PLANAR SPIN GLASSES

by

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ABSTRACT

The infinite–range p–state clock spin–glass model is studied.

The model is investigated at first within replica–symmetric theory. No discontinuous phase transitions are found, contrary to what happens to other known models with absence of reflection symmetry in the spin variable. The Almeida–Thouless instability of the replica–symmetric solution in zero magnetic field, is shown to occur at $O(\epsilon^2)$, except for $p = 3$, in which case this instability appears at $O(\epsilon)$ $(\epsilon = (T_g - T)/T_g)$.

Spin–glass ordering in conventional models normally reflects, on average, the rotational symmetry of the hamiltonian. It is demonstrated that the four–state clock model is exceptional, in that the average spin–glass order is essentially collinear (two–fold symmetric), despite the four–fold symmetry of the hamiltonian. Fluctuation effects are predicted due to replica–symmetry breaking, but these are shown to be relatively small.

The effects of a magnetic field in the p–state clock spin glass are investigated. It is shown that the $p = 3$ case is peculiar in the sense that the critical line associated to the transverse spin–glass freezing changes under reflection of the magnetic field $(\underline{h} \to -\underline{h})$. The case $p = 4$ is similar to $p = 2$, and an Almeida–Thouless line is followed. It is demonstrated that all $p \geq 5$ clock glasses present the conventional XY–like Gabay–Toulouse line.

The role of a four–fold anisotropy field on the XY spin glass is analysed. It is proven that the normal four–fold symmetric spin–glass phase occurs except in the limit of infinite anisotropy.

The Parisi replica–symmetry–breaking scheme is applied to the infinite–range p–state clock spin glass. It is shown that all values of p present the conventional Parisi solution, except the case $p = 3$, for which a step function is

the stable solution. The absence of reflection symmetry in the spin variable is qualitatively irrelevant for all other odd–state clock spin glasses.

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CONTENTS

ABBREVIATIONS AND NOTATION

EA : Edwards and Anderson (1975)

SK : Sherrington and Kirkpatrick (1975)

TAP : Thouless, Anderson and Palmer (1977)

AT : de Almeida and Thouless (1978)

GT : Gabay and Toulouse (1981)

RS : Replica Symmetry (Sherrington and Kirkpatrick 1975)

FC : Field Cooled

ZFC : Zero–Field Cooled

tr_α : trace over a single replica α

Tr_α : trace over all replicas ($\alpha = 1,\ldots,n$)

$\langle\ \rangle_T$: thermal average with respect to the original hamiltonian H

$\langle\ \rangle$: thermal average with respect to the replicated hamiltonian H_{eff}

$[\ ]_{av}$: average over the disorder

$\sum_{ij}$: summation over all sites i and j

$\sum'_{ij}$: summation over all sites i and j with $i \neq j$

$\sum_{(ij)}$: summation over distinct pairs of sites

$\sum_{\langle ij\rangle}$: summation over distinct nearest–neighbour pairs of sites

$\sum_{\alpha\beta}$: summation over all replicas α and β

$\sum'_{\alpha\beta}$: summation over all replicas α and β with $\alpha \neq \beta$

$\sum_{(\alpha\beta)}$: summation over distinct pairs of replicas

CHAPTER 1: INTRODUCTION AND SURVEY

Spin glasses continue to be of much active interest with new features and applications discovered regularly. Despite the complexity of such systems, much progress has been attained in the latest years, from both theoretical and experimental points of view.

Most of the theory has been constructed from the investigation of the simplest model, the Ising spin glass. For the short-range-interaction case (Edwards and Anderson 1975), although it is agreed that the lower critical dimension lies somewhere in between 2 and 3 (Bray and Moore 1984, 1985, 1986, McMillan 1984a, 1984b, 1985a, Bhatt and Young 1985, Ogielski and Morgenstern 1985, Singh and Chakravarty 1986), the nature of the low-temperature phase remains controversial (Fisher and Huse 1986, Bray and Moore 1986, Villain 1986). The mean-field theory, as realized in the infinite-range-interaction model of Sherrington and Kirkpatrick (1975), is now fairly well understood. The spin-glass phase is characterized by an infinite number of order parameters (Parisi 1979, 1980a, 1980b, 1980c, 1980d, 1980e, 1983), related to the existence of many free-energy minima which are separated by infinite-height barriers (Mackenzie and Young 1982). These minima, usually called pure states, are constrained to an ultrametric structure (Mézard *et al.* 1984a, 1984b).

Whether mean-field theory provides a good qualitative picture of the low-temperature phase in real spin glasses, is a point of much debate at the present moment (Fisher and Huse 1986, Bray and Moore 1986, Villain 1986). However, it turns out that infinite-range spin-glass models, besides providing a mean-field treatment to real spin glasses, are closely related to other subjects like complex optimization problems and biological systems (brain models). Therefore, the concepts developed for spin glasses are currently penetrating other areas of research. This gives an stimulating support for further development of its mean–field theory.

One is usually tempted to generalize Ising models in order to give more freedom to the spin variable. Although the Sherrington–Kirkpatrick model is now considered as well understood, generalizations of it bring new features and challenges to the spin–glass theory. In particular, systems in which the spin variable does not present symmetry under reflection, like the Potts spin glass, exhibit a quite curious behaviour (Erzan and Lage 1983, Elderfield and Sherrington 1983a, 1983b, 1983c, Goldbart and Elderfield 1985, Gross *et al.* 1985).

In this thesis we study the infinite–range p–state clock spin glass, for which the spin variables are represented as unit vectors restricted to p equally angularly spaced orientations in a plane. Such a system contains the Sherrington–Kirkpatrick model as the particular case $p = 2$, while interpolates between two very distinct spin glasses, that is, $p = 3$ (3–state Potts) and $p = \infty$ (XY). The spin variable presents (does not present) symmetry under reflection for every even (odd) value of p.

Chapter 2 is an introduction to spin glasses. We do no attempt to give a detailed review on the subject, but rather put emphasis on issues to be explored throughout the thesis. We refer the interested reader to several reviews available (Binder and Young 1986, van Hemmen and Morgenstern 1983, 1986, Mézard *et al.* 1987, Sherrington 1988, Chowdhury and Mookerjee 1984, Moore 1984, Fischer 1983, 1985). We discuss the Sherrington–Kirkpatrick model and some of its generalizations; the Parisi replica–symmetry–breaking scheme is presented. We show the relations of infinite–range spin glasses to complex optimization problems and neural systems. We discuss briefly the short–range spin glasses.

We introduce the p–state clock spin glass in Chapter 3; its relations to well–known models are explored. We apply the replica method to such a system and develop free–energy expansions. Finally, we investigate the replica–symmetric solution and its stability in the spin–glass phase.

Chapter 4 deals with the nature of the low–temperature phase of the 4–state clock spin glass. Within the replica–symmetric approximation we find two distinct solutions corresponding to the same free energy. By means of a mapping to two identical Ising models, we argue that the spin–glass phase is highly anisotropic, that is, collinear. We introduce a higher–order test function able to distinguish between isotropy and collinearity; fluctuations from perfect collinearity are predicted due to replica–symmetry breaking. Finally, we perform a Monte Carlo simulation to support our assertion of anisotropic spin–glass order and show that the fluctuations from perfect collinearity are rather small. Most of the results in this chapter are also described in Nobre *et al.* (1989).

In Chapter 5 we investigate the effects of a magnetic field in the p–state clock spin glass. We show that the $p = 3$ case is peculiar in the sense that the critical line associated to the transverse spin–glass freezing changes under reflection of the magnetic field $(\underline{h} \to -\underline{h})$. The case $p = 4$ is similar to $p = 2$, and despite the four–fold symmetry of the spin variable, a magnetic field induces the order to two–fold symmetric. Finally, we show that all $p \geq 5$ clock glasses are XY like. The results of this chapter are also presented in Nobre and Sherrington (1989).

In Chapter 6 we analyse the role of a four–fold anisotropy field on the XY spin glass. We show that the normal four–fold symmetric spin–glass phase occurs except in the limit of infinite anisotropy. This is also discussed in Nobre *et al.* (1989).

In Chapter 7 we apply the Parisi replica–symmetry–breaking scheme to the infinite–range p–state clock spin glass. We show that all values of p present the conventional Parisi solution, except the case $p = 3$, for which a step function is the stable solution. The absence of reflection symmetry in the spin variable is qualitatively irrelevant for all other odd–state clock spin glasses. Most of the results in this chapter are also described in Nobre and Sherrington (1986).

Finally, in Chapter 8 we conclude.

CHAPTER 2: AN INTRODUCTION TO SPIN GLASSES

2.1. Spin–glass systems

The name "spin glass" stands for disordered magnetic systems in which the interactions between the magnetic moments are random in sign. This terminology was first introduced by Coles (Anderson 1973, Coles 1973) to denote a class of dilute magnetic alloys of a non–magnetic host (noble metal) and a magnetic impurity (transition metal). The impurities, at low concentrations (typically $1 \to 10\%$), are "quenched", that is, the alloy is taken abruptly from high to low temperatures, such that the impurity atoms cannot rearrange themselves to its lowest energy state and the disorder is frozen. This is to be contrasted with "annealing", where the cooling process is slow, allowing the system to relax to its minimum energy state. Annealed systems are less interesting from the theoretical point of view, since they can be related to non–random systems (Thorpe and Beeman 1976). Examples of the "classical", or "canonical", spin glasses are (the host is underlined) $\underline{Au}Fe$, $\underline{Au}Mn$, $\underline{Cu}Mn$ and $\underline{Ag}Mn$.

In these systems, the magnetic moments of the impurities, which are distributed randomly in the host, interact with each other by polarizing the conduction electrons around them. This leads to an indirect exchange interaction, the so called RKKY interaction (Ruderman and Kittel 1954, Kasuya 1956, Yosida 1957), which oscillates with the distance between the magnetic moments at two given sites i and j , R_{ij} ,

$$J_{ij} = J_0 \frac{\cos(2k_F R_{ij} + \phi_0)}{(k_F R_{ij})^3} \quad . \tag{2.1.1}$$

In (2.1.1), J_0 and ϕ_0 are constants and k_F is the Fermi wave vector of the host metal. As can be seen in Figure 2.1 , depending on the separation R_{ij} , the coupling between two given spins can be either ferromagnetic ($J_{ij} > 0$) or antiferromagnetic ($J_{ij} < 0$). Since the distances between the impurities are random, some of the interactions of a given spin with the others will be positive, favouring parallel alignment, some negative, favouring antiparallel alignment, that is, there will be "competition" among the interactions. Both disorder and competition between couplings are believed to be the main requirements for spin–glass behaviour. They lead to a low–temperature phase in which the magnetic moments freeze in random directions as shown in Figure 2.2.

It is important to mention that even for the canonical spin glasses discussed above (Morgownik and Mydosh 1983a, 1983b), an effective magnetic hamiltonian is not known precisely; however, it is clear that the competition between ferromagnetic and antiferromagnetic bonds should be considered in any theoretical approach attempt.

A second class of well–studied systems are the insulating spin glasses such as $Eu_xSr_{1-x}S$, $Eu_{1-x}Gd_xS$ and $Fe_{1-x}Mg_xCl_2$, for which the magnetic interactions are strictly short ranged. The most studied of these systems is $Eu_xSr_{1-x}S$ (Maletta and Felsch 1979, Maletta 1982). In this system, the Eu and Sr ions sit on the sites of an fcc lattice with a fraction x of these sites being occupied at random by the Eu ions. Although this time, the magnetic interactions which occur between the Eu ions are short ranged, there is a competition between ferromagnetic nearest–neighbour couplings and antiferromagnetic next–nearest neighbour couplings; interactions between more distant neighbours are negligibly small. Spin–glass behaviour has been observed for $0.13 \lesssim x \lesssim 0.5$ (Maletta and Felsch 1979).

It should be noted that spin–glass characteristics have also been observed experimentally in several other systems including non–magnetic systems, like dilute ferroelectrics and molecular crystals (for reviews see Binder and Young 1986, van Hemmen and Morgenstern 1983, 1986, Fischer 1985, Chowdhury and Mookerjee 1984).

2.2. Basic experimental properties

In order to classify a given material possessing the requirements mentioned in the previous section (disorder and competing exchange), as a "good" spin glass, usually three basic experimental properties are investigated:

(a) a.c. susceptibility;

(b) magnetic specific heat;

(c) low–temperature magnetization versus field.

If all three measurements present the behaviour described below, then the system may be considered as a conventional spin glass and further study may be followed.

Spin–glass behaviour was first observed by Canella and Mydosh (1972); they found a fairly sharp "cusp" in the low–field a.c. susceptibility of AuFe at a well defined temperature, T_g, suggesting that the system undergoes a phase transition. In fact, the cusp is very sensitively field dependent and gets flattened even at fields as low as 50 G ; however, the curves get sharper and sharper as h is reduced, and the extrapolation to zero field is consistent with such a peak (see Figure 2.3).

A rather curious aspect of the a.c. susceptibility is its dependence on the frequency ω, in which the measurements are performed. The temperature T_g decreases with ω (see Figure 2.4) and then, one could argue that in the limit

$\omega = 0$ (d.c. measurement), $T_g \to 0$ and the "phase transition" is simply a non–equilibrium manifestation.

In contrast to the susceptibility behaviour, the magnetic contribution to the specific heat C_m exhibits no anomaly at T_g but a broad maximum well above T_g (Zimmermann and Hoare 1960, Wenger and Keeson 1975, 1976, Martin 1978, 1979). In addition to that, neutron diffraction experiments indicate the absence of long–range magnetic order below T_g (Arrot 1965).

The third basic property of a canonical spin glass concerns the magnetization in a field. There are two ways of producing a low–temperature magnetization in a field:

(i) Field Cooled (FC) Magnetization: a magnetic field is applied to the sample at high temperatures and the magnetization is tracked as the system is cooled.

(ii) Zero–Field Cooled (ZFC) Magnetization: the sample is cooled in zero magnetic field to a very low temperature ($T \ll T_g$), then a field is applied and the magnetization is followed as the temperature increases.

For a ferromagnet, these two procedures do provide identical results for the magnetization at fixed values of the magnetic field and temperature. For a spin–glass system however, they yield very different results which are shown in Figure 2.5 . Below T_g , the two curves bifurcate (Tholence and Tournier 1974, Nagata *et al.* 1979), the FC one forming a plateau, whereas the ZFC curve jumps to a definite value $\chi = M/h$ equivalent to the a.c. susceptibility. The FC curve is weakly time dependent and is generally accepted as representing the equilibrium magnetization; it is a completely reversible curve. On the other side, the ZFC curve is time dependent and evolves slowly (over many decades) towards the FC curve. It is an irreversible curve since, if at point A one reduces the temperature, the system goes to point B instead of following the ZFC curve; by raising the temperature at B, the system will first go to A before joining the ZFC curve. Along the FC curve the name "thermoremanent magnetization" is currently used for the magnetization, whereas in the ZFC case the term "isothermal remanent magnetization" is employed.

Although the properties described above are enough to characterize experimentally a spin–glass system, they are not very conclusive about the occurrence of a true static phase transition at the temperature T_g . Perhaps the best experimental evidence in favour of a phase transition for spin glasses, comes from measurements of the non–linear susceptibility χ_{nl} , by approaching T_g from above $(T \to T_g^+)$, where the dynamical effects are weak. For a small magnetic field h , the magnetization may be expanded in power series,

$$\frac{M}{h} = \chi_0(T) - h^2\,\chi_{nl}(T) + O(h^4) \quad , \quad (T > T_g) \qquad (2.2.1)$$

where $\chi_0(T)$ is the zero–field susceptibility. The quantity $\chi_{nl}(T)$ is more sensitive to spin–glass order than $\chi_0(T)$, presenting a spectacular increase on approaching T_g , quite comparable to the growth of $\chi_0(T)$ in ordinary magnets. Monod and Bouchiat (1982), by investigating the field–cooled magnetization of <u>Ag</u>Mn , concluded that the resulting non–linear susceptibility was consistent with a critical divergence at T_g ,

$$\chi_{nl} \approx \left[\frac{T - T_g}{T_g}\right]^{-\gamma} \quad , \quad T \to T_g^+ \quad , \qquad (2.2.2)$$

where γ is a critical exponent associated with a static phase transition to a spin–glass state. Although some controversy exists on the observed values of γ

(Barbara *et al.* 1981, 1982, Omari *et al.* 1983, Bouchiat 1986), it is accepted that real spin glasses do present a true phase transition at the temperature T_g .

2.3. The Edwards–Anderson model

As an attempt to describe the spin–glass properties discussed above, Edwards and Anderson (1975) proposed a theoretical model , the EA model, in which one works with a well–defined lattice, and the site randomness as seen in Figure 2.2 , is replaced by a bond randomness. The Edwards–Anderson model consists of a set of N classical vector spins $\underline{S}_i$ distributed on a regular lattice, interacting via the hamiltonian,

$$H = - \sum_{\langle ij \rangle} J_{ij} \, \underline{S}_i \cdot \underline{S}_j \quad , \tag{2.3.1}$$

where the sum $\langle ij \rangle$ is over nearest–neighbour pairs of spins, and the exchange constants J_{ij} are randomly chosen according to a fixed probability distribution $P(J_{ij})$. Two of the most commonly used probability distributions are the gaussian (Edwards and Anderson 1975, Sherrington and Southern 1975).

$$P(J_{ij}) = (1/2\pi J^2)^{\frac{1}{2}} \exp\left[-(J_{ij} - J_0)^2/2J^2\right] \tag{2.3.2a}$$

and the " ± J " (Toulouse 1977),

$$P(J_{ij}) = p\delta(J_{ij} - J) + (1-p)\delta(J_{ij} + J) \quad . \tag{2.3.2b}$$

Both distributions mentioned above lead to a random mixture of positive and negative bonds, presenting the two basic ingredients for a spin glass, that is, disorder and "frustration".

The concept of frustration was introduced by Toulouse (1977), and in order to explain this, let us restrict ourselves to an Ising spin hamiltonian ($S_i = \pm 1$) with $\pm J$ bonds. Consider the smallest closed loop of bonds on a lattice, i.e. a plaquette, which for the two-dimensional square lattice is a square. If one attempts to minimize the energy of a single plaquette, then the number of minimum energy spin configurations depends strongly on the signs of the exchange couplings. For the case in Figure 2.6(a), all bonds are "satisfied", there are two minimum energy configurations and the plaquette is said to be frustrated. In Figure 2.6(b) however, one of the bonds remains "unsatisfied"; there are eight possible ground-state configurations and the plaquette is frustrated. The basic difference between the two plaquettes in Figure 2.6 is contained in the frustration function:

$$\begin{aligned} \Phi = \mathrm{sign}\,(J_{ij}J_{jk}J_{kl}J_{li}) \quad &= -1\,, \quad \text{for a frustrated plaquette,} \\ &= +1\,, \quad \text{otherwise.} \end{aligned} \tag{2.3.3}$$

The existence of frustrated plaquettes leads to a characteristic feature of spin glasses, that is, for a given disorder realization $\{J_{ij}\}$, there exist many ground states which are unrelated by any global symmetry of the system. The frustration concept discussed above can be generalized to more complicated lattices and to higher-dimensionality spins (see the review by Binder and Young 1986).

The disorder average of a quantity $A\{J_{ij}\}$ is then, given by

$$[\, A\{J_{ij}\}\,]_{av} = \int \prod_{\langle ij\rangle} \left[dJ_{ij}\, P(J_{ij})\right] A\{J_{ij}\} \quad , \qquad (2.3.4)$$

where $\{J_{ij}\}$ denotes a particular realization of the disorder, usually called a sample.

It is important to note that for a real system the random variables $\{J_{ij}\}$ may fluctuate with time as a consequence of the diffusion of atoms through the lattice. For annealed systems the observation time is such that these random variables reach thermal equilibrium, and one computes the disorder average in a similar way to the statistical averages. As an example, the free energy per spin is given by

$$-\beta f_{ann} = \lim_{N\to\infty} N^{-1} \ln [\, Z\{J_{ij}\}\,]_{av} \quad , \qquad (2.3.5a)$$

where

$$Z\{J_{ij}\} = \mathrm{tr}\, \exp(-\beta H) \quad , \qquad (2.3.5b)$$

is the partition function for a particular sample $\{J_{ij}\}$. These systems are much easier to deal with (Thorpe and Beeman 1976), but are not very relevant for the spin–glass problem.

For quenched systems however, the random variables are not in thermal equilibrium and averaging like in (2.3.5) is not correct, as will be discussed below. Following an argument due to Brout (1959), one should average only extensive variables. By dividing a large system into smaller subsystems with no interaction between them, he argued that the disorder average of a given extensive variable (normalized per degree of freedom) for the whole system is equal to the average of the values of such a quantity over the subsystems, in the limit when the number of subsystems becomes large. Quantities $O\{J_{ij}\}$ possessing this property are called "self-averaging", in the sense that,

$$O\{J_{ij}\} = [\, O\{J_{ij}\}\,]_{av} \quad , \quad (N \to \infty) \quad . \tag{2.3.6}$$

For finite N the Brout argument leads to fluctuations around the average value of the density of an extensive quantity; such fluctuations obey a gaussian probability distribution of width of order $N^{-\frac{1}{2}}$. For the free energy per spin, as an example,

$$P(f) \propto \exp\left\{ - N\, (f - [f]_{av})^2 / \, 2(\Delta f)^2 \right\} \quad . \tag{2.3.7}$$

Since

$$Z = \exp\,(-N\beta f) \quad , \tag{2.3.8}$$

by using the probability distribution (2.3.7), one gets for the average over the partition function (annealed systems),

$$f_{ann} = [f]_{av} + \beta(\Delta f)^2 \quad . \tag{2.3.9}$$

This shows that $f_{ann} \geq [f]_{av}$, providing an overestimate for the free energy per spin of the quenched system. Therefore, the correct thing to do for quenched systems is to average over the free-energy density,

$$[f]_{av} = -\beta^{-1} \lim_{N\to\infty} N^{-1} [\, \ln Z\{J_{ij}\}\,]_{av} \quad . \tag{2.3.10}$$

The average in (2.3.10) is not an easy task because the random variables occur inside a logarithm. Edwards and Anderson (1975) overcame this difficulty by making use of the replica trick (Kac 1968, Edwards 1970, 1971, Emery 1975),

$$[\, \ln Z\{J_{ij}\}\,]_{av} = \lim_{n\to 0} \frac{1}{n} \left[[Z^n\{J_{ij}\}]_{av} - 1 \right] \quad , \tag{2.3.11}$$

which is easily proved by the expansion,

$$Z^n = \exp(n\ln Z) \approx 1 + n\ln Z \quad , \quad \text{for } n \to 0 \, . \tag{2.3.12}$$

In the equations above Z^n is the product of n identical and independent replicas of the system,

$$Z^n = \prod_\alpha Z_\alpha = \mathrm{Tr}_\alpha \exp\left(-\beta \sum_\alpha H^\alpha\right) \quad , \quad \alpha = 1,\ldots,n \quad , \tag{2.3.13}$$

where Z_α is the partition function of the α-th replica and the trace is to be extended over all replicas. Although Z^n is only defined for positive integer n , it is assumed that the limit $n \to 0$ can be taken. This is a point which caused a lot of debate concerning the reliability of the replica trick (van Hemmen and Palmer 1979). As will be seen in the forthcoming sections, the analytic continuation $n \to 0$ is highly non-trivial, but if done properly, leads to results that whenever possible to be checked against alternative methods, do indeed agree.

As a possible order parameter Edwards and Anderson (1975) suggested the long–time correlation function,

$$q_{EA} = \lim_{t\to\infty} \lim_{N\to\infty} N^{-1} \sum_{i=1}^{N} [\langle \underline{S}_i(0).\underline{S}_i(t) \rangle_T]_{av} \quad , \qquad (2.3.14)$$

where $\langle \, \rangle_T$ denotes a thermal average with respect to the hamiltonian and $[\;]_{av}$ stands for an average over the disorder. EA claimed the spin–glass phase to be associated with a "freezing" of the spins in random directions during long time scales, or in other words, the spin on each site remembers its orientation over long periods of time.

Another equally important quantity on spin glasses is the statistical mechanics order parameter,

$$q = \lim_{h\to 0} \lim_{N\to\infty} N^{-1} \sum_{i=1}^{N} [\langle \underline{S}_i \rangle_T^2]_{av} \quad , \qquad (2.3.15)$$

for which one has to introduce a small symmetry–breaking field $\underline{h}$ to ensure $\langle \underline{S}_i \rangle_T \neq 0$. This could be a uniform field or a random field (in the case of Ising spins) and the limit $h \to 0$ should come after the thermodynamic limit has been taken (Young and Kirkpatrick 1982, Young and Jain 1983).

Some spin–glass systems like the Sherrington–Kirkpatrick model which will be discussed in the following section, present the property of non–ergodicity. This means that the system may get trapped in given states corresponding to a restricted region of phase space, in such a way that a time average is not equivalent to the statistical mechanics ensemble average. For non–ergodic spin glasses it is clear that

$$q_{EA} \neq q \quad , \tag{2.3.16}$$

and it is sometimes useful to define a parameter Δ which is a measure of breakdown of ergodicity (Sommers 1978),

$$\Delta = q_{EA} - q \quad . \tag{2.3.17}$$

Such a quantity appears naturally in the dynamical approaches to spin glasses (Sompolinsky 1981, Sompolinsky and Zippelius 1981, 1982).

The Edwards–Anderson model of spin glass, although a theoretical problem simple to formulate, has proved to be extremely difficult to solve. In the next section we turn to its mean–field treatment for the case of Ising spins.

2.4. The Sherrington–Kirkpatrick model

2.4.1. The Sherrington–Kirkpatrick solution

The infinite–range interaction model in which every spin of the system interacts with all the others via the same exchange is, for the ferromagnetic case, a problem for which mean–field theory is exact (Stanley 1971). The extension of such a model to the spin–glass case was proposed by Sherrington and Kirkpatrick (SK) (1975) as a possible mean–field treatment of the Edwards–Anderson model. The SK hamiltonian has the same form as the one for the EA model for Ising spins ($S_i = \pm 1$), namely

$$H = -\sum_{(ij)} J_{ij} S_i S_j - h \sum_i S_i \quad , \tag{2.4.1}$$

where h is an external magnetic field and the sum (ij) is now over all distinct pairs of spins. The J_{ij}'s are quenched random couplings distributed according to the probability

$$P(J_{ij}) = (N/2\pi J^2)^{\frac{1}{2}} \exp\left[-N(J_{ij} - J_0/N)^2/2J^2\right] \quad , \qquad (2.4.2)$$

where the N dependence (N is the total number of spins of the system) is necessary to ensure a non–trivial thermodynamic limit. In the discussion which follows we take the ferromagnetic offset J_0 to be zero, as a simplification.

To obtain the average free energy per spin, $[f]_{av}$, one makes use of the replica method as discussed in the previous section. Details on the application of this method will be given explicitly in the following chapter for a more general problem, the p–state clock spin glass (which has the SK model as the particular case p = 2).

The average over the disorder is easily performed and in the infinite–range–interaction limit the problem is reduced to a single–site dependence (from now on, the site index will be dropped) (Sherrington and Kirkpatrick 1975, Kirkpatrick and Sherrington 1978). The free energy is obtained from a steepest descents integration which leads to the extremization of a free–energy functional $g(q^{\alpha\beta})$,[1]

$$\beta f = \lim_{n \to 0} \frac{1}{n} \min \{g(q^{\alpha\beta})\} \quad , \qquad (2.4.3)$$

with

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[1]For the remainder of this thesis, the quenched average of the free energy per spin will be denoted simply by f , instead of $[f]_{av}$ as used up to now.

$$g(q^{\alpha\beta}) = -\frac{n}{4}(\beta J)^2 + \frac{(\beta J)^2}{4}\sum_{\alpha\beta}{}'(q^{\alpha\beta})^2 - \ln \mathrm{Tr}_\alpha \exp\{H_{eff}\} \quad , \tag{2.4.4a}$$

$$H_{eff} = \frac{(\beta J)^2}{2}\sum_{\alpha\beta}{}' q^{\alpha\beta} S^\alpha S^\beta + \beta h \sum_\alpha S^\alpha \quad , \tag{2.4.4b}$$

where $\sum'_{\alpha\beta}$ denotes a sum over all $\alpha \neq \beta$. One gets rid of the disorder but the price paid for that is the appearance of couplings between distinct replicas in the effective hamiltonian H_{eff}. There are $n(n-1)/2$ parameters $q^{\alpha\beta}$ which are determined by extremizing $g(q^{\alpha\beta})$,

$$q^{\alpha\beta} = \langle S^\alpha S^\beta \rangle \quad ; \quad \alpha \neq \beta \quad , \tag{2.4.5}$$

where $\langle\ \rangle$ denotes a thermal average with respect to H_{eff}.

The main difficulty of this problem comes on choosing the appropriate parametrization for the parameters $q^{\alpha\beta}$ in the limit $n \to 0$. As long as n is a positive integer, H_{eff} is invariant under permutations of the replica indices and so, what appears naturally as a first attempt is the replica–symmetric solution (RS) (Sherrington and Kirkpatrick 1975),

$$q^{\alpha\beta} = q \quad \text{for all } \alpha \neq \beta \quad . \tag{2.4.6}$$

Such a solution leads to a phase transition from a paramagnetic $(T > T_g\,,\ q = 0)$ to a spin–glass state $(T < T_g\,,\ q \neq 0)$ at a critical temperature,

$$T_g = J \quad . \tag{2.4.7}$$

Curiously, one gets that f is a *maximum* with respect to q in both states and for $T < T_g$, the spin–glass solution ($q \neq 0$) presents a *higher* free energy than the $q = 0$ solution, contrary to expectation (see Figure 2.7). The explanation for this comes from the fact that the number of parameters $q^{\alpha\beta}$, $n(n-1)/2$, becomes *negative* in the limit $n \to 0$. This is responsible for changing the minimum in equation (2.4.3) into a maximum condition. As will be discussed next, the replica–symmetric spin–glass solution is unstable below T_g, but the Parisi solution (Parisi 1979), which is believed to be the correct one, presents an even higher free energy than the SK solution. The minimum condition in (2.4.3) only makes sense when seen as a local stability condition, that is, minimum with respect to each one of the $q^{\alpha\beta}$ parameters. This is done by requiring the stability matrix $\underline{\mathbf{S}}$,with elements (de Almeida and Thouless 1978),

$$S^{(\alpha\beta)(\gamma\delta)} = \frac{\partial^2 g}{\partial q^{\alpha\beta} \partial q^{\gamma\delta}} \quad , \tag{2.4.8}$$

to be positive definite, i.e. all its eigenvalues should be positive for stability.

Unfortunately, the SK solution leads to problems at low temperatures; the entropy becomes negative (Sherrington and Kirkpatrick 1975), which is not acceptable for an Ising system. During sometime this entropy "catastrophe" was attributed to the replica trick itself (van Hemmen and Palmer 1979). In order to avoid that, Thouless, Anderson and Palmer (TAP) (1977) proposed a new solution for the SK model without using the replica trick. For $T > T_g$, by means of a high–temperature expansion they found the same results as Sherrington and Kirkpatrick. For $T < T_g$, TAP used a diagrammatic expansion to derive the mean–field equations for the SK model, namely,

$$m_i = \tanh\left\{\beta \sum_j J_{ij}\, m_j + \beta h_i - \beta^2 \sum_j J_{ij}^2\,(1 - m_j^2)\, m_i\right\} \quad ,$$

$$i = 1,\ldots,N \quad . \qquad (2.4.9)$$

In equations (2.4.9) we have added a local magnetic field h_i, and m_i denotes the magnetization on site i. The first two terms inside the hyperbolic tangent are familiar from the mean–field theory of pure systems, while the last contribution is known as the Onsager reaction–field term (Onsager 1936, Brout and Thomas 1967). This comes essentially as a self–effect correction which subtracts the contribution of the spin at site i to the total mean field experienced by itself. In the corresponding theory for a ferromagnet, such a term is negligible, while for spin glasses with a probability distribution $P(J_{ij})$ as in (2.4.2), this is of the same order of magnitude as the first two terms. Since the J_{ij}'s are elements of a random matrix, the solution of the set of equations (2.4.9) is not an easy task and TAP succeeded in finding solutions only near T_g and $T = 0$. While near T_g their results are essentially those obtained by SK, near $T = 0$ they found very different behaviour and in particular, a positive value for the entropy.

The explanation of why the SK solution went wrong was given by de Almeida and Thouless (AT) (1978). By looking at the eigenvalues of the stability matrix (2.4.8), AT showed that the apparently unharmful choice (2.4.6) is locally unstable for all temperatures below T_g. The nature of the instability indicates that the correct solution must break the permutation symmetry between distinct replicas, since the eigenvalue corresponding to replica–symmetry–breaking fluctuations becomes negative for $T < T_g$. This eigenvalue is inversely proportional to the spin–glass susceptibility (Pytte and Rudnick 1979, Bray and Moore 1979),

$$\chi_{SG} = N^{-1} \sum_{ij} \left[(\langle S_i S_j \rangle_T - \langle S_i \rangle_T \langle S_j \rangle_T)^2 \right]_{av} \quad , \tag{2.4.10}$$

which therefore, becomes negative, making the SK solution unphysical for all $T < T_g$. AT also showed, in the presence of an external magnetic field, the existence of a line in the h–T plane (see Figure 2.8) below which the replica–symmetric solution is unstable. On this line χ_{SG} becomes infinite signaling a phase transition from paramagnet to spin glass, contrary to what happens for ferromagnets where the magnetic field destroys the transition.

Physically, this replica–symmetry breaking leads to strong irreversibility effects, which are directly related to the fact that the TAP equations present a large number of solutions below the AT line. In fact, the number of solutions grows as (Bray and Moore 1980, De Dominicis *et al.* 1980, Tanaka and Edwards 1980),

$$[N_S(h,T)]_{av} \propto \exp\{N\alpha(h,T)\} \quad , \tag{2.4.11}$$

where $\alpha(h,T)$ is non–zero below the AT line. Different solutions of the TAP equations correspond to distinct free–energy minima. This leads to a multi–valley structure for the free–energy surface as illustrated schematically in Figure 2.9 . The minima represent metastable states and the barrier heights between valleys must diverge in the thermodynamic limit, otherwise thermal fluctuations would induce mixing and consequently, destroy the metastable states (Bray and Moore 1981a). This picture is consistent with the infinite relaxation times found through dynamical approaches (Sompolinsky 1981, Sompolinsky and Zippelius 1981). Direct evidence for that was given by Monte Carlo simulations which lead to barrier heights between valleys diverging as $N^{\frac{1}{4}}$ (Mackenzie and Young 1982). The SK model presents then, for $T < T_g$, breakdown of ergodicity corresponding to a non–zero value of the parameter Δ in equation (2.3.17).

Next, the Parisi replica–symmetry–breaking scheme will be introduced and its implications will be discussed.

2.4.2. The Parisi solution

As pointed out by de Almeida and Thouless (1978), the permutation symmetry between distinct replicas should be broken for a stable spin–glass solution. However, it is not obvious how to break such a symmetry in the limit $n \to 0$. Several attempts were initially proposed (Blandin 1978, Bray and Moore 1978, Blandin *et al.* 1980) but all of them failed to resolve the instability. The most successful replica–symmetry–breaking scheme is due to Parisi (1979, 1980a, 1980b, 1980c, 1980d, 1980e). His theory is stable (in fact, it is marginally stable as will be discussed by the end of this section), agrees well with numerical results and is believed to be the exact solution of the SK model.

The Parisi scheme is shown schematically in Figure 2.10 ; it is essentially a hierarchical construction of the $n \times n$ order–parameter matrix $q^{\alpha\beta}$, generated from the replica–symmetric solution. One starts at the zeroth level with the $n \times n$ matrix (we will use the notation $n \equiv m_0$) parametrized such that all elements are equal to $q_0(m_0)$ ($q^{\alpha\alpha}$ is a constant which will be set to zero), or in other words, the SK solution. The first step consists in dividing this matrix into blocks $m_1 \times m_1$, changing the value of the elements in the diagonal blocks from $q_0(m_0)$ to $q_1(m_1)$. Now, one takes each of the diagonal blocks and divide them into smaller $m_2 \times m_2$ blocks, changing once again the elements of the diagonal sub–blocks from $q_1(m_1)$ to $q_2(m_2)$. This procedure is repeated k times on each successive set of diagonals and one gets

$$n = m_0 \geq m_1 \geq m_2 \geq \ldots \geq m_k \geq 1 \quad , \tag{2.4.12}$$

with the corresponding order parameters $q_0(m_0)$, $q_1(m_1),\ldots,q_k(m_k)$. The process is followed indefinitely, that is, $k \to \infty$, and then the analytic continuation $n \to 0$ is performed; equation (2.4.12) becomes

$$0 = m_0 \leq m_1 \leq m_2 \leq \ldots \leq m_k \leq 1 \quad , \tag{2.4.13}$$

and m_i is replaced by a continuous variable x, $0 \leq x \leq 1$. The Parisi order–parameter function, $q(x)$, is obtained in the continuum limit,

$$\lim_{k\to\infty} q_i(m_i) = q(x) \quad ; \quad 0 \leq x \leq 1 \quad , \tag{2.4.14}$$

that is, we have now an infinite number of order parameters.

The shape of the function $q(x)$ can be obtained easily near T_g, for $h = 0$, by expanding the free–energy functional, equations (2.4.4), in powers of $q^{\alpha\beta}$,

$$g(q^{\alpha\beta}) = -n\left[\ln 2 + \frac{(\beta J)^2}{4}\right] - \frac{(\beta J)^2}{4}\left[(\beta J)^2 - 1\right] \sum_{\alpha\beta} (q^{\alpha\beta})^2 - \frac{(\beta J)^6}{6} \sum_{\alpha\beta\gamma} q^{\alpha\beta} q^{\beta\gamma} q^{\gamma\alpha}$$

$$- \frac{(\beta J)^8}{12} \sum_{\alpha\beta} (q^{\alpha\beta})^4 + \ldots \quad , \tag{2.4.15}$$

where the Parisi approximation was taken (Parisi 1980a), in retaining the only quartic term which is responsible for breaking the replica symmetry (Bray and Moore 1978, Pytte and Rudnick 1979). Using Parisi's parametrization, one gets (Parisi 1980a),

$$-\lim_{n\to 0} \frac{1}{n} \sum_{\alpha\beta} (q^{\alpha\beta})^m = \int_0^1 dx\, q^m(x) \quad . \tag{2.4.16}$$

More complicated terms like the third-order contribution in (2.4.15) can also be evaluated. However, the algebra simplifies a lot if one makes use of the rules developed by Parisi (1980b), i.e. associating with a given matrix $\underline{A}$, in the limit $n \to 0$, a pair $[\tilde{a}, a(x)]$. The quantity $\tilde{a}$ is a number defined as

$$\tilde{a} = A^{\alpha\alpha} \quad , \tag{2.4.17}$$

which is set to zero for spin glasses, but will be allowed to be non-zero as a more general situation. The function $a(x)$ is defined on the interval $[0,1]$. In fact, with the matrices defined in this way, the addition and multiplication take a rather simple form. Let us consider as an example, three matrices $\underline{A}$, $\underline{B}$, $\underline{C}$ and associate with each of them a pair,

$$\underline{A} \to [\tilde{a}, a(x)] \quad ; \quad \underline{B} \to [\tilde{b}, b(x)] \quad ; \quad \underline{C} \to [\tilde{c}, c(x)] \quad . \tag{2.4.18}$$

For the addition one has trivially,

$$\underline{C} = \underline{A} + \underline{B} \quad , \tag{2.4.19a}$$

$$\tilde{c} = \tilde{a} + \tilde{b} \quad , \tag{2.4.19b}$$

$$c(x) = a(x) + b(x) \quad , \tag{2.4.19c}$$

whereas the multiplication requires some algebra to show that (Parisi 1980b),

$$\underline{C} = \underline{A}\,\underline{B} \quad , \tag{2.4.20a}$$

$$\tilde{c} = \tilde{a}\,\tilde{b} - \langle ab \rangle \quad , \tag{2.4.20b}$$

$$c(x) = (\tilde{b} - \langle b \rangle)\, a(x) + (\tilde{a} - \langle a \rangle)\, b(x) + \int_0^x dy\, [a(x) - a(y)]$$

$$\times\, [b(x) - b(y)] \quad , \tag{2.4.20c}$$

where

$$\langle e \rangle = \int_0^1 dx\, e(x) \quad . \tag{2.4.21}$$

One also finds,

$$\lim_{n\to 0} \frac{1}{n} \mathrm{tr}(\underline{A}) = \tilde{a} \quad , \tag{2.4.22a}$$

$$\lim_{n\to 0} \frac{1}{n} \sum_{\alpha\beta} (A^{\alpha\beta})^m = (\tilde{a})^m - \langle a^m \rangle \quad . \tag{2.4.22b}$$

Equation (2.4.22b) is the generalization of (2.4.16) for the case of non–zero diagonal matrix elements.

Using the above rules in the functional (2.4.15), the free energy in (2.4.3) will be given by

$$\beta f[q] = -\beta\left[\ln 2 + \frac{(\beta J)^2}{4}\right] + \frac{(\beta J)^2}{4}[(\beta J)^2 - 1]\langle q^2 \rangle - \frac{(\beta J)^6}{6}\int_0^1 dx \left\{ xq^3(x) \right.$$

$$\left. + 3q(x)\int_0^x dy\, q^2(y) \right\} + \frac{(\beta J)^8}{12}\langle q^4 \rangle + \ldots \quad , \tag{2.4.23}$$

where

$$\langle q^m \rangle = \int_0^1 dx\, q^m(x) \quad . \tag{2.4.24}$$

By extremizing equation (2.4.23) with respect to q(x) one gets the shape of the Parisi function in zero magnetic field, as shown in Figure 2.11(a). It is a monotonically increasing function for $0 \lesssim x \lesssim \bar{x}$, followed by a plateau for $\bar{x} \lesssim x \lesssim 1$. Both the breaking point $\bar{x}$ and the plateau height q_m are of order ϵ ($\epsilon = (T_g - T)/T_g$).

It turns out that Parisi's solution maximizes the free energy, contrary to what is required in (2.4.3); indeed, it gives a higher free energy than the SK solution shown in Figure 2.7. This comes from the fact that the quadratic contribution for the free energy, $\sum_{\alpha\beta} (q^{\alpha\beta})^2$, becomes negative in the limit $n \to 0$, as can be seen in equation (2.4.16). Such a change of sign is responsible for changing the minimum condition in equation (2.4.3) into a maximum, similarly to what happened for the SK solution discussed in the last section.

The stability analysis as described in the previous section, for the Parisi solution is, due to the complicated structure of the matrix $q^{\alpha\beta}$, a highly non–trivial task. Eigenvalues in a restricted subspace were found by Thouless *et al.* (1980), and a complete diagonalization of the stability matrix was done by De Dominicis and Kondor (1983). No negative eigenvalues were found, that is, the Almeida–Thouless instability disappeared, but some isolated zero eigenvalues are present through the whole spin–glass phase. This leads to a marginal stability and in particular, to an infinite spin–glass susceptibility for all temperatures below T_g.

Although Parisi's ansatz is only marginally stable, it provides results which are in very good agreement with numerical simulations, and is believed to be the exact solution of the SK model. Its implications and physical interpretation will be presented next.

2.4.3. Implications and physical interpretation of Parisi's solution

The existence of many solutions of the TAP equations leads to the valley structure for the free energy as discussed before. Since the barrier heights between valleys diverge in the limit $N \to \infty$, each minimum corresponds to a different thermodynamic state in which an infinite system will remain forever if initially prepared in that state. Such states are usually called "ergodic components" or "pure states".

A given pure state s, of the spin–glass phase, is characterized by the local magnetization

$$m_i^s = \langle S_i \rangle_s \quad , \tag{2.4.25}$$

at every site, where $\langle\ \rangle_s$ denotes an average over the microstates of s. One can define the overlap between two solutions s and s′ as

$$q^{ss'} = N^{-1} \sum_i \langle S_i \rangle_s \langle S_i \rangle_{s'} \quad , \tag{2.4.26}$$

which is a measure of the "distance" in phase space between states, approaching 1 (0) if s and s′ are "near" ("far") to one another. The probability that there are states with overlap equal to q is given by (Houghton *et al.* 1983, Parisi 1983),

$$P(q) = \Big[\sum_{s,s'} P_s P_{s'} \, \delta(q - q^{ss'}) \Big]_{av} \quad . \tag{2.4.27}$$

In the equation above P_s is the probability for the pure state s, with Boltzmann weight (De Dominicis and Young 1983),

$$P_s = Z^{-1} \exp(-\beta F_s) \quad , \tag{2.4.28a}$$

where F_s is the free energy of state s and Z , the partition function,

$$Z = \sum_s \exp(-\beta F_s) \quad . \tag{2.4.28b}$$

Since the Parisi function is monotonic, one can define its inverse, $x(q)$, which is related to $P(q)$ by (Houghton *et al.* 1983, Parisi 1983),

$$\frac{dx}{dq} = P(q) \quad . \tag{2.4.29}$$

The meaning of the above relation is that the Parisi function $q(x)$ contains information about the distribution of phase–space distances. The probability distribution $P(q)$ for the SK model in zero magnetic field, is shown in Figure 2.11(b). The delta function at q_m corresponds to the plateau of the Parisi function, whereas for the region $q(x)$ varying with x , one gets a structure for $P(q)$ which indicates the existence of multiple states with non–zero overlap between them. The replica–symmetric solution yields trivially, $P(q)$ as a single delta function.

Another important point about (2.4.29) is that it allows one to compute $P(q)$, and consequently $q(x)$, by numerical simulations. This was done (Young 1983) and within the limitations due to the finite size of the samples, the structure predicted from the replica theory was confirmed.

In the pure–state language, any statistical mechanics average should include contributions from each state s with a weight P_s. In particular,

$$\langle S_i \rangle_T = \sum_s P_s \langle S_i \rangle_s \quad . \tag{2.4.30}$$

Hence, the Edwards–Anderson order parameter, equation (2.3.14), which describes ordering in a single state, is given by

$$q_{EA} = N^{-1} \sum_i \left[\sum_s P_s \langle S_i \rangle_s^2 \right]_{av} = \left[\sum_s P_s \, q^{ss} \right]_{av} \quad , \tag{2.4.31}$$

whereas the statistical mechanics order parameter, equation (2.3.15), will be

$$q = N^{-1} \sum_i \left[\left[\sum_s P_s \langle S_i \rangle_s \right]^2 \right]_{av} = \left[\sum_{s,s'} P_s P_{s'} \, q^{ss'} \right]_{av} \quad , \tag{2.4.32}$$

involving interference between different solutions.

Both q_{EA} and q can be obtained from the Parisi function $q(x)$. The parameter q from (2.4.32), is the first moment of the distribution $P(q)$; using (2.4.29), one gets,

$$q = \int dq' \, q' \, P(q') = \int_0^1 dx \, q(x) \quad . \tag{2.4.33}$$

One can also demonstrate that (De Dominicis and Young 1983),

$$q_{EA} = q(x = 1) \quad , \tag{2.4.34}$$

showing that $q_{EA} > q$ for $q(x)$ non–constant, i.e. if many states exist below T_g.

Another interesting consequence of Parisi's solution is related to the concept of ultrametricity (for a review see Rammal *et al.* 1986). If one takes three distinct pure states labelled respectively, 1, 2 and 3 , in principle, no restrictions on the distances between them in phase space, q^{12} , q^{23} and q^{31} should be expected. However, as shown by Mézard *et al.* (1984a, 1984b), they cannot be all different; either they are all equal or the two smaller ones are the same. This result becomes apparent if one considers the Parisi ansatz as a hierarchical structure (see Figure 2.12). Taking the end-points as the states, the overlap between two of them is found by tracing back along the branches of the tree until their nearest common ancestor is reached. The overlap becomes smaller the further back one needs to go. One can verify easily the ultrametric structure for any three arbitrarily chosen states.

In summary, the Sherrington–Kirkpatrick model of spin glass is now well understood. The spin–glass ordering is characterized by a function, that is, an infinite number of order parameters, which are related to the existence of many minima (free–energy valleys) below T_g . The system presents non–ergodic behaviour by remaining trapped in a given state for which it was initially prepared. The distances in phase space between different minima are restricted to an ultrametric space.

The occurrence of such features in more general infinite–range spin–glass models, or in short–range spin glasses, is something not completely understood, as will be discussed briefly in the next sections.

2.5. Generalizations of the Sherrington–Kirkpatrick model

Ising spins ($S_i = \pm 1$) are nice from the theoretical point of view, but they represent quite a crude approximation of nature. One is usually tempted to generalize Ising models in order to give more freedom to the spin variable, by allowing it, let us say, to be a continuous variable in a plane (XY variable) or in a sphere (Heisenberg variable), or even to be discrete but assuming more than two possible values.

As a generalization of the Sherrington–Kirkpatrick model, let us now take the hamiltonian,

$$H = - \sum_{(ij)} J_{ij} \, \underline{S}_i \cdot \underline{S}_j - \underline{h} \cdot \sum_i \underline{S}_i \quad , \tag{2.5.1}$$

where $\underline{S}_i$ are classical vectors with more than one cartesian component.

If the vectors $\underline{S}_i$ are continuous in m dimensions, i.e. are allowed to point in any direction in an m–dimensional space, subject to the normalization condition,

$$\underline{S}_i^2 = m \quad , \tag{2.5.2}$$

one gets the so called m–vector spin glass. As particular cases of the m–vectors, one has the following models:

$m = 1$: Ising ,

$m = 2$: XY ,

$m = 3$: Heisenberg ,

$m = \infty$: Spherical .

By taking $\underline{S}_i$ as unit vectors pointing in each of the p–symmetric directions of a hypertetrahedron in (p–1) dimensions (from the centroid to each of its p vertices), the hamiltonian in (2.5.1) describes the Potts model (for a review see Wu 1982). The Potts hamiltonian is most commonly known in the form

$$H = -\sum_{(ij)} \tilde{J}_{ij}\, \delta_{k_i,k_j} - \tilde{h} \sum_i \delta_{k_i,0} \quad , \tag{2.5.3}$$

where δ denotes a Kronecker delta and $k_i = 0,1,\ldots,p-1$ .Trivially, the SK model is recovered for $p = 2$, by means of the identifications,

$$S_i = \cos\pi k_i \quad ; \quad \delta_{k_i,k_j} = \frac{1}{2}(1 + S_i S_j) \quad ; \quad k_i = 0,1 \; . \tag{2.5.4}$$

Another variation on (2.5.1) is the case where the $\underline{S}_i$ are unit vectors restricted to p equally angularly spaced orientations in a plane. This is called the planar Potts or the clock model, which will be extensively studied throughout this thesis.

For systems described by (2.5.1), the order–parameters space becomes more complicated as one has now tensors in spin space,

$$R^{\alpha}_{\mu\nu} = \langle S^{\alpha}_{\mu}\, S^{\alpha}_{\nu} \rangle \quad , \tag{2.5.5a}$$

$$Q^{\alpha\beta}_{\mu\nu} = \langle S^{\alpha}_{\mu}\, S^{\beta}_{\nu} \rangle \quad ; \quad \alpha \neq \beta \quad , \tag{2.5.5b}$$

where μ,ν refer to spin–cartesian components. The probability distribution for the overlaps between states can be generalized as,

$$P\{Q_{\mu\nu}\} = \left[\sum_{S,S'} P_S P_{S'} \prod_{\mu\nu} \delta(Q_{\mu\nu} - Q_{\mu\nu}^{SS'}) \right]_{av} \quad . \tag{2.5.6}$$

where

$$Q_{\mu\nu}^{SS'} = N^{-1} \sum_i \langle S_{i\mu} \rangle_S \langle S_{i\nu} \rangle_{S'} \quad . \tag{2.5.7}$$

The statistical mechanics order parameter $Q_{\mu\nu}$ is the first moment of the distribution (2.5.6), or in other words,

$$Q_{\mu\nu} = \left[\sum_{S,S'} P_S P_{S'} Q_{\mu\nu}^{SS'} \right]_{av} \quad . \tag{2.5.8}$$

In the next chapter we will discuss how to restrict the order–parameters space by imposing conditions on tensors (2.5.5), reducing the number of parameters.

For the m–vector case, in zero magnetic field, the Almeida–Thouless instability is observed for any finite m, but a curious behaviour is found in the limit $m \to \infty$, where the replica–symmetric solution is stable (de Almeida *et al.* 1978). In this limit then, the spin–glass phase is characterized by a single thermodynamic state, related to the fact that the analogues of the TAP equations present a number of solutions (see equation (2.4.11)) with $\alpha \to 0$ as $m \to \infty$ (Bray and Moore 1981b). The Parisi function in zero magnetic field, shows a similar behaviour to the Ising case (Figure 2.11(a)) for finite m, but $\bar{x} \to 0$ as $m \to \infty$ (Elderfield and Sherrington 1982, Gabay *et al.* 1982).

For $h \neq 0$, the system is biased, and both the magnetization and the spin–glass parameters parallel to the field are always non–zero. There is, however, a transverse freezing signaled by the occurrence of perpendicular spin–glass parameters, which takes place in the h–T plane at the Gabay–Toulouse (GT) line (Gabay and Toulouse 1981). Below this line replica symmetry is unstable (Cragg *et al.* 1982, Moore and Bray 1982), and the Parisi ansatz has to be used (Elderfield and Sherrington 1982, Gabay *et al.* 1982).

The Potts spin glass on the other side, presents a quite curious behaviour (Erzan and Lage 1983, Elderfield and Sherrington 1983a, 1983b, 1983c, Goldbart and Elderfield 1985, Gross *et al.* 1985). The Almeida–Thouless instability takes place at a lower order in perturbation theory when compared to the m–vectors, showing that replica symmetry is an even worse approximation in this case (Elderfield and Sherrington 1983a). By applying the Parisi ansatz in the conventional way, as described in section 2.4.2, one is leaded to unphysical results, like a breaking point $\bar{x} > 1$, or a decreasing $q(x)$ (implying a negative $P(q)$) (Goldbart and Elderfield 1985). Such effects are a direct consequence of the absence of reflection symmetry in the spin variable, that is, the hamiltonian is not invariant under the inversion $\underline{S}_i \to -\underline{S}_i$.

The correct solution for the Potts spin glass was given by Gross *et al.* (1985). They found two transitions as the temperature is lowered. Just below the upper critical temperature, T_{g1}, the Parisi function is two–valued, that is, a step function, whereas below T_{g2} it presents a structure with non–zero overlaps between states. The transitions are continuous for $p = 3,4$, but first order for $p > 4$, with the plateau value jumping discontinuously to zero at T_{g1}.

Generalizations of the Sherrington–Kirkpatrick model, as the ones mentioned above, bring new features and challenges to the spin–glass theory. The planar Potts (or clock) model will be introduced in the next chapter.

2.6. Relations to optimization problems and biological systems

Many problems in science and engineering today are formulated in such a way as to minimize a given function with respect to controllable parameters, subject to fixed constraints. In simple cases, the function in question may have a single or a few local minima, and the optimal solution is easy to select. Complications arise when one is faced with a large number of unrelated local minima, showing a similar structure as the one of the phase space of a spin glass (Figure 2.9). These are usually considered as "hard" or "complex" optimization problems, for in most of them, the finding of the optimal solution takes a computing time which grows exponentially, or in other words, non–polinomially (NP–complete), with the size of the system. The search for the ground state of a spin glass is, for most spin–glass models, an NP–complete problem.

A classic example of such a problem is the travelling salesman. Given N cities randomly distributed in a plane, the salesman is to visit each city once and return to the starting point, in such a way as to minimize the total length of the tour (Lawler *et al.* 1985). Since both the starting point and the direction along each circuit are irrelevant, the number of different routes is given by,

$$N_r = \frac{1}{2}(N-1)! \quad , \tag{2.6.1}$$

and for large N , one can use the Stirling's formula,

$$N! \approx (2\pi N)^{\frac{1}{2}}\, N^N \exp(-N) \quad , \tag{2.6.2}$$

to write,

$$N_r \approx \exp(N \ln N) \quad ; \quad N \text{ large} \quad . \tag{2.6.3}$$

Clearly, as the number of cities becomes large it is impossible to test all possible routes to look for the optimal one.

The travelling salesman problem can be shown to correspond to the limit $m \to 0$ of an m–vector spin glass and can therefore, be studied through spin–glass techniques (Kirkpatrick and Toulouse 1985, Mézard and Parisi 1986).

Another example is the graph–partitioning problem, particularly relevant to microchip design. Consider a set of N vertices, labelled $i = 1,\ldots,N$, connected by edges according to a connectivity matrix $\{a_{ij}\}$, where $a_{ij} = 1$ if there exists an edge, zero otherwise. The problem is to partition the vertices in p subsets such as to minimize the number of edges between the subsets, while avoiding the congestion of any particular subset. Note the correspondence to the microchip design problem throughout the identifications:

vertices $\longleftrightarrow$ circuits

edges $\longleftrightarrow$ connections

subsets $\longleftrightarrow$ boards .

For $p = 2$, the bi–partitioning problem, the objective is to minimize the number of connections between boards,

$$N_c = \sum_{(ij)} a_{ij} \frac{1}{2} (1 - S_i S_j) \quad , \tag{2.6.4}$$

where S_i takes the values ± 1 according to whether a given circuit i is in one board or the other. One usually studies bi–equipartitioning, which corresponds to the minimization of N_c subject to the constraint,

$$m = \sum_i S_i = 0 \quad , \tag{2.6.5}$$

which ensures an equal number of circuits in each board. Since the connectivity matrix elements in (2.6.4) are non–negative, this problem is equivalent to finding the ground–state energy of a dilute ferromagnet subject to a constraint of zero magnetization. However, if one relaxes the constraint (2.6.5) by adding to (2.6.4) the term

$$\delta H = \frac{\lambda}{2}\left(\sum_i S_i \right)^2 = \lambda \sum_{(ij)} S_i S_j + \frac{1}{2}\lambda N \quad , \tag{2.6.6}$$

which restricts fluctuations around $m = 0$ to order $N^{-\frac{1}{2}}$, one gets the hamiltonian,

$$H = - \sum_{(ij)} \left(\frac{1}{2} a_{ij} - \lambda \right)(S_i S_j - 1) + \text{const} \quad . \tag{2.6.7}$$

The "interaction" $\left(\frac{1}{2} a_{ij} - \lambda \right)$ can now assume both positive and negative values at random, and the hamiltonian (2.6.7) describes essentially and Ising spin glass.

The generalization to the p–partitioning problem can easily be done by means of a cost function,

$$H = - \sum_{(ij)} (a_{ij} - \lambda)(\delta_{k_i,k_j} - 1) \quad , \tag{2.6.8}$$

analogous to the hamiltonian of the Potts spin glass, where $k_i = 1,\ldots,p$, labels the location of circuit i in one of the p different boards. The number of possible solutions in this case is,

$$N_s = p^N = \exp(N\ln p) \quad , \qquad (2.6.9)$$

showing the NP–completeness character of the problem.

One conventional strategy for dealing with such problems is iterative improvement. One starts with a randomly chosen initial configuration, performs some rearrangement, and the new configuration is accepted if it corresponds to a lower value for the cost function, rejected otherwise. The procedure is iterated until no further lowering is attained. In problems characterized by many local minima separated by high cost barriers, the system may head towards the nearest local minimum which is usually far from the global optimum, becoming trapped there. In order to avoid that, the optimization scheme must incorporate a mechanism for escaping from such traps.

Motivated by the analogy between spin glass and optimization problems, Kirkpatrick *et al.* (1983) proposed a scheme known as optimization by simulated annealing (OSA), as a method for finding low–lying local minima. The basic idea is to introduce a control parameter in the optimization problem, playing a role analogous to that of the temperature in statistical mechanics. The search for minima is done by using Monte Carlo methods (for reviews see Binder 1979, 1985). One starts at a fairly high "temperature" and the system is "cooled" slowly. Changes which lower the cost are always accepted, but uphill moves, with a controlled probability, can also be accepted and in this way, one avoids getting stuck in a local minimum.

It should be noted that like optimization by simulated annealing, many other techniques developed for spin glasses have been used to study optimization problems (for reviews see van Hemmen and Morgenstern 1986, Mézard *et al.* 1987).

Another recent application of spin–glass techniques goes towards brain research. The brain contains many neurons (of order 10^{10}) which are highly connected through the synapses (of order 10^{15}), representing a suitable system for the application of statistical mechanics. The link is provided by the Hopfield–Little model (Hopfield 1982, 1984, Hopfield *et al.* 1983, Little 1974, Little and Shaw 1978). This model is based on an earlier simple idealization of the neuron, in which it is allowed to be at any time in one of the two states, firing or not firing (McCulloch and Pitts 1943).

In the Hopfield–Little model one represents this by associating to a neuron a two–valued variable $S_i = \pm 1$ according to whether it is firing or not. The synaptic junctions present efficacies, J_{ij}, which determine the contribution of a signal fired by the j–th neuron to the potential at the i–th neuron, and can be either positive (excitatory synapse) or negative (inhibitory synapse). Assuming only pairwise interactions, one can write the total input potential at neuron i as the sum,

$$V_i = \sum_j J_{ij} \frac{1}{2}(S_j + 1) \quad , \tag{2.6.10}$$

where $(S_j + 1)/2$ contributes with unit if neuron j is firing, zero if not. This input potential will affect the neuron at i, which will fire a signal if its potential V_i exceeds a given threshold value U_i. The stable states of the network will be those in which the variables S_i align with their molecular fields,

$$h_i S_i > 0 \quad ; \quad h_i = V_i - U_i \quad , \tag{2.6.11a}$$

or in other words,

$$S_i = \mathrm{sign}(h_i) = \mathrm{sign}\left\{ \sum_j J_{ij} \frac{1}{2}(S_j + 1) - U_i \right\} \quad . \qquad (2.6.11b)$$

The identification with the spin–glass problem is done by assuming symmetric synapses,

$$J_{ij} = J_{ji} \quad ; \quad J_{ii} = 0 \quad , \qquad (2.6.12)$$

in which case, conditions (2.6.11) are equivalent to the requirement that the hamiltonian,

$$H = \sum_i h_i S_i = \sum_{(ij)} J_{ij} S_i S_j + \sum_i b_i S_i \quad , \qquad (2.6.13a)$$

$$b_i = \frac{1}{2} \sum_j J_{ij} - U_i \quad , \qquad (2.6.13b)$$

takes its local minimum. The many local minima correspond to the retrieval states, and similarly to spin glasses where the bond realizations, i.e. the $\{J_{ij}\}$, determine the minima, the retrieval states are defined by the synapses. Hopfield (1982) adopted the form proposed by Hebb (1949),

$$J_{ij} = N^{-1} \sum_{a=1}^{p} \xi_i^a \, \xi_j^a \quad , \qquad (2.6.14)$$

where ξ_i^a, $a = 1,\ldots,p$, are quenched random variables assuming the values ± 1 with equal probabilities. By using this choice of couplings one gets a set of p patterns $\{\xi_i^a\}$ stored simultaneously.

We can recognize the familiar spin–glass hamiltonian in equations (2.6.13). This model has been studied within the framework of spin–glass techniques (Amit *et al.* 1985a, 1985b, 1987), and extensions of it in order to include dilution, asymmetric synapses, correlations between memories and other more realistic features constitute a very active field in spin–glass research nowadays.

2.7. Short–range spin glasses

The spin–glass theory as discussed in sections 2.4 and 2.5 applies to infinite–range models, in which limit mean–field theory is believed to be exact. Although such a theory is fairly well understood, at least for the Sherrington–Kirkpatrick model, much less is known about the more realistic short–range spin glasses. A lot of effort has been done in the study of finite–range models through qualitative arguments, approximate methods and numerical simulations, but a satisfactory understanding is still lacking. A full description of the work done in this area can be found in several reviews (see for example Binder and Young 1986, van Hemmen and Morgenstern 1983, 1986), and here we shall give a compact sketch in what concerns the existence of a finite–temperature phase transition and the character of the low–temperature phase (whenever there exists one).

For the Ising spin glass on a square lattice it is generally accepted that there is no finite–temperature transition. This has been shown by several distinct methods and for different bond distributions with some of them listed below:

a) Real–space renormalization group (Migdal–Kadanoff scheme (Migdal 1975, Kadanoff 1976)): both $\pm$ J and continuous bond distributions (Southern and Young 1977, Kirkpatrick 1977).

b) Monte Carlo simulations: $\pm J$ (Young 1983a, McMillan 1983).

c) Phenomenological scaling: continuous bonds (Bray and Moore 1984, 1985, 1986, McMillan 1984a, 1984b).

d) High–temperature series expansion: $\pm J$ (Singh and Chakravarty 1986).

Although until recently no consensus existed, nowadays it is believed that the Ising spin glass in a cubic lattice does indeed exhibit a finite–temperature phase transition. Some of the works showing this are:

a) Real–space renormalization group (Migdal–Kadanoff scheme): both $\pm J$ and continuous bonds (Southern and Young 1977, Kirkpatrick 1977).

b) Monte Carlo simulations: $\pm J$ (Bhatt and Young 1985, Ogielski and Morgenstern 1985).

c) Phenomenological scaling: continuous bonds (Bray and Moore 1984, 1985, 1986, McMillan 1984a, 1984b, 1985a).

d) High–temperature series expansion: $\pm J$ (Singh and Chakravarty 1986).

Therefore, the "lower critical dimension" d_l , i.e. the dimension where finite temperature ordering disappears, for the short–range Ising spin glass, satisfies $2 < d_l < 3$.

It is a well–known fact that the lower–critical dimension for the m–vector ferromagnet ($d_l = 2$) is higher than the one for the corresponding Ising model ($d_l = 1$). The same seems to happen in spin glasses. For the m–vector spin glasses (XY and Heisenberg), most of the recent work is consistent with $d_l > 3$ (McMillan 1985b, Morris *et al.* 1986, Olive *et al.* 1986, Jain and Young 1986, Chakrabarti and Dasgupta 1986). However, the inclusion of a uniaxial anisotropy induces a finite–temperature transition (Morris and Bray 1984, Chakrabarti and Dasgupta 1987). This may be the reason why, experimentally one measures spin–glass transitions on Heisenberg systems presenting critical exponents closer to those of Ising models, than to isotropic Heisenberg models.

The "upper critical dimension" d_u , i.e. the dimension where mean–field theory becomes exact, is, for most spin glasses, $d_u = 6$, For dimensions between the upper and lower critical dimensions, $d_l < d < d_u$, mean–field theory for the Ising spin glass as realized in the Sherrington–Kirkpatrick model, is only approximate, and one is tempted to ask which of its intriguing features (many valleys, non–trivial structure of P(q), ultrametricity), survives in such range of dimensions. In particular, one would like to know what happens by the time the physical value $d = 3$ is reached. Recent arguments (Fischer and Huse 1986, Bray and Moore 1986) in favour of a much simpler ordered phase for the three–dimensional Ising spin glass (a single thermodynamic state, a trivial delta function for P(q), no Almeida–Thouless line), constitute an issue of much controversy at the the present moment. While the droplet model of Fischer and Huse has been criticized by Villain (1986), the one–parameter scaling theory of Bray and Moore lacks an understanding of how it breaks down at higher dimensions. Numerical simulations to elucidate this point are waited for. Experimentally, the best test would be to look for an Almeida–Thouless line at low temperatures. The experiments done so far which identify Almeida–Thouless lines (or Gabay–Toulouse lines) in the h–T plane, are not very conclusive in showing the divergence of relaxation times. The clarifying of this matter is crucial for a better understanding of real spin–glass systems.

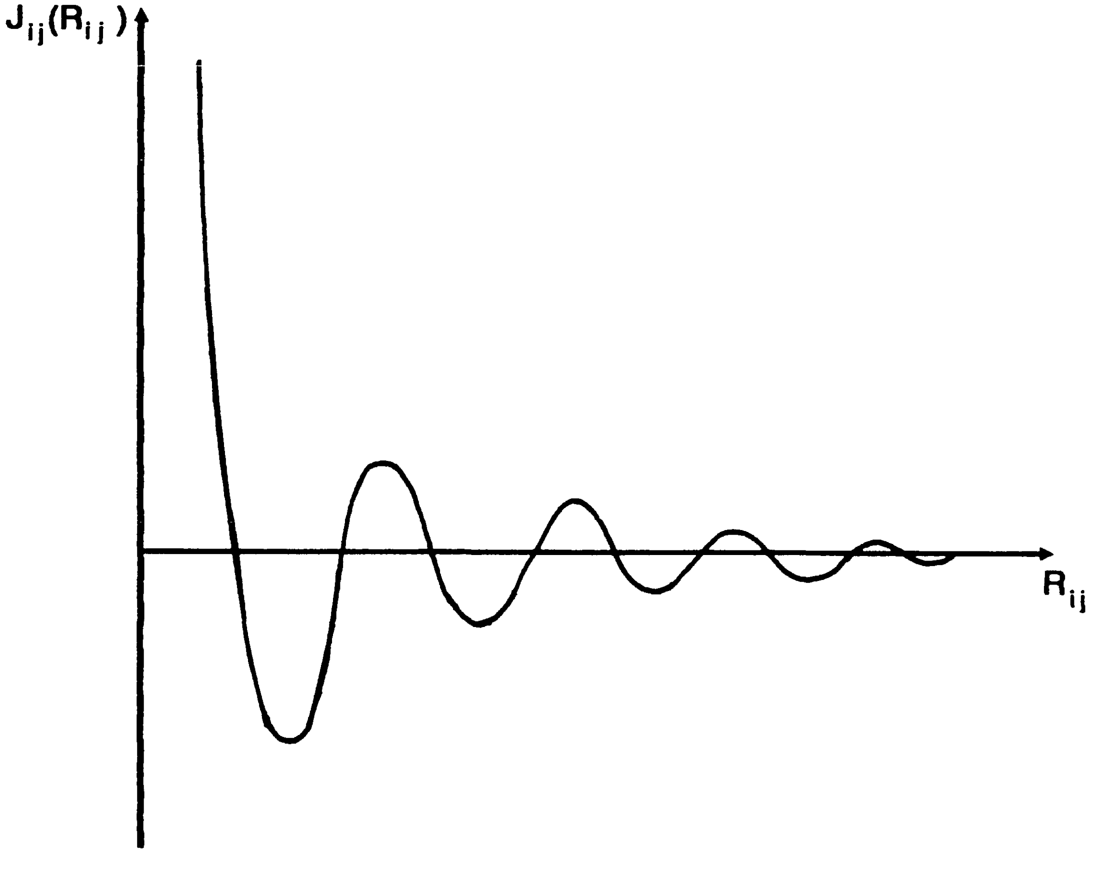

Figure 2.1: The RKKY interaction: depending on the separation R_{ij} between two given impurities, they may experience either a ferromagnetic $(J_{ij} > 0)$ or an antiferromagnetic coupling $(J_{ij} < 0)$.

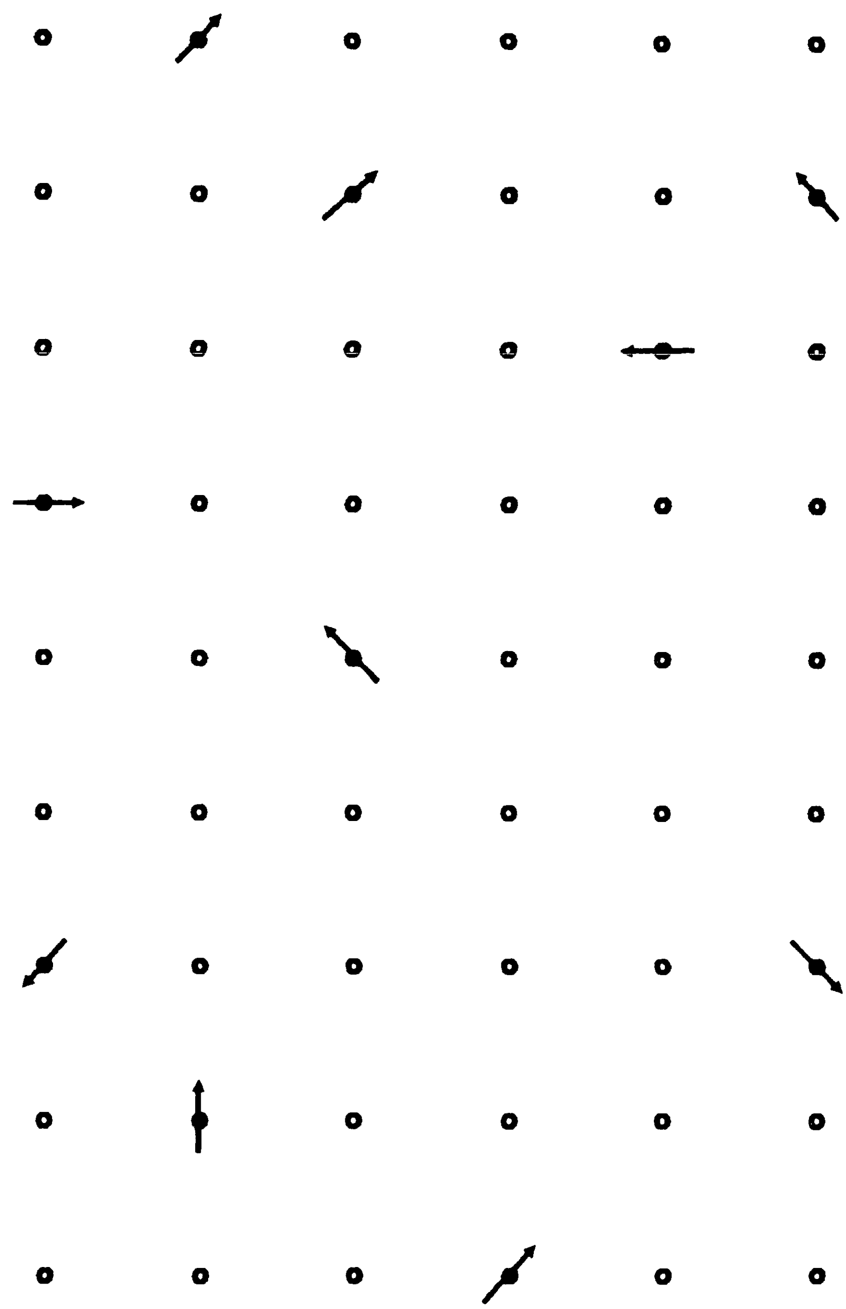

Figure 2.2: Schematic two–dimensional slice showing the magnetic impurities distributed randomly in the host metal. At low temperatures the magnetic moments freeze in random directions.

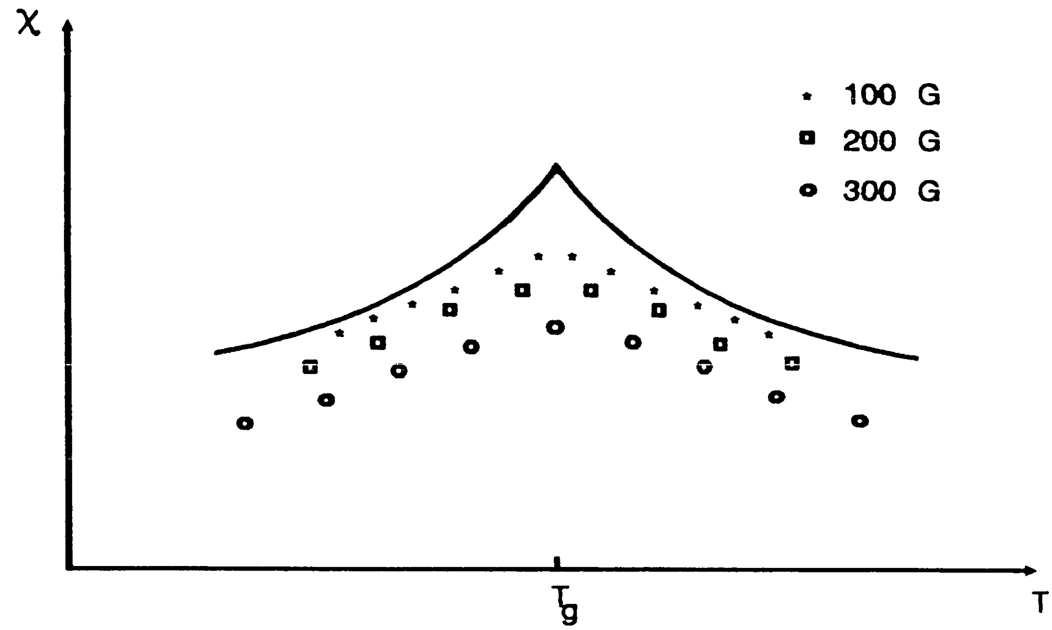

Figure 2.3: Field dependence of the a.c. susceptibility: the curves are flat for non–zero values of the magnetic field but they tend to a cusp as the field is reduced to zero.

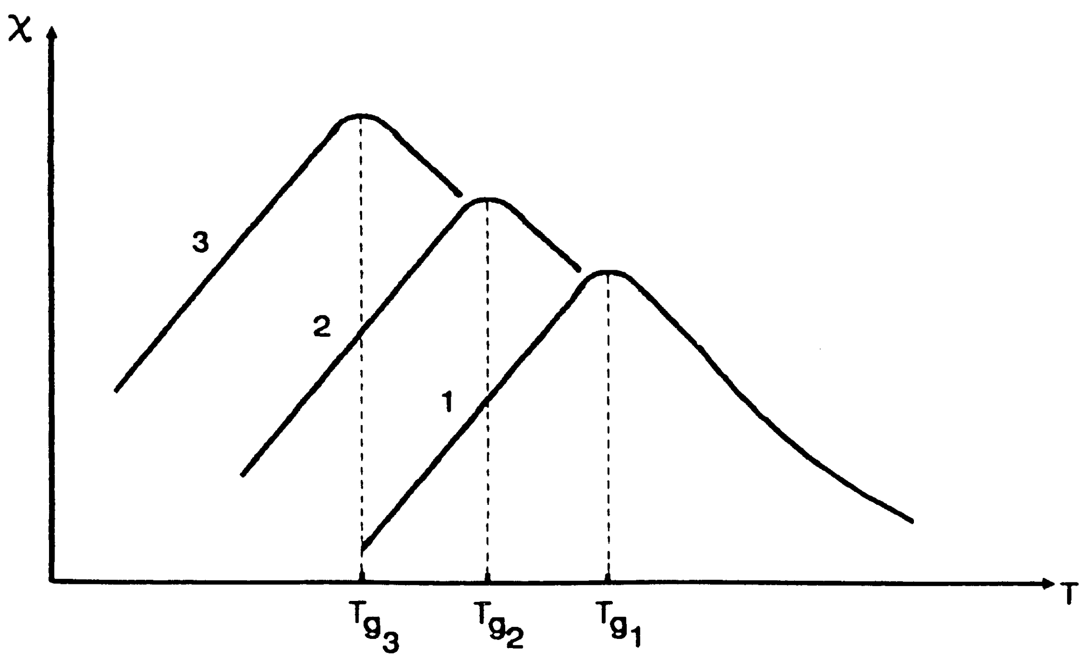

Figure 2.4: Frequency dependence of the a.c. susceptibility: three different curves 1 , 2 and 3 for the a.c. susceptibility are shown schematically at different frequencies ω_1 , ω_2 and ω_3 $(\omega_3 < \omega_2 < \omega_1)$. The freezing temperature decreases with ω .

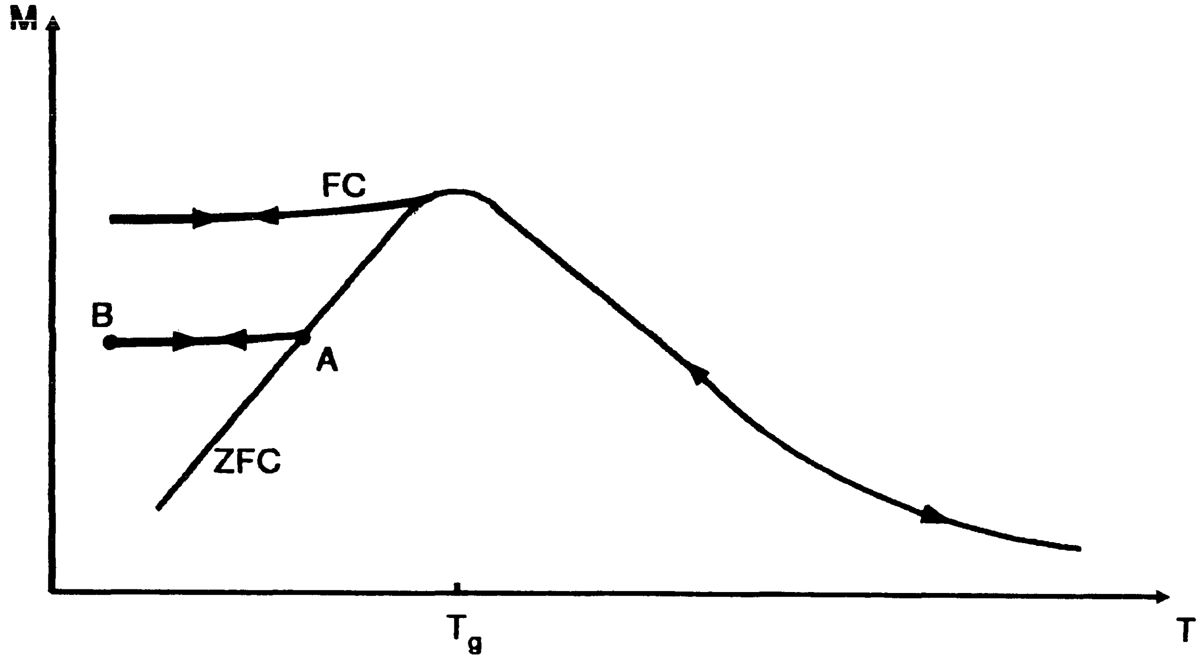

Figure 2.5: The low–temperature magnetization in a field: two distinct situations, the Field Cooled (FC) and the Zero–Field Cooled (ZFC) magnetizations are observed. The FC curve is completely reversible and is believed to represent the equilibrium magnetization. The ZFC curve is irreversible and evolves slowly towards the FC curve.

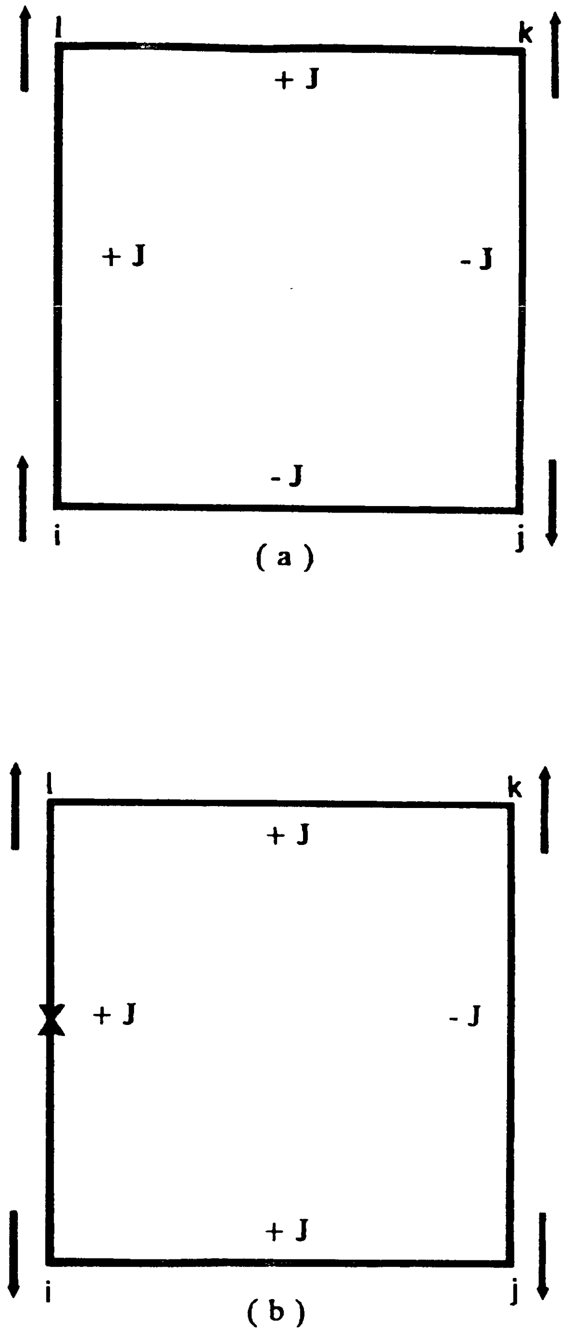

Figure 2.6: An illustration of the frustration effect in a two-dimensional square lattice. Two elementary squares, called "plaquettes" are shown. In (a) all bonds are satisfied at low temperatures, whereas in (b) there is always one unsatisfied bond and the plaquette is said to be frustrated.

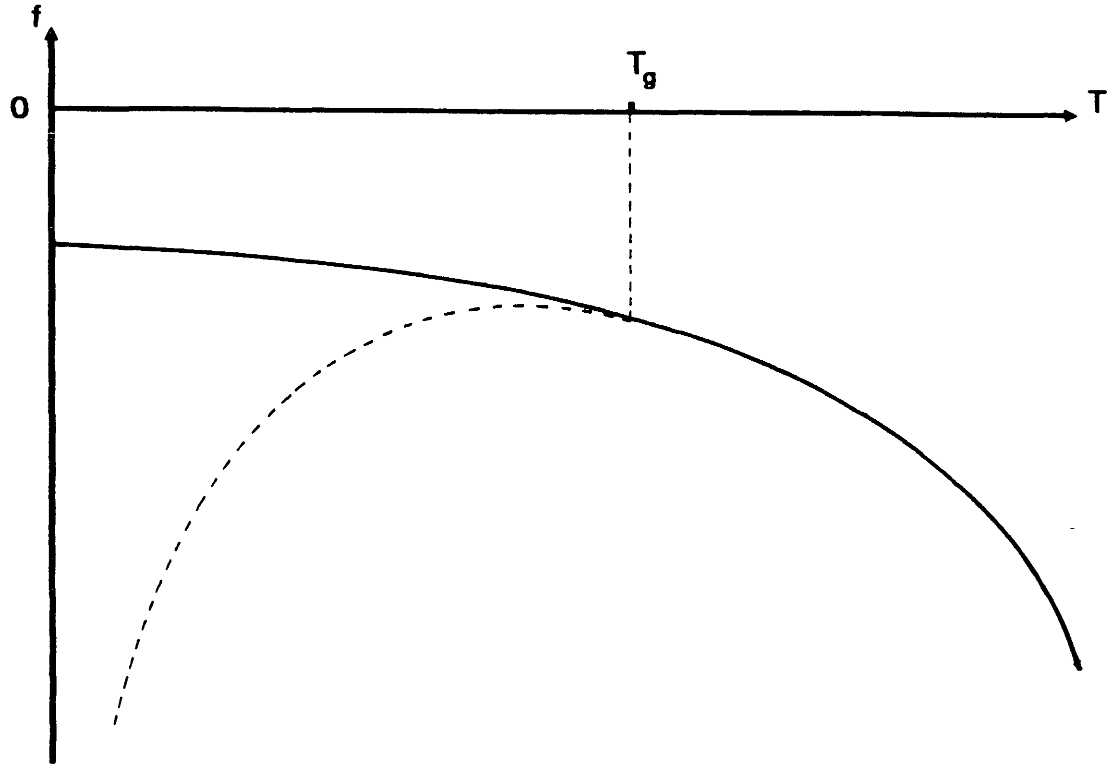

Figure 2.7: The free energy for the Sherrington–Kirkpatrick model in the replica–symmetry ansatz. For $T < T_g$, the spin–glass (full line) presents a higher free energy than the $q = 0$ (dashed line) solution. The Parisi ansatz, which is believed to be the correct one, gives an even higher free energy than replica symmetry.

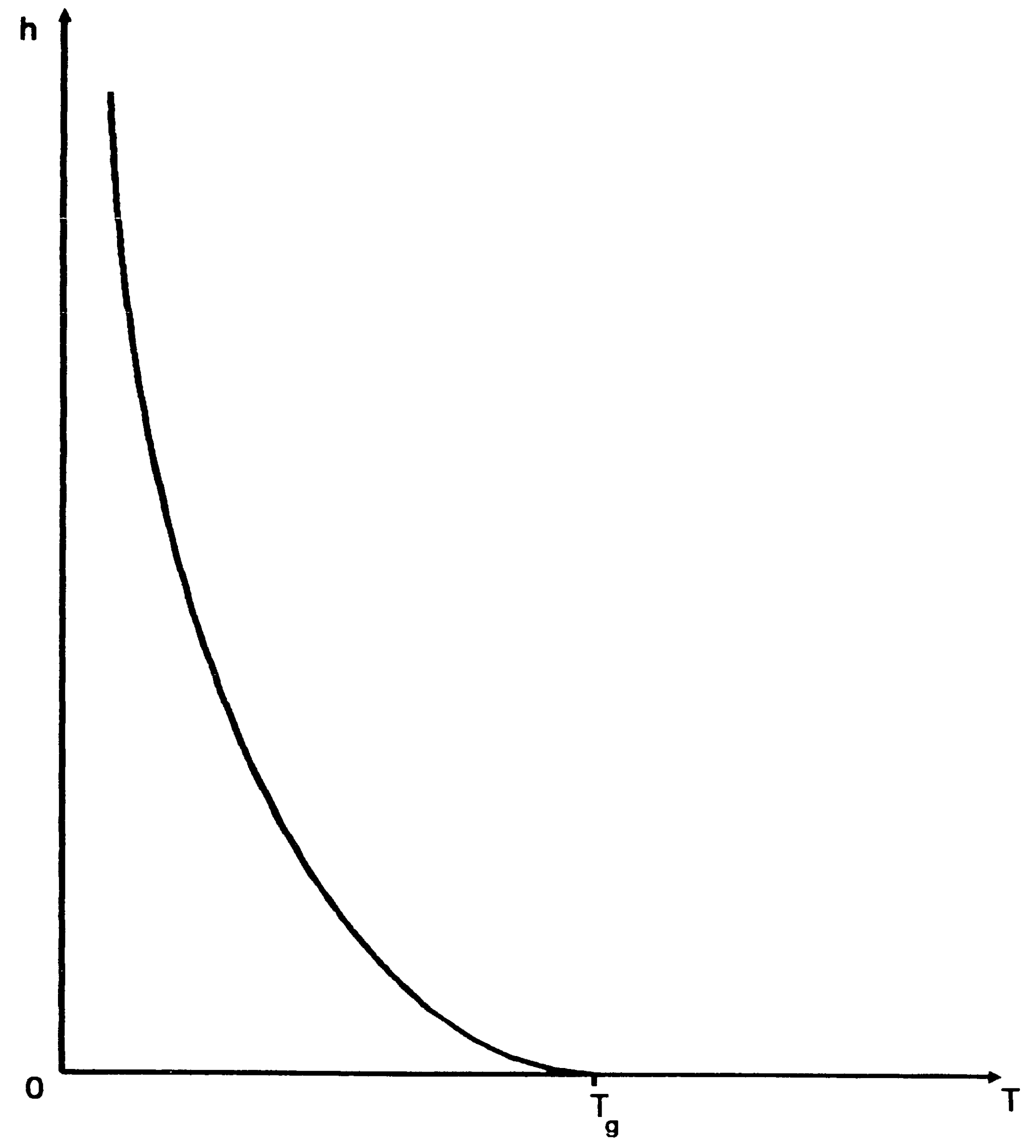

Figure 2.8: The Almeida–Thouless line: a phase transition is predicted in the h–T plane for the Sherrington–Kirkpatrick model. The line signals the instability of the replica–symmetric solution.

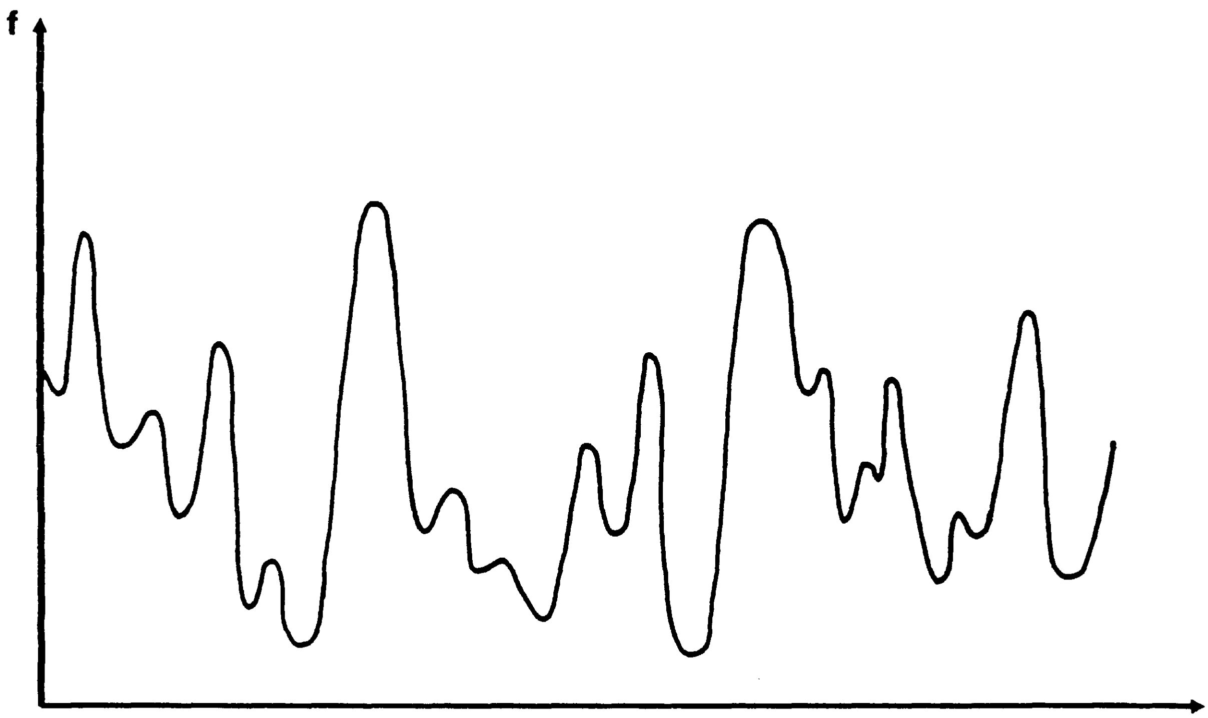

Figure 2.9: The free–energy multivalley structure for the Sherrington–Kirkpatrick model. The figure shows schematically a slice through the free–energy surface. The ordinate is the free energy per spin, and the abcissa should be considered as a one–dimensional qualitative representation of a multidimensional space.

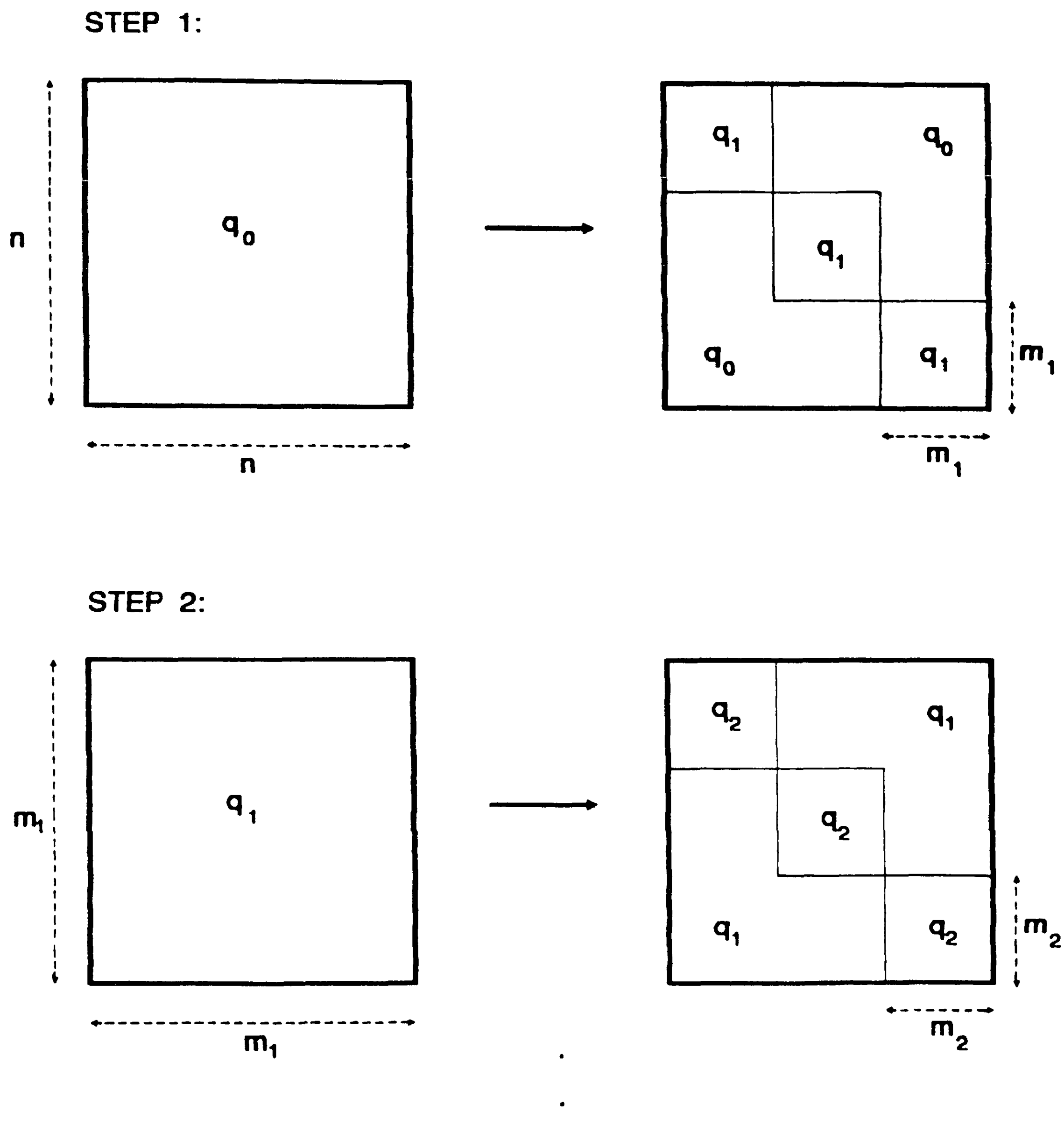

Figure 2.10: The Parisi replica–symmetry–breaking scheme.

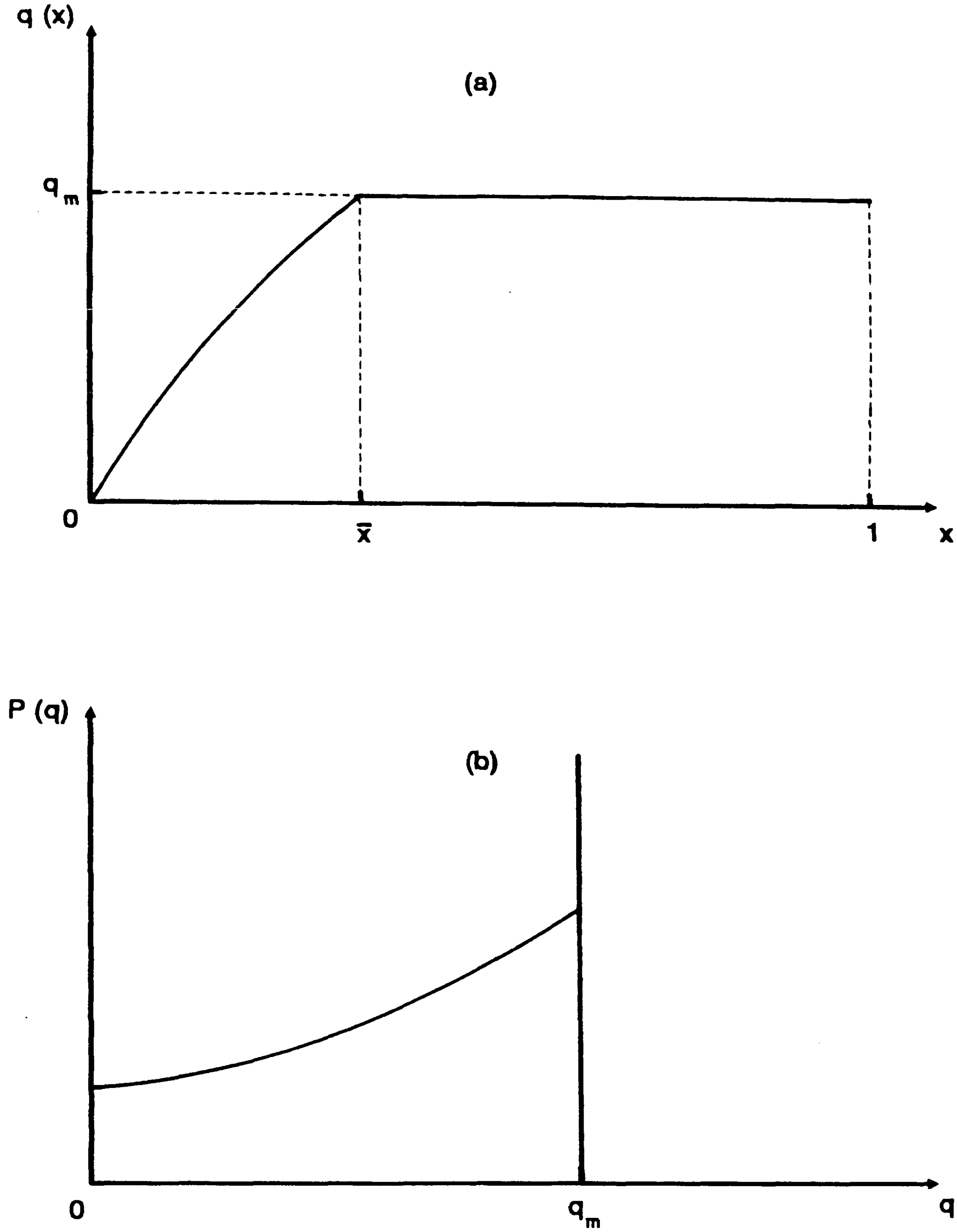

Figure 2.11: (a) The Parisi function for the Sherrington–Kirkpatrick model in zero magnetic field. Both the height of the plateau (q_m) and the breaking point ($\bar{x}$) are of $O(\epsilon)$ ($\epsilon = (T_g - T)/T_g$). (b) The probability distribution $P(q)$ associated to the function $q(x)$ shown in (a).

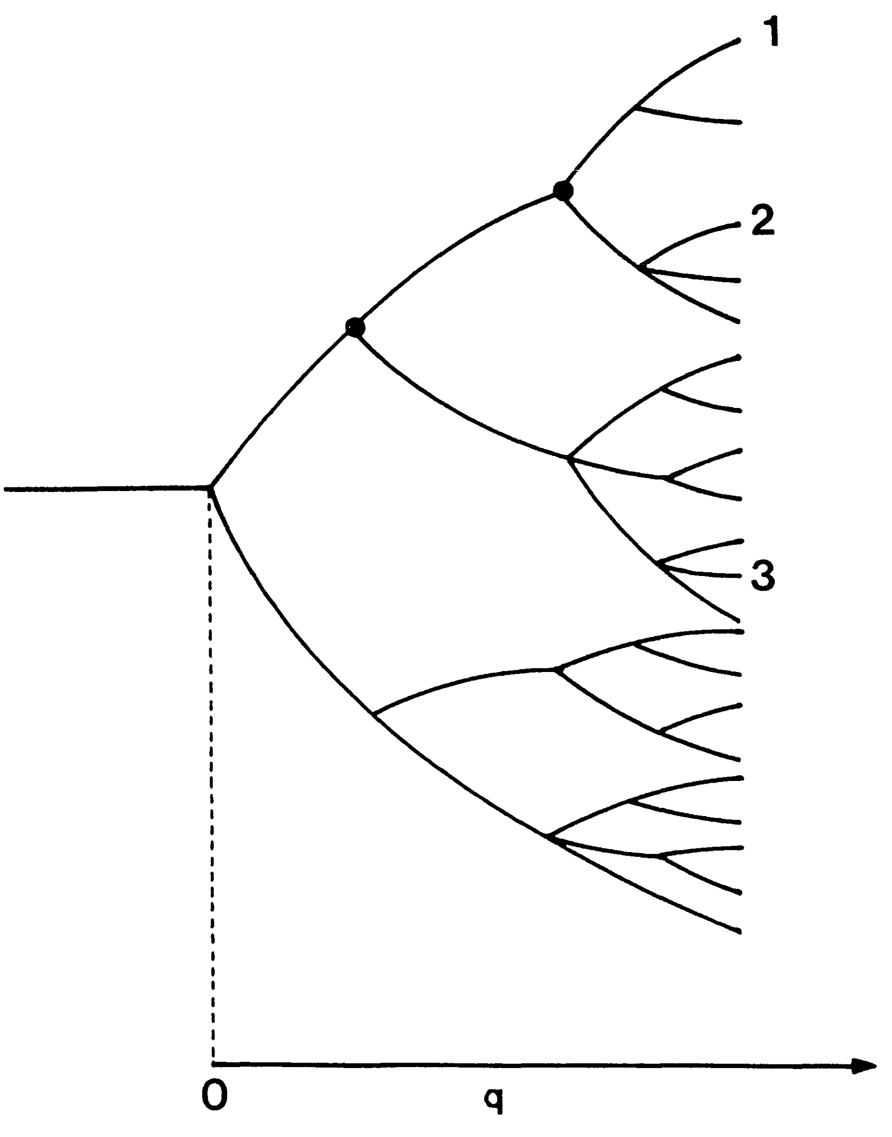

Figure 2.12: The hierarchical structure of the Parisi ansatz showing its ultrametric structure. For any three arbitrarily chosen states, 1 , 2 and 3 , the distances between them in phase space are such that, either they are all equal or the two smaller ones are the same.

CHAPTER 3: THE INFINITE–RANGE P–STATE CLOCK SPIN GLASS

3.1. Introduction

Systems in which the spin variable does not possess reflection symmetry like quadrupolar (Goldbart and Sherrington 1985) and Potts glasses (Erzan and Lage 1983, Elderfield and Sherrington 1983a, b, c, Goldbart and Elderfield 1985, Gross *et al.* 1985), present very different critical behaviour from the well established m–vector spin glasses. In particular for zero magnetic field, the Parisi function, as introduced in the previous chapter, changes drastically; also, discontinuous phase transitions occur as a direct consequence of the absence of reflection symmetry in the spin variable. Whether such "anomalies" are peculiar to Potts and quadrupolar glasses only, or if they also happen in other systems, is not known.

In order to investigate the relevance of reflection symmetry in the spin variable on other systems, it is interesting then, to study a p–state clock spin–glass model. Such a system can be seen as an XY model in an infinite p–fold anisotropy field, such that the spin variables are restricted to p orientations in a plane. Reflection symmetry is absent for every p odd and is present for p even, in which cases one expects the conventional critical behaviour already observed for the m–vector spin glasses.

In section 3.2 we define the infinite–range p–state clock spin glass and explore its relation to well–known models. In section 3.3 we apply the replica method to this model; we present perturbative expansions for the free energy in sections 3.4 and 3.5. In sections 3.6 and 3.7 we deal respectively, with the replica–symmetric solution and its stability in the spin–glass phase.

3.2. The model

As an attempt to investigate magnetic systems where symmetry-breaking crystalline fields are present in addition to the usual exchange couplings, José et al. (1977) introduced the model defined by the hamiltonian,

$$H = -J \sum_{\langle ij \rangle} \underline{S}_i \cdot \underline{S}_j - D \sum_i \cos p\theta_i \quad , \qquad (3.2.1)$$

where i denotes sites of a regular lattice and the symbol $\langle ij \rangle$ indicates a sum over nearest-neighbour lattice sites only. The quantity J is a ferromagnetic exchange coupling ($J > 0$), D represents an anisotropy field, and the $\underline{S}_i$ are continuous spin variables of fixed length restricted to a plane.

For $D = 0$, we have the well known and extensively studied XY model. However, for positive finite D , and at a sufficiently low temperature, the system is forced into a state of broken symmetry in which one of the directions

$$\theta = \frac{2\pi k}{p} \quad ; \quad (k = 0, 1, ..., p-1) \qquad (3.2.2)$$

is especially preferred. In the limit $D = \infty$, the hamiltonian (3.2.1) describes what is called as the p-state clock (or planar Potts) model (Wu 1982).

In this limit, by introducing random couplings, one gets the p-state clock spin glass as defined by

$$H = -\sum_{(ij)} J_{ij} \underline{S}_i \cdot \underline{S}_j - \underline{h} \cdot \sum_i \underline{S}_i \quad , \qquad (3.2.3)$$

where $\underline{h}$ is an external magnetic field and the $\underline{S}_i$ are now, unit vectors restricted to p equally angularly spaced orientations in a plane. Throughout this thesis we will be interested in the infinite–ranged interaction case (Sherrington and Kirkpatrick 1975) for which the summation is over all pairs (ij) and the $\{J_{ij}\}$ are quenched random couplings distributed according to the probability

$$P(J_{ij}) = (N/2\pi J^2)^{\frac{1}{2}} \exp\left(-NJ_{ij}^2/2J^2\right) \quad . \tag{3.2.4}$$

The components of $\underline{S}_i$ can be represented as,

$$S_{ix} = \cos\theta_i \quad , \quad S_{iy} = \sin\theta_i \quad , \tag{3.2.5a}$$

$$\theta_i = \frac{2\pi}{p} k_i \quad (k_i = 0, 1, ..., p-1) \quad . \tag{3.2.5b}$$

The model defined by (3.2.3) is particularly rich, presenting as special limits:

a) $p = 2$

This is the Ising case ($S_{ix} = \pm 1$; $S_{iy} = 0$) introduced by Sherrington and Kirkpatrick (SK model) which was discussed in Chapter 2.

b) $p = 3$

Making use of the identity,

$$\cos\frac{2\pi}{3}(k_i - k_j) = \frac{1}{2}\left[3\delta_{k_i,k_j} - 1\right] \quad ; \quad (k_i, k_j = 0, 1, 2) \tag{3.2.6}$$

equation (3.2.3), for a magnetic field along the x direction ($\underline{h} = h\hat{x}$), can be re–written as a Potts hamiltonian (equation (2.5.3)),

$$H = -\sum_{\langle ij \rangle} \tilde{J}_{ij}\, \delta_{k_i,k_j} - \tilde{h} \sum_i \delta_{k_i,0} + \text{constant} \quad , \tag{3.2.7a}$$

where

$$\{\tilde{J}_{ij}\} = \frac{3}{2}\{J_{ij}\} \quad ; \quad \tilde{h} = \frac{3}{2} h \quad . \tag{3.2.7b}$$

Hence, for $p = 3$, the clock and Potts are identical models, provided the re-scalings (3.2.7b) are performed.

c) $p = \infty$

In this limit the angles θ_i become continuous variables and (3.2.3) describes now the infinite-range XY spin glass.

How does the critical behaviour change when one interpolates between the very distinct limits of low values of p ($p = 2, 3$) to $p = \infty$? This is one of the main questions which will be addressed throughout this thesis. In order to do this we shall start, in the next section, by applying the replica method to such a model.

3.3. The replica method

In this section the replica method (Kac 1968, Edwards 1970, 1971, Emery 1975, Edwards and Anderson 1975) will be used in order to obtain an expression for the disorder averaged free energy of the p–state clock spin glass. From equation (2.3.11),

$$-\beta[F]_{av} = [\ln Z]_{av} = \lim_{n\to 0} \frac{1}{n}([Z^n]_{av} - 1) \quad , \tag{3.3.1}$$

where Z^n is defined for integer n as the trace over n replicas of the original hamiltonian as in (2.3.13).

Similarly to what happened for the SK model, as discussed in Chapter 2, the analytic continuation from integer n to n = 0 is one of the main difficulties of this problem, and this is one of the points to be addressed in detail later. The average over the disorder can now be easily performed,

$$[Z^n]_{av} = \int \prod_{(ij)} dJ_{ij}\, P(J_{ij})\, Z^n\{J_{ij}\} \quad , \tag{3.3.2}$$

in such a way that for a gaussian probability distribution like in (3.2.4), and a magnetic field along the x direction ($\underline{h} = h\hat{x}$), one gets after discarding terms O(1/N),

$$[Z^n]_{av} = \operatorname*{Tr}_{\{S_i^\alpha\}} \exp\left\{ \frac{(\beta J)^2}{4N} \sum_\alpha \sum_{\mu\nu} \left[\sum_i S_{i\mu}^\alpha S_{i\nu}^\alpha\right]^2 + \frac{(\beta J)^2}{2N} \sum_{(\alpha\beta)} \sum_{\mu\nu} \left[\sum_i S_{i\mu}^\alpha S_{i\nu}^\beta\right]^2 \right.$$

$$\left. + \beta h \sum_i \sum_\alpha S_{ix}^\alpha \right\} \quad , \tag{3.3.3}$$

where $(\alpha\beta)$ denotes distinct pairs of replicas and μ, ν are cartesian components (μ, ν = x , y).

Using the identity,

$$e^{\lambda a^2} = \int_{-\infty}^{\infty} \frac{dx}{(2\pi)^{\frac{1}{2}}} \exp\left[-\frac{x^2}{2} + (2\lambda)^{\frac{1}{2}} ax\right] \quad , \qquad (3.3.4)$$

equation (3.3.3) is reduced to a single-site form (from now on, the site index will be dropped),

$$[Z^n]_{av} = \int_{-\infty}^{\infty} \prod_{\alpha} \prod_{\mu\nu} \left[\left[\frac{N}{2\pi}\right]^{\frac{1}{2}} \frac{\beta J}{2^{\frac{1}{2}}} dR^{\alpha}_{\mu\nu}\right] \prod_{(\alpha\beta)} \prod_{\mu\nu} \left[\left[\frac{N}{2\pi}\right]^{\frac{1}{2}} \beta J \, dQ^{\alpha\beta}_{\mu\nu}\right]$$

$$\times \exp\left[-N\, g(R^{\alpha}_{\mu\nu}, Q^{\alpha\beta}_{\mu\nu})\right] \quad , \qquad (3.3.5a)$$

where

$$g(R^{\alpha}_{\mu\nu}, Q^{\alpha\beta}_{\mu\nu}) = \frac{(\beta J)^2}{4} \sum_{\alpha} \sum_{\mu\nu} (R^{\alpha}_{\mu\nu})^2 + \frac{(\beta J)^2}{4} \sum_{\alpha\beta}{}' \sum_{\mu\nu} (Q^{\alpha\beta}_{\mu\nu})^2$$

$$- \ln \mathrm{Tr}_{\alpha} \exp\{H_{eff}\} \quad , \qquad (3.3.5b)$$

$$H_{eff} = \frac{(\beta J)^2}{2} \sum_{\alpha} \sum_{\mu\nu} R^{\alpha}_{\mu\nu} S^{\alpha}_{\mu} S^{\alpha}_{\nu} + \frac{(\beta J)^2}{2} \sum_{\alpha\beta}{}' \sum_{\mu\nu} Q^{\alpha\beta}_{\mu\nu} S^{\alpha}_{\mu} S^{\beta}_{\nu} + \beta h \sum_{\alpha} S^{\alpha}_{x} \quad , \qquad (3.3.5c)$$

with $\sum'_{\alpha\beta}$ denoting a sum over different replicas, $\alpha \neq \beta$.

It is important to remember that one is interested in the thermodynamic limit, $N \to \infty$, and it would be very convenient in this case to use the steepest descents method to evaluate (3.3.5a). Strictly, the $n \to 0$ limit must be taken before the $N \to \infty$ and although no rigorous proof exists it is usually assumed that these limits can be freely interchanged. For some years, it was suspected that this interchange of limits was responsible for the failure of the SK solution at low temperatures (van Hemmen and Palmer 1979, 1982), but it is now believed that this does not really create trouble. Then, interchanging the limits, (3.3.5a) can be evaluated by steepest descents, and the free energy per spin, $f = [F]_{av}/N$, will be given by

$$\beta f = \lim_{n \to 0} \frac{1}{n} \min\{g(R^{\alpha}_{\mu\nu} , Q^{\alpha\beta}_{\mu\nu})\} \quad , \tag{3.3.6}$$

where the $R^{\alpha}_{\mu\nu}$ and $Q^{\alpha\beta}_{\mu\nu}$ are given self-consistently by the saddle-point conditions,

$$\frac{\partial g}{\partial R^{\alpha}_{\mu\nu}} = \frac{\partial g}{\partial Q^{\alpha\beta}_{\mu\nu}} = 0 \tag{3.3.7a}$$

or, in other words,

$$R^{\alpha}_{\mu\nu} = \langle S^{\alpha}_{\mu} S^{\alpha}_{\nu} \rangle = \lim_{n \to 0} \frac{\mathrm{Tr}_{\alpha}\ S^{\alpha}_{\mu}\ S^{\alpha}_{\nu}\ \exp\{H_{eff}\}}{\mathrm{Tr}_{\alpha}\ \exp\{H_{eff}\}} \quad , \tag{3.3.7b}$$

$$Q^{\alpha\beta}_{\mu\nu} = \langle S^{\alpha}_{\mu} S^{\beta}_{\nu} \rangle = \lim_{n \to 0} \frac{\mathrm{Tr}_{\alpha}\ S^{\alpha}_{\mu}\ S^{\beta}_{\nu}\ \exp\{H_{eff}\}}{\mathrm{Tr}_{\alpha}\ \exp\{H_{eff}\}} \ ; \ \alpha \neq \beta \ . \tag{3.3.7c}$$

In each case the $\langle\ \rangle$ brackets denote thermal averaging with respect to H_{eff} . In order to give a finite contribution for the free energy in (3.3.6), the term $\mathrm{Tr}\exp\{H_{eff}\}$ in (3.3.5b) must go to 1 as $n\to 0$ and then, the denominators in equations (3.3.7) can be neglected.

The next step now is to find the correct solutions of equations (3.3.7) in the limit $n\to 0$. It is obvious that H_{eff} is invariant under permutations of the indices of the replicas, as long as n is a positive integer; however, what is not obvious is that this symmetry is preserved when one takes $n\to 0$. This leads to the point of finding a particular parametrization for $R^{\alpha}_{\mu\nu}$ and $Q^{\alpha\beta}_{\mu\nu}$ which gives sensible physics after $n\to 0$.

If $h = 0$, one expects that, on the average, the system will be isotropic in spin space, for $p > 2$, and the solutions of (3.3.7) simplify a lot if one assumes the isotropic conditions,

$$R^{\alpha}_{\mu\nu} = X^{\alpha}\,\delta_{\mu\nu} \quad , \tag{3.3.8a}$$

$$Q^{\alpha\beta}_{\mu\nu} = Q^{\alpha\beta}\,\delta_{\mu\nu} \quad , \tag{3.3.8b}$$

which means that all directions in spin space are equivalent. The choice (3.3.8b) has been criticized based on the fact that, for $h = 0$, one can rotate all the spins in a single replica without any cost of energy, leaving H_{eff} invariant. Therefore, solutions allowing independent rotations between replicas should also be taken into account (Sompolinsky et al. 1984). One can find equivalent solutions,

$$Q^{\alpha\beta}_{\mu\nu} = \sum_{\rho\sigma} T^{\alpha}_{\mu\rho}\,T^{\beta}_{\nu\sigma}\,Q^{\alpha\beta}_{\rho\sigma} \quad , \tag{3.3.9}$$

where $T^{\alpha}_{\mu\rho}$ is a 2×2 rotation matrix for replica α. The isotropic choice assumes that there is always a rotation for which (3.3.8b) is true.

The next chapter will deal with a particular case for which the choices (3.3.8) can not be assumed *a priori* and one should consider as possible solutions in zero field,

$$R^{\alpha}_{xx} \neq R^{\alpha}_{yy} \quad ; \quad Q^{\alpha\beta}_{xx} \neq Q^{\alpha\beta}_{yy} \quad , \tag{3.3.10}$$

as well as the habitual isotropic ones. For the moment, we shall then, take the usual assumption that there is a possible rotation for which the crossed parameters will not contribute even for $h \neq 0$,

$$R^{\alpha}_{\mu\nu} = Q^{\alpha\beta}_{\mu\nu} = 0 \quad , \quad \text{if} \quad \mu \neq \nu \quad . \tag{3.3.11}$$

Making use of (3.3.11) it is convenient to define,

$$R^{\alpha} = R^{\alpha}_{xx} - \frac{1}{2} = \frac{1}{2} - R^{\alpha}_{yy} \quad , \tag{3.3.12}$$

to get,

$$g(R^{\alpha}, Q^{\alpha\beta}_{xx}, Q^{\alpha\beta}_{yy}) = -\frac{n}{8}(\beta J)^2 + \frac{(\beta J)^2}{2}\sum_{\alpha}(R^{\alpha})^2 + \frac{(\beta J)^2}{4}\sum_{\alpha\beta}{}' [(Q^{\alpha\beta}_{xx})^2 + (Q^{\alpha\beta}_{yy})^2]$$

$$- \ln \mathrm{Tr}_{\alpha} \exp \{H_{eff}\} \quad , \tag{3.3.13a}$$

with

$$H_{eff} = (\beta J)^2 \sum_{\alpha} R^{\alpha} [(S_x^{\alpha})^2 - 1/2] + \frac{(\beta J)^2}{2} \sum_{\alpha\beta}' (Q_{xx}^{\alpha\beta} S_x^{\alpha} S_x^{\beta} + Q_{yy}^{\alpha\beta} S_y^{\alpha} S_y^{\beta})$$

$$+ \beta h \sum_{\alpha} S_x^{\alpha} \quad . \tag{3.3.13b}$$

R^{α} is called a quadrupolar parameter and is a measure of anisotropy in the replicated spin space,

$$R^{\alpha} = \langle (S_x^{\alpha})^2 \rangle - 1/2 \quad , \tag{3.3.14a}$$

while $Q_{xx}^{\alpha\beta}$ and $Q_{yy}^{\alpha\beta}$ are respectively the spin–glass parameters, parallel and perpendicular to the external field,

$$Q_{xx}^{\alpha\beta} = \langle S_x^{\alpha} S_x^{\beta} \rangle \quad , \quad Q_{yy}^{\alpha\beta} = \langle S_y^{\alpha} S_y^{\beta} \rangle \quad ; \quad \alpha \neq \beta \quad . \tag{3.3.14b}$$

For $p = 2$, the clock spin glass reduces to the SK model for which $Q_{yy}^{\alpha\beta}$ is zero and R^{α} is a constant. Throughout this thesis, instead of restricting to $p > 2$ only, we shall use the convenience of doing a general calculation for any $p \geq 2$, taking the limit $p = 2$ as a check of the results.

For $h \approx 0$ and T close to T_g (T_g is the paramagnetic/spin glass critical temperature) one expects the parameters R^{α}, $Q_{xx}^{\alpha\beta}$ and $Q_{yy}^{\alpha\beta}$ to be all small. The free–energy functional $g(R^{\alpha}, Q_{xx}^{\alpha\beta}, Q_{yy}^{\alpha\beta})$ can then be expanded perturbatively (Landau and Lifshitz 1980); this expansion is going to be used throughout this thesis and it can be seen in Appendix B.

In Chapter 5 the case $h \neq 0$ will be treated in detail and for the remainder of this chapter we will be concerned with the situation $h = 0$. In the Landau expansion in Appendix B, equation (B.1), one has for $h = 0$,

$$g = -ng_0 - A_2 \sum_{\alpha} (R^{\alpha})^2 - B_2 \sum_{\alpha\beta}{}' (Q_{xx}^{\alpha\beta})^2 - D_2 \sum_{\alpha\beta}{}' (Q_{yy}^{\alpha\beta})^2 - \ldots \quad , \tag{3.3.15a}$$

$$g_0 = \ln p + \frac{(\beta J)^2}{8}(1 + 2\delta_{2,p}) \quad , \tag{3.3.15b}$$

$$A_2 = \frac{(\beta J)^2}{2}\left\{ \frac{(\beta J)^2}{8}(1 - \delta_{2,p} + \delta_{4,p}) - 1 \right\} \quad , \tag{3.3.15c}$$

$$B_2 = \frac{(\beta J)^2}{4}\left\{ \frac{(\beta J)^2}{4}(1 + 3\delta_{2,p}) - 1 \right\} \quad , \tag{3.3.15d}$$

$$D_2 = \frac{(\beta J)^2}{4}\left\{ \frac{(\beta J)^2}{4}(1 - \delta_{2,p}) - 1 \right\} \quad . \tag{3.3.15e}$$

The onset of ordering of quadrupolar or spin–glass type is signaled by the change of sign of the corresponding quadratic contributions. As can be seen, if the temperature is lowered, the coefficient B_2 becomes zero at

$$T_g = \frac{J}{2}(1 + 3\delta_{2,p})^{\frac{1}{2}} \quad , \tag{3.3.16a}$$

and for $p > 2$, D_2 also goes through a zero for

$$T_g = \frac{J}{2} \quad , \tag{3.3.16b}$$

(we work in units $k_B = 1$), signaling the onset of spin–glass order.

For $p > 2$ (and $\neq 4$), the coefficient A_2 of the term quadratic in the quadrupolar parameter does not change sign until a lower temperature,

$$T_a^{(0)} = \frac{J}{8^{\frac{1}{2}}} \qquad (3.3.16c)$$

and the transition at T_g can be considered of the normal isotropic type.

Since $T_a^{(0)} < T_g$, in order to investigate the possibility of occurrence of a second phase transition, from the isotropic to an anisotropic spin–glass state, one needs to take into account the effects of the Q orderings at T_g. This requires a more careful analysis because far from T_g, Q is not small and an expansion like (B.1) does not hold anymore. As argued for the m–vector spin glasses (Toulouse and Gabay 1981), by considering the Q ordering, the temperature $T_a^{(0)}$ is renormalized to a lower value T_a and, although there is no rigorous proof, it is a current belief that $T_a = 0$, for $h = 0$, at least for the m–vectors.

A curious situation happens, however, for the case $p = 4$, where A_2, B_2 and D_2 all become zero for $h = 0$ at the same temperature, namely, $T_g = J/2$. In the next chapter, this problem will be discussed in detail and it will be shown that, besides the usual isotropic choice (3.3.8) another solution is possible, satisfying

$$R^{\alpha}_{xx} \neq R^{\alpha}_{yy} \quad , \qquad (3.3.17a)$$

$$Q^{\alpha\beta}_{\mu\nu} = Q^{\alpha\beta}\delta_{\mu\nu}\delta_{\mu x} \quad (\text{or} = Q^{\alpha\beta}\delta_{\mu\nu}\delta_{\mu y}) \quad . \qquad (3.3.17b)$$

Next, perturbative expansions for the free–energy functional will be developed, first using solutions of the type (3.3.8) for all $p > 2$, including $p = 4$; the anisotropic solution for $p = 4$ will be dealt in separate.

3.4. A perturbative expansion of the free energy for h = 0 (the isotropic solution)

The final discussion in the last section led to the conclusion that as far as the transition at T_g is concerned, for $h = 0^-$, solutions of the type (3.3.8) can be assumed for all $p > 2$ (and $\neq 4$), while in the case $p = 4$ it can be taken as one of the possible solutions. Using that,

$$R^{\alpha}_{xx} + R^{\alpha}_{yy} = \langle (S^{\alpha}_x)^2 \rangle + \langle (S^{\alpha}_y)^2 \rangle = 1 \quad , \tag{3.4.1a}$$

condition (3.3.8a) gives,

$$R^{\alpha}_{xx} = \frac{1}{2}(1 + \delta_{2,p}) \quad ; \quad R^{\alpha}_{yy} = \frac{1}{2}(1 - \delta_{2,p}) \tag{3.4.1b}$$

and from (3.3.12) one gets,

$$R^{\alpha} = \frac{1}{2}\, \delta_{2,p} \quad , \tag{3.4.1c}$$

where the case $p = 2$ has been included for completeness.

Within the space of solutions (3.3.8), the free–energy functional in equations (3.3.13) can be written as

$$g(Q^{\alpha\beta}) = -\frac{n}{8}(\beta J)^2(1 + \delta_{2,p}) + \frac{(\beta J)^2}{4}(2 - \delta_{2,p}) \sum_{\alpha\beta}{}' (Q^{\alpha\beta})^2$$

$$- \ln \mathrm{Tr}_{\alpha} \exp\{H_{\mathrm{eff}}\} \quad , \tag{3.4.2a}$$

$$H_{eff} = \frac{(\beta J)^2}{2} \sum_{\alpha\beta}' Q^{\alpha\beta} (S_x^\alpha S_x^\beta + S_y^\alpha S_y^\beta) \quad , \tag{3.4.2b}$$

where

$$Q^{\alpha\beta} = \frac{1}{2 - \delta_{2,p}} \langle S_x^\alpha S_x^\beta + S_y^\alpha S_y^\beta \rangle \quad . \tag{3.4.3}$$

Close to the phase transition ($Q^{\alpha\beta}$ small), equation (3.4.2a) may be expressed in terms of a Landau expansion,

$$g(Q^{\alpha\beta}) = -n\bar{g}_0 - \frac{1}{2}\bar{A}_2 \sum_{\alpha\beta} (Q^{\alpha\beta})^2 - \frac{1}{3}\bar{A}_3 \sum_{\alpha\beta} (Q^{\alpha\beta})^3 - \frac{1}{3}\bar{B}_3 \sum_{\alpha\beta\gamma} Q^{\alpha\beta} Q^{\beta\gamma} Q^{\gamma\alpha}$$

$$-\frac{1}{12}\bar{A}_4 \sum_{\alpha\beta} (Q^{\alpha\beta})^4 + \frac{1}{12}\bar{C}_4 \sum_{\alpha\beta\gamma} (Q^{\alpha\beta})^2 (Q^{\beta\gamma})^2 - \frac{1}{12}\bar{D}_4 \sum_{\alpha\beta\gamma} Q^{\alpha\beta} (Q^{\beta\gamma})^2 Q^{\gamma\alpha}$$

$$-\frac{1}{12}\bar{B}_4 \sum_{\alpha\beta\gamma\delta} Q^{\alpha\beta} Q^{\beta\gamma} Q^{\gamma\delta} Q^{\delta\alpha} + O(Q^5) \tag{3.4.4}$$

where the summations now are totally unrestricted. The coefficients in equation (3.4.4) are given by

$$\bar{g}_0 = \ln p + \frac{(\beta J)^2}{8}(1 + \delta_{2,p}) \quad , \tag{3.4.5a}$$

$$\bar{A}_2 = (\beta J)^2 \left\{ \frac{(\beta J)^2}{4}(1 + \delta_{2,p}) - \frac{(2 - \delta_{2,p})}{2} \right\} \quad , \tag{3.4.5b}$$

$$\bar{A}_3 = \frac{(\beta J)^6}{16} \delta_{3,p} \quad , \tag{3.4.5c}$$

$$\bar{B}_3 = \frac{(\beta J)^6}{8}(1 + 3\delta_{2,p}) \quad , \tag{3.4.5d}$$

$$\bar{A}_4 = \frac{(\beta J)^8}{32}(3 + 29\delta_{2,p} + \delta_{4,p}) \quad , \tag{3.4.5e}$$

$$\bar{B}_4 = \frac{3}{16}(\beta J)^8(1 + 7\delta_{2,p}) \quad , \tag{3.4.5f}$$

$$\bar{C}_4 = \frac{3}{8}(\beta J)^8(1 + 7\delta_{2,p}) \quad , \tag{3.4.5g}$$

$$\bar{D}_4 = \frac{3}{8}(\beta J)^8\,\delta_{3,p} \quad . \tag{3.4.5h}$$

From (3.4.4) the spin–glass critical temperature can be obtained for all values of p by equating $\bar{A}_2 = 0$,

$$T_g = \frac{J}{2}(1 + 3\delta_{2,p})^{\frac{1}{2}} \quad , \tag{3.4.6}$$

which is the same as the one obtained by setting $B_2 = 0$ from the expansion given by equations (3.3.15).

By looking at equations (3.4.4) and (3.4.5), it can be seen that the 3–state clock spin glass is very special; the terms with coefficients $\bar{A}_3$ (a third–order term) and $\bar{D}_4$ (a fourth–order term) *only appear for* $p = 3$. A qualitative analysis of higher–order terms in the expansion (3.4.4) can show that special contributions also occur for $p = 5$ on terms of 5th or higher order. In general, if one goes to mth order in the expansion (3.4.4), some single extra terms appear for all odd values of p with $p \lesssim m$. These special contributions come as a direct consequence of the absence of reflection symmetry in the spin variable for odd values of p . The identities derived in Appendix A show how these terms happen, where traces which normally vanish for p even, do indeed give a non–zero contribution for some odd values of p . These terms will be responsible for special critical behaviour for odd values of p as will be seen in the forthcoming chapters.

3.5. A perturbative expansion of the free energy for $h = 0$: the case $p = 4$ (anisotropic solution)

As will be discussed in the next chapter, the 4–state clock spin glass presents two equally important solutions: besides the isotropic one used in the last section, an anisotropic solution satisfying (3.3.17) becomes possible. In this section, an expansion of the free energy for such a solution will be developed.

From (3.3.17a) and (3.4.1a) one gets that the quadrupolar parameter R^{α} defined in (3.3.12) is now non–zero. By choosing x as the preferred direction in spin space,[1] equations (3.3.17) lead to

$$R^{\alpha} > 0 \quad ; \quad Q^{\alpha\beta}_{xx} = Q^{\alpha\beta} \quad ; \quad Q^{\alpha\beta}_{yy} = 0 \quad . \tag{3.5.1}$$

For this solution, the free–energy functional will be given by

$$g(R^{\alpha}, Q^{\alpha\beta}) = -\frac{n}{8}(\beta J)^2 + \frac{(\beta J)^2}{2}\sum_{\alpha}(R^{\alpha})^2 + \frac{(\beta J)^2}{4}\sum_{\alpha\beta}{}'(Q^{\alpha\beta})^2$$

$$-\ln \mathrm{Tr}_{\alpha} \exp\{H_{\mathrm{eff}}\} \quad , \tag{3.5.2a}$$

$$H_{\mathrm{eff}} = (\beta J)^2 \sum_{\alpha} R^{\alpha}[(S^{\alpha}_x)^2 - 1/2] + \frac{(\beta J)^2}{2}\sum_{\alpha\beta}{}' Q^{\alpha\beta} S^{\alpha}_x S^{\beta}_x \quad , \tag{3.5.2b}$$

[1]This is done by applying a small symmetry breaking field, $\underline{h} = h\hat{x}$, which is taken to zero after the limit $N \to \infty$.

where

$$R^{\alpha} = \langle (S_x^{\alpha})^2 \rangle - 1/2 \quad ; \quad Q^{\alpha\beta} = \langle S_x^{\alpha} S_x^{\beta} \rangle \quad . \tag{3.5.3}$$

For T just below T_g , R^{α} and $Q^{\alpha\beta}$ are small and (3.5.2a) can be expanded in a power series. The resulting expansion is a particular case of the one in equation (B.1) by taking h = 0 , p = 4 and solutions of type (3.5.1). One gets,

$$\begin{aligned} g(R^{\alpha}, Q^{\alpha\beta}) = & -ng_0 - A_2 \sum_{\alpha} (R^{\alpha})^2 - B_2 \sum_{\alpha\beta} (Q^{\alpha\beta})^2 - D_3 \sum_{\alpha\beta\gamma} Q^{\alpha\beta} Q^{\beta\gamma} Q^{\gamma\alpha} \\ & - G_3 \sum_{\alpha\beta} R^{\alpha} (Q^{\alpha\beta})^2 - A_4 \sum_{\alpha} (R^{\alpha})^4 - B_4 \sum_{\alpha\beta} (Q^{\alpha\beta})^4 \\ & - C_4 \sum_{\alpha\beta\gamma} (Q^{\alpha\beta})^2 (Q^{\beta\gamma})^2 - E_4 \sum_{\alpha\beta\gamma\delta} Q^{\alpha\beta} Q^{\beta\gamma} Q^{\gamma\delta} Q^{\delta\alpha} \\ & - O_4 \sum_{\alpha\beta} R^{\alpha} R^{\beta} (Q^{\alpha\beta})^2 - R_4 \sum_{\alpha\beta\gamma} Q^{\alpha\beta} Q^{\beta\gamma} Q^{\gamma\alpha} R^{\alpha} - \ldots \end{aligned} \tag{3.5.4}$$

The summations are unrestricted and the coefficients in equation (3.5.4) are the same as the ones in Appendix B for h = 0 and p = 4 , namely,

$$g_0 = \ln 4 + \frac{(\beta J)^2}{8} \quad ,$$

$$A_2 = \frac{(\beta J)^2}{2} \left[\frac{(\beta J)^2}{4} - 1 \right] \quad ; \quad B_2 = \frac{(\beta J)^2}{4} \left[\frac{(\beta J)^2}{4} - 1 \right] \quad ,$$

$$D_3 = \frac{(\beta J)^6}{48} \quad ; \quad G_3 = \frac{(\beta J)^6}{16} \quad ,$$

$$A_4 = -\frac{(\beta J)^8}{192} \quad ; \quad B_4 = \frac{(\beta J)^8}{768} \quad ; \quad C_4 = -\frac{(\beta J)^8}{128} \quad ;$$

$$E_4 = \frac{(\beta J)^8}{128} \quad ; \quad O_4 = \frac{(\beta J)^8}{64} \quad ; \quad R_4 = \frac{(\beta J)^8}{32} \quad . \tag{3.5.5}$$

The expansion above will be used in Chapter 7, where the Parisi parametrization will be explored, as well as in the next two sections, where the simplest parametrization will be taken: the so called replica-symmetric ansatz.

3.6. The replica-symmetry ansatz

In order to perform the averaging over the disorder, the replica method, as used in section 3.3, introduced n equivalent and independent copies of the system. It is clear that, as long as n is a positive integer, H_{eff} is invariant under permutations of the replica indices and so, it appears as a natural first attempt to try the replica-symmetric (RS) solutions,

$$R^{\alpha} = R \quad , \quad \text{all } \alpha \quad ; \quad Q^{\alpha\beta}_{\mu\mu} = Q_{\mu\mu} \quad , \quad \mu = x,y \quad , \quad \text{all } \alpha \neq \beta \quad . \tag{3.6.1}$$

Within this space, a straightforward physical interpretation of the parameters in (3.3.14) appears. Let us look for example, at the statistical mechanics order parameter,

$$Q_{\mu\mu} = [\langle S_{\mu} \rangle_T^2]_{av} \quad , \tag{3.6.2}$$

where $\langle \, \rangle_T$ denotes a thermal average with respect to H and $[\,]_{av}$ is an average over the disorder. This can be written as

$$Q_{\mu\mu} = \left[\frac{\mathrm{tr}_\alpha\, S_\mu^\alpha \exp(-\beta H^\alpha)}{Z_\alpha} \; \frac{\mathrm{tr}_\beta\, S_\mu^\beta \exp(-\beta H^\beta)}{Z_\beta} \right]_{av} ,$$

where tr_α denotes a trace over a single replica α. Multiplying both numerator and denominator by Z^{n-2} one gets,

$$Q_{\mu\mu} = \left[\frac{1}{Z^n} \; \mathrm{Tr}_\alpha\, S_\mu^\alpha S_\mu^\beta \exp\left[-\beta \sum_\gamma H^\gamma \right] \right]_{av} \qquad \alpha \neq \beta ,$$

with Tr_α denoting a trace over n replicas. In the limit $n \to 0$ the denominator will give 1 and similar calculations to the ones done in section 3.3. lead to

$$\left[\exp\left[-\beta \sum_\gamma H^\gamma \right] \right]_{av} = \exp\{H_{eff}\}$$

and

$$Q_{\mu\mu} = \lim_{n\to 0} \; \mathrm{Tr}_\alpha\, S_\mu^\alpha S_\mu^\beta \exp\{H_{eff}\} = \langle S_\mu^\alpha S_\mu^\beta \rangle \quad ; \quad \alpha \neq \beta \quad . \qquad (3.6.3)$$

Also, one can show that

$$R = [\langle S_x^2 \rangle_T]_{av} - \frac{1}{2} = \lim_{n\to 0} \; \mathrm{Tr}_\alpha \left[(S_x^\alpha)^2 - 1/2 \right] \exp\{H_{eff}\} \quad , \qquad (3.6.4)$$

which is a measure of anisotropy in the system.

Replica symmetry assumes that whatever the pair of replicas $(\alpha\beta)$ we choose, equation (3.6.3) will be valid. In this space, the spin–glass parameters defined by (3.3.14b) are all of Edwards–Anderson's type, with no distinction from the statistical mechanics parameter (3.6.2). The measure of breakdown of ergodicity, as defined in (2.3.17), is zero in every direction in spin space $(\Delta_\mu = 0\,;\ \mu = x,y)$.

Within the RS approximation, the free energy can be evaluated,

$$\beta f = -\frac{(\beta J)^2}{8} + \frac{(\beta J)^2}{2}(R^2 + R) - \frac{(\beta J)^2}{4}(Q_{xx}^2 + Q_{yy}^2) + \frac{(\beta J)^2}{2} Q_{yy}$$

$$- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{du dv}{2\pi} e^{-u^2/2} e^{-v^2/2} \ln Z \qquad (3.6.5a)$$

where R , Q_{xx} and Q_{yy} are to be determined self–consistently from the saddle point equations,

$$R = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{du dv}{2\pi} e^{-u^2/2} e^{-v^2/2} \left[Z^{-1} \frac{\partial^2 Z}{\partial a_x^2} \right] - \frac{1}{2} \quad , \qquad (3.6.5b)$$

$$Q_{\mu\mu} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{du dv}{2\pi} e^{-u^2/2} e^{-v^2/2} \left[Z^{-1} \frac{\partial Z}{\partial a_\mu} \right]^2 \quad (\mu = x,y) \qquad (3.6.5c)$$

with

$$Z = \mathrm{tr}\,\exp\left(bS_x^2 + a_xS_x + a_yS_y\right) \quad , \tag{3.6.5d}$$

$$a_x = \beta JQ_{xx}^{\frac{1}{2}}u + \beta h \ , \ a_y = \beta JQ_{yy}^{\frac{1}{2}}v \ , \ b = \frac{(\beta J)^2}{2}(2R - Q_{xx} + Q_{yy}) \quad . \tag{3.6.5e}$$

Considering the case $h = 0$, similarly to what was done in section 3.4 , one makes use of the isotropic conditions (3.3.8) and gets,

$$\beta f = -\frac{(\beta J)^2}{8}(1 + \delta_{2,p}) - \frac{(\beta J)^2}{4}(2 - \delta_{2,p})Q^2 + \frac{(\beta J)^2}{2}Q$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dudv}{2\pi}\,e^{-u^2/2}\,e^{-v^2/2}\,\ln Z \tag{3.6.6a}$$

where

$$Q = \frac{1}{2 - \delta_{2,p}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dudv}{2\pi}\,e^{-u^2/2}\,e^{-v^2/2}\left\{\left[Z^{-1}\frac{\partial Z}{\partial a_x}\right]^2 + \left[Z^{-1}\frac{\partial Z}{\partial a_y}\right]^2\right\} \tag{3.6.6b}$$

with

$$Z = \mathrm{tr}\,\exp\left(a_xS_x + a_yS_y\right) \quad , \tag{3.6.6c}$$

$$a_x = \beta JQ^{\frac{1}{2}}u \quad ; \quad a_y = \beta JQ^{\frac{1}{2}}v \quad . \tag{3.6.6d}$$

Near the phase transition, one has the expansion,

$$\beta f = -\bar{g}_0 + \frac{1}{2}\bar{A}_2 Q^2 + \frac{1}{3}\bar{A}_3 Q^3 - \frac{2}{3}\bar{B}_3 Q^3 + O(Q^4) \qquad (3.6.7)$$

where the coefficients are given by (3.4.5). The extremal equation,

$$\frac{\partial(\beta f)}{\partial Q} = 0 \quad , \qquad (3.6.8)$$

gives,

$$Q = \frac{\bar{A}_2}{2\bar{B}_3 - \bar{A}_3} + O(Q^2) \quad , \qquad (3.6.9)$$

or, in other words,

$$Q = \epsilon + O(\epsilon^2) \quad ; \quad p = 2 \quad , \qquad (3.6.10a)$$

$$Q = \frac{2}{3}\epsilon + O(\epsilon^2) \quad ; \quad p = 3 \quad , \qquad (3.6.10b)$$

$$Q = \frac{\epsilon}{2} + O(\epsilon^2) \quad ; \quad p > 3 \quad , \qquad (3.6.10c)$$

where

$$\epsilon = (T_g - T)/T_g \qquad (3.6.11)$$

and T_g is given by (3.4.6).

Equations (3.6.10) give a continuous phase transition at T_g, for all values of p . This is in contrast to what happens for the Potts (Elderfield and Sherrington 1983a) and quadrupolar (Goldbart and Sherrington 1985) glasses, where first order phase transitions are obtained within RS approximation for $p > 6$ (Potts) and $p > 4.6$ (quadrupolar), as a direct consequence of the absence of reflection symmetry in the spin variable. For the clock spin glasses this effect is irrelevant as far as the kind of the transition at T_g is concerned.

Now, the anisotropic solution for $h = 0$ and $p = 4$, as mentioned in section 3.5, will be considered within the replica-symmetry ansatz. Using (3.5.1) one gets,

$$\beta f = -\frac{(\beta J)^2}{8} + \frac{(\beta J)^2}{2}(R^2 + R) - \frac{(\beta J)^2}{4} Q^2 - \int_{-\infty}^{\infty} \frac{du}{(2\pi)^{\frac{1}{2}}} e^{-u^2/2} \ln Z \tag{3.6.12a}$$

where

$$R = \int_{-\infty}^{\infty} \frac{du}{(2\pi)^{\frac{1}{2}}} e^{-u^2/2} \left[Z^{-1} \frac{\partial^2 Z}{\partial a_x^2} \right] - \frac{1}{2} \quad , \tag{3.6.12b}$$

$$Q = \int_{-\infty}^{\infty} \frac{du}{(2\pi)^{\frac{1}{2}}} e^{-u^2/2} \left[Z^{-1} \frac{\partial Z}{\partial a_x} \right]^2 \tag{3.6.12c}$$

$$Z = \mathrm{tr} \exp (b S_x^2 + a_x S_x) \tag{3.6.12d}$$

$$a_x = \beta J Q^{\frac{1}{2}} u \quad ; \quad b = \frac{(\beta J)^2}{2}(2R - Q) \quad . \tag{3.6.12e}$$

For T close to T_g , the free energy becomes,

$$\beta f = -g_0 - A_2R^2 + B_2Q^2 - 2D_3Q^3 + G_3RQ^2 + O(\epsilon^4) \quad , \qquad (3.6.13)$$

where the coefficients are given by (3.5.5). The extremal equations,

$$\frac{\partial(\beta f)}{\partial R} = 0 \quad ; \quad \frac{\partial(\beta f)}{\partial Q} = 0 \quad , \qquad (3.6.14)$$

give respectively,

$$-2A_2R + G_3Q^2 + O(\epsilon^3) = 0 \quad , \qquad (3.6.15a)$$

$$2B_2Q - 6D_3Q^2 + 2G_3RQ + O(\epsilon^3) = 0 \quad . \qquad (3.6.15b)$$

Equations (3.6.15) can be solved,

$$R = \frac{\epsilon}{2} + O(\epsilon^2) \quad , \qquad (3.6.16a)$$

$$Q = \epsilon + O(\epsilon^2) \quad , \qquad (3.6.16b)$$

giving a continuous phase transition at $T = T_g$.

By having the free energy, one can derive the thermodynamics of the model for each value of p and the problem would die at this point. Unfortunately, in a similar way to what was discussed in Chapter 2 for the SK model, the apparently harmless replica–symmetric choice leads to trouble in the spin–glass phase. This will be discussed in the next section.

3.7. Stability analysis for the replica–symmetric solution in zero magnetic field

One question of the most importance concerns the stability of the replica–symmetric solution. As discussed in Chapter 2 for the Ising case, this kind of solution becomes unstable in the spin–glass phase (de Almeida and Thouless 1978), and so, the same could happen for the p–state clock glass with $p > 2$.

The stability analysis for $h \neq 0$ will be carried out in detail in Chapter 5 and throughout this section we will look at the case $h = 0$. First we will deal with the isotropic solution (3.3.8) and afterwards the anisotropic solution for $p = 4$ will be analysed.

a) Isotropic solution

For solutions of the kind (3.3.8), the free–energy functional is given for a general value of p by equations (3.4.2). In order to investigate the validity of the RS ansatz, we will turn to the stability analysis in an Almeida–Thouless (AT) fashion by considering fluctuations around the replica–symmetric solution,

$$Q^{\alpha\beta} = Q + \eta^{\alpha\beta} \quad ; \quad \alpha \neq \beta \quad . \tag{3.7.1}$$

The free–energy functional can be expanded,

$$g(Q^{\alpha\beta}) = g(Q) + (\beta J)^2 \sum_{(\alpha\beta)(\gamma\delta)} S^{(\alpha\beta)(\gamma\delta)} \Big|_{RS} \eta^{(\alpha\beta)} \eta^{(\gamma\delta)} + \ldots \tag{3.7.2a}$$

where the stability matrix $\underset{\sim}{S}$ has dimension $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ and elements

$$S^{(\alpha\beta)(\gamma\delta)} = \frac{\partial^2 g}{\partial Q^{\alpha\beta}\partial Q^{\gamma\delta}} = (2-\delta_{2,p})\delta^{(\alpha\beta)(\gamma\delta)} - (\beta J)^2 \Big[\Big\langle (S_x^\alpha S_x^\beta + S_y^\alpha S_y^\beta)$$

$$\times (S_x^\gamma S_x^\delta + S_y^\gamma S_y^\delta)\Big\rangle - \Big\langle S_x^\alpha S_x^\beta + S_y^\alpha S_y^\beta \Big\rangle \Big\langle S_x^\gamma S_x^\delta + S_y^\gamma S_y^\delta \Big\rangle\Big] \quad . \tag{3.7.2b}$$

The stability matrix elements are to be evaluated in the RS approximation, and $(\alpha\beta)$ denotes distinct pairs of replicas. As discussed in the previous chapter, the minimum condition in equation (3.3.6) requires the matrix $\underset{\sim}{S}$ to be positive definite i.e., all eigenvalues should be positive, for stability.

Three distinct matrix elements can occur (de Almeida and Thouless 1978):

$$S^{(\alpha\beta)(\alpha\beta)} = (2-\delta_{2,p}) - (\beta J)^2 \Big[\Big\langle (S_x^\alpha S_x^\beta + S_y^\alpha S_y^\beta)^2 \Big\rangle$$

$$- \Big\langle S_x^\alpha S_x^\beta + S_y^\alpha S_y^\beta \Big\rangle^2\Big] = U \quad , \tag{3.7.3a}$$

$$S^{(\alpha\beta)(\alpha\gamma)} = -(\beta J)^2 \Big[\Big\langle (S_x^\alpha S_x^\beta + S_y^\alpha S_y^\beta)(S_x^\alpha S_x^\gamma + S_y^\alpha S_y^\gamma)\Big\rangle$$

$$- \Big\langle S_x^\alpha S_x^\beta + S_y^\alpha S_y^\beta \Big\rangle^2\Big] = V \quad , \tag{3.7.3b}$$

$$S^{(\alpha\beta)(\gamma\delta)} = -(\beta J)^2 \Big[\Big\langle (S_x^\alpha S_x^\beta + S_y^\alpha S_y^\beta)(S_x^\gamma S_x^\delta + S_y^\gamma S_y^\delta)\Big\rangle$$

$$- \Big\langle S_x^\alpha S_x^\beta + S_y^\alpha S_y^\beta \Big\rangle^2\Big] = W \quad , \tag{3.7.3c}$$

which hold for $\alpha, \beta, \gamma, \delta$ all different from one another.

One has then, to solve the eigenvalue equation,

$$\underset{\sim}{S}\,\underset{\sim}{\eta} = \lambda\,\underset{\sim}{\eta} \quad , \tag{3.7.4a}$$

where

$$\underline{\eta} = (\{\eta^{(\alpha\beta)}\}) \qquad \alpha = 1,2,...,n \quad , \tag{3.7.4b}$$

is a column vector with $\frac{1}{2}n(n-1)$ elements.

The complete set of eigenvectors of $\underline{S}$ falls into three symmetry classes which will be described below.

1) Vectors which are replica independent:

$$\eta_1^{(\alpha\beta)} = a \quad , \quad \text{all } (\alpha\beta) \quad .$$

The corresponding eigenvalues are, in the limit $n = 0$,

$$\lambda = \frac{1}{2}\{(U - 4V + 3W) \pm |-(U - 4V + 3W)|\} \quad , \tag{3.7.5}$$

which are non-negative throughout the spin-glass phase. This case corresponds to replica-symmetric fluctuations; de Almeida and Thouless argued that in this case, the condition for positive λ is equivalent to the condition that the RS solution should be a saddle point.

2) Vectors which depend on a single replica index:

$$\eta_2^{(\alpha\beta)} = a \quad , \quad \text{for } \alpha \text{ or } \beta = \theta \quad ; \quad \eta_2^{(\alpha\beta)} = b \quad , \quad \text{for } \alpha,\beta \neq \theta \quad .$$

Imposing orthogonality between $\underline{\eta}_2$ and $\underline{\eta}_1$, one gets that the corresponding eigenvalues are, in the limit $n = 0$, the same as the ones found in (3.7.5).

3) Vectors which depend on two replica indices:

$$\eta_3^{(\theta\nu)} = a \quad ; \quad \eta_3^{(\theta\alpha)} = \eta_3^{(\alpha\nu)} = b \quad , \quad \text{for } \alpha \neq \theta,\nu \quad ;$$

$$\eta_3^{(\alpha\beta)} = c \quad , \quad \text{for } \alpha,\beta \neq \theta,\nu \quad .$$

To ensure orthogonality to $\underline{n}_1$ and $\underline{n}_2$ one must take

$$a = (2-n)b \quad ; \quad b = \frac{1}{2}(3-n)c \quad .$$

The substitution of such a vector in the eigenvalue equation (3.7.4a) leads to the eigenvalue,

$$\lambda = U - 2V + W \quad . \tag{3.7.6}$$

For T just below T_g, Q is small and λ can be evaluated perturbatively; one gets,

$$\lambda = -2\bar{A}_3Q - \frac{4}{3}\bar{A}_4Q^2 + O(Q^3) \quad . \tag{3.7.7}$$

Using $\bar{A}_3$ and $\bar{A}_4$ from (3.4.5), one sees that λ is negative, leading to instability of the spin–glass phase.

In the derivation of (3.7.7) only the "most dangerous" fourth–order term, the term with coefficient $\bar{A}_4$, was considered in the Landau expansion (3.4.4). This is known as the Parisi approximation (Parisi 1980a). In looking at the stability of the RS solution for the SK model, one finds that it is the $\Sigma(Q^{\alpha\beta})^4$ term that makes λ negative (Bray and Moore 1978, Pytte and Rudnick 1979). The same happens for the p–state clock spin glass for all values of p except $p = 3$, for which the $\Sigma(Q^{\alpha\beta})^3$ term is responsible for λ negative.

Then, the instability in (3.7.7) appears at order Q, i.e. at order ϵ, for $p = 3$, while for all other values of p it occurs at $O(\epsilon^2)$, as in the SK model. Therefore, the replica–symmetric approximation is even worse for the 3–state clock spin glass. As expected, this instability for $p = 3$, is in agreement with results already known for Potts glasses (Elderfield and Sherrington 1983a), since as shown in section 3.2, the $p = 3$ clock and the 3–state Potts models are isomorphic.

b) Anisotropic solution for $p = 4$

For $p = 4$, a solution like (3.5.1) gives the free–energy functional (3.5.2). The AT stability analysis can be followed by taking,

$$R^{\alpha} = R + \omega^{\alpha} \quad ; \quad Q^{\alpha\beta} = Q + \eta^{\alpha\beta} \quad , \tag{3.7.8}$$

for which,

$$g(R^{\alpha},Q^{\alpha\beta}) = g(R,Q) + (\beta J)^2 \left(\sum_{\alpha\beta} S_{RR}^{\alpha\beta}\Big|_{RS} \omega^{\alpha}\omega^{\beta} + 2 \sum_{(\alpha\beta)\gamma} S_{QR}^{(\alpha\beta)\gamma}\Big|_{RS} \eta^{(\alpha\beta)}\omega^{\gamma} \right.$$

$$\left. + \sum_{(\alpha\beta)(\gamma\delta)} S_{QQ}^{(\alpha\beta)(\gamma\delta)}\Big|_{RS} \eta^{(\alpha\beta)}\eta^{(\gamma\delta)} \right) + \ldots \quad , \tag{3.7.9a}$$

where $\Big|_{RS}$ denotes evaluation of matrix elements within the Replica–Symmetry ansatz.

The stability matrix $\underline{S}$ has dimension $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)$ and elements,

$$S_{RR}^{\alpha\beta} = \frac{\partial^2 g}{\partial R^{\alpha}\partial R^{\beta}} = \delta^{\alpha\beta} - (\beta J)^2 \left[\left\langle (S_x^{\alpha})^2 (S_x^{\beta})^2 \right\rangle - \left\langle (S_x^{\alpha})^2 \right\rangle \left\langle (S_x^{\beta})^2 \right\rangle \right] \quad , \tag{3.7.9b}$$

$$S_{QR}^{(\alpha\beta)\gamma} = \frac{\partial^2 g}{\partial Q^{\alpha\beta}\partial R^{\gamma}} = -(\beta J)^2 \left[\left\langle S_x^{\alpha} S_x^{\beta} (S_x^{\gamma})^2 \right\rangle - \left\langle S_x^{\alpha} S_x^{\beta} \right\rangle \left\langle (S_x^{\gamma})^2 \right\rangle \right] \quad , \tag{3.7.9c}$$

$$S_{QQ}^{(\alpha\beta)(\gamma\delta)} = \frac{\partial^2 g}{\partial Q^{\alpha\beta}\partial Q^{\gamma\delta}} = \delta^{(\alpha\beta)(\gamma\delta)} - (\beta J)^2 \left[\left\langle S_x^{\alpha} S_x^{\beta} S_x^{\gamma} S_x^{\delta} \right\rangle \right.$$

$$\left. - \left\langle S_x^{\alpha} S_x^{\beta} \right\rangle \left\langle S_x^{\gamma} S_x^{\delta} \right\rangle \right] \quad . \tag{3.7.9d}$$

Now, seven distinct elements can occur, namely,

$$S_{RR}^{\alpha\alpha} = 1 - (\beta J)^2 \left[\left\langle (S_x^\alpha)^4 \right\rangle - \left\langle (S_x^\alpha)^2 \right\rangle^2\right] = A \quad , \tag{3.7.10a}$$

$$S_{RR}^{\alpha\beta} = -(\beta J)^2 \left[\left\langle (S_x^\alpha)^2 (S_x^\beta)^2 \right\rangle - \left\langle (S_x^\alpha)^2 \right\rangle^2\right] = B \quad , \tag{3.7.10b}$$

$$S_{QR}^{(\alpha\beta)\alpha} = -(\beta J)^2 \left[\left\langle (S_x^\alpha)^3 S_x^\beta \right\rangle - \left\langle S_x^\alpha S_x^\beta \right\rangle \left\langle (S_x^\alpha)^2 \right\rangle\right] = C \quad , \tag{3.7.10c}$$

$$S_{QR}^{(\alpha\beta)\gamma} = -(\beta J)^2 \left[\left\langle S_x^\alpha S_x^\beta (S_x^\gamma)^2 \right\rangle - \left\langle S_x^\alpha S_x^\beta \right\rangle \left\langle (S_x^\gamma)^2 \right\rangle\right] = D \quad , \tag{3.7.10d}$$

$$S_{QQ}^{(\alpha\beta)(\alpha\beta)} = 1 - (\beta J)^2 \left[\left\langle (S_x^\alpha S_x^\beta)^2 \right\rangle - \left\langle S_x^\alpha S_x^\beta \right\rangle^2\right] = E \quad , \tag{3.7.10e}$$

$$S_{QQ}^{(\alpha\beta)(\alpha\gamma)} = -(\beta J)^2 \left[\left\langle (S_x^\alpha)^2 S_x^\beta S_x^\gamma \right\rangle - \left\langle S_x^\alpha S_x^\beta \right\rangle^2\right] = F \quad , \tag{3.7.10f}$$

$$S_{QQ}^{(\alpha\beta)(\gamma\delta)} = -(\beta J)^2 \left[\left\langle S_x^\alpha S_x^\beta S_x^\gamma S_x^\delta \right\rangle - \left\langle S_x^\alpha S_x^\beta \right\rangle^2\right] = G \quad , \tag{3.7.10g}$$

which hold for all distinct $\alpha, \beta, \gamma, \delta$.

The eigenvalue equation is

$$\underline{S}\,\underline{\sigma} = \lambda\,\underline{\sigma} \quad , \tag{3.7.11a}$$

where

$$\underline{\sigma} = \begin{bmatrix} \{\omega^\alpha\} \\ \{\eta^{(\alpha\beta)}\} \end{bmatrix} \qquad \alpha = 1,2,\ldots,n \quad , \tag{3.7.11b}$$

is a column vector with $\frac{1}{2}n(n+1)$ elements.

Similarly to what was done for the isotropic solution, three types of eigenvectors are possible. The one responsible for replica–symmetry breaking, $\underline{\sigma}_3$, have all ω^α zero and

$$\eta_3^{(\alpha\beta)} = \begin{cases} c & ; \quad (\alpha\beta) = (\theta\nu) \quad , \\ (2-n)^{-1} c & ; \quad \alpha \text{ or } \beta = \theta \text{ or } \nu \quad , \quad \text{but not both} \quad , \\ 2(2-n)^{-1}(3-n)^{-1} c & ; \quad \alpha,\beta \neq \theta,\nu \quad . \end{cases} \tag{3.7.12}$$

The eigenvector (3.7.12) leads to the eigenvalue,

$$\lambda = E - 2F + G \quad . \tag{3.7.13}$$

In order to ensure orthogonality to the other vectors, one must take all $\omega^\alpha = 0$ for the vector $\underline{\sigma}_3$. As a consequence of this, the eigenvalue in (3.7.13) is similar to the one in (3.7.6), for the isotropic solution, showing that the inclusion of the single–replica–dependent parameter R^α, does not affect replica–symmetry breaking. In fact, all quantities involving a single–replica index are replica independent, i.e. are replica symmetric.

Near the phase transition, the eigenvalue in (3.7.13) becomes,

$$\lambda = -16B_4Q^2 + O(Q^3) \quad , \tag{3.7.14}$$

where B_4 is given in (3.5.5) and so, λ is negative, leading to instability of the spin–glass phase. Here also, the only quartic term included in the calculation of λ was the "dangerous" $\Sigma(Q^{\alpha\beta})^4$ term in the expansion (3.5.4).

Hence, as for the Sherrington–Kirkpatrick model, the simple choice preserving the symmetry under permutation of replicas, the replica–symmetric solution, becomes unstable when $T < T_g$ for any p in the p–state clock spin glass. This instability is even stronger in the case $p = 3$. Any satisfactory solution for the spin glass phase must break the symmetry between the replicas and such a solution will be presented later in Chapter 7.

3.8. Conclusion

In this chapter, the infinite–range p–state clock spin glass was introduced. Its relation with well–known systems like the Sherrington–Kirkpatrick, 3–state Potts and XY models was discussed. The standard replica trick was applied to such a model, reducing it to a single site problem. For zero external magnetic field, argumentation was given in favour of a choice of solutions preserving isotropy in spin space for all $p > 2$, but for $p = 4$ an alternative (anisotropic) solution was also considered. Perturbative expansions of the free energy, which will be used in the forthcoming chapters, were developed near the transition point. The replica–symmetric solution was presented, and no discontinuous phase transitions were obtained for any p. This is in contrast to other known models with absence of reflection symmetry in the spin variable, like Potts and quadrupolar glasses, for which first order phase transitions are present. Finally, an Almeida–Thouless stability analysis was carried out, proving the instability of the replica–symmetric solution in the spin–glass phase for any p; it was shown that such an assumption is even more serious for $p = 3$, where the instability occurs at a lower order in perturbation theory.

CHAPTER 4: THE FOUR-STATE CLOCK: A COLLINEAR SPIN GLASS

4.1. Introduction

This chapter introduces a very unexpected feature, an anisotropic spin–glass solution to a spin–glass model with an isotropic hamiltonian. The model in question is a 4–state clock model with symmetric exchange $P\{J_{ij}\} = P\{-J_{ij}\}$. As defined in the previous chapter, a p–state clock model is a special case of a vector–spin model in which the vectors (of constant length) may point only in p equally angularly spaced orientations in a plane.

To date it has been generally considered that vector spin–glass models with isotropic and unbiased exchange can be described in terms of spin–glass solutions which are isotropic in spin space, i.e.

$$Q^{\alpha\beta}_{\mu\nu} = N^{-1} \sum_{i} \langle S^{\alpha}_{i\mu} S^{\beta}_{i\nu} \rangle = Q^{\alpha\beta} \delta_{\mu\nu} \quad , \tag{4.1.1}$$

where α,β denote replicas and μ,ν are cartesian coordinates. In the previous chapter, it was argued that although all other clock glasses are normal in this regard, as far as the transition from paramagnet to spin glass is concerned, for the 4–state clock such a conclusion could not be drawn immediately.

In this chapter we prove that for the 4–state clock the spin–glass state is highly anisotropic, i.e., collinear;

$$Q^{\alpha\beta}_{\mu\nu} = Q^{\alpha\beta} \delta_{\mu\nu} \delta_{\mu x}$$

$$\text{or} \quad Q^{\alpha\beta} \delta_{\mu\nu} \delta_{\mu y} \quad , \tag{4.1.2}$$

with small fluctuations from collinearity only showing up because of replica–symmetry breaking.

In section 4.2, we introduce the 4–state clock spin glass and point out its special character within the realm of p–state clock glasses; an alternative representation in terms of two identical Ising spin glasses follows. In section 4.3, we solve the saddle–point equations within the replica–symmetric approximation and find two distinct solutions corresponding to the same free energy. In section 4.4, we prove the collinearity property (4.1.2) by means of the mapping to two identical Ising models. In section 4.5, we give further support in favour of the anisotropic solution by means of a pure–state analysis. In section 4.6 we introduce a higher–order test function able to distinguish between isotropy and quasi–collinearity, and also (in principle) to show up the fluctuations from perfect collinearity expected from replica–symmetry breaking; we perform a Monte Carlo simulation to test the assertion of anisotropic spin–glass order.

4.2. The four–state clock spin glass

As defined in section 3.2, a four–state clock spin glass is described by the hamiltonian,

$$H = -\sum_{(ij)} J_{ij}\, \underline{S}_i \cdot \underline{S}_j \quad , \tag{4.2.1}$$

where the $\{J_{ij}\}$ are quenched random couplings distributed according to the probability (3.2.4) and the $\underline{S}_i$ are unit vectors with components defined by (3.2.5), i.e.,

$$S_{ix} = 1,0,-1,0 \quad ; \quad S_{iy} = 0,1,0,-1 \quad ; \quad \text{for } k_i = 0,1,2,3 \quad . \tag{4.2.2}$$

The application of the replica method as performed in section 3.3 leads to the free energy per spin in the thermodynamic limit ($N \to \infty$) as the extremal problem

$$\beta f = \lim_{n \to 0} \frac{1}{n} \min\{g(R^\alpha, Q^{\alpha\beta}_{\mu\nu})\} \quad . \tag{4.2.3}$$

The functional $g(R^\alpha, Q^{\alpha\beta}_{\mu\nu})$ is given by

$$g(R^\alpha, Q^{\alpha\beta}_{\mu\nu}) = -\frac{n}{8}(\beta J)^2 + \frac{(\beta J)^2}{2} \sum_\alpha (R^\alpha)^2 + \frac{(\beta J)^2}{4} \sum_{\alpha\beta}{}' \sum_{\mu\nu} (Q^{\alpha\beta}_{\mu\nu})^2$$

$$- \ln \mathrm{Tr}_\alpha \exp\{H_{eff}\} \quad , \tag{4.2.4a}$$

where

$$H_{eff} = (\beta J)^2 \sum_\alpha R^\alpha [(S^\alpha_x)^2 - 1/2] + \frac{(\beta J)^2}{2} \sum_{\alpha\beta}{}' \sum_{\mu\nu} Q^{\alpha\beta}_{\mu\nu} S^\alpha_\mu S^\beta_\nu \quad . \tag{4.2.4b}$$

It is important to notice that in obtaining (4.2.4), no assumption on the parameters in (3.3.7) was done so far. The single-replica-dependent parameter,

$$R^\alpha_{xy} = \langle S^\alpha_x S^\alpha_y \rangle \quad , \tag{4.2.5}$$

is identically zero for $p = 4$, since from (4.2.2) whenever $S^\alpha_x (S^\alpha_y)$ is non–zero, S_y^α (S_x^α) is zero. Therefore, R^α is the quadrupolar parameter, as defined by (3.3.12),

$$R^\alpha = \langle (S_x^\alpha)^2 \rangle - 1/2 \quad , \tag{4.2.6a}$$

while $Q_{\mu\nu}^{\alpha\beta}$ are the usual spin–glass parameters,

$$Q_{\mu\nu}^{\alpha\beta} = \langle S_\mu^\alpha \, S_\nu^\beta \rangle \quad ; \quad \alpha \neq \beta \quad . \tag{4.2.6b}$$

Note that g satisfies the symmetry condition

$$g(R^\alpha , Q_{\mu\nu}^{\alpha\beta}) = g(-R^\alpha , Q_{\bar{\mu}\bar{\nu}}^{\alpha\beta}) \quad , \tag{4.2.7a}$$

where $\bar{\mu}$, $\bar{\nu}$ are the complements of μ , ν ; i.e. if

$$\begin{aligned} \mu = x \quad &\text{then} \quad \bar{\mu} = y \quad , \\ \mu = y \quad &\text{then} \quad \bar{\mu} = x \quad . \end{aligned} \tag{4.2.7b}$$

In the discussion at the end of section 3.3, by analysing the Landau expansion in Appendix B, one was led to the conclusion that for all clock models, with $p > 2$ (and $\neq 4$), the transition at T_g is to the normal isotropic spin–glass state, since the term quadratic in the quadrupolar parameter changes sign at a temperature lower than T_g .

However, for the case $p = 4$, the free-energy functional in (4.2.4) may be expanded,

$$g(R^\alpha, Q_{\mu\nu}^{\alpha\beta}) = -ng_0 - A_2 \sum_\alpha (R^\alpha)^2 - \frac{1}{2} A_2 \sum_{\alpha\beta}' \sum_{\mu\nu} (Q_{\mu\nu}^{\alpha\beta})^2 + \ldots \qquad (4.2.8)$$

with

$$g_0 = \frac{(\beta J)^2}{8} + \ln 4 \quad , \qquad (4.2.9a)$$

$$A_2 = \frac{(\beta J)^2}{2} \left[\frac{(\beta J)^2}{4} - 1 \right] \quad . \qquad (4.2.9b)$$

Therefore, as the temperature is lowered, both quadrupolar and spin-glass quadratic contributions to (4.2.8) become zero at

$$T_g = \frac{J}{2} \quad . \qquad (4.2.10)$$

Another problem in which the matching of these critical temperatures occurs is the m-vector spin glass in the limit $m = \infty$, i.e., the spherical model (Kosterlitz *et al.* 1976, de Almeida *et al.* 1978). In assuming the isotropic conditions (3.3.8), de Almeida *et al.* (1978) obtained the striking result that the replica-symmetric solution becomes stable in the spin-glass phase as $m \to \infty$. Although this result was obtained by setting $R_{\mu\nu}^\alpha = 0$, no changes are expected for non-zero $R_{\mu\nu}^\alpha$, since as discussed in section 3.7, single-replica dependent parameters do not affect replica-symmetry breaking. Another unusual feature presented by this model is in the short-range interaction limit, where the upper critical dimension d_u is equal to 8 (Green *et al.* 1982), as opposed to $d_u = 6$ for other spin–glass models. In such a derivation, fluctuations around mean–field theory are done by means of an ϵ–expansion ($\epsilon = 8$–d), and the retaining of the $R^{\alpha}_{\mu\nu}$ parameters is crucial, as shown by Green *et al.* (1982). Whether the low–temperature phase for $m = \infty$ is anisotropic, or if the quadrupolar effects are totally suppressed by the spin–glass ordering, is not known.

Since R^{α} is an anisotropic order parameter, the above matching of critical temperatures suggests that the four–state clock spin–glass phase is anisotropic. This will be demonstrated explicitly in the forthcoming sections and in order to do this, it is useful to change representation.

As shown in Table 4.1, the four–state clock spin variable $\underline{S}_i \equiv (S_{ix}, S_{iy})$ in equation (4.2.2), is related to Ising variables τ_i, σ_i $(= \pm 1)$ by

$$S_{ix} = \frac{1}{2}(\tau_i + \sigma_i) \quad , \quad S_{iy} = \frac{1}{2}(\tau_i - \sigma_i) \quad . \tag{4.2.11}$$

Within this representation, the hamiltonian in equation (4.2.1) may be re–expressed as

$$H = -\sum_{(ij)} \tilde{J}_{ij}(\tau_i\tau_j + \sigma_i\sigma_j) \quad ; \quad \{\tilde{J}_{ij}\} = \frac{1}{2}\{J_{ij}\} \quad ; \tag{4.2.12}$$

that is, the four state clock model is equivalent to two independent Ising models with identical exchange interactions of strength one half of those of the original clock model.

Table 4.1

k_i	S_{ix}	S_{iy}	τ_i	σ_i
0	1	0	+1	+1
1	0	1	+1	-1
2	–1	0	–1	-1
3	0	–1	–1	+1

Table 4.1: The values of $S_{ix} = \cos \frac{\pi}{2} k_i$ and $S_{iy} = \sin \frac{\pi}{2} k_i$ $(k_i = 0,1,2,3)$, for the four–state clock model; the last two columns show the corresponding values of the Ising variables τ_i and σ_i such that (4.2.11) is satisfied.

The free-energy functional in equations (4.2.4) can be re-written in terms of the Ising variables,

$$g(q_\tau^{\alpha\beta}, q_\sigma^{\alpha\beta}, t^{\alpha\beta}) = -\frac{n}{8}(\beta J)^2 + \frac{(\beta J)^2}{16}\sum_{\alpha\beta}{}' (q_\tau^{\alpha\beta})^2 + \frac{(\beta J)^2}{16}\sum_{\alpha\beta}{}' (q_\sigma^{\alpha\beta})^2$$

$$+ \frac{(\beta J)^2}{8}\sum_{\alpha\beta} (t^{\alpha\beta})^2 - \ln \mathrm{Tr}_\alpha \exp\{H_{eff}\} \quad , \qquad (4.2.13a)$$

$$H_{eff} = \frac{(\beta J)^2}{8}\sum_{\alpha\beta}{}' q_\tau^{\alpha\beta}\, \tau^\alpha \tau^\beta + \frac{(\beta J)^2}{8}\sum_{\alpha\beta}{}' q_\sigma^{\alpha\beta}\, \sigma^\alpha \sigma^\beta + \frac{(\beta J)^2}{4}\sum_{\alpha\beta} t^{\alpha\beta}\, \tau^\alpha \sigma^\beta \quad , \qquad (4.2.13b)$$

where the summations over $t^{\alpha\beta}$ are now totally unrestricted. The $q_\tau^{\alpha\beta}$ and $q_\sigma^{\alpha\beta}$ are Ising spin-glass parameters,

$$q_\tau^{\alpha\beta} = \langle \tau^\alpha \tau^\beta \rangle \quad ; \quad q_\sigma^{\alpha\beta} = \langle \sigma^\alpha \sigma^\beta \rangle \quad ; \quad \alpha \neq \beta \qquad (4.2.14a)$$

and $t^{\alpha\beta}$ is a measure of the correlation between the τ's and σ's in the replica space,

$$t^{\alpha\beta} = \langle \tau^\alpha \sigma^\beta \rangle \quad ; \quad \text{any } \alpha,\beta \quad . \qquad (4.2.14b)$$

The parameters in (4.2.6) are related to the ones in (4.2.14) by,

$$Q_{xx}^{\alpha\beta} = \frac{1}{4}(q_\tau^{\alpha\beta} + q_\sigma^{\alpha\beta} + 2t^{\alpha\beta}) \quad ; \quad \alpha \neq \beta \quad , \qquad (4.2.15a)$$

$$Q_{yy}^{\alpha\beta} = \frac{1}{4}(q_\tau^{\alpha\beta} + q_\sigma^{\alpha\beta} - 2t^{\alpha\beta}) \quad ; \quad \alpha \neq \beta \quad , \qquad (4.2.15b)$$

$$Q_{xy}^{\alpha\beta} = Q_{yx}^{\alpha\beta} = \frac{1}{4}(q_{\tau}^{\alpha\beta} - q_{\sigma}^{\alpha\beta}) \quad ; \quad \alpha \neq \beta \quad , \tag{4.2.15c}$$

$$R^{\alpha} = \frac{1}{2} t^{\alpha\alpha} \quad . \tag{4.2.15d}$$

In this representation, the isotropic solution as described by (3.3.8) requires that

$$q_{\tau}^{\alpha\beta} = q_{\sigma}^{\alpha\beta} \quad , \tag{4.2.16a}$$

$$t^{\alpha\beta} = 0 \quad , \tag{4.2.16b}$$

and anisotropy will be present whenever (4.2.16b) is not satisfied.

From now on, we shall concentrate on the understanding of the low–temperature phase of the four–state clock spin glass. To start with, in the next section, calculations within replica symmetry will be carried out.

4.3. The four–state clock spin glass in the replica–symmetry approximation

In the Ising representation of the four–state clock spin glass, the replica symmetry ansatz is given by

$$q_\tau^{\alpha\beta} = q_\tau \quad ; \quad q_\sigma^{\alpha\beta} = q_\sigma \quad ; \quad \text{all } \alpha \neq \beta \quad , \tag{4.3.1a}$$

$$t^{\alpha\beta} = t \quad ; \quad \text{all } \alpha,\beta \quad . \tag{4.3.1b}$$

Within this approximation, the free energy in (4.2.3) is given by

$$\beta f = -\frac{1}{2}\frac{(\beta J)^2}{4}(1-q)^2 - 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dydz}{2\pi}\,e^{-y^2/2}\,e^{-z^2/2}\ln\left\{2\cosh\left[\frac{\beta J}{2}(q-t)^{\frac{1}{2}}y + \frac{\beta J}{2}t^{\frac{1}{2}}z\right]\right\} \tag{4.3.2}$$

and the saddle–point equations, (4.2.14), become

$$q_\tau = q_\sigma = q \quad , \tag{4.3.3a}$$

$$q = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dydz}{2\pi}\,e^{-y^2/2}\,e^{-z^2/2}\tanh^2\left[\frac{\beta J}{2}(q-t)^{\frac{1}{2}}y + \frac{\beta J}{2}t^{\frac{1}{2}}z\right] \quad , \tag{4.3.3b}$$

$$t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dxdydz}{(2\pi)^{3/2}}\, e^{-x^2/2}\, e^{-y^2/2}\, e^{-z^2/2} \tanh\left[\frac{\beta J}{2}(q-t)^{\frac{1}{2}}x + \frac{\beta J}{2} t^{\frac{1}{2}} z\right]$$

$$\times \tanh\left[\frac{\beta J}{2}(q-t)^{\frac{1}{2}}y + \frac{\beta J}{2} t^{\frac{1}{2}} z\right] \quad . \tag{4.3.3c}$$

Two solutions of equations (4.3.3) are particularly interesting:

(a) $q \neq 0 \quad ; \quad t = 0$

This corresponds to the isotropic spin glass as mentioned in equations (4.2.16).

(b) $q = t \neq 0$

This gives the anisotropic solution (collinear spin glass),

$$Q_{xx} = q \quad ; \quad Q_{yy} = Q_{xy} = Q_{yx} = 0 \quad , \tag{4.3.4a}$$

$$R = \frac{q}{2} \quad . \tag{4.3.4b}$$

The relation $R = Q_{xx}/2$ has been already obtained in Chapter 3, when the anisotropic solution was treated in the original clock representation, by means of a perturbation expansion close to T_g, as shown in equations (3.6.16).

A curious fact about the above solutions is that both provide the same value for the free energy as can be seen by direct substitution in equation (4.3.2). In fact, by changing the variables,

$$\frac{\beta J}{2}\,q^{\frac{1}{2}}u = \frac{\beta J}{2}(q-t)^{\frac{1}{2}}y + \frac{\beta J}{2}\,t^{\frac{1}{2}}z \quad , \tag{4.3.5a}$$

$$\frac{\beta J}{2}\,q^{\frac{1}{2}}v = \frac{\beta J}{2}\,t^{\frac{1}{2}}y - \frac{\beta J}{2}(q-t)^{\frac{1}{2}}z \quad , \tag{4.3.5b}$$

the free energy may be re–expressed as

$$\beta f = -\frac{1}{2}\frac{(\beta J)^2}{4}(1-q)^2 - 2\int_{-\infty}^{\infty}\frac{du}{(2\pi)^{\frac{1}{2}}}\,e^{-u^2/2}\ln\left\{2\cosh\left[\frac{\beta J}{2}\,q^{\frac{1}{2}}u\right]\right\} \tag{4.3.6}$$

and the self–consistent equation for q ,

$$q = \int_{-\infty}^{\infty}\frac{du}{(2\pi)^{\frac{1}{2}}}\,e^{-u^2/2}\tanh^2\left[\frac{\beta J}{2}\,q^{\frac{1}{2}}u\right] \quad . \tag{4.3.7}$$

Equations (4.3.6) and (4.3.7) are very familiar (Sherrington and Kirkpatrick 1975). Equation (4.3.6) gives twice the free energy of the SK spin glass with exchange couplings $\tilde{J}_{ij}$ as defined by (4.2.12), reflecting the fact that the four–state clock is equivalent to two independent Ising models. Equation (4.3.7) is the self–consistent equation for the spin–glass parameter of the SK model with couplings $\tilde{J}_{ij}$.

The fact that the free energy presents no t dependence, makes it difficult to choose the correct low–temperature solution even within the replica–symmetric approximation. In the next section, the four–state clock spin glass will be examined in the full replica space and by applying a small symmetry–breaking field, argumentation will be given in favour of the anisotropic solution as represented by (4.1.2).

4.4. Full replica–space and symmetry–breaking field analysis

The four–state clock spin glass in the n–replica space can also be described in terms of a single Ising model in a 2n–replica space. In order to do this, one may define the Ising variable ξ^a (= ± 1), where a is now a replica index running from 1 to 2n ,

$$\xi^a = \begin{cases} \tau^\alpha & ; \quad a = 1, \ldots, n \quad , \\ \sigma^\alpha & ; \quad a = n+1, \ldots, 2n \quad . \end{cases} \tag{4.4.1}$$

Equations (4.2.13) become

$$g(P^{ab}) = -\frac{n}{8}(\beta J)^2 + \frac{(\beta J)^2}{16} \sum_{ab}{}' (P^{ab})^2 - \ln \mathrm{Tr}_\alpha \exp\left\{\frac{(\beta J)^2}{2} \sum_{ab}{}' P^{ab}\xi^a\xi^b\right\} \tag{4.4.2}$$

In this space, the spin–glass parameters,

$$P^{ab} = \langle \xi^a \xi^b \rangle \quad ; \quad a \neq b \quad , \tag{4.4.3}$$

are the elements of a 2n × 2n matrix,

$$\mathbb{P} = \left[\begin{array}{c:c} q_\tau^{\alpha\beta} & t^{\alpha\beta} \\ \hdashline t^{\beta\alpha} & q_\sigma^{\alpha\beta} \end{array}\right] \quad . \tag{4.4.4}$$

It is interesting to note the relation of the simultaneous mode–softening of both $Q^{\alpha\beta}_{\mu\nu}$ and R^{α} degrees of freedom in the clock representation to this Ising model. In a replication of the Ising system described by equation (4.4.2) there are 2n spins per site, leading to the softening at the transition of $2n(2n-1)/2 = (2n^2-n)$ Ising P^{ab} modes. In the clock representation there are n spins with four combinations, xx , xy , yx , yy , giving $4n(n-1)/2 = 2n^2-2n$ $Q^{\alpha\beta}_{\mu\nu}$ modes. The inclusion of the n R^{α} modes gives a total of $2n^2-n$ as in the Ising representation. This argument shows that the R^{α} modes are completely equivalent to the $Q^{\alpha\beta}_{\mu\nu}$ modes for the 4–state clock and must be included in any description of the ordered state.

It is clear that in applying the replica–symmetry ansatz to the Ising model in a 2n–replica space, one should take

$$P^{ab} = P \quad ; \quad \text{for all } a \neq b \quad , \tag{4.4.5}$$

irrespective to whether P^{ab} is in a q–block or in a t–block of the matrix $\mathbb{P}$. Solutions (4.4.5) correspond to the collinear spin glass described by (4.3.4). Obviously, fluctuations from collinearity should be expected when the replica symmetry is broken, but the anisotropic solution can still be obtained on the average, as it will be discussed next.

Let us now present a simple argument in terms of the effects of an infinitesimal symmetry–breaking field $\underline{h}$, adding to the hamiltonian in equation (4.2.1) the term,

$$H_{field} = -\underline{h} \cdot \sum_i \underline{S}_i = -h_x \sum_i S_{ix} - h_y \sum_i S_{iy} \quad . \tag{4.4.6}$$

If the spin–glass phase is isotropic it should be unaffected by the application of $\underline{h}$ in the limit $h \to 0$. It will be seen that this is not the case. In terms of the Ising variables,

$$H_{field} = -h_\tau \sum_i \tau_i - h_\sigma \sum_i \sigma_i \quad , \tag{4.4.7a}$$

where

$$h_\tau = \frac{1}{2}(h_x + h_y) \quad , \quad h_\sigma = \frac{1}{2}(h_x - h_y) \quad . \tag{4.4.7b}$$

Hence a symmetry–breaking field along the x direction gives $h_\tau = h_\sigma$ and then, because the τ's and σ's are completely equivalent,

$$\langle \tau_i \rangle_T = \langle \sigma_i \rangle_T \quad , \tag{4.4.8a}$$

so

$$\langle S_{iy} \rangle_T = 0 \quad , \tag{4.4.8b}$$

where $\langle \; \rangle_T$ denotes the average with respect to H. Similarly, a field along the y direction gives

$$\langle \tau_i \rangle_T = -\langle \sigma_i \rangle_T \quad , \tag{4.4.9a}$$

so

$$\langle S_{ix} \rangle_T = 0 \quad . \tag{4.4.9b}$$

Although one can argue that (4.4.8b) and (4.4.9b) follow trivially from symmetry reasons, an expression for R in terms of the Ising statistical mechanics order parameter q ($R = \frac{1}{2}[\langle \tau_i \rangle_T^2]_{av} = \frac{q}{2}$) , analogous to (4.3.4b) can be shown to be be true in general. That exhibits the anisotropic behaviour of the low–temperature phase of the four–state clock spin glass, implying on a perfect collinear state ($R = \frac{1}{2}$) at zero temperature.

In the next section, we present a pure–state analysis for the four–state clock spin glass.

4.5. Pure–state analysis

Let us now consider the four–state clock in terms of the pure thermodynamic states of the Ising model. Two situations for the Ising system can be identified:

(i) There are only two pure states, which are global inverses of one another. This is the situation for a conventional Ising ferromagnet and has been argued to be the situation for a short–range Ising spin glass (e.g. Fisher and Huse 1986; Bray and Moore 1986), although that problem remains incompletely solved.

(ii) There are many pure states, which are unrelated by global symmetry (as well as global inverse pairs). This is the situation for the infinite–ranged spin–glass model of Sherrington and Kirkpatrick (1975).

In case (i), as discussed in the previous section, it is immediately clear that all pure states of the four–state clock are perfectly collinear. One has either

(4.4.8a) or (4.4.9a), depending upon whether the τ and σ systems are in the same pure Ising state or global inverses; an arbitrarily small field will determine these states. The former leads to a collinear state in the x direction and the latter to a collinear state in the y direction.

In case (ii), it is only if the τ and σ systems belong to globally identical or inverse Ising states that perfect collinearity results. In general, this is not the case; nevertheless, there does result average collinearity in the sense discussed in section 4.4 . To see this within the pure–state language, let us first separate the complete set of pure states into two groups, each group having a common sign of overall magnetization or positive overlap

$$q^{ss'} = N^{-1} \sum_i \langle \tau_i \rangle_s \langle \tau_i \rangle_{s'} \quad , \tag{4.5.1}$$

where subscripts s,s′ label the pure states. We restrict ourselves to one such group and for definiteness take the same group for the τ and σ systems (one could equally well take the opposite group for each).

Consider now the statistical mechanics order parameter as defined by (2.5.8),

$$Q_{\mu\nu} = \left[\sum_{\tilde{s},\tilde{s}'} \tilde{P}_{\tilde{s}} \tilde{P}_{\tilde{s}'} N^{-1} \sum_i \langle S_{i\mu} \rangle_{\tilde{s}} \langle S_{i\nu} \rangle_{\tilde{s}'} \right]_{av} \quad ; \quad \mu,\nu = x,y \quad , \tag{4.5.2}$$

where the $\tilde{s},\tilde{s}'$ are states of the clock system, with probabilities $\tilde{P}_{\tilde{s}}$, $\tilde{P}_{\tilde{s}'}$. Each state $\tilde{s}$ is made up of two states, s,s′ of the Ising τ and σ systems, with

$$\tilde{P}_{\tilde{s}} = P_s P_{s'} \quad , \tag{4.5.3}$$

where P_s is the probability of state s. Noting that

$$q = \left[\sum_{s,s'} P_s P_{s'} N^{-1} \sum_i \langle \eta_i \rangle_s \langle \xi_i \rangle_{s'} \right]_{av} \tag{4.5.4}$$

takes the same value irrespective of whether η, ξ are τ, σ, collinearity follows in the sense that

$$Q_{xx} = q \quad , \quad Q_{yy} = Q_{xy} = Q_{yx} = 0 \quad . \tag{4.5.5}$$

Explicitly,

$$Q_{\mu\nu} = \frac{1}{4} \left[\sum_{s,s',s'',s'''} P_s P_{s'} P_{s''} P_{s'''} N^{-1} \sum_i \left\{ \langle \tau_i \rangle_s \langle \tau_i \rangle_{s''} \right. \right.$$

$$\left. \left. + \lambda_\mu \langle \sigma_i \rangle_{s'} \langle \tau_i \rangle_{s''} + \lambda_\nu \langle \tau_i \rangle_s \langle \sigma_i \rangle_{s'''} + \lambda_\mu \lambda_\nu \langle \sigma_i \rangle_{s'} \langle \sigma_i \rangle_{s'''} \right\} \right]_{av}$$

$$= \frac{1}{4} q (1+\lambda_\mu)(1+\lambda_\nu) \quad , \tag{4.5.6}$$

where

$$\lambda_\mu = \begin{cases} 1 \; ; \; \mu = x \\ -1 \; ; \; \mu = y \end{cases} \quad . \tag{4.5.7}$$

Similarly to the discussion at the end of the previous section, one can argue that (4.5.5) follows trivially from symmetry reasons. In the next section the collinearity property will be demonstrated by means of a Monte Carlo simulation.

4.6. Monte Carlo simulation

The main limitation in applying numerical simulations to study many–body problems, is the fact that only finite systems can be handled by the computer. In fact, statistical mechanics tells us that the occurrence of phase transitions is intimately linked to the thermodynamic limit, $N \to \infty$. In a finite system, a phase transition in strict sense cannot occur at all; any singularities associated with the transition in the infinite system are washed out for finite N . However, very good agreement with both theory and experiment can be obtained by extrapolating the finite system behaviour to the large N limit.

For the p–state clock spin glass, as an example, the distribution $P\{Q_{\mu\nu}\}$, which gives the probability that there are states s and s′ with overlaps $\{Q_{\mu\nu}^{ss'}\}$ equal to $\{Q_{\mu\nu}\}$, as defined in section 2.5, cannot be obtained in a computer simulation since on finite samples, the barriers are finite and it is not possible to divide the phase space into mutually inaccessible regions corresponding to the various pure states. However, one can define the overlap distribution microscopically (Young 1983b, 1985), without direct reference to thermodynamic states. Consider two independent sets of spins $\{S_i^1\}$, $\{S_i^2\}$, $i = 1,..,N$, with the same interactions, and define

$$P_N\{Q_{\mu\nu}\} = \left[\left\langle \prod_{\mu\nu} \delta(\, Q_{\mu\nu} - Q_{\mu\nu}^{12}\,) \right\rangle_T \right]_{av} \quad , \tag{4.6.1a}$$

where

$$Q_{\mu\nu}^{12} = N^{-1} \sum_{i=1}^{N} S_{i\mu}^1 S_{i\nu}^2 \quad ; \quad \mu , \nu = x , y \quad , \tag{4.6.1b}$$

as the probability that the two systems have overlaps $\{Q^{12}_{\mu\nu}\}$. It is possible to show that (Young 1985),

$$\lim_{N\to\infty} P_N\{Q_{\mu\nu}\} = P\{Q_{\mu\nu}\} \quad . \tag{4.6.2}$$

From equation (4.6.2) it follows that the moments of $P_N\{Q_{\mu\nu}\}$ and $P\{Q_{\mu\nu}\}$ are all equal in the limit $N \to \infty$. In particular, the first moments give the statistical mechanics parameters $\{Q_{\mu\nu}\}$ as defined in section 2.5,

$$\lim_{N\to\infty} \int dQ_{\mu\nu}\, P_N\{Q_{\mu\nu}\}\, Q_{\mu\nu} = \lim_{N\to\infty} N^{-1} \sum_i [\langle S^1_{i\mu}\, S^2_{i\nu}\rangle_T]_{av}$$

$$= \lim_{N\to\infty} N^{-1} \sum_i [\langle S_{i\mu}\rangle_T \langle S_{i\nu}\rangle_T]_{av} = Q_{\mu\nu} \quad ; \quad \mu , \nu = x , y \quad , \tag{4.6.3}$$

since in the thermodynamic limit, the statistical mechanics average corresponds to an average over all pure states ($\langle\ \rangle_T \to \sum_s P_s \langle\ \rangle_s$) .

Equations (4.6.1) have the advantage that they can be used for finite systems, in which the microscopic states are generated by the computer simulation. The $\langle\ \rangle_T$ refer to thermal averages over both sets of spins, which in a Monte Carlo simulation are replaced by time averages, and $[\ ]_{av}$ refers to an average over different realizations of the $\{J_{ij}\}$.

In order to test the assertion of anisotropic spin–glass order by means of computer simulation. one needs to define a test–quantity requiring that it should:

(a) preserve the symmetries of the hamiltonian (inversion and $x \leftrightarrow y$ interchange);

(b) remain invariant under independent rotations between the replicas 1 and 2 as stated in (3.3.9);

(c) provide different results for isotropic and anisotropic order.

Such a quantity is

$$\Psi = \frac{a}{b} \quad , \tag{4.6.4a}$$

where

$$a = 4\,[\langle (Q^{12}_{xx})^2 (Q^{12}_{yy})^2 + (Q^{12}_{xy})^2 (Q^{12}_{yx})^2 \rangle_T\,]_{av} \quad , \tag{4.6.4b}$$

$$b = [\langle (Q^{12}_{xx})^2 + (Q^{12}_{yy})^2 + (Q^{12}_{xy})^2 + (Q^{12}_{yx})^2 \rangle_T^2\,]_{av} \quad . \tag{4.6.4c}$$

Using (4.6.1b) one can write,

$$a = 8\,N^{-4} \sum_{ijkl} [\langle S_{ix}\, S_{jx}\, S_{ky}\, S_{ly} \rangle_T^2\,]_{av} \quad , \tag{4.6.5a}$$

$$b = N^{-4} \left[\left\{ \sum_{ij} \left\langle \sum_{\mu} S_{i\mu}\, S_{j\mu} \right\rangle_T^2 \right\}^2 \right]_{av} \quad , \tag{4.6.5b}$$

which is closely related to χ^2_{SG} , where χ_{SG} is the spin–glass susceptibility,

$$\chi_{SG} = N^{-1} \sum_{ij} \left[\left\langle \sum_{\mu} S_{i\mu}\, S_{j\mu} \right\rangle_T^2 \right]_{av} \quad . \tag{4.6.5c}$$

For $T = \infty$, there are no correlations and the averagings can be evaluated by making use of the identities of Appendix A for p = 4 ,

$$\langle S_{ix}^m \rangle_T = \langle S_{iy}^m \rangle_T = 0 \quad , \quad \text{for m odd} \quad , \tag{4.6.6a}$$

$$\langle S_{ix}^m S_{iy}^n \rangle_T = 0 \quad , \quad \text{for any m,n (m,n} \neq 0) \quad , \tag{4.6.6b}$$

$$\langle S_{ix}^2 \rangle_T = \langle S_{iy}^2 \rangle_T = \frac{1}{2} \quad . \tag{4.6.6c}$$

One can easily show that

$$\Psi = \frac{1}{2}\frac{N-1}{N} \quad , \tag{4.6.7}$$

which gives $\Psi = 1/2$ in the limit $N \to \infty$.

Were the low–temperature order isotropic as stated in (4.1.1), one would get

$$\lim_{N\to\infty} \Psi = 1 \quad , \quad \text{as } T \to 0 \quad , \tag{4.6.8a}$$

whereas perfect collinearity as in (4.1.2), would give

$$\lim_{N\to\infty} \Psi = 0 \quad , \quad \text{for } T < T_g \quad . \tag{4.6.8b}$$

Clearly, these are two very distinct limits.

In the infinite–range case one can again use the Ising mapping to relate to the order parameter space of the Sherrington–Kirkpatrick model. One has, for the anisotropic solution,

$$a = (32N^4)^{-1} \sum_{ijkl} \Big\{ 4\,[\langle \sigma_i\, \sigma_j\, \sigma_k\, \sigma_l \rangle_T^2\,]_{av}$$

$$+\,12\,[\langle \sigma_i\, \sigma_j \rangle_T^2\, \langle \sigma_k\, \sigma_l \rangle_T^2\,]_{av} - 8\,[\langle \sigma_i\, \sigma_j\, \sigma_k\, \sigma_l \rangle_T\, \langle \sigma_i\, \sigma_j \rangle_T\, \langle \sigma_k\, \sigma_l \rangle_T\,]_{av}$$

$$-\,8\,[\langle \sigma_i\, \sigma_j \rangle_T\, \langle \sigma_j\, \sigma_k \rangle_T\, \langle \sigma_k\, \sigma_l \rangle_T\, \langle \sigma_l\, \sigma_i \rangle_T\,]_{av} \Big\} \quad , \tag{4.6.9a}$$

$$b = N^{-4} \sum_{ijkl} [\langle \sigma_i\, \sigma_j \rangle_T^2\, \langle \sigma_k\, \sigma_l \rangle_T^2\,]_{av} \quad . \tag{4.6.9b}$$

The terms in equations (4.6.9) can be written in the replica formulation; as an example,

$$[\langle \sigma_i\, \sigma_j\, \sigma_k\, \sigma_l \rangle_T^2\,]_{av}$$

$$= \left[\frac{\mathrm{tr}_\alpha\, \sigma_i^\alpha\, \sigma_j^\alpha\, \sigma_k^\alpha\, \sigma_l^\alpha\, \exp(-\beta H^\alpha)}{Z_\alpha}\, \frac{\mathrm{tr}_\beta\, \sigma_i^\beta\, \sigma_j^\beta\, \sigma_k^\beta\, \sigma_l^\beta\, \exp(-\beta H^\beta)}{Z_\beta} \right]_{av}$$

$$= \left[\frac{1}{Z^n}\, \mathrm{Tr}_\alpha\, \sigma_i^\alpha\, \sigma_j^\alpha\, \sigma_k^\alpha\, \sigma_l^\alpha\, \sigma_i^\beta\, \sigma_j^\beta\, \sigma_k^\beta\, \sigma_l^\beta\, \exp\Big[-\beta \sum_\gamma H^\gamma\Big] \right]_{av} \qquad \alpha \neq \beta \quad ,$$

where in the last step, both numerator and denominator were multiplied by Z^{n-2}, and tr_α is a single–replica trace whereas Tr_α denotes a trace over the n replicas. In the limit $n \to 0$, Z^n will give 1 and, as in section 3.6,

$$\left[\exp\left[-\beta\sum_{\gamma} H^{\gamma}\right]\right]_{av} = \exp\{H_{eff}\} \quad ,$$

so that the averages are now to be evaluated with respect to H_{eff}. The effective hamiltonian is single–site dependent and then, contributions from different sites decouple. The important contributions from equations (4.6.9) in the thermodynamic limit, are the ones with i,j,k,l all distinct (all other cases give terms O(1/N)). One gets,

$$[\langle\sigma_i\ \sigma_j\ \sigma_k\ \sigma_l\rangle_T^2]_{av} = \langle\sigma_i^\alpha\ \sigma_i^\beta\rangle\ \langle\sigma_j^\alpha\ \sigma_j^\beta\rangle\ \langle\sigma_k^\alpha\ \sigma_k^\beta\rangle\ \langle\sigma_l^\alpha\ \sigma_l^\beta\rangle = (q_\sigma^{\alpha\beta})^4 \quad , \qquad (4.6.10a)$$

where $\langle\ \rangle$ denotes an average with respect to $\exp\{H_{eff}\}$ and $q_\sigma^{\alpha\beta}$, $\alpha \neq \beta$, is the Ising spin–glass parameter in equation (4.2.14a).

Similar analysis can be carried for the other terms,

$$[\langle\sigma_i\ \sigma_j\rangle_T^2\ \langle\sigma_k\ \sigma_l\rangle_T^2]_{av} = (q_\sigma^{\alpha\beta})^2\ (q_\sigma^{\gamma\delta})^2 \quad , \qquad (4.6.10b)$$

$$[\langle\sigma_i\ \sigma_j\ \sigma_k\ \sigma_l\rangle_T\ \langle\sigma_i\ \sigma_j\rangle_T\ \langle\sigma_k\ \sigma_l\rangle_T]_{av} = (q_\sigma^{\alpha\beta})^2\ (q_\sigma^{\alpha\gamma})^2 \quad , \qquad (4.6.10c)$$

$$[\langle\sigma_i\ \sigma_j\rangle_T\ \langle\sigma_j\ \sigma_k\rangle_T\ \langle\sigma_k\ \sigma_l\rangle_T\ \langle\sigma_l\ \sigma_i\rangle_T]_{av} = q_\sigma^{\alpha\beta}\ q_\sigma^{\beta\gamma}\ q_\sigma^{\gamma\delta}\ q_\sigma^{\delta\alpha} \quad , \qquad (4.6.10d)$$

which hold for all distinct $\alpha, \beta, \gamma, \delta$.

Because of replica–symmetry breaking, $q_\sigma^{\alpha\beta}$ depends on the choice of replicas and the correct thing to do now is to sum equations (4.6.10) over all replicas. Therefore, summing both sides of each of equations (4.6.10) over

$$\frac{1}{N(N-1)(N-2)(N-3)}\ \frac{1}{n(n-1)(n-2)(n-3)}\ \sum_{ijkl}{}'\ \sum_{\alpha\beta\gamma\delta}{}'$$

(the Σ' summations denote all indices distinct), and taking the limits $N \to \infty$, $n \to 0$, Ψ may be expressed in terms of the Parisi function for the Ising spin glass, $q(x)$ (Parisi 1979, 1980a, 1980b, 1980c, 1980d, 1980e). Using the rules presented in section 2.4, one gets for $T < T_g$,

$$\Psi[q] = \frac{a[q]}{b[q]} \quad , \tag{4.6.11a}$$

where

$$a[q] = \frac{1}{24}\Big\{ \langle q^4 \rangle + 3\,\langle q^2 \rangle^2 - 4\,\langle q \rangle^2 \langle q^2 \rangle + \int_0^1 dx \int_0^x dy\,[\,q^2(x) - q^2(y)\,]^2$$

$$- 4\,\langle q \rangle \int_0^1 dx\, q(x) \int_0^x dy\,[\,q(x) - q(y)\,]^2$$

$$- \int_0^1 dx \int_0^x dy \int_0^x dz\,[\,q(x) - q(y)\,]^2\,[\,q(x) - q(z)\,]^2 \Big\} \quad , \tag{4.6.11b}$$

$$b[q] = \frac{1}{3}\Big\{ - \langle q^4 \rangle + 4\,\langle q^2 \rangle^2 + 2 \int_0^1 dx \int_0^x dy\,[\,q^2(x) - q^2(y)\,]^2 \Big\} \quad , \tag{4.6.11c}$$

and

$$\langle q^m \rangle = \int_0^1 dx\, q^m(x) \quad . \tag{4.6.11d}$$

Clearly, if $q(x)$ is a non-zero constant, the replica-symmetric situation for $T < T_g$, this indeed leads to $\Psi = 0$. More generally, for the replica-symmetry broken situation, this result holds only in the limits $T \to T_g^-$ and $T \to 0$. For small $\epsilon = (T_g - T)/T_g$, $q(x)$ is given by (Kondor 1983, Thomsen *et al.* 1986, Sommers 1985)

$$q(x) = \frac{1}{2}(1 + 3\epsilon)x + O(\epsilon^3) \quad ; \quad x < x_l$$

$$= q(1) \quad ; \quad x > x_l \quad , \tag{4.6.12a}$$

with

$$q(1) = \epsilon + \epsilon^2 - \epsilon^3 + O(\epsilon^4) \tag{4.6.12b}$$

$$x_l = 2\epsilon - 4\epsilon^2 + O(\epsilon^3) \quad . \tag{4.6.12c}$$

This leads to

$$\Psi(\epsilon) = \frac{4}{45}\,\epsilon\left[1 - \frac{76}{45}\,\epsilon\right] + O(\epsilon^3) \quad , \tag{4.6.13}$$

so $\Psi(\epsilon) \to 0$ as $T \to T_g^-$.

Also, as $T \to 0$, $q(x) \to 1$ so Ψ vanishes in this limit too. Thus, for the anisotropic solution, Ψ is expected to be zero at $T = T_g^-$ and $T = 0$ and non–zero in between, where fluctuations from perfect collinearity are present due to replica–symmetry breaking.

The expected behaviour of Ψ, in the thermodynamic limit, for both isotropic and anisotropic solutions is shown in Figures 4.1a and 4.1b respectively.

To test the above argument, a Monte Carlo simulation of Ψ was performed, using the original clock representation and a "heat–bath" Monte Carlo algorithm (Binder 1979, 1985), for which single spins are changed or not according to the probability,

$$p = \frac{1}{1 + \exp(\beta \Delta E)} \quad , \tag{4.6.14}$$

and ΔE is energy required for the spin to go from its initial to its final state.

The thermal averages were performed as Monte Carlo averages (time averages) over times ranging from t_0 to nt_0 , $n > 2$, where t_0 is the equilibration time as estimated by the method of Bhatt and Young (1985), that is, by looking for the coalescence of upper and lower bounds to the spin–glass susceptibility in equation (4.6.5c). The time–dependent four–spin–correlation function,

$$\chi_{SG}(t) = N^{-1} \Big[\sum_{ij} \sum_{\mu\nu} S_{i\mu}(t_0)\, S_{i\nu}(t + t_0)\, S_{j\mu}(t_0)\, S_{j\nu}(t + t_0) \Big]_{av} \tag{4.6.15a}$$

involving temporal correlations of a single system, averaged over $\{J_{ij}\}$ instances, gives an upper bound estimate,

$$\chi^{U}_{SG} = \chi_{SG}(t_0) \quad , \tag{4.6.15b}$$

approaching χ_{SG} from *above* if t_0 is not long enough. The lower bound involves correlations between the two identical but simultaneously evolving systems 1,2, given by

$$\chi^{L}_{SG} = (Nt_0)^{-1} \Big[\sum_{t=1}^{t_0} \sum_{ij} \sum_{\mu\nu} S^{1}_{i\mu}(t + t_0)\, S^{2}_{i\nu}(t + t_0)$$

$$\times\, S^{1}_{j\mu}(t + t_0)\, S^{2}_{j\nu}(t + t_0) \Big]_{av} \quad . \qquad (4.6.16)$$

Since the two sets of spins are initially uncorrelated, χ^{L}_{SG} approaches χ_{SG} from *below* if t_0 is shorter than the necessary equilibration time. Data was only accepted if χ^{U}_{SG} and χ^{L}_{SG} agreed within the errors. Typical errors were of order 10% .

Naturally, only finite–sized systems can be studied, but an analysis of several different sizes can reveal key features. The results for various sizes and temperatures are shown in Figure 4.2 . They do present finite–size effects, but are clearly in accord with the collinear/anisotropic prediction of a step function at $T_g = J/2$ from $\Psi = 0$ at T_g^- to $\Psi = 1/2$ for $T > T_g$ in the thermodynamic limit, $N \to \infty$. They unequivocally rule out isotropy. They also show that the fluctuations from perfect collinearity for temperatures $0 < T < T_g$, due to replica–symmetry breaking, are relatively small. However, these results are unable to detect the rise of Ψ beneath T_g^- as predicted by the Parisi theory, but this may be because the sizes studied are too small.

4.7. Conclusion

In this chapter, the special character of the 4–state clock spin glass was pointed out within the realm of p–state clock glasses. An alternative representation in terms of two identical Ising spin glasses has allowed one to use knowledge of the infinite–ranged Sherrington–Kirkpatrick model to analyse the 4–state clock glass. Within the replica–symmetric approximation, two distinct low–temperature solutions corresponding to the isotropic and collinear spin glasses were shown to provide the same value for the free energy. It was argued that the average spin–glass order is essentially collinear (two–fold symmetric) despite the four–fold symmetry of the hamiltonian. Fluctuation effects were predicted only for systems with pure states unrelated by symmetry. A Monte Carlo simulation was performed in support of the assertion of anisotropic spin–glass order.

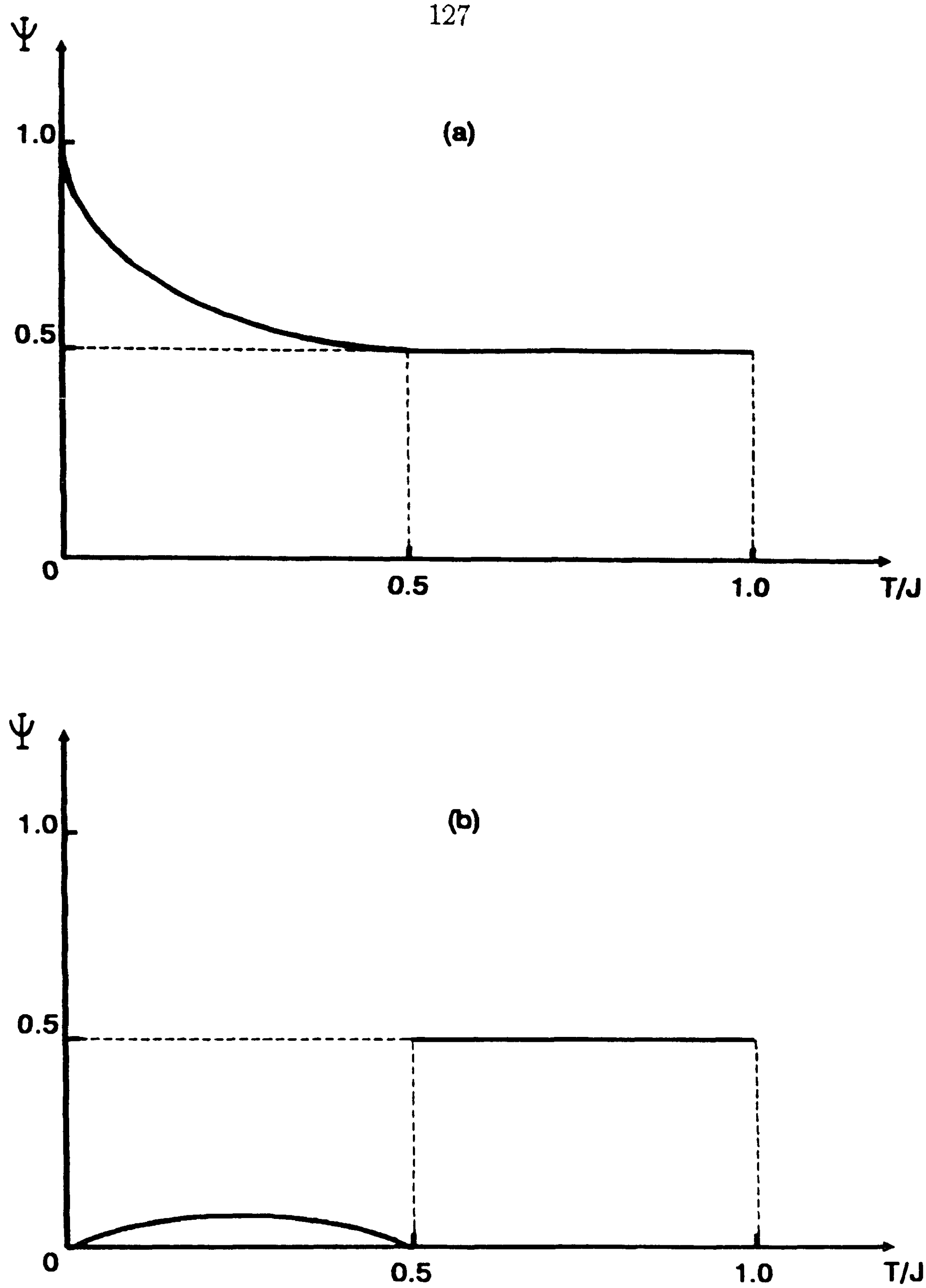

Figure 4.1: The expected behaviour of Ψ in the limit $N \to \infty$:

(a) if the low–temperature phase is isotropic;

(b) if the low–temperature phase is anisotropic (collinear). The rising of Ψ for temperatures $0 < T < T_g$, represents fluctuations from perfect collinearity as predicted from Parisi theory.

In both cases (a) and (b) the transition temperature is $T_g = \frac{J}{2}$.

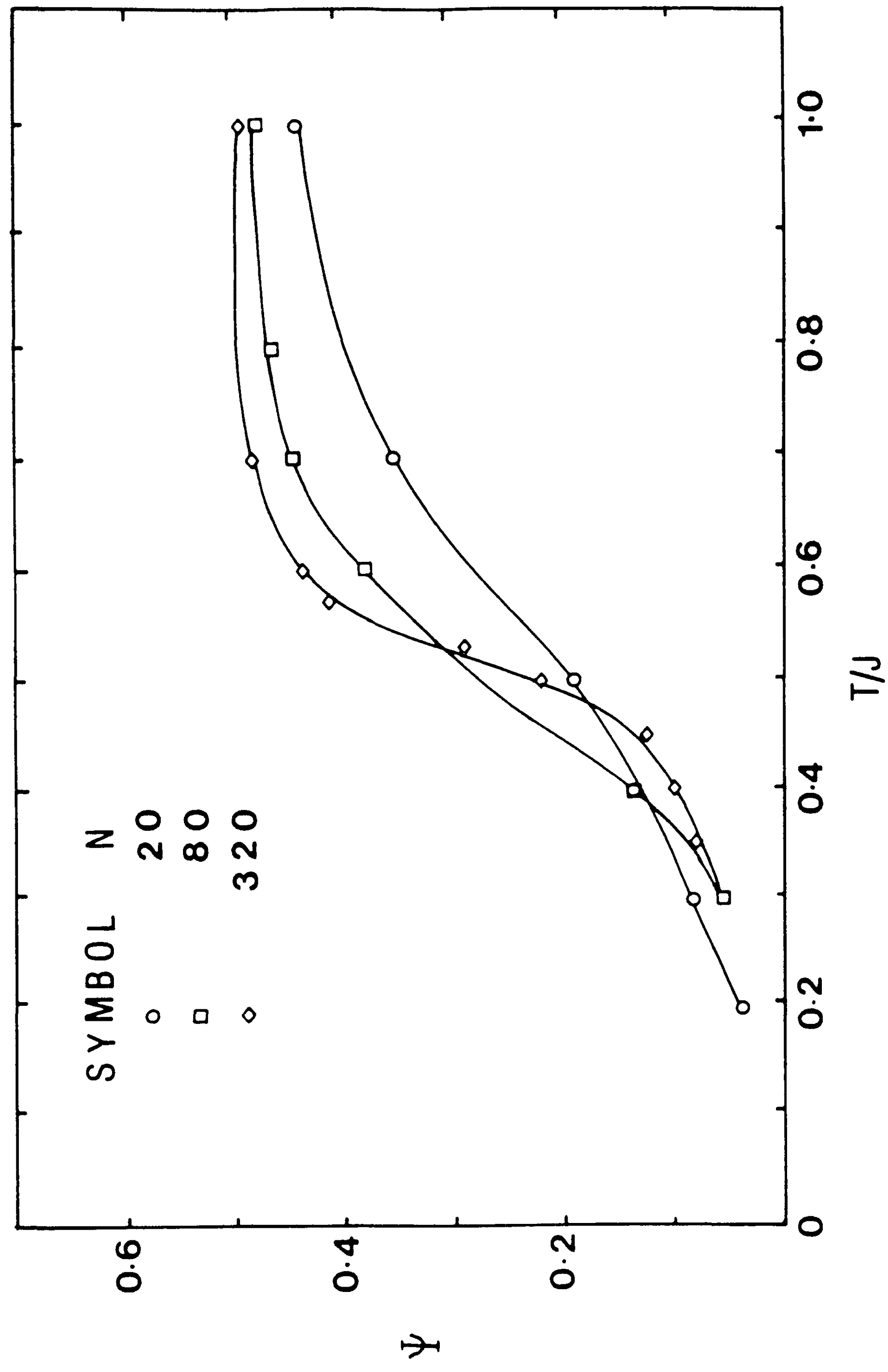

Figure 4.2: Plot of Ψ versus T/J for $N = 20, 80, 320$. The transition temperature is predicted as $T_g = \frac{J}{2}$. These results are clearly in accord with the anisotropic prediction in Figure 4.1 (b). They also show that the fluctuations from perfect collinearity for $0 < T < T_g$ are relatively small.

CHAPTER 5: INSTABILITIES OF CLOCK SPIN GLASSES IN A MAGNETIC FIELD

5.1. Introduction

In a ferromagnetic system the introduction of a magnetic field removes all singularities and consequently, the phase transition is destroyed. However, for spin–glass models (at least within mean–field theory), a magnetic field leads to interesting behaviour, with new phase transitions related to the onset of strong irreversibility, i.e. replica–symmetry breaking (de Almeida and Thouless 1978).

In real spin glasses the relaxation times become very large at low temperatures, suggesting irreversibility, and many experimental observations claim the existence of an Almeida–Thouless line (for a review see Binder and Young 1986). Nevertheless, neither the experimental measurements nor the numerical simulations were able to show a true divergence of relaxation times in three–dimensional spin glasses, being this a controversial issue at the present moment (Fisher and Huse 1986, Bray and Moore 1986, Villain 1986), as discussed in section 2.7 .

The investigation of the effects of a magnetic field in the infinite–range p–state clock spin glass is the main purpose of this chapter. The onset of replica–symmetry breaking for the two extremum cases, $p = 2$ (de Almeida and Thouless 1978) and $p = \infty$ (Gabay and Toulouse 1981, Cragg *et al.* 1982) appears in very different ways, defining two distinct universality classes. We show that the $p = 3$ case is peculiar, lying in a different universality class than the ones mentioned above, and that the results change under reflection of the magnetic field ($\underline{h} \to -\underline{h}$). The $p = 4$ case lies in the same class as $p = 2$ and, despite the four–fold symmetry of the spin variable, a small magnetic field induces the spin–glass order to two–fold symmetric, confirming the results of the previous chapter. Finally, we show that all $p \geq 5$ clock glasses are XY like.

5.2. The p–state clock spin glass in a magnetic field

Throughout this chapter we will work with the p–state clock spin glass as defined in equation (3.2.3), with a magnetic field in the x direction ($\underline{h} = h\hat{x}$). We assume that there is a possible rotation for which condition (3.3.11) is satisfied, that is, the crossed parameters are zero. The free energy per spin, in the thermodynamic limit, $N \to \infty$, is obtained by extremizing the functional $g(R^\alpha, Q_{xx}^{\alpha\beta}, Q_{yy}^{\alpha\beta})$ (see equations (3.3.6), (3.3.13) and (3.3.14)),

$$g(R^\alpha, Q_{xx}^{\alpha\beta}, Q_{yy}^{\alpha\beta}) = -\frac{n}{8}(\beta J)^2 + \frac{(\beta J)^2}{2}\sum_\alpha (R^\alpha)^2 + \frac{(\beta J)^2}{4}\sum_{\alpha\beta}{}' [(Q_{xx}^{\alpha\beta})^2 + (Q_{yy}^{\alpha\beta})^2]$$

$$- \ln \mathrm{Tr}_\alpha \exp\{H_{eff}\} \quad , \qquad (5.2.1a)$$

$$H_{eff} = (\beta J)^2 \sum_\alpha R^\alpha [(S_x^\alpha)^2 - 1/2] + \frac{(\beta J)^2}{2}\sum_{\alpha\beta}{}' (Q_{xx}^{\alpha\beta} S_x^\alpha S_x^\beta + Q_{yy}^{\alpha\beta} S_y^\alpha S_y^\beta)$$

$$+ \beta h \sum_\alpha S_x^\alpha \quad , \qquad (5.2.1b)$$

where,

$$R^\alpha = \langle (S_x^\alpha)^2 \rangle - 1/2 \quad , \qquad (5.2.2a)$$

$$Q_{xx}^{\alpha\beta} = \langle S_x^\alpha S_x^\beta \rangle \quad , \quad Q_{yy}^{\alpha\beta} = \langle S_y^\alpha S_y^\beta \rangle \quad ; \quad \alpha \neq \beta \quad . \qquad (5.2.2b)$$

For p=2 , our model reduces to the well-known Ising spin glass of Sherrington and Kirkpatrick (1975) for which R^{α} and $Q_{yy}^{\alpha\beta}$ are zero (see section 2.4). In the following discussion, we shall restrict ourselves to $p > 2$.

Within the replica-symmetric approximation, $R^{\alpha} = R$, $Q_{xx}^{\alpha\beta} = Q_{xx}$, $Q_{yy}^{\alpha\beta} = Q_{yy}$, any average in the replica space in the limit $n \to 0$, is related to a disorder-averaged product of thermodynamic averages (Kirkpatrick and Sherrington 1978),

$$\lim_{n\to 0}\Big\langle \underbrace{(S_x^{\alpha})^j \ldots (S_x^{\beta})^j}_{\substack{q\ \text{different}\\ \text{replicas}}} \underbrace{(S_y^{\gamma})^k \ldots (S_y^{\delta})^k}_{\substack{r\ \text{different}\\ \text{replicas}}} \underbrace{(S_x^{\rho})^g (S_y^{\rho})^m \ldots (S_x^{\sigma})^g (S_y^{\sigma})^m}_{\substack{t\ \text{different}\\ \text{replicas}}}\Big\rangle$$

$$= \left[\langle S_x^j \rangle_T^q \langle S_y^k \rangle_T^r \langle S_x^g S_y^m \rangle_T^t\right]_{av} \qquad \text{(all replica indices different)}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{du dv}{2\pi} e^{-u^2/2} e^{-v^2/2} \left[Z^{-1}\frac{\partial^j Z}{\partial a_x^j}\right]^q \left[Z^{-1}\frac{\partial^k Z}{\partial a_y^k}\right]^r \left[Z^{-1}\frac{\partial^{g+m} Z}{\partial a_x^g\, \partial a_y^m}\right]^t \qquad (5.2.3a)$$

where

$$Z = \mathrm{tr}\, \exp\,(bS_x^2 + a_x S_x + a_y S_y) \quad , \qquad (5.2.3b)$$

$$a_x = \beta J Q_{xx}^{\frac{1}{2}} u + \beta h \ , \ a_y = \beta J Q_{yy}^{\frac{1}{2}} v \ , \ b = \frac{(\beta J)^2}{2}(2R - Q_{xx} + Q_{yy}) \ . \qquad (5.2.3c)$$

As discussed in section 3.7 for the case of zero magnetic field, one question of the most importance concerns the stability of the replica-symmetric solution (de Almeida and Thouless 1978). As is well known, for the m-vector spin glasses

(Gabay and Toulouse 1981, Cragg *et al.* 1982), a critical line in the h–T plane occurs, below which replica symmetry is unstable. For small h, this line looks in general, like

$$\epsilon \sim h^{\psi} \quad ; \quad \epsilon = (T_g - T)/T_g \quad , \tag{5.2.4}$$

where T_g is the spin–glass critical temperature in zero field.

For two particular limits of the p–state clock glass the above behaviour is found but the critical exponent ψ falls into different universality classes:

a) p = 2 (Ising spin glass) (de Almeida and Thouless 1978): in this case, the critical line is a direct consequence of replica–symmetry breaking and $\psi = 2/3$.

b) p = ∞ (XY spin glass): the ordering of the transverse degrees of freedom takes place at a critical line for which $\psi = 2$ (Gabay and Toulouse 1981), and below this line replica symmetry is unstable (Cragg *et al.* 1982).

The main purpose of this chapter is to investigate how this critical line changes as we interpolate between these two limits.

In order to do this, we turn to the stability analysis in an Almeida–Thouless fashion by taking fluctuations around the replica–symmetric solutions ($R^{\alpha} = R + \omega^{\alpha}$; $Q^{\alpha\beta}_{xx} = Q_{xx} + \eta^{\alpha\beta}$; $Q^{\alpha\beta}_{yy} = Q_{yy} + \varphi^{\alpha\beta}$; $\alpha \neq \beta$). The derivation of Cragg *et al.* (1982) can be reproduced straightforwardly for the p–state clock glass to give (see Appendix C)

$$(1 - \chi^{(2)}_{xx} - \lambda)(1 - \chi^{(2)}_{yy} - \lambda) - (\chi^{(2)}_{xy})^2 = 0 \quad , \tag{5.2.5a}$$

where

$$\chi^{(2)}_{\mu\nu} = (\beta J)^2 \left[(\langle S_\mu S_\nu \rangle_T - \langle S_\mu \rangle_T \langle S_\nu \rangle_T)^2 \right]_{av} \quad (\mu,\nu = x,y) \quad . \tag{5.2.5b}$$

The "correlation functions" $\chi^{(2)}_{\mu\nu}$ are to be evaluated in the replica–symmetric approximation and stability requires all eigenvalues λ to be positive.

For large h, by solving equations (5.2.5), the softening to zero of λ is satisfied by small T and one gets (see Appendix D)

$$\frac{T}{J} = \frac{a_p}{(2\pi)^{\frac{1}{2}}} \exp(-h^2/2J^2) \qquad (T \sim 0) \quad , \qquad (5.2.6)$$

which is valid in general for any value of p . The coefficient a_p is a number depending only on the value of p , and typical values are listed in Table D.1 (Appendix D).

For small h however, different values of p lead to distinct behaviour with different values for the exponent ψ, and this will concern the analysis which follows.

5.3. p = 3

As discussed in section 3.2 , this case is identical to the 3–state Potts glass. As is well known, in the isotropic situation (h=0) , the absence of reflection symmetry in the spin variable plays a crucial role in this problem, changing radically the critical behaviour (Gross *et al.* 1985, Goldbart and Elderfield 1985, Nobre and Sherrington 1986). Special properties can also be noticed for $h \neq 0$.

For small h, by solving equations (B.14) (Appendix B), one gets the critical line associated with the transverse spin–glass freezing,

$$\epsilon = \pm\frac{1}{2}(4 \mp \alpha_\pm)\frac{|h|}{J} + \left[13 \mp \frac{11}{2}\alpha_\pm + \frac{7}{4}\alpha_\pm^2 - \frac{1}{2}\beta_\pm\right]\frac{h^2}{J^2} + O(h^3/J^3) \quad , \qquad (5.3.1)$$

at which,[1]

$$R = \frac{h}{J} + O(h^3/J^3) \quad , \tag{5.3.2a}$$

$$Q_{xx} = \alpha_\pm \frac{|h|}{J} + \beta_\pm \frac{h^2}{J^2} + O(h^3/J^3) \quad . \tag{5.3.2b}$$

In the equations above,

$$\alpha_\pm = \frac{1}{9}(2\sqrt{34} \pm 8) \quad ; \quad (\alpha_+ \approx 2.18 \ ; \ \alpha_- \approx 0.41) \quad , \tag{5.3.3a}$$

$$\beta_\pm = \frac{\pm 96 + 104\alpha_\pm \mp 222\alpha_\pm^2 + 53\alpha_\pm^3}{18\alpha_\pm \mp 16} \ ; \ (\beta_+ \approx -7.88 \ ; \ \beta_- \approx -0.57) \tag{5.3.3b}$$

and the upper (lower) signs refer to $h > 0$ $(h < 0)$.

In contrast to the m–vector spin glass, the results vary under the operation $\underline{h} \to -\underline{h}$ for small $|h|$ and in particular, the parallel spin–glass parameter Q_{xx} decreases,

$$Q_{xx}(h>0) \ > \ Q_{xx}(h<0) \quad , \tag{5.3.4}$$

---

[1]The quadrupolar parameter turns out to be a magnetization for $p = 3$ (Elderfield and Sherrington 1983a). Trivially one has,

$$R = [\langle S_x^2\rangle_T]_{av} - \frac{1}{2} = \left[\left\langle \cos^2\frac{2\pi k}{3} - \frac{1}{2}\right\rangle_T\right]_{av} = \frac{1}{2}\left[\left\langle\cos\frac{4\pi k}{3}\right\rangle_T\right]_{av} = \frac{1}{2}[\langle S_x\rangle_T]_{av} \quad ,$$

$$(k = 0,1,2).$$

whereas the transverse spin–glass–freezing temperature T_f ,

$$T_f = T_g \left\{ 1 \mp \frac{1}{2}(4 \mp \alpha_\pm)\frac{|h|}{J} \right.$$

$$\left. - \left[13 \mp \frac{11}{2}\alpha_\pm + \frac{7}{4}\alpha_\pm^2 - \frac{1}{2}\beta_\pm \right] \frac{h^2}{J^2} + \right\} + O(h^3/J^3) \quad , \qquad (5.3.5)$$

satisfies

$$T_f(h>0) < T_g < T_f(h<0) \quad . \qquad (5.3.6)$$

From equation (5.3.1) one gets that for $h \geq 0$, $\epsilon = 0$ only if $h = 0$. However, for $h \leq 0$, one gets two solutions with $\epsilon = 0$, namely,

$$\frac{|h|}{J} = 0 \quad , \quad \text{or} \quad \frac{|h|}{J} \approx 0.14 \quad , \qquad (5.3.7)$$

providing the small $|h|$ behaviour shown in Figure 5.1 .

Thus, the operation $\underline{h} \to -\underline{h}$ tends to weaken the parallel spin–glass ordering, while enhancing the perpendicular ordering. This is reflected in the change of sign of the parameter R and is a direct consequence of the absence of reflection symmetry in the spin variable. The fact that R is negative for $h < 0$, makes the transition in Q_{yy} more Ising like, in contrast to the usual transition found for the m–vectors.

For T just below T_f , equations (5.2.5) can be solved and the lowest eigenvalue is

$$\lambda = -\frac{1}{8 - \alpha_\pm^2} \left[2\alpha_\pm^2 \frac{Q_{yy}^2}{Q_{xx}} + (24 \pm 16\alpha_\pm - 15\alpha_\pm^2)Q_{yy}^2 \right] + \ldots \quad (h \neq 0) \qquad (5.3.8)$$

where Q_{yy} grows as

$$Q_{yy} = \frac{\epsilon}{2} \mp \frac{|h|}{J} + \frac{Q_{xx}}{4} \quad , \tag{5.3.9}$$

and again the upper (lower) signs refer to $h > 0$ $(h < 0)$. Then, replica symmetry becomes unstable at an even lower order than the usual m–vector glasses for which the instability occurs at order Q_{yy}^2 on λ . For the 3–state clock spin glass in a magnetic field, this instability occurs at order $\epsilon^2 J/|h|$, or using (5.3.1), at order ϵ . Such a lowering in the order of the instability has already been noticed for the case of a Potts glass in zero magnetic field (Elderfield and Sherrington 1983a).

For large $|h|$, the transverse spin–glass transition is given by (5.2.6) for both h positive and negative. This is expected physically, since for either $h > 0$ (XY–like transition) (Gabay and Toulouse 1981) or $h < 0$ (Ising–like transition) (de Almeida and Thouless 1978), the large $|h|$ behaviour is the same.

Therefore, the 3–state clock lies in a different universality class than the Ising and XY spin glasses, obeying an equation like (5.2.4) but with exponent $\psi = 1$. The absence of reflection symmetry on the spin variable plays a crucial role for small $|h|$, where the inversion of the field changes drastically the critical line, but becomes irrelevant for large $|h|$. The critical lines for both $h > 0$ and $h < 0$ are shown in Figure 5.1 ; they signal the transverse spin–glass ordering and below them, replica symmetry is broken.

5.4. p = 4

This case has been extensively discussed in the previous chapter. It was shown that this model is essentially collinear, in the sense that an infinitesimal field suffices to orient all the spins along the same axis (that of the field). Therefore, one expects the replica-symmetry breaking in the presence of a magnetic field to be Ising like, i.e. an Almeida–Thouless line should be obtained.

Introducing the Ising variables as defined by (4.2.11), the hamiltonian in equation (3.2.3) with a magnetic field in the x direction ($\underline{h} = h\hat{x}$), may be re-written as

$$H = -\sum_{(ij)} \tilde{J}_{ij}\,(\tau_i\tau_j + \sigma_i\sigma_j) - \tilde{h}\sum_i(\tau_i + \sigma_i) \quad ; \tag{5.4.1a}$$

$$\{\tilde{J}_{ij}\} = \frac{1}{2}\{J_{ij}\} \quad ; \quad \tilde{h} = \frac{h}{2} \quad . \tag{5.4.1b}$$

Hence, the 4–state clock model is equivalent to two independent Ising models, each with exchange interactions and magnetic field re–scaled by a factor of one half with respect to those of the original clock model.

The onset of replica-symmetry breaking can be obtained within this representation, for which an Almeida–Thouless line follows for each of the independent Ising models. Alternatively, in the original clock representation (equation (3.2.3)), equations (5.2.5) can be solved with the "correlation functions" $\chi^{(2)}_{\mu\nu}$ evaluated in the replica-symmetric approximation, by making use of equations (4.3.4),

$$2R = Q_{xx} \quad ; \quad Q_{yy} = 0 \quad . \tag{5.4.2}$$

One gets an Almeida–Thouless line, taking into account the proper re–scalings,

$$\epsilon^3 = 6\left[\frac{h}{J}\right]^2 \quad ; \quad (\text{h small}) \quad , \tag{5.4.3a}$$

$$\frac{T}{J} = \frac{2}{3}(2\pi)^{-\frac{1}{2}}\, e^{-h^2/2J^2} \quad ; \quad (\text{h large}) \quad . \tag{5.4.3b}$$

The Gabay–Toulouse type of behaviour, where the ordering of the transverse degrees of freedom takes place already in the replica–symmetric space, can not occur for this case as a consequence of equation (5.4.2). Despite the four–fold symmetry of the spin variable, a magnetic field sets the spin–glass order to two–fold symmetric, in further support to the results obtained in the last chapter.

5.5. $\underline{p \geq 5}$

For the isotropic case $(h = 0)$, $p_c = 5$ is the clock dimension at and above which, the effect of the absence of reflection symmetry in the spin variable becomes irrelevant and the critical behaviour is XY like (Nobre and Sherrington 1986). The same happens for the case $h \neq 0$, where corrections due to this effect appear only as higher–order terms in the perturbation expansion. Solving equations (B.14) (Appendix B) to leading order, one gets the critical line associated with the transverse spin–glass freezing,

$$T_f = T_g\left[1 - \frac{7}{16}\left[\frac{h}{J}\right]^2\right] \quad , \tag{5.5.1a}$$

close to which,

$$R = \frac{1}{4}\left[\frac{h}{J}\right]^2 \quad ; \quad Q_{xx} = \frac{1}{2}\frac{|h|}{J} \quad . \tag{5.5.1b}$$

The stability analysis can now be done by solving equations (5.2.5); for T just below T_f and small h , the lowest eigenvalue is given by

$$\lambda = -3Q_{yy}^2 - \frac{1}{8}\frac{|h|}{J}Q_{yy}^2 + O(h^2Q_{yy}^2) \quad , \tag{5.5.2}$$

which is negative, signaling instability. For large h , the onset of replica–symmetry breaking is given by equation (5.2.6). Therefore, the clock glasses for $p \geq 5$ all lie in the same universality class as the XY spin glass (Gabay and Toulouse 1981, Cragg *et al.* 1982).

5.6. Conclusion

By studying the p–state clock spin glass in a magnetic field we have found that the $p = 3$ case (3–state Potts) is very peculiar. The onset of replica–symmetry breaking is of the Gabay–Toulouse type but with a different exponent. The results depend on the sign of the magnetic field and in particular, $h > 0$ $(h < 0)$ enhances the parallel (perpendicular) spin–glass ordering. For $p = 4$, the critical behaviour is Ising like, despite the four–fold symmetry of the spin variable. An Almeida–Thouless type of instability results in this case, in further support to the results obtained in Chapter 4 . For $p \geq 5$, the effects due to the absence of reflection symmetry in the spin variable appear only as higher–order terms in a perturbation expansion, and the dominant behaviour is XY like.

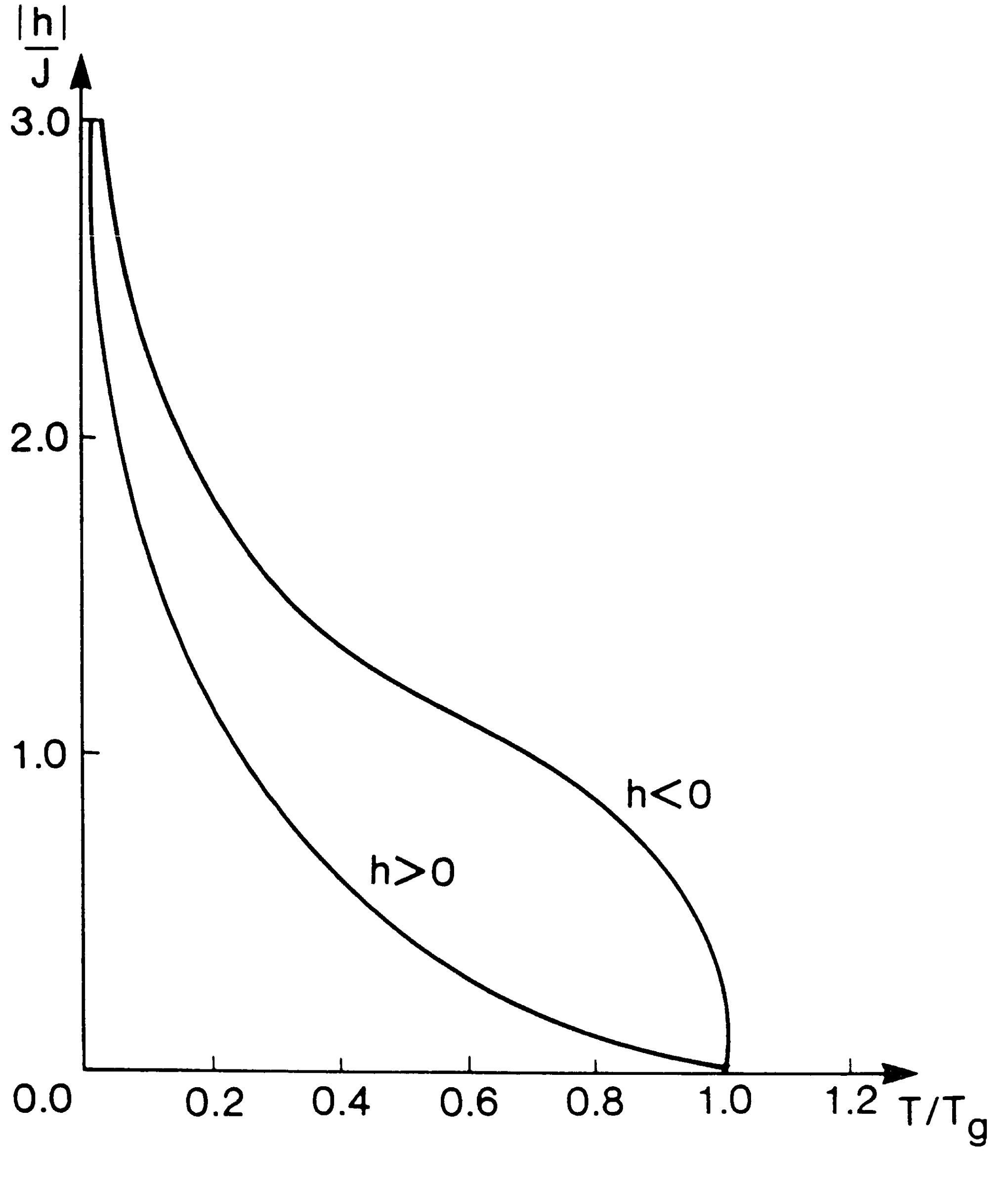

Figure 5.1: Phase diagram of the 3-state clock spin glass in a magnetic field h . The lower line is for $h > 0$ and its shape is much like the Almeida–Thouless line for the Sherrington–Kirkpatrick model (see Figure 2.8). The upper line is for $h < 0$, and for small $|h|$ one has $T_f > T_g$. In either case, the transition is of the Gabay–Toulouse type, signaling the transverse spin–glass ordering; below each line, replica symmetry is unstable.

CHAPTER 6: THE XY SPIN GLASS IN A FOUR–FOLD ANISOTROPY FIELD

6.1. Introduction

As discussed in section 3.3 , as the temperature is lowered, most clock spin glasses ($p \neq 2,4$) in zero magnetic field, go through a phase transition from a paramagnetic to an isotropic spin–glass state at a given temperature T_g. By further decreasing of the temperature, a second instability becomes possible, signaled by the onset of a quadrupolar parameter. This would occur between the isotropic and an anisotropic spin–glass state at a temperature T_a ($T_a < T_g$). The investigation of such an instability is a difficult task, since below T_g one needs to deal with an infinite number of order parameters, which are not necessarily small near T_a , and an analytical approach through series expansions is not valid anymore. Computer simulations, probably the most appropriate tool for that, become very difficult at low temperatures due to long equilibration times. Although there is no rigorous proof, it is a current belief that $T_a = 0$, for $h = 0$, for most spin glasses (Toulouse and Gabay 1981).

In the last two chapters we have argued that the 4–state clock glass is exceptional regarding this matter: one has that $T_a = T_g$ and the low–temperature phase is uniaxial with small fluctuations occurring only due to replica–symmetry breaking. Therefore, it is interesting to investigate the role of a four–fold anisotropy field on the XY model and to look at how the glass phase behaves as one changes the strength of this anisotropy. This is the main purpose of this chapter. We show that the normal four–fold symmetric spin–glass phase occurs except in the limit of infinite anisotropy.

6.2. The XY spin glass in a four–fold anisotropy field

Let us consider the model defined by the hamiltonian,

$$H = -\sum_{(ij)} J_{ij} \cos(\theta_i - \theta_j) - D \sum_i \cos 4\theta_i \quad , \qquad (6.2.1)$$

where the θ_i are continuous angles varying from 0 to 2π and the $\{J_{ij}\}$ obey the same probability distribution as in equation (3.2.4). Our analysis will be restricted to $D \geq 0$ and clearly, $D=0$ and $D=\infty$ correspond respectively to the XY and 4–state clock spin–glass limits.

Standard calculations lead to a free–energy functional,

$$g(R^\alpha, Q_{xx}^{\alpha\beta}, Q_{yy}^{\alpha\beta}) = -\frac{n}{8}(\beta J)^2 + \frac{(\beta J)^2}{2} \sum_\alpha (R^\alpha)^2 + \frac{(\beta J)^2}{4} \sum_{\alpha\beta}{}' [(Q_{xx}^{\alpha\beta})^2 + (Q_{yy}^{\alpha\beta})^2]$$

$$- \ln \mathrm{Tr}_\alpha \exp\{H_{eff}\} \quad , \qquad (6.2.2a)$$

$$H_{eff} = (\beta J)^2 \sum_\alpha R^\alpha [(S_x^\alpha)^2 - 1/2] + \frac{(\beta J)^2}{2} \sum_{\alpha\beta}{}' (Q_{xx}^{\alpha\beta} S_x^\alpha S_x^\beta + Q_{yy}^{\alpha\beta} S_y^\alpha S_y^\beta)$$

$$+ \beta D \sum_\alpha \cos 4\theta^\alpha \quad , \qquad (6.2.2b)$$

and again we are assuming that condition (3.3.11) is valid.

Making use of the traces evaluated in Appendix E, the above free–energy functional can be expanded in a power series,

$$g = -ng_0 - (\beta J)^2 \left\{ a_2 \sum_\alpha (R^\alpha)^2 + b_2 \sum_{\alpha\beta}{}' \sum_\mu (Q_{\mu\mu}^{\alpha\beta})^2 \right\} + \ldots \quad \mu = x,y \qquad (6.2.3)$$

with

$$g_0 = \frac{(\beta J)^2}{8} + \ln[I_0(\beta D)] \quad , \tag{6.2.4a}$$

$$a_2 = \frac{1}{2}\left[\frac{(\beta J)^2}{8}(1+\Delta_1) - 1\right] \quad , \tag{6.2.4b}$$

$$b_2 = \frac{1}{4}\left[\frac{(\beta J)^2}{4} - 1\right] \quad , \tag{6.2.4c}$$

where

$$\Delta_1 = \frac{I_1(\beta D)}{I_0(\beta D)} \quad , \tag{6.2.4d}$$

and the $I_k(\beta D)$ denote modified Bessel functions of the first kind of order k. For D large we can expand the Bessel functions,

$$\Delta_1 = 1 - \frac{1}{2\beta D} - \frac{1}{8(\beta D)^2} + O((\beta D)^{-3}) \quad . \tag{6.2.5}$$

Thus, for D finite we always have $\Delta_1 < 1$ and the quadrupolar parameter critical temperature, associated with $a_2 = 0$, is always lower than the Q–ordering temperature $T_g = J/2$,[1] associated with $b_2 = 0$, resulting in four–fold symmetric spin–glass order for T just below T_g. However, as T is lowered further, a

[1]Note that T_g is unrelated to D through the corresponding independence of b_2. As shown in Appendix E, this is due to the independence of D of the ratio,

$$\frac{\int_0^{2\pi} d\theta \cos^2\theta \exp(\beta D \cos 4\theta)}{\int_0^{2\pi} d\theta \exp(\beta D \cos 4\theta)} \quad .$$

second transition becomes possible, at a temperature $T_a(D)$ $(T_a(D) \lesssim T_g)$. If the renormalizing effects of the four–fold symmetric spin–glass ordering were ignored, a second transition to a partially collinear state would occur at the lower temperature where $a_2=0$. However, when such effects are included, modifications result, lowering (or removing) the second transition temperature (Toulouse and Gabay 1981).

In order to investigate whether a transition from four–fold to two–fold–symmetric order occurs, let us define

$$Q^{\alpha\beta} = \frac{Q^{\alpha\beta}_{xx} + Q^{\alpha\beta}_{yy}}{2} \quad ; \quad Z^{\alpha\beta} = \frac{Q^{\alpha\beta}_{xx} - Q^{\alpha\beta}_{yy}}{2} \quad (\alpha \neq \beta) \quad , \qquad (6.2.6)$$

by means of which, equations (6.2.2) may be re–expressed as

$$g(R^{\alpha}, Q^{\alpha\beta}, Z^{\alpha\beta}) = -\frac{n}{8}(\beta J)^2 + \frac{(\beta J)^2}{2}\sum_{\alpha}(R^{\alpha})^2 + \frac{(\beta J)^2}{2}\sum_{\alpha\beta}{}'(Q^{\alpha\beta})^2$$

$$+ \frac{(\beta J)^2}{2}\sum_{\alpha\beta}{}'(Z^{\alpha\beta})^2 - \ln \mathrm{Tr}_{\alpha} \exp\{H_{eff}\} \quad , \qquad (6.2.7a)$$

$$H_{eff} = (\beta J)^2 \sum_{\alpha} R^{\alpha}[(S^{\alpha}_x)^2 - 1/2] + \frac{(\beta J)^2}{2}\sum_{\alpha\beta}{}' Q^{\alpha\beta}(S^{\alpha}_x S^{\beta}_x + S^{\alpha}_y S^{\beta}_y)$$

$$+ \frac{(\beta J)^2}{2}\sum_{\alpha\beta}{}' Z^{\alpha\beta}(S^{\alpha}_x S^{\beta}_x - S^{\alpha}_y S^{\beta}_y) + \beta D \sum_{\alpha} \cos 4\theta^{\alpha} \quad . \qquad (6.2.7b)$$

If the transition at T_a is of the continuous type, then for T just below T_a both R and Z are small. We can then, expand the free–energy functional,

$$g(R^{\alpha}, Q^{\alpha\beta}, Z^{\alpha\beta}) = g(0, Q^{\alpha\beta}, 0) + \frac{(\beta J)^2}{2}\left[\sum_{\alpha\beta} S_{RR}^{\alpha\beta}\Big|_{R=Z=0} R^{\alpha}R^{\beta}\right.$$

$$+ 2\sum_{(\alpha\beta)\gamma} S_{ZR}^{(\alpha\beta)\gamma}\Big|_{R=Z=0} Z^{\alpha\beta}R^{\gamma}$$

$$\left. + \sum_{(\alpha\beta)(\gamma\delta)} S_{ZZ}^{(\alpha\beta)(\gamma\delta)}\Big|_{R=Z=0} Z^{\alpha\beta}Z^{\gamma\delta}\right] + \dots \quad . \tag{6.2.8}$$

The stability matrix (de Almeida and Thouless 1978) $\underline{S}$ has dimension $\frac{1}{2}n(n+1)$ by $\frac{1}{2}n(n+1)$ and elements,

$$S_{RR}^{\alpha\beta} = \frac{\partial^2 g}{\partial R^{\alpha}\partial R^{\beta}} = \delta^{\alpha\beta} - (\beta J)^2 \left[\left\langle (S_x^{\alpha})^2 (S_x^{\beta})^2 \right\rangle - \left\langle (S_x^{\alpha})^2 \right\rangle \left\langle (S_x^{\beta})^2 \right\rangle\right] \quad , \tag{6.2.9a}$$

$$S_{ZR}^{(\alpha\beta)\gamma} = \frac{\partial^2 g}{\partial Z^{\alpha\beta}\partial R^{\gamma}} = -(\beta J)^2 \left[\left\langle (S_x^{\alpha} S_x^{\beta} - S_y^{\alpha} S_y^{\beta})(S_x^{\gamma})^2 \right\rangle\right.$$

$$\left. - \left\langle S_x^{\alpha} S_x^{\beta} - S_y^{\alpha} S_y^{\beta} \right\rangle \left\langle (S_x^{\gamma})^2 \right\rangle\right] \quad , \tag{6.2.9b}$$

$$S_{ZZ}^{(\alpha\beta)(\gamma\delta)} = \frac{\partial^2 g}{\partial Z^{\alpha\beta}\partial Z^{\gamma\delta}} = 2\delta^{(\alpha\beta)(\gamma\delta)} - (\beta J)^2 \left[\left\langle (S_x^{\alpha} S_x^{\beta} - S_y^{\alpha} S_y^{\beta})\right.\right.$$

$$\left. \times (S_x^{\gamma} S_x^{\delta} - S_y^{\gamma} S_y^{\delta}) \right\rangle - \left\langle S_x^{\alpha} S_x^{\beta} - S_y^{\alpha} S_y^{\beta} \right\rangle \left\langle S_x^{\gamma} S_x^{\delta} - S_y^{\gamma} S_y^{\delta} \right\rangle\Big] \quad . \tag{6.2.9c}$$

Within the replica–symmetric approximation, continuous phase transitions are signaled by the softening to zero of an eigenvalue of the replica–symmetric eigenfunction of $\underline{S}$ in the limit $n \to 0$.

The replica–symmetric eigenfunctions of $\underline{S}$ give, for $n=0$, the eigenvalues

$$\lambda_{\pm} = -a_2' - 2b_2' \pm \left[(a_2' - 2b_2')^2 - 2c_2'^2\right]^{\frac{1}{2}} \quad , \tag{6.2.10}$$

where

$$a_2' = \frac{1}{2}\left\{(\beta J)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{du dv}{2\pi} e^{-u^2/2} e^{-v^2/2} \left[\tilde{G}_{40} - (\tilde{G}_{20})^2 \right] - 1 \right\} \quad , \qquad (6.2.11a)$$

$$b_2' = \frac{1}{2}\left\{ \frac{(\beta J)^2}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{du dv}{2\pi} e^{-u^2/2} e^{-v^2/2} \left[(\tilde{G}_{20})^2 - 2(\tilde{G}_{11})^2 + (\tilde{G}_{02})^2 \right.\right.$$

$$- 4\tilde{G}_{20}(\tilde{G}_{10})^2 + 8\tilde{G}_{11}\tilde{G}_{10}\tilde{G}_{01} - 4\tilde{G}_{02}(\tilde{G}_{01})^2$$

$$\left.\left. + 3(\tilde{G}_{10})^4 - 6(\tilde{G}_{01})^2(\tilde{G}_{10})^2 + 3(\tilde{G}_{01})^4 \right] - 1 \right\} \quad , \qquad (6.2.11b)$$

$$c_2' = (\beta J)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{du dv}{2\pi} e^{-u^2/2} e^{-v^2/2} \left[\tilde{G}_{30}\tilde{G}_{10} - \tilde{G}_{01}\tilde{G}_{21} \right.$$

$$\left. - (\tilde{G}_{10})^2 \tilde{G}_{20} + (\tilde{G}_{01})^2 \tilde{G}_{20} \right] \quad . \qquad (6.2.11c)$$

and

$$\tilde{G}_{km} = G_{km}/G_{00} \quad , \qquad (6.2.11d)$$

$$G_{km} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \cos^k\theta \sin^m\theta \exp(\beta J Q^{\frac{1}{2}} u\cos\theta + \beta J Q^{\frac{1}{2}} v\sin\theta + \beta D\cos 4\theta) \quad , (6.2.11e)$$

with the mean spin–glass parameter satisfying the self–consistency equation,

$$Q = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dudv}{2\pi} e^{-u^2/2} e^{-v^2/2} \left[(\tilde{G}_{10})^2 + (\tilde{G}_{01})^2 \right] \quad . \tag{6.2.11f}$$

The coefficients in equations (6.2.11) are related to the free–energy functional in its replica–symmetric form by

$$a_2' = -\frac{1}{2(\beta J)^2} \lim_{n\to 0} \frac{1}{n} \frac{\partial^2 g(R,Q,Z)}{\partial R^2} \Big|_{R=Z=0} \quad , \tag{6.2.12a}$$

$$b_2' = \frac{1}{2(\beta J)^2} \lim_{n\to 0} \frac{1}{n} \frac{\partial^2 g(R,Q,Z)}{\partial Z^2} \Big|_{R=Z=0} \quad , \tag{6.2.12b}$$

$$c_2' = \frac{1}{(\beta J)^2} \lim_{n\to 0} \frac{1}{n} \frac{\partial^2 g(R,Q,Z)}{\partial R \partial Z} \Big|_{R=Z=0} \quad , \tag{6.2.12c}$$

and for $Q=0$,

$$a_2' \,\Big|_{Q=0} = a_2 \quad ; \quad b_2' \,\Big|_{Q=0} = 2b_2 \quad ; \quad c_2' \,\Big|_{Q=0} = 0 \quad . \tag{6.2.13}$$

Within this replica–symmetric approach, the onset of anisotropy is signaled by one of the eigenvalues $\lambda_\pm$ becoming negative. Instead of analysing $\lambda_\pm$ directly, it is much easier if one looks at the quantities,

$$\gamma = \frac{1}{2}(\lambda_+ + \lambda_-) = -a_2' - 2b_2' \quad , \tag{6.2.14a}$$

$$\delta = \frac{1}{2} \lambda_+ \lambda_- = 4a_2' b_2' + c_2'^2 \quad . \tag{6.2.14b}$$

The stability condition for the isotropic spin–glass phase, i.e. $\lambda_+ , \lambda_- > 0$, is now translated into $\gamma , \delta > 0$.

Near the transition paramagnetic/four-fold-symmetric spin glass, Q is small and equations (6.2.11) can be expanded in powers of ϵ (ϵ is given by (3.6.11)),

$$a_2' = \frac{1}{4}(\Delta_1 - 1) + \frac{1}{2}(1 + \Delta_1)\epsilon + \frac{1}{8}(5 + 4\Delta_1 - \Delta_1^2)\epsilon^2 + O(\epsilon^3) \quad , \tag{6.2.15a}$$

$$b_2' = -\epsilon - \frac{1}{2}\Delta_1 \left[3 + \frac{1}{3}\Delta_1\right]\epsilon^2 - \alpha\epsilon^3 + O(\epsilon^4) \quad , \tag{6.2.15b}$$

$$c_2' = (1 + \Delta_1)\epsilon + \frac{1}{4}(1 + \Delta_1)\left[5 + \frac{1}{3}\Delta_1^2\right]\epsilon^2 + O(\epsilon^3) \quad . \tag{6.2.15c}$$

The third-order coefficient in (6.2.15b), $\alpha = \alpha(D)$, was not evaluated explicitly, but it is required to provide a finite value for the 4-state clock ($D \to \infty$), such that the series expansion for b_2' converges in this limit, or in other words, it should not present a $(1 - \Delta_1)^{-1}$ factor.

The quantities γ and δ can be computed within this expansion,

$$\gamma = \frac{1}{4}(1 - \Delta_1) + \frac{1}{2}(3 - \Delta_1)\epsilon + \frac{1}{8}\left[-5 + 20\Delta_1 + \frac{11}{3}\Delta_1^2\right]\epsilon^2 + O(\epsilon^3) , \tag{6.2.16a}$$

$$\begin{aligned}\delta = (1 - \Delta_1)\Big\{ & \epsilon - \frac{1}{6}(6 - 3\Delta_1 - \Delta_1^2)\epsilon^2 \\ & + \left[\alpha - \frac{1}{6}\Delta_1^2(1 + \Delta_1)\right]\epsilon^3 \Big\} + O(\epsilon^4) \quad .\end{aligned} \tag{6.2.16b}$$

As can be seen from equations (6.2.16), one gets $\gamma , \delta > 0$ for any finite D $(\Delta_1 < 1)$, but up to $O(\epsilon^3)$, δ goes to zero in the 4-state clock limit $(D \to \infty , \Delta_1 \to 1)$. Therefore, the four-fold-symmetric spin-glass phase is stable except in the clock limit.

The above result has been checked numerically (Nobre *et al.* 1989) and no instability of the four–fold–symmetric state has been found at non–zero temperatures and finite D . One concludes that the clock limit is very special, because only in this case the model can be described in terms of two independent Ising models. For any finite D , such a decomposition is not possible and the system then prefers the four–fold–symmetric state.

6.3. Conclusion

We have studied an XY spin glass with a four–fold anisotropy. In searching for an instability of the four–fold–symmetric spin–glass state, we have analysed the fluctuations of the quadrupolar, and the difference between the spin–glass parameters in the x and y directions, within the replica–symmetry ansatz. We have shown that the four–fold–symmetric spin–glass phase is stable, except in the limit of infinite anisotropy (the four–state clock limit).

CHAPTER 7: THE PARISI FUNCTION FOR CLOCK SPIN GLASSES

7.1. Introduction

The Parisi replica–symmetry–breaking scheme (Parisi 1979, 1980a, 1980b, 1980c, 1980d, 1980e) as presented in section 2.4 , is believed to be the exact solution for the Sherrington–Kirkpatrick model (Sherrington and Kirkpatrick 1975). Although being a marginally stable solution (Thouless *et al.* 1980, De Dominicis and Kondor 1983), it provides results which are in very good agreement with numerical simulations. Its generalization to the infinite–range m–vector spin glasses is very similar to the SK model, i.e. the Parisi function is continuous and monotonically increasing (Gabay *et al.* 1982, Elderfield and Sherrington 1982). All models studied so far, in which the spin variables are symmetric under reflection $(\underline{S}_i \to -\underline{S}_i)$, show such "conventional" behaviour in the Parisi function.

However, the same procedure when applied to systems in which the spins do not present symmetry under reflection, like quadrupolar (Goldbart and Sherrington 1985) and Potts glasses (Erzan and Lage 1983, Elderfield and Sherrington 1983a, 1983b, 1983c, Goldbart and Elderfield 1985, Gross *et al.* 1985), turns out to provide rather surprising behaviour. Discontinuities in the Parisi function as well as first–order phase transitions are observed. Whether these effects are peculiar to Potts and quadrupolar glasses only, or if they also happen on other systems, is not known.

An interesting problem is, therefore, to investigate the Parisi solution for the p–state clock spin–glass model as defined in Chapter 3 , for which the spins present (do not present) reflection symmetry for every even (odd) value of p . An analysis of the behaviour of the Parisi function as p varies, is the main purpose of this chapter. In section 7.2 we present the replica–symmetry–breaking theory for the clock spin glasses. We show that although the $p = 3$ case has the peculiar behaviour already observed for Potts models, the absence of reflection symmetry is qualitatively irrelevant for all other odd–state clock glasses. In section 7.3 we analyse the solution for $p = 3$ in detail. In section 7.4 we deal with the anisotropic solution for the 4–state clock.

7.2. Replica–symmetry breaking for clock spin glasses

In this section we apply the Parisi scheme to the p–state clock spin glass in zero magnetic field. We restrict analysis of the ordered phase to ϵ small (see equation (3.6.11)) using a Landau expansion. First, we deal with the isotropic solution (3.3.8) for all p . The anisotropic solution for $p = 4$ will be treated later in section 7.4 .

Near the phase transition ($Q^{\alpha\beta}$ small) the free–energy functional within the isotropic conditions, can be written as a power series (see section 3.4). The Parisi parametrization may easily be implemented by using the rules (2.4.17) → (2.4.22) in expansion (3.4.4). In doing that, the free energy in (3.3.6) will be expressed as

$$\beta f[Q] = -\bar{g}_0 + \frac{1}{2}\bar{A}_2 \langle Q^2 \rangle + \frac{1}{3}\bar{A}_3 \langle Q^3 \rangle$$

$$-\frac{1}{3}\bar{B}_3 \int_0^1 dx \left[xQ^3(x) + 3\,Q(x) \int_0^x dy\; Q^2(y) \right] + \frac{1}{12}\bar{A}_4 \langle Q^4 \rangle$$

$$-\frac{1}{12}\bar{C}_4 \left\{ \langle Q^4 \rangle - 2\,\langle Q^2 \rangle^2 - \int_0^1 dx \int_0^x dy\; [\, Q^2(x) - Q^2(y)\,]^2 \right\}$$

$$-\frac{1}{12}\bar{D}_4 \left\{ 2\,\langle Q\rangle\,\langle Q^3\rangle + \int_0^1 dx\; Q^2(x) \int_0^x dy\,[\,Q(x)-Q(y)\,]^2 \right\}$$

$$-\frac{1}{12}\bar{B}_4 \left\{ \langle Q^2\rangle^2 - 4\,\langle Q\rangle^2\,\langle Q^2\rangle - 4\,\langle Q\rangle \int_0^1 dx\; Q(x) \int_0^x dy\,[\,Q(x)-Q(y)\,]^2 \right.$$

$$\left. - \int_0^1 dx \int_0^x dy \int_0^x dz\,[\,Q(x)-Q(y)\,]^2\,[\,Q(x)-Q(z)\,]^2 \right\} + \ldots \quad , \qquad (7.2.1)$$

where the coefficients are given in equations (3.4.5) . In equation (7.2.1) the Parisi function $Q(x)$ is such that

$$Q_{\mu\nu}(x) = Q(x)\,\delta_{\mu\nu} \quad ; \quad \mu,\nu = x,y \quad , \qquad (7.2.2)$$

and

$$\langle Q^m\rangle = \int_0^1 dt\; Q^m(t) \quad . \qquad (7.2.3)$$

In order to find the shape of $Q(x)$, one needs to solve the extremal equation,

$$\frac{\delta(\beta f\,[Q])}{\delta Q(t\,)} = 0 \quad , \qquad (7.2.4a)$$

or in other words,

$$\bar{A}_2\, Q(t) + \bar{A}_3\, Q^2(t) - \frac{1}{3}\bar{B}_3 \left\{ 3tQ^2(t) + 3\int_0^t dy\, Q^2(y) + 6\, Q(t) \int_t^1 dy\, Q(y) \right\}$$

$$+ \frac{1}{3}\bar{A}_4\, Q^3(t) + \frac{1}{3}\bar{C}_4 \langle Q^2\rangle\, Q(t) - \frac{1}{12}\bar{D}_4 \Big\{ 2\langle Q^3\rangle + 6\langle Q\rangle\, Q^2(t) + 4tQ^3(t)$$

$$- 6\, Q^2(t) \int_0^t dy\, Q(y) - 2\int_t^1 dy\, Q^3(y) + 2\, Q(t)\langle Q^2\rangle \Big\}$$

$$- \frac{1}{12}\bar{B}_4 \Big\{ 4\langle Q^2\rangle\, Q(t) - 16\langle Q\rangle^2 Q(t) - 12\langle Q^2\rangle\langle Q\rangle - 12t\langle Q\rangle\, Q^2(t) - 4t^2Q^3(t)$$

$$- 4\int_0^1 dy\, Q(y) \int_0^y dz\, [\, Q(y) - Q(z)\,]^2 + 12\langle Q\rangle \int_t^1 dy\, Q^2(y)$$

$$+ 24\langle Q\rangle\, Q(t) \int_0^t dy\, Q(y) + 12tQ^2(t) \int_0^t dy\, Q(y) + 4\int_t^1 dy\, yQ^3(y)$$

$$- 4tQ(t) \int_0^t dy\, Q^2(y) - 4\, Q(t) \int_t^1 dy\, yQ^2(y) - 8\, Q(t) \int_0^t dy\, Q(y) \int_0^t dz\, Q(z)$$

$$- 8\int_t^1 dy\, Q^2(y) \int_0^y dz\, Q(z) + 4\int_0^t dy\, Q(y) \int_0^t dz\, Q^2(z)$$

$$+ 4\int_t^1 dy\, Q(y) \int_0^y dz\, Q^2(z) + 8\, Q(t) \int_t^1 dy\, Q(y) \int_0^y dz\, Q(z)$$

$$- 4\, Q(t) \int_t^1 dy \int_0^y dz\, Q^2(z) \Big\} + \ldots = 0 \quad . \qquad (7.2.4b)$$

We can now take the derivative of (7.2.4a) and write it in the form,

$$\frac{d}{dt}\frac{\delta(\beta f\,[Q])}{\delta Q(t)} = Q'(t)\, \Phi[Q] = 0 \quad , \qquad (7.2.5)$$

where

$$\Phi[Q] = \bar{A}_2 + 2\bar{A}_3\, Q(t) - 2\bar{B}_3 \left\{ tQ(t) + \int_t^1 dy\, Q(y) \right\} + \bar{A}_4\, Q^2(t) + \frac{1}{3}\bar{C}_4 \langle Q^2 \rangle$$

$$- \frac{1}{6}\bar{D}_4 \left\{ 6 \langle Q \rangle\, Q(t) + 6tQ^2(t) - 6\, Q(t) \int_0^t dy\, Q(y) + \langle Q^2 \rangle \right\}$$

$$- \frac{1}{3}\bar{B}_4 \left\{ \langle Q^2 \rangle - 4 \langle Q \rangle^2 - 6t\langle Q \rangle\, Q(t) - 3t^2Q^2(t) + 6 \langle Q \rangle \int_0^t dy\, Q(y) \right.$$

$$+ 6tQ(t) \int_0^t dy\, Q(y) - t \int_0^t dy\, Q^2(y) - \int_t^1 dy\, yQ^2(y)$$

$$- 2 \int_0^t dy\, Q(y) \int_0^t dz\, Q(z) + 2 \int_t^1 dy\, Q(y) \int_0^y dz\, Q(z)$$

$$\left. - \int_t^1 dy \int_0^y dz\, Q^2(z) \right\} + \ldots \quad . \qquad (7.2.6)$$

Equation (7.2.5) has two types of solutions,

1) $Q'(t) \neq 0 \quad ; \quad \Phi[Q] = 0 \quad ,$ (7.2.7a)

2) $Q'(t) = 0 \quad ; \quad \Phi[Q] \neq 0 \quad .$ (7.2.7b)

Solutions of type 2 are the replica–symmetric ones which are known to be unstable, as shown in section 3.7 . Let us restrict ourselves for the moment to a search for type–1 solutions. Then, from equation (7.2.5) we get

$$\frac{1}{Q'(t)} \frac{d}{dt} \frac{1}{Q'(t)} \frac{d}{dt} \frac{\delta(\beta f\,[Q])}{\delta Q(t)} = 2\bar{A}_3 - 2\bar{B}_3 t + O(\epsilon) = 0 \quad , \qquad (7.2.8)$$

which tells us that the function $Q(t)$ can only vary in the neighbourhood of the point $\bar{t}$ with width $O(\epsilon)$,

$$\bar{t} = \frac{\bar{A}_3}{\bar{B}_3} + O(\epsilon) = \frac{\delta_{3,p}}{2} + O(\epsilon) \quad . \tag{7.2.9}$$

Taking one further derivative on (7.2.5), one gets

$$\frac{d}{dt}\frac{1}{Q'(t)}\frac{d}{dt}\frac{1}{Q'(t)}\frac{d}{dt}\frac{\delta(\beta f[Q])}{\delta Q(t)}$$

$$= -2\bar{B}_3 + 2(\bar{A}_4 - \bar{D}_4 t + \bar{B}_4 t^2)\, Q'(t) + O(\epsilon) = 0 \quad , \tag{7.2.10}$$

which gives

$$Q'(t) = \frac{\bar{B}_3}{\bar{B}_4 t^2 - \bar{D}_4 t + \bar{A}_4} + O(\epsilon) \quad . \tag{7.2.11}$$

Using equations (7.2.4), (7.2.9) and (7.2.11) the Parisi function can be obtained, to leading order in ϵ, for any value of p. Let us now look at some special cases.

$\underline{p = 2}$

In this case our model reduces to the spin glass of Sherrington and Kirkpatrick discussed in Chapter 2. Within our truncated approximation we have (see Figure 7.1(a)),

$$Q_m = \epsilon + O(\epsilon^2) \quad ; \quad \bar{x} = 2\epsilon + O(\epsilon^2) \quad , \tag{7.2.12a}$$

$$Q'(x) = \frac{1}{2} + O(\epsilon) \quad ; \quad \text{for} \quad 0 < x < \bar{x} \quad . \tag{7.2.12b}$$

p = 3

This is equivalent to the 3–state Potts glass as shown in section 3.2 . A type–1 solution leads to $Q'(x) < 0$ and therefore, to a negative probability (see equation (2.4.29)), which is clearly incorrect. As will be discussed in detail in the next section, the right solution in this case turns out to be a step function (Gross *et al.* 1985). This is a type–2 solution for every $x \in [0,1]$ except at the breaking point, at which $Q'(x) = \infty$ (see Figure 7.1(b)).

p = 4

One gets a solution as shown in Figure 7.1(a) with,

$$Q_m = \frac{\epsilon}{2} + O(\epsilon^2) \quad ; \quad \bar{x} = 2\epsilon + O(\epsilon^2) \quad , \tag{7.2.13a}$$

$$Q'(x) = \frac{1}{4} + O(\epsilon) \quad ; \quad \text{for} \quad 0 < x < \bar{x} \quad . \tag{7.2.13b}$$

This was obtained within the space of isotropic solutions (equations (3.3.8)); the anisotropic solutions, equations (3.3.17), will be treated in section 7.4 .

p ≥ 5

Stopping expansion (3.4.4) at the fourth–order terms, we have that the p–state clock with $p \geq 5$ is identical to the corresponding truncation for the XY glass. Looking at higher–order terms one concludes that in truncating the expansion at m–th order, deviations from the XY glass occur only for $p \leq m$. As can be seen from the traces derived in Appendix A , these deviations arise just as a change in some coefficients in the Landau expansion if p is even, while they also appear as new extra terms if p is odd. For any $p > m$, one gets exactly the same terms and coefficients as for the XY glass.

Let us now analyse what changes would result from the introduction of the possible "dangerous term"

$$\sum_{\alpha\beta} (Q^{\alpha\beta})^5 \quad ,$$

in the expansion (3.4.4) for the case $p = 5$. Equations (3.4.4) and (7.2.1) would be respectively,

$$\tilde{g}(Q^{\alpha\beta}) = g(Q^{\alpha\beta}) - \frac{1}{60} \bar{A}_5 \sum_{\alpha\beta} (Q^{\alpha\beta})^5 \quad , \qquad (7.2.14a)$$

$$\beta \tilde{f}[Q] = \beta f[Q] + \frac{1}{60} \bar{A}_5 \langle Q^5 \rangle \quad , \qquad (7.2.14b)$$

where

$$\bar{A}_5 = \frac{(\beta J)^{10}}{64} \quad , \quad \text{for} \quad p = 5 \,. \qquad (7.2.14c)$$

In equation (7.2.8) this term would give a correction $O(\epsilon^2)$ on $\bar{t}$, while in (7.2.11) this would appear as

$$Q'(t) = \frac{\bar{B}_3}{\bar{B}_4 t^2 - \bar{D}_4 t + \bar{A}_4 + \bar{A}_5 Q(t)} \quad , \qquad (7.2.15)$$

contributing as a higher–order correction to the constant term $\bar{A}_4$ in the denominator.

Therefore, the inclusion of higher–order terms in (3.4.4) do not change qualitatively the shape of the Parisi function and for all $p \geq 5$ one gets the "conventional" behaviour shown in Figure 7.1(a). The Parisi function is similar to the one of an XY glass with spins of unitary length $(\underline{S}_i^2 = 1)$. In Figure 7.1(a) one has,

$$Q_m = \frac{\epsilon}{2} + O(\epsilon^2) \quad ; \quad \bar{x} = \frac{3}{2}\,\epsilon + O(\epsilon^2) \quad , \tag{7.2.16a}$$

$$Q'(x) = \frac{1}{3} + O(\epsilon) \quad ; \quad \text{for} \quad 0 < x < \bar{x} \quad . \tag{7.2.16b}$$

Let us now turn to the stability of the above solutions. We do that in a restricted space, based on the analysis by Thouless *et al.* (1980) for the SK model. The full stability analysis, as performed by De Dominicis and Kondor (1983) for the SK model, is quite a difficult task and is still lacking for other infinite–range spin glasses.

As discussed in section 2.4 , the condition for a minimum of the free energy, as required in (3.3.6), only makes sense when seen as a local stability condition in the full replica space, i.e. minimum with respect to each of the $Q^{\alpha\beta}$'s . After the limit $n \to 0$ is performed, this changes into a maximum condition. Therefore, in the analysis which follows, a maximum of the free–energy functional (7.2.1) will be required for stability.

Let us consider the stability functional,

$$\Sigma[Q] = \frac{\delta^2(\beta f[Q])}{\delta Q(s)\,\delta Q(t)} \quad , \tag{7.2.17}$$

which must be negative definite for the Parisi solution to be stable. Taking two functional derivatives of (7.2.1) one gets

$$\Sigma[Q] = \delta(t-s)\Phi[Q] + \Omega[Q] \quad , \tag{7.2.18}$$

where $\delta(t-s)$ is a Dirac-delta function, $\Phi[Q]$ is given by (7.2.6), and

$$\Omega[Q] = -2\bar{B}_3 \{Q(s)\theta(t-s) + Q(t)\theta(s-t)\} + O(\epsilon^2) \quad , \tag{7.2.19}$$

with

$$\theta(x) = \begin{cases} 0 & , \quad x < 0 \\ 1 & , \quad x > 0 \end{cases} \quad . \tag{7.2.20}$$

Therefore, for any $p \neq 3$, the type-1 solution in (7.2.7a), gives

$$\Sigma[Q] = \Omega[Q] \quad , \tag{7.2.21}$$

which is clearly non-positive for the function $Q(x)$ as shown in Figure 7.1(a). The eigenvalue equation

$$\int_0^1 ds\, f(s)\, \Sigma[Q] = \lambda\, f(t) \quad , \tag{7.2.22}$$

presents however, zero eigenvalues (Thouless *et al.* 1980) showing the marginal stability of the Parisi solution.

7.3. p = 3

The procedure developed in the previous section leads, for $p = 3$, to a negative $Q'(x)$ which is an unphysical result according to the interpretation given for the Parisi function in section 2.4.3 . In this section we show that the correct solution in this case turns out to be a step function (Gross *et al.* 1985), as seen in Figure 7.1(b) ,

$$Q(t) = Q_m \theta (t-\bar{t}) \quad . \tag{7.3.1}$$

Except for the discontinuity at the breaking point, this is a type–2 solution in equations (7.2.7).

Substituting (7.3.1) into (7.2.1) one gets the free energy as

$$\begin{aligned}\beta f(Q_m,\bar{t}) = &- \bar{g}_0 + \frac{1}{2}\bar{A}_2 Q_m^2 (1-\bar{t}) + \frac{1}{3}\bar{A}_3 Q_m^3 (1-\bar{t}) - \frac{1}{3}\bar{B}_3 Q_m^3 (1-\bar{t})(2-\bar{t}) \\ &+ \frac{1}{12}\bar{A}_4 Q_m^4 (1-\bar{t}) + \frac{1}{12}\bar{C}_4 Q_m^4 (1-\bar{t})^2 - \frac{1}{12}\bar{D}_4 Q_m^4 (1-\bar{t})(2-\bar{t}) \\ &+ \frac{1}{12}\bar{B}_4 Q_m^4 (1-\bar{t}) \{3 - 3\bar{t} + (\bar{t})^2\} + O(\epsilon^5) \quad .\end{aligned} \tag{7.3.2}$$

The equilibrium conditions,

$$\frac{\partial(\beta f(Q_m,\bar{t}))}{\partial Q_m} = 0 \quad ; \quad \frac{\partial(\beta f(Q_m,\bar{t}))}{\partial \bar{t}} = 0 \quad , \tag{7.3.3}$$

give respectively,

$$\bar{A}_2 + \bar{A}_3\,Q_m - \bar{B}_3\,Q_m\,(2-\bar{t}) + \frac{1}{3}\,\bar{A}_4\,Q_m^2 + \frac{1}{3}\,\bar{C}_4\,Q_m^2\,(1-\bar{t}) - \frac{1}{3}\,\bar{D}_4\,Q_m^2\,(2-\bar{t})$$

$$+ \frac{1}{3}\,\bar{B}_4\,Q_m^2\,\{3 - 3\bar{t} + (\bar{t})^2\} + O(\epsilon^3) = 0 \quad , \qquad (7.3.4a)$$

$$-\frac{1}{2}\,\bar{A}_2 - \frac{1}{3}\,\bar{A}_3\,Q_m - \frac{1}{3}\,\bar{B}_3\,Q_m\,(-3+2\bar{t}) - \frac{1}{12}\,\bar{A}_4\,Q_m^2 + \frac{1}{12}\,\bar{C}_4\,Q_m^2\,(-2+2\bar{t})$$

$$-\frac{1}{12}\,\bar{D}_4\,Q_m^2\,(-3+2\bar{t}) + \frac{1}{12}\,\bar{B}_4\,Q_m^2\,\{-6 + 8\bar{t} - 3(\bar{t})^2\} + O(\epsilon^3) = 0 \quad . \qquad (7.3.4b)$$

By solving equations (7.3.4),

$$\bar{t} = \frac{\bar{A}_3}{\bar{B}_3} + O(\epsilon) = \frac{1}{2} + O(\epsilon) \quad , \qquad (7.3.5a)$$

$$Q_m = \frac{\bar{A}_2}{2(\bar{B}_3 - \bar{A}_3)} + O(\epsilon^2) = \epsilon + O(\epsilon^2) \quad , \qquad (7.3.5b)$$

giving a continuous phase transition at T_g, i.e. Q_m goes continuously to zero as $T \to T_g$.

The next question to address concerns the stability of the present solution. In order to do this, let us substitute (7.3.1) into the stability functional $\Sigma[Q]$ as defined in the previous section. Substituting the step–function solution into (7.2.18), one gets

$$\Phi(Q_m,\bar{t}) = \bar{A}_2 + 2\bar{A}_3\,Q_m\theta\,(t-\bar{t}) - 2\bar{B}_3\,Q_m\,\{(1-\bar{t}) + \bar{t}\,\theta\,(t-\bar{t})\} + \bar{A}_4\,Q_m^2\,\theta\,(t-\bar{t})$$

$$+ \frac{1}{3}\,\bar{C}_4\,Q_m^2\,(1-\bar{t}) - \frac{1}{6}\,\bar{D}_4\,Q_m^2\,\{(1-\bar{t}) + 6\theta\,(t-\bar{t})\}$$

$$+ \bar{B}_4\, Q_m^2\, \{(1-\bar{t})^2 + \bar{t}(2-\bar{t})\theta(t-\bar{t})\} + O(\epsilon^3) \quad , \tag{7.3.6a}$$

$$\Omega(Q_m,\bar{t}) = -2\bar{B}_3\, Q_m \theta(t-\bar{t})\theta(s-\bar{t}) + O(\epsilon^2) \quad . \tag{7.3.6b}$$

The quantity $\Phi(Q_m,\bar{t})$ can be decomposed in two parts, i.e. for $t > \bar{t}$ and $t < \bar{t}$, respectively. Making use of equations (7.3.4), these may be expressed as

$$\Phi_\pm = \frac{1}{6}\bar{A}_4\, Q_m^2 - \frac{1}{6}\bar{D}_4\, Q_m^2\, \bar{t} + \frac{1}{6}\bar{B}_4\, Q_m^2\, (\bar{t})^2 + O(\epsilon^3) \quad , \tag{7.3.7}$$

where the $+$ $(-)$ sign refer to $t > \bar{t}$ $(t < \bar{t})$. The stability functional becomes,

$$\Sigma[Q] = \{\Phi_+\theta(t-\bar{t}) + \Phi_-\theta(\bar{t}-t)\}\delta(t-s) - 2\bar{B}_3\, Q_m\theta(t-\bar{t})\theta(s-\bar{t}) \quad . \tag{7.3.8}$$

To find the eigenvalues associated to (7.3.8) one needs to solve equation (7.2.22). In doing this, one gets the following eigenvalues and their corresponding eigenfunctions,

$$\begin{array}{ll}
\Phi_+ & f_+(t) \quad , \\
\Phi_- & f_-(t) \;\; ; \;\; \kappa_-\theta(\bar{t}-t) \quad , \\
\Phi_+ - 2\bar{B}_3\, Q_m\, (1-\bar{t}) & \kappa_+\theta(t-\bar{t}) \quad ,
\end{array}$$

where κ_+, κ_- are constants and $f_+(t)$ $(f_-(t))$ vanishes for $t < \bar{t}$ $(t > \bar{t})$, non-zero otherwise, restricted to

$$\int_0^1 dt\ f_+(t) = \int_0^1 dt\ f_-(t) = 0 \quad . \tag{7.3.9}$$

It is clear that the above eigenfunctions form a complete set, since they are orthogonal and any arbitrary function $f(t)$, $0 \leq t \leq 1$, may be expressed as a linear combination,

$$f(t) = f_-(t) + f_+(t) + \kappa_- \theta(\bar{t}-t) + \kappa_+ \theta(t-\bar{t}) \quad . \tag{7.3.10}$$

Hence, one has the eigenvalues,

$$\lambda_1 = \frac{1}{6} \{ \bar{A}_4 - \bar{D}_4\, \bar{t} + \bar{B}_4\, (\bar{t})^2 \}\, Q_m^2 + O(\epsilon^3) \quad , \tag{7.3.11a}$$

$$\lambda_2 = -\,2\bar{B}_3\, Q_m\, (1-\bar{t}) + O(\epsilon^2) \quad . \tag{7.3.11b}$$

It is interesting to note that for any $p \neq 3$, equation (7.2.9) gives $\bar{t} = O(\epsilon)$ and then,

$$\lambda_1 = \frac{1}{6}\, \bar{A}_4\, Q_m^2 + O(\epsilon^3) \quad ; \quad p \neq 3 \quad , \tag{7.3.12}$$

which is positive, signaling the instability of the step–function solution. However, for $p = 3$, one has $\bar{t} = 1/2 + O(\epsilon)$, and using the coefficients given in equations (3.4.5) ,

$$\lambda_1 = -\frac{(\beta J)^8}{128}\, Q_m^2 + O(\epsilon^3) \quad ; \quad \lambda_2 = -\frac{(\beta J)^6}{8}\, Q_m + O(\epsilon^2) \quad , \tag{7.3.13}$$

providing stability to solution (7.3.1). Notice that there are no zero eigenvalues.

This is to be contrasted with the marginal stability obtained from conventional Parisi solutions for the cases $p \neq 3$, as discussed in the last section. The step–function solution, as shown in Figure 7.1(b), represents just the first stage of Parisi's replica–symmetry–breaking scheme. The probability distribution for the overlaps, $P(q)$, is simply given by two delta functions, and different states can either present overlaps 0 or Q_m.

7.4. $p = 4$: the anisotropic solution

In the previous sections the Parisi functions for clock glasses, in zero magnetic field, were investigated within the space of isotropic solutions, equations (3.3.8). However, as demonstrated in Chapter 4, the case $p = 4$ in highly anisotropic and instead of (3.3.8), one has (3.3.17) with small fluctuations from perfect collinearity. In this section we shall investigate the Parisi solution resulting from (3.3.17). We do this by using the rules (2.4.17) → (2.4.22) in expansion (3.5.4); note that the single–replica–dependent parameter R^{α} does not affect replica–symmetry breaking, as discussed in section 3.7. We get,

$$\beta f[R,Q] = -g_0 - A_2R^2 + B_2 \langle Q^2\rangle - D_3 \int_0^1 dx \left[xQ^3(x) + 3\,Q(x) \int_0^x dy\; Q^2(y) \right]$$

$$+ G_3R \langle Q^2\rangle - A_4R^4 + B_4 \langle Q^4\rangle - R_4R \int_0^1 dx \left[xQ^3(x) + 3\,Q(x) \int_0^x dy\; Q^2(y) \right]$$

$$- C_4 \left\{ - \langle Q^4\rangle + 2 \langle Q^2\rangle^2 + \int_0^1 dx \int_0^x dy \,[\, Q^2(x) - Q^2(y) \,]^2 \right\} + O_4R^2 \langle Q^2\rangle$$

$$- E_4 \left\{ \langle Q^2\rangle^2 - 4 \langle Q\rangle^2 \langle Q^2\rangle - 4 \langle Q\rangle \int_0^1 dx\, Q(x) \int_0^x dy \,[\, Q(x) - Q(y) \,]^2 \right.$$

$$- \int_0^1 dx \int_0^x dy \int_0^x dz \, [\, Q(x) - Q(y)\,]^2 \, [\, Q(x) - Q(z)\,]^2 \Big\} + \dots \quad , \tag{7.4.1}$$

where the coefficients are given in equations (3.5.5). In equation (7.4.1) the Parisi function $Q(x)$ is such that

$$Q_{\mu\nu}(x) = Q(x)\, \delta_{\mu\nu}\, \delta_{\mu x} \quad ; \quad \mu,\nu = x,y \quad , \tag{7.4.2}$$

and $\langle Q^m \rangle$ is defined as in (7.2.3).

At the extrema the replica-symmetric parameter R can be eliminated to give

$$\frac{\delta(\beta f)}{\delta Q(t)} = 2B_2\, Q(t) - D_3 \Big\{ 3tQ^2(t) + 3 \int_0^t dy\, Q^2(y) + 6\, Q(t) \int_t^1 dy\, Q(y) \Big\}$$

$$+ \frac{G_3}{A_2} Q(t) \Big\{ G_3 \langle Q^2 \rangle - \frac{G_3^3 A_4}{2A_2^3} \langle Q^2 \rangle^3 + \frac{G_3 O_4}{A_2} \langle Q^2 \rangle^2$$

$$- R_4 \int_0^1 dx \Big[xQ^3(x) + 3\, Q(x) \int_0^x dy\, Q^2(y) \Big] \Big\}$$

$$+ 4\, B_4\, Q^3(t) - 4\, C_4 \langle Q^2 \rangle Q(t) + \frac{G_3^2 O_4}{2A_2^2} \langle Q^2 \rangle Q(t)$$

$$- \frac{G_3 R_4}{2A_2} \langle Q^2 \rangle \Big\{ 3tQ^2(t) + 3 \int_0^t dy\, Q^2(y) + 6\, Q(t) \int_t^1 dy\, Q(y) \Big\}$$

$$- E_4 \Big\{ 4\, \langle Q^2 \rangle Q(t) - 16\, \langle Q \rangle^2 Q(t) - 12\, \langle Q^2 \rangle \langle Q \rangle - 12t \langle Q \rangle Q^2(t) - 4t^2 Q^3(t)$$

$$-4\int_0^1 dy\, Q(y)\int_0^y dz\,[\,Q(y)-Q(z)\,]^2 + 12\,\langle Q\rangle \int_t^1 dy\, Q^2(y)$$

$$+ 24\,\langle Q\rangle\, Q(t)\int_0^t dy\, Q(y) + 12tQ^2(t)\int_0^t dy\, Q(y) + 4\int_t^1 dy\, yQ^3(y)$$

$$-4tQ(t)\int_0^t dy\, Q^2(y) - 4\,Q(t)\int_t^1 dy\, yQ^2(y) - 8\,Q(t)\int_0^t dy\, Q(y)\int_0^t dz\, Q(z)$$

$$-8\int_t^1 dy\, Q^2(y)\int_0^y dz\, Q(z) + 4\int_0^t dy\, Q(y)\int_0^t dz\, Q^2(z)$$

$$+ 4\int_t^1 dy\, Q(y)\int_0^y dz\, Q^2(z) + 8\,Q(t)\int_t^1 dy\, Q(y)\int_0^y dz\, Q(z)$$

$$-4\,Q(t)\int_t^1 dy\int_0^y dz\, Q^2(z)\,\Big\} + \dots = 0 \quad . \qquad (7.4.3)$$

One can now follow the same steps as in section 7.2 to find a type–1 solution (equation (7.2.7a)), as shown in Figure 7.1(a) , with

$$Q_m = \epsilon + O(\epsilon^2) \quad ; \quad \bar{x} = \epsilon + O(\epsilon^2) \quad , \qquad (7.4.4a)$$

$$Q'(x) = 1 + O(\epsilon) \quad ; \quad \text{for} \quad 0 < x < \bar{x} \quad . \qquad (7.4.4b)$$

The stability analysis can also be performed to give

$$\Sigma[Q] = -6D_3\,\{Q(s)\theta(t{-}s) + Q(t)\theta(s{-}t)\} + 2\,\frac{G_3^2}{A_2}\,Q(t)\,Q(s) + O(\epsilon^2) \quad . \qquad (7.4.5)$$

and using equations (3.5.5),

$$\Sigma[Q] = -8\left\{ Q(s)\left[1-\frac{1}{\epsilon}Q(t)\right]\theta(t-s)\right.$$

$$\left. + Q(t)\left[1-\frac{1}{\epsilon}Q(s)\right]\theta(s-t)\right\} + O(\epsilon^2) \quad , \qquad (7.4.6)$$

which is clearly non-positive for the function $Q(x)$ in equations (7.4.4). Note the appearance of an extra term in (7.4.5) as compared to the stability functional for the isotropic solution (equations (7.2.19) → (7.2.21)). This term makes the stability functional even closer to zero, as seen in equation (7.4.6), and a higher marginality is expected for the collinear solution, within the realm of our simplified stability analysis.

7.5. Conclusion

The Parisi replica-symmetry-breaking scheme was applied to the p-state clock spin-glass model near the transition temperature. The case $p = 3$ has presented an anomalous behaviour: a step function, that is, the first stage of the Parisi process is the stable solution. No zero eigenvalues were found, in contrast to the marginal stability obtained from the conventional Parisi solution in the Sherrington-Kirkpatrick model. The absence of reflection symmetry in the spin variable was shown to be qualitatively irrelevant for all other odd-state clock glasses. All $p \neq 3$ cases behave in the conventional way and for $p \geq 5$, the Parisi function is to leading order in ϵ, the same as for the XY spin glass.

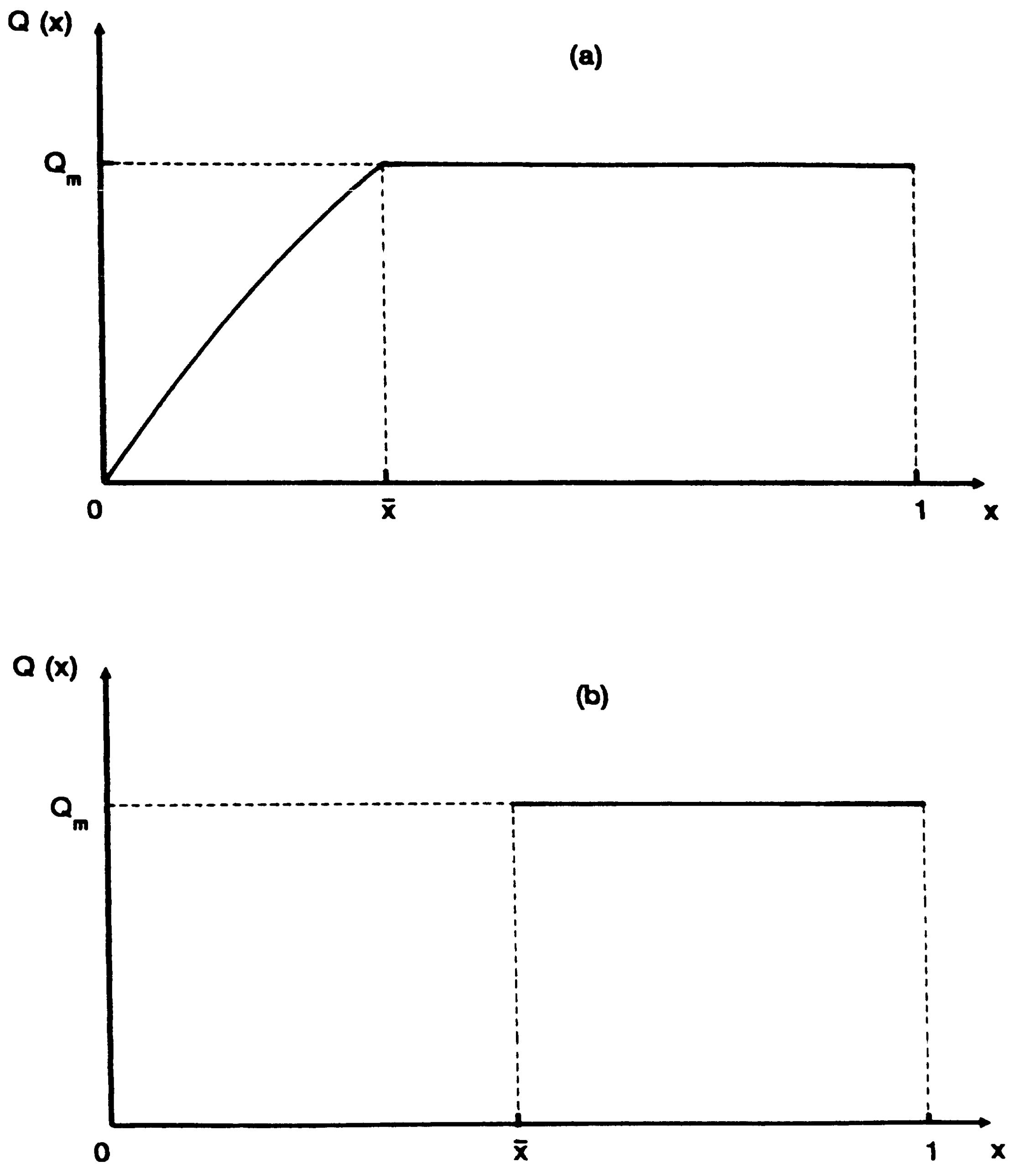

Figure 7.1: The Parisi function for the clock spin glasses.

(a) Cases $p \neq 3$;

(b) case $p = 3$.

CHAPTER 8: CONCLUSION

The infinite-range p-state clock spin-glass model has been studied. Such a system can be seen as a realization of an XY spin glass in an infinite-strength p-fold anisotropy field, presenting as particular cases the Sherrington-Kirkpatrick $(p = 2)$, the 3-state Potts $(p = 3)$ and the XY $(p = \infty)$ models. The spin variable exhibits (does not exhibit) symmetry under reflection for every even (odd) value of p. Special attention has been devoted to the relevance of reflection symmetry for this model.

It has been shown that the lack of reflection symmetry in the spin variable plays an important role for $p = 3$, but it is qualitatively irrelevant for all other odd-state clock glasses. Deviations from the XY spin-glass behaviour appear for $p \geq 5$, as higher-order corrections and the dominant behaviour is XY like.

The Almeida-Thouless instability of the replica-symmetric solution, in zero magnetic field, has been demonstrated to occur at $O(\epsilon^2)$ for any p, except for $p = 3$, in which case this instability appears at $O(\epsilon)$ $(\epsilon = (T_g - T)/T_g)$. For a non-zero magnetic field, it has been shown that the critical line associated to the transverse spin-glass freezing changes under reflection of the field $(\underline{h} \to -\underline{h})$ for $p = 3$, but all $p \geq 5$ clock glasses present the conventional XY-like Gabay-Toulouse line. By applying the Parisi replica-symmetry-breaking scheme, it has been proven that all values of p present the conventional Parisi solution, except $p = 3$, for which a step function is the stable solution. All these peculiar properties for $p = 3$ come as a direct consequence of the absence of reflection symmetry in the spin variable.

Particular emphasis has also been given to the investigation of the nature of the low-temperature phase of the 4-state clock spin glass. It has been demonstrated that the average spin–glass order is essentially collinear (two–fold symmetric), despite the four–fold symmetry of the hamiltonian. Fluctuations from perfect collinearity are present due to replica–symmetry breaking but these have been shown to be relatively small. The role of a four–fold anisotropy field on the XY spin glass has also been analysed. It has been argued that the normal four–fold spin–glass phase occurs except in the limit of infinite anisotropy.

Appendix A: Bare averagings for a p-state clock model

Whenever doing perturbation expansions in the order parameters for a p-state clock model, one needs to evaluate quantities like

$$\langle S_x^m S_y^n \rangle_0 = \frac{1}{\mathrm{tr}_0(1)} \mathrm{tr}_0 \left[S_x^m S_y^n \right] = \frac{1}{p} \sum_{k=0}^{p-1} \cos^m \frac{2\pi k}{p} \sin^n \frac{2\pi k}{p} \quad , \qquad \text{(A.1)}$$

where m, n, are positive integers and $\langle\ \rangle_0$ denotes a "bare" average i.e. an averaging over a system with zero hamiltonian.

In order to evaluate such quantities we will make use of the identity,

$$\sum_{k=0}^{p-1} e^{i2\pi km/p} = p\delta_{m,0}\,(\mathrm{mod}\ p) \quad , \qquad \text{(A.2a)}$$

which can also be written as

$$\sum_{k=0}^{p-1} \cos \frac{2\pi k}{p} m = p\delta_{m,0}\,(\mathrm{mod}\ p) \quad ; \quad \sum_{k=0}^{p-1} \sin \frac{2\pi k}{p} m = 0 \quad , \qquad \text{(A.2b)}$$

where

$$\delta_{m,0}\,(\mathrm{mod}\ p) = \begin{cases} 1, \text{ if } m = 0,p,2p,\ldots \\ 0, \text{ if } m \neq 0,p,2p,\ldots \end{cases} \quad . \qquad \text{(A.2c)}$$

Using (A.2b) together with basic trigonometric relations, averages like (A.1) can easily be evaluated. Here follows a list of the ones used throughout this thesis, valid for any p integer ($p \geq 2$):

$$\langle S_x^2 \rangle_0 = \frac{1}{2}(1 + \delta_{2,p}) \quad , \tag{A.3a}$$

$$\langle S_y^2 \rangle_0 = \frac{1}{2}(1 - \delta_{2,p}) \quad , \tag{A.3b}$$

$$\langle S_x^3 \rangle_0 = \frac{1}{4}\,\delta_{3,p} \quad , \tag{A.4}$$

$$\langle S_x^4 \rangle_0 = \frac{1}{8}(3 + 5\delta_{2,p} + \delta_{4,p}) \quad , \tag{A.5a}$$

$$\langle S_y^4 \rangle_0 = \frac{1}{8}(3 - 3\delta_{2,p} + \delta_{4,p}) \quad , \tag{A.5b}$$

$$\langle S_x^5 \rangle_0 = \frac{1}{16}(5\delta_{3,p} + \delta_{5,p}) \quad , \tag{A.6}$$

$$\langle S_x^6 \rangle_0 = \frac{1}{32}(10 + 22\delta_{2,p} + \delta_{3,p} + 6\delta_{4,p} + \delta_{6,p}) \quad , \tag{A.7a}$$

$$\langle S_y^6 \rangle_0 = \frac{1}{32}(10 - 10\delta_{2,p} - \delta_{3,p} + 6\delta_{4,p} - \delta_{6,p}) \quad , \tag{A.7b}$$

$$\langle S_x^7 \rangle_0 = \frac{1}{64}(21\delta_{3,p} + 7\delta_{5,p} + \delta_{7,p}) \quad , \tag{A.8}$$

$$\langle S_x^8 \rangle_0 = \frac{1}{128}(35 + 93\delta_{2,p} + 8\delta_{3,p} + 29\delta_{4,p} + 8\delta_{6,p} + \delta_{8,p}) \quad , \tag{A.9a}$$

$$\langle S_y^8 \rangle_0 = \frac{1}{128}(35 - 35\delta_{2,p} - 8\delta_{3,p} + 29\delta_{4,p} - 8\delta_{6,p} + \delta_{8,p}) \quad , \tag{A.9b}$$

$$\langle\, S_x\, S_y^2 \,\rangle_0 = -\tfrac{1}{4}\, \delta_{3,p} \quad , \tag{A.10}$$

$$\langle\, S_x\, S_y^4 \,\rangle_0 = \frac{1}{16}\,(\,-3\delta_{3,p} + \delta_{5,p}) \quad , \tag{A.11}$$

$$\langle\, S_x^3\, S_y^2 \,\rangle_0 = \frac{1}{16}\,(\,-\,\delta_{3,p} - \delta_{5,p}) \quad , \tag{A.12}$$

$$\langle\, S_x^3\, S_y^4 \,\rangle_0 = \frac{1}{64}\,(\,-3\delta_{3,p} - \delta_{5,p} + \delta_{7,p}) \quad , \tag{A.13}$$

$$\langle\, S_x^2\, S_y^2 \,\rangle_0 = \frac{1}{8}\,(1 - \delta_{2,p} - \delta_{4,p}) \quad , \tag{A.14}$$

$$\langle\, S_x^2\, S_y^4 \,\rangle_0 = \frac{1}{32}\,(2 - 2\delta_{2,p} + \delta_{3,p} - 2\delta_{4,p} + \delta_{6,p}) \quad , \tag{A.15a}$$

$$\langle\, S_x^4\, S_y^2 \,\rangle_0 = \frac{1}{32}\,(2 - 2\delta_{2,p} - \delta_{3,p} - 2\delta_{4,p} - \delta_{6,p}) \quad , \tag{A.15b}$$

$$\langle\, S_x^2\, S_y^6 \,\rangle_0 = \frac{1}{128}\,(5 - 5\delta_{2,p} + 4\delta_{3,p} - 5\delta_{4,p} + 4\delta_{6,p} - \delta_{8,p}) \quad , \tag{A.16a}$$

$$\langle\, S_x^6\, S_y^2 \,\rangle_0 = \frac{1}{128}\,(5 - 5\delta_{2,p} - 4\delta_{3,p} - 5\delta_{4,p} - 4\delta_{6,p} - \delta_{8,p}) \quad , \tag{A.16b}$$

$$\langle\, S_x^4\, S_y^4 \,\rangle_0 = \frac{1}{128}\,(3 - 3\delta_{2,p} - 3\delta_{4,p} + \delta_{8,p}) \quad , \tag{A.17}$$

$$\langle\, S_y^m \,\rangle_0 = 0 \quad , \quad \text{for m odd} \quad , \tag{A.18a}$$

$$\langle\, S_x^m\, S_y^n \,\rangle_0 = 0 \quad , \quad \text{for n odd} \quad (\text{any m}) \quad . \tag{A.18b}$$

Let us now make a few comments on equations (A.3) → (A.18):

a) As it should be, all averages involving the component S_y vanish for $p = 2$.

b) Averages which would normally vanish for systems possessing reflection symmetry, e.g. (A.4), (A.6), (A.8), (A.10) → (A.13), do contribute for some odd values of p. In particular, (A.4) and (A.10) are responsible for the appearing of special terms in expansion (B.1) which do lead to special behaviour for $p = 3$ as described in Chapters 3, 5 and 7. Contributions of the type (A.6), (A.8), (A.11) → (A.13) only occur in higher order terms and are insufficient to change radically the behaviour of other odd values of p.

c) As can be seen in equations (A.3) → (A.18), the invariance $x \leftrightarrow y$ is respected for all p even ($p > 2$) but $p = 6$. This can be explained by looking at Table A.1, where it is shown that the 6–state clock spin variable S_μ can be written in terms of an Ising variable τ and a 3–state clock spin variable ξ_μ,

$$S_\mu = \tau \xi_\mu \qquad (\tau = \pm 1 \;\; ; \;\; \mu = x,y) \quad , \tag{A.19}$$

$$S_x = \cos \frac{\pi}{3} k \;\; ; \;\; S_y = \sin \frac{\pi}{3} k \qquad (k = 0,1,\ldots,5) \quad , \tag{A.20a}$$

$$\xi_x = \cos \frac{2\pi j}{3} \;\; ; \;\; \xi_y = \sin \frac{2\pi j}{3} \qquad (j = 0,1,2) \quad . \tag{A.20b}$$

From (A.19) it follows that

$$\langle S_x^m S_y^n \rangle_0 = \langle \xi_x^m \xi_y^n \rangle_0 \qquad (\text{m,n even numbers}) \quad , \tag{A.21}$$

and this explains why the case p = 6 in equations (A.7), (A.9), (A.15) and (A.16), instead of respecting the invariance $x \leftrightarrow y$, gives similar results to p = 3 . These terms however, only contribute as higher–order terms in expansion (B.1) and are unable to change the XY behaviour for p = 6 .

Table A.1

k	S_x	S_y	j	τ
0	1	0	0	+1
1	1/2	$\sqrt{3}/2$	2	−1
2	−1/2	$\sqrt{3}/2$	1	+1
3	−1	0	0	−1
4	−1/2	$-\sqrt{3}/2$	2	+1
5	1/2	$-\sqrt{3}/2$	1	−1

Table A.1: The values of S_μ (μ = x,y) for the 6–state clock model; the last two columns show the corresponding values of the three–valued variable j and the Ising variable τ such that (A.19) is satisfied.

Appendix B: Power–series expansions for a p–state clock spin glass

In this appendix, we will develop most of the power–series expansions used throughout this thesis. Close to the point h, ϵ small ($\epsilon = (T_g - T)/T_g$), R^{α}, $Q_{xx}^{\alpha\beta}$, $Q_{yy}^{\alpha\beta}$, are all small and using the traces evaluated in Appendix A, i.e. equations (A.3) → (A.18), the free–energy functional $g(R^{\alpha}, Q_{xx}^{\alpha\beta}, Q_{yy}^{\alpha\beta})$ in equation (3.3.13a) can be written as,

$$
\begin{aligned}
g(R^{\alpha}, Q_{xx}^{\alpha\beta}, Q_{yy}^{\alpha\beta}) = &- n g_0 - A_1 \sum_{\alpha} R^{\alpha} - B_1 \sum_{\alpha\beta} Q_{xx}^{\alpha\beta} - A_2 \sum_{\alpha} (R^{\alpha})^2 - B_2 \sum_{\alpha\beta} (Q_{xx}^{\alpha\beta})^2 \\
&- C_2 \sum_{\alpha\beta\gamma} Q_{xx}^{\alpha\beta} Q_{xx}^{\beta\gamma} - D_2 \sum_{\alpha\beta} (Q_{yy}^{\alpha\beta})^2 - E_2 \sum_{\alpha\beta} R^{\alpha} Q_{xx}^{\alpha\beta} - A_3 \sum_{\alpha} (R^{\alpha})^3 - B_3 \sum_{\alpha\beta} (Q_{xx}^{\alpha\beta})^3 \\
&- C_3 \sum_{\alpha\beta\gamma} (Q_{xx}^{\alpha\beta})^2 Q_{xx}^{\beta\gamma} - D_3 \sum_{\alpha\beta\gamma} Q_{xx}^{\alpha\beta} Q_{xx}^{\beta\gamma} Q_{xx}^{\gamma\alpha} - E_3 \sum_{\alpha\beta\gamma} Q_{yy}^{\alpha\beta} Q_{yy}^{\beta\gamma} Q_{yy}^{\gamma\alpha} \\
&- F_3 \sum_{\alpha\beta} Q_{xx}^{\alpha\beta} (Q_{yy}^{\alpha\beta})^2 - G_3 \sum_{\alpha\beta} R^{\alpha} (Q_{xx}^{\alpha\beta})^2 - I_3 \sum_{\alpha\beta} R^{\alpha} (Q_{yy}^{\alpha\beta})^2 - K_3 \sum_{\alpha\beta} (R^{\alpha})^2 Q_{xx}^{\alpha\beta} \\
&- L_3 \sum_{\alpha\beta} R^{\alpha} R^{\beta} Q_{xx}^{\alpha\beta} - H_3 \sum_{\alpha\beta\gamma} Q_{xx}^{\alpha\beta} Q_{xx}^{\beta\gamma} R^{\gamma} - A_4 \sum_{\alpha} (R^{\alpha})^4 - B_4 \sum_{\alpha\beta} (Q_{xx}^{\alpha\beta})^4 \\
&- C_4 \sum_{\alpha\beta\gamma} (Q_{xx}^{\alpha\beta})^2 (Q_{xx}^{\beta\gamma})^2 - D_4 \sum_{\alpha\beta\gamma} Q_{xx}^{\alpha\beta} (Q_{xx}^{\beta\gamma})^2 Q_{xx}^{\gamma\alpha} - E_4 \sum_{\alpha\beta\gamma\delta} Q_{xx}^{\alpha\beta} Q_{xx}^{\beta\gamma} Q_{xx}^{\gamma\delta} Q_{xx}^{\delta\alpha} \\
&- F_4 \sum_{\alpha\beta} (Q_{yy}^{\alpha\beta})^4 - G_4 \sum_{\alpha\beta\gamma} (Q_{yy}^{\alpha\beta})^2 (Q_{yy}^{\beta\gamma})^2 - H_4 \sum_{\alpha\beta\gamma\delta} Q_{yy}^{\alpha\beta} Q_{yy}^{\beta\gamma} Q_{yy}^{\gamma\delta} Q_{yy}^{\delta\alpha} \\
&- I_4 \sum_{\alpha\beta} (Q_{xx}^{\alpha\beta})^2 (Q_{yy}^{\alpha\beta})^2 - K_4 \sum_{\alpha\beta\gamma} (Q_{xx}^{\alpha\beta})^2 (Q_{yy}^{\beta\gamma})^2 - L_4 \sum_{\alpha\beta\gamma} Q_{xx}^{\alpha\beta} Q_{xx}^{\beta\gamma} (Q_{yy}^{\alpha\gamma})^2
\end{aligned}
$$

$$-M_4\sum_{\alpha\beta\gamma} Q_{yy}^{\alpha\beta} Q_{yy}^{\beta\gamma} Q_{yy}^{\gamma\alpha} Q_{xx}^{\alpha\beta} - N_4\sum_{\alpha\beta}(R^{\alpha})^2 (Q_{xx}^{\alpha\beta})^2 - O_4\sum_{\alpha\beta} R^{\alpha} R^{\beta} (Q_{xx}^{\alpha\beta})^2$$

$$-P_4\sum_{\alpha\beta} R^{\alpha} (Q_{xx}^{\alpha\beta})^3 - Q_4\sum_{\alpha\beta\gamma} (Q_{xx}^{\alpha\beta})^2 Q_{xx}^{\beta\gamma} R^{\gamma} - R_4\sum_{\alpha\beta\gamma} Q_{xx}^{\alpha\beta} Q_{xx}^{\beta\gamma} Q_{xx}^{\gamma\alpha} R^{\alpha}$$

$$-S_4\sum_{\alpha\beta}(R^{\alpha})^2 R^{\beta} Q_{xx}^{\alpha\beta} - T_4\sum_{\alpha\beta}(R^{\alpha})^2 (Q_{yy}^{\alpha\beta})^2 - U_4\sum_{\alpha\beta} R^{\alpha} R^{\beta} (Q_{yy}^{\alpha\beta})^2$$

$$-V_4\sum_{\alpha\beta\gamma} Q_{yy}^{\alpha\beta} Q_{yy}^{\beta\gamma} Q_{yy}^{\gamma\alpha} R^{\alpha} - W_4\sum_{\alpha\beta} R^{\alpha} Q_{xx}^{\alpha\beta} (Q_{yy}^{\alpha\beta})^2$$

$$-X_4\sum_{\alpha\beta\gamma} Q_{xx}^{\alpha\beta} Q_{xx}^{\beta\gamma} R^{\alpha} R^{\gamma} - \dots \qquad . \tag{B.1}$$

In equation (B.1) the summations over replicas are unrestricted and the coefficients are given by,

$$g_0 = \ln p + \frac{(\beta J)^2}{8} + \frac{(\beta h)^2}{4}(1 + \delta_{2,p}) + \frac{(\beta h)^3}{24}\delta_{3,p}$$

$$+ \frac{(\beta h)^4}{192}(-3 - 13\delta_{2,p} + \delta_{4,p}) + O(h^5) \quad , \tag{B.2}$$

$$A_1 = \frac{(\beta J)^2}{2}\Big[\delta_{2,p} + \frac{\beta h}{2}\delta_{3,p} + \frac{(\beta h)^2}{8}(1 - \delta_{2,p} + \delta_{4,p})$$

$$+ \frac{(\beta h)^3}{48}(-3\delta_{3,p} + \delta_{5,p})\Big] + O(h^4) \quad , \tag{B.3a}$$

$$B_1 = \frac{(\beta J)^2(\beta h)^2}{8}\Big[1 + 3\delta_{2,p} + \frac{\beta h}{2}\delta_{3,p}\Big] + O(h^4) \quad , \tag{B.3b}$$

$$A_2 = \frac{(\beta J)^2}{2}\left\{\frac{(\beta J)^2}{8}\left[(1-\delta_{2,p}+\delta_{4,p})+\frac{\beta h}{2}(\delta_{3,p}+\delta_{5,p})\right.\right.$$

$$\left.\left.+\frac{(\beta h)^2}{8}(-3\delta_{3,p}+\delta_{6,p})\right]-1\right\}+O(h^3) \quad , \qquad \text{(B.4a)}$$

$$B_2 = \frac{(\beta J)^2}{4}\left\{\frac{(\beta J)^2}{4}\left[(1+3\delta_{2,p})+\beta h\,\delta_{3,p}\right.\right.$$

$$\left.\left.+\frac{(\beta h)^2}{4}(-3-29\delta_{2,p}+\delta_{3,p}+\delta_{4,p})\right]-1\right\}+O(h^3) \quad , \qquad \text{(B.4b)}$$

$$C_2 = \frac{(\beta J)^4(\beta h)^2}{16}(1+7\delta_{2,p})+O(h^3) \quad , \qquad \text{(B.4c)}$$

$$D_2 = \frac{(\beta J)^2}{4}\left\{\frac{(\beta J)^2}{4}\left[(1-\delta_{2,p})-\beta h\,\delta_{3,p}\right.\right.$$

$$\left.\left.+\frac{(\beta h)^2}{4}(-1+\delta_{2,p}+\delta_{3,p}-\delta_{4,p})\right]-1\right\}+O(h^3) \quad , \qquad \text{(B.4d)}$$

$$E_2 = \frac{(\beta J)^4\,\beta h}{8}\left[\delta_{3,p}+\frac{\beta h}{4}(2-2\delta_{2,p}+\delta_{3,p}+2\delta_{4,p})\right]+O(h^3) \quad , \qquad \text{(B.4e)}$$

$$A_3 = \frac{(\beta J)^6}{192}\left[(\delta_{3,p}+\delta_{6,p})+\frac{\beta h}{2}(-3\delta_{3,p}+\delta_{5,p}+\delta_{7,p})\right]+O(h^2) \qquad \text{(B.5a)}$$

$$B_3 = \frac{(\beta J)^6}{192}(1-3\beta h)\delta_{3,p}+O(h^2) \quad , \qquad \text{(B.5b)}$$

$$C_3 = \frac{(\beta J)^6\,\beta h}{32}\,\delta_{3,p}+O(h^2) \quad , \qquad \text{(B.5c)}$$

$$D_3 = \frac{(\beta J)^6}{48}\left[1 + 7\delta_{2,p} + \frac{3}{2}\beta h\delta_{3,p}\right] + O(h^2) \quad , \tag{B.5d}$$

$$E_3 = \frac{(\beta J)^6}{48}\left[1 - \delta_{2,p} - \frac{3}{2}\beta h\delta_{3,p}\right] + O(h^2) \quad , \tag{B.5e}$$

$$F_3 = \frac{(\beta J)^6}{64}(1 - \beta h)\delta_{3,p} + O(h^2) \quad , \tag{B.5f}$$

$$G_3 = \frac{(\beta J)^6}{64}\left[2(1 - \delta_{2,p} + \delta_{4,p}) + \beta h(-2\delta_{3,p} + \delta_{5,p})\right] + O(h^2) \ , \tag{B.5g}$$

$$H_3 = \frac{(\beta J)^6\ \beta h}{16}\delta_{3,p} + O(h^2) \quad , \tag{B.5h}$$

$$I_3 = \frac{(\beta J)^6}{64}\left[2(-1 + \delta_{2,p} - \delta_{4,p}) + \beta h(\delta_{3,p} - \delta_{5,p})\right] + O(h^2) \ , \tag{B.5i}$$

$$K_3 = \frac{(\beta J)^6\ \beta h}{64}(\delta_{3,p} + \delta_{5,p}) + O(h^2) \quad , \tag{B.5k}$$

$$L_3 = \frac{(\beta J)^6}{32}(1 + \beta h)\delta_{3,p} + O(h^2) \quad , \tag{B.5l}$$

$$A_4 = \frac{(\beta J)^8}{3072}(-3 + 3\delta_{2,p} - 13\delta_{4,p} + \delta_{8,p}) + O(h) \quad , \tag{B.6a}$$

$$B_4 = \frac{(\beta J)^8}{3072}(9 + 247\delta_{2,p} - 5\delta_{4,p}) + O(h) \quad , \tag{B.6b}$$

$$C_4 = \frac{(\beta J)^8}{256}(-3 - 61\delta_{2,p} + \delta_{4,p}) + O(h) \quad , \tag{B.6c}$$

$$D_4 = \frac{(\beta J)^8}{128}\delta_{3,p} + O(h) \quad , \tag{B.6d}$$

$$E_4 = \frac{(\beta J)^8}{128}(1 + 15\delta_{2,p}) + O(h) \quad , \tag{B.6e}$$

$$F_4 = \frac{(\beta J)^8}{3072}(9 - 9\delta_{2,p} - 5\delta_{4,p}) + O(h) \quad , \tag{B.6f}$$

$$G_4 = \frac{(\beta J)^8}{256}(-3 + 3\delta_{2,p} + \delta_{4,p}) + O(h) \quad , \tag{B.6g}$$

$$H_4 = \frac{(\beta J)^8}{128}(1 - \delta_{2,p}) + O(h) \quad , \tag{B.6h}$$

$$I_4 = \frac{(\beta J)^8}{512}(1 - \delta_{2,p} + 3\delta_{4,p}) + O(h) \quad , \tag{B.6i}$$

$$K_4 = \frac{(\beta J)^8}{128}(-1 + \delta_{2,p} - \delta_{4,p}) + O(h) \quad , \tag{B.6j}$$

$$L_4 = \frac{(\beta J)^8}{128}\delta_{3,p} + O(h) \quad , \tag{B.6k}$$

$$M_4 = \frac{(\beta J)^8}{64}\delta_{3,p} + O(h) \quad , \tag{B.6l}$$

$$N_4 = \frac{(\beta J)^8}{256}(-3\delta_{3,p} + \delta_{6,p}) + O(h) \quad , \tag{B.6m}$$

$$O_4 = \frac{(\beta J)^8}{256}(1 - \delta_{2,p} + 3\delta_{4,p}) + O(h) \quad , \tag{B.6n}$$

$$P_4 = -\frac{(\beta J)^8}{128}\delta_{3,p} + O(h) \quad , \tag{B.6o}$$

$$Q_4 = \frac{(\beta J)^8}{64}\delta_{3,p} + O(h) \quad , \tag{B.6p}$$

$$R_4 = \frac{(\beta J)^8}{64}(1 - \delta_{2,p} + \delta_{4,p}) + O(h) \quad , \tag{B.6q}$$

$$S_4 = \frac{(\beta J)^8}{128}\,\delta_{3,p} + O(h) \quad , \tag{B.6r}$$

$$T_4 = \frac{(\beta J)^8}{256}(-\delta_{3,p} - \delta_{6,p}) + O(h) \quad , \tag{B.6s}$$

$$U_4 = \frac{(\beta J)^8}{256}(1 - \delta_{2,p} + 3\delta_{4,p}) + O(h) \quad , \tag{B.6t}$$

$$V_4 = \frac{(\beta J)^8}{64}(-1 + \delta_{2,p} - \delta_{4,p}) + O(h) \quad , \tag{B.6u}$$

$$W_4 = -\frac{(\beta J)^8}{128}\,\delta_{3,p} + O(h) \quad , \tag{B.6v}$$

$$X_4 = \frac{(\beta J)^8}{64}\,\delta_{3,p} + O(h) \quad . \tag{B.6w}$$

In the replica-symmetric space ($R^\alpha = R$, $Q_{xx}^{\alpha\beta} = Q_{xx}$, $Q_{yy}^{\alpha\beta} = Q_{yy}$), the free-energy functional in (B.1) gives the free energy per spin in equation (3.3.6) as,

$$\beta f(R , Q_{xx} , Q_{yy}) = -\alpha_0 - \alpha_1 R + \beta_1 Q_{xx} - \alpha_2 R^2 + \beta_2 Q_{xx}^2 + \gamma_2 Q_{yy}^2 + \delta_2 R Q_{xx}$$

$$- \alpha_3 R^3 - \beta_3 Q_{xx}^3 - \gamma_3 Q_{yy}^3 + \delta_3 Q_{xx} Q_{yy}^2 + \epsilon_3 R Q_{xx}^2 + \zeta_3 R Q_{yy}^2$$

$$+ \eta_3 R^2 Q_{xx} - \alpha_4 R^4 + \beta_4 Q_{xx}^4 + \gamma_4 Q_{yy}^4 + \delta_4 Q_{xx}^2 Q_{yy}^2 + \epsilon_4 Q_{xx} Q_{yy}^3$$

$$+ \zeta_4 R^2 Q_{xx}^2 + \eta_4 R Q_{xx}^3 + \kappa_4 R^3 Q_{xx} + \lambda_4 R^2 Q_{yy}^2 + \mu_4 R Q_{yy}^3$$

$$+ \nu_4 R Q_{xx} Q_{yy}^2 + \dots \quad , \tag{B.7}$$

where the coefficients above are related to the ones in equations (B.2) → (B.6) by,

$$\alpha_0 = g_0 \quad , \tag{B.8}$$

$$\alpha_1 = A_1 \quad ; \quad \beta_1 = B_1 \quad , \tag{B.9}$$

$$\alpha_2 = A_2 \quad , \tag{B.10a}$$

$$\beta_2 = B_2 - C_2 = \frac{(\beta J)^2}{4} \left\{ \frac{(\beta J)^2}{4} \left[(1 + 3\delta_{2,p}) + \beta h \delta_{3,p} \right.\right.$$

$$\left.\left. + \frac{(\beta h)^2}{4} (-7 - 57\delta_{2,p} + \delta_{3,p} + \delta_{4,p}) \right] - 1 \right\} + O(h^3) \quad , \tag{B.10b}$$

$$\gamma_2 = D_2 \quad ; \quad \delta_2 = E_2 \quad , \tag{B.10c}$$

$$\alpha_3 = A_3 \quad , \tag{B.11a}$$

$$\beta_3 = -(B_3 - C_3 - 2D_3)$$

$$= \frac{(\beta J)^6}{192}\left[8 + 56\delta_{2,p} - (1 - 21\beta h)\delta_{3,p}\right] + O(h^2) \quad , \tag{B.11b}$$

$$\gamma_3 = 2E_3 \quad ; \quad \delta_3 = F_3 \tag{B.11c}$$

$$\epsilon_3 = G_3 - H_3$$

$$= \frac{(\beta J)^6}{64}\left[2(1 - \delta_{2,p} + \delta_{4,p}) + \beta h(-6\delta_{3,p} + \delta_{5,p})\right] + O(h^2) , \tag{B.11d}$$

$$\zeta_3 = I_3 \quad , \tag{B.11e}$$

$$\eta_3 = K_3 + L_3 = \frac{(\beta J)^6}{64}\left[2\delta_{3,p} + \beta h(3\delta_{3,p} + \delta_{5,p})\right] + O(h^2) \quad , \tag{B.11f}$$

$$\alpha_4 = A_4 \quad , \tag{B.12a}$$

$$\beta_4 = B_4 - C_4 - 2D_4 + 3E_4$$

$$= \frac{(\beta J)^8}{3072}(117 + 2059\delta_{2,p} - 48\delta_{3,p} - 17\delta_{4,p}) + O(h) \quad , \tag{B.12b}$$

$$\gamma_4 = F_4 - G_4 + 3H_4 = \frac{(\beta J)^8}{3072}(117 - 117\delta_{2,p} - 17\delta_{4,p}) + O(h) \quad , \tag{B.12c}$$

$$\delta_4 = I_4 - K_4 - 2L_4 = \frac{(\beta J)^8}{512}(5 - 5\delta_{2,p} - 8\delta_{3,p} + 7\delta_{4,p}) + O(h) \ , \tag{B.12d}$$

$$\epsilon_4 = -2M_4 \quad , \tag{B.12e}$$

$$\zeta_4 = N_4 + O_4 - X_4$$

$$= \frac{(\beta J)^8}{256}(1 - \delta_{2,p} - 7\delta_{3,p} + 3\delta_{4,p} + \delta_{6,p}) + O(h) \quad , \tag{B.12f}$$

$$\eta_4 = P_4 - Q_4 - 2R_4 = \frac{(\beta J)^8}{128}(-4 + 4\delta_{2,p} - 3\delta_{3,p} - 4\delta_{4,p}) + O(h) \tag{B.12g}$$

$$\kappa_4 = S_4 \quad , \tag{B.12h}$$

$$\lambda_4 = T_4 + U_4 = \frac{(\beta J)^8}{256}(1 - \delta_{2,p} - \delta_{3,p} + 3\delta_{4,p} - \delta_{6,p}) + O(h) \ , \tag{B.12i}$$

$$\mu_4 = -2V_4 \quad ; \quad \nu_4 = W_4 \quad . \tag{B.12j}$$

The equilibrium conditions,

$$\frac{\partial\beta f(R,Q_{xx},Q_{yy})}{\partial R} = 0 \ ; \ \frac{\partial\beta f(R,Q_{xx},Q_{yy})}{\partial Q_{xx}} = 0 \ ; \ \frac{\partial\beta f(R,Q_{xx},Q_{yy})}{\partial Q_{yy}} = 0 \ , \qquad (B.13)$$

give the self consistent equations for R , Q_{xx} and Q_{yy} respectively,

$$\alpha_1 + 2\alpha_2 R - \delta_2 Q_{xx} + 3\alpha_3 R^2 - \epsilon_3 Q_{xx}^2 - \zeta_3 Q_{yy}^2 - 2\eta_3 R Q_{xx} + 4\alpha_4 R^3 - 2\zeta_4 R Q_{xx}^2$$

$$- \eta_4 Q_{xx}^3 - 3\kappa_4 R^2 Q_{xx} - 2\lambda_4 R Q_{yy}^2 - \mu_4 Q_{yy}^3 - \nu_4 Q_{xx} Q_{yy}^2 + \ldots = 0 \quad , \qquad (B.14a)$$

$$\beta_1 + 2\beta_2 Q_{xx} + \delta_2 R - 3\beta_3 Q_{xx}^2 + \delta_3 Q_{yy}^2 + 2\epsilon_3 R Q_{xx} + \eta_3 R^2 + 4\beta_4 Q_{xx}^3 + 2\delta_4 Q_{xx} Q_{yy}^2$$

$$+ \epsilon_4 Q_{yy}^3 + 2\zeta_4 R^2 Q_{xx} + 3\eta_4 R Q_{xx}^2 + \kappa_4 R^3 + \nu_4 R Q_{yy}^2 + \ldots = 0 \quad , \qquad (B.14b)$$

$$2\gamma_2 Q_{yy} - 3\gamma_3 Q_{yy}^2 + 2\delta_3 Q_{xx} Q_{yy} + 2\zeta_3 R Q_{yy} + 4\gamma_4 Q_{yy}^3 + 2\delta_4 Q_{xx}^2 Q_{yy}$$

$$+ 3\epsilon_4 Q_{xx} Q_{yy}^2 + 2\lambda_4 R^2 Q_{yy} + 3\mu_4 R Q_{yy}^2 + 2\nu_4 R Q_{xx} Q_{yy} + \ldots = 0 \quad . \qquad (B.14c)$$

Appendix C: Stability analysis for the clock spin glass in the presence of a magnetic field

In this appendix, the Almeida–Thouless stability analysis will be derived for the p–state clock spin glass in the presence of a magnetic field. The calculation which follows is based on the derivation of Cragg *et al.* (1982) for the m–vector spin glass.

The free–energy functional in (3.3.13) can be expanded around its replica–symmetric value by taking,

$$R^{\alpha} = R + \omega^{\alpha} \quad ; \quad Q_{xx}^{\alpha\beta} = Q_{xx} + \eta^{\alpha\beta} \quad ; \quad Q_{yy}^{\alpha\beta} = Q_{yy} + \varphi^{\alpha\beta} \quad ; \quad \alpha \neq \beta \,. \qquad (C.1)$$

One gets,

$$\begin{aligned} g(R^{\alpha}, Q_{xx}^{\alpha\beta}, Q_{yy}^{\alpha\beta}) = g(R, Q_{xx}, Q_{yy}) + (\beta J)^2 \Bigg(& \sum_{\alpha\beta} S_{RR}^{\alpha\beta}\Big|_{RS} \omega^{\alpha}\omega^{\beta} \\ & + \sum_{(\alpha\beta)(\gamma\delta)} S_{Q_{xx}Q_{xx}}^{(\alpha\beta)(\gamma\delta)}\Big|_{RS} \eta^{(\alpha\beta)}\eta^{(\gamma\delta)} + \sum_{(\alpha\beta)(\gamma\delta)} S_{Q_{yy}Q_{yy}}^{(\alpha\beta)(\gamma\delta)}\Big|_{RS} \varphi^{(\alpha\beta)}\varphi^{(\gamma\delta)} \\ & + 2\sum_{(\alpha\beta)\gamma} S_{Q_{xx}R}^{(\alpha\beta)\gamma}\Big|_{RS} \eta^{(\alpha\beta)}\omega^{\gamma} + 2\sum_{(\alpha\beta)\gamma} S_{Q_{yy}R}^{(\alpha\beta)\gamma}\Big|_{RS} \varphi^{(\alpha\beta)}\omega^{\gamma} \\ & + 2\sum_{(\alpha\beta)(\gamma\delta)} S_{Q_{xx}Q_{yy}}^{(\alpha\beta)(\gamma\delta)}\Big|_{RS} \eta^{(\alpha\beta)}\varphi^{(\gamma\delta)}\Bigg) + \ldots \quad , \end{aligned} \qquad (C.2)$$

where $\Big|_{RS}$ denotes evaluation of matrix elements within the Replica–Symmetry ansatz.

The stability matrix $\underline{S}$ has dimension $n^2 \times n^2$ and elements,

$$S_{RR}^{\alpha\beta} = \frac{\partial^2 g}{\partial R^{\alpha} \partial R^{\beta}} = \delta^{\alpha\beta} - (\beta J)^2 \left[\left\langle (S_x^{\alpha})^2 (S_x^{\beta})^2 \right\rangle - \left\langle (S_x^{\alpha})^2 \right\rangle \left\langle (S_x^{\beta})^2 \right\rangle \right] \quad , \tag{C.3a}$$

$$S_{Q_{\mu\mu}R}^{(\alpha\beta)\gamma} = \frac{\partial^2 g}{\partial Q_{\mu\mu}^{\alpha\beta} \partial R^{\gamma}} = -(\beta J)^2 \left[\left\langle S_{\mu}^{\alpha} S_{\mu}^{\beta} (S_x^{\gamma})^2 \right\rangle - \left\langle S_{\mu}^{\alpha} S_{\mu}^{\beta} \right\rangle \left\langle (S_x^{\gamma})^2 \right\rangle \right] \quad , \tag{C.3b}$$

$$S_{Q_{\mu\mu}Q_{\nu\nu}}^{(\alpha\beta)(\gamma\delta)} = \frac{\partial^2 g}{\partial Q_{\mu\mu}^{\alpha\beta} \partial Q_{\nu\nu}^{\gamma\delta}} = \delta^{(\alpha\beta)(\gamma\delta)} \delta_{\mu\nu} - (\beta J)^2 \left[\left\langle S_{\mu}^{\alpha} S_{\mu}^{\beta} S_{\nu}^{\gamma} S_{\nu}^{\delta} \right\rangle \right.$$

$$\left. - \left\langle S_{\mu}^{\alpha} S_{\mu}^{\beta} \right\rangle \left\langle S_{\nu}^{\gamma} S_{\nu}^{\delta} \right\rangle \right] \quad ; \quad \mu , \nu = x , y \quad . \tag{C.3c}$$

The stability matrix elements are to be evaluated in the replica–symmetric approximation and stability requires the matrix $\underline{S}$ to be positive definite i.e., all eigenvalues should be positive.

The eigenvalue equation is

$$\underline{S}\, \underline{\sigma} = \lambda\, \underline{\sigma} \quad , \tag{C.4a}$$

where

$$\underline{\sigma} = \begin{bmatrix} \{\omega^{\alpha}\} \\ \{\eta^{(\alpha\beta)}\} \\ \{\varphi^{(\alpha\beta)}\} \end{bmatrix} \quad \alpha = 1,2,\ldots,n \quad , \tag{C.4b}$$

is a column vector with n^2 elements. As in section 3.7, the eigenvectors responsible for replica–symmetry breaking, $\underline{\sigma}_3$, have all ω^{α} zero and

$$\eta_3^{(\alpha\beta)} = \begin{cases} c_x & ; \quad (\alpha\beta) = (\theta\nu) \quad , \\ (2-n)^{-1} c_x & ; \quad \alpha \text{ or } \beta = \theta \text{ or } \nu \quad , \quad \text{but not both} \quad , \\ 2(2-n)^{-1}(3-n)^{-1} c_x & ; \quad \alpha,\beta \neq \theta,\nu \quad . \end{cases} \tag{C.5a}$$

$$\varphi_3^{(\alpha\beta)} = \begin{cases} c_y & ; \quad (\alpha\beta) = (\theta\nu) \quad , \\ (2-n)^{-1} c_y & ; \quad \alpha \text{ or } \beta = \theta \text{ or } \nu \quad , \quad \text{but not both} \quad . \\ 2(2-n)^{-1}(3-n)^{-1} c_y & ; \quad \alpha,\beta \neq \theta,\nu \quad . \end{cases} \tag{C.5b}$$

Therefore, the only contributing blocks of the matrix $\underline{S}$ are the ones in (C.3c); they present the following possible matrix elements:

$$S^{(\alpha\beta)(\alpha\beta)}_{Q_{\mu\mu}Q_{\nu\nu}} = \delta_{\mu\nu} - (\beta J)^2 \left[\left\langle S^\alpha_\mu S^\beta_\mu S^\alpha_\nu S^\beta_\nu \right\rangle - \left\langle S^\alpha_\mu S^\beta_\mu \right\rangle \left\langle S^\alpha_\nu S^\beta_\nu \right\rangle \right] = E_{\mu\nu} \quad , \tag{C.6a}$$

$$S^{(\alpha\beta)(\alpha\gamma)}_{Q_{\mu\mu}Q_{\nu\nu}} = -(\beta J)^2 \left[\left\langle S^\alpha_\mu S^\beta_\mu S^\alpha_\nu S^\gamma_\nu \right\rangle - \left\langle S^\alpha_\mu S^\beta_\mu \right\rangle \left\langle S^\alpha_\nu S^\gamma_\nu \right\rangle \right] = F_{\mu\nu} \quad , \tag{C.6b}$$

$$S^{(\alpha\beta)(\gamma\delta)}_{Q_{\mu\mu}Q_{\nu\nu}} = -(\beta J)^2 \left[\left\langle S^\alpha_\mu S^\beta_\mu S^\gamma_\nu S^\delta_\nu \right\rangle - \left\langle S^\alpha_\mu S^\beta_\mu \right\rangle \left\langle S^\gamma_\nu S^\delta_\nu \right\rangle \right] = G_{\mu\nu} \quad , \tag{C.6c}$$

which hold for all distinct $\alpha,\beta,\gamma,\delta$.

Equation (C.4a) yields for the vector $\underline{\sigma}_3$,

$$\sum_\nu (E_{\mu\nu} - 2F_{\mu\nu} + G_{\mu\nu})c_\nu = \lambda c_\mu \quad , \tag{C.7}$$

or in other words,

$$\sum_\nu (\delta_{\mu\nu} - \chi^{(2)}_{\mu\nu})c_\nu = \lambda c_\mu \quad , \tag{C.8a}$$

where

$$\chi_{\mu\nu}^{(2)} = (\beta J)^2 \left[(\langle S_\mu S_\nu \rangle_T - \langle S_\mu \rangle_T \langle S_\nu \rangle_T)^2 \right]_{av} \quad ; \quad \mu, \nu = x, y \quad . \tag{C.8b}$$

Equation (C.8a) is valid for arbitrary c_μ, giving the characteristic equation,

$$(1 - \chi_{xx}^{(2)} - \lambda)(1 - \chi_{yy}^{(2)} - \lambda) - (\chi_{xy}^{(2)})^2 = 0 \quad . \tag{C.9}$$

Appendix D: The low–temperature replica–symmetry–breaking line for the clock spin glass in the presence of a magnetic field

In this appendix, the low–temperature replica–symmetry breaking line behaviour, for a p–state clock spin glass in the presence of a magnetic field will be derived. As discussed in Chapter 5 , the onset of replica–symmetry breaking takes place along the line $Q_{yy} = 0$, for which equations (5.2.5) give

$$\lambda = 1 - \chi^{(2)}_{xx} = 1 - (\beta J)^2 \left[\langle S_x^2 \rangle_T^2 - 2 \langle S_x^2 \rangle_T \langle S_x \rangle_T^2 + \langle S_x \rangle_T^4 \right]_{av} \quad . \qquad (D.1)$$

The function $\chi^{(2)}_{xx}$ is to be evaluated in the replica–symmetric approximation and stability requires λ to be positive. On this line one has,

$$\lambda = 1 - (\beta J)^2 \int_{-\infty}^{\infty} \frac{du}{(2\pi)^{\frac{1}{2}}} e^{-u^2/2} \left\{ \left[Z^{-1} \frac{\partial^2 Z}{\partial a_x^2} \right]^2 - 2 Z^{-1} \frac{\partial^2 Z}{\partial a_x^2} \left[Z^{-1} \frac{\partial Z}{\partial a_x} \right]^2 + \left[Z^{-1} \frac{\partial Z}{\partial a_x} \right]^4 \right\} \quad , \qquad (D.2)$$

where

$$Z = \mathrm{tr}\, \exp\,(b S_x^2 + a_x S_x) \quad , \qquad (D.3a)$$

$$a_x = \beta J Q_{xx}^{\frac{1}{2}} u + \beta h \quad ; \quad b = \frac{(\beta J)^2}{2} (2R - Q_{xx}) \quad . \qquad (D.3b)$$

Making use of the identity,

$$e^{\alpha a^2} = \int_{-\infty}^{\infty} \frac{dy}{(2\pi)^{\frac{1}{2}}} \exp\left[-\frac{y^2}{2} + (2\alpha)^{\frac{1}{2}} a y\right] \quad , \tag{D.4}$$

equations (D.3) can be written as

$$Z = \int_{-\infty}^{\infty} \frac{dy}{(2\pi)^{\frac{1}{2}}} e^{-y^2/2} \operatorname{tr} \exp(cS_x) \quad , \tag{D.5a}$$

$$c = \beta J(2R - Q_{xx})^{\frac{1}{2}} y + \beta J Q_{xx}^{\frac{1}{2}} u + \beta h \quad . \tag{D.5b}$$

In the analysis which follows, the case $p = 3$, $h < 0$, will be treated separately.

(a) Cases $p \neq 3$ (h positive or negative) ; $p = 3$ $(h > 0)$

At low temperatures, one expects that

$$Q_{xx} = 1 + O(T) \quad ; \quad R = \frac{1}{2} + O(T) \quad , \tag{D.6}$$

and then,

$$\beta J Q_{xx}^{\frac{1}{2}} u \gg \beta J(2R - Q_{xx})^{\frac{1}{2}} y \quad ; \quad \beta h \gg \beta J(2R - Q_{xx})^{\frac{1}{2}} y \quad , \tag{D.7}$$

which gives

$$Z \approx \mathrm{tr}\,\exp\left(a_x S_x\right) = \sum_{k=0}^{p-1} \exp\left[a_x \cos\frac{2\pi k}{p} \right] \quad . \tag{D.8}$$

Equation (D.2) may now be written with a_x as an integration variable, by using

$$u = \frac{T a_x - h}{J Q_{xx}^{\frac{1}{2}}} \quad , \tag{D.9}$$

or in other words,

$$\lambda = 1 - \frac{\beta J}{Q_{xx}^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{da_x}{(2\pi)^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} \frac{(T a_x - h)^2}{J^2 Q_{xx}} \right\} \left\{ \left[Z^{-1} \frac{\partial^2 Z}{\partial a_x^2} \right]^2 \right.$$

$$\left. - 2 Z^{-1} \frac{\partial^2 Z}{\partial a_x^2} \left[Z^{-1} \frac{\partial Z}{\partial a_x} \right]^2 + \left[Z^{-1} \frac{\partial Z}{\partial a_x} \right]^4 \right\} \quad . \tag{D.10}$$

As $T \to 0$, $Q_{xx} \to 1$ and one gets

$$\lambda = 1 - \beta J \frac{a_p}{(2\pi)^{\frac{1}{2}}} \exp(-h^2/2J^2) \quad , \tag{D.11}$$

which gives the replica-symmetry-breaking line ($\lambda = 0$) ,

$$\frac{T}{J} = \frac{a_p}{(2\pi)^{\frac{1}{2}}} \exp(-h^2/2J^2) \quad . \tag{D.12}$$

In equations (D.11) and (D.12) a_p is a number depending only on the value of p ,

$$a_p = \int_{-\infty}^{\infty} da_x \left\{ \left[Z^{-1} \frac{\partial^2 Z}{\partial a_x^2} \right]^2 - 2\, Z^{-1} \frac{\partial^2 Z}{\partial a_x^2} \left[Z^{-1} \frac{\partial Z}{\partial a_x} \right]^2 + \left[Z^{-1} \frac{\partial Z}{\partial a_x} \right]^4 \right\} \quad , \qquad \text{(D.13)}$$

and Z is given by (D.8). The coefficient a_p can easily be evaluated analytically for $p = 2,4$ or numerically for any p, and typical values are listed on Table D.1 .

(b) Case $p = 3$ $(h < 0)$

As discussed in Chapter 5, the quadrupolar parameter R^{α} is for $p = 3$, a magnetization parameter. Therefore, the inversion $\underline{h} \to -\underline{h}$ must come together with $R^{\alpha} \to -R^{\alpha}$; similar calculations as the ones in case (a) follow and equations (D.5) become,

$$Z = \int_{-\infty}^{\infty} \frac{dy}{(2\pi)^{\frac{1}{2}}} e^{-y^2/2} \operatorname{tr} \exp (cS_x) \quad , \qquad \text{(D.14)}$$

$$c = \beta J(-2R - Q_{xx})^{\frac{1}{2}} y + \beta J Q_{xx}^{\frac{1}{2}} u + \beta h \quad . \qquad \text{(D.15)}$$

At low temperatures, one has

$$Q_{xx} = 1 + O(T) \quad ; \quad R = -\frac{1}{2} + O(T) \quad , \qquad \text{(D.16)}$$

and then, equation (D.8) is obtained.

One gets the same replica–symmetry–breaking line as the one in equation (D.12).

Table D.1

p	2	3	4	5	6	10	12	15
a_p	4/3	0.562	2/3	0.571	0.583	0.571	0.571	0.571

Table D.1: The coefficient a_p (equations (D.12) and (D.13)) for several values of p ; it oscillates for small p , but converges to a constant value as p gets large.

Appendix E: Averagings for an XY model in a four–fold anisotropy field

When doing perturbation expansions in Chapter 6, for an XY model with four–fold anisotropy, one needs to evaluate quantities like

$$\langle S_x^m S_y^n \rangle_D = \frac{1}{tr_D(1)} tr_D \left[S_x^m S_y^n \right] =$$

$$= \frac{\frac{1}{2\pi}\int_0^{2\pi} d\theta \cos^m\theta \sin^n\theta \exp(\beta D\cos 4\theta)}{\frac{1}{2\pi}\int_0^{2\pi} d\theta \exp(\beta D\cos 4\theta)} \tag{E.1}$$

where m, n, are positive integers and $\langle\ \rangle_D$ denotes an average with respect to $\exp(\beta D\cos 4\theta)$.

In order to evaluate such quantities we will make use of the identities,

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \sin m\theta \exp(\beta D\cos 4\theta) = 0 \quad , \tag{E.2a}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \cos m\theta \exp(\beta D\cos 4\theta) = \delta_{m,4k}\, I_k(\beta D) \quad , \tag{E.2b}$$

where $I_k(\beta D)$ are modified Bessel functions of the first kind of order k ,

$$I_k(z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \cos k\theta \exp(z\cos\theta) \quad ; \quad k = 0,1,2,\ldots \quad . \tag{E.3}$$

Equations (E.2) together with basic trigonometric relations makes it possible to evaluate averages like (E.1). Here follows a list of the ones used in Chapter 6:

$$\langle S_x^2 \rangle_D = \langle S_y^2 \rangle_D = \frac{1}{2} \quad , \tag{E.4a}$$

$$\langle S_x^4 \rangle_D = \langle S_y^4 \rangle_D = \frac{3}{8} + \frac{1}{8}\Delta_1 \quad , \tag{E.4b}$$

$$\langle S_x^2 S_y^2 \rangle_D = \frac{1}{8} - \frac{1}{8}\Delta_1 \quad , \tag{E.4c}$$

$$\langle S_x^6 \rangle_D = \langle S_y^6 \rangle_D = \frac{5}{16} + \frac{3}{16}\Delta_1 \quad , \tag{E.4d}$$

$$\langle S_x^4 S_y^2 \rangle_D = \langle S_x^2 S_y^4 \rangle_D = \frac{1}{16} - \frac{1}{16}\Delta_1 \quad , \tag{E.4e}$$

$$\langle S_x^8 \rangle_D = \langle S_y^8 \rangle_D = \frac{35}{128} + \frac{7}{32}\Delta_1 + \frac{1}{128}\Delta_2 \quad , \tag{E.4f}$$

$$\langle S_x^6 S_y^2 \rangle_D = \langle S_x^2 S_y^6 \rangle_D = \frac{5}{128} - \frac{1}{32}\Delta_1 - \frac{1}{128}\Delta_2 \quad , \tag{E.4g}$$

$$\langle S_x^4 S_y^4 \rangle_D = \frac{3}{128} - \frac{1}{32}\Delta_1 + \frac{1}{128}\Delta_2 \quad , \tag{E.4h}$$

$$\langle S_x^m \rangle_D = \langle S_y^m \rangle_D = 0 \quad , \quad \text{for m odd} \quad , \tag{E.4i}$$

$$\langle S_x^m S_y^n \rangle_D = 0 \quad , \quad \text{for m or n or both odd} \quad , \tag{E.4j}$$

where

$$\Delta_k = \frac{I_k(\beta D)}{I_0(\beta D)} \quad . \tag{E.5}$$

Equations (E.4) can be checked in two particular limits with equations (A.3) → (A.18) from Appendix A:

a) $D = \infty$: 4–state clock limit

One has $\Delta_k = 1$ and equations (E.4) reproduce the equations from Appendix A for $p = 4$.

b) $D = 0$: XY limit

One has $\Delta_k = 0$ and equations (E.4) reproduce the equations from Appendix A for $p = \infty$.

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