq N_1$ *and let* $\{T_1, T_2\}$ *defines a partition of* T *such that*

$$\alpha\big(G[S]\big) \leq \left\lfloor \frac{d}{c} \right\rfloor + p,$$

$$\alpha\big(G[T_1 \setminus N_1(\bar{S})]\big) \leq \left\lfloor \frac{d}{c} \right\rfloor + p - |\bar{S}|, \ \forall \bar{S} \in \mathcal{I}(S), \tag{18}$$

$$\alpha\big(G[T_2 \setminus N_1(\bar{S})]\big) \leq \left(p - |\bar{S}|\right)^+, \ \forall \bar{S} \in \mathcal{I}(S). \tag{19}$$

Then the following inequality is valid for X.

$$s + c \sum_{i \in N_1 \setminus T} x_i + r \sum_{i \in T} x_i \geq r \left\lceil \frac{d}{c} \right\rceil + (c - r)\Big(\sum_{i \in S} x_i - p + \sum_{i \in T_2} x_i \Big). \tag{cMIR 4}$$

Proof. Let $(s, x) \in X$. Let $x_i = 1, i \in \bar{S} \subseteq S$, and $x_i = 0, i \in S \setminus \bar{S}$. If $|\bar{S}| < p$, then

$$\sum_{i \in S} x_i - p + \sum_{i \in T_2} x_i = |\bar{S}| - p + \sum_{i \in T_2} x_i \leq |\bar{S}| - p + \alpha\big(G[T_2 \setminus N_1(\bar{S})]\big) \leq 0,$$

where the last inequality follows from (19). Hence, inequality (cMIR 4) is implied by the MIR inequality

$$s + r \sum_{i \in N_1 \setminus T} x_i + r \sum_{i \in T} x_i \geq r \left\lceil \frac{d}{c} \right\rceil.$$

Now, let $|\bar{S}| \geq p$. Then, from (19) it follows that $x_i = 0, i \in T_2$. The proof is now similar to the proof of Proposition 3.8 for case $\sum_{i \in S} x_i \geq 1$. $\qquad \square$

Example 3.5. *Let* $d = 14, c = 9, N = \{1, \ldots, 10\}, N_1 = \{1, \ldots, 6\}$ *and consider the conflict graph* G *shown in Figure 7. Using the software PORTA (see [8]) the following description of* P *is obtained.*
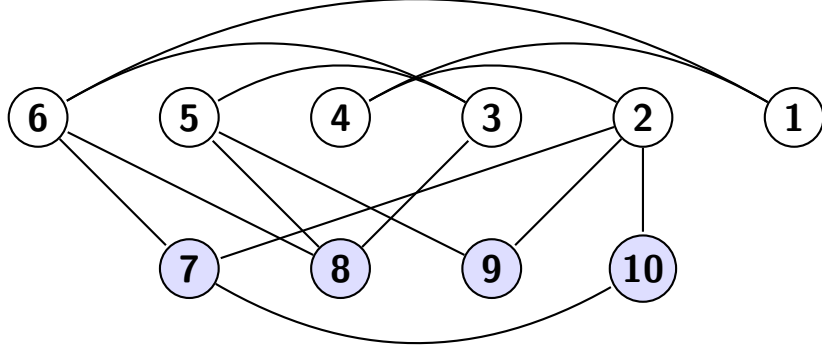
Figure 7: Conflict graph corresponding with Example 3.5.

$(I1)$ $s + 9x_1 + 9x_2 + 9x_3 + 9x_4 + 9x_5 + 9x_6 \geq 14,$

$(I2)$ $x_1 \geq 0,$

$(I3)$ $x_2 \geq 0,$

$(I4)$ $x_3 \geq 0,$

$(I5)$ $x_4 \geq 0,$

$(I6)$ $x_5 \geq 0,$

$(I7)$ $x_6 \geq 0,$

$(I8)$ $x_7 \geq 0,$

$(I9)$ $x_8 \geq 0,$

$(I10)$ $x_9 \geq 0,$

$(I11)$ $x_{10} \geq 0,$

$(I12)$ $s \geq 0,$

$(I13)$ $x_6 + x_7 \leq 1,$

$(I14)$ $x_5 + x_9 \leq 1,$

$(I15)$ $x_2 + x_9 \leq 1,$

$(I16)$ $x_2 + x_4 \leq 1,$

$(I17)$ $x_1 + x_6 \leq 1,$

$(I18)$ $x_1 + x_4 \leq 1,$

$(I19)$ $x_3 + x_6 + x_8 \leq 1,$

$(I20)$ $x_3 + x_5 + x_8 \leq 1,$

$(I21)$ $x_2 + x_7 + x_{10} \leq 1,$

$(I22)$ $x_1 + x_2 + x_4 + x_6 + x_7 \leq 2,$

$(I23)$ $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_8 + x_9 \leq 3,$

$(I24)$ $s + 5x_1 + 5x_2 + 5x_3 + 5x_4 + 5x_5 + 5x_6 \geq 10,$

$(I25)$ $s + 9x_1 + 5x_2 + 9x_3 + 9x_4 + 9x_5 + 5x_6 \geq 4x_7 + 10,$

$(I26)$ $s + 9x_1 + 9x_2 + 5x_3 + 9x_4 + 5x_5 + 5x_6 \geq 4x_8 + 10,$

$(I27)$ $s + 9x_1 + 5x_2 + 9x_3 + 9x_4 + 5x_5 + 9x_6 \geq 4x_9 + 10,$

$(I28)$ $s + 9x_1 + x_2 + 9x_3 + 9x_4 + 5x_5 + 5x_6 \geq 4x_7 + 4x_9 + 6,$

$(I29)$ $s + 9x_1 + 5x_2 + 5x_3 + 9x_4 + 5x_5 + x_6 \geq 4x_7 + 4x_8 + 6,$

$(I30)$ $s + 9x_1 + 5x_2 + 5x_3 + 9x_4 + x_5 + 5x_6 \geq 4x_8 + 4x_9 + 6,,$

$(I31)$ $s + 9x_1 + x_2 + 5x_3 + 9x_4 + x_5 + x_6 \geq 4x_7 + 4x_8 + 4x_9 + 2,$

$(I32)$ $s + 5x_1 \geq 5x_8,$

$(I33)$ $s + 5x_2 \geq 5x_8,$

$(I34)$ $s + 5x_3 + 5x_4 \geq 5x_9,$

$(I35)$ $s + 5x_3 + 5x_6 \geq 5x_9,$

$(I36)$ $s + 5x_1 + 5x_4 \geq 5x_8 + 5x_9,$

$(I37)$ $s + 5x_1 + 5x_4 \geq 5x_7 + 5x_8,$

$(I38)$ $s + 5x_1 + 5x_3 + 5x_4 \geq 5x_7 + 5x_9,$

$(I39)$ $s + 9x_1 + 4x_3 + 9x_4 \geq 4x_7 + 5x_8 + 5x_9,$

$(I40)$ $s + 9x_1 + 4x_3 + 9x_4 \geq 5x_7 + 5x_8 + 4x_9,$

$(I41)$ $s + 9x_1 + 5x_3 + 9x_4 \geq 5x_7 + 4x_8 + 5x_9,$

$(I42)$ $s + 5x_1 + 5x_3 + 5x_4 \geq 5x_7 + 5x_8 + 5x_9,$

$(I43)$ $s + 4 + 9x_1 + 9x_4 \geq 4x_6 + 9x_7 + 9x_8,$

$(I44)$ $s + 4 + 9x_1 + 9x_4 \geq 4x_5 + 9x_8 + 9x_9,$

$(I45)$ $s + 4 + 9x_1 + 9x_3 + 9x_4 \geq 4x_2 + 9x_7 + 9x_9,$

$(I46)$ $s + 5 \geq 5x_6 + 5x_7 + 5x_8,$

$(I47)$ $s + 5 \geq 5x_5 + 5x_8 + 5x_9,$

$(I48)$ $s + 5 + 5x_3 \geq 5x_2 + 5x_7 + 5x_9.$

Inequalities (I1)–(I18) are trivial inequalities discussed in Proposition 2.3, Proposition 2.6 and Proposition 2.7. Inequalities (I19)–(I23) stem from P_{VP}. (I24) is the MIR inequality, (I25)–(I27) are of type (cMIR 1), (I28)–(I31) are of type (cMIR 4), (I32)–(I38) are of type (lift 2), and (I43)–(I45) are of type (residual 3). Note that inequalities (I39)–(I42) do not belong to any of the families of valid inequalities derived in this paper.

3.4. Valid inequalities for case $d = c$

Notice that all the inequalities discussed previously are valid when $d \geq c$. Below a new class of valid inequalities is introduced for X which defines facets only when $d = c$.

Proposition 3.13. *Let $S \subseteq N_0$, $\alpha(G[S]) = p$, and define*

$$T = \bigcap_{\bar{S} \in \mathcal{I}_p(S)} N_1(\bar{S}), \quad \overline{T} = \bigcap_{\bar{S} \in \mathcal{I}_{p-1}(S)} N_1(\bar{S}).$$

Let $T' \subseteq \overline{T}$ such that T' defines a clique. The following inequality is valid for X.

$$s + c \sum_{i \in N_1 \setminus T} x_i \geq c \Big(\sum_{i \in S} x_i - p + 1 \Big) + c \sum_{i \in T'} x_i. \tag{20}$$

Proof. To prove validity of (20), consider the following cases. Let $(s, x) \in X$.

Case *I.* Let $p = 1$. It implies that S is a clique, $T = \widetilde{N}_1(S)$ and $\overline{T} = T' = \emptyset$. If $\sum_{i \in S} x_i = 0$, then the validity follows from nonnegativity of s and $x_i, i \in N_1 \setminus T$. Assume $\sum_{i \in S} x_i = 1$. Then inequality (1) implies

$$s + c \sum_{i \in N_1} x_i = s + c \sum_{i \in N_1 \setminus T} x_i + c \sum_{i \in T} x_i = s + c \sum_{i \in N_1 \setminus T} x_i + c \sum_{i \in \widetilde{N}_1(S)} x_i$$

$$= s + c \sum_{i \in N_1 \setminus T} x_i \geq c = c \sum_{i \in S} x_i.$$

Case *II.* Let $p \geq 2$. If $\sum_{i \in S} x_i = 0$ then validity of (20) is implied by nonnegativity of $s, x_i, i \in N_1 \setminus T$, and properties $\sum_{i \in T'} x_i \leq 1$ and $1 - p \leq -1$. Suppose $\sum_{i \in S} x_i = |\bar{S}|$ where $\bar{S}$ is an independent set. If $1 \leq |\bar{S}| \leq p - 2$, then

$$\sum_{i \in S} x_i - p + 1 = |\bar{S}| - p + 1 \leq p - 2 - p + 1 = -1,$$

which implies that $c \big(\sum_{i \in S} x_i - p + 1 \big) + c \sum_{i \in T'} x_i \leq 0$. Thus, the validity is implied by nonnegativity of s and $x_i, i \in N_1 \setminus T$.

Now, let $p - 1 \leq |\bar{S}| \leq p$. Then, it results from the definition of T' that this condition implies $\sum_{i \in T'} x_i = 0$. So, for the case $|\bar{S}| = p - 1$, the validity follows from nonnegativity of s and $x_i, i \in N_1 \setminus T$. For $|\bar{S}| = p$, it can be concluded that $\sum_{i \in T} x_i = 0$. So inequality (1) implies

$$s + c \sum_{i \in N_1} x_i = s + c \sum_{i \in N_1 \setminus T} x_i + c \sum_{i \in T} x_i = s + c \sum_{i \in N_1 \setminus T} x_i \geq c = c \Big(\sum_{i \in S} x_i - p + 1 \Big).$$

$\square$

Next, sufficient conditions for inequality (20) to define a facet of P are presented.

Proposition 3.14. *Let $S \subseteq N_0$ be an independent set. Inequality (20) is facet-defining for P if the following conditions hold.*

(i) *For each $i \in T \setminus T'$, there exists at least one $\bar{S} \in \mathcal{I}_{p-1}(S)$ such that $i \in T \setminus (T' \cup N_1(\bar{S}))$.*

(ii) *For each $i \in T'$, there exists at least one $\bar{S} \in \mathcal{I}_{p-2}(S)$ such that $i \in T' \setminus N_1(\bar{S})$.*

(iii) *For each $i \in N_0 \setminus S$, there exists at least one $\bar{S} \in \mathcal{I}(S)$ where $p - 1 \leq |\bar{S}| \leq p$ such that $i \in N_0 \setminus (S \cup N_0(\bar{S}))$.*
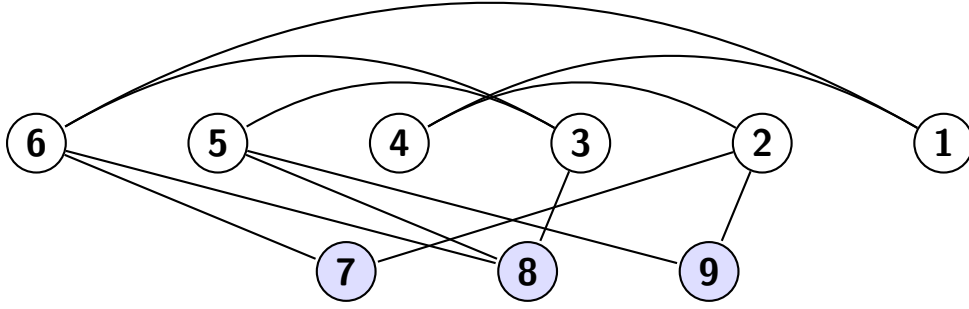
The proof is left to the Appendix.

Figure 8: Conflict graph considered in Example 3.6.

Example 3.6. *Assume* $d = c = 15, N = \{1, \ldots, 9\}, N_1 = \{1, \ldots, 6\}$, *and consider the conflict graph G given in Figure 8. Then the following inequalities of type* (20) *are facet-defining for P.*

$$s + 15x_1 + 15x_4 \geq 15(x_7 + x_8 - 1) + 15x_6,$$
$$s + 15x_1 + 15x_4 \geq 15(x_8 + x_9 - 1) + 15x_5,$$
$$s + 15x_1 + 15x_3 + 15x_4 \geq 15(x_7 + x_9 - 1) + 15x_2.$$

Observe that, as discussed in Section 2, inequality (20) under the foregoing conditions defines a facet of P if $c > d$.

4. Separation

This section discusses the separation problems associated with the families of inequalities (lift 1), (cMIR 1), (lift 2) and (residual 2), used in the computational tests reported in Section 5.

Consider a point $(s^*, x^*) \in \mathbb{R}_+ \times [0,1]^n$. Then for each family, $\mathcal{V}$, of valid inequalities the separation problem is to find an inequality in $\mathcal{V}$ that is violated by the point (s^*, x^*) or show that there is no such inequality. All the separation problems discussed here are NP-hard since they include the computation of the independence number of a graph as a subproblem.

The separation problems are discussed in detail for inequalities (lift 1) and (cMIR 1), and also a brief discussion on the separation of (lift 2) and (residual 2) is given at the end of this section.

First consider inequalities (lift 1). For a clique $S \subseteq N$ and $T \subseteq N_1 \setminus S$, these inequalities can be written as follows.

$$\sum_{i \in S}(d - p_i c)^+ x_i \leq s + c \sum_{i \in T} x_i$$
$$\iff \sum_{i \in S}(d - p_i c)^+ x_i + c \sum_{i \in N_1 \setminus T} x_i \leq s + c \sum_{i \in N_1} x_i.$$

Hence, for a given solution (s^*, x^*), inequality (lift 1) is violated if and only if the maximum of the LHS,

$$\max_{S \subseteq N, T \subseteq N_1 \setminus S} \left\{ \sum_{i \in S}(d - p_i c)^+ x_i^* + c \sum_{i \in N_1 \setminus T} x_i^* \mid S \text{ is a clique} \right\}, \tag{21}$$

is greater than the constant $s^* + c \sum_{i \in N_1} x_i^*$. Recall that $p_i = \alpha(G[N_1 \setminus (N_1(i) \cup T)])$ and, therefore, it depends on the choice of set T.

Exact separation of inequalities (lift 1)

In order to solve this separation problem exactly, define the binary variables $y_i, i \in N_1$ such that y_i is 1 if $i \in N_1 \setminus T$, and 0 otherwise, and the binary variables $z_i, i \in N$ indicating whether $i \in S$ or not. For each $i \in N$, also define the non-negative integer variables γ_i which are 0 if $z_i = 0$ and are lower bounded by p_i if $z_i = 1$. The maximization problem (21) can be solved by solving the following MIP problem.

$$\max \quad \sum_{i \in N_1} cx_i^* y_i + \sum_{i \in N} dx_i^* z_i - \sum_{i \in N} cx_i^* \gamma_i \tag{22}$$

$$\text{s.t.} \quad z \text{ defines a clique in } N, \tag{23}$$

$$\gamma_i \geq \sum_{j \in I} y_j z_i, i \in N, I \in \mathcal{I}(N_1 \setminus N_1(i)), \tag{24}$$

$$z_i \leq y_i, i \in N_1, \tag{25}$$

$$y_i \in \{0,1\}, \ i \in N_1, \tag{26}$$

$$z_i \in \{0,1\}, \ i \in N, \tag{27}$$

$$\gamma_i \in \mathbb{Z}_0^+, \ i \in N. \tag{28}$$

Constraints (23) can be modeled in many different ways. For a discussion and comparison of formulations for clique problems see [12]. Following [18], define the variables $z_{ij}, \{i,j\} \in E$ indicating whether both nodes i and j belong to the clique. Then constraints (23) can be modeled as follows:

$$z_{ij} \leq z_i, z_{ij} \leq z_j, \ \{i,j\} \in E,$$
$$z_i + z_j \leq 1 + z_{ij}, \ \{i,j\} \in E,$$
$$z_i + z_j \leq 1, \ \{i,j\} \notin E,$$
$$z_{ij} \in \{0,1\}, \ \{i,j\} \in E,$$
$$z_i \in \{0,1\}, \ i \in N.$$

Constraints (24) ensure that γ_i must be greater than the cardinality of each independent set defined by variables y, hence it must be greater than the maximum cardinality set. Clearly, in any optimal solution to (22)–(28), constraint (24) will be satisfied as equation, that is, $\gamma_i = p_i$. Since (24) are nonlinear, they can be linearized by introducing new binary variables $w_{ij} = y_j z_i$. For each $i \in N$, constraints (24) can be replaced by the following set of constraints.

$$\gamma_i \geq \sum_{j \in I} w_{ij}, I \in \mathcal{I}(N_1 \setminus N_1(i)), \tag{29}$$

$$w_{ij} \leq z_i, \ j \in N_1, \tag{30}$$

$$w_{ij} \leq y_j, \ j \in N_1, \tag{31}$$

$$w_{ij} \geq z_i + y_j - 1, \ j \in N_1, \tag{32}$$

$$w_{ij} \in \{0,1\}, \ j \in N_1. \tag{33}$$

Finally, constraints (25) impose that each element in S that also belongs to N_1 must be in $N_1 \setminus T$, that implies S and T are disjoint.

As the set of inequalities (29) is large (increases exponentially with the number of nodes of G), then for each $i \in N$, these inequalities can be added dynamically by determining the maximum independent set on the graph $G[N_1(W_i)]$, where $N_1(W_i) = \{j \in N_1 \setminus N_1(i)| \ w_{ij} = 1\}$.

Algorithm 1 Separation heuristic for inequalities (lift 1).

$T \leftarrow \{j \in N_1 : x_j^* = 0\}$

for all $i \in N \setminus T$ **do**

    Compute an upper bound $\overline{p}_i$ on $\alpha(G[N_1 \setminus (N_1(i) \cup T)])$ using the sequential elimination algorithm given in [11] for the complement of graph G

    $c_i \leftarrow (d - \overline{p}_i c)^+$

end for

Sort the values of $x_j^*, j \in N \setminus T$ in a decreasing order. Let $j_1, \ldots, j_r$ denote the indices of the resulting order.

$S \leftarrow \emptyset$

for all $j_1, \ldots, j_r$ **do**

    if $S \cup \{j_1\}$ is a clique **then**

        $S \leftarrow S \cup \{j_1\}$

    end if

end for

if $\sum_{i \in S} c_i x_i^* > s^* + c \sum_{i \in T} x_i^*$ **then**

    Add inequality (lift 1) for the given S and T.

end if

Heuristic separation of inequalities (lift 1)

The exact separation procedure can hardly be used in practice. Here a heuristic procedure to separate inequalities (lift 1) is proposed, which is given in Algorithm 1.

Next, the separation of inequality (cMIR 1) is examined. For $S \subseteq N_0$ and $T \subseteq N_1$, this inequality can be written as follows.

$$r\left\lceil\frac{d}{c}\right\rceil + (c - r)\sum_{i \in S} x_i \leq s + c\sum_{i \in N_1 \setminus T} x_i + r\sum_{i \in T} x_i$$

$$\iff (c - r)\sum_{i \in S} x_i - c\sum_{i \in N_1 \setminus T} x_i - r\sum_{i \in T} x_i \leq s - r\left\lceil\frac{d}{c}\right\rceil$$

$$\iff (c - r)\sum_{i \in S} x_i + (c - r)\sum_{i \in T} x_i \leq s - r\left\lceil\frac{d}{c}\right\rceil + c\sum_{i \in N_1} x_i$$

$$\iff \sum_{i \in S} x_i + \sum_{i \in T} x_i \leq \frac{s - r\lceil\frac{d}{c}\rceil + c\sum_{i \in N_1} x_i}{c - r}. \tag{34}$$

Notice that condition (17) is equivalent to the following condition.

$$\bar{\alpha}\big(G[T \cup S]\big) \leq \left\lfloor\frac{d}{c}\right\rfloor. \tag{35}$$

Consider a fractional solution (s^*, x^*) and the graph G where the weight of node $i \in N$ is given by x_i^*. To find the most violated inequality, one needs to maximize the LHS of inequality (34) by determining S and T that satisfy condition (35):

$$\max_{S \subseteq N_0, T \subseteq N_1} \left\{ \sum_{i \in S} x_i^* + \sum_{i \in T} x_i^* \mid \bar{\alpha}\big(G[T \cup S]\big) \leq \left\lfloor\frac{d}{c}\right\rfloor \right\}.$$

Therefore, the separation problem is equivalent to find the maximum-weight subset of N such that the maximum independence number of the subgraph induced by that subset is less than or equal to $\lfloor\frac{d}{c}\rfloor$, and this independent set must include at least one node from set N_0.

Exact separation of inequalities (cMIR 1)

A possible approach to solve this separation problem exactly is to formulate it as a binary problem. To achieve this goal, define the binary variables $z_i, i \in N$, that indicate, for $i \in N_1$, whether $i \in T$, and for $i \in N_0$, whether $i \in S$. Let $\mathcal{C}$ be the family of all subsets in N whose independence number is greater than $\lfloor \frac{d}{c} \rfloor$, that is $\mathcal{C} = \{C \subseteq N \mid \alpha(G[C]) > \lfloor \frac{d}{c} \rfloor\}$. Then the separation problem can be solved by solving the following binary problem.

$$\max \quad \sum_{i \in N} x_i^* z_i \tag{36}$$

$$\text{s.t.} \quad \sum_{j \in C} z_j \leq |C| - 1, \forall C \in \mathcal{C}, \tag{37}$$

$$\sum_{j \in N_0} z_j \geq 1, \tag{38}$$

$$z_i \in \{0, 1\}, \ i \in N. \tag{39}$$

Inequalities (37) increase exponentially with the size of the graph. Hence, these inequalities should be included dynamically using a separation routine to find the maximum cardinality independent set.

Heuristic separation of inequalities (cMIR 1)

A heuristic procedure is now described to separate (cMIR 1). A greedy heuristic is proposed to form set $S \cup T$. First, the nodes are sorted accordingly to the value $x_j^* \times |\delta(j)|$, where $\delta(j)$ denotes the set of arcs incident to node j. Then, following that order (starting from a node in N_0) the nodes are selected if the independence number of the resulting induced graph does not exceed $\lfloor \frac{d}{c} \rfloor$. In order to ensure that this condition holds, a node j is selected if there are at most $\lfloor \frac{d}{c} \rfloor - 1$ selected nodes that are not neighbors of j, that is, if $\overline{N}_C(j) \leq \lfloor \frac{d}{c} \rfloor - 1$, where $\overline{N}_C(j)$ denotes the set of neighbors of j in $C \subseteq N$, in the complement of graph G.

The separation algorithm is given in Algorithm 2.

Algorithm 2 Separation heuristic for inequalities (cMIR 1).

$L_j \leftarrow x_j^* \times |\delta(j)|, \ j \in N$.
Sort L_j in a decreasing order. Let $j_1, \ldots, j_n$ denote the indices of the resulting order.
$C \leftarrow \{j^*\}$ where $j^* = \min\{i : j_i \in N_0\}$
for all $i \in N | j_i \notin C$ **do**
    if $|\overline{N}_C(j_i)| \leq \lfloor \frac{d}{c} \rfloor - 1$ **then**
        $C \leftarrow C \cup \{j_i\}$
    end if
end for
$S \leftarrow C \cap N_0; \ T \leftarrow C \cap N_1; \ RHS \leftarrow \frac{s^* - r\lceil \frac{d}{c} \rceil + c \sum_{i \in N_1} x_i^*}{c - r}$
if $\sum_{i \in S \cup T} x_i^* > RHS$ **then**
    Add inequality (cMIR 1) for the given S and T.
end if

Separation of inequalities (lift 2) *and* (residual 2)

As a final remark, the separation of inequalities (lift 2) and (residual 2) is discussed. It is similar to the separation of inequalities (cMIR 1). For each $S \subseteq N_0$, and $T \subseteq N_1$, inequalities (lift 2) can be rewritten as

$$r \sum_{i \in S} x_i \leq s + r \sum_{i \in N_1 \setminus T} x_i \iff \sum_{i \in S \cup T} x_i \leq \frac{s}{r} + \sum_{i \in N_1} x_i.$$
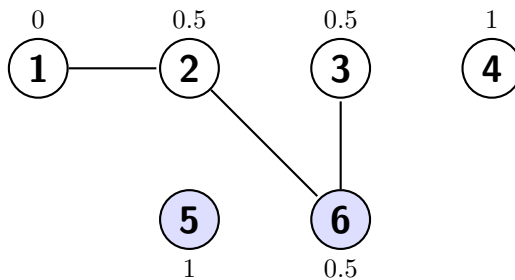
Figure 9: Conflict graph corresponding to the fractional solution given in Example 4.1.

So inequality (lift 2) resembles inequality (34). Hence, the separation problem amounts to maximize $\sum_{i \in S \cup T} x_i$ over a set which is very similar to (35).

Now, consider the case of inequalities (residual 2). For each $S \subseteq N_0$, and $T \subseteq N_1$, these inequalities can be rewritten as

$$c \sum_{i \in S} x_i \leq s + c \sum_{i \in N_1 \setminus T} x_i + (c - r) \iff c \sum_{i \in S \cup T} x_i \leq s + c \sum_{i \in N_1} x_i + (c - r).$$

Again, the separation problem becomes very similar to the one of inequalities (cMIR 1) and (lift 2).

Example 4.1. *Consider set Y with $N = \{1, \ldots, 6\}, N_1 = \{1, \ldots, 4\}, d = 12, c = 5$ and $E = \{\{1, 2\}, \{2, 6\}, \{6, 3\}\}$. Also consider the problem of minimizing an objective function over set Y. For a given objective function, the following fractional solution of the linear relaxation is obtained.*

$$s = 2, \ y_1 = 0, y_2 = 2.5, y_3 = 2.5, y_4 = 5, y_5 = 5, y_6 = 2.5,$$
$$x_1 = 0, x_2 = 0.5, x_3 = 0.5, x_4 = 1, x_5 = 1, x_6 = 0.5.$$

The corresponding conflict graph is presented in figure 9, where the weight of node $i \in N$ is given by the value of x_i in the fractional solution. In order to separate inequality (cMIR 1), as explained in Section 4, set $S = \{6\}$ and $T = \{2, 3, 4\}$ where $S \cup T$ is the maximum-weight subset of N satisfying condition (35). This gives 2.5 for the left-hand side of inequality (34), while the right-hand side is equal to 2, and so inequality (cMIR 1) is violated for the proposed sets S and T.

5. Computational experiments

In Section 3 several families of valid inequalities have been introduced and sufficient conditions for defining facets of P have been provided, showing that these inequalities are relevant from a theoretical point of view. From a practical point of view, applying these inequalities to general mixed integer problems raises several questions, namely, to find the most efficient inequalities, to find efficient separation algorithms, and to test different relaxations of those problems since, for some problems as the ones discussed in [2], set X can be obtained through different relaxations. Given all these difficulties, this paper aims at providing only preliminary computational tests, using the intermediate set Y, to test, from a practical point of view, the inclusion of such inequalities. Thus, the goals of the computational experiments are (a) to evaluate how these inequalities approximate the convex hull of Y, and (b) to test whether these inequalities can improve the performance of a commercial solver to solve IP instances.

All computations are performed using the optimization software Xpress-Optimizer Version 23.01.03 with Xpress Mosel Version 3.4.0 [20], on a computer with processor Intel Core i7, 2.4 GHz and with 32 GB RAM.

Only inequalities (lift 1), (lift 2), (residual 2), and (cMIR 1), representing the three major families of inequalities presented in Section 3 are tested. Section 5.1 reports the integrality gap reduction obtained with the inclusion of these inequalities, while Section 5.2 reports the improvement obtained with the inclusion of these inequalities as cuts to solve a set of instances to optimality using a commercial solver.

5.1. Integrality gap reduction

In this section, the integrality gap reduction obtained with the addition of inequalities (lift 1), (lift 2), (residual 2), and (cMIR 1) is tested.

A set of instances of the minimization problem over the single node fixed-charge set are generated as follows. For each $d \in \{55, 80, 95, 110, 130\}$ and each $c \in \{25, 35, 45\}$ five instances are randomly generated. The conflict graph $G = (N, E)$ with $|N| = 20$ is randomly generated with density 25% and 50%. Elements in N_1 are randomly chosen from N with probability $\frac{1}{2}$. The coefficients of s in the objective function are randomly generated in the interval $[3, 5)$; the coefficients of $y_i, i \in N_1$, in the objective function are randomly generated in the interval $[0, 1)$; and the coefficients of x_i are randomly generated in the interval $[0, 20)$ if $i \in N_1$, and in the interval $(-20, 0]$ otherwise.

For each pair (d, c) the following average values are computed:

- the average initial integrality gap denoted by IG;

- the average closed gap using known inequalities for X_{SMI} (the MIR inequality) and for X_{VP} (Clique and Odd hole inequalities), denoted by MCO;

- the average closed gap using the *new* inequalities (lift 1), (lift 2), (residual 2), (cMIR 1), denoted by *New*;

- the average closed gap using MCO and the *New* cuts, denoted by *All*.

For MCO inequalities, the MIR inequality is included *a priori* while clique and odd hole inequalities are introduced as cuts using the separation routines given in [15]. For the *New* inequalities, the exact separation schemes discussed in Section 4 are implemented. Initial gaps are computed as $\frac{OPT - LR}{max\{|OPT|, |LR|\}} \times 100$ where OPT denotes the optimal value and LR indicates the linear relaxation value. Furthermore, closed gaps are calculated as $\frac{ILR - LR}{OPT - LR} \times 100$ where ILR denotes the value of the linear relaxation after the inclusion of the corresponding cuts. Moreover, the closed gap obtained by the MIR, clique, odd hole inequalities and inequality (lift 1) is denoted by $MCO+$(lift 1), and also the similar notation is used for inequalities (lift 2), (residual 2), and (cMIR 1). The computational results are reported in Tables 1-3.

Table 1: Average integrality gaps and closed gaps on 75 randomly generated instances with graph density 25%.

(d,c)	IG	MCO	New	All
(55,25)	47.91	81.62	71.72	87.54
(55,35)	39.35	82.71	59.32	94.12
(55,45)	24.28	95.55	54.74	97.32
(80,25)	75.34	91.54	91.62	96.52
(80,35)	38.64	94.88	87.82	97.21
(80,45)	27.61	81.56	48.42	95.52
(95,25)	89.86	90.02	92.47	95.46
(95,35)	48.98	78.89	74.88	94.54
(95,45)	34.99	96.71	79.56	98.91
(110,25)	107.01	93.09	96.12	96.88
(110,35)	62.45	85.83	92.33	95.82
(110,45)	38.85	82.70	60.80	89.58
(130,25)	113.15	95.33	98.63	100
(130,35)	102.94	90.85	95.08	99.61
(130,45)	48.06	80.71	80.34	87.97
Average	**59.96**	**88.13**	**78.92**	**95.13**

Table 2: Average integrality gaps and closed gaps on 75 randomly generated instances with graph density 50%.

(d,c)	IG	MCO	New	All
(55,25)	105.46	87.64	90.55	97.75
(55,35)	73.04	85.81	64.74	98.77
(55,45)	69.53	83.67	58.70	94.67
(80,25)	142.43	87.75	89.49	100
(80,35)	99.79	92.19	94.57	99.72
(80,45)	69.21	73.64	44.18	88.95
(95,25)	138.44	78.86	89.86	99.22
(95,35)	116.20	87.78	96.20	99.45
(95,45)	92.91	80.25	68.45	94.65
(110,25)	103.73	86.26	76.84	99.91
(110,35)	141.95	88.16	94.45	99.92
(110,45)	113.13	90.31	94.21	99.52
(130,25)	93.10	90.90	59.45	94.90
(130,35)	175.50	79.25	85.93	99.31
(130,45)	132.18	89.02	93.49	98.72
Average	**111.1**	**85.43**	**80.07**	**97.69**

It can be seen from Tables 1-2 that the addition of the *New* cuts to the linear relaxation allowed to improve the integrality gap closed by *MCO* inequalities of all tested instances. Moreover, those tables also show that the improvement on the integrality gap obtained by adding the *New* cuts to the linear relaxation of the instances with graph density 50% is slightly greater than the improvement obtained for the instances with lower graph densities. Such behaviour is somehow expected since most inequalities introduced in the paper are based on conditions stating that when a given set of variables is selected from N_0, then the maximum number of variables that can be selected from N_1 times c is not enough to cover d, forcing s to be positive. These conditions are satisfied when there are many edges between nodes in N_0 and nodes in N_1. Additional tests on graphs with density of 10% were performed. Such tests, not reported here, showed that for such small size instances MCO inequalities were able to reduce the integrality gap in 100% in almost all the instances. These results seem to indicate that the inequalities introduced here should be applied to subsets of more general sets where the conflict graph should not be too sparse.

Table 3 shows that inequality (lift 1) was ineffective, while (lift 2) was the most effective inequality for 10 pairs of (d, c), and inequality (cMIR 1) was the most effective one for the remaining pairs.

5.2. Inclusion of cuts to solve a set of instances

This section reports the results obtained to test the use of the proposed inequalities as cuts to solve a new set of instances. The objective of this experiment is different from the one in the previous section. Here, the purpose is to avoid exact separation, as it is too time consuming, and to tackle more difficult instances (in the previous section all the instances were solved to optimality). To this end, the new set of instances is generated in a very similar way to the one given in the previous section for a density of 50%, but with two differences: the number of nodes considered is set to $\mid N \mid= 400$ and the coefficient of s is taken in the interval $(0, 1]$.

For each pair (d, c), three instances are generated. Each instance is solved by Xpress Optimizer twice. First the instance is solved with the default options. Then the instance is solved with the addition of cuts at the root node using the separation heuristics described in Section 4. An overall time limit of 1800 seconds is assumed. The average results are reported in Table 4. Columns *Time* give the running time in seconds. For almost all the pairs of (d, c) at least one instance could not be solved within the time limit. Columns *Nodes* indicate the number of nodes generated during the branch-and-cut algorithm. Columns *Gap* indicate the integrality gap at the end of the running time (it is zero if the instance is solved to optimality).

Table 3: Average closed gaps by inequalities (lift 1), (lift 2), (residual 2), and (cMIR 1) which are taken individually on 75 randomly generated instances with graph density 50%.

(d,c)	MCO	MCO + (lift 1)	MCO + (lift 2)	MCO + (residual 2)	MCO + (cMIR 1)
(55,25)	87.64	87.64	94.30	**94.56**	94.09
(55,35)	85.81	85.81	**97.92**	86.31	85.81
(55,45)	83.67	83.67	**93.54**	83.67	83.67
(80,25)	87.75	87.75	93.22	95.27	**98.77**
(80,35)	92.19	92.19	**98.23**	94.51	96.31
(80,45)	73.64	73.64	**87.72**	78.16	73.64
(95,25)	78.86	78.86	**99.22**	91.34	80.75
(95,35)	87.78	87.78	**99.45**	93.20	88.16
(95,45)	80.25	80.25	84.02	86.21	**90.89**
(110,25)	86.26	86.26	92.16	90.85	**98.96**
(110,35)	88.16	88.16	92.48	95.02	**98.75**
(110,45)	90.31	90.31	**99.52**	91.18	90.95
(130,25)	90.90	90.90	90.90	91.04	**94.90**
(130,35)	79.25	79.25	**99.31**	87.25	80.97
(130,45)	89.02	89.02	**98.66**	96.23	89.35
Average	**85.43**	**85.43**	**94.71**	**90.32**	**89.73**

It can be readily seen that both the running times and the average gap decreased substantially with the inclusion of the proposed cuts. These results should be regarded as illustrative examples where the cuts proposed in the paper can be useful to solve IP problems. Other sets of instances generated with other parameters have been tested where no significant impact was observed, such as the instances using the coefficient of s generated as in the previous section.

6. Conclusion

This paper investigated a mixed integer set that intersects a simple mixed integer set, defined for a single constraint, with a vertex packing set, resulting from a conflict graph. It was shown that many new facet-defining inequalities appear when the intersection of the two sets is considered. Such inequalities cannot be obtained from original sets individually. In particular, the conflict MIR inequalities were proposed, which extend the well-known MIR inequalities to the case where incompatibilities between binary variables are considered. The new families were effective in solving and in reducing the integrality gap of a single node fixed-charge set with arc incompatibilities,when the conflict graph is dense.

Observe that identifying relevant sets X as substructure of general feasible sets is an open question that depends on the problem at hand. Another research direction is the study of related mixed integer sets, such as the intersection of X_{VP} with the following multiple simple mixed integer sets

$$\left\{ (s,x) \in \mathbb{R}_+^r \times \mathbb{B}^n : s_k + \sum_{i \in N_k} c_i x_i \geq d_k, k \in \{1, \ldots, r\} \right\},$$

arising when several 0-1 mixed integer constraints are considered simultaneously.

Table 4: Computational results with and without the inclusion of the new cuts at the root node.

(d,c)	Without cuts			With cuts		
	Time (s)	Nodes	Gap	Time (s)	Nodes	Gap
(55,25)	1444	156857	2.47	941	123452	1.08
(55,35)	1678	156651	7.18	952	123515	3.24
(55,45)	1697	153661	3.45	761	162406	0.80
(80,25)	1512	159775	3.41	768	145346	1.97
(80,35)	1499	154722	6.44	731	139510	1.73
(80,45)	1417	155827	1.94	816	142776	1.04
(95,25)	1649	166961	2.86	709	136488	1.67
(95,35)	1507	166912	6.64	825	137997	2.09
(95,45)	1634	168664	2.38	733	157304	1.64
(110,25)	1526	161720	4.83	779	162567	1.32
(110,35)	1517	154936	6.39	674	130098	1.53
(110,45)	1575	160989	2.4	712	143534	1.82
(130,25)	1466	149827	2.79	758	159055	1.41
(130,35)	1663	144488	7.97	725	142073	2.85
(130,45)	1643	145206	5.80	731	153750	1.25
Average	**1561.8**	**156746.4**	**4.46**	**774.3**	**143991.4**	**1.70**

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Appendix

Proof of Proposition 3.4. Consider the equation

$$s + r \sum_{i \in N_1 \setminus T} x_i = r \sum_{i \in S} x_i, \tag{A.1}$$

and let $K = P \cap \{(s,x) \mid (s,x) \text{ satisfies (A.1)}\}$. Now assume inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ is valid for X and satisfies the condition

$$\gamma s + \sum_{i \in N} \beta_i x_i = \gamma_0, \forall (s,x) \in K. \tag{A.2}$$

Next, it is shown that equality (A.2) is a multiple of (A.1). In order to achieve this goal, generate the points belonging to K as follows.

Condition (i) implies $\alpha(G[T]) \geq \lceil \frac{d}{c} \rceil$. So the following points exist and are in K.

(P1) $\forall \overline{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(T), s = 0; x_i = 1, i \in \overline{T}; x_i = 0, i \in N \setminus \overline{T}$.

In addition, condition (i) shows that for each $j \in T$, there exist $T_j \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(T)$ such that $j \notin N_1(T_j)$. So the following points are in K.

(P2) $\forall j \in T, s = 0; x_i = 1, i \in T_j; x_j = 1; x_i = 0, i \in N \setminus (T_j \cup \{j\})$.

Condition (ii) ensures the existence of the following points.

(P3) $\forall j \in N_0 \setminus S, s = 0; x_i = 1, i \in T_j \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(T); x_j = 1; x_i = 0, i \in N \setminus (T_j \cup \{j\})$.

Conditions (iii) ensures the existence of the following points.

(P4) $\forall \bar{S} \in \mathcal{S}, \overline{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil - |\bar{S}|}(T \setminus N_1(\bar{S})), \forall\, I \in \mathcal{I}_{|\bar{S}|-1}(N_1 \setminus (T \cup N_1(\bar{S}) \cup N_1(\overline{T}))), s = r; x_i = 1, i \in (\bar{S} \cup \overline{T} \cup I); x_i = 0, i \in N \setminus (\bar{S} \cup \overline{T} \cup I);$

(P5) $\forall \bar{S} \in \mathcal{S}, \overline{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil - |\bar{S}|}(T \setminus N_1(\bar{S})), \forall\, I \in \mathcal{I}_{|\bar{S}|}(N_1 \setminus (T \cup N_1(\bar{S}) \cup N_1(\overline{T}))), s = 0; x_i = 1, i \in (\bar{S} \cup \overline{T} \cup I); x_i = 0, i \in N \setminus (\bar{S} \cup \overline{T} \cup I).$

Substituting points (P1) and (P2) in equation (A.2) and subtracting the resultant equations imply $\beta_j = 0, j \in T$. Similarly, using points (P1) and (P3) gives $\beta_j = 0, j \in N_0 \setminus S$. Then replacing any points of type (P1) in equation (A.2) gives $\gamma_0 = 0$. So equation (A.2) can be written as

$$\gamma s + \sum_{i \in N_1 \setminus T} \beta_i x_i + \sum_{i \in S} \beta_i x_i = 0. \tag{A.3}$$

Let $i, j \in N_1 \setminus T$ and assume that they are adjacent in $G' = (N_1 \setminus T, E')$. So condition (iv) implies that there exist $\bar{S} \in \mathcal{S}$ and exists an independent set $I \subseteq N_1 \setminus (T \cup N_1(\bar{S}))$ such that $|I| = |\bar{S}|$, $i \in I$, $j \notin I$, and $I' = (I \setminus \{i\}) \cup \{j\}$ is an independent set. Substituting points (P4) or (P5), depending on the cardinality of the independent set, corresponding to sets I and I' in equation (A.3) and subtracting them imply $\beta_i = \beta_j, i, j \in N_1 \setminus T$. It follows from connectivity of graph $G' = (N_1 \setminus T, E')$ that $\beta_i = \beta_1, i \in N_1 \setminus T$.

Similarly to the justification of the previous part, one can check, using condition (v), that $\beta_i = \beta_2, i \in S$. Then replacing points (P4) or (P5) (depending on the cardinality of the independent set) in equation (A.3) it follows that $\beta_2 = -\beta_1$. Finally, substituting points (P4) in equation (A.3) gives $\beta_1 = \gamma r$. $\qquad\square$

Proof of Proposition 3.9. In order to prove that inequality (cMIR 1) defines a facet, consider the equation

$$s + c \sum_{i \in N_1 \setminus T} x_i + r \sum_{i \in T} x_i = r \left\lceil \frac{d}{c} \right\rceil + (c - r) \sum_{i \in S} x_i, \tag{A.4}$$

and let $K = P \cap \{(s, x) \mid (s, x) \text{ satisfies (A.4)}\}$. Now assume inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ is valid for X and satisfies the condition

$$\gamma s + \sum_{i \in N} \beta_i x_i = \gamma_0, \forall (s, x) \in K. \tag{A.5}$$

One can justify that equality (A.5) is a multiple of (A.4) as follows. Consider the following points in K.

(P1) $\forall\, T_1 \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(T), s = 0; x_i = 1, i \in T_1; x_i = 0, i \in N \setminus T_1;$

(P2) $\forall\, T_2 \in \mathcal{I}_{\lfloor \frac{d}{c} \rfloor}(T), s = r; x_i = 1, i \in T_2; x_i = 0, i \in N \setminus T_2;$

(P3) $\forall \bar{S} \in \mathcal{S}_1, \forall\, \overline{T} \in \mathcal{I}_{\lfloor \frac{d}{c} \rfloor - |\bar{S}|}(T \setminus N_1(\bar{S})), s = c|\bar{S}| + r; x_i = 1, i \in \bar{S}; x_i = 1, i \in \overline{T}; x_i = 0, i \in N \setminus (\bar{S} \cup \overline{T}).$

Note that condition (ii) implies that for each $k \in N_1 \setminus T$, there exist sets $\bar{S} \in \mathcal{S}_1$ and $\overline{T} \in \mathcal{I}_{\lfloor \frac{d}{c} \rfloor - |\bar{S}|}(T \setminus N_1(\bar{S}))$ such that $k \in N_1 \setminus (T \cup N_1(\overline{T} \cup \bar{S}))$. So the following points are in K.

(P4) $\forall k \in N_1 \setminus T, s = c(|\bar{S}| - 1) + r; x_i = 1, i \in \bar{S}; x_i = 1, i \in \overline{T}; x_k = 1; x_i = 0, i \in N \setminus (\bar{S} \cup \overline{T} \cup \{j\}).$

In addition, it follows from condition (iii) that for each $k \in N_0 \setminus S$, there exist sets $\bar{S} \in \mathcal{S}_1$ and $\overline{T} \in \mathcal{I}_{\lfloor \frac{d}{c} \rfloor - |\bar{S}|}(T \setminus N_1(\bar{S}))$ such that $i \in N_0 \setminus (S \cup N_0(\overline{T} \cup \bar{S}))$. Thus, the following points belong to K.

(P5) $\forall k \in N_0 \setminus S, s = c\left|\bar{S}\right| + r; x_i = 1, i \in \bar{S}; x_i = 1, i \in \overline{T}; x_k = 1; x_i = 0, i \in N \setminus (\bar{S} \cup \overline{T} \cup \{j\})$.

Now, let $i \in N_0 \setminus S$. Considering points of type (P3) and (P5), and substituting them in equation (A.5) and subtracting the resultant equations, it follows that $\beta_i = 0, i \in N_0 \setminus S$. Thus, equality (A.5) can be written as

$$\gamma s + \sum_{i \in N_1 \setminus T} \beta_i x_i + \sum_{i \in T} \beta_i x_i + \sum_{i \in S} \beta_i x_i = \gamma_0. \tag{A.6}$$

Consider $i, j \in T$ and suppose i and j are adjacent in $G'_{\lfloor \frac{d}{c} \rfloor} = (T, E')$. So there exists an independent set $I \subseteq T$ such that $i \in I$, $j \notin I$, $|I| = \lfloor \frac{d}{c} \rfloor$, and $I' = (I \setminus \{i\}) \cup \{j\}$ is independent. Using point (P2) corresponding to sets I and I' and equation (A.6) it follows that $\beta_i = \beta_j, i, j \in T$. It can be concluded from connectivity of $G'_{\lfloor \frac{d}{c} \rfloor} = (T, E')$ that $\beta_i = \beta_1, i \in T$.

Next, take $i, j \in S$ and assume that they are connected in $G'' = (S, E'')$. Therefore, there exists an independent set J such that $J \subseteq S$, $\alpha(G[T \setminus N_1(J)]) = \lfloor \frac{d}{c} \rfloor - |J|$, $i \in J$, $j \notin J$, $J' = (J \setminus \{i\}) \cup \{j\}$ is an independent set, and $\alpha(G[T \setminus N_1(J')]) = \lfloor \frac{d}{c} \rfloor - |J|$. Using points (P3) corresponding to J and J', and (A.6) implies $\beta_i = \beta_j, i, j \in S$. It follows from connectivity of $G'' = (S, E'')$ that $\beta_i = \beta_2, i \in S$.

Let $i \in N_1 \setminus T$. Substituting points of type (P3) and (P4) in equation (A.6) and subtracting them gives $\beta_i = \gamma c, i \in N_1 \setminus T$.

It follows from replacing points (P1) and (P2) in equation (A.6) that $\gamma_0 = \beta_1 \lceil \frac{d}{c} \rceil$ and $\gamma r + \beta_1 \lfloor \frac{d}{c} \rfloor = \gamma_0$ which implies $\beta_1 = \gamma r$, $\gamma_0 = \gamma r \lceil \frac{d}{c} \rceil$. Finally, substituting points (P3) in (A.6) gives $\beta_2 = -\gamma(c - r)$. $\qquad \square$

Proof of Proposition 3.14. First, observe that since S is an independent set, then $T = N_1(S)$. Now consider an equation

$$s + c \sum_{i \in N_1 \setminus T} x_i = c \sum_{i \in S} x_i + c \sum_{i \in T'} x_i + c(1 - p), \tag{A.7}$$

and let $K = P \cap \{(s, x) \mid (s, x) \text{ satisfies (A.7)}\}$. Now assume inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ is valid for X and satisfies the condition that

$$\gamma s + \sum_{i \in N} \beta_i x_i = \gamma_0, \forall (s, x) \in K. \tag{A.8}$$

One can prove that equality (A.8) is a multiple of (A.7) by introducing the following points belonging to K.

(P1) $s = c; x_i = 1, i \in S; x_i = 0, i \in N \setminus S$;

(P2) $\forall j \in N_1 \setminus T, s = 0; x_j = 1; x_i = 1, i \in S; x_i = 0, i \in N \setminus (S \cup \{j\})$;

(P3) $\forall \bar{S} \in \mathcal{I}_{p-1}(S), s = 0; x_i = 1, i \in \bar{S}; x_i = 0, i \in N \setminus \bar{S}$;

(P4) $\forall \bar{S} \in \mathcal{I}_{p-1}(S), \forall j \in T \setminus T', s = 0; x_i = 1, i \in \bar{S}; x_j = 1; x_i = 0, i \in N \setminus (\bar{S} \cup \{j\})$;

(P5) $\forall \bar{S} \in \mathcal{I}_{p-2}(S), \forall j \in T', s = 0; x_i = 1, i \in \bar{S}; x_j = 1; x_i = 0, i \in N \setminus (\bar{S} \cup \{j\})$.

Now let $i \in T \setminus T'$. Then, using points of type (P3) and (P4) corresponding to set $\bar{S}$ and equation (A.8) gives $\beta_i = 0, i \in T \setminus T'$.

Let $i \in N_0 \setminus S$. Condition (iii) implies that there exists at least one $\bar{S} \in \mathcal{I}(S)$ with $|\bar{S}| \in \{p-1, p\}$ such that $i \notin N_0(\bar{S})$. Therefore, depending on the cardinality of $\bar{S}$, either of points (P1) and (P3) in addition with setting $x_i = 1$ belongs to K as well. Substituting this new point with points (P1) or (P3) in equation (A.8) and subtracting the resultant equations imply $\beta_i = 0, i \in N_0 \setminus S$.

Substituting points (P1) and (P2) in equation (A.8) and subtracting the resultant equations give $\beta_i = \gamma c, i \in N_1 \setminus T$. Additionally, replacing points (P1) and (P3) in equation (A.8) and subtracting the resultant equations imply $\beta_i = -\gamma c, i \in S$.

Let $i, j \in T'$. As a consequence of condition (ii), there exist $\bar{S}_1, \bar{S}_2 \in \mathcal{I}_{p-2}(S)$ such that $i \in T' \setminus N_1(\bar{S}_1)$ and $j \in T' \setminus N_1(\bar{S}_2)$. Replacing points (P5) corresponding to subsets $\bar{S}_1$ and $\bar{S}_2$ in equation (A.8) implies $\beta_i = \beta_j, i, j \in T'$ and so $\beta_i = \beta, i \in T'$. Next, substituting points (P1) in equation (A.8) gives $\gamma_0 = \gamma c(1-p)$. Finally, $\beta = -\gamma c$ can be obtained by replacing points (P5) in equation (A.8). $\qquad\square$