NON-COMMUTATIVE STOPPING TIMES

by

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ABSTRACT

A theory of stopping times in a Von Neumann algebra is presented. Stopping is discussed in both tracial and non-tracial settings. Stopping processes, in particular stopping martingales, are studied, where it is shown that for a certain class of martingales our definition of stopping is equivalent to the usual definition (i.e. that given in the commutative theory). It is shown that stopping preserves the martingale property and we prove the Doob's optional stopping theorem. It is also shown that stopping a L^2-bounded martingale is equivalent to applying a certain projection to the element closing the martingale. We discuss certain algebraic relationships between these projections and establish that stopping preserves L^2-boundedness for martingales. We further discuss stopping times that lie in the commutant, where it is shown that these projections, mentioned above, are indeed conditional expectations. As a concrete example of the tracial case, we work in the Clifford algebra arising in quantum statistical mechanics. Here we are interested in stopping increasing L^1-processes associated with the Doob-Meyer decomposition of L^2-martingales. We also give a characterisation of these stopped processes just as in the commutative theory.

For the non-tracial case, we prefer to work in the Canonical Anticommutation Relations algebra. The main result here is the random stopping time convergence theorem for martingales, thus extending the existing result in the tracial setting.

The probability gauge space given by $L^{\infty}(\Omega,\Sigma,P) \otimes M_2(C)$ is also studied. Here the Doob-Meyer decomposition for L^2-bounded martingales

is obtained and stochastic integrals are constructed. Concrete examples of stopping times are given and stopping processes is discussed. The concept of local martingales is introduced, where a Doob-Meyer type decomposition for a certain class of local martingales is obtained. A stochastic integral with respect to a local martingale is constructed.

CONTENTS

For Penny

"And time, that takes survey of

all the world, must have a stop"

William Shakespeare

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CHAPTER ONE

INTRODUCTION

Ever since the initiating work of Dixmier in 1953 [20] and Segal in 1956 [35] the theory of non-commutative probability has been developing extensively (see for example [1,2,27] and references therein). In particular in the last decade or so, there has been great interest in the sub-branch: quantum probability. This is partially due, on the one hand, to the fact that the quantum theory is already a probabilistic theory and hence it is natural to study probability theory with quantum objects, (which are non-commuting in general). Whilst on the other hand, like Dixmier in 1953 [20], it is natural, at least mathematically, to extend the ideas of classical (or commutative) probability theory to a non-commutative setting. Whilst making this transition, the quantum theory is not only often a motivation, but also presents several concrete non-commutative models with desirable properties to work in. We may think of this as the analogue of the classical theory, where probabilists often prefer to establish their results in a concrete model furnished by the Brownian motion process as opposed to developing their theorems in a general arbitary model directly. Indeed, the analogy between the model given by the Brownian motion process in the classical theory and quantum probability as an example of the non-commutative theory, is far deeper than a mere coincidence in their desirable properties. Indeed a result of Segal [36] says that we may identify the Brownian motion process with a certain quantum mechanical process called the quantum Brownian motion process [25].

Some of the recent contributions to non-commutative probability have included constructing martingales with certain properties and subsequently developing an integral with respect to them. In particular when restricting to a concrete model from quantum theory far-reaching analogies with the properties of the usual Ito integral are observed. Indeed several results engendered by the model generated by the Brownian motion process have had parallels in models from the quantum theory [1,2,3,8,9,10,11,12,13,16,25,26]. However, the greatest motivation for establishing the non-commutative probability theory as a generalisation of the classical theory is the following simple observation:

The starting point for studying the classical theory of probability is the triple (Ω,Σ,P), where Ω is a sample space, Σ is a σ-algebra of subsets of Ω (the algebra of events) and P is a probability measure on Σ so that we may assume that $P(\Omega) = 1$. Objects of interest are the random variables. These are precisely the Σ-measurable functions on Ω and taking values in $\mathbb{C}$. These random variables form a *-algebra, of which the bounded ones form $L^\infty(\Omega,\Sigma,P)$, which is a commutative Von Neumann algebra of bounded operators in the Hilbert space $L^2(\Omega,\Sigma,P)$. The action of $L^\infty(\Omega,\Sigma,P)$ on $L^2(\Omega,\Sigma,P)$ is given by multiplication of course and the adjoint is simply the complex conjugation. Now following Segal [35] we conclude that the *-algebra of random variables is precisely the *-algebra of measurable operators over the Von Neumann algebra $L^\infty(\Omega,\Sigma,P)$. That is each random variable is a closed, densely defined operator on $L^2(\Omega,\Sigma,P)$ and affiliated with $L^\infty(\Omega,\Sigma,P)$. Another important objects are the elements of Σ, the algebra of events. Thus given a set $E \in \Sigma$, we form the function χ_E, which is Σ-measurable, bounded and satisfies $\chi_E\chi_E = \chi_E$. That is, to every event E, we associate a projection operator χ_E in $L^\infty(\Omega,\Sigma,P)$. Conversely, every projection in $L^\infty(\Omega,\Sigma,P)$ is of this form. Thus there is no loss in replacing an event E by a projection χ_E. The expectation E, of $F \in L^\infty(\Omega,\Sigma,P)$ is given by:

$$\int_\Omega F(w)dP(w)$$

and satisfies

(i) $\mathsf{E} : L^\infty(\Omega,\Sigma,P)^+ \to [0,\infty]$

(ii) $\mathsf{E}(\lambda F + \mu G) = \lambda \mathsf{E}(F) + \mu \mathsf{E}(G)$

(iii) $\mathsf{E}(FG) = \mathsf{E}(GF)$

(iv) $F \in L^\infty(\Omega,\Sigma,P)^+$ and $\mathsf{E}(F) = 0$ then $F = 0$

(v) $(F_\alpha) \subseteq L^\infty(\Omega,\Sigma,P)^+$ and $F_\alpha \uparrow F$, $F \in L^\infty(\Omega,\Sigma,P)$ then $\mathsf{E}(F_\alpha) \uparrow \mathsf{E}(F)$.

Thus E is a faithful normal trace on $L^\infty(\Omega,\Sigma,P)$ which extends to the *-algebra of measurable operators over $L^\infty(\Omega,\Sigma,P)$ [35] and is equal to the usual expectation of a random variable [35]. For $1 \leq p < \infty$, the L^p-spaces over $L^\infty(\Omega,\Sigma,P)$ are then just the equivalent classes of random variables equal P - a.e. and satisfying $\mathsf{E}(|F|^p) < \infty$. These spaces are denoted by $L^p(\Omega,\Sigma,P)$.

Now if Σ' is a sub-σ-field of Σ then $L^\infty(\Omega,\Sigma',P)$ is a Von-Neumann subalgebra of $L^\infty(\Omega,\Sigma,P)$ and conversely every Von Neumann subalgebra of $L^\infty(\Omega,\Sigma,P)$ is of this form. The conditional expectation of a random variable (operator!) $F \in L^\infty(\Omega,\Sigma,P)$ with respect to Σ' is a Σ'-measurable bounded function $\mathsf{E}(F|\Sigma')$ such that

$$\int_E E(F|\Sigma')dP = \int_E F\, dP$$

for any $E \in \Sigma'$. That is (if and only if)

$$E(E(F|\Sigma')P) = E(FP)$$

for any projection P in $L^\infty(\Omega,\Sigma',P)$.

Furthermore the conditional expectation of I $(= \chi_\Omega)$ is I and $E(HE(F|\Sigma')G) = E(HFG)$ for any H , G in $L^\infty(\Omega,\Sigma',P)$ and $F \in L^\infty(\Omega,\Sigma,P)$. Thus we think of the conditional expectation as a map from $L^\infty(\Omega,\Sigma,P)$ onto $L^\infty(\Omega,\Sigma',P)$ with the above properties. Then the conditional expectation is a contraction, positive preserving, faithful normal projection mapping of $L^\infty(\Omega,\Sigma,P)$ onto $L^\infty(\Omega,\Sigma',P)$ with the properties that $E(HFG|\Sigma') = HE(F|\Sigma')G$ for $H,G \in L^\infty(\Omega,\Sigma',P)$ and $F \in L^\infty(\Omega,\Sigma',P)$, and $E(F|\Sigma')^*E(F|\Sigma') \leq E(F^*F|\Sigma')$. It is known that this map extends to a positive faithful, linear contraction from $L^p(\Omega,\Sigma,P)$ onto $L^p(\Omega,\Sigma',P)$ of norm one and satisfying $E(E(F|\Sigma')G) = E(FE(G|\Sigma'))$ where $F \in L^p(\Omega,\Sigma,P)$ and $G \in L^q(\Omega,\Sigma,P)$ with $p^{-1} + q^{-1} = 1$. Thus we think of the conditional expectation as a map from $L^\infty(\Omega,\Sigma,P)$ onto $L^\infty(\Omega,\Sigma',P)$ satisfying $E(HE(F|\Sigma')G) = E(HFG)$ for $F \in L^\infty(\Omega,\Sigma,P)$ and $H,G \in L^\infty(\Omega,\Sigma',P)$ and $E(I|\Sigma') = I$.

Finally the commutative theory of stochastic processes depends on a filtration of sub-σ-fields of Σ . Thus if $(\Sigma_t)_{t \in R^+}$ is such a filtration then a stochastic process $(X_t)_{t \in R^+}$ is a family of random variables indexed by R^+ such that for each $t \in R^+$, X_t is a Σ_t-measurable function. Thus in our formulation we may think of X_t as a closed densely defined operator affiliated with $L^\infty(\Omega,\Sigma_t,P)$.

We now summarise the formulation of the classical theory at an operator theoretical level.

{Bounded random variables} $\leftrightarrow$ $L^{\infty}(\Omega,\Sigma,P)$

{Random variables} $\leftrightarrow$ measurable operators over $L^{\infty}(\Omega,\Sigma,P)$

Event $\leftrightarrow$ Projection $P \in L^{\infty}(\Omega,\Sigma,P)$

Expectation $\leftrightarrow$ Trace

Conditional Expectation w.r.t. Σ' $\leftrightarrow$ a map from $L^{\infty}(\Omega,\Sigma,P)$ onto $L^{\infty}(\Omega,\Sigma',P)$ with certain properties

Stochastic process (X_t) $\leftrightarrow$ (X_t) : such that X_t is a measurable operator over $L^{\infty}(\Omega,\Sigma_t,P)$.

It is now clear that the essential ingredients to study classical probability theory are the Von Neumann algebra $L^{\infty}(\Omega,\Sigma,P)$ over the Hilbert space $L^2(\Omega,\Sigma,P)$, a filtration of Von Neumann subalgebras $(L^{\infty}(\Omega,\Sigma_t,P))_{t \in R^+}$ and a faithful normal trace E .

Thus a natural generalisation of the classical theory is to consider a non-commutative Von Neumann algebra A over a Hilbert space H , a filtration $(A_t)_{t \in R^+}$ of Von Neumann subalgebras of A and a finite faithful normal trace ϕ on A . We can now construct the non-commutative L^p-spaces associated with (A,ϕ) [38]. The random variables are then just the elements of the *-algebra of measurable operators over A [35] and a stochastic process $(X_t)_{t \in R^+}$ is a family of random variables such that for each t , X_t is a measurable operator over A_t . Thus instead of thinking of a stochastic base [28,30] $(\Omega,\Sigma,P,\Sigma_t,R^+)$ we think of (A,H,ϕ,A_t,R^+) in the non-commutative setting. The existence of the conditional expectation denoted by M_t , $M_t : A \to A_t$ for each $t \in R^+$ with the properties $M_t(yxz) = yM_t(x)z$ where $y,z \in A_t$, $x \in A$ and $M_t(I) = I$ is assured by Takesaki [37].

There exists yet another extension to the model discussed above. This is to consider a Von Neumann algebra with only a faithful normal state. Such models occur quite naturally in quantum statistical mechanics [18]. Thus we are now not only concerned with extending the ideas of classical probability theory to a finite Von Neumann algebra (with a trace), but if we intend to solve quantum analogues, of say the Langevin equation, we have to extend our theory to type III Von Neumann algebras. Indeed several notions and objects of interest in the classical theory have now been extended to a non-commutative setting to include both the tracial and the non-tracial cases [13,26,31].

In this thesis we extend the notion of stopping time to a non-commutative setting. In chapter two we study stopping times in a finite Von Neumann algebra [14]. We define "stopping" for non-commutative processes and show that our definition of stopping is equivalent to the commutative one for a certain class of processes. In particular we are interested in stopping martingales, where it is shown that stopping preserves the martingale property. Certain algebraic relations between stopping times are discussed and these are used to prove the non-commutative analogue of the Doobs optional stopping theorem. We also consider a concrete model from quantum theory, the Clifford algebra generated by the free Fermion fields over the Hilbert space of square integrable functions on the positive reals. Here we show how to stop increasing processes associated with Doob-Meyer decomposition of certain martingales [8]. We also give a characterisation of these stopped processes. In chapter three we continue as in chapter two but here we do not require the existence of a trace. We require only a faithful normal state [15]. Once again stopping is discussed, but here we prefer to work in the Hilbert space given by the G.N.S. representation associated with the state. Thus we obtain results about stopping operator valued martingales whilst working at a Hilbert space level. We also look at a concrete model from quantum theory, the C.A.R. algebra. Here we prove a random time convergence theorem analogues to the tracial case given in [7]. In chapter four we work in the type I_2 Von Neumann algebra: $L^\infty(\Omega,\Sigma,P) \otimes M_2(C)$. Here certain objects such as martingales, stochastic integrals etc. are briefly discussed and a Doob-Meyer decomposition theorem is proved for L^2-bounded martingales. The main purpose of this chapter however is to give examples of non-commutative local martingales and consequently develop a stochastic integral with respect to them. This is achieved by defining stopping in a slightly weaker sense than that given in chapter two.

There has been some development in non-commutative stopping times. In [24] the strong Markov property of the Boson Brownian motion is proved. However, there the stopping times are "Pre" adapted and the Brownian motion "Future" adapted so that the two commute. The Brownian motion is also a strongly continuous process and there exists a total set in the underlying Hilbert space (Symmetric Fock space) factorising the "pre" and the "future" algebras. In [3] the Fermion strong Markov property is proved, again under similar conditions to those in [24]. We shall follow the definition of stopping as given in [7]. There a more abstract account of stopping is given though most results are obtained in the Clifford algebra. Inspired by [24] and using finite stopping times, a strong factorisation of the Boson Fock space is given in the recent pre-print [34]. In the pre-print [4], using the results of [24] and [34], certain cocycle identities associated with a unitary processes satisfying a quantum stochastic differential equation [25,26] have been obtained for finite stopping times. Finally in [6], concrete examples of stopping times have been obtained. Necessary and sufficient conditions have also been given for a process to be a stopping time in the gauge space given by $L^{\infty}(\Omega,\Sigma,P) \otimes M_2(C)$.

Finally a point about notation: We shall often abbreviate a family indexed by R^+ , such as, a filtration of Von Neumann algebras, processes etc., as (A_t) , (X_t) etc. instead of $\{A_t : t \in R^+\}$ $\{X_t : t \in R^+\}$ etc.

CHAPTER TWO

STOPPING TIMES IN A FINITE VON NEUMANN ALGEBRA

2.1 Introduction

In this chapter we study stopping times in a finite non-commutative Von Neumann algebra. First we reformulate stopping times and stopping processes in the commutative probability theory.

Thus let $(\Omega,\Sigma,P,(\Sigma_t),R^+)$ be a stochastic base and for simplicity we assume that the filtration of sub-σ-fields (Σ_t) of Σ is right continuous. That is, for each $t \in R^+$,

$$\bigcap_{s > t} \Sigma_s = \Sigma_{t+} = \Sigma_t \quad .$$

A stopping time τ is a Σ-measurable $R^+ \cup \{\infty\}$ $(= \overline{R^+})$ valued function on Ω with the property that for each $t \in R^+$,

$$\{\omega \in \Omega : \tau(\omega) \leq t\} \in \Sigma_t \quad .$$

In view of the right continuity of the filtration we can replace this last condition by requiring that

$$\{\omega \in \Omega : \tau(\omega) < t\} \in \Sigma_t \quad .$$

We observe that τ is a measurable operator over $L^\infty(\Omega,\Sigma,P)$ [35].

Now, with τ we can associate an increasing family of sets in Σ and indexed by R^+. That is, for each $t \in R^+$, we consider the set $\{\tau < t\}$. Then clearly $\{\tau < t\} \subseteq \{\tau < t\}$ if $s \leq t$. (We have written $\{\tau < t\}$ instead of $\{\omega \in \Omega : \tau(\omega) < t\}$ as is customary in literature.) Now $\{\tau < t\} \in \Sigma_t$ by hypothesis, hence the function $\omega \to \chi_{\{\tau<t\}}(\omega)$ is Σ_t-measurable, an idempotent and hence belongs to $L^\infty(\Omega,\Sigma_t,P)$. Thus we can associate with τ, an increasing projection valued process [28,29], $\tilde{\tau}$, given by

$$\tilde{\tau}(t) = \chi_{\{\tau<t\}} \quad .$$

Clearly $\tilde{\tau}(0) = 0$ and we set $\tilde{\tau}(\infty) = I$ since $\{\tau \leq \infty\} = \Omega$. We observe that if τ is bounded, say there is a $T \in R^+$ such that $\tau(\omega) \leq T \;\forall\, \omega \in \Omega$, then clearly $\tilde{\tau}(T) = I$. Also if σ is another stopping time and $\sigma \leq \tau$ then clearly $\{\tau < t\} \subseteq \{\sigma \leq t\}$ for each $t \in R^+$ and hence $\tilde{\tau}(t) \leq \tilde{\sigma}(t)$. Moreover $\sigma \wedge \tau$ and $\sigma \vee \tau$ are stopping times given by $\omega \to \sigma(\omega) \wedge \tau(\omega)$ and $\omega \to \sigma(\omega) \vee \tau(\omega)$ respectively. Now since $\{\sigma \wedge \tau < t\} = \{\sigma < t\} \cup \{\tau < t\}$ and $\chi_{\{\sigma<t\} \cup \{\tau<t\}} = \chi_{\{\sigma<t\}} \vee \chi_{\{\tau<t\}}$, the corresponding projection valued process associated with $\sigma \wedge \tau$ is given by $\widetilde{\sigma \wedge \tau}(t) = \tilde{\sigma}(t) \vee \tilde{\tau}(t)$ for each $t \in \overline{R^+}$. Similarly $\widetilde{\sigma \vee \tau}(t) = \tilde{\sigma}(t) \wedge \tilde{\tau}(t)$ for each $t \in \overline{R^+}$. Also if $\tau(\omega) = t$, $\forall\, \omega \in \Omega$, that is τ is the deterministic time given by $t \in R^+$. Then $\{\tau < s\} = \phi$ if $s \leq t$ and $\{\tau < s\} = \Omega$ if $s > t$. Hence the projection valued process associated with τ, $\tilde{\tau}$, is given by $\tilde{\tau}(s) = 0$ if $s \leq t$ and $\tilde{\tau}(s) = I$ if $s > t$.

On the other hand, given an increasing family of projections indexed by $\overline{R^+}$, (χ_{E_t}) and such that $\chi_{E_0} = 0$, $\chi_{E_\infty} = I$ and for each $t \in R^+$, $\chi_{E_t} \in L^\infty(\Omega,\Sigma_t,P)$, we can associate a stopping time τ_n by setting

$$\tau_n = \sum_{k=1}^{\infty} \frac{k}{2^n} (\chi_{E_{\frac{k}{2^n}}} - \chi_{E_{\frac{k-1}{2^n}}})$$

for each $n \in \mathbb{N}$.

Another example is

$$\tau(\omega) = \begin{cases} \operatorname{Inf}\{t : \chi_{E_t}(\omega) = 1\} & \text{if inf exists} \\ \infty & \text{otherwise} \end{cases} .$$

Thus we may think of a stopping time as an increasing projection valued process, starting with the zero-projection at $t = 0$ and finishing with the identity at $t = \infty$.

Before we go on to the discussion about stopping processes, we recall one more definition from the commutative theory. Let Σ_τ denote the σ-field of "events prior to τ ". That is

$$\Sigma_\tau = \{A \in \Sigma : A \cap \{\tau<t\} \in \Sigma_t \ \forall t \in R^+\} .$$

Now let $X = (X_t)$ be a L^2-bounded martingale, so that there is a $X \in L^2(\Omega,\Sigma,P)$ such that $X_t = E(X|\Sigma_t)$ [5]. Let X_τ denote the stopped random variable defined by

$$\omega \to X_{\tau(\omega)}(\omega)$$

where τ is a stopping time. We observe that if τ is simple, say $\tau = \sum_i t_i \chi_{E_i}$, then

$$X_\tau = \sum_i X_{t_i} (\tilde{Q}_{t_i} - \tilde{Q}_{t_{i-1}})$$

where $\tilde{Q}_{t_i} = \bigvee_{j\leq i} \chi_{E_j}$.

Now let $\theta \in P[0,\infty]$ - the space of partitions of $[0,\infty]$. Say $\theta = \{t_0,\ldots,t_n\}$ then

$$\tau(\theta) = \sum t_i \, \chi_{\{t_{i-1}\leq\tau<t_i\}}$$

defines a stopping time, hence

$$X_{\tau(\theta)} = \sum X_{t_i} \chi_{\{t_{i-1}\leq\tau<t_i\}}$$

$$= \sum X_{t_i} (\chi_{\{\tau<t_i\}} - \chi_{\{\tau<t_{i-1}\}})$$

$$= \sum_i E(X|\Sigma_{t_i})\Delta Q_{t_i}$$

where $Q_{t_i} = \chi\,\{\tau < t_i\}$.

Writing $E_t(\cdot)$ instead of $E(\cdot|\Sigma_t)$ we have that

$$X_{\tau(\theta)} = \sum_i E_{t_i}(X)\Delta Q_{t_i}$$

$$= E_{\tau(\theta)}\,(X)$$

where $E_{\tau(\theta)}(\cdot) = \sum_i E_{t_i}(\cdot)\Delta Q_{t_i}$.

It is not difficult to show that $E_{\tau(\theta)}$ is a self-adjoint projection on $L^2(\Omega,\Sigma,P)$ and if θ' is another partition of $[0,\infty]$ and $\theta' \supseteq \theta$ (θ' is finer than θ) then $E_{\tau(\theta')} \leq E_{\tau(\theta)}$. Thus let

$$E_\tau = \inf_{\theta \in P[0,\infty]} E_{\tau(\theta)} \quad .$$

Then E_τ leaves $L^2(\Omega,\Sigma_\tau,P)$ invariant, for let $A \in \Sigma_\tau$. Then

$$\begin{aligned} E_\tau(\chi_A) &= \operatorname*{Lim}_\theta E_{\tau(\theta)}(\chi_A) \\ &= \operatorname*{Lim}_\theta \sum_\theta E_{t_i}(\chi_A)\Delta Q_{t_i} \\ &= \operatorname*{Lim}_\theta \sum_\theta E_{t_i}(\chi_A \Delta Q_{t_i}) \\ &= \operatorname*{Lim}_\theta \sum_\theta \chi_A \Delta Q_{t_i} \\ &\qquad \text{since } \chi_A Q_{t_i} \in L^\infty(\Omega,\Sigma_{t_i},P) \\ &= \chi_A \quad . \end{aligned}$$

Indeed, it can be verified that E_τ is the conditional expectation map of $L^2(\Omega,\Sigma,P)$ onto $L^2(\Omega,\Sigma_\tau,P)$.

Now, it is known that there exists an increasing sequence (T_n) of partitions such that the corresponding stopping times $\tau(T_n)$ converges to τ (a.s.) from above [28]. Since $X_t = E_t(X)$ is also right continuous we have that

$$\begin{aligned} X_\tau &= \operatorname{Lim} X_{\tau_n} \qquad \text{a.s.} \\ &= \operatorname{Lim} X_{\tau(T_n)} \qquad \text{a.s.} \\ &= \operatorname{Lim} E_{\tau(T_n)}(X) \qquad \text{a.s.} \\ &= E_\tau(X) \quad . \end{aligned}$$

That is $E_\tau(X) = X_\tau$, and since $\|E_{\tau(n)}(X)\|_2 \leq \|X\|_2$, we have that

$$E_{\tau(T_n)}(X) \to X_\tau$$

is the L^2-norm. Thus given $\varepsilon > 0$ there exists T_N such that for all $T_n \supseteq T_N$

$$\|E_{\tau(T_N)}(X) - E_\tau(X)\|_2 < \varepsilon \quad .$$

Now let θ be such that $\theta \supseteq T_N$ then

$$\begin{aligned}
\|X_\tau - X_{\tau(\theta)}\|_2 &= \|E_\tau(X) - E_{\tau(\theta)}(X)\|_2 \\
&= \|E_{\tau(T_N)}(E_\tau(X) - E_{\tau(\theta)}(X))\|_2 \\
&= \|E_{\tau(\theta)}(E_\tau(X) - E_{\tau(T_N)}(X))\|_2 \\
&\leq \|E_\tau(X) - E_{\tau(T_N)}(X)\|_2 \\
&< \varepsilon \quad .
\end{aligned}$$

Thus we have that

$$X_\tau = L^2 - \operatorname*{Lim}_\theta \sum_\theta X_{t_i} \, \Delta \, Q_{t_i} \quad .$$

Likewise, a similar argument shows that if $X = (X_t)$ is a uniformly integrable L^1-martingale then [14]

$$X_\tau = L^1 - \operatorname*{Lim}_\theta \sum_\theta X_{t_i} \, \Delta \, Q_{t_i} \quad .$$

If τ is a stopping time and (X_t) is a process then the process stopped by τ is defined as

$$X_{\tau \wedge t} : \Omega \to C$$

$$\omega \to X_{\tau(\omega) \wedge t}(\omega)$$

for each $t \in R^+$.

Now $\tau \wedge t$ defines a stopping time and setting $Q'_s = \chi\ \{\tau \wedge t < s\}$, we have that $Q'_s = I$ if $s > t$, thus $\tau \wedge t$ is bounded. Now arguing as above we observe that if $X = (X_t)$ is a L^2-bounded martingale then

$$X_{\tau \wedge t} = L^2 - \underset{\theta}{\mathrm{Lim}} \sum X_{t_i} \, \Delta \, Q'_{t_i}$$

$$= L^2 - \underset{\theta}{\mathrm{Lim}} \sum_{t_i \le t} X_{t_i} \, \Delta \, Q'_{t_i} + X_{t_{i+1}}(1 - Q'_{t_i}) \quad .$$

But $\{\tau \wedge t < s\} = \{\tau < s\}$ if $s \le t$. Hence

$$\chi_{\tau \wedge t} = L^2 - \underset{\theta}{\mathrm{Lim}} \left(\sum_{t_i \le t} X_{t_i} \, \Delta \, Q_{t_i} + X_{t_{i+1}}(1 - Q_{t_i}) \right) \quad .$$

We shall often write

$$X_\tau = \int_0^\infty X_s \, dQ_s$$

$$X_{\tau \wedge t} = \int_0^t X_s \, dQ_s + X_t(1 - Q_t) \quad .$$

Having briefly reviewed stopping in the commutative theory, we now move onto a non-commutative setting.

2.2 The Non-Commutative Setting

In this section we introduce the context in which we shall be working and make some definitions.

Let A be a finite Von Neumann algebra of bounded operators on some Hilbert space K. Let ϕ be a faithful normal trace on A, $(A_t)_{t \in R^+}$ be a filtration of sub Von Neumann algebra of A with conditional expectations $(M_t)_{t \in R^+}$. That is M_t maps A onto A_t with the properties that $M_t(I) = I$ and $\phi(yM_t(x)z) = \phi(yxz)$ for all $y,z \in A_t$ and $x \in A$ [27]. Finally we suppose the family (A_t) satisfies

(i) $A_s \subseteq A_t$ if $s \leq t$

(ii) $\bigcap_{s > t} A_s = A_t$

(iii) $(\bigcup_t A_t)'' = A$.

For $1 \leq p \leq \infty$, let $L^p(A)$ denote the non-commutative L^p-spaces associated with A and ϕ [38]. Then it is known that the conditional expectation can be extended to a positive linear faithful map, which we again denote by M_t from $L^p(A)$ onto $L^p(A_t)$ and of norm one and satisfies $M_t(M_t(X)Y) = M_t(XM_t(Y))$ for all $X \in L^p(A)$, $Y \in A$ [27].

2.21 Definitions

(i) Let $1 \leq p \leq \infty$. Then an L^p-process is a family $\{X_t : t \in R^+\} \subseteq L^p(A)$ such that for each $t \in R^+$, $X_t \in L^p(A_t)$.

(ii) An L^p-process is called a martingale if (in addition to (i)) it satisfies

$$M_s(X_t) = X_s \qquad \text{for all} \quad s \leq t \quad .$$

(iii) An L^p-martingale is L^p-bounded if (in addition to (ii)) it satisfies

$$\sup_t \|X_t\|_p < \infty \quad .$$

In this case it is known [5] that if $1 < p \leq \infty$, there is $X \in L^p(A)$ such that

(a) $X_t = M_t(X)$ for all $t \in R^+$

(b) $L^p - \text{Lim } X_t = X$.

(iv) An L^1-process is called weakly relatively compact or uniformly integrable if it is L^1-bounded and satisfies

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : R \in A \ , \ \|R\|_\infty \leq 1 \ , \ \|R\|_1 < \delta$$

then $|\phi(RX_t)| < \varepsilon$ for all $\forall\, t \in R^+$.

2.22 Remark

If (X_t) is a L^p-bounded martingale and $1 < p \leq \infty$ so that $X_t = M_t(X)$ then we may regard (X_t) as a process indexed by $\overline{R^+}$ since we can set $X_\infty = X$.

2.23 Definition

(i) A process $\tau = (Q_t)$ indexed by $\overline{R^+}$ is called a stopping time if $\tau(0) = 0$, $\tau(\infty) = I$, $\tau(t) = Q_t$ is a projection in A_t for each $t \in R^+$ and $Q_s \leq Q_t$ for all $s \leq t$ [7,14].

(ii) Let $\tau = (Q_t)$ and $\sigma = (Q_t')$ be stopping times. Then we say $\tau \leq \sigma$ if and only if $Q_t \geq Q_t'$ for all $t \in R^+$.

(iii) The deterministic time given by say $t \in R^+$ is the process $\tilde{t}$:

$$\tilde{t}(s) = \begin{cases} 0 & s \leq t \\ I & s > t \end{cases} .$$

(iv) A stopping time $\tau = (Q_t)$ is bounded if there is a $T \in R^+$ such that

$$\tau(s) = Q_s = I \qquad \text{for all} \quad s \geq T \qquad .$$

(v) If $\tau = (Q_t)$ and $\sigma = (Q_t')$ are stopping times then we define the stopping times $\tau \vee \sigma$ and $\tau \wedge \sigma$ by setting

$$\tau \vee \sigma(s) = Q_s \wedge Q_s'$$

$$\tau \wedge \sigma(s) = Q_s \vee Q_s'$$

where $\wedge$ denotes the infimum and $\vee$ the supremium of projections. We observe that the stopping time $\tau \wedge t$ is given by:

$$\tau \wedge t(s) = \begin{cases} Q_s & s \leq t \\ I & s > t \end{cases}$$

hence $\tau \wedge t$ is bounded.

In the introduction to this chapter (2.1) we observed that stopping a L^2-bounded martingale in the commutative theory is equivalent to

$$L^2 - \underset{\theta \in P[0,\infty]}{\text{Lim}} \sum_{\theta} X_{t_i} \; \Delta \, Q_{t_i}$$

where $Q_t = X_{\{\tau<t\}}$. That is

$$X_\tau = L^2 - \underset{\theta}{\text{Lim}} \sum X_{t_i} \; \Delta \, Q_{t_i}$$

$$= L^2 - \underset{\theta}{\text{Lim}} \; E_{\tau(\theta)}(X)$$

where $X_t = E_t(X)$.

It is now natural to ask if this limit can be extended to a non-commutative setting. That is if

$$\underset{\theta \in P[0,\infty]}{\text{Lim}} \sum_{\theta} M_{t_i}(\cdot) \; \Delta \, Q_{t_i}$$

exists in the non-commutative setting.

In the next section we show that the limit exists, thus giving us a way of stopping, at least, L^2-bounded martingales.

2.3 Time Projections in a Von Neumann Algebra

2.31 Definition

Let $\tau = (Q_t)$ be a stopping time and $\theta = \{t_0,\ldots,t_n\} \in P[0,\infty]$ we then define $M_{\tau(\theta)}$ by

$$M_{\tau(\theta)} = \sum_{i=0}^{n} M_{t_i}(\cdot)\Delta Q_{t_i} \qquad .$$

Indeed, $M_{\tau(\theta)}$ is the non-commutative extension of $E_{\tau(\theta)}$.

We now have the following:

2.32 Theorem (Properties of $M_{\tau(\theta)}$) [14]

Let $\tau = (Q_t)$ be a stopping time. Then

(i) $M_{\tau(\theta)}$ is a self-adjoint projection on $L^2(A)$ for any $\theta \in P[0,\infty]$.

(ii) If $\theta_1 \subseteq \theta_2$ then $M_{\tau(\theta_2)} \leq M_{\tau(\theta_1)}$.

(iii) If $\sigma \leq \tau$ then $M_{\sigma(\theta)} \leq M_{\tau(\theta)}$ for any $\theta \in P[0,\infty]$.

Proof [14]

We give a proof merely as a completion as we give a similar proof for the non-tracial case in the next chapter.

(i) Let $X \in L^2(A)$ and $\theta \in P[0,\infty]$ say $\theta = \{t_0,\ldots,t_n\}$. Then

$$M_{\tau(\theta)}(X) = \sum_{i=0}^{n} M_{t_i}(X)\Delta Q_{t_i} .$$

Hence

$$M_{\tau(\theta)}(M_{\tau(\theta)}(X))$$

$$= M_{\tau(\theta)}(\sum_i M_{t_i}(X)\Delta Q_{t_i})$$

$$= \sum_j M_{t_j}(\sum_i M_{t_i}(X)\Delta Q_{t_i})\Delta Q_{t_j}$$

$$= \sum_j M_{t_j}(\sum_i M_{t_i}(X)\Delta Q_{t_i}\Delta Q_{t_j}) \quad \text{since } Q_t \in A_t$$

$$= \sum_i M_{t_i}(M_{t_i}(X)\Delta Q_{t_i}) \quad \text{since } \Delta Q_{t_i}\Delta Q_{t_j} = 0 \text{ if } i \neq j$$

$$= \sum_i M_{t_i}(X)\Delta Q_{t_i}$$

$$= M_{\tau(\theta)}(X) .$$

Hence $M_{\tau(\theta)}$ is a projection on $L^2(A)$. To show that $M_{\tau(\theta)}$ is self-adjoint we first note that the inner product in $L^2(A)$ is given by

$$\langle X,Y\rangle = \phi(X^*Y) .$$

Thus

$$\langle M_{\tau(\theta)}(X),Y\rangle = \phi(M_{\tau(\theta)}(X)^*Y)$$

$$= \phi(\sum_i \Delta Q_{t_i} M_{t_i} (X)^*Y)$$

$$= \sum_i \phi(\Delta Q_{t_i} M_{t_i} (X^*)Y)$$

since $M_t(X)^* = M_t(X^*)$

$$= \sum_i \phi(M_{t_i} (X^*)Y\Delta Q_{t_i})$$

since ϕ is a trace

$$= \sum_i \phi(X^*M_{t_i} (Y\Delta Q_{t_i}))$$

from the property of M_t

$$= \sum_i \phi(X^*M_{t_i} (Y)\Delta Q_{t_i})$$

$$= \phi(X^* \sum_i M_{t_i} (Y)\Delta Q_{t_i})$$

$$= \phi(X^*M_{\tau(\theta)}(Y))$$

$$= \langle X , M_{\tau(\theta)}(Y)\rangle .$$

Hence $M_{\tau(\theta)}$ is self-adjoint on $L^2(A)$.

(ii) We suppose that $\theta_2 = \theta_1 \cup \{\gamma\}$. Say $\theta_1 = \{t_0,\ldots,t_n\}$ and $\theta_2 = \{t_0,\ldots,t_r,\gamma,t_{r+1},\ldots,t_n\}$. Let $X \in L^2(A)$. Then

$$M_{\tau(\theta_2)}(X)M_{\tau(\theta_1)}(X) = \sum_{j=1}^{r} M_{t_j}(M_{\tau(\theta_1)}(X))\Delta Q_{t_i}$$

$$+ M_\gamma(M_{\tau(\theta_1)}(X))(Q_\gamma - Q_r)$$

$$+ M_{t_{r+1}}(M_{\tau(\theta_1)}(X))(Q_{t_{r+1}} - Q_\gamma)$$

$$+ \sum_{j=r+2}^{n} M_{t_j}(M_{\tau(\theta_1)}(X))\Delta Q_{t_j}$$

$$= \sum_{j=0}^{r} M_{t_j}\left(\sum_{i=0}^{n} M_{t_i}(X)\Delta Q_{t_i}\right)\Delta Q_{t_j}$$

$$+ M_\gamma\left(\sum_{i=0}^{n} M_{t_i}(X)\Delta Q_{t_i}\right)(Q_\gamma - Q_r)$$

$$+ M_{t_{r+1}}\left(\sum_{i=0}^{n} M_{t_i}(X)\Delta Q_{t_i}\right)(Q_{t_{r+1}} - Q_\gamma)$$

$$+ \sum_{j=r+2}^{n} M_{t_j}\left(\sum_{i=0}^{n} M_{t_i}(X)\Delta Q_{t_i}\right)\Delta Q_{t_j}$$

$$= \sum_{j=0}^{r} M_{t_j}(X)\Delta Q_{t_j} + M_\gamma(X)(Q_\gamma - Q_r)$$

$$+ M_{t_{r+1}}(X)(Q_{t_{r+1}} - Q_\gamma)$$

$$+ \sum_{j=r+2}^{r} M_{t_j}(X)\Delta Q_{t_j}$$

$$= M_{\tau(\theta_2)}(X) \quad .$$

Since $\Delta Q_{t_i}\Delta Q_{t_j} = 0$ if $i \neq j$ and the tower property of conditional expectation: $M_s M_t = M_s$ for $s \leq t$. It is now clear that $M_{\tau(\theta_1)} \geq M_{\tau(\theta_2)}$ for any $\theta_2 \supseteq \theta_1$.

(iii) Let $\tau = (Q_t)$ and $\sigma = (Q_t')$ and $\theta = \{t_0,\ldots,t_n\} \in P[0,\infty)$. Then

$$\sum_{i=1}^{n} \Delta Q_{t_i} = \sum_{i=1}^{n} \Delta Q'_{t_i} = I$$

and $\sigma \leq \tau$ means that $Q_t' \geq Q_t$ for each $t \in R^+$. Let $X \in L^2(A)$ and consider

$$M_{\sigma(\theta)} \circ M_{\tau(\theta)}(X) = \sum_{j=0}^{n} M_{t_j}\Big(\sum_{i=0}^{n} M_{t_i}(X)\Delta Q_{t_i}\Big)\Delta Q'_{t_j} \quad .$$

Now consider

$$M_{t_j}\Big(\sum_{i=0}^{n} M_{t_i}(X)\Delta Q_{t_i}\Big)\Delta Q'_{t_i}$$

$$= M_{t_j}\Big(\sum_{i=0}^{j-1} M_{t_i}(X)\Delta Q_{t_i}\Big)\Delta Q'_{t_j}$$

$$+ M_{t_j}\Big(\sum_{i=j}^{n} M_{t_i}(X)\Delta Q_{t_i}\Big)\Delta Q'_{t_j}$$

$$= M_{t_j}\Big(\sum_{i=j}^{n} M_{t_i}(X)\Delta Q_{t_i}\Big)\Delta Q'_{t_j}$$

since $\Delta Q_{t_i}\Delta Q'_{t_j} = 0$ for all $i \leq j-1$.

But $M_{t_j}\Big(\sum_{i=j}^{n} M_{t_i}(X)\Delta Q_{t_i}\Big)\Delta Q'_{t_j}$

$$= M_{t_j}\Big(\sum_{i=j}^{n-1} M_{t_i}(X)\Delta Q_{t_i}\Big)\Delta Q'_{t_j} + M_{t_j}(X(I - Q_{t_{n-1}}))\Delta Q'_{t_j}$$

$$= M_{t_j}\Big(\sum_{i=j}^{n-1} M_{t_i}(X)\Delta Q_{t_i}\Big)\Delta Q'_{t_j} + M_{t_j}\Big(X\Big(I - \sum_{k=0}^{n-1} \Delta Q_{t_k}\Big)\Big)\Delta Q'_{t_j}$$

$$= M_{t_j}\Big(X \sum_{i=j}^{n-1} \Delta Q_{t_i}\Big)\Delta Q'_{t_j} + M_{t_j}(X)\Delta Q'_{t_j} - M_{t_j}\Big(X \sum_{k=j}^{n-1} \Delta Q_{t_k}\Big)\Delta Q'_{t_j}$$

since $\Delta Q_{t_k}\Delta Q'_{t_j} = 0$ for all $k \leq j - 1$

$$= M_{t_j}(X)\Delta Q'_{t_j} \qquad .$$

Hence $$M_{\sigma(\theta)} \circ M_{\tau(\theta)}(X) = \sum_{j=0}^{n} M_{t_j}(X)\Delta Q'_{t_j}$$

$$= M_{\sigma(\theta)}(X) \qquad .$$

We have thus shown that if τ is a stopping time then $\{M_{\tau(\theta)} : \theta \in P[0,\infty]\}$ defines a decreasing net of self-adjoint projections on $L^2(A)$. Thus let

$$M_\tau = \inf_{\theta \in P[0,\infty]} M_{\tau(\theta)}$$

so that $M_{\tau(\theta)} \downarrow M_\tau$ strongly. We call M_τ the time projection (associated with τ) on $L^2(A)$.

Now let $X = (X_t)$ be a L^2-bounded martingale, say $X_t = M_t(X)$ and τ be a stopping time. Let $\theta \in P[0,\infty]$. Then

$$\sum_\theta X_{t_i}\Delta Q_{t_i} = \sum_\theta M_{t_i}(X)\Delta Q_{t_i}$$

$$= M_{\tau(\theta)}(X) \qquad .$$

Thus $\quad L^2 - \underset{\theta}{\mathrm{Lim}} \sum X_{t_i} \Delta Q_{t_i} = L^2 - \underset{\theta}{\mathrm{Lim}}\, M_{\tau(\theta)}(X)$

$$= M_\tau(X) \quad .$$

It is now natural to define the "stopped operator" associated with (X_t) and a stopping time τ analogous to the stopped random variable in the commutative theory. To this end we make the following definition.

2.33 Definition

Let (X_t) be a L^2-bounded martingale. Then define

$$X_\tau = L^2 - \underset{\theta \in P[0,\infty]}{\mathrm{Lim}} \sum_\theta X_{t_i} \Delta Q_{t_i} \quad .$$

We call X_τ is the stopped operator. Indeed this definition coincides with that of the stopped random variable when we restrict to the commutative setting.

Similarly we define the process stopped by τ, $(X_{\tau \wedge t})$ by

$$X_{\tau \wedge t} = \underset{\theta \in P[0,\infty]}{L^2 - \mathrm{Lim}} \sum_\theta X_{t_i} \Delta R_{t_i}$$

for each $t \in R^+$. Here (R_t) is the process giving the stopping time $\tau \wedge t$. That is

$$R_s = \begin{cases} Q_s & s \leq t \\ I & s > t \end{cases} \quad .$$

We see that

$$X_{\tau \wedge t} = L^2 - \operatorname*{Lim}_{\theta} \sum_{t_i \leq t} X_{t_i} \Delta Q_{t_i} + X_{t_{i+1}} (I - Q_t) \quad .$$

Now since (X_t) is L^2-bounded martingale and hence right continuous, we have that

$$X_{\tau \wedge t} = \operatorname*{Lim}_{\theta \in P[0,t]} \sum_{\theta} X_{t_i} \Delta Q_{t_i} + X_t (I - Q_t)$$

hence $\quad X_{\tau \wedge t} \in L^2(A_t)$.

Thus we have that

(i) $\quad X_\tau = \operatorname*{Lim}_{\theta} \sum_{\theta} X_{t_i} \Delta Q_{t_i} = M_\tau(X)$

(ii) $\quad X_{\tau \wedge t} = M_{\tau \wedge t}(X)$

where $M_{\tau \wedge t}$ is the time-projection corresponding to the stopping time $\tau \wedge t$.

We sometimes express stopping as an integral. Thus

$$X_\tau = \int_0^\infty X_s dQ_s$$

$$X_{\tau \wedge t} = \int_0^t X_s dQ_s + X_t (I - Q_t)$$

where

$$\int_0^t X_s dQ_s = L^2 - \operatorname*{Lim}_{\theta \in P[0,t]} \sum_{\theta} X_{t_i} \Delta Q_{t_i}$$

for each $t \in [0,\infty]$.

2.34 Remark

The stopped operator and the process stopped by τ , as described above should be thought of as the right-stopped operator by τ and the right-stopped process by τ . The left-stopped operator is defined as

$$_{\tau}X = L^2 - \operatorname*{Lim}_{\theta} \sum_{\theta} \Delta Q_{t_i} X_{t_i}$$

and the left-stopped process by

$$_{\tau \wedge t}X = L^2 - \operatorname*{Lim}_{\theta \in P[0,t]} \sum_{\theta} \Delta Q_{t_i} X_{t_i} + (I - Q_t)X_t \quad .$$

Arguing as in our previous analysis for right-stopping, shows that these limits exist . In the integral form we have

$$_{\tau}X = \int_0^{\infty} dQ_s X_s$$

$$_{\tau \wedge t}X = \int_0^{t} dQ_s X_s + (I - Q_t)X_t \quad .$$

The following theorem gives some properties of these (stopping-time) integrals.

2.35 Theorem

Let (X_t) and (Y_t) be L^2-bounded martingales and $\tau = (Q_t)$ be a stopping time. Then

(i) $\int_0^t (X_s + Y_s)dQ_s = \int_0^t X_s dQ_s + \int_0^t Y_s dQ_s$

for all $t \in \overline{R^+}$.

(ii) $\int_0^T X_s dQ_s = \int_0^t X_s dQ_s + \int_t^T X_s dQ_s$

for all $0 \leq t < T \leq \infty$.

(iii) $\left(\int_0^t X_s dQ_s \right)^* = \int_0^t dQ_s X_s^*$

for all $t \in \overline{R^+}$.

(iv) If $X_t Q_s = Q_s X_t$ and $Y_t Q_s = Q_s Y_t$ for all s and t in R^+ then

$$\int_0^t X_s Y_s dQ_s = \int_0^t X_s dQ_s \int_0^t Y_s dQ_s$$

where the integral on the left is a L^1 - Lim of $\sum X_{t_i} Y_{t_i} \Delta Q_{t_i}$.

<u>Proof</u>.

It is clear that all the integrals in (i)-(iii) exist. For (i) consider

$$\sum_{\theta \in P[0,t]} (X_{t_i} + Y_{t_i})\Delta Q_{t_i}$$

$$= \sum_\theta X_{t_i} \Delta Q_{t_i} + \sum_\theta Y_{t_i} \Delta Q_{t_i} .$$

Taking the L^2-limit as θ refines gives the result.

(ii) Let $\theta = \{t_0 < t_1 < \ldots t_r = t < t_{r+1} < \ldots t_n = T\}$ then

$$\sum_{\theta} X_{t_i} \Delta Q_{t_i} = \sum_{t_i \leq t} X_{t_i} \Delta Q_{t_i} + \sum_{t_i > t} X_{t_i} \Delta Q_{t_i} \quad .$$

Taking the limit as θ refines shows that

$$\int_0^T X_s dQ_s = \int_0^t X_s dQ_s + \int_t^T X_s dQ_s \quad .$$

(iii) Let $\varepsilon > 0$ then there is a $\theta_0 \in \mathcal{P}[0,t]$ such that for all $\theta \supseteq \theta_0$:

$$\| \sum_{\theta} X_{t_i} \Delta Q_{t_i} - \int_0^t X_s dQ_s \|_2 < \varepsilon \quad .$$

Hence [38]

$$\| \sum_{\theta} \Delta Q_{t_i} X^*_{t_i} - \left(\int_0^t X_s dQ_s \right)^* \|_2 < \varepsilon \quad .$$

Thus

$$\int_0^t dQ_s X^*_s = \left(\int_0^t X_s dQ_s \right)^* \quad .$$

(iv) Let $\theta \in \mathcal{P}[0,t]$. Then

$$\sum_{\theta} X_{t_i} Y_{t_i} \Delta Q_{t_i} = \sum_{\theta} X_{t_i} \Delta Q_{t_i} Y_{t_i} \Delta Q_{t_i}$$

$$= \sum_{\theta} X_{t_i} \Delta Q_{t_i} \cdot \sum_{\theta} Y_{t_j} \Delta Q_{t_j}$$

since $\Delta Q_{t_i} \Delta Q_{t_j} = 0$ if $i \neq j$.

Now

$$\| \sum_{\theta} X_{t_i} \Delta Q_{t_i} \sum_{\theta} Y_{t_j} \Delta Q_{t_j} - \int_0^t X_s dQ_s \cdot \int_0^t Y_s dQ_s \|_1$$

$$\leq \| \sum_{\theta} X_{t_i} \Delta Q_{t_i} \|_2 \| \sum_{\theta} Y_{t_j} \Delta Q_{t_j} - \int_0^t Y_s dQ_s \|_2$$

$$+ \| \sum_{\theta} X_{t_i} \Delta Q_{t_i} - \int_0^t X_s dQ_s \|_2 \| \int_0^t Y_s dQ_s \|_2$$

by [38].

The result now follows from a standard theorem of analysis. That is

$$L^1 - \lim_{\theta \in P[0,t]} \sum_{\theta} X_{t_i} Y_{t_i} \Delta Q_{t_i} = \int_0^t X_s dQ_s \int_0^t Y_s dQ_s \quad .$$

2.36 Corollary

We have that

(i) $(X_\tau)^* = {}_\tau X^*$

(ii) $(X_{\tau \wedge t})^* = {}_{\tau \wedge t} X^*$ for each $t \in R^+$.

So far we have managed to stop L^2-bounded martingales. However if τ is bounded then we can stop any L^2-martingale.

2.37 Theorem

Let τ be a bounded stopping time and (X_t) be a L^2-martingale. Then X_τ and $X_{\tau \wedge t}$ exist for any $t \in R^+$.

Proof.

Suppose $\tau(t) = I$ for all $t > T$ and let $\theta \in P[0,\infty]$ and we may suppose θ contains the point T. Then

$$\sum_\theta X_{t_i} \Delta Q_{t_i} = \sum_{t_i \le T} X_{t_i} \Delta Q_{t_i} + X_{t_{i+1}}(I - Q_T)$$

$$= \sum_{t_i \le T} M_{t_i}(X_T) \Delta Q_{t_i} + X_{t_{i+1}}(I - Q_T) \quad .$$

As θ refines, our previous analysis shows that the first term on the right converges in L^2 to $\int_0^T X_s dQ_s$. For the second term the right continuity of the L^2-martingale gives $X_T(I - Q_T)$. Thus

$$L^2 - \lim_\theta \sum_\theta X_{t_i} \Delta Q_{t_i} = \int_0^T X_s dQ_s + X_T(I - Q_T) \quad .$$

Similarly if $t \le T$, we have that

$$X_{\tau \wedge t} = \int_0^t X_s dQ_s + X_t(I - Q_t)$$

and for $t > T$

$$X_{\tau \wedge t} = \int_0^T X_s dQ_s + X_T(I - Q_T) \quad .$$

2.37 Theorem (Doob's optional stopping theorem)

Let (X_t) be a L^2-bounded martingale and $\sigma \le \tau$. Then $M_\sigma(X_\tau) = X_\sigma$.

Proof.

From 2.32(iii) we have that

$$M_{\sigma(\theta)} \leq M_{\tau(\theta)}$$

for any $\theta \in P[0,\infty]$. Hence $M_\sigma \leq M_\tau$. Now

$$X_\tau = M_\tau(X)$$

hence

$$M_\sigma(X_\tau) = M_\sigma \circ M_\tau(X)$$

$$= M_\sigma(X)$$

$$= X_\sigma \qquad .$$

We now state some properties of the time projections.

2.38 Theorem

Let $\tau = (Q_t)$ and $\sigma = (Q_t')$ be stopping times such that $Q_t Q_t' = Q_t' Q_t$ for each $t \in R^+$. Then

(i) $M_\sigma M_\tau = M_\tau M_\sigma$

(ii) $M_\sigma \wedge M_\tau = M_{\sigma \wedge \tau}$

(iii) $M_\sigma \vee M_\tau = M_{\sigma \vee \tau}$.

Proof.

The stopping time $\sigma \wedge \tau$ is given by $\sigma \wedge \tau(t) = Q_t \vee Q_t'$ for each $t \in R^+$. Thus for $\theta \in P[0,\infty]$:

$$M_{\sigma\wedge\tau}(\theta) = \sum_{\theta} M_{t_i}(\ )\Delta(Q_{t_i} \vee Q'_{t_i})$$

$$= \sum_{\theta} M_{t_i}(\ )\Delta(Q_{t_i} + Q'_{t_i} - Q_{t_i}Q'_{t_i}) \quad .$$

Taking the limit as $\theta\uparrow$ we have

$$M_{\sigma\wedge\tau} = M_{\sigma} + M_{\tau} - M_{\sigma\vee\tau} \quad .$$

Now $\sigma \geq \sigma \wedge \tau$ hence

$$M_{\sigma}M_{\sigma\wedge\tau} = M_{\sigma} + M_{\sigma}M_{\tau} - M_{\sigma}M_{\sigma\vee\tau}$$

$$M_{\sigma\wedge\tau} = M_{\sigma} + M_{\sigma}M_{\tau} - M_{\sigma}$$

$$= M_{\sigma}M_{\tau} \quad .$$

Likewise $M_{\tau}M_{\sigma\wedge\tau} = M_{\tau}M_{\sigma} + M_{\tau} - M_{\tau}M_{\sigma\vee\tau}$

i.e. $M_{\sigma\wedge\tau} = M_{\tau}M_{\sigma}$.

Hence $M_{\tau}M_{\sigma} = M_{\sigma}M_{\tau} = M_{\sigma\wedge\tau}$.

Also $M_{\sigma\vee\tau} = M_{\tau} + M_{\sigma} - M_{\sigma\wedge\tau}$

$$= M_{\tau} + M_{\sigma} - M_{\sigma}M_{\tau}$$

$$= M_{\tau} \vee M_{\sigma} \quad .$$

2.39 Theorem

Let (X_t) be a L^2-bounded martingale. Then $(X_{\tau \wedge t})$ defines a L^2-bounded martingale.

Proof.

Say $X_t = M_t(X)$. Then

$$X_{\tau \wedge t} = M_{\tau \wedge t}(X)$$

$$= M_\tau M_t(X)$$

$$= M_t M_\tau(X) \quad \text{from above.}$$

Hence

$$M_s(X_{\tau \wedge t}) = M_s M_t M_\tau(X)$$

$$= M_s M_\tau(X)$$

$$= M_{\tau \wedge s}(X)$$

$$= X_{\tau \wedge s} \quad .$$

Thus $(X_{\tau \wedge t})$ is a L^2-martingale, in particular $X_{\tau \wedge t} = M_t M_\tau(X)$ $= M_t(X_\tau)$.

So far we have discussed stopping for L^2-martingales. Here we defined the stopped operator X_τ as the $L^2 - \lim_\theta \sum X_{t_i} \Delta Q_{t_i}$. Motivated by this we make the following definition.

2.310 Definition

Let (X_t) be a L^p-process indexed by $\overline{R^+}$ and let $\tau = (Q_t)$ be a stopping time. We set $X_{\tau(\theta)} = \sum_\theta X_{t_i} \Delta Q_{t_i}$ for each $\theta \in P[0,\infty]$.

If the L^p-limit as θ refines of $X_{\tau(\theta)}$ exists, we call it the stopped operator and denote it by X_τ . (Of course for L^2-bounded martingales this limit exists in the L^2-norm as we have shown.) Likewise the stopped process by τ is defined by

$$X_{\tau \wedge t} = L^p - \lim_{\theta} X_{\tau \wedge t(\theta)}$$

for each $t \in R^+$ (provided the limit exists!) where

$$\tau \wedge t(s) = \begin{cases} Q_s & s \leq t \\ I & s > t \end{cases} .$$

We have the following Lemma.

2.311 Lemma

Let (X_t) be a right-continuous L^p-process. Then $X_{\tilde{t}} = X_t$ for each $t \in R^+$ where $\tilde{t}$ is given in 2.23(iii).

Proof.

Since (X_t) is right continuous at $t \in R^+$. We have for all $\varepsilon > 0$ there exists $\delta > 0 : t \leq s < t + \delta$

$$\|X_t - X_{t+\delta}\|_p < \varepsilon .$$

Now let $\theta_0 = \{0 , t + \delta/2 , \infty\} \in P[0,\infty]$. Then

$$X_{\tilde{t}(\theta_0)} = X_{t+\delta/2} .$$

Hence

$$\|X_t - X_{\tilde{t}(\theta_0)}\|_p < \varepsilon$$

and this holds for any $\theta \supseteq \theta_0$.

2.312 Theorem

Let (X_t) be a L^p-bounded martingale if $1 < p \leq \infty$ (and weakly relatively compact if $p = 1$) . If $X_{\tau \wedge t}$ exists for each $t \in [0,\infty]$, then $(X_{\tau \wedge t})$ defines a L^p-bounded martingale.

Proof.

We have that $M_t(X) = X_t$ for some $X \in L^p(A)$ [5]. Let $t \in R^+$ and $\theta \in P[0,\infty]$. We may suppose θ contains the point t $(= t_r)$. Then

$$X_{\tau \wedge t(\theta)} = \sum_{i=1}^{r} X_{t_i} \Delta Q_{t_i} + X_{t_{r+1}}(I - Q_t)$$

$$X_{\tau(\theta)} = \sum_{i=1}^{n} X_{t_i} \Delta Q_{t_i} .$$

Then

$$M_t(X_{\tau(\theta)}) = M_t \sum_{i=1}^{n} M_{t_i}(X) \Delta Q_{t_i}$$

$$= \sum_{i=1}^{r} X_{t_i} \Delta Q_{t_i} + X_t(I - Q_t)$$

$$= M_t \left(\sum_{i=1}^{r} X_{t_i} \Delta Q_{t_i} + X_{t_{r+1}}(I - Q_t)\right)$$

$$= M_t(X_{\tau \wedge t(\theta)}) .$$

2

Now taking the L^p-limit as θ refines and observing that M_t is L^p-continuous, we have

$$M_t(X_\tau) = M_t(X_{\tau \wedge t}) \quad .$$

Now all that remains to show is that $X_{\tau \wedge t} \in L^p(A_t)$. To this end consider

$$X_{\tau \wedge t(\theta)} = \sum_{t_i \leq t} X_{t_i} \Delta Q_{t_i} + X_{t_{i+1}}(I - Q_t) \quad .$$

We know that as θ refines $X_{\tau \wedge t(\theta)} \to X_{\tau \wedge t}$. However the first term on the right lies in $L^p(A_t)$, whilst the right continuity of the martingale gives us that $X_{t_{i+1}}(I - Q_t) \to X_t(I - Q_t) \in L^p(A_t)$. Hence

$$M_t(X_\tau) = X_{\tau \wedge t} \quad .$$

Again if X_τ and $X_{\tau \wedge t}$ exist in the L^p-sense, we express them as integrals:

$$X_\tau = \int_0^\infty X_s dQ_s$$

$$X_{\tau \wedge t} = \int_0^t X_s dQ_s + X_t(I - Q_t)$$

where $\displaystyle\int_0^t X_s dQ_s = L^p - \lim_{\theta \in P[0,t]} \sum_\theta X_{t_i} \Delta Q_{t_i}$

for $t \in [0,\infty]$.

Then this integral has the following properties.

2.313 Theorem

Let (X_t) and (Y_t) be L^p-process and suppose that both

$$\int_0^t X_s dQ_s \qquad \text{and} \qquad \int_0^t Y_s dQ_s$$

exist for all $t [0,\infty]$. Then

$$\text{(i)} \quad \int_0^t (X_s + Y_s) dQ_s = \int_0^t X_s dQ_s + \int_0^t Y_s dQ_s$$

for all $t \in \overline{R^+}$.

$$\text{(ii)} \quad \int_0^T X_s dQ_s = \int_0^t X_s dQ_s + \int_t^T X_s dQ_s$$

for all $0 \leq t < T \leq \infty$.

$$\text{(iii)} \quad \left(\int_0^t X_s dQ_s \right)^* = \int_0^t dQ_s X_s^*$$

for all $t \in \overline{R^+}$.

(iv) If (Z_s) is a L^q-process where $p^{-1} + q^{-1} = 1$ and $\int_0^t Z_s dQ_s$ exists for all $t \in \overline{R^+}$ and $Z_s Q_t = Q_t Z_s$ for all $s,t \in \overline{R^+}$, then

$$\int_0^t X_s Z_s dQ_s \qquad \text{exists as a}$$

$$L^1 - \operatorname{Lim}_{\theta} \sum X_{t_i} Z_{t_i} \Delta Q_{t_i} \qquad \text{and equals}$$

$$\int_0^t X_s dQ_s \int_0^t Z_s dQ_s$$

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for all $t \in R^+$.

Proof.

This is exactly same as that of theorem 2.35.

Before we leave this section and go on to look at stopping times that lie in the commutant A' , we make some definitions and prove a theorem related to local martingales. However we shall discuss local martingales in chapter four.

2.314 Definition

Let (τ_n) be a monotonic increasing sequence of stopping times given by $\tau_n = (Q_t^{(n)})$. Then we say $\tau_n \uparrow \infty$ as $n \uparrow \infty$ if for each $t \in R^+$

$$\phi(Q_t^{(n)}) \to 0 \qquad \text{as} \qquad n \to \infty \quad .$$

That is for each $t \in R^+$, for all $\varepsilon > 0$ there exists N : for all $n \geq N$

$$\phi(Q_t^{(n)}) < \varepsilon \quad .$$

2.315 Definition

An L^p-process (X_t) is called a L^p-local martingale if there exists a sequence (τ_n) of stopping times as in 2.314 such that $X_{\tau_n \wedge t}$ exists for each $t \in R^+$ and for each n $(X_{\tau_n \wedge t})$ defines a L^p-bounded martingale.

2.316 Theorem

Let (X_t) be a uniformly bounded L^1 right continuous local martingale. Then (X_t) is a bounded A-valued martingale.

Proof.

We have that $X_{\tau_n \wedge t} = L^1 - \operatorname{Lim}_{\theta} X_{\tau_n \wedge t}(\theta)$. That is

$$X_{\tau_n \wedge t} = \int_0^t X_s dQ_s^{(n)} + X_t(1 - Q_t^{(n)})$$

$$\in L^1(A_t) \quad .$$

Now

$$\|X_{\tau_n \wedge t} - X_t\|_1 = \|\int_0^t X_s dQ_s^{(n)} - X_t Q_t^{(n)}\|_1$$

$$\leq \|\int_0^t X_s dQ_s^{(n)}\|_1 + \|X_t Q_t^{(n)}\|_1$$

$$\leq \|\int_0^t X_s dQ_s^{(n)}\|_1 + \|X_t\|_\infty \phi(Q_t^{(n)}) \quad .$$

Now $\int_0^t X_s dQ_s^{(n)} = \operatorname{Lim}_{\theta \in \mathcal{P}[0,t]} \sum X_{t_i} \Delta Q_{t_i}^{(n)}$.

But $\|\sum_\theta X_{t_i} \Delta Q_{t_i}^{(n)}\| \leq \sum \|X_{t_i}\|_\infty \|\Delta Q_{t_i}^{(n)}\|_1$

$$\leq M\phi(Q_t^{(n)}) \quad \text{for all } \theta$$

where $M = \operatorname{Sup}_t \|X_t\|_\infty$.

Hence

$$\|\int_0^t X_s dQ_s^{(n)}\|_1 \leq \|\int_0^t X_s dQ_s^n - \sum_\theta X_{t_i} \Delta Q_{t_i}^{(n)}\|_1$$

$$+ M\phi(Q_t^{(n)}) \quad \text{for all } \theta \quad .$$

Now let $\varepsilon > 0$ be given and let $N \in \mathbb{N}$ so that for all $n \geq N$ $\phi(Q_t^{(n)}) < \frac{\varepsilon}{2M}$.

For each $n > N$ let $\theta_n \in P[0,\infty]$ such that for all $\theta \subseteq \theta_n$:

$$\| \int_0^t X_s dQ_s^{(n)} - \sum_\theta X_{t_i} \Delta Q_{t_i}^{(n)} \|_1 < \frac{\varepsilon}{2} \quad .$$

Thus for all $n \geq N$

$$\| \int_0^t X_s dQ_s^{(n)} \|_1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad .$$

Hence for all $n \geq N$

$$\| X_{\tau_n \wedge t} - X_t \|_1 < \varepsilon + \varepsilon = 2\varepsilon \quad .$$

Thus $X_{\tau_n \wedge t} \to X_t$ in L^1 as $n \to \infty$, for each $t \in R^+$. The conditional expectation is L^1-continuous thus by hypothesis

$$M_s(X_{\tau_n \wedge t}) = X_{\tau_n \wedge s}$$

gives

$$L^1 - \text{Lim}\ M_s(X_{\tau_n \wedge t}) = L^1 - \text{Lim}\ X_{\tau_n \wedge s}$$

i.e.

$$M_s(X_t) = X_s \quad .$$

Hence (X_t) is a martingale. But (X_t) is uniformly bounded hence $X_t \doteq M_t(X)$ for some $X \in A$.

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2.4 Stopping Times in the Commutant

In this section we study stopping times that lie in the commutant A' of A. Here we show that the time projections M_τ is the conditional expectation map onto A_τ - the Von Neumann algebra of "events prior to τ ".

2.41 Definition

Let τ be a stopping time (not necessarily in A'). Then the algebra of "events prior to τ " is defined by

$$A_\tau = \{R \in A : R = R^* = R^2 , RQ_t \in A_t \text{ for each } t\} .$$

2.42 Proposition

If $\sigma \leq \tau$ then $A_\sigma \subseteq A_\tau$.

Proof.

Let $\sigma = (Q_t')$ and $\tau = (Q_t)$. Then $\sigma \leq \tau$ gives $Q_t' \geq Q_t$ for each $t \in R^+$. Let $R \in A_\sigma$ so that $RQ_t' \in A_t$. Then

$$RQ_t = RQ_t'Q_t = (RQ_t')Q_t \in A_t .$$

Hence $A_\sigma \subseteq A_\tau$.

Now let $\theta = \{t_0,\ldots,t_n\} \in P[0,\infty]$ and $\tau = (Q_t)$ be a stopping time. Then

$$\tau(\theta)(s) = \sum Q_{t_i} \chi_{[t_i,t_{i+1})}(s)$$

defines a stopping time and $\tau(\theta) \geq \tau$, hence $A_{\tau(\theta)} \supseteq A_\tau$.

2.43 Proposition

We have that

$$A_\tau = \bigcap_\theta A_{\tau(\theta)} .$$

Proof.

$$A_\tau \subseteq A_{\tau(\theta)} \qquad \text{for all} \quad \theta \in P[0,\infty] .$$

Hence

$$A_\tau \subseteq \bigcap_\theta A_{\tau(\theta)} .$$

Conversely suppose $X \in \bigcap_\theta A_{\tau(\theta)}$. Now it is known [7] that there exists a sequence (θ_n) of partitions such that $\tau(\theta_n) \downarrow \tau$ pointwise strongly. Now $X \in A_{\tau(\theta_n)}$ for all n , hence

$$X\tau(\theta_n)(s) \in A_s .$$

But $\tau(\theta_n)(s) \to \tau(s)$ strongly and A_s is strongly closed. Hence $X\tau(s) = XQ_s \in A_s$ for each $s \in R^+$. That is $X \in A_\tau$.

In the rest of this section we shall assume that $\tau(s) \in A'$ for all $s \in R^+$.

Let $\theta = \{t_0,\ldots,t_n\} \in P[0,\infty]$, and recall that

$$M_{\tau(\theta)} = \sum_\theta M_{t_i}(\ )\Delta Q_{t_i} .$$

2.44 Proposition

$M_{\tau(\theta)}$ is the conditional expectation map of A onto $A_{\tau(\theta)}$.

Proof.

Suppose $X \in A_{\tau(\theta)}$ so that $X\tau(\theta)(s) \in A_s$ for each $s \in R^+$.

$\tau(\theta)(s) = \sum_i Q_{t_i} \chi_{[t_i, t_{i+1})}(s)$. Now

$$M_{\tau(\theta)}(X) = \sum_{i=0}^{n} M_{t_i}(X)\Delta Q_{t_i}$$

$$= \sum_{i=0}^{n} M_{t_i}(X\Delta Q_{t_i})$$

$$= X \quad .$$

Since $Q_{t_i} = \tau(\theta)(t_i)$ for all $i = 1,\ldots,n$. Conversely let $X \in A$ and consider $M_{\tau(\theta)}(X)$. Then

$$M_{\tau(\theta)}(X)\tau(\theta)(s) = \sum_i M_{t_i}(X)\Delta Q_{t_i}\tau(\theta)(s) \quad .$$

Suppose $t_k \leq s < t_{k+1}$ then

$$M_{\tau(\theta)}(X)\tau(\theta)(s) = \sum_{i=1}^{k} M_{t_i}(X)\Delta Q_{t_i} \in A_{t_k} \subseteq A_s \quad .$$

Hence $M_{\tau(\theta)} : A \to A_{\tau(\theta)}$ is onto. Clearly $M_{\tau(\theta)}(I) = I$. Now let $Z,Y \in A_{\tau(\theta)}$ and $X \in A$. Then

$$\phi(ZM_{\tau(\theta)}(X)Y) = \sum_{\theta} \phi(ZM_{t_i}(X)\Delta Q_{t_i}Y)$$

$$= \sum_{\theta} \phi(Z\Delta Q_{t_i} M_{t_i}(X)Y\Delta Q_{t_i})$$

$$= \sum_{\theta} \phi(Z\Delta Q_{t_i} XY\Delta Q_{t_i})$$

$$= \sum_{\theta} \phi(ZXY\Delta Q_{t_i})$$

$$= \phi(ZXY) \quad .$$

Since $Z\Delta Q_{t_i} \in A_{t_i}$ and $Y\Delta Q_{t_i} \in A_{t_i}$. Hence $M_{\tau(\theta)}$ is the conditional expectation map.

We now wish to show that the time projection M_τ is the conditional expectation map of A onto A_τ .

2.45 Theorem

We have that M_τ is the conditional expectation map of A onto A_τ .

Proof.

It is clear that M_τ maps A into $L^2(A)$. Let $X \in A_\tau$ and $\theta \in P[0,\infty]$ so that $\tau(\theta) \geq \tau$ and $X \in A_{\tau(\theta)}$. Now

$$M_\tau(X) = L^2 - \text{Lim}\, M_{\tau(\theta)}(X)$$

$$= L^2 - \text{Lim}\, X$$

$$= X \quad .$$

Hence $M_\tau(X) = X$ for all $X \in A_\tau$. Similarly $M_\tau(I) = I$.

Now let $Y \in A_\tau$ ($\subseteq A$) and consider

$$\phi(M_\tau(X)Y) \quad .$$

Then

$$|\phi(M_\tau(X)Y) - \phi(XY)|$$

$$\leq |\phi(M_{\tau(\theta)}(X)Y) - \phi(M_\tau(X)Y)| + |\phi(M_{\tau(\theta)}(X)Y) - \phi(XY)|$$

$$= |\phi(M_{\tau(\theta)}(X)Y) - \phi(M_\tau(X)Y)|$$

since $M_{\tau(\theta)}$ is a conditional expectation

$$\leq \|Y\|_\infty \|M_{\tau(\theta)}(X) - M_\tau(X)\|_2 \to 0 \quad \text{as} \quad \theta \uparrow \quad .$$

Hence

$$\phi(M_\tau(X)Y) = \phi(XY) \quad .$$

Thus all that remains to show is that

$$M_\tau : A \to A_\tau \quad .$$

We first show if $X \in A$, then $M_{\tau(\theta)}(X)$ converges to $M_\tau(X)$ strongly. To this end, since ϕ is normal and faithfull , we may assume that A acts in its G.N.S. space $\mathcal{H}$ $(= L^2(A))$ with cyclic and separating vector Ω . Hence $A'\Omega$ is dense in $\mathcal{H}$. Let $X \in A$.

We know that $M_{\tau(\theta)} \to M_\tau$ on $L^2(A)$ $(= \mathcal{H})$. Consider $M_{\tau(\theta)}(X)$, then $\|M_{\tau(\theta)}(X)\|_\infty \leq \|X\|_\infty$ since $M_{\tau(\theta)}$ is the conditional expectation. Let $Y' \in A'$ then

$$M_{\tau(\theta)}(X)Y'\Omega = Y'M_{\tau(\theta)}(X)\Omega$$

$$\rightarrow Y'M_{\tau}(X)\Omega$$

$$= M_{\tau}(X)Y'\Omega \quad .$$

Thus $M_{\tau(\theta)}(X)$ converges strongly to $M_{\tau}(X)$ on a dense set. But $(M_{\tau(\theta)}(X))$ is uniformly bounded. Hence $M_{\tau(\theta)}(X) \rightarrow M_{\tau}(X)$ strongly everywhere on $\mathcal{H}$ and thus $M_{\tau}(X) \in A$. Since $M_{\tau(\theta)}(X) \in A_{\tau(\theta)} \subseteq A$. Now $M_{\tau}(X)Q_t$ is:

$$\text{strong-}\lim_{\theta} M_{\tau(\theta)}(X)Q_t = \text{strong-}\lim_{\theta} \sum_{t_i \leq t} M_{t_i}(X)\Delta Q_{t_i}$$

$$\in A_t \quad .$$

Thus $M_{\tau}(X) \in A_{\tau}$ and $M_{\tau} : A \rightarrow A_{\tau}$ is the conditional expectation.

2.46 Proposition

Let (X_t) be a A-valued process indexed by $\overline{R^+}$ and suppose X_{τ} exists as a uniform limit of $X_{\tau(\theta)}$ (see definition 2.310). Then $X_{\tau} \in A_{\tau}$.

Proof.

$X_{\tau(\theta)} \rightarrow X_{\tau}$ uniformly, hence $X_{\tau(\theta)} \rightarrow X_{\tau}$ strongly and $X_{\tau(\theta)}Q_t \rightarrow X_{\tau}Q_t$ strongly. But $X_{\tau(\theta)}Q_t \in A_t$ for all $\theta \in P[0,\infty]$, and A_t is strongly closed, hence $X_{\tau}Q_t \in A_t$ for each $t \in R^+$. Thus $X_{\tau} \in A_{\tau}$.

Now for each $t \in R^+$, $\tau \wedge t$ denotes a stopping time and $A_{\tau \wedge t}$ denotes the algebra of events prior to $\tau \wedge t$. For each t suppose $X_{\tau \wedge t}$ exists as a uniform limit of $X_{\tau \wedge t(\theta)}$ where (X_t) is a A-valued process. We have then the theorem:

2.47 Theorem

$(X_{\tau \wedge t})$ is a martingale relative to (A_t) and (M_t) if and only if it is a martingale relative to $(A_{\tau \wedge t})$ and $(M_{\tau \wedge t})$.

Proof.

Since $X_{\tau \wedge t} \in A_{\tau \wedge t}$ we have that $X_{\tau \wedge t} \in A_t$ as $\tau \wedge t \leq t$. Thus $(X_{\tau \wedge t})$ is adapted to (A_t). Now suppose

$$M_s(X_{\tau \wedge t}) = X_{\tau \wedge s} \qquad \text{for all} \quad s \leq t \quad .$$

Then

$$M_{\tau \wedge s}(X_{\tau \wedge t}) = M_{\tau \wedge s} \circ M_s(X_{\tau \wedge t}) \qquad \text{by 2.38}$$

$$= M_{\tau \wedge s}(X_{\tau \wedge s})$$

$$= X_{\tau \wedge s}$$

and $(X_{\tau \wedge t})$ is a martingale relative to $(M_{\tau \wedge t})$.

Conversely suppose

$$M_{\tau \wedge s}(X_{\tau \wedge t}) = X_{\tau \wedge s} \qquad \text{for all} \quad x \leq t \quad .$$

To show that $(X_{\tau \wedge t})$ is a martingale relative to (A_t) and (M_t) it is enough to show that

$$\phi(X_{\tau \wedge t} R) = \phi(X_{\tau \wedge s} R)$$

for any $R \in A_s^{Proj}$. Thus let $R \in A_s^{Proj}$ then $R(1 - Q_s) \in A_{\tau \wedge s}$. Now

$$\phi(X_{\tau \wedge t} R) = \phi(X_{\tau \wedge t} RQ_s) + \phi(X_{\tau \wedge t} R(I - Q_s))$$

$$= \phi(X_{\tau \wedge t} RQ_s) + \phi(X_{\tau \wedge s} R(I - Q_s)) \quad .$$

Since $R(1 - Q_s) \in A_{\tau \wedge s}^{Proj}$ and $(X_{\tau \wedge t})$ is a martingale relative to $(A_{\tau \wedge t})$ and $(M_{\tau \wedge t})$. Now consider

$$\phi(X_{\tau \wedge t(\theta)} RQ_s) = \phi(X_{\tau \wedge s(\theta)} RQ_s) \quad .$$

As θ refines we have that

$$\phi(X_{\tau \wedge t} RQ_s) = \phi(X_{\tau \wedge s} RQ_s) \quad .$$

Hence

$$\phi(X_{\tau \wedge t} R) = \phi(X_{\tau \wedge s} RQ_s) + \phi(X_{\tau \wedge s} R(I - Q_s))$$

$$= \phi(X_{\tau \wedge s} R) \quad .$$

That is $(X_{\tau \wedge t})$ is a martingale relative to (A_t) and (M_t) .

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2.48 Remark

We mention that in general the time projection M_τ need not be a conditional expectation. We shall illustrate this fact in the next section.

2.5 The Clifford Algebra

In this section we study stopping times in a concrete model from the quantum theory. We shall look at the Clifford algebra. Stopping has been studied in this model [7]. There the main interest was in stopping L^2-martingales. In the present section we shall be interested in stopping the increasing process associated with the Doob-Meyer decomposition of a L^2-martingale. For a L^2-martingale (X_t), we give a characterisation of the stopped martingale $(X_{\tau \wedge t})$. First we give some preliminaries (taken from [8,11]).

Let $\mathcal{H}$ denote the Hilbert space $L^2(R^+)$ and F the anti-symmetric Fock space over $\mathcal{H}$. That is

$$F = C \oplus \mathcal{H} \oplus (\mathcal{H} \otimes_a \mathcal{H}) \oplus (\mathcal{H} \otimes_a \mathcal{H} \otimes_a \mathcal{H}) \oplus \ldots$$

where $\otimes_a$ denotes the anti-symmetric tensor product. We abbreviate F to

$$C \oplus \mathcal{H} \oplus \mathcal{H}^2 \oplus \mathcal{H}^3 \oplus \ldots$$

where $$\mathcal{H}^n = \underbrace{(\mathcal{H} \otimes_a \mathcal{H} \otimes_a \ldots \otimes_a \mathcal{H})}_{n\text{-times}} .$$

For each $u \in \mathcal{H}$, let $C(u)$ denote the creation operator on F and $C(u)^* = A(u)$ be the annihilation operator on F. The free fermion field is then defined as

$$\psi(u) = C(u) + A(\bar{u}) \quad .$$

Then for $u,v \in \mathcal{H}$ we have [8]:

$$\psi(u)\psi(v) + \psi(v)\psi(u) = 2\langle\bar{u},v\rangle I \quad .$$

Let $\mathcal{C}$ denote the Von Neumann algebra generated by $\{\psi(u) : u \in \mathcal{H}\}$. Then $\mathcal{C}$ is called the weakly closed Clifford algebra of bounded operators on $\mathcal{F}$ and over $\mathcal{H}$. It is known that the Fock vacuum vector Ω, that is

$$\Omega = 1 \oplus 0 \oplus 0 \oplus \ldots$$

is a faithful normal finite tracial vector on $\mathcal{C}$ and we let $\phi(\cdot) = \langle\Omega,\cdot\Omega\rangle$ so that $(\mathcal{C},\phi,\mathcal{F})$ is a regular probability gauge space [35]. For each $t \in R^+$ let $\mathcal{C}_t$ denote the Von Neumann algebra generated by $\{\psi(u) : u \in L^2(R^+) \text{ and } \mathrm{supp}(u) \subseteq [0,t]\}$. Then we have a filtration of Von Neumann algebras $\{\mathcal{C}_t : t \in R^+\}$. Let M_t denote the conditional expectation map of $\mathcal{C}$ onto $\mathcal{C}_t$ for each $t \in R^+$ so that M_t extends to the conditional expectation map of $L^p(\mathcal{C})$ onto $L^p(\mathcal{C}_t)$ for all $1 \leq p \leq \infty$.

Now consider the family $\{\psi(\chi_{[0,t]}) : t \in R^+\}$. Then for each $t \in R^+$, $\psi(\chi_{[0,t]}) \in \mathcal{C}_t$. Hence the family $\{\psi(\chi_{[0,t]}) : t \in R^+\}$ is a $\mathcal{C}$-valued process. In fact it is shown in [8] that it is a martingale with the properties that

(i) $\phi(\psi(\chi_{[0,t]})) = 0 \qquad$ for all $t \in R^+$

(ii) $\phi(\psi(\chi_{[0,t]})\psi(\chi_{[0,s]})) = s \wedge t$

(iii) $\phi((\psi(\chi_{[0,t]}) - \psi(\chi_{[0,s]}))(\psi(\chi_{[0,r]}) - \psi(\chi_{[0,q]})))$

$$= \phi(\psi(\chi_{[0,t]}) - \psi(\chi_{[0,s]}))\phi(\psi(\chi_{[0,r]}) - \psi(\chi_{[0,q]}))$$

$$\text{if } 0 \leq t \leq s \leq r \leq q \quad .$$

Property (iii) says that the family $(\psi(\chi_{[0,t]}))$ has independent increments. We call $(\psi(\chi_{[0,t]}))$ the Clifford process.

Now let $X = (X_t)$ be a $L^2(C)$ valued process. That is for each $t \in R^+$, $X_t \in L^2(C_t)$. Suppose X is simple, so that

$$X_t = \sum X_{t_{i-1}} \chi_{[t_{i-1},t_i)}(t)$$

and write $\psi_t = \psi(\chi_{[0,t]})$. Then the stochastic integral

$$\int_0^t X_s d\psi_s$$

is defined as

$$\sum_{i=1}^{r} X_{t_{i-1}}(\psi_{t_i} - \psi_{t_{i-1}}) + X_{t_r}(\psi_t - \psi_{t_r})$$

where $t \in [t_r, t_{r+1})$. Then it is shown [8] that

$$\|\int_0^t X_s d\psi_s\|_2^2 = \sum_{i=1}^{r} \phi(|X_{t_{i-1}}|^2)(t_i - t_{i-1}) + \phi(|X_{t_r}|^2)(t - t_r)$$

$$= \int_0^t \|X_s\|_2^2 \, ds \quad .$$

Let $\mathcal{S} \equiv \mathcal{S}([0,T],C)$ be the space of all simple processes on $[0,T]$ and let $h = h([0,T],ds;L^2(C))$ be the space of $L^2(C)$-valued processes on $[0,T]$ measurable and square integrable w.r.t. ds. Then h is a Hilbert space and for each $f \in h$

$$\int_0^t f(s) d\psi_s$$

exists as a L^2 - Lim of $\int_0^t f_n(s) d\psi_s$ where $(f_n) \subseteq \mathcal{S}$. Then

$$\|\int_0^t f(s) d\psi_s\|_2^2 = \int_0^t \|f(s)\|_2^2 \, ds$$

and

$$\{ \int_0^t f(s) d\psi_s : s \in [0,T] \}$$

is a L^2-centred martingale [8]. Conversely given a L^2-martingale (X_t), there is a unique process $(\tilde{X}_s) \in h$ such that

$$X_t = \int_0^t \tilde{X}_s d\psi_s$$

and

$$\|X_t\|_2^2 = \int_0^t \|\tilde{X}_s\|_2^2 \, ds \quad .$$

Thus the function $t \to \int_0^t \|\tilde{X}_s\|_2^2\, ds$ defines a Borel measure on $[0,T]$ given by μ_X :

$$\mu_X([a,b]) = \int_a^b \|\tilde{X}_s\|_2^2\, ds \quad .$$

If $f \in S$ then the stochastic integral $\int_0^t f(s)dX_s$ is defined as before. Now let $K = K([0,T],\mu_X)$ denote the closure of S in the norm given by

$$\|f\| = \int_0^T \|f(s)\|_\infty^2\, d\mu_X(s) \quad .$$

Now if $f \in S$ then

$$\|\int_0^t f(s)dX_s\|_2^2 \leq \int_0^t \|f(s)\|_\infty^2\, d\mu_X(s)$$

and hence if $f \in K$ then there exists a sequence $(f_n) \subseteq S$ such that

$$L^2 - \text{Lim} \int_0^t f_n(s)dX_s$$

exists and we denote it by $\int_0^t f(s)dX_s$. Furthermore

$$\|\int_0^t f(s)dX_s\|_2^2 \leq \int_0^t \|h(s)\|_\infty^2\, d\mu_X(s)$$

and

$$\int_0^t f(s)dX_s = \int_0^t f(s)\tilde{X}_s d\psi_s$$

[11].

We now state two lemmas from [7].

2.51 Lemma [7]

Let $f : [0,T] \to C$ be a self-adjoint process such that there exists a sequence (f_n) of uniformly bounded self-adjoint processes with $f_n \to f$ strongly ds - a.e. on $[0,T]$. Then if (X_t) is an L^2-martingale with:

$$X_t = \int_0^t \tilde{X}_s d\psi_s$$

we have

(i) $f\tilde{X} \in h$

(ii) $\int_0^t f_n dX_s \xrightarrow{L^2} \int_0^t f dX_s = \int_0^t f\tilde{X}_s d\psi_s$.

2.52 Lemma [7]

Let $\tau : R^+ \to C$ be a stopping time. Then there exists a sequence (τ_n) of simple stopping times such that $\tau_n \to \tau$ strongly ds - a.e. (Indeed $\tau_n = \tau(\theta_n)$ for some partition θ_n such that $\tau(\theta_n)(s) \leq \tau(\theta_{n+1})(s) \leq \tau(s)$.)

2.53 Proposition

Let (X_t) be a L^2-martingale. Then

$$X_{\tau \wedge t} = X_t - \int_0^t dX_s Q_s$$

where the stochastic integral on the right is the "left" version of that given in Lemma 2.51. $\tau = (Q_t)$ is a stopping time.

Proof.

We have that

$$X_t = \int_0^t \tilde{X}_s d\psi_s$$

for some $(\tilde{X}_s) \in h$ [8]. Hence using the parity operator β [11]

$$X_t = \int_0^t d\psi_s \beta(\tilde{X}_s) \quad .$$

(This is the left version of the Ito-Clifford integral given in [8,11].)

We know that $X_{\tau \wedge t}$ exists and equals

$$\int_0^t X_s dQ_s + X_t(I - Q_t) \quad .$$

Now

$$\int_0^t X_s dQ_s = L^2 - \underset{\theta \in P[0,t]}{\text{Lim}} \sum_\theta X_{t_i} \Delta Q_{t_i}$$

$$= X_t Q_t - L^2 - \underset{\theta}{\text{Lim}} \sum_\theta \Delta X_{t_i} Q_{t_{i-1}} \quad .$$

Hence

$$X_{\tau \wedge t} = X_t - L^2 - \underset{\theta}{\text{Lim}} \sum \Delta X_{t_i} Q_{t_{i-1}} \quad .$$

Now

$$\Delta X_{t_i} Q_{t_{i-1}} = \int_{\Delta t_i} d\psi_s \beta(\tilde{X}_s) \cdot Q_{t_{i-1}}$$

and since $Q_{t_{i-1}} : L^2(C) \to L^2(C)$ is a continuous operator by multiplication,

$$\Delta X_{t_i} Q_{t_{i-1}} = \int_{\Delta t_i} d\psi_s \beta(\tilde{X}_s) Q_{t_{i-1}} \quad .$$

Thus

$$\sum \Delta X_{t_i} Q_{t_{i-1}} = \int_0^t d\psi_s \beta(\tilde{X}_s) \tau(\theta)(s)$$

where

$$\tau(\theta)(s) = \sum_{\theta} Q_{t_{i-1}} \chi_{[t_{i-1},t_i)}(s) \quad .$$

But we know from Lemma 2.51 and 2.52 that

$$\int_0^t d\psi_s \beta(\tilde{X}_s) \tau(s)$$

exists as a $L^2 - \mathrm{Lim} \int_0^t d\psi_s \beta(\tilde{X}_s) \tau(\theta_n)(s)$.

Now consider

$$\| \int_0^t d\psi_s \beta(\tilde{X}_s)(\tau(\theta)(s) - \tau(s)) \|_2^2$$
$$= \int_0^t \| \beta(\tilde{X}_s)(\tau(\theta)(s) - \tau(s)) \|_2^2 \, ds$$

by the isometry property [8].

We know that given $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$ and hence $\theta_n \supseteq \theta_{N(\varepsilon)}$ we have

$$\int_0^t \|\beta(\tilde{X}_s)(\tau(\theta_n)(s) - \tau(s))\|_2^2 < \varepsilon \quad .$$

Thus let $\theta \supseteq \theta_{N(\varepsilon)}$ so that $\tau(s) \geq \tau(\theta)(s) \geq \tau(\theta_{N(\varepsilon)})(s)$.

Then

$$\int_0^t \|\beta(\tilde{X}_s)(\tau(\theta)(s) - \tau(s))\|_2^2 \, ds$$

$$= \int_0^t \|\beta(\tilde{X}_s)(\tau(\theta_{N(\varepsilon)})(s) - \tau(s))(\tau(\theta) - \tau(s))\|_2^2 \, ds$$

$$\leq \int_0^t \|\beta(\tilde{X}_s)(\tau(\theta_{N(\varepsilon)})(s) - \tau(s))\|_2^2 \, ds$$

$$< \varepsilon \quad .$$

Hence

$$L^2 - \underset{\theta}{\text{Lim}} \sum \Delta X_{t_i} Q_{t_{i-1}} = \int_0^t d\psi_s \beta(\tilde{X}_s) Q_s \quad .$$

That is

$$X_{\tau \wedge t} = X_t - \int_0^t d\psi_s \beta(\tilde{X}_s) Q_s$$

$$= \int_0^t d\psi_s \beta(\tilde{X}_s) - \int_0^t d\psi_s \beta(\tilde{X}_s) Q_s$$

$$= \int_0^t d\psi_s \beta(\tilde{X}_s)(I - Q_s)$$

$$= \int_0^t dX_s (I - Q_s)$$

by 2.51 and 2.52.

If (X_t) is an L^2-martingale so that $X_t = \int_0^t \tilde{X}_s d\psi_s$, then the increasing process associated with (X_t) in the Doob-Meyer decomposition is denoted by $<X>_t$ where

$$<X>_t = \int_0^t |\beta\tilde{X}_s|^2 \, ds$$

[8.11].

We are now interested in finding the relationship between the increasing processes associated with the stopped martingale $(X_{\tau \wedge t})$ and the stopped increasing process, $<X>_{\tau \wedge t}$. In the commutative theory we have that $<X^\tau>_t = <X>_{\tau \wedge t}$ where $X^\tau = (X_{\tau \wedge t})$.

We first state a corollory of proposition 2.53.

2.54 Corollary

We have that

$$<X^\tau>_t = \int_0^t |(\beta(\tilde{X}_s))(I - Q_s)|^2 \, ds \quad .$$

Proof.

This follows immediately from the definition of the increasing process associated with the Ito-Clifford integral [8,11].

Our aim now is to stop the increasing process associated with $X_t = \int_0^t \tilde{X}_s d\psi_s$. We first show that

$$\lim_{\theta \in P[0,t]} \sum_\theta <X>_{t_i} \Delta Q_{t_i}$$

exists in the L^1-sense.

2.55 Proposition

We have that

$$L^1 - \lim_{\theta \in P[0,t]} \sum_{\theta} \langle X\rangle_{t_i} \Delta Q_{t_i}$$

$$= \langle X\rangle_t Q_t - \int_0^t |\beta\tilde{X}_s|^2 Q_s ds \quad .$$

Proof.

We first show that the integral on the right exists. To show that the function $s \to |\beta\tilde{X}_s|^2 Q_s$ is Lebesgue measurable, we consider the case when the function $s \to |\beta\tilde{X}_s|^2$ is elementary, say $\tilde{X}_s = \tilde{X}_0$ for all $s \in [0,T]$. By 2.52 there is a sequence τ_n of stopping times such that $\tau_n \to \tau$ strongly ds - a.e. Now let $\varepsilon > 0$ then $|\beta\tilde{X}_0|^2 = h + K$ where $h \in C$ and $K \in L^1(C)$ with $\|K\|_1 < \varepsilon$. Then

$$\| |\beta\tilde{X}_0|^2(\tau_n(s) - \tau(s))\|_1 \leq \|h(\tau_n(s) - \tau(s))\|_1$$

$$+ \|K(\tau_n(s) - \tau(s))\|_1$$

$$\leq 2\|h\|_1 + \|K\|_\infty \|\tau_n(s) - \tau(s)\|_1$$

$$\to 0 \quad \text{as} \quad n \to \infty \quad ds - a.e.$$

Hence the function $s \to |\beta\tilde{X}_s|^2 Q_s$ is Lebesgue measurable. Furthermore

$$\| |\beta\tilde{X}_s|^2 Q_s \|_1 \leq \| |\beta\tilde{X}_0|^2 \|_1 < \infty \quad .$$

Hence by the dominated convergence theorem $\int_0^t |\beta\tilde{X}_s|^2 Q_s ds$ exists as a L^1 - Lim of $\int_0^t |\beta\tilde{X}_s|^2 \tau_n(s)ds$, when (X_s) is elementary. For general (X_s) the result follows by linearity. Now

$$\sum_{\theta \in P[0,t]} \langle X\rangle_{t_i} \Delta Q_{t_i}$$

$$= \langle X\rangle_t Q_t - \sum_\theta \Delta\langle X\rangle_{t_i} Q_{t_{i-1}}$$

$$= \langle X\rangle_t Q_t - \sum_\theta \Big(\int_{\Delta t_i} ds |\beta\tilde{X}_s|^2 \Big) Q_{t_{i-1}}$$

$$= \langle X\rangle_t Q_t - \int_0^t |\beta\tilde{X}_s|^2 \tau(\theta)(s)ds \quad .$$

Arguing as in the proof of proposition 2.53

$$= \int_0^t |\beta\tilde{X}_s|^2 (Q_t - \tau(\theta)(s))ds \quad .$$

We now wish to show that as θ refines

$$\int_0^t |\beta\tilde{X}_s|^2 (Q_t - \tau(\theta)(s))ds$$

converges in L^1 to

$$\int_0^t |\beta\tilde{X}_s|^2 (Q_t - \tau(s))ds \quad .$$

We know that

$$\left\| \int_0^t |\beta \tilde{X}_s|^2 (Q_t - \tau(\theta_n)(s)) ds \right\|_1$$

$$\leq \int \| |\beta \tilde{X}_s|^2 (Q_t - \tau(\theta_n)(s) \|_1 \, ds$$

$\to 0$ as $n \to \infty$ by the dominated convergence theorem. Thus arguing as in the proof of 2.53 it is easy to see that

$$L^1 - \operatorname*{Lim}_{\theta} \int_0^t |\beta \tilde{X}_s|^2 \tau(\theta)(s) ds = \int_0^t |\beta \tilde{X}_s|^2 Q_s ds \quad .$$

Hence

$$L^1 - \operatorname*{Lim}_{\theta \in P[0,t]} \sum \langle X \rangle_{t_i} \Delta Q_{t_i}$$

$$= \langle X \rangle_t \, Q_t - \int_0^t |\beta \tilde{X}_s|^2 Q_s ds$$

$$= \int_0^t |\beta \tilde{X}_s|^2 (Q_t - Q_s) ds \quad .$$

2.56 Theorem

$$\langle X \rangle_{\tau \wedge t} = \int_0^t |\beta \tilde{X}_s|^2 (I - Q_s) ds \quad .$$

Proof.

$$\langle X \rangle_{\tau \wedge t} = L^1 - \operatorname*{Lim}_{\theta} \sum_{t_i \leq t} \langle X \rangle_{t_i} \Delta Q_{t_i} + \langle X \rangle_{t_{i+1}} (1 - Q_t) \quad .$$

From proposition 2.55 and the right continuity of $t \to \langle X \rangle_t$ [8,11] we have

$$\langle X\rangle_{\tau\wedge t} = \langle X\rangle_t - \int_0^t |\beta\tilde{X}_s|^2 Q_s ds$$

$$= \int_0^t |\beta\tilde{X}_s|^2 (I - Q_s) ds \qquad .$$

We now write $\langle X\rangle^\tau$ to denote the process $(\langle X\rangle_{\tau\wedge t})$.

2.57 Corollary

(i) $$_{\tau\wedge t}(\langle X\rangle^\tau) = \int_0^t (I - Q_s)|\beta\tilde{X}_s|^2 (I - Q_s) ds$$

(ii) $$(\langle X\rangle^\tau)^*_t = \int_0^t (I - Q_s)|\beta\tilde{X}_s|^2 \, ds$$

(iii) $$_{\tau\wedge t}(\langle X\rangle^\tau) = (\langle X\rangle^\tau)^*_{\tau\wedge t}$$

where $_{\tau\wedge t}\langle X\rangle$ is left-stopping the process $\langle X\rangle_t$.

Proof.

Left stopping is given by

$$_{\tau\wedge t}\langle X\rangle = \int_0^t (I - Q_s)|\beta\tilde{X}_s|^2 \, ds$$

arguing as in the right stopping case and the corollary follows.

2.58 Theorem

We have that

$$<X^{\tau}>_t = {}_{\tau\wedge t}(<X>^{\tau}) = (<X>^{\tau})^*_{\tau\wedge t} \qquad .$$

Proof.

$$<X^{\tau}>_t = \int_0^t |\beta\tilde{X}_s(I - Q_s)|^2\, ds$$

$$= \int_0^t (I - Q_s)|\beta\tilde{X}_s|^2(I - Q_s)ds$$

$$= (<X>^{\tau})^*_{\tau\wedge t} \qquad .$$

Now if $X_t = \int_0^t \tilde{X}_s d\psi_s$ and $Y_t = \int_0^t \tilde{Y}_s d\psi_s$ are L^2-martingales. Then the pointed bracket process is defined as [11]:

$$<X,Y>_t = \int_0^t \beta(\tilde{X}_s)^*\beta(Y_s)ds$$

so that $<X,X>_t = <X>_t$. Hence by polarisation, it is easy to see that

$$<X,Y>_{\tau\wedge t} = \int_0^t \beta(\tilde{X}_s)^*\beta(Y_s)(I - Q_s)ds \qquad .$$

In proposition 2.55 we showed that

$$\int_0^t d<X>_s Q_s = \int_0^t |\beta\tilde{X}_s|^2 Q_s ds$$

exists as a L^1 - Lim of $\int_0^t |\beta\tilde{X}_s|^2 \tau(\theta_n) ds$. By polarisation it is easy to see that

$$\int_0^t d\langle X,Y\rangle_s Q_s$$

exists.

In [11] it is shown that $(\int_0^t dX_s f(s))$ is the unique centred L^2-martingale (Z_t) say such that

$$\int_0^t d\langle X,Y\rangle_s f(s) = \langle Z,Y\rangle_s$$

for any L^2-martingale (Y_t) , and f is a $\mathcal{C}$-valued process which is ds - a.e. limit of uniformly bounded sequence in $\mathcal{S}$. Now using Lemma 2.51 (see [7]) we conclude that this result holds for

$$\int_0^t dX_s \tau(s) \qquad \left(= \int_0^t d\psi_s \beta\tilde{X}_s \tau(s) \right) .$$

Bearing this in mind, we have the following characterisation of $(X_{\tau \wedge t})$.

2.59 Theorem

We have that $(X_{\tau \wedge t})$ is the unique centred L^2-martingale (X^τ) such that

$$\langle X,Y\rangle_{\tau \wedge t} = \langle X^\tau Y\rangle_t$$

for any L^2-martingale (Y_t) .

Proof.

$$<X^\tau,Y>_t = <\int_0^t dX_s(I - Q_s) - Y>_t$$

$$= \int_0^t d<X,Y>_s(I - Q_s) \qquad \text{from [11]}$$

$$= <X,Y>_{\tau \wedge t}$$

by polarisation of the result in proposition 2.55.

The uniqueness follows from the fact that [11] if

$$<X^\tau,Y>_t = <Z^\tau,Y>_t$$

for any L^2-martingale (Y_t) . Then

$$<X^\tau-Z^\tau,Y>_t = 0$$

for all $t \in [0,T]$, and any L^2-martingale (Y_t) . In particular

$$<X^\tau - Z^\tau , X^\tau - Z^\tau>_t = 0 \quad .$$

Hence

$$\int_0^t |\beta\tilde{X}_s(I - Q_s) - \beta\tilde{Z}_s^\tau|^2 ds = 0$$

for all $t \in [0,T]$. That is $\beta\tilde{X}_s(I - Q_s) - \beta\tilde{Z}_s^\tau = 0$ ds — a.e

. Hence $\tilde{X}_s\beta(I - Q_s) - \tilde{Z}_s^\tau = 0$ ds – a.e

i.e $\tilde{X}_s\beta(I - Q_s) = \tilde{Z}_s^\tau$ ds – a.e .

Hence

$$\int_0^t \tilde{X}_s\beta(I - Q_s)d\psi_s = \int_0^t \tilde{Z}_s^\tau d\psi_s$$

i.e.

$$X_{\tau \wedge t} = Z_t^\tau \qquad \text{for each } t \in [0,T] .$$

Finally we prove the strong Markov property of the Clifford process (ψ_t) . The strong Markov property is one of the basic properties of the Brownian motion process in the classical probability theory. The quantum analogue for the Boson Brownian motion is established in [24] and that of the Fermion Brownian motion in [3]. We shall follow the description given in [3] to suit our needs.

2.510 Theorem

Let τ be a finite stopping time. Then $\psi_{\tau+t}$ exists strongly for any $t \geq 0$ and $\{\psi_{\tau+t} - \psi_\tau : t \in R^+\}$ is a Clifford process.

Proof.

Since τ is finite, we can assume $\tau(t) = I$ for all $t > T$. Hence by 2.37 we have that

$$\psi_\tau = \int_0^T \psi_s dQ_s + \psi_T(I - Q_T)$$

$$= L^2 - \lim_{\theta \in P[0,T]} M_{\tau(\theta)}(\psi_T) + \psi_T(I - Q_T) .$$

Now for $f \in F$, $\|M_{\tau(\theta)}(\psi_T)f\| \leq \sum_i \|M_{t_i}(\psi_T)\Delta Q_{t_i} f\|$

$$\leq \sum_i \|M_{t_i}(\psi_T)\|_\infty \|\Delta Q_{t_i} f\|$$

$$\leq \|\psi_T\|_\infty \sum_i \|\Delta Q_{t_i} f\|$$

$$= \|\psi_T\|_\infty \|Q_T f\|$$

$$\leq \|\psi_T\|_\infty \|f\| \quad .$$

Hence

$$\|M_{\tau(\theta)}(\psi_T)\|_\infty \leq \|\psi_T\|_\infty \quad .$$

Thus from [27] $M_{\tau(\theta)}(\psi_T)$ converges to $M_\tau(\psi_T)$ strongly. Thus the integral

$$\int_0^T \psi_s dQ_s + \psi_T(I - Q_T)$$

exists in the strong sense. Likewise

$$\psi_{\tau+t} = \int_0^T \psi_{t+s} dQ_s + \psi_{T+t}(I - Q_T)$$

exists strongly. Now

$$\psi_{\tau+t} - \psi_\tau = \int_0^T \psi(\chi_{[s,t+s]})dQ_s + \psi(\chi_{(T,T+t]})(I - Q_T)$$

and let $\Phi_t = \psi_{\tau+t} - \psi_\tau$. Then

$$\Phi_t\Phi_r + \Phi_r\Phi_t =$$

$$\left(\int_0^T \psi(\chi_{(s,t+s]})dQ_s + \psi(\chi_{(T,T+t]})(I-Q_T)\right) \times$$

$$\times \left(\int_0^T \psi(\chi_{(s,r+s]})dQ_s + \psi(\chi_{(T,T+r]})(I-Q_T)\right)$$

$$+ \left(\int_0^T \psi(\chi_{(s,r+s]})dQ_s + \psi(\chi_{(T,T+r]})(I-Q_T)\right) \times$$

$$\times \left(\int_0^T \psi(\chi_{(s,t+s]})dQ_s + \psi(\chi_{(T,T+t]})(I-Q_T)\right) \quad .$$

We now note that for each $t \in R^+$, the antisymmetric Fock space F decomposes as

$$F = F_t \otimes F^t$$

where F_t is the antisymmetric Fock space over $L^2([0,t])$ and F^t is that over $L^2((t,\infty))$. If C_t and C^t denote the corresponding Clifford algebras over $L^2([0,t])$ and $L^2((t,\infty))$ respectively then $C_t \subseteq B(F_t) \otimes I$ and $C^t \subseteq I \otimes B(F^t)$ [3]. Thus C_t and C^t commute. Thus by 2.35(iv)

$$\int_0^T \psi(\chi_{(s,t+s]})dQ_s \int_0^T \psi(\chi_{(s,r+s]})dQ_s$$

$$= \int_0^T \psi(\chi_{(s,t+s]})\psi(\chi_{(s,r+s]})dQ_s$$

and

$$\int_0^T \psi(\chi_{(s,r+s]})dQ_s \cdot \int_0^T \psi(\chi_{(s,t+s]})dQ_s$$

$$= \int_0^T \psi(\chi_{(s,r+s]})\psi(\chi_{(s,t+s]})dQ_s \quad .$$

On adding and using the canonical anticommutation relationship of (ψ_t) stated at the beginning of Section 2.5, we get

$$\int_0^T \langle\chi_{(s,t+s]},\chi_{(s,r+s]}\rangle dQ_s$$

$$= \int_0^T (t \wedge r)dQ_s$$

$$= (t \wedge r)Q_T \quad . \tag{2.510a}$$

Likewise

$$\psi(\chi_{(T,T+t]})(I-Q_T)\psi(\chi_{(T,T+r]})(I-Q_T) +$$

$$\psi(\chi_{(T,T+r]})(I-Q_T)\psi(\chi_{(T,T+t]})(I-Q_T)$$

$$= t \wedge r(I - Q_T) \quad . \tag{2.510b}$$

Finally, the "cross" terms are all zero. For example

$$\int_0^T \psi(\chi_{(s,t+s]})dQ_s \cdot \psi(\chi_{(T,T+r]})(I - Q_T)$$

$$= \int_0^T \psi(\chi_{(s,t+s]})dQ_s \cdot (I - Q_T)\psi(\chi_{(T,T+r]})$$

$$= 0 \quad .$$

We end this chapter with an example to illustrate that the time-projections are not in general conditional expectations.

<u>2.511 Example</u> (M_τ is not a Conditional Expectation)

We work in the Clifford algebra. Let Q be a projection in C_t and let τ be the stopping time:

$$\tau(s) = \begin{cases} 0 & s < t \\ Q & s > t \\ I & s = \infty \end{cases} .$$

Then if $M_\tau : L^1 \to L^1$ <u>is</u> a conditional expectation we have

$$M_\tau(X)^* = M_\tau(X^*) .$$

In particular if $X = X^*$, we have that

$$X(I - Q) + M_t(X)Q = QM_t(X) + (I - Q)X$$

i.e.

$$[M_t(X),Q] = [X,Q]$$

for any projection $Q \in C_t$.

Now let $s \geq t$ and $X = \psi_s$. Then $[\psi_t,Q] = [\psi_s,Q]$ and consider $\frac{\psi_t}{\sqrt{t}}$, which is a self-adjoint unitary. Hence there is a projection $P \in C_t$ such that:

$$\frac{\psi_t}{\sqrt{t}} = 2P - 1 ,$$

We let $Q = \frac{1}{2}\left(\frac{\psi_t}{\sqrt{t}} + 1\right)$. Then

$$\frac{1}{2}[\psi_t , \frac{\psi_t}{\sqrt{t}} + 1] = \frac{1}{2}[\psi_s , \frac{\psi_t}{\sqrt{t}} + 1] .$$

Hence $[\psi_t,\psi_s] = 0$ for all $s \geq t$, i.e.

$$[\psi_t,\psi_t+\psi(\chi_{(t,s]})] = 0 .$$

Hence

$$[\psi_t,\psi(\chi_{(t,s]})] = 0 .$$

That is

$$\psi_t\psi(\chi_{(t,s]}) - \psi(\chi_{(t,s]})\psi_t = 0 . \quad (2.511a)$$

Also from the C.A.R., we have that

$$\psi_t\psi(\chi_{(t,s]}) + \psi(\chi_{(t,s]})\psi_t = 0 . \quad (2.511b)$$

Thus adding we get

$$\psi_t\psi(\chi_{(t,s]}) = 0 . \quad (2.511c)$$

Multiplying (2.511c) to (2.511b) gives

$$\psi_t\psi(\chi_{(t,s]})\psi(\chi_{(t,s]})\psi_t = 0 .$$

Hence

$$\psi_t(s-t)^2 I\psi_t = 0 \quad .$$

Hence

$$t^2(s-t)^2 I = 0 \quad .$$

That is

$$t^2(s-t)^2 = 0$$

for all $t \leq s$. We have a contradiction and M_τ is not a conditional expectation map.

CHAPTER THREE

STOPPING TIMES IN A VON NEUMANN ALGEBRA WITH A STATE

3.0 Introduction

In this chapter we study stopping times in a Von Neumann algebra with a faithful normal state. This is a natural extension of the tracial case studied in the last chapter to include certain type III factors [18]. Once again our work here is motivated by examples from quantum mechanics [18].

3.1 Preliminaries

Let K be a complex Hilbert space, $B(K)$ the bounded linear operators on K and $A \subseteq B(K)$ be a Von Neumann algebra with a faithful normal state ω. For each non-negative real t, let A_t be a Von Neumann subalgebra of A and suppose the family $\{A_t : t \in R^+\}$ satisfies;

(i) if $t_1, t_2 \in R^+$ and $t_1 \leq t_2$ then A_{t_1} is a Von Neumann subalgebra of A_{t_2}

(ii) the Von Neumann algebra A is generated by $\bigcup_{t \geq 0} A_t$

(iii) $\bigcap_{t>s} A_t = A_s$

(iv) $\{ \bigcup_{s<t} A_s \}'' = A_t$,

Finally suppose there exists a family $\{M_t : t \in \mathbb{R}^+\}$ of conditional expectations:

$$M_t : \mathcal{A} \to \mathcal{A}_t$$

such that:

$$\text{(i)} \quad \omega \circ M_t = \omega \qquad \text{for all } t \in \mathbb{R}^+$$

$$\text{(ii)} \quad M_t(AXB) = AM_t(X)B \qquad \text{for all } A,B \in \mathcal{A}_t \ , \ X \in \mathcal{A}$$

$$\text{(iii)} \quad M_t(A) = A \qquad \text{for } A \in \mathcal{A}_t \ .$$

Since ω is faithful, we may, without loss of generality, assume that $\mathcal{A}$ and for each $t \in \mathbb{R}^+$, $\mathcal{A}_t$ act in their G.N.S. spaces $\mathcal{H}$ and $\mathcal{H}_t$ respectively. Here, $(\mathcal{H},\Pi,\Omega)$ is the G.N.S. triple associated with $(\mathcal{A},\omega)$ and for each $t \in \mathbb{R}^+$, $(\mathcal{H}_t,\Pi,\Omega)$ is that associated with $(\mathcal{A}_t,\omega)$. For each $t \in \mathbb{R}^+$, let P_t denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_t$, the subspace generated by $\mathcal{A}_t\Omega$. The following straight forward Lemmas will be used subsequently.

3.11 Lemma

For $A \in \mathcal{A}$, we have that

$$P_t A\Omega = M_t(A)\Omega \qquad \text{for all } t \in \mathbb{R}^+ \ .$$

Proof.

Let $B \in \mathcal{A}_t$, so that $M_t(B) = B$. Now consider

$$\begin{aligned} \langle B\Omega, P_t A\Omega \rangle &= \langle B\Omega, A\Omega \rangle \\ &= \omega(B^*A) \\ &= \omega(M_t(B^*A)) \\ &= \omega(B^*M_t(A)) \\ &= \langle B\Omega, M_t(A)\Omega \rangle \quad . \end{aligned}$$

Since $A_t\Omega$ is dense in $\mathcal{H}_t$, the result follows.

3.12 Lemma

For each $t \in R^+$, P_t lies in the commutant of $\mathcal{A}_t$.

Proof.

Let $B \in \mathcal{A}_t$, $A \in \mathcal{A}$. Then

$$\begin{aligned} P_t BA\Omega &= M_t(BA)\Omega \\ &= BM_t(A)\Omega \\ &= BP_t A\Omega \quad . \end{aligned}$$

Since $\mathcal{A}\Omega$ is dense in $\mathcal{H}$, the result follows.

3.13 Theorem

We have that the map $s \to P_s$ is strongly continuous.

Proof.

We first establish the left continuity of the map $s \to P_s$. We wish to show that $P_s X\Omega \to P_t X\Omega$ as $s \uparrow t$, $X \in \mathcal{A}$. Now since

$\bigcup_{s<t} A_s$ is strongly dense in A_t , we have that: given $\varepsilon > 0$, there is a $Z \in \bigcup_{s<t} A_s$, say $Z \in A_{s_0}$, $s_0 < t$, such that

$$\|M_t(X)\Omega - Z\Omega\| = \|P_t X\Omega - P_{s_0} Z\Omega\| < \varepsilon/2 \quad .$$

Now for $t \geq s \geq s_0$, we have

$$\begin{aligned}\|P_t X\Omega - P_s X\Omega\| &\leq \|P_t X\Omega - Z\Omega\| + \|Z\Omega - P_s X\Omega\| \\ &< \frac{\varepsilon}{2} + \|P_s Z\Omega - P_s P_t X\Omega\| \qquad \text{since } P_s P_t = P_s \\ &\leq \frac{\varepsilon}{2} + \|P_s\|_\infty \|Z\Omega - P_t X\Omega\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\end{aligned}$$

and the left continuity is established.

To show the right continuity, it is enough to show $\bigcap_{t>s} \mathcal{H}_t = \mathcal{H}_s$. Now let $P_{s+} = \bigwedge_{t>s} P_t$. Then $P_s \mathcal{H} = \mathcal{H}_s \subseteq P_{s+}\mathcal{H} = \bigcap_{t>s} \mathcal{H}_t$. To show the reverse inclusion, let $h \in \bigcap_{t>s} A_t \Omega$, then there is $A \in A$ such that $h = P_t A\Omega$ for all $t > s$. (Since $A_{t_1}\Omega = A_{t_2}\Omega = h$ gives $A_{t_1} = A_{t_2}$ since Ω is separating) and $h = P_{s+}A\Omega$. Now, $P_t A\Omega = M_t(A)\Omega$ by 3.11 and $\|M_t(A)\| \leq \|A\|$ for all $t > s$, hence the family $\{M_t(A) \; ; \; t>s\}$ is weakly relatively compact in A . Thus we can find a subnet such that $M_{t_\alpha}(A)$ converges to Y weakly as $t \downarrow s$. Thus we have,

$$<M_{t_\alpha}(A)\Omega, B\Omega> \to <Y\Omega, B\Omega> = <P_s Y\Omega, B\Omega> \quad .$$

Also,

$$M_{t_\alpha}(A)\Omega = P_{t_\alpha} A\Omega \to P_{s+} A\Omega = h$$

strongly (hence weakly). Hence

$$<h, B\Omega> = <P_s Y\Omega, B\Omega>$$

and $h = P_s Y\Omega$. Hence $h \in \mathcal{H}_s$.

Now if $h \in \bigcap_{t>s} \mathcal{H}_t$, then given $\varepsilon > 0$, there is $A \in \mathcal{A}$ such that

$$\|h - M_t(A)\Omega\| < \varepsilon \qquad \text{for all } t > s$$

since $\mathcal{A}\Omega$ is dense in $\mathcal{H}$ so that there is $A \in \mathcal{A}$ such that $\|h - A\Omega\| < \varepsilon$ and hence

$$\|h - M_t(A)\Omega\| = \|P_t h - P_t A\Omega\| \leq \|h - A\Omega\| < \varepsilon \quad .$$

Hence arguing as above shows that there is a $Y \in \mathcal{A}_s$ such that $\|h - Y\Omega\| < \varepsilon$ and hence $h \in \mathcal{H}_s$.

3.2 Stopping Times

Before we define stopping times in this model, we define vector and operator valued martingales in this model.

3.21 Definition

An $\mathcal{A}$-valued martingale with respect to the filtration $\{\mathcal{A}_t : t \in R^+\}$ is a family $\{A_t : t \in R^+\}$ with $A_t \in \mathcal{A}_t$ for each $t \in R^+$ and $M_s(A_t) = A_s$ if $0 \leq s \leq t$. Likewise, an $\mathcal{H}$-valued martingale with respect to the filtration $\{\mathcal{H}_t : t \in R^+\}$ is a family $\{\eta_t : t \in R^+\}$ with $\eta_t \in \mathcal{H}_t$ and $P_s\eta_t = \eta_s$ if $0 \leq s \leq t$. An $\mathcal{H}$-valued martingale is called simple if it is of the form $\eta_t = P_t\eta$ for some $\eta \in \mathcal{H}$.

3.22 Remark

It is evident that given an $\mathcal{A}$-valued martingale (A_t) we can construct an $\mathcal{H}$-valued martingale (α_t) by defining

$$\alpha_t = A_t\Omega \qquad \text{for each} \quad t \in R^+ \quad .$$

Conversely, given a $\mathcal{H}$-valued martingale, the following prescription shows how to construct an operator valued martingale [31].

First define the vector spaces U_η and $U_\eta(t)$ as follows:

$$U_\eta \quad = \{R \;\eta\; \mathcal{A} \;;\; \Omega \in \text{Domain } (R)\}$$

$$U_\eta(t) = \{R \;\eta\; \mathcal{A}_t \;:\; \text{Domain } (R) = \mathcal{A}_t'\Omega\}$$

where η means affiliation. That is $R \;\eta\; \mathcal{A} \Leftrightarrow A'$ Domain $(R) \subseteq$ Domain (R) and $A'R \subseteq RA'$ for all $A' \in \mathcal{A}'$.

We observe that since $\mathcal{A}' \subseteq \mathcal{A}_t'$ for all $t \in R^+$, $U_\eta(t) \subseteq U_\eta$. The conditional expectation is now extended to U_η by :

$$M_t(R)T\Omega = TP_tR\Omega$$

where $T \in A_t'$, $R \in U_\eta$.

Thus given an $\mathcal{H}$-valued martingale (α_t) say, we define a U_η-valued martingale (X_t) by:

$$X_tT\Omega = T\alpha_t$$

where $T \in A_t'$. Then for $s \leq t$,

$$\begin{aligned} M_s(X_t)R\Omega &= RP_sX_t\Omega \\ &= RP_s\alpha_t \\ &= R\alpha_s \\ &= X_sR\Omega \end{aligned}$$

where $R \in A_s'$.

Thus $M_s(X_t) = X_s$ on a dense set and the fact that $X_t \in U_\eta(t)$ is clear. Furthermore if (X_t) is a A-valued process and (α_t) is the corresponding $\mathcal{H}$-valued process:

$$\alpha_t = X_t\Omega \qquad .$$

Then defining $(\tilde{X}_t)$ by

$$\tilde{X}_tT\Omega = T\alpha_t \qquad\qquad T \in A_t'$$

gives

$$T\alpha_t = TX_t\Omega$$

$$= X_t T\Omega \quad .$$

Hence $\tilde{X}_t$ and X_t agree on a dense set and hence are equal. As in the tracial case in the last chapter we make the following definitions about stopping times [7].

3.23 Definitions

(i) A stopping time, τ , is an increasing family of projections (Q_t) such that $\tau(t) = Q_t \in A_t$ for each $t \in R^+$, $\tau(0) = 0$ and $\tau(\infty) = I$.

(ii) Let P denote the set of finite partitions of $[0,\infty]$. Then for $\theta \in P$, say $\theta = \{t_0, t_1, \ldots, t_n\}$ we define an operator $P_{\tau(\theta)}$ on $\mathcal{H}$ as:

$$P_{\tau(\theta)} = \sum_{i=1}^{n} (Q_{t_i} - Q_{t_{i-1}})P_{t_i}$$

$$= \sum_{i=1}^{n} \Delta Q_{t_i} P_{t_i} \quad .$$

(iii) If $\tau = (Q_t)$ and $\sigma = (Q_t')$ are two stopping times then we say $\sigma \geq \tau$ if and only if $Q_t \geq Q_t'$ for each $t \in R^+$.

The following theorem is the analogue of stopping L^2-bounded martingales with simple stopping times in the commutative theory, and is an extension of theorem 2.32 to the non-tracial case.

3.24 Theorem

Let τ be a stopping time. Then

(i) $P_{\tau(\theta)}$ is a self-adjoint projection on $\mathcal{H}$ for any $\theta \in \mathcal{P}$.

(ii) If $\theta_1, \theta_2 \in \mathcal{P}$ with θ_2 finer then θ_1 then $P_{\tau(\theta_1)} \geq P_{\tau(\theta_2)}$.

(iii) If σ is another stopping time with $\sigma \geq \tau$ then $P_{\sigma(\theta)} \geq P_{\tau(\theta)}$ for any $\theta \in \mathcal{P}$.

Proof.

Let $\tau = (Q_t)$, $\xi \in \mathcal{H}$ and $\theta = \{t_0, t_1, \ldots, t_n\} \in \mathcal{P}$. Then

$$P_{\tau(\theta)} P_{\tau(\theta)}(\xi) = \sum_{j=1}^{n} \Delta Q_{t_j} P_{t_j} \left(\sum_{i=1}^{n} \Delta Q_{t_i} P_{t_i} \xi \right)$$

$$= \sum_j \sum_i \Delta Q_{t_j} P_{t_j} \Delta Q_{t_i} P_{t_i} \xi$$

$$= \sum_j \sum_i P_{t_j} \Delta Q_{t_j} \Delta Q_{t_i} P_{t_i} \xi \qquad \text{by 3.12}$$

$$= \sum_i P_{t_i} \Delta Q_{t_i} \xi$$

$$= P_{\tau(\theta)} \xi \quad ,$$

since $\Delta Q_{t_i} \Delta Q_{t_j} = 0$ for $i \neq j$. Thus $P_{\tau(\theta)}$ is an idempotent.

Now let $\xi, \eta \in \mathcal{H}$. Then

$$\langle P_{\tau(\theta)}\eta,\xi\rangle = \sum_i \langle \Delta Q_{t_i} P_{t_i}\eta,\xi\rangle$$

$$= \sum_i \langle \eta, P_{t_i}\Delta Q_{t_i}\xi\rangle$$

$$= \sum_i \langle \eta, \Delta Q_{t_i} P_{t_i}\xi\rangle \qquad \text{by 3.12}$$

$$= \langle \eta, \sum_i \Delta Q_{t_i} P_{t_i}\xi\rangle$$

$$= \langle \eta, P_{\tau(\theta)}\xi\rangle \quad .$$

Hence $P_{\tau(\theta)}$ is self-adjoint and (i) is established.

(ii) Suppose $\theta_2 = \theta_1 \cup \{q\}$ where $\theta_1 = \{t_0,\ldots,t_n\}$ with $t_i < t_{i+1}$ and $q \in (t_r,t_{r+1})$, $r + 1 < n$. Then for $\xi \in \mathcal{H}$,

$$\begin{aligned} P_{\tau(\theta_1)} \cdot P_{\tau(\theta_2)}(\xi) = {} & \sum_{j=1}^{n} \Delta Q_{t_j} P_{t_j} \Big(\sum_{i=1}^{r} \Delta Q_{t_i} P_{t_i}(\xi)\Big) \\ & + \sum_{j=1}^{n} \Delta Q_{t_j} P_{t_j} (Q_q - Q_{t_r}) P_q(\xi) \\ & + \sum_{j=1}^{n} \Delta Q_{t_j} P_{t_j} (Q_{t_{r+1}} - Q_q) P_{t_{r+1}}(\xi) \\ & + \sum_{j=1}^{n} \Delta Q_{t_j} P_{t_j} \Big(\sum_{i=r+2}^{n} \Delta Q_{t_i} P_{t_i}(\xi)\Big) \quad . \end{aligned}$$

Using 3.12 and the orthogonality of ΔQ_{t_i} and ΔQ_{t_j} for $i \neq j$ we get

$$P_{\tau(\theta_1)} \; P_{\tau(\theta_2)}(\xi) = \sum_{j=1}^{r} \Delta Q_{t_i} P_{t_i}(\xi) + (Q_q - Q_r)P_q(\xi)$$

$$+ (Q_{t_{r+1}} - Q_q)P_{t_{r+1}}(\xi)$$

$$+ \sum_{j=r+2}^{n} \Delta Q_{t_j} P_{t_j}(\xi)$$

$$= P_{\tau(\theta_2)}(\xi) \qquad .$$

Hence $P_{\tau(\theta_1)} \geq P_{\tau(\theta_2)}$ for $\theta_2 \supseteq \theta_1$. The result for arbitrary $\theta_2 \supseteq \theta_1$ is now clear.

(iii) Given $\sigma \geq \tau$, let $\sigma = (Q_t')$ so that $Q_t' \leq Q_t$ for each $t \in R^+$. Let $\theta \in P$ be as in 3.23(ii) say. Then

$$I = \sum_{i=1}^{n} \Delta Q_{t_i} = \sum_{i=1}^{n} \Delta Q'_{t_i} \qquad .$$

Now,

$$P_{\tau(\theta)} \; P_{\sigma(\theta)} = \sum_{i=1}^{n} \Delta Q_{t_i} P_{t_i} \Big(\sum_{j=1}^{n} \Delta Q'_{t_j} P_{t_j} \Big) \qquad .$$

But

$$\Delta Q_{t_i} P_{t_i} \sum_{j=1}^{n} \Delta Q'_{t_j} P_{t_j} = \sum_{j=i}^{n} \Delta Q_{t_i} P_{t_i} \Delta Q'_{t_j}$$

by 3.12 and observing that $\Delta Q'_{t_j}$ is orthogonal to ΔQ_{t_i} for $j \leq i - 1$. Hence

$$P_{\tau(\theta)} \; P_{\sigma(\theta)} = \sum_{i=1}^{n} \sum_{j=i}^{n} \Delta Q_{t_i} P_{t_i} \Delta Q'_{t_j}$$

$$= \sum_{i=1}^{n} \Big(\sum_{j=i}^{n-1} \Delta Q_{t_i} P_{t_i} \Delta Q'_{t_j} + \Delta Q_{t_i} P_{t_i} (I - \sum_{K=1}^{n-1} \Delta Q'_K)\Big)$$

$$= \sum_{i=1}^{n} \Delta Q_{t_i} P_{t_i}$$

$$= P_{\tau(\theta)} \quad .$$

Hence $\quad P_{\tau(\theta)} \leq P_{\sigma(\theta)} \quad .$

3.25 Remark

We think of $P_{\tau(\theta)}$ as the "time-Projection" corresponding to the simple time

$$\tau(\theta) = \sum_{i=1}^{n} P_{t_{i-1}} \chi_{[t_{i-1}, t_i)} \quad .$$

In the last chapter this was defined as $M_{\tau(\theta)}$. Here we work with $P_{\tau(\theta)}$ because we choose to work on the Hilbert space $\mathcal{H}$ whilst establishing results about A (or U_η) -valued martingales.

3.26 Definitions

(i) For a stopping time $\tau = (Q_t)$ we define the time projection at τ , P_τ by

$$P_\tau = \inf_{\theta \in P} P_{\tau(\theta)}$$

$$\Big(= \inf_{\theta \in P} \sum_i \Delta Q_{t_i} P_{t_i} \Big) \quad .$$

(ii) For any $\mathcal{H}$-valued martingale (ξ_t), we define the "martingale stopped by τ" as:

$$\xi_\tau = \operatorname*{Lim}_{\mathcal{P}} \sum_i \Delta Q_{t_i} \xi_{t_i} \quad .$$

3.27 Theorem

Let (ξ_t) be a simple $\mathcal{H}$-valued martingale. Then $\xi_\tau = P_\tau(\xi)$.

Proof.

Since (ξ_t) is simple. $\xi_t = P_t(\xi)$ for some $\xi \in \mathcal{H}$. Now let $\theta \in \mathcal{P}$

$$\sum_\theta \Delta Q_{t_i} \xi_{t_i} = \sum_\theta \Delta Q_{t_i} P_{t_i}(\xi)$$

$$= P_{\tau(\theta)}(\xi) \quad .$$

Thus on taking the limit over $\mathcal{P}$, we get

$$\xi_\tau = P_\tau(\xi) \quad .$$

As before the deterministic times are given by $\tilde{t}$:

$$\tilde{t}(s) = \begin{cases} 0 & s \leq t \\ I & s > t \end{cases} \qquad \ldots\ldots \qquad (3.27a)$$

then we have the following consistency lemma.

3.28 Lemma

Let (ξ_t) be any right continuous $\mathcal{H}$-valued process. Then $\xi_t = \xi_t^{\sim}$.

Proof.

Let $\theta = \{0,t,t+\varepsilon,\infty\}$. Then $\xi^{\sim}_{t(\theta)} = \xi_{t+\varepsilon}$. Hence as θ refines, the right continuity gives the result.

3.29 Theorem

Let τ and σ be stopping times with $\sigma \geq \tau$ and let (ξ_t) be a simple $\mathcal{H}$-valued martingale. Then

$$P_\tau(\xi_\sigma) = \xi_\tau \qquad .$$

Proof.

Since (ξ_t) is a simple martingale,

$$\xi_t = P_t(\xi)$$

for some $\xi \in \mathcal{H}$. From 3.24(iii) we have that

$$P_{\tau(\theta)} \leq P_{\sigma(\theta)}$$

for any $\theta \in \mathcal{P}$. Hence $P_\tau \leq P_\sigma$. Now,

$$\xi_\tau = P_\tau \xi$$

$$= P_\tau \circ P_\sigma \xi$$

$$= P_\tau(\xi_\sigma) \qquad .$$

Theorem 3.28 is known as Doob's optional stopping theorem in commutative probability.

3.210 Definition

If τ and σ are stopping times we define the stopping times $\sigma \vee \tau$ and $\sigma \wedge \tau$ as:

$$\sigma \vee \tau(s) = \sigma(s) \wedge \tau(s) \qquad s \in R^+$$

$$\sigma \wedge \tau(s) = \sigma(s) \vee \tau(s) \qquad s \in R^+ \qquad .$$

3.211 Theorem

If τ and σ commute pointwise then we have

(i) $P_{\sigma \wedge \tau} = P_\sigma \circ P_\tau$

(ii) $P_{\sigma \vee \tau} = P_\sigma \vee P_\tau$.

Proof.

Let $\tau = (Q_t)$ and $\sigma = (Q'_t)$. Then

$$\sigma \wedge \tau(s) = Q_s + Q'_s - Q_s Q'_s$$

for each $s \in R^+$.

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Let $\theta \in \mathcal{P}$ and $\xi \in \mathcal{H}$. Then

$$P_{\sigma \wedge \tau(\theta)}(\xi) = \sum_i \Delta(Q_{t_i} + Q'_{t_i} - Q_{t_i}Q'_{t_i})P_{t_i}(\xi)$$

$$= P_{\tau(\theta)}(\xi) + P_{\sigma(\theta)}(\xi) - P_{\sigma \vee \tau(\theta)}(\xi) \quad .$$

Hence

$$P_{\sigma \wedge \tau(\theta)} = P_{\tau(\theta)} + P_{\sigma(\theta)} - P_{\sigma \vee \tau(\theta)}$$

for any $\theta \in \mathcal{P}$, and taking the infimum over $\mathcal{P}$, we get

$$P_{\sigma \wedge \tau} = P_\tau + P_\sigma - P_{\sigma \vee \tau} \quad . \qquad \ldots \qquad (3.211a)$$

Now applying P_σ on both sides and using theorem 3.29 we get

$$P_{\sigma \wedge t} = P_\sigma \circ P_\tau$$

and (i) is established.

We observe that applying P_τ on both sides of (3.211a) we get

$$P_{\sigma \wedge \tau} = P_\tau \circ P_\sigma$$

hence

$$P_\tau \circ P_\sigma = P_\sigma \circ P_\tau \quad .$$

From (3.211a) we get

$$P_{\sigma \vee \tau} = P_\tau + P_\sigma - P_{\sigma \wedge \tau}$$

$$= P_\tau + P_\sigma - P_\tau \circ P_\sigma$$

$$= P_\tau \vee P_\sigma$$

and (ii) is shown.

Observing that the deterministic times commute with any stopping time we have:

3.212 Theorem

If (ξ_t) is a simple $\mathcal{H}$-valued martingale, then $(\xi_{\tau \wedge t})$ defines a simple $\mathcal{H}$-valued martingale for any stopping time τ .

Proof.

We consider "t" and "s" as stopping times given in (3.27a). Let $\xi_t = P_t(\xi)$ for some $\xi \in \mathcal{H}$. Then

$$\xi_{\tau \wedge t} = P_{\tau \wedge t}(\xi)$$

$$= P_\tau \circ P_t(\xi) \qquad \text{by 3.26 and 3.210} \quad .$$

Thus for $s \leq t$

$$P_s(\xi_{\tau\wedge t}) = P_s \circ P_{\tau\wedge t}(\xi)$$

$$= P_s \circ P_t \circ P_\tau(\xi)$$

$$= P_s \circ P_\tau(\xi)$$

$$= P_{\tau\wedge s}(\xi)$$

$$= P_s(\xi_\tau) \quad .$$

Theorem 3.211 is the analogue of the result which says that the process obtained by stopping a martingale with time $\tau \wedge t$ gives a martingale.

3.213 Remark

As in the last chapter $P_\tau(\xi)$ can be written in the integral form as $\int_0^\infty dQ_s P_s(\xi)$ and since $P_{\tau\wedge t}(\theta) = \sum_{i=1}^{n} \Delta Q_{t_i} P_{t_i} + (I-Q_t)P_{t+\varepsilon}$ where $\theta = \{t_0, t_1, \ldots, t_n = t, t+\varepsilon\} \in \mathcal{P}$. As $\varepsilon \to 0$, $(I-Q_t)P_{t+\varepsilon} \to (I-Q_t)P_t$ since $s \to P_s$ is continuous, hence we have

$$P_{\tau\wedge t} = \int_0^t dQ_s P_s + (I - Q_t)P_t \quad .$$

That is for $\xi \in \mathcal{H}$,

$$P_{\tau\wedge t}(\xi) = \int_0^t dQ_s P_s(\xi) + (I - Q_t)P_s(\xi) \quad .$$

3.214 Theorem

We have that

(i) $$\int_0^t dQ_sP_s(a\xi) = a\int_0^t dQ_sP_s(\xi)$$

for $a \in C$, $\xi \in \mathcal{H}$,

(ii) $$\int_0^t dQ_sP_s(\xi+\xi') = \int_0^t dQ_sP_s(\xi) + \int_0^t dQ_sP_s(\xi')$$

for ξ , $\xi' \in \mathcal{H}$,

(iii) $$\int_0^t dQ_sP_s(\xi) = \int_0^{t_0} dQ_sP_s(\xi) + \int_{t_0}^t dQ_sP_s(\xi)$$

for $0 \le t_0 < t$, $\xi \in \mathcal{H}$.

Proof.

(i) $$\int_0^t dQ_sP_s(\xi) = \Big(\operatorname*{Inf}_{\theta \in P[0,t]} \sum \Delta Q_{t_i}P_{t_i}\Big)(\xi) \quad .$$

Now $\sum_{\theta \in P[0,t]} \Delta Q_{t_i}P_{t_i}$ converges strongly to $\int_0^t dQ_sP_s$ as θ refines in $P[0,t]$ and $\sum_\theta \Delta Q_{t_i}P_{t_i}(a\xi) = a\sum \Delta Q_{t_i}P_{t_i}(\xi)$ hence the result follows.

(ii) Let $\theta \in P[0,t]$ then

$$\sum_\theta \Delta Q_{t_i}P_{t_i}(\xi+\xi') = \sum_\theta \Delta Q_{t_i}P_{t_i}(\xi) + \sum_\theta \Delta Q_{t_i}P_{t_i}(\xi') \quad .$$

Taking the limit over $P[0,t]$ gives:

$$\int_0^t dQ_s P_s(\xi+\xi') = \int_0^t dQ_s P_s(\xi) + \int_0^t dQ_s P_s(\xi') \quad .$$

(iii) Let $0 \leq t_0 < t$ and $\theta \in P[0,t]$ be such that $\theta = \{t_0, t_1, \ldots, t_r = t_0, t_{r+1} = t_0 + \varepsilon, t_{r+2}, \ldots, t_n\}$

$$\sum_\theta \Delta Q_{t_i} P_{t_i}(\xi) = \sum_{i=1}^{r} \Delta Q_{t_i} P_{t_i}(\xi) + \sum_{i=r+1}^{n} \Delta Q_{t_i} P_{t_i}(\xi) \quad .$$

Taking the limit over $P[0,t]$ gives

$$\int_0^t dQ_s P_s(\xi) = \int_0^{t_0} dQ_s P_s(\xi) + \int_{t_0}^{t} dQ_s(\xi) \quad .$$

We saw in the last chapter that the Clifford algebra furnishes an example of a "tracial case" in quantum statistical mechanics [8]. We study the C.A.R. algebra (Canonical Anti Commutation Relations algebra) in the next section to illustrate yet another example from the quantum theory. It is worth mentioning that all the results obtained in the present chapter apply equally to the C.A.R. model, however, due to certain desirable properties we are further able to prove a "random stopping theorem" for a certain class of martingales.

3.3 The C.A.R. Algebra [18]

Let K denote the complex Hilbert space $L^2(R^+)$ and A be the unital C^*-algebra generated by $\{b^*(f), b(f) : f \in K\}$ satisfying:

$$b(f)b^*(g) + b^*(g)b(f) = \langle f,g\rangle I$$

$$b(f)b(g) + b(g)b(f) = 0$$

$$b^*(\lambda f+g) = \lambda b^*(f) + b^*(g)$$

$$\|b^*(f)\| = \|b(f)\| = \|f\|$$

for all $f,g \in K$ and $\lambda \in \mathbb{C}$.

Let $R \in B(K)$ with $0 < R \leq 1$ and ω be the gauge invariant quasi-free state on A determined by $\omega(b^*(f)b(g)) = \langle f,Rg\rangle$ [18]. Let $(\mathcal{H},\pi,\Omega)$ be the G.N.S. triple associated with (A,ω) and for each $t \in \mathbb{R}^+$ $(\mathcal{H}_t,\pi,\Omega)$ be that associated with (A_t,ω) where A_t is the C.A.R. algebra over $K_t = L^2([0,t])$. Then there exists a family of conditional expectations (M_t) satisfying the properties listed in section 3.1 and constructed as in [23]. Here $M_t : A \to A_t$ is given by the equation:

$$M_t = \theta_t \circ \gamma_t$$

where θ_t is the completely positive map from $A_t \otimes B(\mathcal{H}^t)$ onto A_t given by $\theta_t(A \otimes B) = A\langle\Omega,B\Omega\rangle$ and $\gamma_t : A \to A_t \otimes B(\mathcal{H}^t)$ is an injective *-homomorphism determined by

$$\gamma_t(b(f)) = \gamma_t(b(f_t \oplus f^t))$$

$$= b(f_t) \otimes \Gamma + I \otimes \Pi(b(f^t)) \quad ,$$

where $(\mathcal{H}^t,\pi,\Omega)$ is the G.N.S. triple of the C.A.R. algebra over $K^t = L^2((t,\infty))$ and $f_t = \chi_{[0,t]}f$, $f^t = \chi_{(t,\infty)}f$. Γ is the unitary operator on $\mathcal{H}$ satisfying $\Gamma\Omega = \Omega$ and $\pi(\beta(x)) = \Gamma\pi(x)\Gamma^{-1}$ where β is the *-automorphism of A determined by $\beta(b^*(f)) = -b^*(f)$. [23].

Since $R > 0$, π is faithful, hence we may suppose A acts on $\mathcal{H}$. Now, let $f \in K$ then

$$X_s = \lambda_1 b^*(\chi_{[0,s]} f) + \lambda_2 b(\chi_{[0,s]} f)$$

defines a A-valued martingale for $\lambda_1, \lambda_2 \in C$. Thus we can define stochastic integrals with respect to (X_s) [13]. Thus if $(h(s))$ is a simple process, that is for each $s \in R^+$, $h(s) \in A_s$ and $h(s)$ takes the form

$$h(s) = \sum_t h_{i-1}\chi_{[t_{i-1},t_i)}(s) \quad .$$

Then

$$\int h(s)dX_s = \sum_i h_{i-1}\Delta X_{t_i}\Omega \quad .$$

Then the integral obeys the isometry property [13]:

$$\| \int_0^\infty h(s)dX_s \|^2 = \int_0^\infty \|h(s)\Omega\|^2 d\mu(s) \quad , \qquad \ldots \quad (3.3a)$$

where $d\mu(s) = (|\lambda_1|^2(1-\rho(s))+|\lambda_2|^2\rho(s))|f(s)|^2 ds)$ and ρ is the multiplication operator on K corresponding to $R \in B(K)$.

As usual the stochastic integral is extended to all processes that lie in the Hilbert space completion h , of simple A-valued processes with respect to the norm

$$\|g\|^2 = \int_0^\infty \|g(s)\Omega\|^2 d\mu(s)$$

where $g : R^+ \to A$ is a simple process.

Again, it was observed in [13] that the Hilbert space h is isomorphic to the subspace of μ - a.e. adapted elements of $L^2(R^+,d\mu;\mathcal{H})$. Hence, for $F \in L^2(R^+,d\mu;\mathcal{H})$ and adapted, the stochastic integral $\int dX_s F(s)$ can be defined as $\int dX_s f(s)$ where $f \in h$ corresponds to F in the isomorphism. The right stochastic integral is defined as $\int F(s)dX_s = \int dX_s \Gamma F(s)$. Then the family $(\int_0^t F(s)dX_s)_{t \in R^+}$ defines a $\mathcal{H}$-valued martingale.

We are now in the position to state the main theorem of this section.

<u>3.31 Theorem</u> (A random stopping theorem)

Let $\xi_t = \int_0^t F(s)dX_s$ define a $\mathcal{H}$-valued martingale, where $F \in L^2(R^+,\mu,\mathcal{H})$ and adapted to $(\mathcal{H}_t)$. Let $\tau_n = (Q_t^n)$ be a sequence of stopping times converging pointwise strongly to the stopping time $\tau = (Q_t)$. Then $\xi_{\tau_n} \to \xi_\tau$.

Before we prove the theorem we need the following results.

<u>3.32 Proposition</u>

Let (ξ_t) be as in the theorem. Then $\xi_t = P_t(\xi_\infty)$.

Proof.

Since $F \in L^2(R^+,\mu,\mathcal{H})$, $\int_0^\infty \|F(s)\Omega\|^2 d\mu(s) < \infty$. Hence we set

$$\xi_\infty = \int_0^\infty F(s)dX_s$$

and the stochastic integral exists.

3.33 Lemma [7]

If $\tau = (Q_t)$ is a stopping time, there exists a sequence (τ_n) of stopping times such that

(i) τ_n is simple for all $n \in N$

(ii) $\tau_n \geq \tau_{n+1}$ for all $n \in N$

(iii) $\tau_n(s) \to \tau(s)$ strongly for all $s \in R^+$.

Explicitly, there exists an increasing sequence (θ_n) of partitions of $[0,\infty]$ such that

$$\tau_n(s) = \sum_{\theta_n} Q_{t^n_{i-1}} \chi_{[t^n_{i-1},t^n_i)}(s) .$$

We write $\tau_n = (Q_s^{\theta_n})$, so that for each $s \in R^+$:

$$Q_s^{\theta_n} = \tau_n(s) \to \tau(s)$$

strongly as $n \to \infty$.

3.34 Proposition

Let (ξ_t) be as in the statement of the theorem and $\tau = (Q_t)$ be any stopping time. Then

(i) $\int Q_s F(s)dX_s$ is a well defined stochastic integral

(ii) Let $\theta \in \mathcal{P}[0,\infty]$, then

$$\xi_{\tau(\theta)} = P_{\tau(\theta)}(\xi) = \int_0^\infty (I-Q_s^\theta)F(s)dX_s$$

where if $\theta = \{t_0,\ldots,t_n\}$ then $Q_s^\theta = \sum_{i=1}^n Q_{t_{i-1}}\chi_{[t_{i-1},t_i)}(s)$.

(iii) $$P_{\tau(\theta)}\int_0^\infty (I-Q_s)F(s)dX_s = \int_0^\infty (I-Q_s)F(s)dX_s \quad .$$

(iv) $$\xi_\tau = \int_0^\infty (I-Q_s)F(s)dX_s \quad .$$

Proof.

(i) Let $h : R^+ \to \mathcal{H}$ be given by $h(s) = Q_s F(s)$. Then from 3.12 and the fact that F is adapted we have:

$$\begin{aligned} P_s h(s) &= P_s Q_s F(s) \\ &= Q_s P_s F(s) \\ &= Q_s F(s) \\ &= h(s) \quad . \end{aligned}$$

Hence h is adapted. We now want to show that $h \in L^2(R^+,\mu;\mathcal{H})$. To show that h is measurable with respect to μ, it is enough to consider F elementary. The general case follows by linearity and continuity. Now suppose $F(s) = F_0$ and (τ_n) be a sequence of stopping times approximating τ as in 3.33. Then

$$\int_0^\infty \|\tau_n(s)F_0 - \tau(s)F_0\|^2 d\mu(s)$$
$$= \int_0^\infty \|(Q_s^{\theta_n} - Q_s)F_0\|^2 d\mu(s) \qquad .$$

But

$$\|(Q_s^{\theta_n} - Q_s)F_0\|^2 \to 0 \qquad \text{as } n \to \infty$$

since $Q_s^{\theta_n} \to Q_s$ strongly and

$$\|(Q_s^{\theta_n} - Q_s)F_0\|^2 \leq 4\|F_0\|^2$$

F is in $L^2(R^+,d\mu,\mathcal{H})$ and furthermore $d\mu$ is a finite measure. Hence the dominated convergence theorem is applicable [22] and we have

$$\tau_n(s)F_0 \to \tau(s)F_0$$

in $\mathcal{H}$ pointwise. Thus there is a sequence of simple processes, $(\tau_n F_0)_{n \in N}$ converging to τF pointwise in $\mathcal{H}$. Hence the $\mathcal{H}$-valued function $s \to \tau(s)F(s)$ is measurable [22]. Moreover

$$\|h\|^2 = \int_0^\infty \|Q_s F(s)\|^2 d\mu(s)$$

$$\leq \int_0^\infty \|F(s)\|^2 d\mu(s)$$

$$< \infty \quad .$$

Hence the process $s \to h(s)$ lies in $L^2(R^+, d\mu, \mathcal{H})$ and the stochastic integral $\int_0^\infty h(s)dX_s$ is well defined [13].

(ii) Since ξ_∞ exists (ξ_t) is a simple $\mathcal{H}$-valued martingale, i.e. $\xi_t = P_t \xi_\infty$ [5]. Thus for $\theta \in P$

$$\xi_{\tau(\theta)} = P_{\tau(\theta)} \xi_\infty$$

$$= \left(\sum_{i=1}^{n} \Delta Q_{t_i} P_{t_i}\right)\xi_\infty$$

$$= \left(I - \sum_{i=1}^{n} Q_{t_{i-1}} \Delta P_{t_i}\right)\xi_\infty$$

$$= \int_0^\infty F(s)dX_s - \sum_{i=1}^{n} Q_{t_{i-1}} \int_{t_{i-1}}^{t_i} F(s)dX_s \quad .$$

Now,

$$Q_{t_{i-1}} \int_{\Delta t_i} F(s)dX_s = Q_{t_{i-1}} \operatorname*{Lim}_n \sum_j \Delta X_{s_j} \Gamma F^n(s_{j-1}) \quad \ldots \quad (3.33a)$$

Here $(F^n)_{n \in N}$ is a sequence of simple $\mathcal{H}$-valued processes converging to F in the norm given by (3.3a).

Now, $Q_{t_{i-1}}$ is continuous and linear, hence (3.33a) becomes

$$\operatorname*{Lim}_n \sum_j Q_{t_{i-1}} \Delta X_{s_j} \Gamma F^n(s_{j-1})$$

$$= \operatorname*{Lim}_n \sum_j \Delta X_{s_j} \Gamma Q_{t_{i-1}} F^n(s_{j-1}) \qquad \text{by [13]} \quad .$$

But $Q_{t_{i-1}} F^n \to Q_{t_{i-1}} F$ in the norm given by (3.3a) and $Q_{t_{i-1}} F$ is μ-measurable from (i), hence (3.33a) becomes

$$\int_{t_{i-1}}^{t_i} Q_{t_{i-1}} F(s) dX_s \quad .$$

Thus

$$\sum_{i=1}^{n} Q_{t_{i-1}} \int_{t_{i-1}}^{t_i} F(s) dX_s = \int_0^\infty Q_s^\theta F(s) dX_s$$

and

$$\xi_{\tau(\theta)} = \int_0^\infty (I - Q_s^\theta) F(s) dX_s \quad .$$

(iii) Let $Z = \int_0^\infty (I-Q_s)F(s)dX_s$. Then

$$P_{\tau(\theta)} Z = \int_0^\infty (I-Q_s^\theta)(I-Q_s)F(s)dX_s$$

from (ii). But by construction if $\theta = \{t_0, \ldots, t_n\}$ then

$$Q_s^\theta = \sum_{i=1}^{n} Q_{t_{i-1}} \chi_{[t_{i-1}, t_i)}(s) \quad .$$

Thus $Q_s^\theta \leq Q_s$ for all $s \in R^+$ and hence $(I-Q_s^\theta)(I-Q_s) = I - Q_s$. Thus $P_{\tau(\theta)}Z = Z$.

(iv) Let θ_n and θ be any partitions of $[0,\infty]$ with θ_n coarser than θ , so that

$$P_{\tau(\theta)} \circ P_{\tau(\theta_n)} = P_{\tau(\theta)} \qquad \text{by } 3.24 ,$$

and

$$\begin{aligned}
\|Z - P_{\tau(\theta)}\xi_\infty\|^2 &= \|P_{\tau(\theta)}(Z - P_{\tau(\theta_n)}\xi_\infty)\|^2 \qquad \text{by (iii)} \\
&\leq \|Z - P_{\tau(\theta_n)}\xi_\infty\|^2 \\
&= \left\|\int_0^\infty (Q_s - Q_s^{\theta_n})F(s)dX_s\right\|^2 \\
&= \int_0^\infty \|(Q_s - Q_s^{\theta_n})F(s)\|^2 d\mu(s)
\end{aligned}$$

by the isometry property [13].

Now let (θ_n) be a sequence of partitions given in the approximation of τ in lemma 3.33. That is

$$\begin{aligned}
\tau_n(s) &= \sum_{\theta_n} Q_{t_{i-1}} \chi_{[t_{i-1},t_i)}(s) \\
&= Q_s^{\theta_n}
\end{aligned}$$

and $Q_s^{\theta_n} \to Q_s$ strongly. Thus $\|(Q_s - Q_s^{\theta_n})F(s)\|^2 \to 0$ for each $s \in R^+$ and $\|(Q_s - Q_s^{\theta_n})F(s)\|^2 \leq 4\|F(s)\|$. Hence the dominated convergence theorem is applicable:

$$\int_0^\infty \|(Q_s - Q_s^{\theta_n})F(s)\|^2 d\mu(s) \to 0$$

as $n \to \infty$. That is given $\varepsilon > 0$ there is a $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon)$

$$\left\|\int_0^\infty (I - Q_s)F(s)dX_s - \int_0^\infty (I - Q_s^{\theta_n})F(s)dX_s\right\|^2$$
$$= \int_0^\infty \|(Q_s - Q_s^{\theta_n})F(s)\|^2 d\mu(s) < \varepsilon \quad .$$

Now set $\theta = \theta_{N(\varepsilon)} \in \mathcal{P}$, then for any partition θ' finer than θ, we have

$$\|Z - P_{\tau(\theta')}\xi_\infty\|^2 \leq \|Z - P_{\tau(\theta)}\xi_\infty\|^2 < \varepsilon \quad .$$

Hence

$$Z = P_\tau \xi_\infty$$
$$= \int_0^\infty (I - Q_s)F(s)dX_s \quad .$$

<u>Proof of theorem 3.31</u>

From the proposition above we know that

$$\xi_\tau = \int_0^\infty (I - Q_s)F(s)dX_s$$

and

$$\xi_{\tau_n} = \int_0^\infty (I - Q_s^n)F(s)dX_s \quad .$$

Thus

$$\|\xi_\tau - \xi_{\tau_n}\|^2 = \int_0^\infty \|(Q_s - Q_s^n)F(s)\|^2 d\mu(s) \qquad .$$

As $Q_s^n \to Q_n$ strongly and $\|(Q_s - Q_s^n)F(s)\|^2 \leq 4\|F(s)\|^2$, the dominated convergence theorem is applicable and the result follows.

CHAPTER FOUR

A PROBABILITY GAUGE SPACE

4.0 Introduction

In this chapter we study probability theory in the gauge space given by $L^{\infty}(\Omega,\Sigma,P) \otimes M_2(C)$. The purpose of this is to give examples of non-commutative local martingales and hence develop a stochastic integration theory with respect to those processes. We shall first develop the necessary theory which is of interest in its own right and open to exploitations but we shall not pursue this here.

4.1 Preliminaries

Let $(\Omega,\Sigma,P,\Sigma_{\alpha},R^{+})$ be a stochastic base and $L^{\infty}(\Omega,\Sigma,P)$ be the space of equivalence classes of bounded measurable complex valued functions over the probability space (Ω,Σ,P) . Then, as mentioned earlier, $L^{\infty}(\Omega,\Sigma,P)$ is a commutative Von Neumann algebra of bounded operators on $L^{2}(\Omega,\Sigma,P)$, where for $\infty > p > 0$, $L^{p}(\Omega,\Sigma,P)$ is the space of measurable functions over (Ω,Σ,P) with $\int_{\Omega}|f|^{p}\, dP < \infty$, and the action of $L^{\infty}(\Omega,\Sigma,P)$ on $L^{2}(\Omega,\Sigma,P)$ is given by multiplication.

Let $L^{\infty}(\Omega,\Sigma,P) \otimes M_2(C)$ denote the tensor product of $L^{\infty}(\Omega,\Sigma,P)$ and $M_2(C)$, the space of 2×2 matrices with entries in C [21] . Then $L^{\infty} \otimes M_2$ is a finite Von Neumann algebra with normalised trace given by ϕ :

$$\phi(F \otimes N) = \tfrac{1}{2} \int_{\Omega} F dP \cdot (N(1,1) + N(2,2))$$

where $F \in L^{\infty}(\Omega,\Sigma,P)$ and $N = (N(i,j)) \in M_2(C)$. We shall abbreviate $L^{\infty}(\Omega,\Sigma,P)$ to L^{∞} and $M_2(C)$ to M_2. It is now clear that $\phi = E \otimes \tau$ where E is the usual expectation on L^{∞} and τ the normalised trace on M_2.

4.2 The Probability Gauge Space $(L^{\infty} \otimes M_2, \phi)$

In this section we study the gauge space [35] given by $L^{\infty} \otimes M_2$ and ϕ. We first identify the Von Neumann algebra $L^{\infty} \otimes M_2$ with $M_2(L^{\infty})$, the space of 2×2 matrices with entries from L^{∞}. The elements of $M_2(L^{\infty})$ are clearly bounded operators on the Hilbert space $L^2(\Omega,\Sigma,P) \oplus L^2(\Omega,\Sigma,P)$, which is isomorphic to $L^2(\Omega,\Sigma,P) \otimes C^2$, the Hilbert space on which the tensor product $L^{\infty} \otimes M_2$ act [21]. We contract $L^p(\Omega,\Sigma,P)$ to L^p, where $1 \leq p \leq \infty$.

Now let $\begin{pmatrix} f \\ g \end{pmatrix}$ denote an element of $L^2 \oplus L^2$, then the norm of $\begin{pmatrix} f \\ g \end{pmatrix}$ is given by:

$$\left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|_2^2 = \|f\|_2^2 + \|g\|_2^2$$

where the norms on the right hand side are the usual norms on L^2.

4.21 Lemma

We have that $M_2(L^{\infty})$ is a Von Neumann algebra of bounded operators on $L^2 \oplus L^2$.

Proof.

It is clear that $M_2(L^\infty)$ is a unital *-algebra of bounded operators on $L^2 \oplus L^2$ with

$$A^* = \begin{pmatrix} \overline{A(1,1)} & \overline{A(2,2)} \\ \overline{A(1,2)} & \overline{A(2,2)} \end{pmatrix}$$

where

$$A = \begin{pmatrix} A(1,1) & A(1,2) \\ A(2,1) & A(2,2) \end{pmatrix}$$

and $A(i,j) \in L^\infty$, $i = 1,2$; $j = 1,2$.

Now all that remains to show is that $M_2(L^\infty)$ is strongly closed. To this end let (A_α) be a net in $M_2(L^\infty)$ converging strongly to A , a bounded operator on $L^2 \oplus L^2$. Then $E_1A_\alpha E_1$ belongs to $M_2(L^\infty)$ and converges strongly to E_1AE_1 where E_1 is the 2×2 matrix whose top left entry is the identity (χ_Ω or I) and the rest zero. Then $(E_1A_\alpha E_1)$ is a strongly Cauchy net. That is

$$\left\| \left(E_1A_\alpha E_1 - E_1A_\beta E_1\right)\begin{pmatrix} x \\ y \end{pmatrix}\right\|_2 \to 0$$

as α and β increases for all $\begin{pmatrix} x \\ y \end{pmatrix}$ in $L^2 \oplus L^2$. Hence

$$\|(A_\alpha(1,1) - A_\beta(1,1))x\|_2 \to 0$$

as α and β increases for all $x \in L^2$. Thus $(A_\alpha(1,1))$ is a strongly Cauchy net in L^∞, and there exists $A(1,1) \in L^\infty$ which is a strong limit of $A_\alpha(1,1)$. Likewise considering $E_i A_\alpha E_j$, $i = 1,2$; $j = 1,2$ where E_2 is the 2×2 matrix whose bottom right entry is the identity and others zero, we get the strong convergence of each $A_\alpha(i,j)$ to $A(i,j)$. Now let $\tilde{A}$ denote the 2×2 matrix given by $A(i,j)$ $i = 1,2$; $j = 1,2$ and consider

$$\left\| (A - \tilde{A}) \begin{pmatrix} x \\ y \end{pmatrix} \right\|_2$$

$$\leq \left\| (A_\alpha - A) \begin{pmatrix} x \\ y \end{pmatrix} \right\|_2 + \left\| (A_\alpha - \tilde{A}) \begin{pmatrix} x \\ y \end{pmatrix} \right\|_2 .$$

The first term on the right of the inequality sign tends to zero as α increases, whilst the second term squared is dominated by

$$\|(A_\alpha(1,1) - A(1,1))x\|_2^2 + \|(A_\alpha(1,s) - A(1,2))y\|_2^2$$

$$+ \|(A_\alpha(2,1) - A(2,1))x\|_2^2 + \|(A_\alpha(2,2) - A(2,2))y\|_2^2$$

which again tends to zero as α increases. Thus $A = (A(i,j))$. That is $M_2(L^\infty)$ is strongly closed. Hence $M_2(L^\infty)$ is a Von Neumann algebra.

4.22 Corollary

Let $(A_\alpha) \subseteq M_2(L^\infty)$ be a net. Then A_α converges strongly to A as α increases if and only if $A_\alpha(i,j)$ converges strongly to

$A(i,j)$ for each $i = 1,2$; $j = 1,2$.

Proof.

Suppose $A_\alpha \begin{pmatrix} x \\ y \end{pmatrix} \to A \begin{pmatrix} x \\ y \end{pmatrix}$ for all $\begin{pmatrix} x \\ y \end{pmatrix} \in L^2 \oplus L^2$. In particular, setting $y = 0$, gives

$$\begin{pmatrix} A_\alpha(1,1)x \\ A_\alpha(2,1)x \end{pmatrix} \to \begin{pmatrix} A(1,1)x \\ A(2,1)x \end{pmatrix}$$

for all $x \in L^2$. Hence $A_\alpha(1,1) \to A(1,1)$ strongly and $A_\alpha(2,1) \to A(2,1)$ strongly. Likewise, setting $x = 0$, shows that the other entries converge strongly.

The converse follows from Lemma 4.21.

The next result says that we may identify $L^\infty \times M_2$ with $M_2(L^\infty)$.

4.23 Proposition

Let $L^\infty \times M_2$ denote the algebraic tensor product of L^∞ and M_2 [21]. Then there is a strongly continuous *-isomorphism from $L^\infty \times M_2$ onto $M_2(L^\infty)$.

Proof.

Let $\Pi : L^\infty \times M_2 \to M_2(L^\infty)$ be given by

$$\Pi\left(\sum_{K=1}^{N} F_K \otimes N_K\right) = \sum_{K=1}^{N} F_K \cdot N_K$$

where $F_K \in L^\infty$, $N_K \in M_2$ $K = 1,2,\ldots,N$ and $F \cdot N = (FN(i,j))$.

Then it is clear that Π is a linear, *-preserving isomorphism. The strong continuity of Π follows from:

$$\left\| \Pi(F \otimes N) \begin{pmatrix} x \\ y \end{pmatrix} \right\|_2 = \| (F \otimes N)(x \otimes e_1) + (F \otimes N)(y \otimes e_2) \|_2$$

where $\{e_1, e_2\}$ is the usual basis for C^2 .

4.24 Corollory

We have that $L^\infty \otimes M_2$ is a Von Neumann algebra.

Proof.

Let $(T_\alpha) \subseteq L^\infty \otimes M_2$ converge strongly to T , a bounded operator on $L^2 \otimes C^2$ $(= L^2 \oplus L^2)$. Then by 4.21 and 4.23, $\pi(T_\alpha)$ converges to $S \in M_2(L^\infty)$. Now,

$$\| (\pi^{-1}(S) - T)(f \otimes Z) \|_2$$

$$\leq \| (\pi^{-1}(S) - T_\alpha)(f \otimes Z) \|_2 + \| (T_\alpha - T)(f \otimes Z) \|_2$$

where $f \otimes Z \in L^2 \otimes C^2$.

The second term on the right tends to zero as α increases, whilst the first term on the right is

$$\left\| (S - \pi(T_\alpha)) \begin{pmatrix} Z_1 f \\ Z_2 f \end{pmatrix} \right\|_2 ,$$

which tends to zero. Here we have set $Z = Z_1 e_1 + Z_2 e_2$. Thus $\pi^{-1}(S) = T$ and $L^\infty \otimes M_2$ *is* the Von Neumann algebra $L^\infty \otimes M_2$.

It is now clear that we may identify $L^\infty \otimes M_2$ with $M_2(L^\infty)$ with trace, which we again denote by ϕ , given by

$$\phi(A) = \tfrac{1}{2} \int_\Omega (A(1,1) + A(2,2))dP \quad ,$$

where $A = (A(i,j))$, $A(i,j) \in L^\infty$, $i = 1,2$; $j = 1,2$.

We now proceed to identify the measurable operators and the non-commutative L^p-spaces associated with $(L^\infty \times M_2 , \phi)$ [38].

4.25 Definition [35]

Let A be a finite Von Neumann algebra of bounded operators on a Hilbert space $\mathcal{H}$. Then a (possibly unbounded) operator T on $\mathcal{H}$ is said to be measurable with respect to A if:

(i) T is closed

(ii) T has a strongly dense domain

(iii) $TU = UT$ for all $U \in A'_{\text{unitary}}$.

We denote the algebra of measurable operators over A by $M(A)$.

4.26 Theorem

We have that

$$M(M_2(L^\infty)) = M_2(M(L^\infty)) \quad .$$

That is the algebra of measurable operators over $M_2(L^\infty)$ is same as the algebra of 2×2 matrices whose entries are equivalence classes of measurable functions over (Ω,Σ,P) .

Before we prove theorem 4.26, we need a lemma:

4.27 Lemma

Let $E \oplus F$ be dense in $L^2 \oplus L^2$. Then both E and F are dense in L^2.

Proof.

Let $f \in L^2$, so that $\begin{pmatrix} f \\ f \end{pmatrix} \in L^2 \oplus L^2$. By hypothesis, there exists a sequence $\begin{pmatrix} f_n \\ g_n \end{pmatrix} \subseteq E \oplus F$ such that

$$\left\| \begin{pmatrix} f_n \\ g_n \end{pmatrix} - \begin{pmatrix} f \\ g \end{pmatrix} \right\|_2^2 \to 0 \qquad .$$

That is

$$\|f_n - f\|_2^2 + \|g_n - f\|_2^2 \to 0$$

and the result follows.

Proof of 4.26.

Let $T \in M_2(\mathcal{M}(L^\infty))$ so that $T = (T(i,j))$ and $T(i,j) \in \mathcal{M}(L^\infty)$. If $\mathcal{D}(i,j)$ denotes the strongly dense domain of $T(i,j)$, we set

$$\mathcal{D} = (\mathcal{D}(1,1) \cap \mathcal{D}(1,2)) \oplus (\mathcal{D}(2,1) \cap \mathcal{D}(2,2)) \qquad .$$

It is known that the intersection of two strongly dense domains is strongly dense [35], hence $\mathcal{D}$ is strongly dense in $L^2 \oplus L^2$.

Now let $\left\{\begin{pmatrix} f_n \\ g_n \end{pmatrix}\right\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{D}$ and suppose

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} \to \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} f_n \\ g_n \end{pmatrix} \to \begin{pmatrix} h \\ K \end{pmatrix}$$

as $n \to \infty$. Then

$$\begin{pmatrix} T(1,1)f_n + T(1,2)g_n \\ T(2,1)f_n + T(2,2)g_n \end{pmatrix} \to \begin{pmatrix} h \\ K \end{pmatrix} .$$

But $f_n \to f$ in L^2 and hence in measure, and $T(1,1)f_n \to T(1,1)f$ in measure. By passing to a subsequence, we may assume $T(1,1)f_n \to T(1,1)f$ (a.s.). Similarly $T(1,2)g_n \to T(1,2)g$ (a.s.) hence $T(1,1)f_n + T(1,2)g_n \to T(1,1)f + T(1,2)g$ (a.s.). Thus $T(1,1)f + T(1,2)g = h$ and similarly $T(2,1)f + T(2,2)g = K$. We now have $T\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} h \\ K \end{pmatrix}$ and T is closed.

Now let U be a unitary operator in the commutant of $M_2(L^\infty)$. Then

$$U = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$$

where $u \in L^\infty$ with $|u| = 1$.

Let $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}$ and consider $TU\begin{pmatrix} f \\ g \end{pmatrix}$.

$$TU\begin{pmatrix} f \\ g \end{pmatrix} = T\begin{pmatrix} uf \\ ug \end{pmatrix} .$$

This is well defined since $u\mathcal{D}(i,j) \subseteq \mathcal{D}(i,j)$. Now

$$T\begin{pmatrix} uf \\ ug \end{pmatrix} = \begin{pmatrix} T(1,1)uf + T(1,2)ug \\ T(2,1)uf + T(2,2)ug \end{pmatrix}$$

$$= \begin{pmatrix} uT(1,1)f + uT(1,2)g \\ uT(2,1)f + uT(2,2)g \end{pmatrix}$$

$$= UT\begin{pmatrix} f \\ g \end{pmatrix} .$$

We have used the fact that $T(i,j)$ is measurable. We have $M_2(\mathcal{M}(L^\infty)) \subseteq \mathcal{M}(M_2(L^\infty))$.

Conversely, suppose that $T \in \mathcal{M}(M_2(L^\infty))$. We want to show $T = (T(i,j))$ $i = 1,2$; $j = 1,2$ and $T(i,j) \in \mathcal{M}(L^\infty)$. Now since $\mathcal{M}(M_2(L^\infty))$ is an algebra [35], E_iTE_j is in $\mathcal{M}(M_2(L^\infty))$. (Note, that we take strong products here [35].) Let V_1 and V_2 be maps from L^2 to $L^2 \oplus L^2$ defined by:

$$V_1(f) = \begin{pmatrix} f \\ 0 \end{pmatrix} .$$

$$V_2(f) = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

Then if $E \oplus F$ is the domain of E_iTE_j , we have from Lemma 4.27 that both $V_1^*(E \oplus F)$ and $V_2^*(E \oplus F)$ are dense in L^2 . We now define $T(i,j)$ on $V_j^*(E \oplus F)$ by

$$T(i,j)f = V_i^* E_i T E_j V_j(f) \quad .$$

Then $T(i,j)$ is densely defined.

To show that $T(i,j)$ is closed, suppose $(f_n) \subseteq V_j^*(E \oplus F)$, $f_n \to f$ and $T(i,j)f_n \to \gamma$. That is,

$$V_i^* E_i T E_j V_j(f_n) \to \gamma \quad ,$$

hence

$$E_i T E_j V_j(f_n) \to V_i(\gamma) \quad .$$

Now, $E_i T E_j$ is measurable and $V_j(f_n) \to V_j(f)$, it follows that $E_i T E_j V_j(f) = V_i(\gamma)$.

That is

$$V_i^* E_i T E_j V_j(f) = \gamma$$

and $T(i,j)$ is closed.

Now let $u \in L^\infty$ with $|u| = 1$, and $f \in V_j^*(E \oplus F)$. Then

$$\begin{aligned} T(i,j)uf &= V_i^* E_i T E_j V_j uf \\ &= V_i^* E_i T E_j V_j(uf) \\ &= V_i^* E_i T E_j (u \otimes I) V_j f \\ &= V_i^* (u \otimes I) E_i T E_j V_j f \end{aligned}$$

since $u \otimes I \in M_2(L^\infty)'$

$$= uV_i^*E_iTE_jV_jf \quad .$$

Hence $T(i,j)$ is affiliated to L^∞ [35]. Hence $T(i,j)$ is measurable operator with respect to L^∞. That is $T(i,j) \in M(L^\infty)$. Hence there is a measurable function $t(i,j)$ on (Ω,Σ,P) such that

$$(T(i,j)f)(w) = t(i,j)(w)f(w)$$

where $f \in V_j^*(E \oplus F)$ [35].

Now set

$$T_0 = \begin{pmatrix} t(1,1) & t(1,2) \\ t(2,1) & t(2,2) \end{pmatrix}$$

then the domain of T_0 is $E \oplus F$ and

$$T_0\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} t(1,1)f + t(1,2)g \\ t(2,1)g + t(2,2)g \end{pmatrix}$$

$$= \begin{pmatrix} V_1^*E_1TE_1V_1(f) + V_1^*E_1TE_2V_2(g) \\ V_2^*E_2TE_1V_1(f) + V_2^*E_2TE_2V_2(g) \end{pmatrix}$$

$$= V_1^*E_1T\begin{pmatrix} f \\ g \end{pmatrix} \oplus V_2^*E_2T\begin{pmatrix} f \\ g \end{pmatrix}$$

$$= T\begin{pmatrix} f \\ g \end{pmatrix} \quad .$$

Thus $T = (t(i,j))$ $i = 1,2$; $j = 1,2$ and $T(i,j) \in \mathcal{M}(L^\infty)$ and the result follows.

We are now in a position to define the non-commutative L^p-spaces associated with $M_2(L^\infty)$ [38].

4.28 Definition

Let (T_n) be a sequence in $\mathcal{M}(M_2(L^\infty))$ and T be in $\mathcal{M}(M_2(L^\infty))$. We say T_n converges to T metrically nearly everywhere (m.n.e.) if for each $\varepsilon > 0$, there is projections $\{E_n : n \in \mathbb{N}\}$ such that

$$E_n \uparrow I \quad , \quad \|(T_n - T)E_n\|_\infty < \varepsilon \quad \text{and} \quad \phi(I - E_n) \to 0 \quad .$$

4.29 Definition

We say that $T \in \mathcal{M}(M_2(L^\infty))$ is integrable if there is a sequence $(T_n) \subseteq M_2(L^\infty)$ such that T_n converges to T m.n.e. and $\phi(|T_n - T_m|) \to 0$ as $m,n \to \infty$. The space of all integrable operators in $\mathcal{M}(M_2(L^\infty))$ is denoted by $L^1(M_2(L^\infty))$ (or simply L^1 when there is no confusion). The trace of $T \in L^1$ is given by

$$\phi(T) = \lim_{m\to\infty} \phi(T_n) \quad .$$

4.210 Definition

Let $1 \leq p < \infty$, define

(i) $L^p(M_2(L^\infty)) = \{T \in \mathcal{M}(M_2(L^\infty)) : \phi(|T|^p) < \infty\}$,

(ii) $\|T\|_p = \phi(|T|^p)^{1/p}$ where $T \in L^p(M_2(L^\infty))$.

We shall abbreviate $L^p(M_2(L^\infty))$ to L^p .

4.211 Lemma

The extension of ϕ to L^1 is given by

$$\phi(T) = \tfrac{1}{2}\int_{\Omega} (T(1,1) + T(2,2))dP$$

where $T = (T(i,j)) \quad i = 1,2 \ ; \quad j = 1,2$.

Proof.

Let $T \in L^1$ then $E_i T E_j \in L^1$ for $i = 1,2$; $j = 1,2$ [38].

Now

$$E_1 T E_1 = \begin{pmatrix} T(1,1) & 0 \\ 0 & 0 \end{pmatrix}$$

and $T(1,1) \in M(L^\infty)$. Thus if $T(1,1) = \omega|T(1,1)|$ be the polar decomposition of $T(1,1)$ and

$$|T(1,1)| = \int_0^\infty \lambda dE_\lambda$$

be the spectral decomposition of $|T(1,1)|$ then

$$E_1 T E_1 = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |T(1,1)| & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$|E_1 T E_1| = \int_0^\infty \lambda d \begin{pmatrix} E_\lambda & 0 \\ 0 & 0 \end{pmatrix} .$$

Now,

$$\phi(|E_1TE_1|) = \lim_{n\to\infty} \phi\left[\int_0^n \lambda d\begin{pmatrix} E_\lambda & 0 \\ 0 & 0 \end{pmatrix}\right]$$

$$= \lim_{n\to\infty} \phi\left[\begin{pmatrix} \int_0^n \lambda dE_\lambda & 0 \\ 0 & 0 \end{pmatrix}\right]$$

$$= \tfrac{1}{2}\int_\Omega |T(1,1)|d\Omega \quad .$$

Hence $T(1,1) \in L^1(\Omega,\Sigma,P)$. Also

$$T_n \equiv \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix} \int_0^n \lambda d\begin{pmatrix} E_\lambda & 0 \\ 0 & 0 \end{pmatrix}$$

converges m.n.e. to E_1TE_1 and is Cauchy in L^1 . It follows that

$$\phi(E_1TE_1) = \lim_{n\to\infty} \phi(T_n)$$

$$= \lim_{n\to\infty} \varphi\left[\begin{pmatrix} \omega\int_0^n \lambda dE_\lambda & 0 \\ 0 & 0 \end{pmatrix}\right]$$

$$= \tfrac{1}{2}\int_\Omega T(1,1)dP \quad .$$

Likewise $\phi(E_2TE_2) = \frac{1}{2}\int_\Omega T(2,2)dP$ and $\phi(E_1TE_2) = \phi(E_2TE_1) = 0$,

it follows that

$$\phi(T) = \frac{1}{2}\int_\Omega (T(1,1) + T(2,2))dP \quad .$$

4.212 Theorem

For $1 \leq p \leq \infty$, we have that

$$L^p(M_2(L^\infty(\Omega,\Sigma,P)) = M_2(L^p(\Omega,\Sigma,P)) \quad .$$

That is the L^p space associated with $M_2(L^\infty)$ is the same as the space of 2×2 matrices with entries from $L^p(\Omega,\Sigma,P)$.

Proof.

First suppose $T = (T(i,j)) \in M_2(L^p)$ (we have abbreviated $M_2(L^p(\Omega,\Sigma,P))$ to $M_2(L^p)$). Then

$$\|T\|_p \leq \|T(1,1)\|_p + \|T(2,2)\|_p + \|T(1,2)\|_p + \|T(2,1)\|_p$$

and

$$T \in L^p(M_2(L^\infty)) \quad .$$

Conversely, suppose $T \in L^p(M_2(L^\infty))$. Then $E_1TE_1 \in L^p(M_2(L^\infty))$. Hence $|E_1TE_1|^p \in L^1(M_2(L^\infty))$. That is,

$$\begin{pmatrix} |T(1,1)|^p & 0 \\ 0 & 0 \end{pmatrix} \in L^1(M_2(L^\infty)) \qquad ,$$

and

$$\infty > \phi(|E_1TE_1|^p) = \tfrac{1}{2}\int_\Omega |T(1,1)|^p \, dP \qquad .$$

Thus $T(1,1) \in L^p(\Omega,\Sigma,P)$. Likewise, considering E_iTE_j $i = 1,2$; $j = 1,2$ we get $T(i,j) \in L^p(\Omega,\Sigma,P)$ and the result follows.

We now wish to identify the Von Neumann subalgebras of $L^\infty(\Omega,\Sigma,P)$. To this end, let Σ' be a sub-σ-field of Σ and let $L^\infty(\Omega,\Sigma',P)$ denote the space of bounded, Σ'-measurable functions. Let $E_{\Sigma'}$ denote the conditional expectation from $L^p(\Omega,\Sigma,P)$ onto $L^p(\Omega,\Sigma',P)$ for $1 \leq p \leq \infty$. Then it is known that $E_{\Sigma'}$ is weakly continuous. Now, the kernel of the map $I - E_{\Sigma'} : L^\infty(\Omega,\Sigma,P) \to L^\infty(\Omega,\Sigma',P)$ is precisely $L^\infty(\Omega,\Sigma',P)$ and the weak continuity of $I - E_{\Sigma'}$ gives the weak closedness of $L^\infty(\Omega,\Sigma',P)$ and hence it is a Von Neumann algebra.

Conversely if $\mathcal{B} \subseteq L^\infty(\Omega,\Sigma,P)$ is a Von Neumann subalgebra, then set $\Sigma' = \{A \in \Sigma : \chi_A \in \mathcal{B}\}$. Hence $L^\infty(\Omega,\Sigma',P) \subseteq \mathcal{B}$. Also since $\mathcal{B}$ is a Von Neumann algebra, $\chi_{\{f<\lambda\}} \in \mathcal{B}$ for any self-adjoint $f \in \mathcal{B}$ and $\lambda \in R$. Hence $\{f<\lambda\} \in \Sigma'$ and $\mathcal{B} = L^\infty(\Omega,\Sigma',P)$. We have:

4.213 Theorem

Let $\mathcal{B} \subseteq L^\infty(\Omega,\Sigma,P)$. Then $\mathcal{B}$ is a Von Neumann subalgebra of $L^\infty(\Omega,\Sigma,P)$ if and only if $\mathcal{B} = L^\infty(\Omega,\Sigma',P)$ for some sub-σ-field Σ' of Σ .

4.214 Theorem

We have that

$$L^p(M_2(L^\infty(\Omega,\Sigma',P))) = M_2(L^p(\Omega,\Sigma',P)) \quad ,$$

for $1 \leq p \leq \infty$ and any sub-σ-field Σ' of Σ .

Proof.

This is similar to the proof of 4.212 after observing that $L^\infty(\Omega,\Sigma,P) \subseteq L^\infty(\Omega,\Sigma',P)'$.

We now define the conditional expectation $M_{\Sigma'}$ from $M_2(L^p(\Omega,\Sigma,P))$ onto $M_2(L^p(\Omega,\Sigma',P))$ for $1 \leq p \leq \infty$.

Let $E_{\Sigma'}$ denote the conditional expectation from $L^\infty(\Omega,\Sigma,P)$ onto $L^\infty(\Omega,\Sigma',P)$ [28]. Then,

$$\int_G gE_{\Sigma'}(f)dP = \int_G gfdP \quad ,$$

for all $f \in L^\infty(\Omega,\Sigma,P)$, $g \in L^\infty(\Omega,\Sigma',P)$ and $G \in \Sigma'$.

4.215 Proposition

The linear map $E_{\Sigma'} \otimes I$ from $L^\infty(\Omega,\Sigma,P) \otimes M_2(C)$ onto $L^\infty(\Omega,\Sigma',P) \otimes M_2(C)$ given by

$$E_{\Sigma'} \otimes I(f \otimes N) = E_{\Sigma'}(f) \otimes N$$

is the conditional expectation map.

Proof.

It is clear that $M_{\Sigma'} \equiv E_{\Sigma'} \otimes I$ preserves the identity and a straight forward calculation shows that

$$\phi((g \otimes R)(f \otimes N)(h \otimes T))$$

$$= \phi((g \otimes R)M_{\Sigma'}(f \otimes N)(h \otimes T))$$

for all $g \otimes R$, $h \otimes T \in L^{\infty}(\Omega,\Sigma',P)$ and $f \otimes N \in L^{\infty}(\Omega,\Sigma,P)$.

It follows from [27] that $M_{\Sigma'}$ is the conditional expectation map of $M_2(L^{\infty}(\Omega,\Sigma,P))$ onto $M_2(L^{\infty}(\Omega,\Sigma',P))$.

We now wish to construct an increasing filtration of Von Neumann algebras. To this end, let $(\Omega,\Sigma,P,\Sigma_{\alpha},R^{+})$ be a stochastic base and assume that the family of σ-fields $\{\Sigma_{\alpha} : \alpha \in R^{+}\}$ is right continuous:

$$\bigcap_{\beta>\alpha} \Sigma_{\beta} = \Sigma_{\alpha} .$$

Then $\{L^{\infty}(\Omega,\Sigma_{\alpha},P) : \alpha \in R^{+}\}$ gives an increasing family of Von Neumann algebras and $\{M_2(L^{\infty}(\Omega,\Sigma_{\alpha},P)) : \alpha \in R^{+}\}$ gives an increasing family of non-commutative Von Neumann algebras, i.e.
$M_2(L^{\infty}(\Omega,\Sigma_{\beta},P)) \subseteq M_2(L^{\infty}(\Omega,\Sigma_{\alpha},P))$ for $\beta \leq \alpha$. Writing $M_2(L^{\infty}_{\alpha})$ for $M_2(L^{\infty}(\Omega,\Sigma_{\alpha},P))$ let M_{α} be the conditional expectation from $M_2(L^{\infty})$ onto $M_2(L^{\infty}_{\alpha})$. That is

$$M_{\alpha}(N(i,j)) = (E_{\alpha}N(i,j)) \qquad i = 1,2; \quad j = 1,2$$

where E_{α} is the conditional expectation from $L^{\infty}(\Omega,\Sigma,P)$ onto

$L^{\infty}(\Omega,\Sigma_\alpha,P)$.

We are now in a position to define stochastic processes and do some constructions with them.

4.3 Stochastic Processes

4.31 Definition

A stochastic process X is a family $\{X_\alpha : \alpha \in R^+\}$ such that for each $\alpha \in R^+$, $X_\alpha \in M_2(\mathcal{M}(L^\infty_\alpha))$. That is the entries of the matrix given by X_α are measurable functions with respect to (Ω,Σ_α,P) .

We call a stochastic process X a L^p-process if for each $\alpha \in R^+$, $X_\alpha \in M_2(L^p_\alpha)$. As usual X is called a L^p-martingale if X is a L^p-process and $M_\alpha(X_\beta) = X_\alpha$ for all $\alpha \leq \beta$, X is called L^p-bounded martingale if X is a L^p-martingale and

$$\sup_{\alpha \in R^+} \|X_\alpha\|_p < \infty \qquad\qquad 1 \leq p \leq \infty \quad .$$

Note that as usual we shall denote a family X by (X_α) instead of $\{X_\alpha : \alpha \in R^+\}$.

4.32 Proposition

Let $X = (X_\alpha)$ be a L^p-process. Then X is a martingale if and only if the entries of X form martingales.

Proof.

This follows directly from the definition of the conditional expectation.

4.33 Definition

An L^p-process $A = (A_\alpha)$ is called:

(i) Positive if $A_\alpha \geq 0$ for all $\alpha \in R^+$

(ii) Increasing if $A_\alpha \geq A_\beta$ for all $\alpha \geq \beta$.

The inequality here is in the sense of operators.

4.34 Definition

A L^p-process $A = (A_\alpha)$ is called natural [16], if for any $\beta > \alpha \geq 0$ and any sequence (θ_n) of partitions of $[\alpha,\beta]$ such that $\text{mesh}(\theta_n) \to 0$ as $n \to \infty$, we have

$$\phi(\sum_j M_{\alpha_j^n}(Y)(A_{\alpha_{j+1}^n} - A_{\alpha_j^n})) \to \phi(Y(A_\beta - A_\alpha))$$

for any $Y \in M_2(L^\infty)$.

The main theorem we wish to prove in this section is the Doob-Meyer decomposition of L^2-bounded martingales [28,29].

4.35 Theorem

Let $X = (X_\alpha)$ be a L^2-bounded martingale. Then

$$|X_\alpha|^2 = U_\alpha + A_\alpha$$

where $U = (U_\alpha)$ is a L^1-martingale and $A = (A_t)$ is a positive, increasing natural L^1-process.

Before we prove this theorem, we shall need some results about commutative (complex) processes. Thus in the next four propositions we shall assume that $x = (x_\alpha)$, $y = (y_\alpha)$ and $z = (z_\alpha)$ are L^2-bounded martingales in $L^2(\Omega,\Sigma,P)$.

4.36 Proposition

Let $x \subseteq L^2(\Omega,\Sigma,P)$ be a L^2-bounded martingale. Then

$$|x_\alpha|^2 = u_\alpha + a_\alpha$$

where $u = (u_\alpha)$ is a L^1-martingale and $a = (a_\alpha)$ is a positive, natural increasing L^1-process.

Proof.

$$x_\alpha = \frac{x_\alpha + \bar{x}_\alpha}{2} + i\,\frac{(x_\alpha - \bar{x}_\alpha)}{2i}$$

$$= R_\alpha + iT_\alpha \qquad \text{say} \quad .$$

Then (R_α) and (T_α) are clearly R-valued L^2-bounded martingales hence have a Doob-Meyer decomposition:

$$|R_\alpha|^2 = m_\alpha + b_\alpha$$

$$|T_\alpha|^2 = n_\alpha + c_\alpha$$

where (m_α) and (n_α) are L^1-martingales and (b_α) and (c_α) are positive natural increasing L^1-process. Thus

$$|x_\alpha|^2 = (m_\alpha + n_\alpha) + (b_\alpha + c_\alpha)$$

$$= u_\alpha + a_\alpha$$

since the sum of two natural processes is natural. We shall often write $\langle x\rangle_t$ to denote the increasing part a_t .

4.37 Proposition

Let $x = (x_\alpha)$ and $y = (y_\alpha)$ be L^2-bounded martingales in $L^2(\Omega,\Sigma,P)$. Then there exists a L^1-process $\langle x,y\rangle$ such that

$$x_\alpha^* y_\alpha - \langle x,y\rangle_\alpha$$

defines a L^1-martingale.

Proof.

Note that:

$$x_\alpha^* y_\alpha = |x_\alpha + y_\alpha|^2 + i|x_\alpha + iy_\alpha|^2 - |x_\alpha - y_\alpha|^2 - i|x_\alpha - iy_\alpha|^2 \quad .$$

Hence from 4.36 we have:

$$x_\alpha^* y_\alpha = \langle x,y\rangle_\alpha + u_\alpha$$

where $\langle x,y\rangle_\alpha = \langle x+y\rangle_\alpha + i\langle x+iy\rangle_\alpha - \langle x-y\rangle_\alpha - i\langle x-iy\rangle_\alpha$ and (u_α) is a L^1-martingale.

4.38 Proposition

We have that:

(i) $<x,y+z>_\alpha = <x,y>_\alpha + <x,z>_\alpha$

(ii) $<x+y,z>_\alpha = <x,z>_\alpha + <y,z>_\alpha$

(iii) $<\lambda x,y>_\alpha = \overline{\lambda}<x,y>_\alpha$

(iv) $<x,\lambda y>_\alpha = \lambda<x,y>_\alpha$.

For all L^2-bounded martingales $x = (x_\alpha)$, $y = (y_\alpha)$, $z = (z_\alpha)$ and $\lambda \in C$.

4.39 Proposition

We have that

$$|<x,y>_\alpha|^2 \leq <x>_\alpha<y>_\alpha$$

for any L^2-bounded martingales $x = (x_t)$ and $y = (y_\alpha)$.

Proof.

Let $\lambda \in C$, $\alpha \in R^+$, then $<x+\lambda y>_\alpha \geq 0$. P - a.s.

Hence,

$$<x+\lambda y \, , \, x+\lambda y>_\alpha \geq 0 \quad .$$

That is,

$$\langle x\rangle_\alpha + \bar{\lambda}\langle y,x\rangle_\alpha + \lambda\langle x,y\rangle_\alpha + |\alpha|^2\langle y\rangle_\alpha \geq 0 \quad .$$

If either $\langle x\rangle_\alpha = 0$ or $\langle y\rangle_\alpha = 0$ then there is nothing to prove. Suppose $\langle y\rangle_\alpha \neq 0$ and set

$$\lambda = -\frac{\langle y,x\rangle_\alpha}{\langle y\rangle_\alpha}$$

and the result follows.

Proof of 4.35

Let $X_\alpha = \begin{pmatrix} x_\alpha & y_\alpha \\ z_\alpha & w_\alpha \end{pmatrix}$ where $x = (x_\alpha),\ldots,w = (w_\alpha)$ are L^2-bounded martingales. That is they lie in $L^2(\Omega,\Sigma,P)$. Then

$$|X_\alpha|^2 = \begin{pmatrix} |x_\alpha|^2 + |z_\alpha|^2 & \bar{x}_\alpha y_\alpha + \bar{z}_\alpha w_\alpha \\ \bar{y}_\alpha x_\alpha + \bar{w}_\alpha z_\alpha & |y_\alpha|^2 + |w_\alpha|^2 \end{pmatrix} \quad .$$

Now from 4.36 and 4.37 we get $|X_\alpha|^2 = U_\alpha + A_\alpha$ where $U = (U_\alpha)$ is a L^1-martingale and A_α is given by

$$A_\alpha = \begin{pmatrix} \langle x\rangle_\alpha + \langle z\rangle_\alpha & \langle x,y\rangle_\alpha + \langle z,w\rangle_\alpha \\ \langle y,x\rangle_\alpha + \langle w,z\rangle_\alpha & \langle y\rangle_\alpha + \langle w\rangle_\alpha \end{pmatrix} \quad .$$

We now need to show that

(i) $A_\alpha \geq 0$ for all $\alpha \in R$

(ii) $A_\alpha \geq A_\beta$ for all $\alpha \geq \beta$

(iii) (A_α) is natural.

To show (i), we consider A_α as a densely defined operator on $L^2 \oplus L^2$. Let $\begin{pmatrix} f \\ g \end{pmatrix}$ belong to the domain of A_α. Then

$$<A_\alpha \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}>$$

$$= \int_\Omega \{<x>_\alpha f + <z>_\alpha f + <x,y>_\alpha g + <z,w>_\alpha g\}\overline{f}dP$$

$$+ \int_\Omega \{<y,x>_\alpha f + <w,z>_\alpha f + <y>_\alpha g + <w>_\alpha g\}\overline{g}dP$$

$$= \int_\Omega \{<x>_\alpha |f|^2 + <y>_\alpha |g|^2 + 2\mathrm{Re}<x,y>_\alpha g\overline{f}\}dP$$

$$+ \int_\Omega \{<z>_\alpha |f|^2 + <w>_\alpha |g|^2 + 2\mathrm{Re}<w,z>f\overline{g}\}dP$$

$$\geq \int_\Omega \{<x>_\alpha |f|^2 + <y>_\alpha |g|^2 - 2<x>_\alpha^{\frac{1}{2}}<y>_\alpha^{\frac{1}{2}}|g||f|\}dP$$

$$+ \int_\Omega \{<z>_\alpha |f|^2 + <w>_\alpha |g|^2 - 2<w>_\alpha^{\frac{1}{2}}<z>_\alpha^{\frac{1}{2}}|g||f|\}dP \qquad \text{by 4.39}$$

$$\geq \int_\Omega \{\langle x\rangle_\alpha^{\frac{1}{2}}|f| - \langle y\rangle_\alpha^{\frac{1}{2}}|g|\}^2 dP + \int_\Omega \{\langle z\rangle_\alpha^{\frac{1}{2}}|f| - \langle w\rangle_\alpha^{\frac{1}{2}}|g|\}^2 dP$$

$$\geq 0 \quad .$$

Hence for each $\alpha \in R^+$, $A_\alpha \geq 0$.

To show (ii), we make use of the fact that if $x = (x_\alpha)$ and $y = (y_\alpha)$ are L^2-bounded martingales in $L^2(\Omega,\Sigma,P)$ then

$$|\langle x,y\rangle_\alpha - \langle x,y\rangle_\beta| \leq (\langle x\rangle_\alpha - \langle x\rangle_\beta)(\langle y\rangle_\alpha - \langle y\rangle_\beta)$$

for all $\alpha \geq \beta$. This follows directly from 4.39, and the proof of (ii) follows along the same lines as that of (i).

To show (iii), that is (A_α) is natural, let $\alpha > 0$ and (θ_n) be a sequence of partitions of $[0,\alpha]$ such that $\text{mesh}(\theta_n) \to 0$ as $n \to \infty$ and $G \in M_2(L^\infty)$. Then

$$\phi(\sum_j M_{\alpha_j^n}(G)\,(A_{\alpha_{j+1}^n} - A_{\alpha_j^n}))$$

$$= \tfrac{1}{2}E\{\sum_j (E_{\alpha_j^n}(g(1,1))(\Delta\langle x\rangle_{\alpha_{j+1}^n} + \Delta\langle z\rangle_{\alpha_{j+1}^n}) +$$

$$E_{\alpha_j^n}(g(1,2))(\Delta\langle y,x\rangle_{\alpha_{j+1}^n} + \Delta\langle w,z\rangle_{\alpha_{j+1}^n})\}$$

$$+ \tfrac{1}{2}E\{\sum_j (E_{\alpha_j^n}(g(2,1))(\Delta\langle x,y\rangle_{\alpha_{j+1}^n} + \Delta\langle z,w\rangle_{\alpha_{j+1}^n}) +$$

$$E_{\alpha_j^n}(g(2,2))(\Delta\langle y\rangle_{\alpha_{j+1}^n} + \Delta\langle w\rangle_{\alpha_{j+1}^n})\} \quad .$$

Taking the limit as $n \to \infty$ and using 4.36 and 4.37 we get

$$\tfrac{1}{2}E\{E_\alpha(g(1,1))(\langle x\rangle_\alpha + \langle z\rangle_\alpha) + E_\alpha(g(1,2))(\langle y,x\rangle_\alpha + \langle w,z\rangle_\alpha)\}$$

$$+ \tfrac{1}{2}E\{E_\alpha(g(2,1))(\langle x,y\rangle_\alpha + \langle z,w\rangle_\alpha) + E_\alpha(g(2,2))(\langle y\rangle_\alpha + \langle w\rangle_\alpha)\}$$

$$= \phi(M_\alpha(G)A_\alpha)$$

$$= \phi(GA_\alpha) \quad .$$

Hence (A_t) is natural and the theorem is proved.

4.310 Corollary

The Doob-Meyer decomposition given in theorem 4.35 is unique.

Proof.

Suppose $|X_\alpha|^2 = U_\alpha + A_\alpha = V_\alpha + B_\alpha$ where (U_α) and (V_α) are L^1-martingales and (A_t) and (B_t) are positive natural increasing processes. Hence $A_\alpha - B_\alpha$ defines a martingale which is also a natural process. Thus for any $\alpha > 0$ we have that $\phi(G(A_\alpha - B_\alpha)) = 0$ for all $G \in M_2(L^\infty)$.

Now since $M_2(L^\infty)$ is the dual of $M_2(L^1)$, it follows that $A_\alpha - B_\alpha = 0$ for each $\alpha \in R^+$. Hence $A_\alpha = B_\alpha$ and the result is now clear.

Again writing $\langle X\rangle_\alpha$ instead of A_α we have:

4.311 Corollary

Let (X_α) and (Y_α) be L^2-martingales. Then there exists an L^1-martingale (U_α) and an L^1-process $(\langle X,Y\rangle_\alpha)$ such that $X^*_\alpha Y_\alpha = U_\alpha + \langle X,Y\rangle_\alpha$.

4.4 Stochastic Integrals

In this section we shall define stochastic integrals of the form $\int_0^t F_s dX_s$, where $X = (X_s)$ is a L^2-bounded martingale, whose matrix entries have right continuous paths and $F = (F_s)$ is a process with some desirable properties so that the family of stochastic integrals:

$$\{ \int_0^t F_s dX_s : t \in R^+ \}$$

is a L^2-bounded martingale.

4.41 Commutative Stochastic Integrals

We first review the construction of the commutative stochastic integral.

Let $(\Omega, \Sigma, P, \Sigma_\alpha, R^+)$ be a stochastic base with (Σ_α) being right continuous. Let $\mathcal{R}$ denote the collection of all sets of the form $\{0\} \times F_0$ and $(s,t] \times F$ where $F \in \Sigma_s$ for $s < t$ and $F_0 \in \Sigma_0$. The σ-field $\mathcal{P}$ generated by $\mathcal{R}$ is called the predictable σ-field. A function $f : R^+ \times \Omega \to C$ is called predictable if both $Re(f)$ and $Im(f)$ are $\mathcal{P}$-measurable.

Now let $A \in \mathcal{R}$, then $\chi_A(t,\cdot)$ is Σ_t measurable for any $t \in R^+$ and hence $\chi_{A'}(t,\cdot)$ is Σ_t measurable where $A' = \Omega - A$. Thus it follows that $\chi_A(t,\cdot)$ is Σ_t measurable for any A in the field generated by $\mathcal{R}$, and by a monotone class argument $\chi_A(t,\cdot)$ is Σ_t measurable for any A in $\mathcal{P}$. Thus for any A in $\mathcal{P}$, $(\chi_A(t,\cdot))$ is a process. Now, any $\mathcal{P}$-measurable function f, is the limit of linear combinations of elementary functions of sets in $\mathcal{P}$, it follows that $f(t,\cdot)$ is Σ_t measurable and hence $(f(t,\cdot))$ is a process. Such a process is called a predictable process.

Let $x = (x_t)$ be a L^2-bounded right continuous martingale and f be simple predictable function (or process!). Suppose $f = \sum_j \lambda_j \chi_{(s_{m-1}, s_j) \times A_{j-1}}$ then the stochastic integral $\int f dx$ is defined as

$$\sum_j \lambda_j \chi_{A_{j-1}} (X_{s_j} - X_{s_{j-1}}) \quad .$$

Then

$$\| \int f dx \|_2^2 = E \int |f|^2 d\langle x \rangle \quad . \tag{4.41a}$$

It is known that if f is a predictable function such that

$$E \int_{R^+} |f(s,\cdot)|^2 d\langle x \rangle_s < \infty \tag{4.41b}$$

where the integral is understood in the sense of Lebesgue-Stieltjes, then there exists a sequence of simple predictable processes $(f^n(t,\cdot))$ such that

$$E \int_{R^+} |f^n(t,\cdot) - f(t,\cdot)|^2 d\langle x \rangle_s \to 0$$

and we set

$$\int f dx = L^2 - \mathrm{Lim} \int f^n dx \quad .$$

For f as above the stochastic integral:

$$\int_0^t f_s dx_s$$

is the same as

$$\int f\chi_{[0,t]\times\Omega}dx \quad .$$

Now if $L^2(x)$ denotes the Hilbert space of all predictable processes f such that

$$E\int_{R^+}|f_s|^2 d\langle x\rangle_s < \infty$$

then the stochastic integral $\int fdx$ is defined as the image of f under the isometry given by equation 4.41a, from $L^2(x)$ into $L^2(\Omega,\Sigma,P)$.

Another equivalent way of looking at the construction of stochastic integral is to employ the method of Bartle [17]. Thus let μ_x be the vector valued measure on $\mathcal{R}$:

$$\mu_x((s,t]\times A) = (x_t - x_s)\chi_A \quad .$$

We extend μ_x to be a finitely additive measure on the ring generated by $\mathcal{R}$ by defining

$$\mu_x(A) = \sum_{i=1}^{n} \mu_x(A_i)$$

where $A = \bigcup_1^n A_i$ and A_i are disjoint sets in $\mathcal{R}$. Then it is known that μ_x extends to a measure on $\mathcal{P}$ [32], with semi-variation given by the function

$$\|\cdot\|_{sv(\mu_x)} : \mathcal{P} \to R^+$$

where

$$\|E\|_{sv(\mu_x)} = \text{Sup } \|\sum_i \lambda_i \mu_x(E_i)\|_2$$

where the supremum is taken over all finite partitions of E in P and $(\lambda_i) \subseteq C$ with $|\lambda_i| \leq 1$. Then the semi-variation $\|\cdot\|_{sv(\mu_x)}$ is finite. For, consider $R^+ \times \Omega$ and let (E_i) be a partition of $R^+ \times \Omega$ over R then

$$\|\Sigma \lambda_i \mu_x(E_i)\|_2$$

$$= \{ \sum_i |\lambda_i|^2 E(\chi_{A_i}(\langle x\rangle_{t_i} - \langle x\rangle_{t_{i-1}})\}^{\frac{1}{2}}$$

$$\leq \{ \sum_i E(\chi_{A_i}(\langle x\rangle_{t_i} - \langle x\rangle_{t_{i-1}}))\}^{\frac{1}{2}}$$

$$= \|\mu_x(R^+ \times \Omega)\|_2 < \infty \qquad .$$

Since R generates P, the result follows for any finite partition in P. Then the stochastic integral is just the Bartle integral [17]:

$$\int f dx_s = \int f d\mu_x$$

of a predictable function f with respect to μ_x. (The class of predictable functions for which the Bartle integral exists is given in [30]. It is precisely the predictable functions for which 4.41b holds.) The stochastic integral $\int_0^t f_s dx_s$ is just the Bartle integral $\int f \chi_{[0,t] \times \Omega} d\mu_x$. Now if f is any predictable function which is bounded then it is a pointwise limit of simple predictable functions and hence it is μ_x measurable for any L^2-bounded martingale $x = (x_t)$. Since f is bounded it is clear that

$$E\int|f|^2 d\langle x\rangle < \infty$$

hence the Bartle integral $\int f d\mu_x$ exists.

We now proceed to extend the definition of the stochastic integral to a non-commutative setting.

Let $X = (X_t)$ be a L^2-bounded centred martingale, i.e. $X_t \in M_2(L^2)$, $\phi(X_t) = 0$ and $\sup_t \|X_t\|_2 < \infty$, and let F be $M_2(\mathbb{C})$-valued function on $\mathbb{R}^+ \times \Omega$ such that each $F(i,j) : \mathbb{R}^+ \times \Omega \to \mathbb{C}$ are predictable functions $i = 1,2$; $j = 1,2$. Then F defines a process whose matrix entries are predictable processes. Suppose $F = (F_t)$ is a simple process:

$$F_t = \sum_i F_{t_{i-1}} \chi_{[t_{i-1},t_i)}(t) \quad .$$

Then the entries of F are simple predictable processes, we define the stochastic integral:

$$\int_0^\infty F_s dX_s \equiv \sum F_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})$$

$$= \left(\int F_s(i,k)dX_s(k,j)\right)_{i=1,2;j=1,2} \quad .$$

We have then the following.

4.42 Proposition

(i) $\int(\lambda F + \mu G)_s dX_s = \lambda\int F_s dX_s + \mu\int G_s dX_s$

(ii) $\phi(\int F_s dX_s) = 0$

(iii) $\|\int F_s dX_s\|_2^2 = \phi(|F_s|^2 d\langle x\rangle_s)$.

For all simple processes F and G whose entries are predictable functions and $\lambda,\mu \in \mathbb{C}$.

Proof.

(i) Suppose $F(t) = F\chi_{[t_1,t_2)}(t)$ and $G(t) = G\chi_{[s_1,s_2)}(t)$ with $x_{.} < t_1 < s_2 < t_2$. Then

$$(F + G)_t = (F + G)\chi_{[t_1,s_2)} + G\chi_{[s_1,t_1)} + F\chi_{[s_2,t_2)}$$

and

$$\int(F + G)_s dX_s = (F + G)(X_{s_2} - X_{t_1}) + G(X_{t_1} - X_{s_1}) + F(X_{t_2} - X_{s_2})$$

$$= F(X_{t_2} - X_{t_1}) + G(X_{s_2} - X_{s_1})$$

$$= \int F dX_s + \int G dX_s \quad .$$

The result for a general simple process follows likewise.

(ii) $\int F_s dX_s = \sum F_{t_{i-1}} \Delta X_{t_i}$

$$\Rightarrow \phi(\int F_s dX_s) = \phi(\sum F_{t_{i-1}} \Delta X_{t_i})$$

$$= \sum_i \phi(M_{t_{i-1}} F_{t_{i-1}} \Delta X_{t_i})$$

$$= \sum_i \phi(F_{t_{i-1}} M_{t_{i-1}} \Delta X_{t_i})$$

$$= 0 \quad .$$

(iii) $\|\int F_s dX_s\|_2^2 = \phi(\sum_{i,j} F^*_{t_{i-1}} F_{t_{j-1}} \Delta X_{t_j} \Delta X^*_{t_i}) \quad .$

For $i \neq j$, suppose $i < j$ then

$$\phi(F^*_{t_{i-1}} F_{t_{j-1}} \Delta X_{t_j} \Delta X^*_{t_i}) = \phi(F^*_{t_{i-1}} F_{t_{j-1}} M_{t_{j-1}} (\Delta X_{t_j}) \Delta X^*_{t_i})$$

$$= 0 \quad .$$

Hence the contribution of the non-diagonal terms is zero and

$$\|\int F_s dX_s\|_2^2 = \phi(\sum_i |F_{t_{i-1}}|^2 \Delta X_{t_i} \Delta X^*_{t_i})$$

$$= \phi(\sum_i |F_{t_{i-1}}|^2 (|X^*_{t_i}|^2 - |X^*_{t_{i-1}}|^2))$$

$$= \phi(\sum_i |F_{t_{i-1}}|^2 (\langle X^* \rangle_{t_i} - \langle X^* \rangle_{t_{i-1}}))$$

$$= \phi(\int |F_s|^2 d\langle X^* \rangle_s) \quad .$$

We have used the Doob-Meyer decomposition given in 4.35.

We now extend the stochastic integral to include more general integrands. Let $F = (F_t)$ be a process such that its entries form predictable processes. Suppose furthermore that each $F_t(i,j)$ is bounded, i.e. $\sup_{(t,w)} |F_t(i,j)(w)| < \infty$. Then we know that given any L^2-bounded martingale $(X_t(j,k)) \subseteq L^2(\Omega,\Sigma,P)$ the stochastic integral

$$\int F_t(i,j)dX_t(j,k)$$

exists and is the L^2-limit of

$$\int F_t^n(i,j)dX_t(j,k)$$

where $F_t^n(i,j)$ are bounded predictable simple functions.

Now let $X = (X_t)$ be a L^2-bounded martingale then the entries of X , $X(i,j) = (X_t(i,j))$ form L^2-bounded martingales in $L^2(\Omega,\Sigma,P)$.

Now set $F^n = (F^n(i,j))$, where $F^n(i,j)$ are the simple predictable functions converging to $F(i,j)$ in μ_x-measure for any L^2-bounded martingale $x \subseteq L^2(\Omega,\Sigma,P)$. Then

$$\int F^n dX = (\int F^n(i,k)dX(k,j))_{i,j}$$

and

$$\|\int F^n dX - \int F^m dX\|_2^2$$

$$= \|\int (F^n-F^m)(1,1)dX(1,1) + \int (F^n-F^m)(1,2)dX(2,1)\|_2^2$$

$$+ \|\int (F^n-F^m)(1,1)dX(1,2) + \int (F^n-F^m)(1,2)dX(2,2)\|_2^2$$

$$+ \|\int (F^n-F^m)(2,1)dX(1,1) + \int (F^n-F^m)(2,2)dX(2,1)\|_2^2$$

$$+ \|\int (F^n-F^m)(2,1)dX(1,2) + \int (F^n-F^m)(2,2)dX(2,2)\|_2^2 \quad . \qquad (4.42a)$$

It is now clear that $(\int F^n dX)_{n \in N}$ is a Cauchy sequence in L^2 since the right hand side of equation 4.42a converges to zero as $m,n \to \infty$.

We have

$$\int FdX = L^2 - \text{Lim} \int F^n dX \quad . \qquad (4.42b)$$

It is now clear that

$$\int FdX = (\int F(i,k)dX(k,j))_{i=1,2;j=1,2} \quad .$$

4.43 Definition

For $t > 0$ we define $\int_0^t F_s dX_s$ by $\int F(\chi_{[0,t]\times\Omega} \otimes I)dX$.

4.44 Theorem

The family of stochastic integrals $\{\int_0^t F_s dX_s \; ; \; t \in R^+\}$ is a L^2-bounded centred martingale.

Proof.

First suppose F is a simple process: $F_t = \Sigma F_{t_{i-1}} \chi_{[t_{i-1},t_i)}(t)$ and $t \in [t_{k-1},t_k)$. Then

$$\int_0^t F_s dX_s = \sum_{i=1}^{k-1} F_{t_{i-1}} \Delta X_{t_i} + F_{t_{k-1}}(X_t - X_{t_{k-1}}) \quad .$$

Taking expectations on both sides shows that $\phi(\int_0^t F_s dX_s) = 0$ from 4.42(ii). Now let $r \leq t$ say $r \in [t_{j-1},t_j)$. Then

$$M_r\Big(\sum_{i=1}^{k-1} F_{t_{i-1}} \Delta X_{t_i} + F_{t_{k-1}}(X_t - X_{t_{k-1}})\Big)$$

$$= \int_0^r F_s dX_s + \sum_{k \geq i > j} M_r F_{t_{i-1}}(X_{t \wedge t_i} - X_{t \wedge t_{i-1}}) \quad .$$

We wish to show $M_r F_{t_{i-1}}(X_{t \wedge t_i} - X_{t \wedge t_{i-1}}) = 0$.

Let $G \in M_2(L^2_r)$ then

$$\phi(GM_r(F_{t_{i-1}}(X_{t \wedge t_i} - X_{t \wedge t_{i-1}})))$$

$$= \phi(GF_{t_{i-1}}(X_{t \wedge t_i} - X_{t \wedge t_{i-1}}))$$

$$= \phi(GF_{t_{i-1}} M_{t \wedge t_{i-1}}(X_{t \wedge t_i} - X_{t \wedge t_{i-1}}))$$

$$= 0 \quad .$$

Hence $(\int_0^t F_s dX_s)$ is a martingale for a simple process (F_t) .

Now for $X \in M_2(L^2)$ we have

$$\|M_s(X)\|_2 \leq \|X\|_2 \qquad \text{for all} \quad s \in R^+ \quad .$$

Hence

$$\|\int_0^r F_s dX_s - M_r \int_0^t F_s dX_s\|_2$$

$$\leq \|\int_0^r F_s^n dX_s - \int_0^r F_s dX_s\|_2 + \|\int_0^r F_s^n dX_s - M_r \int_0^t F_s^n dX_s\|_2$$

$$+ \|M_r \int_0^t F_s^n dX_s - M_r \int_0^t F_s dX\|_2$$

where $(F^n)_n$ is a sequence of simple processes satisfying equation 4.42b.

It is now clear that each term on the right converges to zero as $n \to \infty$, and the martingale property is established. Now consider

$$|\phi(\int_0^t F_s dX_s)|$$

$$\leq |\phi(\int_0^t (F_s - F_s^n) dX_s)| + |\phi(\int_0^t F_s^n dX_s)|$$

$$\leq \|\int_0^t (F_s - F_s^n) dX_s\|_2 + 0 \qquad \text{by } 4.42(ii)$$

$$\to 0$$

as $n \to \infty$. Hence the martingale $\{\int_0^t F_s dX_s : t \in R^+\}$ is centred.

The L^2-bounded ness of $\{\int_0^t F_s dX_s : t \in R^+\}$ follows from the fact that

$$M_t \int_0^\infty F_s dX_s = \int_0^t F_s dX_s$$

and

$$\int_0^\infty F_s dX_s \in M_2(L^2) \qquad . \qquad [5]$$

We now give a characterisation of the stochastic integral. To this end, we define a vector valued measure on P by:

$$\mu_{<X,Y>}([s,t) \times A) = (<X,Y>_t - <X,Y>_s)(\chi_A \otimes I)$$

where X and Y are L^2 bounded martingales, whose entries have right continuous paths. The semivariation of $\mu_{<x,y>}$ is given by, for $E \in P$,

$$\|E\|_{sv(\mu_{<X,Y>})} = \mathrm{Sup}\|\Sigma Z_i \mu_{<X,Y>}(E_i)\|_1$$

where the supremum is taken over all finite partitions $\{E_i\}$ of E and all finite collections of $\{Z_i\}_i \subseteq M_2(C)$.

From proposition 4.37 we have

$$\|E\|_{sv(\mu_{<X,Y>})} \leq \sum_{k=0}^{3} \|E\|_{sv(\mu_{<X+i^k Y>})} < \infty$$

where $\mu_{<X+iY>}$ is the measure on P given by:

$$\mu_{<X+iY>}([s,t] \times A) = (<X+iY>_t - <X+iY>_s) \cdot (\chi_A \otimes I) \quad .$$

We let $\int F d\mu_{<X,Y>}$ denote the Bartle integral of a process F whose entries are bounded predictable processes $F(i,j)$ $i = 1,2$; $j = 1,2$. It is clear that this Bartle integral will exist since there is a sequence of simple processes F^n , converging pointwise to F and hence F is $\mu_{<X,Y>}$ measurable, being bounded. F is integrable too; and we have

$$L^1 - \text{Lim} \int F^n d\mu_{<X,Y>} = \int F d\mu_{<X,Y>} \quad ,$$

where the integral on the left is defined as

$$\sum F^n_{t_{i-1}} (<X,Y>_{t_i} - <X,Y>_{t_{i-1}})$$

where

$$F^n_t = \sum F^n_{t_{i-1}} \chi_{[t_{i-1},t_i)}(t) \quad .$$

Likewise

$$\int_0^t F_s d\mu_{<X,Y>_s} = \int_0^t F(\chi_{[0,t] \times \Omega} \otimes I) d\mu_{<X,Y>}$$

and we denote the integral on the left by

$$\int_0^t F_s d<X,Y>_s \quad .$$

4.45 Theorem

The stochastic integral $\int_0^t F_s dX_s^*$ is the unique L^2-bounded martingale such that

$$\phi(<(\int_0^{\cdot} F_s dX_s^*)^*, Y>_t) = \phi(\int_0^t F_s d<X,Y>_s) \quad .$$

For any L^2-bounded martingale (Y_t) .

Proof.

Suppose F is a simple process:

$$F_t = \sum F_{t_{i-1}} \chi_{[t_{i-1},t_i)}(t) \quad .$$

Then

$$\int_0^t F_s d<X,Y>_s = \sum_{i=1}^{k-1} F_{t_{i-1}}(<X,Y>_{t_i} - <X,Y>_{t_{i-1}})$$

$$+ F_{t_{k-1}}(<X,Y>_t - <X,Y>_{t_{k-1}})$$

where we assume $t \in [t_{k-1},t_k)$. Now,

$$\phi(\int_0^t F_s d<X,Y>_s) = \sum_{i=1}^{k-1} \phi(F_{t_{i-1}}(X_{t_i}Y_{t_i} - X_{t_{i-1}}Y_{t_{i-1}}))$$

$$+ \phi(F_{t_{k-1}}(X_tY_t - X_{t_{k-1}}Y_{t_{k-1}}))$$

where we have used the fact that $X_t^* Y_t = U_t + \langle X,Y\rangle_t$ as in proposition 4.37, where U_t is a L^1-martingale. Now

$$\phi(F_{t_{i-1}}(X_{t_i}^* Y_{t_i} - X_{t_{i-1}}^* Y_{t_{i-1}}))$$

$$= \phi(M_{t_i}(F_{t_{i-1}} X_{t_i}^* Y_t)) - \phi(M_{t_{i-1}}(F_{t_{i-1}} X_{t_{i-1}}^* Y_t))$$

$$= \phi(F_{t_{i-1}} \Delta X_{t_i}^* Y_t) \qquad .$$

Hence

$$\phi(\int_0^t F_s d\langle X,Y\rangle_s) = \phi(\int_0^t F_s dX_s^* \cdot Y_t)$$

$$= \phi(\langle(\int_0^{\bullet} F_s dX_s^*)^* , Y\rangle_t) \qquad .$$

Since $\phi(\langle A,B\rangle_t) = \phi(A_t^* B_t)$ for L^2-bounded martingales (A_t) , (B_t) .

Now suppose (F_t) is a process whose entries are bounded predictable processes. Hence there is a sequence (F^n) of bounded predictable simple processes such that

$$\int_0^t F_s d\langle X,Y\rangle_s = L^1 - \mathrm{Lim} \int_0^t F_s^n d\langle X,Y\rangle \qquad .$$

Now,

$$\| \int_0^t (F_s - F_s^n) dX_s^* \cdot Y_t \|_2 \leq \| \int_0^t (F_s - F_s^n) dX_s \|_2 \|Y_t\|_2 \rightarrow 0$$

as $n \to \infty$ by 4.42b. Hence

$$\int_0^t F_s^n dX_s^* \cdot Y_t \overset{L^1}{\to} \int_0^t F_s dX_s^* Y_t \qquad ,$$

hence

$$\phi(\int_0^t F_s^n dX_s^* \cdot Y_t) \to \phi(\int_0^t F_s dX_s^* \cdot Y_t) \qquad .$$

But the left hand side here is:

$$\phi(\int F_s^n d<X,Y>_s)$$

which converges to $\phi(\int_0^t F_s d<X,Y>_s)$ and we have

$$\phi(\int_0^t F_s d<X,Y>_s) = \phi(<(\int_0^{\cdot} F_s dX_s^*)^* \cdot Y>_t) \qquad .$$

To show the uniqueness property, suppose there is another L^2-bounded martingale $Z = (Z_t)$ such that

$$\phi(<(\int_0^{\cdot} F_s dX_s^*)^* , Y>_t) = \phi(<Z^*,Y>_t)$$

for all L^2-bounded martingales (Y_t). Then, $\phi(\int_0^t F_s dX_s^* \cdot Y_t) =$ $= \phi(Z_t Y_t)$. That is

$$\phi((\int_0^t F_s dX_s^* - Z_t)Y_t) = 0 \qquad .$$

Hence

$$\phi\left(\left(\int_0^t F_s dX_s^* - Z_t\right)Y\right) = 0 \qquad \text{for all} \quad Y \in M_2(L_t^2) \quad .$$

Hence

$$\int_0^t F_s dX_s^* = Z_t \qquad \text{for each} \quad t \quad .$$

4.46 Theorem

Let (F_t) and (G_t) be processes whose entries are bounded predictable processes. Then

$$\int_0^t G_s dY_s = \int_0^t G_s F_s dX_s$$

where $Y_t = \int_0^t F_s dX_s$.

Proof.

If $G_t = G\chi_{[\alpha,\beta]}(t)$ say, then

$$\int_0^t G_s dY_s = G(Y_\beta - Y_\alpha)$$

$$= G \int_\alpha^\beta F_s dX_s \quad .$$

Now $G \in M_2(L^\infty)$ (since its entries are bounded functions) and multiplication by such operator is a continuous map on $M_2(L^2)$ hence

$$\int_0^t G_s dY_s = \int_0^t G_s F_s dX_s \quad .$$

By linearity the result follows for a simple process (G_t). Now for (G_t) as in the statement of the theorem there exists a sequence (G^n) whose entries are bounded predictable simple functions such that

$$\int_0^t G_s dY_s = L^2 - \text{Lim} \int G_s^n dY_s \quad .$$

Now

$$\left\| \int_0^t G_s dY_s - \int_0^t G_s F_s dX_s \right\|_2$$

$$\leq \left\| \int_0^t G_s dY_s - \int_0^t G_s^n dY_s \right\|_2 + \left\| \int_0^t G_s^n dY_s - \int_0^t G_s F_s dX_s \right\|_2 \quad .$$

The first term tends to zero as $n \to \infty$, and the second term is

$$\tfrac{1}{2}E\{\int_0^t \overline{F}(k,1)(\overline{G}^n(i,k)-\overline{G}(i,k))(G^n(i,\varepsilon)-G(i,\varepsilon))F(\varepsilon,P)d\langle x\rangle(P,1)\}$$

$$+ \tfrac{1}{2}E\{\int_0^t \overline{F}(k,2)(\overline{G}^n(i,k)-\overline{G}(i,k))(G^n(i,\varepsilon)-G(i,\varepsilon))F(\varepsilon,P)d\langle x\rangle(P,2)\}$$

where we sum over repeated indices, $k , i , \varepsilon , P = 1 , 2$. Let us consider an arbitrary term in the above sum, say

$$\left| E(\int_0^t \overline{F}(k,1)(\overline{G}^n(i,k)-\overline{G}(i,k))(\overline{G}^n(i,\varepsilon)-G(i,\varepsilon))F(\varepsilon,P)d\langle x\rangle(P,1) \right| \quad .$$

Then this is of the form

$$\left| E\int_0^t f_1 f_2 (g_1^n - g_1)(g_2^n - g_2)d\langle x,y\rangle \right| \qquad (4.46a)$$

where f_i , g_i , g_i^n $i = 1,2$, are bounded predictable processes and (g_i^n) is a sequence of simple bounded processes converging to g_i pointwise on $R^+ \times \Omega$.

Now 4.46a is dominated by

$$E\left|\int_0^t f_1 f_2 (g_1^n - g_1)(g_2^n - g_2)d\langle x,y\rangle\right|$$

$$= \left\| \int_0^t f_1 f_2 (g_1^n - g_1)(g_2^n - g_2)d\langle x,y\rangle \right\|_1$$

$$\leq 2M \left\| \int_0^t (g_2^n - g_2)d\langle x,y\rangle \right\|_1 \qquad (4.46b)$$

where

$$M = \sup_{R^+ \times \Omega} |f_1 f_2 g_1| < \infty$$

since each f_1 , f_2 and g_1 are bounded and hence (g_1^n) is a uniformly bounded sequence, and

$$\int g_2 d\langle X,Y\rangle = L' - \text{Lim} \int_0^t g_2^n d\langle X,y\rangle$$

hence 4.46b converges to zero as $n \to \infty$.

Note that we could have used the bounded convergence theorem [22] to show

$$L^2 - \operatorname{Lim} \int_0^t G_s^n F_s dX_s = \int_0^t G_s F_s dX_s \quad .$$

4.5 Stopping Times

Recall that a stopping time is a projection values process τ , such that $\tau(0) = 0$, $\tau(\infty) = 1$ and $\tau(s) \leq \tau(t)$ for $s \leq t$. (We denote τ by (P_s) , i.e. $\tau(s) = P_s$ for each $s \in R^+$.)

4.51 Examples of Stopping Times

(i) Let $\tau_0 : \Omega \to R^+$ be a stopping time relative to $(\Omega,\Sigma,P,\Sigma_\alpha,R^+)$. Then for each $t \in R^+$,

$$\{\tau_0 < t\} \in \Sigma_t$$

hence

$$\chi_{\{\tau_0<t\}} \in L^\infty(\Omega,\Sigma,P) \quad .$$

Now set $P_t = \chi_{\{\tau_0<t\}}$ and define τ as

$$\tau(t) = \begin{cases} \begin{pmatrix} P_t & 0 \\ 0 & P_t \end{pmatrix} & t \in R^+ \\ I & t = \infty \end{cases} \quad .$$

Then τ defines a $M_2(L^\infty)$ valued stopping time.

(ii) Let $\sigma_0 : \Omega \to R^+$ be another stopping time and set $Q_t = \chi_{\{\sigma_0 < t\}}$ then

$$\tau(t) = \begin{cases} \begin{pmatrix} P_t & 0 \\ 0 & Q_t \end{pmatrix} & t \in R^+ \\ I & t = \infty \end{cases}$$

defines a $M_2(L^\infty)$ valued stopping time.

(iii) More generally let $f : R^+ \to [0,1)$ and $u : R^+ \to \{z \in C : |z| = 1\}$ be Borel measurable functions. Let $\tau_0 : \Omega \to R^+$ be a stopping time relative to $(\Omega,\Sigma,P,\Sigma_\alpha,R^+)$ and set $P_s = \chi_{\{\tau_0 < s\}}$. Then for $t \in R^+$

$$\tau(t) = \begin{pmatrix} \int_0^t f(s)dP_s & \int_0^t u(s)(f(s)-f(s)^2)^{\frac{1}{2}}dP_s \\ \int_0^t \overline{u}(s)(f(s)-f(s)^2)dP_s & \int_0^t (1-f(s))dP_s \end{pmatrix}$$

and $\tau(\infty) = I$ defines a $M_2(L^\infty)$ valued stopping time. The integrals in the matrix are the usual spectral integrals relative to the spectral family $\{P_s\}_{s \in R^+}$.

Recall that if $X = (X_t)$ is a right continuous L^p-process then we define the stopped operator X_τ by

$$L^p - \operatorname{Lim}_\theta X_{\tau(\theta)}$$

where θ is a finite partition of $[0,\infty]$, and if $\theta = \{t_1,\ldots,t_n\}$ say, then

$$X_{\tau(\theta)} = \sum_{i=1}^{n} X_{t_i}(P_{t_i} - P_{t_{i-1}}) \quad .$$

Likewise the stopped process is defined as

$$X_{\tau\wedge t} = L^p - \lim_{\theta} X_{\tau\wedge t(\theta)}$$

for each $t \in R^+$, where $\tau \wedge t$ is the stopping time:

$$\tau\wedge t(s) = \begin{cases} P_s & s \leq t \\ I & s > t \end{cases}$$

where $\tau = (P_s)$.

4.52 Proposition

Let $X = (X_t)$ be a right continuous L^p-process. If X_τ exists for any stopping time τ and $\phi(X_\tau) = 0$ then (X_t) is a centred martingale.

Proof.

Let $X_t = \begin{pmatrix} x_t & y_t \\ z_t & w_t \end{pmatrix}$. Then the stopping time $\tilde{t}$ given by

$$\tilde{t}(s) = \begin{cases} 0 & 0 \le s \le t \\ I & s > t \end{cases}$$

gives $X_{\tilde{t}} = X_t$, hence by hypothesis $\phi(X_{\tilde{t}}) = \phi(X_t) = 0$. That is X is a centred process, or

$$E(x_t + w_t) = 0 \qquad \text{for all} \quad t \in R^+ \quad . \tag{4.52a}$$

Now let $t \in R^+$ be given, and let P , Q be projections in $L^\infty(\Omega,\Sigma_t,P)$ and set

$$\tau(s) = \begin{cases} 0 & 0 \le s \le t \\ \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} & T \ge s > t \\ I & s > T \end{cases}$$

where $T > t$.

Then since (X_t) is L^p-right continuous, we have

$$X_\tau = \begin{pmatrix} x_t & y_t \\ z_t & w_t \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} + \begin{pmatrix} x_T & y_T \\ z_T & w_T \end{pmatrix}\begin{pmatrix} I - P & 0 \\ 0 & I - Q \end{pmatrix}$$

and $\phi(X_\tau) = 0$ gives

$$E(x_tP + x_T(I-P)) + E(w_tQ + w_T(I-Q)) = 0 \quad .$$

Taking $Q = 0$, gives

$$E((x_t - x_T)P) + E(x_T + w_T) = 0 \quad .$$

That is $E((x_t - x_T)P) = 0$ by 4.52a for all projections $P \in L^{\infty}(\Omega,\Sigma_t,P)$. Hence

$$E((x_t - E_t(x_T))P) = 0$$

for any projection $P \in L^{\infty}(\Omega,\Sigma_t,P)$. Hence

$$E_t(x_T) = x_t \qquad \text{for all} \quad T \geq t \quad .$$

Similarly taking $P = 0$ gives

$$E_t(w_T) = w_t \qquad \text{for all} \quad T \geq t$$

and hence (x_t) and (w_t) are martingales.

Now let P be a projection in $L^{\infty}(\Omega,\Sigma_t,P)$ for some $t > 0$. Define the stopping time σ by:

$$\sigma(s) = \begin{cases} 0 & 0 \leq s \leq t \\ \frac{1}{2}\begin{pmatrix} P & P \\ P & P \end{pmatrix} & t < s \leq T \\ I & s > T \end{cases} \quad .$$

Then

$$X_\sigma = \tfrac{1}{2}\begin{pmatrix} x_t & y_t \\ w_t & z_t \end{pmatrix}\begin{pmatrix} P & P \\ P & P \end{pmatrix} + \begin{pmatrix} x_T & y_T \\ w_T & z_T \end{pmatrix}\begin{pmatrix} I - \frac{P}{2} & -\frac{P}{2} \\ -\frac{P}{2} & I - \frac{P}{2} \end{pmatrix} .$$

Now $\phi(X_\sigma) = 0$ gives, using 4.52a and the fact that (x_t) and (w_t) are martingales,

$$0 = E(((y_t + z_t) - (y_T + z_T))P)$$

for all projections $P \in L^\infty(\Omega,\Sigma_t,P)$. Hence

$$0 = E((y_t + z_t - E_t(y_T + z_T))P)$$

and it follows that

$$E_t(y_T + z_T) = y_t + z_t \qquad \text{for all } T \geq t \quad .$$

That is $(y_t + z_t)$ is a martingale. Now defining $\tilde{\sigma}$ by:

$$\tilde{\sigma}(s) = \begin{cases} 0 & 0 \leq s \leq t \\ \tfrac{1}{2}\begin{pmatrix} P & iP \\ -iP & P \end{pmatrix} & t < s \leq T \\ I & s > T \end{cases}$$

gives

$$X_\sigma^{\sim} = \tfrac{1}{2}\begin{pmatrix} x_t & y_t \\ z_t & w_t \end{pmatrix}\begin{pmatrix} P & iP \\ -iP & P \end{pmatrix} + \begin{pmatrix} x_T & y_T \\ z_T & w_T \end{pmatrix}\begin{pmatrix} I - \frac{P}{2} & -\frac{i}{2}P \\ i\frac{P}{2} & I - \frac{P}{2} \end{pmatrix}$$

and $\phi(X_\sigma^{\sim}) = 0$ gives

$$E((z_t - y_t - (z_T - y_T))P) = 0 \quad .$$

Since P is an arbitrary projection in $L^\infty(\Omega,\Sigma_t,P)$ we conclude that

$$E_t(z_T - y_T) = z_t - y_t \qquad \text{for all } T \geq t \quad .$$

Hence $(z_t - y_t)$ is a martingale. Thus it is now clear that all (x_t), (y_t), (z_t), (w_t) are martingales and hence (X_t) is a martingale.

4.53 Proposition

Let $X = (X_t)$ be a L^p-bounded centred martingale. Then $\phi(X_\tau) = 0$ for any stopping time τ .

Proof.

Let $\tau = (P_s)$ and θ be a partition of $[0,\infty]$. Then

$$X_{\tau(\theta)} = \sum M_{t_i}(X)\Delta P_{t_i}$$

since X is L^p-bounded. Hence

$$\phi(X_{\tau(\theta)}) = \sum\phi(M_{t_i}(X)\Delta P_{t_i})$$

$$= \sum\phi(X\Delta P_{t_i})$$

$$= \phi(X) = 0$$

since X is centred. Now

$$|\phi(X_\tau)| \leq |\phi(X_\tau - X_{\tau(\theta)})| + |\phi(X_{\tau(\theta)})|$$

$$\leq \|X_\tau - X_{\tau(\theta)}\|_p \quad .$$

Choosing θ fine enough, we can make right hand side as small as we like. Hence $\phi(X_\tau) = 0$.

4.54 Corollary

Let $X = (X_t)$ be a L^p-process. Then X is a centred martingale if and only if $\phi(X_\tau) = 0$ for any finite stopping time τ .

4.55 Stopping as a limit of convergence in measure

We now define stopping in a slightly different sense than that given in the earlier chapter and in the last section. First we give some preliminaries.

4.56 Definition

Let A be a finite Von Neumann algebra with trace ϕ . Then a sequence $(X_n) \subseteq M(A)$ converges in measure to $X \in M(A)$ if for all $\varepsilon > 0$ there exists N : for all $n \geq N$ $X_n - X \in N(\varepsilon)$, where

$$N(\varepsilon) = \{T \in M(A) : \text{there exists a projection } P \in A , \text{ s.t. } \|TP\|_\infty < \varepsilon \text{ and } \phi(I-P) < \varepsilon\} .$$

It is known that if (T_n) and (S_n) are sequences in $M(A)$ converging to T and S in measure respectively then we have [33] :

(i) $T_n + S_n \overset{m}{\to} T + S$

(ii) $XT_n \overset{m}{\to} XT$ for all $X \in M(A)$

(iii) $T_nX \overset{m}{\to} TX$ for all $X \in M(A)$

(iv) $T_n^* \overset{m}{\to} T^*$

where m above the arrow indicates convergence is in measure. (Sometimes we write m-Lim .)

It is worth mentioning that the definition of convergence given in 4.56 is equivalent to the "usual" definition of convergence in measure when $A = L^\infty(\Omega,\Sigma,P)$. Indeed let $(X_n) \subseteq M(L^\infty(\Omega,\Sigma,P))$ converge to $X \in M(L^\infty(\Omega,\Sigma,P))$ in measure. Then for all $\varepsilon > 0$ there exists N : for all $n \geq N$

$$P\{w : |X_n(w) - X(w)| \geq \varepsilon\} < \varepsilon .$$

That is

$$E(\chi_{\{|x_n - x| \geq \varepsilon\}}) < \varepsilon \quad .$$

Let

$$P = \chi_{\{|x_n - x| \geq \varepsilon\}} \quad .$$

Then

$$\| (X_n - X)P \|_\infty < \varepsilon$$

and

$$E(I - P) = E(\chi_{\{|X_n - X| \geq \varepsilon\}}) < \varepsilon \quad .$$

Thus the "usual" definition of convergence implies definition 4.56.

Conversely given definition 4.56, that is for all $\varepsilon > 0$ there exists N : for all $n \geq N$ $X_n - X \in N(\varepsilon)$. Thus there exists $B \in \Sigma$ such that

$$\| (X_n - X)\chi_B \|_\infty < \varepsilon \quad \text{and} \quad E(I - \chi_B) < \varepsilon \quad .$$

Let $P = \chi_{\{|X_n - X| < \varepsilon\}}$ then

$$\chi_B \leq P \quad .$$

Hence $I - \chi_B \geq I - P$. That is

$$E(I - P) < \varepsilon \quad .$$

The equivalence of the two definitions is now established.

4.57 Definition [27]

Convergence in measure is equivalent to:

for all $\varepsilon > 0$, $\phi(e_{[\varepsilon,\infty]}|X_n - X|) \to 0$

where $e_{[\varepsilon,\infty]}|X_n - X|$ is the spectral projection of $|X_n - X|$ corresponding to $[\varepsilon,\infty]$.

4.58 Lemma

Let $T = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$, $t \in M(M_2(L^\infty))$. Then

$$e_\lambda|T| = \begin{pmatrix} e_\lambda|t| & 0 \\ 0 & I \end{pmatrix} .$$

Proof.

$e_\lambda|T|$ is the orthogonal projection onto the null space $N(|T|_\lambda^+)$ of

$$|T|_\lambda^+ \equiv (|T| - \lambda I)^+ .$$

It is clear that

$$|T|_\lambda^+ = \begin{pmatrix} |t|_\lambda^+ & 0 \\ 0 & 0 \end{pmatrix} .$$

Hence $N(|T|_\lambda^+) = N(|t|_\lambda^+) \oplus L^2$

$$= e_\lambda |t| L^2 \oplus L^2 \quad .$$

That is

$$e_\lambda |T| (L^2 \oplus L^2) = e_\lambda |t| L^2 \oplus L^2 \quad .$$

From the uniqueness of the orthogonal projection, it follows that

$$e_\lambda |T| = \begin{pmatrix} e_\lambda |t| & 0 \\ 0 & I \end{pmatrix} \quad .$$

<u>4.59 Corollary</u>

(i) If $S = \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}$ then $e_\lambda |S| = \begin{pmatrix} I & 0 \\ 0 & e_\lambda |s| \end{pmatrix}$

(ii) If $R = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}$ then $e_\lambda |R| = \begin{pmatrix} I & 0 \\ 0 & e_\lambda |r| \end{pmatrix}$

(iii) If $W = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}$ then $e_\lambda |W| = \begin{pmatrix} e_\lambda |w| & 0 \\ 0 & I \end{pmatrix}$

where $s, r, w \in M(M_2(L^\infty))$.

4.510 Theorem

Let $(X_n) \subseteq M(M_2(L^\infty))$ and $X \in M(M_2(L^\infty))$. Then X_n converges to X in measure if and only if $X_n(i,j)$ converges in measure to $X(i,j)$ $i = 1,2$; $j = 1,2$.

Proof.

We may take $X = 0$. First suppose each $X_n(i,j)$ converges to zero in measure. For all $\varepsilon > 0$ there exists N : for all $n \geq N$ $X_n(i,j) \in N(\frac{\varepsilon}{8})$ for all $i = 1,2$; $j = 1,2$. That is there exist projections $P^n(i,j) \in L^\infty(\Omega,\Sigma,P)$ such that

$$\|X_n(i,j)P^n(i,j)\|_\infty < \frac{\varepsilon}{8}$$

and

$$E(I - P^n(i,j)) < \frac{\varepsilon}{8} \quad .$$

Let $P_0^n = \bigwedge_{\substack{i=1,2\\ j=1,2}} P^n(i,j)$ and set

$$P^n = \begin{pmatrix} P_0^n & 0 \\ 0 & P_0^n \end{pmatrix} \quad .$$

Then for $n \geq N$

$$\left\| \begin{pmatrix} X_n(1,1) & X_n(1,2) \\ X_n(2,1) & X_n(2,2) \end{pmatrix} \begin{pmatrix} P_0^n & 0 \\ 0 & P_0^n \end{pmatrix} \right\|_\infty < \frac{\varepsilon}{2} < \varepsilon$$

and $\phi(I - P^n) \leq \sum_{i,j=1}^{2} E(I - P^n(i,j)) < \frac{\varepsilon}{2} < \varepsilon$. Hence $X_n \overset{m}{\to} 0$.

Conversely suppose $X_n \overset{m}{\to} 0$. Then $E_1X_nE_1 \overset{m}{\to} 0$ by 4.56. That is for all $\varepsilon > 0$:

$$\phi(e_{[\varepsilon,\infty]}|E_1X_nE_1|) \to 0$$

i.e.

$$\phi(I - e_\varepsilon|E_1X_nE_1|) \to 0$$

i.e. $$\phi\left(I - \begin{pmatrix} e_\varepsilon|X_n(1,1)| & 0 \\ 0 & I \end{pmatrix}\right) \to 0 \qquad \text{by Lemma 4.58.}$$

That is

$$E(e_{[\varepsilon,\infty]}|X_n(1,1)|) \to 0$$

hence $X_n(1,1)$ converges to 0 in measure. Likewise $X_n(i,j) \overset{m}{\to} 0$ follows.

4.511 Remark

For the gauge space $(M_2(L^\infty),\phi)$, we observe that $(X_n) \subseteq M(M_2(L^\infty))$ converges to $X \in M(M_2(L^\infty))$ in measure is equivalent to : for all $\varepsilon > 0$ there exists N : for all $n \geq N$ there exists a projection P_n in the commutant of $M_2(L^\infty)$ such that

$$\|(X_n - X)P_n\|_\infty < \varepsilon \quad \text{and} \quad \phi(1 - P_n) < \varepsilon \quad .$$

It is clear that if this condition is satisfied then $X_n \overset{m}{\to} X$ by definition 4.56. Conversely suppose $X_n \overset{m}{\to} X$. We take $X = 0$ for simplicity. Now $X_n(i,j) \overset{m}{\to} 0$ for all $i,j = 1,2$. Hence for all $\varepsilon > 0$ there exists N for all $n \geq N$ there exist projections $P_n^0(i,j) \in L^\infty(\Omega,\Sigma,P)$ such that

$$\|X_n(i,j)P_n^0(i,j)\|_\infty < \frac{\varepsilon}{8}$$

and

$$E(I - P_n^0(i,j)) < \frac{\varepsilon}{8} \qquad \text{for all } i = 1,2 \ ; \ j = 1,2 \ .$$

For each $n \geq N$, set

$$P_n = \begin{pmatrix} \bigwedge\limits_{\substack{i=1,2\\ j=1,2}} P_n^0(i,j) & 0 \\ 0 & \bigwedge\limits_{\substack{i=1,2\\ j=1,2}} P_n^0(i,j) \end{pmatrix}$$

so that $P_n \in M_2(L^\infty)'$ and

$$\|X_nP_n\|_\infty \leq \sum_{i,j=1}^{2} \|X_n(i,j)P_n^0(i,j)\|_\infty < \frac{\varepsilon}{2}$$

and

$$\phi(I - P_n) \leq \sum_{i,j=1}^{2} E(I - P_n^0(i,j)) < \frac{\varepsilon}{2} \quad .$$

Hence the equivalence is established.

4.512 Lemma

Let $1 \leq p < \infty$, $(X_n) \subseteq L^p(A)$ and $X_n \overset{m}{\rightarrow} X$, $X \in L^p(A)$. If $\{|X_n|^p : n \in \mathbb{N}\}$ is uniformly integrable, then $X_n \rightarrow X$ in L^p.

Proof.

We may suppose $X = 0$. Since $\{|X_n|^p : n \in \mathbb{N}\}$ is uniformly integrable, we have: for all $\varepsilon > 0$ there exists $\delta > 0$ s.t. if $A \in \mathcal{A}$, $\|A\|_\infty \leq 1$ and $\|A\|_1 < \delta$ then $|\phi(|X_n|^p A)| < \frac{\varepsilon}{2}$ for all n. Let $\varepsilon > 0$ and define

$$P_n = e_{[0,(\frac{\varepsilon}{2})^{1/p})}(|X_n|) \quad .$$

Then

$$\|X_n\|_p^p = \phi(|X_n|^p P_n) + \phi(|X_n|^p (I - P_n))$$

$$< \frac{\varepsilon}{2} + \phi(|X_n|^p (I - P_n)) \quad .$$

By hypothesis and 4.57 $\phi(I - P_n) \rightarrow 0$. Hence there exists N, for all $n \geq N$: $\phi(I - P_n) < \delta$. Hence

$$\phi(|X_n|^p (I - P_n)) < \frac{\varepsilon}{2} \quad .$$

That is for all $n \geq N$ $\|X_n\|_p^p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, and the result follows.

4.5A Stopping Processes

We now give a discussion on stopping processes in the gauge space $(M_2(L^\infty),\phi)$. Throughout the rest of this chapter we shall follow stopping in the sense given in the definition below.

4.514 Definition

Let $X = (X_t) \subseteq \mathcal{M}(M_2(L^\infty))$ be a process and $\tau = (P_s)$ be a $M_2(L^\infty)$-valued stopping time. Then for each $n \in \mathbb{N}$ set

$$X_{\tau(n)} = \sum_{k=1}^{\infty} X_{\frac{k}{2^n}} (P_{\frac{k}{2^n}} - P_{\frac{k-1}{2^n}}) \quad .$$

If the limit as $n \to \infty$ of $X_{\tau(n)}$ exists in measure, we denote it by X_τ and call it the stopped operator. Likewise the stopped process [28,29] is defined as

$$m - \text{Lim}\, X_{\tau \wedge t(n)} = X_{\tau \wedge t}$$

for each $t \in R^+$.

4.515 Remark

We note that if $x = (x_t) \subseteq \mathcal{M}(L^\infty(\Omega,\Sigma,P)))$, is a right continuous process and $\tau_0 : \Omega \to R^+$ is a stopping time relative to $(\Omega,\Sigma,P,R^+,\Sigma_\alpha)$ then

$$\tau(n) = \sum_{k=1}^{\infty} \frac{k}{2^n} \chi_{\{\frac{k}{2^n} > \tau \geq \frac{k-1}{2^n}\}}$$

$$= \sum_{k=1}^{\infty} \frac{k}{2^n} (\chi_{\{\tau < \frac{k}{2^n}\}} - \chi_{\{\tau < \frac{k-1}{2^n}\}})$$

converges from above to τ a.s. and hence in measure. Now

$$X_{\tau(n)} = \sum_{k=1}^{\infty} X_{\frac{k}{2^n}} (P_{\frac{k}{2^n}} - P_{\frac{k-1}{2^n}})$$

where $P_i = \chi_{\{\tau<i\}}$. Since (x_t) is right continuous and $\tau(n) \downarrow \tau$ (a.s.) it follows that $x_{\tau(n)} \to x_\tau$ in measure. It is also clear that $\chi_{\{\tau_n<t\}}$ converges to $\chi_{\{\tau<t\}}$ in measure. Thus the definition of stopped operator in 4.514 is equivalent to that of the stopped random variables for a certain class of processes.

4.516 Definition

Just as for convergence in measure, we say that a process $X = (X_t) \subseteq M(M_2(L^\infty))$ is continuous in measure at $t \in R^+$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|t - s| < \delta \Rightarrow X_t - X_s \in N(\varepsilon) \quad .$$

The right continuity in measure is defined similarly.

4.517 Theorem

Let $X = (X_t) \subseteq \mathcal{M}(M_2(L^\infty))$. Then (X_t) is (right) continuous in measure if and only if $(X_t(i,j))$ is (right) continuous in measure for all $i,j = 1,2$. (That is $(X_t(i,j))$ have, in measure, (right) continuous paths.)

Proof.

This is similar to 4.510.

4.518 Proposition

Let $X = (X_t)$ be a process in $\mathcal{M}(M_2(L^\infty))$, which is right continuous in measure. Let $\alpha_0, \sigma_0 : \Omega \to R^+$ be stopping times relative to $(\Omega, \Sigma, P, \Sigma_\alpha, R^+)$ and define the stopping time τ be

$$\tau(s) = \begin{pmatrix} p_s & 0 \\ 0 & q_s \end{pmatrix} = P_s \qquad \text{say,}$$

where $p_s = \chi_{\{\sigma_0 < s\}}$ and $q_s = \chi_{\{\alpha_0 < s\}}$. Then X_τ exists and equals

$$\begin{pmatrix} X_{\sigma_0}(1,1) & X_{\alpha_0}(1,2) \\ X_{\sigma_0}(2,1) & X_{\alpha_0}(2,2) \end{pmatrix} .$$

Proof.

$$X_{\tau(n)} = \sum_{k=1}^{\infty} X_{\frac{k}{2^n}} \Delta P_{\frac{k}{2^n}}$$

$$= \sum_{k=1}^{\infty} \begin{pmatrix} X_{\frac{k}{2^n}}(1,1)\Delta p_{\frac{k}{2^3}} & X_{\frac{k}{2^n}}(1,2)\Delta q_{\frac{k}{2^n}} \\ X_{\frac{k}{2^n}}(2,1)\Delta p_{\frac{k}{2^n}} & X_{\frac{k}{2^n}}(2,2)\Delta q_{\frac{k}{2^n}} \end{pmatrix}$$

$$= \begin{pmatrix} X_{\sigma_0(n)}(1,1) & X_{\alpha_0(n)}(2,1) \\ X_{\sigma_0(n)}(2,1) & X_{\alpha_0(n)}(2,2) \end{pmatrix} .$$

By 4.515 and 4.510, we have on taking the limit in measure

$$X_{\tau} = \begin{pmatrix} X_{\sigma_0}(1,1) & X_{\alpha_0}(1,2) \\ X_{\sigma_0}(2,1) & X_{\alpha_0}(2,2) \end{pmatrix} .$$

4.6 Local Martingales

In this section we shall develop a brief theory of local martingales in the gauge space $(M_2(L^{\infty}),\phi)$ and hence construct stochastic integrals with respect to them.

4.61 Definition

For $1 \leq p \leq \infty$ a m-right continuous process $X = (X_t)$ in $M_2(L^p)$ is called a local-L^p-martingale if and only if there exists a monotonic increasing sequence (τ_n) of $M_2(L^\infty)$-valued stopping times such that $\tau_n \uparrow \infty$ and for each $n \in N$ $X_{\tau_n \wedge t}$ exists for all $t \in R^+$ and $(X_{\tau_n \wedge t})_{t \in R^+}$ is a right continuous L^p-bounded martingale.

The sequence (τ_n) is called a localising sequence for X. By $\tau_n \uparrow \infty$ we mean that for each $t \in R^+$:

$$\forall \varepsilon > 0 \quad \exists N(\varepsilon,t) \in N \quad : \quad \phi(P_n(t)) < \varepsilon$$

$$\forall n \geq N(\varepsilon,t) \qquad \text{where} \qquad \tau_n(s) = P_n(s) \qquad .$$

It is clear that if X is a right continuous L^p-bounded martingale then it is a local L^p-martingale. Simply take

$$\tau_n(s) = \begin{cases} 0 & s \leq n \\ I & s > n \end{cases}$$

as the localising sequence. Sufficient conditions implying the converse are given below:

4.62 Proposition

Let $X = (X_t)$ be a m-right continuous local L^p-martingale with localising sequence (τ_n) given by

$$\tau_n(s) = \begin{pmatrix} p_n(s) & 0 \\ 0 & q_n(s) \end{pmatrix}$$

where, for each $n \in \mathbb{N}$, $\{p_n(s)\}_{s \in R^+}$ and $\{q_n(s)\}_{s \in R^+}$ from the spectral projections of stopping times $\tau_n^0 : \Omega \to R^+$ and $\sigma_n^0 : \Omega \to R^+$.

If for each $t \geq 0$ we have that

$$\{|X_{\tau_n \wedge t}|^p : n \in \mathbb{N}\}$$

is uniformly integrable then (X_t) is a L^p-bounded martingale.

Proof.

From 4.518 we have

$$X_{\tau_n \wedge t} = \begin{pmatrix} X_{\tau_n^0 \wedge t}(1,1) & X_{\sigma_n^0 \wedge t}(1,2) \\ X_{\tau_n^0 \wedge t}(2,1) & X_{\sigma_n^0 \wedge t}(2,2) \end{pmatrix}$$

which defines a L^p-bounded martingale for each $n \in \mathbb{N}$. Since $\tau_n \uparrow \infty$, we may assume that $\tau_n^0 \uparrow \infty$ and $\sigma_n^0 \uparrow \infty$ a.s. (If necessary we may take a subsequence.) Hence for a fixed $t \in R^+$, each $X_{\tau_n^0 \wedge t}(1,i)$ and $X_{\sigma_n^0 \wedge t}(2,i)$ converges to $X_t(1,i)$ and $X_t(2,i)$ in measure as $n \to \infty$, $i = 1,2$. By 4.510,

$$X_{\tau_n \wedge t} \overset{m}{\to} X_t \quad .$$

Now since $\{|X_{\tau_n \wedge t}|^p : n \in \mathbb{N}\}$ is uniformly integrals, we have

$$X_{\tau_n \wedge t} \to X_t \qquad \text{is } L^p \text{ by } 4.512 \quad .$$

Now since the conditional expectation is L^p-continuous, we have

$$\begin{aligned} X_s = L^p - \text{Lim } X_{\tau_n \wedge s} &= L^p - \text{Lim } M_s(X_{\tau_n \wedge t}) \\ &= M_s(L^p - \text{Lim } X_{\tau_n \wedge t}) \\ &= M_s(X_t) \qquad \text{for all } s \leq t \ . \end{aligned}$$

Hence $X = (X_t)$ is a L^p-martingale. The fact that X is L^p-bounded follows from looking at the entries of $X_{\tau_n \wedge t}$.

4.63 Lemma

Let $(X_t(i,j)) \subseteq L^1(\Omega,\Sigma,P)$ be local L^p-martingales, m-right continuous, for $i = 1,2$; $j = 1,2$. Then the $M_2(L^1)$-valued process (X_t) given by

$$X_t = \begin{pmatrix} X_t(1,1) & X_t(1,2) \\ X_t(2,1) & X_t(2,2) \end{pmatrix}$$

defines a local L^p-martingale.

Proof.

For each (i,j) , let $(\tau_n^0(i,j))_{n \in N}$ be the sequence of stopping times localising $X_t(i,j)$ $i = 1,2$; $j = 1,2$. Then it is known that $(\tau_n^0(1,1) \wedge \tau_n^0(2,1))_{n \in N}$ localises both $(X_t(1,1))$ and $(X_t(2,1))$ [19]. Whilst $(\tau_n^0(2,2) \wedge \tau_n^0(1,2))_{n \in N}$ localises $(X_t(2,2))$ and $(X_t(1.2))$. Setting

$$p_n(t) = \chi_{\{\tau_n^0(1,1) \wedge \tau_n^0(2,1) < t\}}$$

and

$$q_n(t) = \chi_{\{\tau_n^0(2,2) \wedge \tau_n^0(2,2) < t\}}$$

and defining

$$\tau_n(t) = \begin{pmatrix} p_n(t) & 0 \\ 0 & q_n(t) \end{pmatrix}$$

gives an increasing sequence of stopping times (τ_n) such that $\tau_n \uparrow \infty$. Now from 4.518 we have:

$$X_{\tau_n \wedge t} = \begin{pmatrix} X^{(1,1)}_{\tau_n^0(1,1) \wedge \tau_n^0(2,1) \wedge t} & X^{(1,2)}_{\tau_n^0(1,2) \wedge \tau_n^0(2,2) \wedge t} \\ X^{(2,1)}_{\tau_n^0(1,1) \wedge \tau_n^0(2,1) \wedge t} & X^{(2,2)}_{\tau_n^0(1,2) \wedge \tau_n^0(2,2) \wedge t} \end{pmatrix}$$

Each entry in the matrix above defines a L^p-bounded martingale. Hence for each $n \in N$, $(X_{\tau_n \wedge t})$ is a L^p-bounded martingale.

4.64 Lemma

Let $X = (X_t)$ be a m-right continuous L^2-bounded martingale and τ be a stopping time such that $\tau(s) \in M_2(L^\infty)'$ for all $s \in R^+$. Then

$$\text{(i)} \quad |X_{\tau \wedge t}|^2 = |X|^2_{\tau \wedge t}$$

$$\text{(ii)} \quad \langle X \rangle_{\tau \wedge t} = \langle X^\tau \rangle_t$$

where X^τ is the process $(X_{\tau \wedge t})$.

Proof.

First note that since $\tau(s) \in M_2(L^\infty)'$ it must take the form:

$$\tau(s) = \begin{pmatrix} p_s & 0 \\ 0 & p_s \end{pmatrix}$$

where (p_s) form a spectral resolution of a stopping time $\tau_0 : \Omega \to R^+$. For simplicity take

$$X_t = \begin{pmatrix} x_t & y_t \\ 0 & 0 \end{pmatrix} \qquad \text{for all } t$$

where (x_t) and (y_t) are L^2-bounded martingales in $L^2(\Omega,\Sigma,P)$. From 4.518 we have, for each t,

$$X_{\tau\wedge t} = \begin{pmatrix} x_{\tau^0\wedge t} & y_{\tau^0\wedge t} \\ 0 & 0 \end{pmatrix} .$$

The matrix entries form L^2-bounded martingales [19,28] , hence $(X_{\tau\wedge t})$ is L^2-bounded martingale. Now

$$|X_{\tau\wedge t}|^2 = \begin{pmatrix} |x_{\tau^0\wedge t}|^2 & \bar{x}_{\tau^0\wedge t}\, y_{\tau^0\wedge t} \\ x_{\tau^0\wedge t}\, \bar{y}_{\tau^0\wedge t} & |y_{\tau^0\wedge t}|^2 \end{pmatrix}$$

$$= \begin{pmatrix} |x|^2_{\tau^0\wedge t} & (\bar{x}y)_{\tau^0\wedge t} \\ (x\bar{y})_{\tau^0\wedge t} & |y|^2_{\tau^0\wedge t} \end{pmatrix}$$

$$= |X|^2_{\tau\wedge t} \qquad \text{by [28]} .$$

The identity $\langle X^\tau\rangle_t = \langle X\rangle_{\tau\wedge t}$ is proved similarly using 4.35 and [28] .

<u>Lemma 4.65</u>

Let (A_t) be any m-right continuous process and τ be a stopping time. Then

$$A_{\tau\wedge t}(I - P_t) = A_t(I - P_t)$$

for all $t \in R^+$, where $\tau = (P_s)$.

Proof.

$$A_{\tau \wedge t} = m - \text{Lim } A_{\tau \wedge t(n)} \quad .$$

Now,

$$A_{\tau \wedge t(n)} = \sum_{k=1}^{[2^n t]} A_{\frac{k}{2^n}} \Delta P_{\frac{k}{2^n}} + A_{\frac{[2^n k]+1}{2^n}} (I - P_{\frac{[2^n t]}{2^n}}) \quad .$$

(Where $[x]$ denotes the integral part of x .) Hence

$$A_{\tau \wedge t(n)}(I - P_t) = A_{\frac{[2^n t]+1}{2^n}} (I - P_t) \quad .$$

Taking the limit in measure gives

$$A_{\tau \wedge t}(I - P_t) = A_t(I - P_t)$$

since (A_t) is m-right continuous.

In Lemma 4.63 we showed how to construct examples of non-commutative local martingales in the gauge space $(M_2(L^\infty),\phi)$. In the rest of this chapter we shall look at local L^2-martingales constructed as in Lemma 4.63. Thus let $(X_t(i,j))$ be a.s. right continuous local L^2-martingales relative to $(\Omega,\Sigma,P,\Sigma_\alpha,R^+)$ with localising sequence $(\sigma_n^0(i,j))$ for each $i = 1,2$; $j = 1,2$. Then $\tau_n^0 = \bigwedge_{i,j=1}^{2} \sigma_n^0(i,j)$ defines a sequence of stopping times localising all $(X_t(i,j))$ $i = 1,2$; $j = 1,2$ [19,28]. Now setting $p_n(t) = \chi_{\{\tau_n^0 < t\}}$ and defining

$$\tau_n(t) = \begin{pmatrix} P_n(t) & 0 \\ 0 & P_n(t) \end{pmatrix} \equiv P_n(t)$$

say we get, for each $t \in R^+$,

$$X_{\tau_n \wedge t} = \begin{pmatrix} X_{\tau_n^0 \wedge t}(1,) & X_{\tau_n^0 \wedge t}(1,2) \\ X_{\tau_n^0 \wedge t}(2,1) & X_{\tau_n^0 \wedge t}(2,2) \end{pmatrix}$$

from 4.518. Thus $(X_{\tau_n \wedge t})$ is a L^2-bounded martingale. That is, the process, $X^{\tau_n} = (X_{\tau_n \wedge t})$ is a L^2-bounded martingales for each $n \in N$. That is:

$$X^{\tau_n} = \begin{pmatrix} X^{\tau_n^0}(1,1) & X^{\tau_n^0}(1,2) \\ X^{\tau_n^0}(2,1) & X^{\tau_n^0}(2,2) \end{pmatrix} .$$

The process $X^{\tau_{n+1}}$ is m-right continuous and using 4.518 we get

$$X^{\tau_{n+1}}_{\tau_n \wedge t} = \begin{pmatrix} X^{\tau^0_{n+1}}_{\tau^0_n \wedge t}(1,1) & X^{\tau^0_{n+1}}_{\tau^0_n \wedge t}(1,2) \\ X^{\tau^0_{n+1}}_{\tau^0_n \wedge t}(2,1) & X^{\tau^0_{n+1}}_{\tau^0_n \wedge t}(2,2) \end{pmatrix}$$

$$= \begin{pmatrix} X_{\tau_n^0 \wedge t}(1,1) & X_{\tau_n^0 \wedge t}(1,2) \\ X_{\tau_n^0 \wedge t}(2,1) & X_{\tau_n^0 \wedge t}(2,2) \end{pmatrix} .$$

Since

$$X^{\tau^0_{n+1}}_{\tau_n^0 \wedge t}(i,j) = X_{\tau_n^0 \wedge \tau^0_{n+1} \wedge t}(i,j) = X_{\tau_n^0 \wedge t}(i,j) \quad [28] .$$

Hence

$$X^{\tau_{n+1}}_{\tau_n \wedge t} = X_{\tau_n \wedge t} ,$$

i.e.

$$(X^{\tau_{n+1}})^{\tau_n} = X^{\tau_n} .$$

Thus,

$$<(X^{\tau_{n+1}})^{\tau_n}>_t = <X^{\tau_n}>_t$$

hence

$$<X^{\tau_{n+1}}>_{\tau_n \wedge t} = <X^{\tau_n}>_t$$

from Lemma 4.64, and from lemma 4.65 we have

$$\langle X^{\tau_{n+1}}\rangle_t (I - P_n(t)) = \langle X^{\tau_n}\rangle_t (I - P_n(t)) \quad . \qquad (4.65a)$$

Now for each $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$, $\langle X^{\tau_n}\rangle_t$ is in $M_2(L^1)$ (more precisely in $M_2(L^1(\Omega,\Sigma_t,P))$) and hence has a strongly dense domain $\mathcal{D}(\tau_n,t)$ say. Then

$$\bigcap_{n=1}^{\infty} \mathcal{D}(\tau_n,t)$$

is a strongly dense domain [35], which we denote by:

$$\mathcal{D}_0(\tau,t) \quad . \qquad (4.65b)$$

Then

$$\mathcal{D}_0(\tau,t) \cap \bigcup_{n=1}^{\infty} (I - P_n(t))\mathcal{H}$$

is strongly dense since $I - P_n(t) \to I$ as $n \to \infty$, and $\mathcal{H} = L^2 \oplus L^2$. Writing

$$\mathcal{D}(\tau,t) = \mathcal{D}_0(\tau,t) \cap (\bigcup_{n=1}^{\infty} (I - P_n(t))\mathcal{H})$$

we define A_t, for each $t \in \mathbb{R}^+$, on $\mathcal{D}(\tau,t)$ by

$$A_t\psi = \langle X^{\tau_n}\rangle_t \, \psi$$

whenever $\psi \in (I - P_n(t))\mathcal{H}$.

Then A_t is well defined, for $\psi \in (I - P_{n+1}(t))\mathcal{H}$ too. But

$$<X^{\tau_{n+1}}>_t\psi = <X^{\tau_{n+1}}>_t(I - P_n(t))\psi$$

$$= <X^{\tau_n}>_t(I - P_n(t))\psi \qquad \text{by (4.65a)}$$

$$= <X^{\tau_n}>_t\psi \quad .$$

A_t is clearly densely defined.

We observe that equivalently A_t can be defined as

$$A_t\psi = \lim_{n\to\infty} <X^{\tau_n}>_t\psi$$

for each $\psi \in \mathcal{D}(\tau,t)$.

The limit exists since there exists $N \in \mathbb{N}$ such that $\psi \in (I - P_N(t))\mathcal{H}$ and for all $n \geq N$

$$<X^{\tau_N}>_t\psi = <X^{\tau_n}>_t\psi$$

by (4.65a). Hence $A_t\psi = <X^{\tau_N}>_t\psi$.

4.66 Proposition

The A_t constructed above is a closed operator.

Proof.

Let $(\psi_n) \subseteq \mathcal{D}(\tau,t)$ with $\psi_n \to \psi$ and $A_t\psi_n \to \gamma$ say. For each ψ_n , there is $r_n \in \mathbb{N}$ such that $\psi_n \in (I - P_{r_n}(t))\mathcal{H}$. We may take $r_n \leq r_{n+1}$ for all $n \in \mathbb{N}$. Now

$$A_t\psi_n = <X^{\tau_{r_n}}>_t\psi_n \to \gamma \quad .$$

Hence for any $K \in \mathbb{N}$,

$$(I - P_K(t))\langle X^{\tau_{r_n}}\rangle_t \psi_n \to (I - P_K(t))\gamma$$

since $P_K(t) \in M_2(L^\infty)'$ we get

$$\langle X^{\tau_{r_n}}\rangle_t (I - P_K(t))\psi_n \to (I - P_K(t))\gamma \quad .$$

Thus for $r_n \geq K$, we get using (4.65a)

$$\langle X^{\tau_K}\rangle_t (I - P_K(t))\psi_n \to (I - P_K(t))\gamma \quad .$$

Now $\langle X^{\tau_K}\rangle_t$ is closed and $(\psi_n) \subseteq \mathcal{D}(\tau_K, t)$ hence

$$\langle X^{\tau_K}\rangle_t (I - P_K(t))\psi = (I - P_K(t))\gamma \quad .$$

Now, taking the limit as $K \to \infty$ gives

$$A_t \psi = \gamma \quad .$$

<u>4.67 Proposition</u>

We have that

$$A_t = m - \mathrm{Lim}\langle X^{\tau_n}\rangle_t \quad .$$

Proof.

Let $\psi \in \mathcal{D}(\tau,t)$ then $(I - P_K(t))\psi \in (I - P_K(t))\mathcal{H}$.

Hence

$$A_t(I - P_K(t))\psi = \langle X^{\tau_K}\rangle_t(I - P_K(t))\psi \quad .$$

Thus on a strongly dense domain

$$A_t(I - P_K(t)) = \langle X^{\tau_K}\rangle_t(I - P_K(t)) \tag{4.67a}$$

for all $K \in \mathbb{N}$.

Now $\phi(P_K(t)) \to 0$ as $K \to \infty$. Hence for all $\varepsilon > 0$, there is $K_0 \in \mathbb{N}$, such that for all $K \geq K_0$ $\phi(P_K(t)) < \frac{\varepsilon}{2}$.

Now (4.67a) implies: For each $K \geq K_0$ there exists a projection $Q_K \in M_2(L^\infty)$ such that $\phi(I - Q_K) < \frac{\varepsilon}{2}$ and $\|(A_t(I - P_K(t)) - \langle X^{\tau_K}\rangle_t(I - P_K(t)))Q_K\|_\infty = 0$. That is $\|(A_t - \langle X^{\tau_K}\rangle)(I - P_K(t)) \wedge Q_K\|_\infty = 0$ since $I - P_K(t) \in M_2(L^\infty)'$, and

$$\phi(I - (I - P_K(t)) \wedge Q_K) \leq \phi(P_K(t)) + \phi(I - Q_K)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad .$$

Hence for all $K \geq K_0$ $A_t - \langle X^{\tau_K}\rangle_t \in N(\varepsilon)$.

4.68 Proposition

A_t is m-right continuous at each $t \in R^+$.

Proof.

By construction $<X^{\tau_n}>_t$ is m-right continuous at each t. Let $t \in R^+$ be fixed and $\varepsilon > 0$ be given. Then there exists N such that $\phi(P_N(t + t_0)) < \frac{\varepsilon}{3}$, $t_0 > 0$. Since $<X^{\tau_N}>_t$ is m-right continuous, there is a $\delta > 0$ such that $s - t < \delta$ $(s > t)$ implies

$$<X^{\tau_N}>_t - <X^{\tau_N}>_s \in N(\tfrac{\varepsilon}{3}) \quad .$$

Let $\Delta = t_0 \wedge \delta$, then for $t < s < \Delta + t$ there is a projection $R_s \in M_2(L^\infty)$ such that $\phi(I - R_s) < \varepsilon/3$ and

$$\|(<X^{\tau_N}>_t - <X^{\tau_N}>_s)R_s\|_\infty < \frac{\varepsilon}{3} \quad .$$

Also (4.67a) gives us a Projection Q_s such that $\phi(I - Q_s) < \frac{\varepsilon}{3}$ and

$$\|(A_s - <X^{\tau_N}>_s)(I - P_N(s))Q_s\|_\infty = 0 \quad .$$

Now set $Z_s = Q_s \wedge (I - P_N(s)) \wedge R_s$ so that $\phi(I - Z_s) < \varepsilon$ and

$$\|(A_t - A_s)Z_s\|_\infty$$

$$\leq \|(A_t - <X^{\tau_N}>_t)Z_s\|_\infty + \|(<X^{\tau_N}>_t - <X^{\tau_N}>_s)Z_s\|_\infty$$

$$+ \|(<X^{\tau_N}>_s - A_s)Z_s\|_\infty < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon \quad .$$

Thus we have shown that for any $\varepsilon > 0$, there exists a $\Delta > 0$, such that for $t < s < \Delta + t$, $A_t - A_s \in N(\varepsilon)$. That is A_t is m-right continuous.

4.69 Proposition

We have that

$$A_{\tau_n \wedge t} = \langle X^{\tau_n} \rangle_t$$

for all $n \in \mathbb{N}$.

Proof.

First we observe that if $\tau : \Omega \to \mathbb{R}^+$ is a stopping time and τ_n are approximations of τ as given in Remark 4.515 then $\chi_{\{\tau_n < t\}} \to \chi_{\{\tau < t\}}$ in measure. Now

$$\begin{aligned} A_{\tau_n \wedge t}(I - P_n(t)) &= A_t(I - P_n(t)) && \text{by } 4.65 \\ &= \langle X^{\tau_n} \rangle_t (I - P_n(t)) && \text{by } 4.67a \quad . \end{aligned}$$

Now consider

$$\begin{aligned} & A_{\tau_n \wedge t}(I - P_{n+1}(t)) \\ &= A_{\tau_n \wedge t}\{(I - P_n(t)) + (P_n(t) - P_{n+1}(t))\} \\ &= \langle X^{\tau_n} \rangle_t (I - P_n(t)) + A_{\tau_n \wedge t}(P_n(t) - P_{n+1}(t)) \quad . \end{aligned}$$

Now,

$$A_{\tau_n \wedge t}(P_n(t) - P_{n+1}(t)) \tag{4.69a}$$

$$= m - \operatorname*{Lim}_{m\to\infty} A_{\tau_n \wedge t(m)}(P_n(t) - P_{n+1}(t))$$

$$= m - \operatorname*{Lim}_{m\to\infty} \sum_{K=1}^{[2^m t]} A_{\frac{k}{2^m}} (P_n(\tfrac{k}{2^m}) - P_n(\tfrac{k-1}{2^m}) + A_{\frac{[2^m t]+1}{2^m}} (I-P_n(\tfrac{[2^m t]}{2^m})$$

$$\times (P_n(t) - P_{n+1}(t)) \quad .$$

Now

$$P_n(\frac{[2^m t]}{2^m}) = \chi_{\{\tau_n < \frac{[2^m t]}{2^m}\}} \overset{m}{\to} \chi_{\{\tau_n < t\}} \quad .$$

Hence (4.69a) equals

$$m - \operatorname*{Lim}_{m\to\infty} \sum_{K=1}^{[2^m t]} (A_{\frac{k}{2^m}}(\Delta P_n(\tfrac{k}{2^m}))\left(1 - P_{n+1}(t)\right) \quad .$$

Now, using the fact that $P_{n+1}(t) \in M_2(L^\infty)'$ and (4.67a) we get that (4.69a) is

$$m - \operatorname*{Lim}_{m\to\infty}\left(\sum_{K=1}^{[2^m t]} \langle X^{\tau_{n+1}}\rangle_{\frac{k}{2^m}} \Delta P_n(\tfrac{k}{2^m})\right) (I - P_{n+1}(t))$$

$$= m - \operatorname*{Lim}_{m\to\infty}\{\langle X^{\tau_{n+1}}\rangle_{\tau_n \wedge t} - \langle X^{\tau_{n+1}}\rangle_{\frac{[2^m t]+1}{2^m}}(I-P_n(\tfrac{[2^m t]}{2^m}))\}(I - P_{n+1}(t))$$

$$= \langle X^{\tau_{n+1}} \rangle_{\tau_n \wedge t}(1 - P_{n+1}(t)) - \langle X^{\tau_{n+1}} \rangle_{\tau_n \wedge t}(I-P_n(t))(I-P_{n+1}(t))$$

by 4.65

$$= \langle X^{\tau_n} \rangle_t (I - P_{n+1}(t)) - \langle X^{\tau_n} \rangle_t (I - P_n(t)) \quad .$$

Hence

$$A_{\tau_n \wedge t}(I - P_{n+1}(t)) = \langle X^{\tau_n} \rangle_t (I - P_{n+1}(t)) \quad .$$

Thus for all $m \geq n$:

$$A_{\tau_n \wedge t}(I - P_m(t)) = \langle X^{\tau_n} \rangle_t (I - P_m(t)) \quad .$$

Now $I - P_m(t) \uparrow I$ as $m \to \infty$, hence on a dense domain

$$A_{\tau_n \wedge t} = \langle X^{\tau_n} \rangle_t \quad .$$

4.69 Theorem

We have that $Y_t \equiv |X_t|^2 - A_t$ defines a local L^2-martingale.

Proof.

$$Y_{\tau_n \wedge t} = |X_{\tau_n \wedge t}|^2 - A_{\tau_n \wedge t}$$

$$= |X|^2_{\tau_n \wedge t} - \langle X^{\tau_n} \rangle_t$$

by 4.64 and 4.69. Since $X_{\tau_n \wedge t}$ is a L^2-bounded martingale with increasing process $\langle X^{\tau_n}\rangle_t$, the result follows as in 4.35.

4.610 Proposition

We have that A_t is an increasing process. That is on a dense domain

$$A_t \geq A_s \qquad \text{if} \qquad t \geq s \quad .$$

Proof.

$\mathcal{D}(\tau,s) \cap \mathcal{D}(\tau,t)$ is strongly dense domain [35]. Let ψ belong to this common domain, then there is a $N \in \mathbb{N}$ such that ψ is in $(I - P_N(t))\mathcal{H}$ and hence in $(I - P_N(s))\mathcal{H}$, so that

$$A_t\psi = \langle X^{\tau_N}\rangle_t\psi$$

$$A_s\psi = \langle X^{\tau_N}\rangle_s\psi \quad .$$

But $\langle X^{\tau_N}\rangle_s \leq \langle X^{\tau_N}\rangle_t$ on the common domain, hence result.

4.611 Remark

We note that (A_t) does not depend on the sequence of stopping times localising $X = (X_t)$. If $(\sigma_n) \subseteq M_2(L^\infty)'$, is another sequence localising X then from our previous analysis:

$$X_{\tau_n \wedge \sigma_n \wedge t} = X_{\sigma_n \wedge \tau_n \wedge t}$$

i.e. $$X^{\tau_n}_{\sigma_n \wedge t} = X^{\sigma_n}_{\tau_n \wedge t} \quad .$$

Hence

$$<X^{\tau_n}>_{\sigma_n \wedge t} = <X^{\sigma_n}>_{\tau_n \wedge t}$$

by 4.64. Thus,

$$<X^{\tau_n}>_t (I-P_n(t) \vee Q_n(t)) = <X^{\sigma_n}>_t (I-P_n(t) \vee Q_n(t))$$

and

$$(\mathcal{D}_0(\tau,t) \cap \mathcal{D}_0(\sigma,t)) \cap (\bigcup_{n=1}^{\infty} (I-P_n(t) \vee Q_n(t))\mathcal{H})$$

gives a dense domain on which

$$<X^{\tau_n}>_t = <X^{\sigma_n}>_t \quad .$$

Thus we have shown:

<u>4.612 Theorem</u> (Doob-Meyer decomposition)

Let $X = (X_t)$ be m-right continuous local martingale whose entries are local L^2-martingales. Then there exists a process $(A_t) \subseteq M(M_2(L^\infty))$ such that

(i) $A_t \geq A_s$ if $t \geq s$

(ii) $|X_t|^2 - A_t$ is a local martingale

(iii) A_t is m-right continuous.

4.613 Proposition

Let (X_t) be a local L^2-martingale as above. If $\sup_t \phi(A_t) < \infty$, then X is a L^2-bounded martingale.

Proof.

Since $|X_{\tau_n \wedge t}|^2 - A_{\tau_n \wedge t}$ is centred, we have:

$$\|X_{\tau_n \wedge t}\|_2^2 = \phi(A_{\tau_n \wedge t}) \leq \phi(A_t) < \infty$$

since A_t is increasing.

Thus for each $t \in R^+$ $\{X_{\tau_n \wedge t} : n \in N\}$ is L^2-bounded. The result now follows from 4.512.

4.7 Stochastic Integrals with respect to a Local Martingale

In this section we shall define the stochastic integral for an integrator which is a local L^2-martingale. Thus, let $X = (X_t)$ be a m-right continuous local L^2-martingale as described in the last section. We shall construct stochastic integrals of the form

$$\int_0^t F_s dX_s$$

where (F_s) is a process, whose entries are bounded predictable processes. We first state a result concerning stochastic integrals with respect to a L^2-bounded martingale.

4.71 Proposition

Let X be a L^2-bounded martingale and τ be a stopping time such that $\tau(s) \in M_2(L^\infty)'$ for all s. Let (F_s) be a process as described above. Then

$$\int_0^t F_s dX_{\tau \wedge s} = \int_0^{\tau \wedge t} F_s dX_s \quad .$$

Proof.

In Section 4.4 we showed that

$$\int_0^t F_s dX_s = \left(\int_0^t F_s(i,k) dX_s(k,j)\right)_{i,j} \quad .$$

From 4.518 and [28] the result follows.

4.72 Remark

In the last proposition, the restriction $\tau(s) \in M_2(L^\infty)'$ is not necessary. In fact $\tau(s)$ can take the form

$$\begin{pmatrix} P_s & 0 \\ 0 & Q_s \end{pmatrix}$$

where (P_s) and (Q_s) form the spectral families of stopping times $\tau_0, \sigma_0 : \Omega \to R^+$.

Now let $X = (X_t)$ be a m-right continuous local L^2-martingale then for each $n \in N$, $(X_{\tau_n \wedge t})$ is a L^2-bounded martingale and the stochastic integral:

$$\int_0^t F_s dX_{\tau_n \wedge s}$$

exists for a class of integrands described at the beginning of this section. For each $n \in \mathbb{N}$, let

$$Y_n(t) = \int_0^t F_s dX_{\tau_n \wedge s} .$$

Then $Y_n(t) \in M_2(L^2)$ hence has a strongly dense domain $\mathcal{D}(t,n)$. For $m \leq n$:

$$Y_n(\tau_m \wedge t) = \int_0^t F_s dX_{\tau_n \wedge \tau_m \wedge s} \qquad \text{by } 4.71 .$$

As in Section 4.6, we get

$$Y_n(\tau_m \wedge t) = \int_0^t F_s dX_{\tau_m \wedge s}$$

$$= Y_m(t) .$$

That is for all $m \leq n$

$$Y_n(\tau_m \wedge t) = Y_m(t) .$$

Hence

$$Y_n(t)(I-P_m(t)) = Y_m(t)(I-P_m(t)) \qquad \text{by } 4.65 \quad \ldots (4.72a)$$

Let $\mathcal{D}(t) = \bigcap_{n=1}^{\infty} \mathcal{D}(t,n)$ and $\mathcal{H}_n = (I - P_n(t))\mathcal{H}$.

For $\psi \in \bigcup_{n=1}^{\infty} \mathcal{H}_n \cap \mathcal{D}(t)$ we define

$$Y_t\psi = \lim_{n\to\infty} Y_n(t)\psi \quad .$$

This limit exists since there is a $N \in \mathbb{N}$ such that $\psi \in I - P_N(t)$ and hence for $n \geq N$, $Y_n(t)\psi = Y_N(t)\psi$ from (4.72a). Hence $Y_t\psi = Y_N(t)\psi$.

4.73 Proposition

We have that

$$m - \text{Lim}\, Y_n(t) = Y_t \quad .$$

Proof.

For all $n \in \mathbb{N}$

$$Y_n(t)(I-P_n(t)) \;=\; Y_t(I-P_n(t)) \quad .$$

Given $\varepsilon > 0$, there exists N : for all $n \geq N$, $\phi(P_n(t)) < \varepsilon$ and the result follows.

4.74 Definition

For each $t \in R^+$, we define:

$$Y_t \equiv \int_0^t F_s dX_s = m - \text{Lim} \int_0^t F_s dX_{\tau_n \wedge s} \quad .$$

The limit $\int_0^t F_s dX_s$ is called the stochastic integral of (F_s) with respect to the local L^2-martingale (X_s).

4.75 Remark

The definition of (Y_t) is independent of the localising sequence $\tau_n(s) \in M_2(L^\infty)'$. This assertion follows as in 4.611.

4.76 Theorem

For each t, Y_t is m-right continuous. That is for all $\varepsilon > 0$ there exists $\delta > 0$ such that $t < s < \delta + t$ then $X_t - X_s \in N(\varepsilon)$.

Proof.

This follows just as in 4.68 after observing that $Y_n(t)$ is m-right continuous at each $t \in R^+$.

4.77 Theorem

Let τ be a stopping time of the form

$$\tau(s) = \begin{pmatrix} P_s & 0 \\ 0 & Q_s \end{pmatrix}$$

where (P_s) and (Q_s) form the spectral families of stopping times $\tau_0, \sigma_0 : \Omega \to R^+$. Then

$$\int_0^{\tau \wedge t} F_s dX_s = \int_0^t F_s dX_{\tau \wedge t} .$$

Proof.

The sequence (τ_n) localises $X_{\tau \wedge t}$, hence $(X_{\tau \wedge t})$ is a local L^2-martingale. Thus

$$\int_0^t F_s dX_{\tau \wedge t} = m - \mathrm{Lim} \int_0^t F_s dX_{\tau \wedge \tau_n \wedge s}$$

$$= m - \mathrm{Lim} \int_0^{\tau \wedge t} F_s dX_{\tau_n \wedge s} \qquad \text{by } 4.71 \quad .$$

Now

$$\int_0^t F_s dX_s (I-P_n(t)) = \int_0^t F_s dX_{\tau_n \wedge s} (I-P_n(t))$$

hence

$$\int_0^r F_s dX_s (I-P_n(t)) = \int_0^r F_s dX_{\tau_n \wedge s} (I-P_n(t))$$

for all $r \in [0,t]$.

That is

$$Y_r(I-P_n(t)) = Y_n(r)(I-P_n(t)) \qquad \text{for all } r \in [0,t] \quad .$$

Now consider the stopping time $\tau \wedge t$; then

$$(Y_{\tau \wedge t(m)} - Y_n(\tau \wedge t(m)))(I-P_n(t))$$

$$= Y_{\frac{[t2^m]+1}{2^m}} (I - P_{\frac{[t2^m]}{2^m}})(I-P_n(t))$$

$$- Y_n(\tfrac{[t2^m]+1}{2^m})(I - P_{\frac{[t2^m]}{2^m}})(I-P_n(t)) \quad .$$

As $m \to \infty$, left hand side converges to

$$(Y_{\tau \wedge t} - Y_n(\tau \wedge t))(I - P_n(t))$$

in measure. Whilst the right hand side converges to

$$Y_t(I-P_t)(I-P_n(t)) - Y_n(t)(I-P_t)(I-P_n(t))$$

$$= (Y_t - Y_n(t))(I - P_n(t))(I - P_t) = 0 \quad .$$

Thus we have,

$$Y_{\tau \wedge t}(I - P_n(t)) = Y_n(\tau \wedge t)(I - P_n(t)) \quad .$$

Taking the limit in measure gives the result.

4.78 Theorem (Some properties of Y_t .)

(i) The stochastic integral with respect to a local L^2-martingale is linear.

(ii) $Y_t = \int_0^t F_s dX_s$ is a local L^2-martingale.

Proof.

(i) Let F and G be processes whose entries are bounded predictable processes. Then

$$\int_0^t F_s dX_s = m - \lim_{n\to\infty} \int_0^t F_s dX_{\tau_n \wedge s}$$

$$\int_0^t G_s dX_s = m - \lim_{n\to\infty} \int_0^t G_s dX_{\tau_n \wedge s} \quad .$$

Thus

$$\int_0^t F_s dX_s + \int_0^t G_s dX_s$$

$$= m - \lim \int_0^t F_s dX_{\tau_n \wedge s} + m - \lim \int_0^t G_s dX_{\tau_n \wedge s}$$

$$= m - \lim \int_0^t (F_s + G_s) dX_{\tau_n \wedge s} \qquad \text{by } 4.42$$

$$= \int_0^t (F_s + G_s) dX_s \quad .$$

(ii) From 4.77,

$$\int_0^{\tau_K \wedge t} F_s dX_s = \int_0^t F_s dX_{\tau_K \wedge s}$$

$$= Y_K(t)$$

which is a L^2-bounded martingale.

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