OPTIMISATION METHODS IN STRUCTURAL SYSTEMS

RELIABILITY

A Thesis presented for the degree of
Doctor of Philosophy in the Faculty of Engineering
of the University of London

by

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October 1991

ABSTRACT

The main aim of this thesis is to investigate the use of optimisation methods in the assessment of structural system reliability and to present improvements for the estimation of probability of failure of framed structures. This thesis gives a critical review of the methods available for the enumeration and identification of all possible failure modes or failure regions and discusses a multiple objective linear programming method for framed structures.

The evaluation of the joint probability of failure presents numerical difficulties and some progress has been made in this direction. Existing techniques for evaluating system reliability bounds are reviewed from a different angle and a new ordering algorithm has been developed to get the best second-order lower bound. For gaussian safety margins, a new fast and accurate method for the evaluation of this joint probability is developed and results are presented.

The applicability of Monte Carlo techniques in failure probability estimation is critically appraised and the use of stochastic multiextremal optimisation methods for dominant failure modes identification is thoroughly investigated. A rational combination of these two concepts is developed to generate an efficient method for simultaneous probability integration and dominant failure modes identification. The method is designed for correlated and/or non-normal variables without transformation to the standardised normal space. Used in conjunction with first-order reliability methods (FORM) and/or bounding techniques, this method offers a comprehensive and reliable way of estimating the probability of failure of structural systems. The evaluation of the sensitivity factors used in reliability-based design, with respect to design variables, has also been critically addressed within this Monte-Carlo framework.

ACKNOWLEDGEMENTS

The present work has been accomplished under the supervision of Dr K. Ramachandran. His skilful guidance and his continuous support from the very beginning until the end of this thesis have been very helpful. The long discussions with him on the subject were very stimulating and productive. I would like to thank him very much for all his efforts.

I would like also to thank Prof. P.J. Dowling, Head of the Civil Engineering Department, and all the department staff who directly or indirectly have been helpful and made my life easier in this department. The friendly and cooperative attitude of the Civil Engineering Library staff is highly appreciated.

Many thanks to all my colleagues in room 424 whose amicable behaviour gave a very pleasant side to the working atmosphere. Special thanks to Khurram Wadee and Pariatmono for helping by reading the final draft of this thesis.

I would like also to thank Prof. Steuer, from the University of Georgia (USA), for kindly providing a copy of his Adbase package.

Finally, I would like to thank the British Council and the Moroccan Ministry of Education for their joint sponsorship. Their financial help is very appreciated.

CONTENTS

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NOTATION

In this thesis, the standard notations of probability theory
have been followed whenever possible. These notations concern
mainly the choice of upper or lower case letters for
distribution functions and probability density functions on
one hand and a random variable and its value on the other. In
general, variables and functions are defined in the
corresponding sections and the following list contains some of
the variables which have been used throughout the thesis.

M	Vector of bending moments at critical sections.
p	Vector of bi-action moments.
B	matrix of influence coefficients due to unit bi-actions.
B_0	matrix of influence coefficients due to unit loads.
λ	Vector of loading variables.
M_p	Vector of plastic moments.
θ	Vector of rotations at critical sections.
δ	Vector of nodal displacements.
μ_p	Vector of mean plastic moments.
μ_λ	Vector of mean external loads.
β	Reliability index.
$P(.)$	Probability of ...
p_f	probability of failure.
P_i	probability of occurrence of the ith event.
$P_{ij..k}$	probability of joint occurrence of the $i^{th}, j^{th}, ..k^{th}$ events (intersection).
ϱ	correlation matrix.
n	number of basic variables.
N	sample size.
A	random unit vector.
ψ	critical angle for clustering.
φ	standard normal density function.
Φ	standard normal cumulative distribution function.

Ω_n area of the n-sphere surface.

$R(\alpha)$ distance from the origin to limit state surface in the direction α.

CHAPTER I

INTRODUCTION

The design of a structural system involves many parameters governing the kinematic and static behaviour of the structure and a mathematical model of analysis. The parameters values, concerning the material, the geometry or the external loads, contain always some element of uncertainty and they are referred to as basic variables. This variability makes the so-called 'allowable stresses', used in the earlier codes of practice, may be exceeded during the structure's lifetime. The risk of structural failure cannot, therefore, be eliminated and could only be reduced by a good design practice. The term 'safe structures' is now being replaced by 'low risk structures' for the above reason. This can only be achieved by a rational assessment of the reliability due to the interaction between the various uncertainties in the basic variables, resulting in a *probabilistic design*.

Consideration of the physical uncertainties in the basic variables is itself a computationally expensive problem; the uncertainty on the statistical parameters (due to lack of sufficient data), makes the problem even costlier. Multiple performance requirements (ultimate limit state ULS, serviceability limit state SLS,...) add a great deal of difficulty and the introduction of model uncertainty gives

another dimension to the complexity of the problem. This explains why much of the progress in structural reliability is in the treatment of physical uncertainties in the basic variables.

The impact of a probabilistic treatment of the basic variables on the structural codes, which should be operational, reflects the difficulties mentioned above. The old codes, solely based on 'good practice', are gradually replaced by new scientifically based ones, where theoretical and experimental research results are embodied; although no reliability analysis is explicitly involved in the codes, some level of reliability is sought through the use of partial safety factors. This is a safety checking approach and referred to as *level 1 method*.

The so-called *level 2 methods* are based on approximations of the failure domain and/or the joint probability distribution. The first approximations of the joint probability distribution used are based on *second moment theory.*

Second moment concepts can be traced back in early works on structural reliability [Mayer 1926, Freudenthal 1956, Rzhanitzyn 1957, Basler 1961]. Cornell [1969a] introduced the concept of *reliability index*, defined in terms of the mean and standard deviation of a linearized limit-state, as a reliability measure. Due to the lack of invariance in Cornell's formulation, Hasofer and Lind [1974] proposed an invariant expression by nonhomogeneous linear mapping of the

basic variables into a normalized and uncorrelated set of variables. This becomes to be known as the *Hasofer-Lind Reliability Index*. It was thought that the reliabilities of different structural elements may be compared with this reliability index, without actual evaluation of their failure probabilities. In structural reliability theory, this is referred to as the *first order second moment method (FOSM)*.

The FOSM approach has been later extended to allow for different probability distributions of the basic variables by a transformation into standard normal ones. This gave rise to the *first-order reliability method (FORM)*. This has been also referred to as 'extended FOSM' or 'advanced FOSM' in the literature. In an attempt to improve the accuracy in estimating the failure probability, a second order approximation of the limit state surface was used in what is referred to as the *second order reliability method (SORM)*.

The so-called *level 3 methods* attempt the estimation of the failure probability at the element and/or system level, using the joint probability distribution of the basic variables. This implies the integration of the joint probability density over the failure domain, which is a computationally expensive operation. As a structure can fail in many interdependent modes, the system failure can also be seen as the union of a number of failure modes whose safety margins are correlated. These two possible approaches for the estimation of system reliability are critically studied and improved in this thesis.

1.1 Scope of the thesis

The overall objective of this thesis is to critically assess existing methods of structural system reliability for framed structures under time-invariant loads, to discuss the applicability of some optimisation methods and to contribute to the improvement of the failure probability estimation. Various aspect of reliability analyses are examined. These aspects cover the identification of stochastically dominant failure modes or dominant regions of a failure domain, the variables transformations and their approximations and the probability integral in normal and original space.

A crucial problem, from the computational point of view, is to find the parts of the failure domain which contribute most to the overall probability of failure. Efficient methods for such purpose would improve dramatically the overall efficiency of the failure probability estimation both in computing time and accuracy. Various linear and nonlinear optimisation techniques, used or of potential use in failure modes identification, are discussed. Special emphasis has been given to stochastic methods.

With identified dominant failure modes/failure regions, accurate probability estimation remains a very difficult task; attempts to improve probability estimations in that respect are made, using both deterministic and stochastic methods. Deterministic methods are investigated through improvements of

the multinormal integral over polyhedral domains on one hand and system bounds on the other. Stochastic estimation by Monte Carlo methods and the possibilities of improvement by variance reduction techniques have been addressed. Both the so-called hit-or-miss Monte Carlo and sample-mean methods are discussed. The problem of reliability-based optimisation of structural systems has also been addressed in a critical manner, with assessment and the impact of reliability analysis efficiency on optimum designs.

1.2 Layout of thesis

The thesis comprise three main parts :

• The problem of failure modes identification for plastic structures is reviewed in chapter 2. It contains a critical appraisal of different heuristic and conventional optimisation techniques for identifying the stochastically most dominant modes. In chapter 3, the transformations to normal space and related approximations are assessed, and a failure probability preserving transformation of the limit state surface for highly reliable structural systems is proposed. Chapter 3 reviews also the linear and nonlinear failure surface approximations. The probability of failure estimation for a given set of gaussian safety margins is tackled in chapter 4, where a new fast and accurate method is proposed, along with a critical assessment of the existing methods. Structural system bounds on the probability estimation are thoroughly treated in chapter 5, with emphasis

on the optimisation of the events ordering.

- In the second part, chapter 6 comprise a review of multiextremal optimisation methods with its potential applicability to the identification of dominant regions of a failure domain in non-normal space. Stochastic methods are particularly scrutinised and a suitable algorithm for the mentioned problem is proposed. Monte-carlo methods for probability integration are analyzed in detail in chapter 7. An integrated algorithm for integration and dominant regions identification, based on the results of chapter 6, is proposed.

- The third part consists of a bibliographical and critical review of the reliability based optimisation techniques for structural systems in chapter 8.

Finally, chapter 9 gives general conclusions and recommendations for future researches.

CHAPTER II

FAILURE MODES IDENTIFICATION FOR STRUCTURAL

SYSTEMS

2.0 INTRODUCTION

Different methods can be used for the estimation of the
probability of failure of structural systems. These methods
may involve first-order approximations, bounding techniques or
direct integration by some manageable approach such as
simulation. In all cases the outcome is a close approximation
of the integral of the joint probability density over the
failure domain. Because of the sharp decaying nature of the
tails of the probability density functions, most of the
contribution to the overall integral is concentrated around
the locally high density points on the boundaries of the
failure domain. These local concentration areas are the
stochastically dominant regions with regard to the probability
of failure. For the multinormal standardised distribution, the
dominant regions are the neighbourhoods of the origin-
projection points on the failure surface, due to its
rotational symmetry. For structural systems with normal basic
variables, each local minimum corresponds to a different
failure mode of the structure such as plastic mechanism or
buckling mode. The global minimum corresponds to the
structure's reliability index in the FORM sense. For non-

normal basic variables, often more than one local minimum might correspond to a single failure mode, once the variables are transformed to normal space. In all cases, good estimate of the probability of failure cannot be obtained without identification of these local minima. In structural systems, the number of failure modes is usually large for all of them to be identified, and so is the number of local minima. However, all of them are not required for a good estimation of the probability of failure. In most practical situations, if β_{min} is the global minimum, only few other local minima would have β values with probabilities exceeding 1% or so of that of β_{min}.

For rigid-plastic structural systems, safety margins (failure modes) are linear in terms of the basic variables. Within the FOSM theory, the local minima of β correspond to different hyperplanes of the failure surface and therefore to different failure modes. With the distributional approach, this is also the case if the basic variables are normally distributed. A failure mode is entirely described by its reliability index β and its correlation with the other (significant) failure modes. The search for the dominant regions coincides with that of the failure modes. The failure event is expressed as the union of discrete events corresponding to those failure modes. This explains why a considerable amount of research in structural reliability, has been devoted to the identification of dominant collapse mechanisms of rigid-plastic structures.

Several methods have been developed for generating all

dominant failure mechanisms. These methods are based on :

- Heuristico-combinatorial techniques

- Linear programming

- Non-linear programming

- Random search or stochastic optimisation

In the following sections are critically reviewed some of the known methods, with particular attention given to a new vector maximisation approach.

2.1 HEURISTIC METHODS

2.1.1 The β-unzipping method

This method was proposed by Thoft-Christensen [1982] and was further developed by Thoft-Christensen and Sorensen [1982a]. The failure of a structure with n elements is modeled as that of a series system of selected parallel systems, each one defined by the joint failure of k out of n elements. The k-subsets are selected such that their joint reliability index lies within the interval $[\beta_{min}, \beta_{min} + \delta]$, where β_{min} is the smallest value among all the k-subsets, and δ is a parameter that would dictate which subsets have to be discarded due to negligible contribution. The selected k-subsets have been referred to as the critical k-subsets, and the resulting probability estimate as that of the systems reliability at level k. It has been claimed by the above authors that in general it is not necessary to go beyond level 3. But it is

believed that the maximum level for a good estimate must depend on the degree of redundancy of the structure, and cannot be preset by the investigator.

The viability of the method depends very much on the size and complexities of the structure, and the level at which the failure is arbitrarily defined. If this level is defined to correspond to the formation of a mechanism, the number of structural analyses required may become extremely high. If the level is set too low, 2 or 3 for instance, the estimate may be too conservative, as one gets only an upper bound on the probability of failure of some mechanism that includes the given *k-subset*. The main advantage of the method is that it can handle structures with brittle elements.

For fully ductile structures, the β-unzipping idea is extended to plastic mechanisms, and referred to as *systems reliability analysis at mechanism level*. Each elementary mechanism is taken in turn and combined with others to form a new combined mechanism. The combined mechanisms are ordered in increasing values of their reliability indices. The ones with reliability index exceeding some prefixed threshold level are discarded, and the others are combined again with the elementary mechanisms, to yield a new set of mechanisms. This new set is treated in the same way, by discarding high reliability mechanisms, and the procedure is repeated. Although it is claimed that this method gives good results, the discarding procedure of the mechanisms might prevent the identification of significant ones. There is no reason to believe that a low

reliability index mechanism cannot yield a significant one, if combination is carried out further without discarding at an early stage. This could be the case if the combination is carried out further, with elementary mechanisms, which have low correlation with the discarded one. It can therefore be argued that the β-unzipping approach is not very suitable at the mechanism level.

2.1.2 Branch-and-bound method

The branch-and-bound method was first applied to structural reliability problems by Murotsu et al. [1981,1983]. More or less the same approach has been later proposed by Tang & Melchers [1984], under the heading of *truncated enumeration approach.*

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The aim of the method is to generate sequences of potential plastic hinges, leading to failure mechanisms with non negligible contribution to the probability of failure. Each sequence is referred to as *a failure path,* and if it corresponds to a mechanism it is called *a complete failure path.* In order to optimise the search, the build up of the sequence is continued with a new hinge candidate, if the resulting reliability of the sequence does not exceed some current threshold level; otherwise the hinge is not added to the sequence. The probability of failure associated with a given sequence is defined as that of the joint occurrence of failure of its members. The threshold probability level for

discarding a new candidate from joining a sequence is set to that of a fixed fraction of the highest sequence probability found so far. The automatic updating of the threshold constitutes the *bounding operation* and the *branching* consists of selecting the path of maximum probability. The complete failure paths are stored as dominant failure modes; the sequence corresponding to the bounding threshold, at the end of the search, is taken as the most dominant failure mode.

The method can generate a set of dominant failure modes with a fair level of confidence, especially if there are only few dominant ones. However, it is difficult to prove that this method can always select all significant mechanisms, and there is a high probability that it would miss some significant failure modes. As presented by Murotsu et Al. [1981, 1983], the bounding procedure is carried out right from the start at the first hinge level. The critical threshold based on a single member failure (or even two members) may be too high and could cause the premature elimination of paths corresponding to some significant failure modes. An initial bounding threshold is better selected, in the author's opinion, as a fraction γ of the probability of union of the failure events of significant elementary mechanisms, or its optimal lower bound (see Chapter 5). This would give a threshold which is likely to yield a value close to the overall probability of failure, and the fraction γ would be more related to it than to some single or pair of members failure. The updating can be carried out by adding newly found failure modes to the union and the new probability threshold

can be obtained at a relatively low extra cost. This would reduce the number of unnecessary branchings in a favourable situation, when one of the elementary mechanisms is the most dominant.

The branch-and-bound method has the flexibility for adaptation to deal with structures with various material behaviours, including ductile, brittle, strain-softening and strain-hardening [Melchers & Tang 1984]. But its main weak points are

- Most of the discardable mechanisms are generated several times, unnecessarily increasing the computing cost.

- The number of branchings increases sharply with the size of the structure; the effect is worse if the most dominant modes are not found first.

- There is no guarantee that all dominant mechanisms are generated.

- If the variables are not normal, the evaluation of the failure path probabilities can be very costly and the overall cost may be prohibitive.

2.2 LINEAR PROGRAMMING BASED METHODS

2.2.1 Linear programming (LP) formulation of limit-state analysis

Early attempts of such formulation are due to Charnes and

Greenberg [1951], Dorn and Greenberg [1957], Charnes et al [1959]. These formulations are mainly based on combination of elementary mechanisms, which is optimised to yield the load factor for a given proportioning of the external forces acting on the structure. The primal-dual relationship between kinematic (minimisation) and static (maximisation) formulations was first suggested by Charnes et al [1959]. The constraints of the primal LP are expressed as compatibility equations, and the elementary mechanisms are presented as a particular technique of getting these equations. This duality is further developed with explicit interpretation of the physical meaning of the variables for both programs by Gavarini [1966]. The compatibility equations (or the equilibrium equations for the dual program) are derived using an independent set of mechanisms, referred to as 'basic mechanisms' by Munro [1965]. A basic mechanism is obtained by first transforming an indeterminate structure into a determinate one by inserting appropriate hinges and then inserting a further hinge.

Based on Gavarini's approach of basic mechanisms and from the results of Jenkins [1961], a comprehensive study of the linear programming duality in plastic analysis was carried out by Smith and Munro [1972]. This study includes both the theoretical bases of the duality and the practical aspects of the simplex formulation.

Both elementary mechanisms and basic mechanisms approaches lead to the same results; however it is important to mention

the following points:

- In the elementary mechanism formulation, *m* more variables are needed (mechanisms combining variables),for the primal approach, *m* being the number of elementary mechanisms. This leads to *m* more constraints for the dual program (static). This has an impact on both storing space and computing time.

- Identification of elementary mechanisms can be difficult for complex structures with complex patterns of loading. For basic mechanisms formulation, it is often possible to select a suitable release system without much difficulty [Munro 1965].

- It is possible to transform an elementary mechanism formulation into a basic mechanism one, by the elimination of the combining variables in the primal program, or by the elimination of *m* dual variables in the dual program, thus removing all equality constraints from the latter.

It follows therefore that the basic mechanism approach is much more suitable for practical applications.

The ideal case of a hyperstatic structure, subject to concentrated loads described by *l* random parameters, is considered in the following. If *n* is the number of critical sections, and *c* the static indeterminacy, then the moments vector *M* in the critical sections can be expressed in terms of an indeterminate forces vector *p* (at appropriately selected sections) and the loading parameters (equilibrium conditions) as

$$M = \begin{pmatrix} I & 0 \\ B & B_\bullet \end{pmatrix} \cdot \begin{pmatrix} p \\ \lambda \end{pmatrix} \qquad (2.1)$$

where B and B_0 are the influence coefficients matrices due to unit bi-actions at the critical sections and the loading positions. The rotations are linked by the compatibility conditions, and the nodal displacements can be expressed in terms of the rotations as

$$\begin{pmatrix} 0 \\ \delta \end{pmatrix} = \begin{pmatrix} I & B^T \\ 0 & B_\bullet^T \end{pmatrix} \cdot \theta \qquad (2.2)$$

Let M_p be the vector of plastic moment capacities. Two possibilities are to be considered. The first one assumes that the loads are proportional to a single parameter and the plastic moments are known (or proportional). This yields the known single objective formulation of the static and kinematic LP's. The second one assumes several independent loading parameters together with proportional or independent plastic moments, and yields a multiobjective static LP.

The geometrical interpretation of the LP problem, and the physical interpretation of the simplex tableau content, are necessary for efficient identification of failure modes. In the simplex tableau of the kinematic program, the decision variables correspond to the rotations (or axial displacements for trusses), while the reduced costs at the bottom correspond to the moments at the critical sections. In the dual program, the variables correspond to the moments and the reduced costs to the rotations. The feasible domains of both primal and dual

problems are polyhedra, i.e. subsets bounded by hyperplanes (constraints) which intersect at vertices, or extreme points. By duality, extreme points of the primal program correspond to facets of the dual programm and vice-versa. At least one vertex constitutes an optimal solution. For the primal program, this corresponds to a specific failure mode (set of rotations), which is represented in the dual (static) by a facet. An extreme point is entirely defined by the intersection of n hyperplanes(n being the dimension of the problem). If the number of active constraints n_a at this point exceeds n, many *basic feasible solutions* correspond to this extreme point, and is said to be *degenerate*. The problem of identifying the failure modes becomes therefore a problem of locating vertices and facets of polyhedra.

Single objective formulation

With a single loading parameter λ , the matrix B_0 is reduced to a column vector b_0 , and the static LP program is

$$
\begin{aligned}
\textit{Maximise} \quad & \lambda \\
\textit{s.t.:} \quad & [B \ b_0] \cdot \begin{pmatrix} p \\ \lambda \end{pmatrix} \leq M_p
\end{aligned}
$$

(2.3)

The corresponding kinematic LP is

$$
\begin{aligned}
\textit{Minimise} \quad & M_p^T \cdot \theta \\
\textit{s.t.:} \quad & \begin{pmatrix} B^T \\ b_0 \end{pmatrix} \cdot \theta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

(2.4)

where the last equality constraint scales the vector of rotations.

Multiobjective formulation

The case of independent loading parameters can be formulated as a multiple objectives linear programming problem (MOLP), if the plastic moments are assumed fully correlated,

$$
\text{Maximise} \quad (\lambda_1, \lambda_2, \dots \lambda_l)^T
$$

$$
s.t.: \quad -M_p \preceq \begin{pmatrix} I & 0 \\ B & B_\circ \end{pmatrix} \cdot \begin{pmatrix} p \\ \lambda \end{pmatrix} \preceq M_p \tag{2.5}
$$

In the above program, the optimality of a solution is replaced by the concept of *noninferior* or *nondominated* or *efficient* solution, or *Pareto-optimal* solution [Pareto 1896]. With several objectives, a feasible solution is said to be efficient, if no improvement on one objective can be obtained, by changing the current decisions variables' values, without loosing in at least one other objective. This is known as *strong noninferiority*. A much broader definition is the *weak noninferiority*, where improvement on one objective implies nonimprovement in at least another one (Fig. 2.1). In the MOLP discussed above, the set of all weakly noninferior objective values coincide with the failure surface. It corresponds to the optimal solutions obtained by replacing the set of objectives by their linear combinations.

BCD : strongly efficient set

ABCDE : weakly efficient set
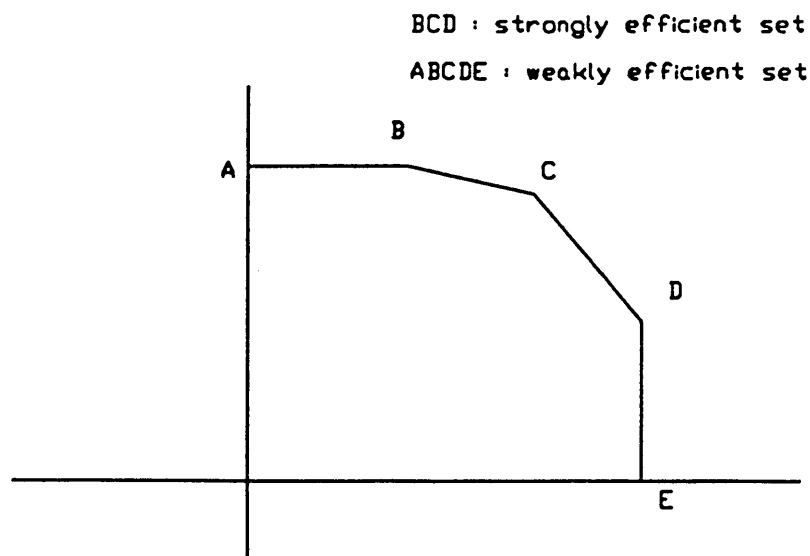
Fig 2.1   Strong and weak efficient sets

Solutions of MOLP problems have been developed by several authors during the sixties and seventies. The extension of the simplex algorithm to *the multicriteria simplex algorithm* allows the generation of all *efficient extreme points* [Evans and Steuer 1972; Yu and Zeleni 1975; Ecker and Kouada 1978]. Methods of generating the whole *efficient set* are also available. [Yu and Zeleni 1976; Izermann 1977; Ecker and Kouada 1980].

One should bear in mind that MOLP methods have been developed, in the first place, to solve multidecision making problems encountered mainly in Economics and Management. The size of the problems addressed in these fields, in terms of number of variables, objectives and constraints, is generally much smaller than that one usually face in structural analysis. The aim of the methods mentioned in the previous paragraph is to

find the whole efficient set. In structural systems this leads to finding all failure modes regardless of their stochastic importance and for real structures, this is unrealistic and unnecessary. From the structural point of view, it has been already pointed out that the number of failure mechanisms may be too high for an entire set to be identified. In linear programming terms, this leads to a large number of decision variables, objectives and constraints, resulting in a huge number of efficient extreme points and as many-folds of corresponding efficient basis. In single objective linear programming the optimality criterion, for an extreme point, is simply the negativity of the reduced cost coefficients within the simplex tableau. The efficiency in MOLP is much more involved; it must be evaluated for each non-basic variable which necessitates each time the solution of a linear programming sub-problem. The number of variables in the sub-problem is at least equal to $(2n+1-c+d)$, where d is the number of degenerate basic variables at the current extreme point and the number of constraints is $(1+d)$ [Evans and Steuer 1975]. Additional calculations and storage are needed for generating the maximal facets corresponding to the failure modes. The increase in the computing cost, with the increasing size of the problem, will therefore be very high.

Investigations on the applicability of these methods have been carried out using the Adbase code [Steuer 1986]. This Fortran code generates only the efficient extreme points. The facets generation corresponding to failure modes, has been carried out by the author by parametric decomposition of the

normalised weight space [Yu and Zeleni 1976], using the reduced cost matrix of each extreme point generated.

The outcome of these investigations is that this method has limited use in structural reliability at the present stage of its development due to the restriction on the size of the problem and cannot be used beyond simple single or double storey frame. The number of efficient bases is very high, and there is no way of predicting this number before hand. For the case of Adbase package, an upper bound on the number of efficient bases is part of the necessary input data. An inputed value lower than the correct one would terminate prematurely the program.

The use of the MOLP method in failure modes generation has been also investigated by Nafday et al. [1988b]. Their conclusions about the practical feasibility of the method are not believed to be realistic, even with full correlation between resistance parameters and very limited number of loading parameters (3 or 4 say).

A different use of the MOLP concept, for a reliability index minimisation, is discussed section 2.3 .

2.2.2 Dominant modes generation

2.2.2.1 Lower and upper bound safety margins concepts

This concept was introduced by Ditlevsen and Bjerager [1984]. The constraints of the system (2.5) are written as a set of $2n$ linear safety margins $Z_i^{s_i}$

$$Z_i^{s_i} = M_{p_i}^{s_i} - s_i p_i, \quad i=1,\ldots,c$$

$$Z_i^{s_i} = M_{p_i}^{s_i} - s_i \sum_{j=1}^{c} b_{i-c,j} p_j - s_i \sum_{j=1}^{n-c} b_{0_{i-c,j}} \lambda_j, \quad i=c+1,\ldots,n$$

$$(2.6)$$

where s_i represents the + or - sign. This set is referred to as *lower bound safety margins.* The structure is in a safe state, if a choice of p is made such that all the above safety margins are non-negative. For a fixed choice of p, the reliability is a lower bound on the overall reliability corresponding to all possible choices of p. This lower bound is optimised by maximising the Hasofer-Lind reliability index β_{HL} corresponding to the smallest reliability index of the $2n$ safety margins. This optimisation is formulated as follows

$$\text{Maximise} \quad \beta$$
$$\text{s.t.} : \quad \beta \le \beta_i^{s_i} \quad (2.7)$$

where $\beta_i^{s_i}$ is the reliability index of the lower bound safety margin $Z_i^{s_i}$. The alternative optima of the above program are referred as the *dominant lower bounds safety margins.* Numerical experience [Ditlevsen and Bjerager 84] shows that the resulting lower bound is dependent on the original choice of the statically determinate system. No general strategy is available for an optimal choice which would maximise the reliability index.

Upper bounds on the reliability index are shown to be obtained from positive linear combinations of lower bound safety margins (equations 2.6) which are independent of p. Those particular combinations yield full plastic failure modes expressed as a difference between internal and external work. This obviously constitutes an upper bound to the Hasofer-Lind reliability index. Generation of optimal combinations, which would lead to the most dominant modes, seem to be very difficult.

2.2.2.2 Linear program for dominant failure modes

A vertex enumeration technique based on kinematic theorem

The parametric linear programming approach for finding collapse modes, has been attempted by different authors [Casciati 1978], [Nafday et al. 1987]. In the latter reference, the primal program (2.4) is used and the extreme points (i.e failure modes) are ranked according to their collapse loads, using Murty's ranking theorem [1968]. This theorem states that if the extreme points $x_1, x_2, .., x_l$ of a given polyhedron $Ax=b$, are the best with respect to a given linear objective function $z(x)=c^Tx$, then the next best extreme point is adjacent to one of these points. By pivoting from one basic feasible solution to an adjacent one successively, failure modes are generated. The method has been referred to as *the polytope extreme point method (PEP)*. This idea of ranking the extremes points which happen to correspond to the failure mechanisms is very appealing one, however the

following points have to be raised :

- The failure mechanisms generated correspond to a prefixed proportioning of the loading parameters, and no provisions where made for pivoting to other mechanisms which would correspond to a different proportion of the loads. This implies that the whole operation is to be repeated for other proportions from scratch, as the failure domains are no longer the same, unless there is full correlation between the loads.

- More important is the criterion for ranking the extreme points, which seems to be based on the load factor. As the problem is formulated in terms of the original variables, even assumed normally distributed, the load factor (linear objective function of the LP) appears to be confused with the reliability index of the corresponding failure mode. It would be more appropriate in reliability analysis to rank the extreme points according to the distance to the origin in *u-space*, obtained from the transformation of the load and resistance variables. But the ranking criterion function becomes non-linear in the space of rotations, with the expression for uncorrelated moments capacities as

$$\beta_i = \frac{\sum_{j=1}^{h} \theta_{ij} \mu_{Pj} - \sum_{j=1}^{l} \mu_{\lambda j} \delta_j}{\sqrt{\sum_{j=1}^{h} \sigma_{Pj}^2 \theta_{ij}^2 + \sum_{j=1}^{l} \sigma_{\lambda j}^2 \delta_j^2}} \qquad (2.8)$$

where μ and σ refer to the mean value and standard deviation of the plastic moments and the loading parameters. With this modified nonlinear objective function Murty's theorem cannot be applied. This problem, which consists now of minimising a

non-linear function over a polyhedron, is in fact an NP-hard problem of nonconvex constrained optimisation. It can be tackled using directly or adapting existing methods in the mathematical programming literature relevant to the problem. An outline of the different methods which have been used, or are of potential use, is given in the next section.

2.3 DOMINANT MODES GENERATION BY QUADRATIC PROGRAMMING

2.3.1 Kinematic formulation

Using the kinematic formulation of the limit-state, it is possible to derive a general expression for the reliability index in terms of the rotations/displacements (Equation 2.8). Dominant failure modes are generated by directly minimising such expressions, in a constrained quadratic program, where the constraints are derived from equation 2.2. This approach has been used by Ishikawa and Iizuka [1987], within the so-called *PNET* method [Ang and Ma 1982]. In this method, the failure modes are generated in a sequence such that the correlation between two consecutive modes does not exceed some prefixed limit. The resulting set of failure modes is then assumed independent, thus simplifying the evaluation of the overall probability of failure. The problem is expressed in the space of the mechanisms combination coefficients as decision variables, and the reliability index as the objective function. A first representative mode is obtained by a sequential linear programming algorithm, and the following

modes are obtained by minimisation with an extra correlation constraint. The problem of nonconvexity does not seem to have been addressed; the failures modes generated are not necessarily the most dominant ones, as the search path is fixed only by the initial starting point and the demarcating correlation value.

The direct minimisation of the reliability index as expressed in (2.8), can present some difficulties due to the shape of the objective function. A better formulation is

$$
\textit{Maximise} \quad \sum_{i=1}^{h} (\sigma_i \theta_i)^2 + \sum_{i=1}^{l} (\sigma_i \delta_i)^2
$$

$$
s.t.:
$$

$$
\mu_p^T \cdot \theta - \mu_\lambda^T \cdot \delta = 1 \tag{2.9}
$$

$$
\begin{pmatrix} I & B^T \\ 0 & B_\bullet^T \end{pmatrix} \cdot \theta - \begin{pmatrix} 0 \\ \delta \end{pmatrix} = 0
$$

with a quadratic objective function, and linear constraints. As the failure modes in the feasible domain correspond to the vertices, the global optimum is necessarily a vertex. A possible method for solving this program is the one proposed by Cabot and Francis [1970], based on Murty's theorem on ranking the extreme points using an auxiliary LP, whose objective function is an overestimate of that of the original program subject to the same constraints. The extreme points are ranked according to the auxiliary objective which provides an upper bound to the main objective function at the current best solution, while the previous best solution constitutes the current lower bound. A backward pivoting away from the

optimal solution of the auxiliary LP leads to a sequence of improved solutions. It is not explicitly proven in the latter reference that the sequence converges to the global optimum . The method proposed by Falk [1973], also seems to be suitable for this problem. In this method a branch-and-bound approach is adopted, and the simplex algorithm is used to generate the vertices, which provides bounds for the objective. Another branch-and-bound type approach, is due to Falk and Soland [1969]. This approach minimises *convex envelope* functions, which closely overestimate the objective, over partitions of the feasible domain. The partitions are refined until the global optimum is isolated in a small area. This is similar to the domain-partitioning approach used in global optimisation of Lipschitz functions [Meewella and Maine 1988]. The main difference is that the convex envelope is less conservative than the global Lipschitz constant, and much easier to find. Very recently, Simoes [1990], addressed the dominant failure modes identification as a nonconvex quadratic programming problem using this method, and claimed some success. A different approach to this problem can be the class of *cutting plane methods* [Tui 1964, Ritter 1966, Konno 1976], where a local optimum is found and its neighbourhood is eliminated from the feasible region by a cutting plane.

2.3.2. Static theorem formulation

The above remarks suggest the extension of the vertex ranking idea to the multiparametric formulation of the static theorem

LP, into a 'facets ranking algorithm'.

Here the vertex ranking is replaced by *a facets ranking* in the objective space, according to their reliability indices. The pivoting from one vertex to an adjacent one is such that the latter is incident to a better facet. A sequence of extreme points is generated following a path of decreasing reliability index. The last point of the sequence is incident to a locally optimal facet with all its adjacent facets having higher reliability indices.

The incident facets to a given extreme point are found by convex parametric decomposition. Let W be the simplex

$$W = \{\, w \in \mathbf{R}^k \mid w_i \geq 0 \,,\, \sum_{i=1}^{k} w_i = 1 \,\} \qquad (2.10)$$

It is shown in Yu and Zeleni [1975, 1976] that this simplex can be decomposed into subsets W_i of polyhedral shape corresponding to each feasible basis J_i (or extreme point x_i for the case of nondegeneracy), such that for each w from W_i, x_i maximises $w^t C x_i$ over the feasible domain. If Z_i is the reduced cost matrix corresponding to J_i, the subset W_i is defined by the optimality condition, which implies the non-positivity of the reduced cost coefficients obtained by convex combination of the rows of Z_i

$$w^T Z_i \leq 0 \qquad (2.11)$$

A solution of equation 2.11 above corresponding to an optimal set equal to a maximal facet of dimension $(n-1)$, must lie on the boundaries of W_i with another subset W_j. As this solution is shared by all extreme points incident to this facet, it also lies on the intersection of the boundaries of their w-set. As this solution must yield a gradient of the combined objective normal to that of the facet, it is unique for the whole facet. It follows that the solution can only be a vertex of W_i. Thus, finding the optimal convex parameters for an incident facet to a given extreme point amounts to finding a vertex of the polyhedron defined by equation 2.11 and the simplex W. This can be obtained by solving an auxiliary linear program

$$Maximise \quad w^T.1$$
$$s.t.:$$
$$w^T Z \leq 0 \qquad\qquad (2.12)$$
$$w^T.1 = 1$$

The above auxiliary LP is to be used within the main LP to generate incident facets to the current extreme point. Solutions are obtained by 'free' pivoting over all vertices. Other methods of identifying incident facets to a given extreme point can also be used [Izermann 1977; Ecker et al. 1980].

The investigations carried out suggest that the method cannot be of practical use. The main reason is that the method needs, for each extreme point on the path, the solution a single

objective LP for all alternative optima. The number of these
alternative optima increases dramatically with the number of
parameters w_i, due to the highly degenerate nature of the main
MOLP.

2.4 RANDOM SEARCH AND SIMULATION TECHNIQUES

Due to lack or limited success of the failure modes
identification deterministic methods discussed above, and
because most of them cannot deal with non-normal variables,
stochastic techniques are becoming the focus of interest in
system reliability research. Stochastic techniques and their
applications in structural reliability are discussed in detail
in Chapters 6 and 7.

CHAPTER III

VARIABLE TRANSFORMATIONS AND LIMIT-STATE APPROXIMATIONS

3.1 VARIABLE TRANSFORMATIONS

3.1.1 Basic principles

Given a set of random variables $X = (X_1, X_2, \ldots, X_n)$, with known joint and continuous probability distribution F_X over a subset S of R^n, a transformation $T: X \longrightarrow U$, where U is standardized uncorrelated normal n-dimensional vector, is said to be probability preserving if

$$P(X \in S) = P(U \in T(S))$$
$$E[X] \in S \leftrightarrow E[U] \in T(S)$$

(3.1)

and the image of any connected subset of S is a connected subset in $T(S)$. This last property is very important for reliability analysis, as the safe set (or the failure set) remains connected after transformation and the probability of failure can be written in terms of the u-space variables as

$$P_f = \int_{T(S)} \prod_{i=1}^{n} \varphi(u_i) \, du_1 du_2 \ldots du_n$$

(3.2)

and the limit-state function $G(x) = 0$ becomes

$$G(T^{-1}(u)) = 0$$

(3.3)

The jacobian matrix of the transformation is denoted J_T

$$J_T = \{a_{ij}\} = \{\frac{\partial T_i^{-1}(u)}{\partial u_j}\} \tag{3.4}$$

which allows the evaluation of the gradient and the hessian of the limit-state function in the u-space

$$\nabla_u G[T^{-1}(u)] = (J_T^{-1})^T \nabla_x G(x) \tag{3.5}$$

$$H_u[G(T^{-1}(u))] = \{J_T\}^T H_x[G(x)]\{J_T\} + \{ \nabla_x G(x) H_T(i,j) \}$$

$$\tag{3.6}$$

where H_x represents the hessian matrix of the limit-state function in the x-space, and $H_T(i,j)$ is the n-dimensional vector

$$H_T(i,j) = [\frac{\partial^2 T^{-1}_k(u)}{\partial u_i \partial u_j}, \; k=1,2,\ldots,n]^T \tag{3.7}$$

For a given limit-state function, there are an infinite number of transformations T from x-space to u-space. Each transformation results in a different image of the failure set, or a different transformed limit-state surface. A transformation can have two aims :

    - The reliability assessment, through the evaluation of the probability of failure, or the reliability index,

    - The easy identification of the most contributing parts to failure, or dominant failure modes.

An ideal transformation would be the one which has the two following features:

• The images of the locally dominant regions in x-space are themselves locally dominant in u-space, in more or less the 'same proportions',

• The image of any connected subset is independent of the subscripting of the original variables, so that the image of the failure domain is independent of the investigator's numbering preference of the variables. In the FORM theory, this would lead to the same set of the so-called β-points, and the same reliability measure for all investigators.

If the first of the two features described above is met, the dominant regions in x-space can be qualitatively identified through their images, but their quantitative contributions to the overall failure probability would not be known with acceptable accuracy without the second feature.

With these basic principles in mind, some existing transformations used in structural reliability, are outlined in the next section, for comparison purposes with a proposed new type of proportion-preserving transformation.

As the aim is to facilitate the evaluation (or estimation) of the probability of failure or the reliability index at low cost, exact transformations are often replaced by approximate ones, which coincide at the β-points.

3.1.2 The Rosenblatt transformation

Rosenblatt [1952] proposed a partial conditioning approach for

generating uniformly distributed random vectors between 0 and 1, from non-uniform joint distributions. This uniform set can be transformed into another set of standardised normal vectors by inversion of the gaussian distribution function for each component separately. This has been first used in structural reliability by Hohenbichler and Rakwitz [1981]. The marginal distribution density of the first i variables irrespective of the others is

$$f_i(x_1, x_2, \ldots, x_i) = \int \ldots \int_{-\infty}^{+\infty} f_X(x_1, x_2, \ldots, x_i, t_{i+1}, \ldots, t_n) \, dt_{i+1} \ldots dt_n$$

$$(3.8)$$

and the conditional distribution function of the ith variable, conditioned on the first $(i-1)$ ones is

$$F_i(x_i | x_1, x_2, \ldots, x_{i-1}) = \frac{\int_{-\infty}^{x_i} f_i(x_1, x_2, \ldots, x_{i-1}, t_i) \, dt_i}{\int_{-\infty}^{+\infty} f_i(x_1, x_2, \ldots, x_{i-1}, t_i) \, dt_i} \quad \text{for } i \geq 2$$

$$(3.9)$$

Then the Rosenblatt transformation T_R is

$$u_1 = \Phi^{-1}(F_1(x_1))$$
$$u_i = \Phi^{-1}(F_i(x_i | x_1, x_2, \ldots, x_{i-1}) \quad i=2,3,\ldots,n$$

$$(3.10)$$

and its inverse transformation is straightforward

$$x_1 = F_1^{-1}(\Phi(u_1))$$
$$x_i = F_i^{-1}(\Phi(u_i) | x_1, x_2, \ldots, x_{i-1}) \quad i=2,3,\ldots,n$$

$$(3.11)$$

For the case of independent variables, the transformation and its inverse simplify to

$$u_i - \Phi^{-1}(F_{X_i}(x_i))$$
$$x_i - F^{-1}{}_{X_i}(\Phi(u_i))$$

For correlated normal variables the transformation and its inverse are linear, and amounts to a an $L.L^T$ factorisation of the correlation matrix C

$$C - L.L^T$$
$$U - L.X$$
$$X - L^{-1}.U$$

For log-normal variables, also the transformation is simple, and it consists of applying the above linear transformation to the logarithms of the variables.

Apart from the above particular cases, the Rosenblatt transformation is often very costly process, as the evaluation of conditional distributions needs to be done numerically if the joint distribution function is not tractable analytically.

As a first-order approximation, the Rosenblatt transformation is applied only at the origin-projection points of the limit-state function in u-space. Both the density and the distribution are equated between the β-points and their reciprocal images. This means a first-order expansion of the Jacobian matrix about the β-points. However, those points are

not known in advance. An algorithm for their evaluation, referred to as the ' β-algorithm ', has been proposed by Fisseler and Rakwitz [1979], for independent variables and generalised to dependents ones [Hohenbichler & Rakwitz, 1981], using the Rosenblatt transformation. This algorithm consists of sequential linearizations of the transformation, leading to a sequence of x-points converging to a reciprocal image of a local minimum of β in u-space.

Ditlevsen [1981], reported more or less the same ideas under the heading of 'normal tail approximation' (NTA). Cheng and Lind [1983], suggested a slightly improved version of the 'normal tail approximation' by equating also the slopes of the densities at the approximation points. An 'equivalent' set of independent normal variables $N(\mu_i, \sigma_i)$ is derived by introducing a third parameter as follows

$$F_X(x) = \alpha \Phi\left(\frac{x-\mu_x}{\sigma_x}\right)$$

$$f_X(x) = \frac{\alpha}{\sigma_x} \phi\left(\frac{x-\mu_x}{\sigma_x}\right) \qquad (3.14)$$

$$f'_X(x) = -\frac{\alpha(x-\mu_x)}{\sigma^3_x} \phi\left(\frac{x-\mu_x}{\sigma_x}\right)$$

Both the full-blown Rosenblatt transformation, and its approximated forms, have two undesirable characteristics

   -The dependence on the original variables ordering which can be easily checked in simple two dimension problems by reversing of the ordering of the variables, which results in two different transformed structural functions, with different sets of β-points. If the lowest β value is to be used as

reliability measure all *n!* possibilities should be investigated. This can be onerous and even unfeasible for high values of n. A polyhedral or quadratic approximations of the transformed limit-state, using the obtained set of β-points, are necessary if the outcome of the transformation is to be credible.

-The coordinates of the β-point in u-space have no physical meaning, as the ith variable depends the first *i* variables in x-space, unless the x-variables are independent. And the wording 'sensitivity factors' used by some authors when referring to the cosine directors of the β-point can be misleading.

The fact that the β-point obtained from the above algorithm does not correspond to the local maximum joint density, has been recognised [Horne & Price 1977], but this should not be the aim of any transformation as will be discussed in the next section, where a new approach towards 'proportions preserving' is developed, ensuring the two features described in the previous section.

3.1.3 The Radial limit-state surface transformation

If the probability of failure is evaluated through some variable transformation, it can make the evaluation easier. However, in practice one is not interested only in the value of the probability no matter how accurate is the estimate, but also in the influence of each of the original variables on

this value. If those influences are not faithfully reflected in the transformed variables, one cannot identify the critical parts of the x-space with most contribution to the probability of failure. This is very important for the sensitivity analysis, as one can manipulate the parameters of some of the original variables in order to meet some target probability. It can be argued that those critical parts can be identified through the reciprocal images of the β-points, as this reciprocal set is independent of the subscripting of the original variables. But the use of the β values for first-order approximations may yield different estimates from different investigators, and this would always shed some doubt over the final results.

It has been more or less implicitly believed that the local maxima of the joint density on the failure surface are the most /'critical points ', and thus the heading of 'design points' seemed to be legitimate, especially within the first-order second moment framework. Any transformation can then be approximated by linearization at those points. But, beyond the second-moment theory, and with the exact joint densities assumed available, this idea can be misleading as a local maximum of the density is not always good indicator of the contribution of its neighbourhood to the probability of failure, except in normal space. The extension of this analogy to non-normal variables is not believed to be valid.

More informative on the critical parts in x-space, are the contributions to the failure probability for fixed

proportioning of the variables. For both normal and non-normal variables, its expression is

$$P(r \leq R \mid \alpha) = \int_{R(\alpha)}^{\infty} f_X(t\alpha) \, t^{n-1} dt \qquad (3.15)$$

where α is a unit vector, characterising the proportioning of the variables. For standardised normal variables, the above radial integral can be expressed in a closed form

$$\int_{R(\alpha)}^{\infty} \phi_n(t\alpha) \, t^{n-1} dt = \frac{1- \chi_n^2(\beta^2)}{\Omega_n} \qquad (3.16)$$

where Ω_n is the area of the unit hyperesphere. This suggests that the analogy with the normal distribution should be equating these radial contributions in each direction α, at each point of the limit-state surface. The result is a transformed limit-state where only the distance to the origin is changed, but the cosine directors vectors α_i are unchanged. The limit-state transformation consists only in 'correcting' the radial distances β_i to the origin, and assumes that the variables are standardised independent normal ones. The transformation consists mainly in shrinking or expanding the radial distances of the failure set boundaries, in order to maintain the probability content of the failure set.

The value of the new distance β to the origin is such that

$$\chi_n^2(\beta^2) = 1 - \Omega_n \int_{R(\alpha)}^{\infty} f_X(t\alpha) \, t^{n-1} dt \qquad (3.17)$$

The above equation can be solved for β only if its right hand side is positive. This is the case for highly reliable systems, such as structures. The solution for β_i is therefore

$$\beta = \sqrt{(\chi_n^2)^{-1}[1 - \Omega_n \int_{R(\alpha)}^{\infty} f_X(t\alpha)\, t^{n-1} dt]} \qquad (3.18)$$

and the ith components of the transformed vector u is

$$u_i = \beta_i \frac{x_i}{\|x\|} \qquad (3.19)$$

From the above formulas, it can be seen that if the original density is such that the radial integral can be performed analytically, the computing cost is expected to be very low. However, in some situations the analytical integral can be a lengthy expression more expensive than a numerical integration. Globally speaking, the computing cost is much lower than that of the Rosenblatt transformation where many multidimensional integrals for the marginal and the conditional distributions of the variables are to be carried out, against one integral for what can be termed as the radial 'shrinking-expansion' approach.

The search for the β-points can be carried out by maximising the radial probability over the unit sphere, using any optimisation package available on the site.

The advantages of this approach can be listed as follows:

- The image of the failure set of R^n is uniquely defined by the transformation regardless of the ordering of the variables. This means that the set of β-points of the transformed failure domain is unique for all orderings of the variables.

- The neighbourhood of the β-points are the worst proportioning of the original variables. The cosine directors at those points can be termed legitimately sensitivity factors.

- Beyond the first-order approximation, this transformation is much more suitable for the probability integration by Monte Carlo methods than the Rosenblatt transformation, as developed in Chapter 7.

Illustrative examples

The following parabolic limit-state function

$$G(x,y) = \frac{5xy^2}{2} + \frac{102y^2}{5} + \frac{3x^2y}{2} - \frac{575y}{2} + x^2 - \frac{223x}{3} - 34$$

is studied for two different non-normal distributions

$$f_{X,Y}(x,y) = e^{-x-y-e^{-x}-e^{-y}}$$

$$g_{X,Y}(x,y) = \frac{(2 + 4x^2 + 2y^2)\,e^{-x^2-y^2}}{5\pi}$$

The graphical representation of the limit-state function in x-space and in u-space using respectively the radial and Rosenblatt transformation is given in Fig (3.1) and Fig (3.2). The two spaces are more conveniently represented with the same

set of coordinates axes. In Table (3.1) are given the results for exact integration and first-order estimates (using a single β-point) based on both transformations.

	Radial Transform.		Rosenblatt Transform.		Exact Probability	
	β	P_f	β	P_f	β	P_f
f_x	1.256	.1045	1.624	.0521	1.295	.0977
g_x	3.549	1.928 E-4	3.504	2.288 E-4	3.551	1.920 E-4

Table 3.1 First-order estimate of the probability and reliability index using the radial and Rosenblatt transformation.

3.2 FAILURE SURFACE APPROXIMATIONS

Most of the limit-state approximations reviewed in this section has been used in the so-called level 2 methods as *FOSM* or *FORM*. In many cases, the probability corresponding to the approximated limit-state is close enough to that of the exact one to yield an acceptable estimate. In general, the corresponding estimate cannot be a poor one; however this approximate limit-state is very useful in Monte Carlo methods for constructing good sampling distributions (See chapter 7).

3.2.1 Linear approximation

The most common is the tangent hyperplane approximation, at the most likely failure point. The probability of failure is simply approximated by that of the tangent hyperplane :

$$p_f = \Phi^{-1}(-\beta) \qquad (3.20)$$

where β is the reliability index of this hyperplane.

In systems reliability, the *representative hyperplane approximation* is more convenient (see Chapter IV). It assumes that the probability of failure corresponding to a given safety margin is known or can be approximated, and the representative hyperplane is defined by the equation

$$\beta - \eta_{av}^T u = 0 \qquad (3.21)$$

where β is defined by

$$\Phi(-\beta) = p_f \qquad (3.22)$$

and η_{av} is the unit normal vector. Various choices for the direction of this normal vector have been proposed. The direction of the most likely failure point (tangent hyperplane) has been adopted by Lind [1980]. For the case of more than one such point, the average direction may be adopted. A parallel translation based approach has been proposed by Hohenbichler [1982]. It is shown that, if δ is an arbitrary translation vector, and $\beta(\delta)$ is defined by

$$\Phi(-\beta(\delta)) = P[(U-\delta) \in \mathscr{F}] \tag{3.23}$$

then an appropriate choice of $\eta_{a\mathscr{F}}$ is

$$\eta_{a\mathscr{F}} = \frac{\nabla\beta(0)}{|\nabla\beta(0)|} \tag{3.24}$$

The viability of such choice depend very much on the evaluation cost of the gradient of $\beta(\delta)$.

A third possibility for a choice of $\eta_{a\mathscr{F}}$ is such that the error

$$\Delta P = P[U \in (H \cup \mathscr{F})] - P[U \in (H \cap \mathscr{F})] \tag{3.25}$$

is minimised, where H is the half-space delimited by the representative hyperplane. A great difficulty with this choice is that the expression above is not easy to evaluate, and its minimisation may lead to several optima.

3.2.2 Polyhedral approximation

The failure surface is approximated by a discrete set of hyperplanes to form a polyhedral failure set. As for the single hyperplane approximation, the choice of the location and the number of hyperplanes can be carried using different methods.

A natural choice is the hyperplanes tangent to the failure surface at the points of locally maximum likelihood. The crucial problem is the identification of those *'checking*

points'. This problem is discussed in other chapters (2, 6 and 7).

If the failure surface is compound of discrete set of continuously differentiable failure surfaces (i.e finite set of failure modes), it can be approximated by a set of *representative gaussian safety margins*. The corresponding reliability indices are defined as in equation (3.21), and the correlation between any couple of safety margins is determined by the equation

$$\Phi_2(-\beta_i, \beta_j; \rho_{ij}) = P[U \in (\mathcal{F}_i \cap \mathcal{F}_j)]$$ (3.26)

In this approach, there are two levels of approximations :

• Approximation of the reliability indices of the different failure modes and that of the probabilities of the mutual intersection, if they are not gaussian;

/• A systematic approximation, inherent to the method, which uses only first and second moment information of the original safety margins;

The two types of approximations have a cumulative effect on the resulting overall error. The representative gaussian safety margins method is used in Chapter 4, for the multinormal integral evaluation, and an approach for reversing the cumulative effect of the two types of errors is proposed.

If the original safety margins or their gaussian representatives are numerous, the computing cost can be substantially reduced by reducing their number, without much

increase in the error. This can be done by *a clustering technique* of the safety margins according to their correlations. Several clustering techniques has been proposed by different authors [Lind, 1980], [Chou et al., 1983], [Ang et al., 1975]. However, the use of such clustering techniques may lead to unacceptably high errors, and their use should be restricted to the cases where the number is very high, say more than 200 or so. With the fast methods of multinormal integration developed in Chapter 4, the cost of probability integration becomes marginal compared to that of the identification of the safety margins. The cost saving with a number reduction by clustering will not be worth the risk of undermining the overall accuracy.

3.2.3 Quadratic approximations

The failure surface is approximated by a quadratic hypersurface at the most likely failure point. This assumes that the probability content of the domain bounded by this quadratic hypersurface can be evaluated exactly or at least can be approximated more easily and cheaply than that corresponding to the original failure surface.

There are at least three known possibilities for generating this quadratic hypersurface :

- the straightforward choice of a quadratic surface defined by the second order Taylor expansion of the failure function about the most likely failure point,

- a curvature fitted parabolic approximation,

- a point fitted parabolic approximation,

In the first case, the resulting approximating function is a general quadratic form with non-zero cross terms coefficients. The corresponding probability cannot be approximated directly. More tractable is the curvature fitted parabola such that its principal curvatures are the same as those of the original failure surface. The probability estimation can be obtained from the asymptotic result [Breitung 1982, 1984]

$$\lim_{\beta \to \infty} p_f = \Phi(-\beta) \prod_{i=1}^{n-1} (1-\beta\kappa_i)^{-1/2} \qquad (3.27)$$

where β is the minimal distance to the origin and $\kappa_1, \kappa_2, .., \kappa_{n-1}$ are the principal curvatures of the limit-state surface at the checking point. More involved approximating expressions have been proposed by Tvedt [1983,1988].

A different approach has been adopted by Der Kiureghian et al.[1987]. It is based on a paraboloid approximation obtained by fitting to the failure surface at discrete points in the neighbourhood of the most likely failure point, in orthogonal directions. The method is claimed to have several advantages over the curvature fitting approach. Among those advantages are the insensitivity to noise in the failure surface, and a better account for higher order terms. However, it does not have the asymptotic behaviour of curvature fitting [Tvedt,88].
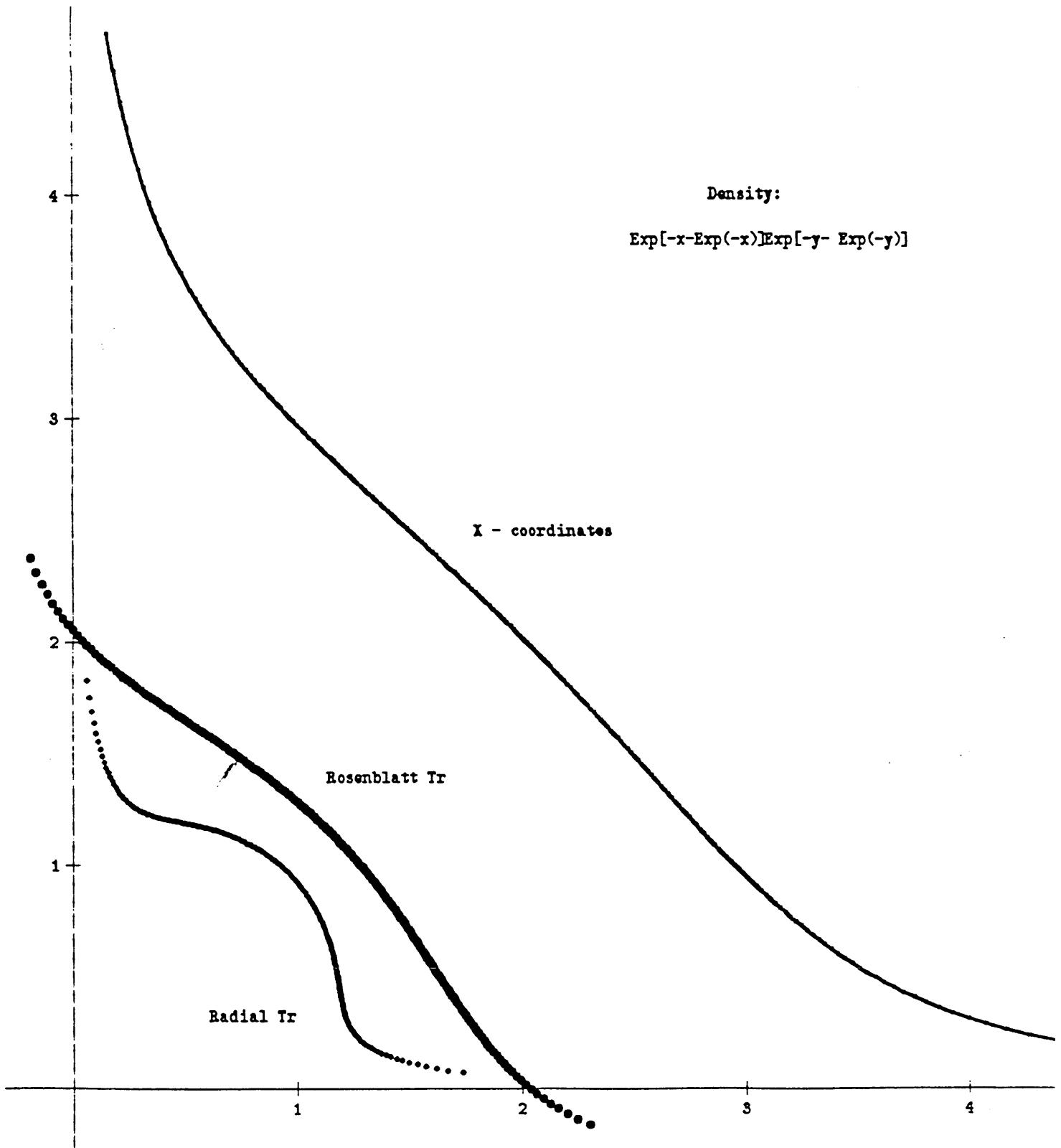
Fig 3.1 . Limit-state transformations for f_x.

Density

$(2 + 4x^2 + 2y^2)Exp(-x^2-y^2) / 5pi$

Radial Tr

Rosenblatt Tr

x-variables
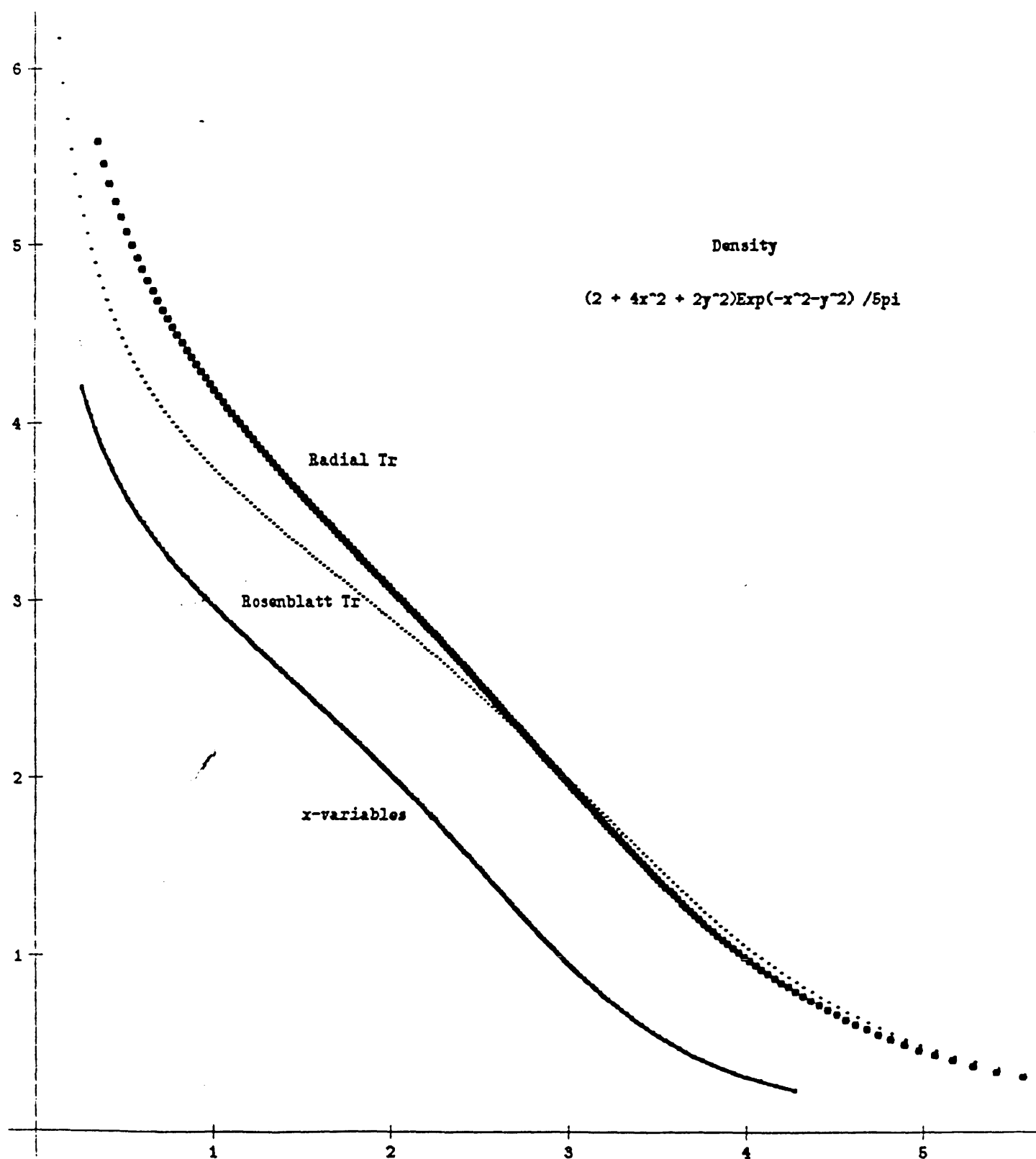
Fig 3.2 . Limit-state transformations for g_x

CHAPTER IV

PROBABILITY APPROXIMATION WITH GAUSSIAN SAFETY MARGINS

4.1 METHOD DESCRIPTION

The probability of failure of systems described by linear gaussian safety margins { $Z_i = X_i + \beta_i, i=1,n$ }, with reliability index vector β and correlation matrix ϱ, can be expressed for parallel systems as

$$P(\bigcap_{i-1}^{n} X_i \leq -\beta_i) - \Phi_n(-\beta;\varrho) \qquad (4.1)$$

and

$$P(\bigcup_{i-1}^{n} X_i \leq -\beta_i) - 1 - \Phi_n(\beta,\varrho) \qquad (4.2)$$

for series systems. The problem is therefore the numerical evaluation of the expression in the right hand side of equation 4.1, where β is taken positive for parallel systems and negative for series systems. This integral was first solved for the particular case where $\rho_{ij}=v_i v_j$ (with $|v_i|$ and $|v_j| \leq 1$) by Dunett & Sobel [1955], where the expression is reduced to a single dimension integral

$$\Phi_n(-\beta;\varrho) - \int_{-\infty}^{\infty} \varphi(t) \prod_{i-1}^{n} \Phi(\frac{-\beta_i - v_i t}{\sqrt{1-v_i^2}}) \, dt \qquad (4.3)$$

which can be easily evaluated numerically. For equicorrelated sets of margins, v_i equals the square root of the common correlation ρ, and the above expression becomes

$$\Phi_n(-\beta;\varrho) - \int_{-\infty}^{\infty} \varphi(t) \prod_{i-1}^{n} \Phi\left(\frac{-\beta_i - \sqrt{\rho}\,t}{\sqrt{1-\rho}}\right) dt \qquad (4.4)$$

Thoft-Christensen and Sorensen [1982] suggested the use of this expression to approximate the general case, by replacing the original correlation matrix by an equicorrelated one, with a weighted average of the correlation coefficients. Their numerical studies seem to suggest that this approximation yields an upper bound on the exact value. Moreover, for equireliable margins, the Taylor expansion of the probability as function of the correlation coefficients, about the equicorrelated point corresponding to the uniform average of all off-diagonal coefficients, presents some attractive features [Ditlevsen 1984a]. Its first degree term is zero, and if the common reliability index exceeds a certain value, the second degree term is negative. In the latter reference, Ditlevsen suggested the use of a modified Taylor expansion about an equicorrelation point, and derived its first and second degree terms, as functions of the margins reliability indices, and the deviations of the original correlation coefficients from the common coefficient of the expansion point. The method gives results of good accuracy for equireliable margins (first degree term vanishes). But for general cases, it is difficult to assess its accuracy. As the

method is exact for equicorrelated cases, there is no exact method left for checking its accuracy. Moreover, for large values of n, the result are unlikely to be satisfactory, as pointed out to by Ditlevsen.

The general case can be better handled by successive conditioning and equivalent gaussian margins representation. The variables X_i are first transformed into an independent set of a standard normal variables U_i, by an LL^T decomposition of the correlation matrix

$$X_i - \sum_{j=1}^{i} \alpha_{ij} U_j \tag{4.5}$$

The first reduction in dimension is carried out by introducing the (n-1)-dimensional conditional margins, given that $\alpha_{11}=1$

$$\Phi_n(\beta;\varrho) - P[\bigcap_{i=2}^{n} \{ \sum_{j=1}^{i} \alpha_{ij} U_j + \beta_i \leq 0 \} \mid U_1 \leq -\beta_1] \cdot P(U_1 \leq -\beta_1)$$

$$\tag{4.6}$$

Because the conditioning on U_1 does not affect the remaining variables, the first factor in the product in the latter equation can be replaced by another expression involving a new variable $\tilde{U}_1$

$$P(\bigcap_{i=2}^{n} \{\alpha_{i1} \tilde{U}_1 + \sum_{j=2}^{i} \alpha_{ij} U_j + \beta_i \leq 0\}) \tag{4.7}$$

where the density of $\tilde{U}_1$ is

$$f_1(\tilde{u}_1) = \frac{\varphi(\tilde{u}_1)}{\Phi(-\beta_1)} \quad \text{for} \quad \tilde{u}_1 \leq -\beta_1$$

$$= 0 \quad \text{for} \quad \tilde{u}_1 > -\beta_1 \tag{4.8}$$

it follows that

$$\tilde{U}_1 = F_1^{-1}[\Phi(U_1|U_1 \leq -\beta_1)] = \Phi^{-1}[\Phi(-\beta_1)\Phi(U_1)] \tag{4.9}$$

and by substitution in equation 4.6

$$\Phi_n(-\beta, \varrho) = P[\bigcap_{i=2}^{n} \{\alpha_{i1}\Phi^{-1}(\Phi(-\beta_1)\Phi(U_1)) + \sum_{j=2}^{i} \alpha_{ij}U_j + \beta_j \leq 0\}]\Phi(-\beta_1)$$

$$= P[\bigcap_{i=2}^{n} (X_i^{(1)} + \beta_i \leq 0)]\Phi(-\beta_1)$$

$$= P[\bigcap_{i=2}^{n} (M_i^{(1)i} \leq 0)]\Phi(-\beta_1)$$

$$\tag{4.10}$$

The first factor in the right-hand side of Equation 4.10 corresponds to the intersection of *n-1* margins defined by a set of non-linear functions. The ideal solution would be to find a set of probability-wise equivalent gaussian variables to this new set of non-gaussian margins, so that the dimension can be reduced to *n-2* margins, and so on. Unfortunately there is no known method for that, and it is necessary to resort to approximate methods. The non-gaussian margins are linear functions of random variables of known distributions (normal and truncated-normal), and therefore their moments (up to some order i) can be easily evaluated. This suggests the idea of deriving a new set of gaussian variables with the same moments as an approximation. The most realistic one is the second-moment approximation.

In the following, U_1 will refer to the β_1-truncated-normal variable. The ith non-linear safety margin expression at the 1th stage of dimension reduction is

$$Z_i^{(1)} = \alpha_{i1}U_1 + \sum_{k=2}^{i} \alpha_{ik}U_k + \beta_i^{(1-1)} \qquad (4.11)$$

Its expectation is

$$E[Z_i^{(1)}] = \alpha_{i1}E[U_1] + \beta_i^{(1-1)} \qquad (4.12)$$

and its covariance with the jth safety margin can be easily found to be

$$Covar(Z_i, Z_j) = \alpha_{i1}\alpha_{j1}E[U_1^2] + (\alpha_{i1}\beta_j + \alpha_{j1}\beta_i)E[U_1] + \beta_i\beta_j +$$

$$\sum_{k=2}^{i} \alpha_{ik}\alpha_{jk} - (\alpha_{i1}E[U_1] + \beta_i)(\alpha_{j1}E[U_1] + \beta_j)$$

$$= \alpha_{i1}\alpha_{j1}(E[U_1^2] - E[U_1]^2) + \sum_{k=2}^{i} \alpha_{ik}\alpha_{jk}$$

$$(4.13)$$

Using the fact that

$$\sum_{k=1}^{i} \alpha_{ik}\alpha_{jk} = \rho_{ij}^{(1-1)} \qquad (4.14)$$

it follows

$$Covar[Z_i, Z_j] = \rho_{i1}^{(1-1)}\rho_{j1}^{(1-1)}\{ E[U_1^2] - E[U_1]^2 - 1 \} + \rho_{ij}^{(1-1)}$$

$$(4.15)$$

The expectations of U_1 and U_1^2 can be derived by the integration of the truncated density

$$E[U_1] = -\frac{\varphi(\beta_1)}{\Phi(-\beta_1)}$$

$$E[U_1^2] = 1 + \beta_1 \frac{\varphi(\beta_1)}{\Phi(-\beta_1)} \qquad \text{(4.16)}$$

The expressions of the correlation coefficients for the equivalent gaussian margins at the lth stage are

$$\rho_{ij}^{(l)} = \frac{\rho_{ji}^{(l-1)}\rho_{ji}^{(l-1)}[\;\eta(\beta_1) + \rho_{ij}^{(l-1)}\;]}{\sqrt{[\;\rho_{ji}^{(l-1)}\eta(\beta_1) + 1\;][\;\rho_{ji}^{(l-1)}\eta(\beta_1) + 1\;]}} \qquad \text{(4.17)}$$

where

$$\eta(\beta_1) = \beta_1 \frac{\varphi(\beta_1)}{\Phi(-\beta_1)} - [\frac{\varphi(\beta_1)}{\Phi(-\beta_1)}]^2 \qquad \text{(4.18)}$$

The corresponding reliability indices can be approximated at least by two different methods

-by assuming that the margins are gaussian and taking their reliability indices as

$$\beta_i^{(l)} = \frac{E[Z_i^{(l)}]}{\sqrt{Var(Z_i^{(l)})}} \qquad \text{(4.19)}$$

-or by "exact" evaluation of their probabilities as an intersection of two gaussian margins

$$P(Z_i^{(l)} \leq 0) = \frac{P(\;Z_i^{(l-1)} \leq 0 \cap Z_1^{(l-1)} \leq 0\;)}{P(Z_1^{(l-1)} \leq 0\;)} \qquad \text{(4.20)}$$

(using their correlation ρ_{11} and their R.I.'s $\beta_1^{(1-1)}$ and $\beta_1^{(1-1)}$)
and the corresponding reliability index

$$\beta_i^{(1)} = \Phi^{-1} [\, P(Z_i^{(1)} \leq 0\,) \,] \qquad (4.21)$$

The first method will be referred to in the sequel as 'method 1', and the second one as 'method 2'.

The above correlation matrix and reliability indices are used to generate a problem identical to the original one with $1-1$ dimensions and the approximation

$$\Phi_n(-\beta;\varrho) \sim P[\bigcap_{i-2}^{n-1} \{ \sum_{j-1}^{i} \alpha_{ij}^{(1)} U_j + \beta_i^{(1)} \leq 0 \}] \, \Phi(-\beta_1^{(1)}) \, \ldots \, \Phi(-\beta_1)$$

$$(4.22)$$

is adopted. This overall approximation comes out as

$$\Phi_n(-\beta;\varrho) \sim \Phi(-\beta_1)\Phi(-\beta_1^{(2)}) \ldots \Phi(-\beta_i^{(n)}) \qquad (4.23)$$

The accuracy of this approximation depends very much on how close is the distribution of the non-linear margins, at different stages, to the gaussian distribution. This closeness can be assessed by a parametric analysis of the skewness of their density distribution functions. The sign and the absolute value of such skewness govern how underestimated or overestimated are the partial reliability indices $\beta^{(1)}_i$. Its expression for the ith safety margin is (the superscript is dropped for convenience)

$$\gamma_1 = \frac{E[(Z_i - E[Z_i])^3]}{Var[Z_i]^{3/2}} \qquad (4.24)$$

Its evaluation can be made easier if the expression of the ith margin Z_i is modified, as follows

$$Z_i = \rho_{ii}U_1 + \sqrt{1-\rho_{ii}^2}\, W + \beta_i \qquad (4.25)$$

where W is

$$W = \frac{\sum_{j=2}^{i} \alpha_{ij}U_j}{\sqrt{1-\alpha_{ii}^2}} \qquad (4.26)$$

and using the fact that $\alpha_{ii} = \rho_{ii}$ (from the Choleski factorisation). Substitution of the new expression of the margin Z_i in (4.25) and simplifications yield the skewness coefficient

$$\gamma_1 = \frac{\rho_{ii}^3 \left[\frac{-2}{\Phi(-\beta_1)} + (3-\beta_1^2)\frac{\phi(\beta_1)}{\Phi(-\beta_1)} + 3\beta_1 \left(\frac{\phi(\beta_1)}{\Phi(-\beta_1)}\right)^2 - 2\left(\frac{\phi(\beta_1)}{\Phi(-\beta_1)}\right)^3 \right]}{\left[\rho_{ii}^2 \left(1 + \beta_1\frac{\phi(\beta_1)}{\Phi(-\beta_1)} - \left(\frac{\phi(\beta_1)}{\Phi(-\beta_1)}\right)^2\right) + \sqrt{1-\rho_{ii}^2}\right]^{3/2}}$$

$$(4.27)$$

as a function of the reliability index β_1 of Z_i and its correlation with Z_i. The parametric study of the above function (Fig 4.1) leads to the following observations

- The skewness has the opposite sign of the correlation. This means that if all correlations are positive, a build-up effect of the error is inevitable, by successive underestimation (or overestimation) of $\beta_i^{(1)}$ if evaluated as a mean-standard deviation ratio.

- For small values of β_1, the magnitude of the skewness is small, and one can expect the normal approximation to yield good estimates. This also suggests that the estimate is best if the margins are numbered in increasing order of their reliability indices. The influence of the correlation could also be taken into account for the ordering, but the coupling of the two ordering criteria may become a combinatorial problem of out of proportion complexity. Except in particular situations where some of the margins can be singled out as 'almost' independent of the rest, ordering based on the correlation is better avoided.

- For series systems, the influence of β on the skewness is significant only for small values of β and becomes negligible for β exceeding 3. The effect of the correlation is much more important, as it increases significantly the absolute value of the skewness coefficient and the increase becomes very sharp in the upper range of the correlation spectrum, say above 0.6.

- For parallel systems, the effect of β_1 on the skewness coefficient is much stronger than that of the correlation , as the increase in magnitude is very sharp. This is hardly surprising from a distribution which is the sum of a symmetrical distribution and a normal distribution truncated at the negative tail. The effect becomes worse if the weighting (i.e. ρ and $(1-\rho^2)^{.5}$) are in favour of the truncated one (ρ).

4.2 NUMERICAL EXPERIENCE

From the above results, one would expect the best estimations to be obtained for small values of β, roughly between -1.5 and 1.5. Extensive numerical experiments tend to support the trend suggested by the study of the skewness of the conditional margins distributions. Samples of such numerical experiments are shown in Tables (4.1) through (4.5). The availability of an exact solution for equicorrelated systems, gives a way of 'severe' testing of the accuracy of the method described above. If its accuracy can be considered acceptable for highly equicorrelated systems, then it would be expected, for a practical case spanning the whole range of correlation, to be also acceptable. Different values of the correlation coefficient have been tested, and the results are compared to the exact ones evaluated numerically. The choice of the reliability indices for testing the method is dictated by two considerations

- On one hand the test must be severe enough to ensure
  better estimates for practical problems. The most severe
  is the equireliable case;

- On the other hand, a method that can deliver acceptable
  estimates should not be dismissed because it fails in
  unrealistic problems.

A compromise must be found, based on the fact that the accuracy increases with the dispersion of the relative importance of the probabilities corresponding to the reliability indices, with respect to each other, and the probability gap between the highest and the lowest values.

Such a compromise is believed to be a certain partial equireliability, i.e. to choose a fraction among the lowest RI's with the same value, and the others are such that the probability of the highest is not less than, say, 1% of that of the highest. An equireliable fraction of 20% is believed to be a reasonable choice.

Non-equicorrelated cases have been also studied, and comparison is made with the bounds evaluated by the methods of chapter 5, on one hand, and an estimation based on directional importance sampling simulation (see chapter 7), on the other hand. In the examples studied, the gradients of the safety margins and their reliability indices are randomly generated. In all cases, the computing time is recorded, and corresponds to runs on a 20Mhz-386/387 microcomputer. More details on different cases are given in the following sections.

Series Systems :

Method 1 and method 2 of evaluation of the partial reliability indices $\beta^{(1)}$, are used separately, for β less than 1.5 and β higher than 1.5. For equicorrelated series systems, Tables (4.1) and (4.2) show typical results. For $\beta < 1.5$, both methods yield good estimates although slightly underestimated. For $\beta > 1.5$, method one shows an overestimation which increases sharply with the correlation. This supports the fact that the partial reliability indices are underestimated, due to negative skewness. Method 2 shows a moderate under-estimation, which also increases with the correlation. This

can only be explained by an underestimation of the correlation coefficients, which seem to be inherent to the equivalent hyperplane representation in which only first and second moment information are used. The expression (4.17) of the current stage correlation, in terms of the previous one, suggests that a build-up process of a negative error leading to an overall underestimation. This process is reversed in method 1 by an overestimation of the β values right from the second stage.

The results of the above equicorrelated cases are confirmed by those of the non-equicorrelated ones of Table (4.3). For method 1, the estimates are very close to or higher than the upper bound, while for method 2, and the estimates can be seen to be mostly close to the lower bound.

The magnitude of the error with method 2 is not very high, but because it is unconservative, any improvement on its estimate is welcome. The above results on series systems suggest that some interpolation of the value of the partial reliability indices between equations (4.19) and (4.21) can yield more accurate estimations. This interpolation takes the following form

$$\Phi(\beta^{(1)}) = (1-\tau)\Phi(^{1}\beta^{(1)}) + \tau\,\Phi(^{2}\beta^{(1)}) \qquad (4.28)$$

where the left superscripts refer to methods 1 and 2, and τ a positive interpolation coefficient less than 1. The value of τ depends on the correlation coefficient ρ_{11} and to a lesser

extent on β_1 and β_i. Its evaluation as a function needs extensive parametric study in terms of the above variables. However numerical experiments show that good results can obtained by assuming τ constant throughout the algorithm. Its global value depends on the number of margins n. Good results are obtained using the following expression for τ for which Table (4.3) shows typical results

$$\tau = -0.0056\,n + \frac{\rho_{ii} - 0.5}{25} + 0.78 \qquad for \quad \rho_{ii} \geq 0.5$$

$$\tau = 1 \qquad\qquad for \quad \rho_{ii} < .5$$

$$(4.29)$$

The above expression for the interpolation coefficient is valid for values of a reliability indices range for which method 1 and 2 bracket the exact result. This seems to be the case for moderate and high values of the β's. It should be emphasized that the method is developed for highly reliable systems for which the individual margins themselves have at least moderate reliability with β not less than 1.5 to 2. For low reliability systems, the estimate of both method 1 and 2 are good and the use the interpolation is not relevant. The choice should therefore go to the faster one, i.e method 1 which requires no bisections integrations (Table 4.1).

Parallel systems

The results for parallel systems are presented for equicorrelated parallel systems, in Tables (4.4) and (4.5). Methods 1 and 2 show the same performance. For $\beta < 1.5$, the relative error on the probability is slightly higher than its

series systems counterpart, but is still reasonable. For $\beta >$ 1.5 , the error is higher as expected (high skewness), but the case corresponds to very highly reliable systems for which the absolute error is more relevant. The importance of the error is better assessed on the reliability index for which the relative error is much smaller, and is exact up to two or three digits. As both method 1 and 2, show the same sign for the error, the interpolation do not appear to be feasible, but on the other hand one might expect the error to be smaller for general cases where the correlation coefficients span evenly the correlation spectrum, and the cumulative effect is partially cancelled out. A thorough parametric study is necessary if one is to be affirmative on this issue.

4.3 COMPARISON WITH EXISTING METHODS

The idea of conditioning and equivalent hyperplane representation has been used by different authors. Hohenbichler and Rakwitz [1981] proposed the use of this idea within the first-order reliability framework. The nonlinear margins are approximated by their tangent hyperplanes at their origin projection points. The partial reliability indices and the correlation matrix at different stages correspond to those tangent hyperplanes. The accuracy reported concerns the reliability index and not the probability of failure, and seemed satisfactory for a first-order approximation. A slight improvement is obtained, by replacing the nonlinear curve by a polygon formed by the tangent hyperplane and the two

asymptotes obtained by letting U_1 tend to minus and plus infinity respectively, and using the same correlation matrix (i.e. that of the tangent hyperplanes).

Using the correlation matrix of the tangent hyperplanes, Tang and Melchers [1987], improved the method one step further by proposing the 'exact' integration of the conditional probabilities (equations 4.19 and 4.21). The corresponding results are given in the right of each table under the M-T method heading. The accuracy seems to be between those of Method 2 and the Interpolation method for series systems, but its performance is less than both methods 1 and 2 for parallel systems. As presented in reference [Tang & Melchers, 1987], the method shows some instability : if the system includes safety margins of correlation exceeding 0.7 or so, sometimes the algorithm either fails to converge, or converges to a completely wrong estimate. This is illustrated in Tables (4.2) and (4.5). The asterisks in the tables correspond to the cases where Tang-Melchers algorithm failed to converge. The method is about 50% slower than method 2 or the interpolation method, and about 2 to 3 times slower than method 1.

A different approach, based on the use of bivariate and trivariate normal integrals of the conditional nonlinear margins, has been proposed by Ramachandran [1987]. The method evaluates the conditional probabilities as in Equation 4.20. However the correlation between two conditional margins is evaluated such that the trisections are also exact, given that the margins of the previous stage are gaussian or assumed to

be so. Hereafter is an outline of the method.

The probability of the intersection of n events is expanded as

$$P(\bigcap_{i=1}^{n} F_i) = P(\bigcap_{i=1}^{n-1} F_i \mid F_n) P(F_n)$$
$$= P(\bigcap_{i=1}^{n-1} [F_i|F_n]) P(F_n) \tag{4.30}$$

which becomes, after denoting $[F_i \mid F_n]$ as F_i^n

$$P(\bigcap_{i=1}^{n} F_n) = P(\bigcap_{i=1}^{n-1} F_i^n) P(F_n) \tag{4.31}$$

The conditional probabilities $P(F_i^n)$ and their intersections $P(F_i^n \cap F_j^n)$ are

$$P(F_i^n) = \frac{P(F_i \cap F_n)}{P(F_n)} \tag{4.32}$$

$$P(F_i^n \cap F_j^n) = \frac{P(F_i \cap F_j \cap F_n)}{P(F_n)} \tag{4.33}$$

The correlations between the conditional events F_i^n and F_j^n are evaluated in such way that Equation 4.33 is satisfied. This amounts to solving the following equation for ρ_{ij} [Owen, 1956]:

$$P(F_i^n) + \frac{1}{2\pi P(F_j^n)} \int_0^{\rho_{ij}} \frac{1}{\sqrt{1-t^2}} EXP[\frac{-((\beta_i^n)^2 + (\beta_j^n)^2 - \beta_i^n \beta_j^n t)}{2(1-t^2)}] du -$$
$$\frac{P(F_i \cap F_j \cap F_n)}{P(F_n)} = 0$$

$$\tag{4.34}$$

where

$$\beta^i_n = \Phi^{-1}(P(F^n_i)) \qquad (4.35)$$

The probability of intersection of the *n-1* conditional events F_{in} is reduced to *n-2* in the same way using the estimated probabilities, bisections and correlations in the previous iteration. The process is repeated until the dimension is reduced to three events. The final expression of the probability becomes

$$P(\bigcap_{i=1}^{n} F_i) = P(F^4_{1,5} \cap F^4_{2,5} \cap F^4_{3,5})\, P(F^5_{4,5})\, P(F^6_{5,6}) \ldots$$
$$\ldots P(F^n_{n-1,n})\, P(F_n)$$

$$(4.36)$$

Table (4.6) shows typical results for equicorrelated systems. The accuracy is comparable to that of method 1 . The method do not need any linearization or a search for the origin projection points. However, the evaluation of the intersection probabilities between the events and all the trisections involving the event candidate for removal, at each iteration, and the extra bisection evaluations to find the correlations between the conditional events, makes the computational effort very high compared to the methods described above, especially for high dimensional problems. Its extensive use in some iterative process cannot be recommended.

4.4 CONCLUSIONS

The method discussed in section 4.1 above, is believed to be the most efficient with regard to both accuracy and computing time. The study of skewness of the conditional margins distributions indicates where the method can be expected to be accurate. Moreover, the method is virtually free from numerical instability and convergence problems, as no optimisation algorithm is needed. Its relatively small computing time and its stability make it suitable for incorporation into reliability-based optimisation packages.

However, the above analysis is based on a qualitative study of the skewness and the correlations expressions in equations 4.17 and 4.27 and numerical experiments. An analytical study of the overall behaviour with the aim of bounding the error in sign and magnitude remains a challenge for future research.
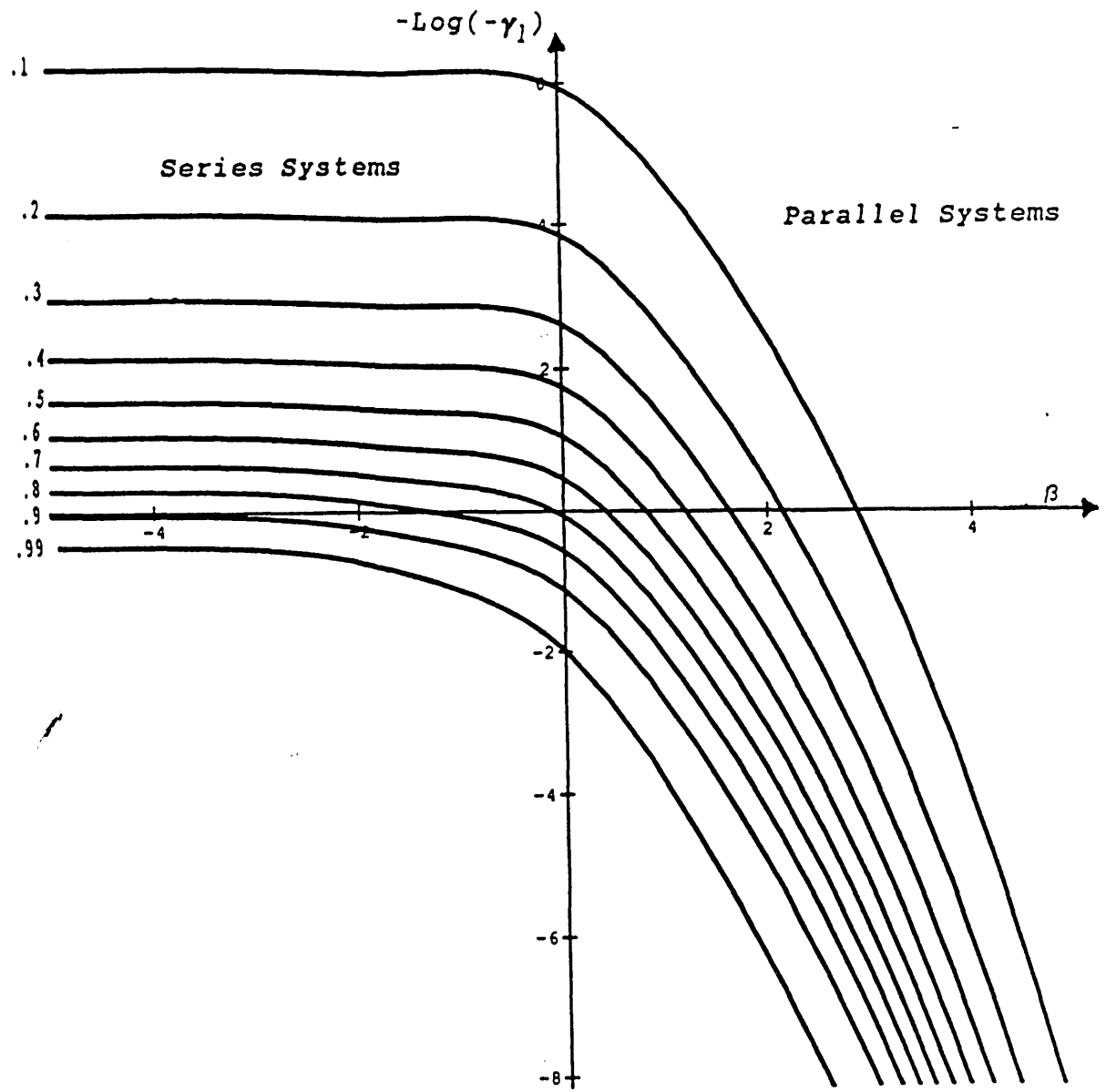
Fig. 4.1 - Skewness coefficient. Effect of β_1 for different correlations

N	c	Exact value		Method 1				Method 2				M-T Method			
		Prob.	R.I.	Prob.	R.I.	Err %	C.T.	Prob.	R.I.	Err %	C.T.	Prob.	R.I.	Err %	C.T.
10	.30	.8064E+00	.8647	.8051E+00	.8600	-.2	.11	.8047E+00	.8587	-.2	.33	.8031E+00	-.8528	-.4	.49
10	.50	.7104E+00	.5546	.7065E+00	.5432	-.6	.05	.7035E+00	.5345	-1.0	.33	.7049E+00	-.5384	-.8	.49
10	.70	.6075E+00	.2729	.6044E+00	.2647	-.5	.11	.5925E+00	.2340	-2.5	.33	.6016E+00	-.2574	-1.0	.49
10	.90	.4893E+00	.0268	.5065E+00	.0163	3.5	.11	.4770E+00	.0576	-2.5	.33	.4868E+00	.0332	-.5	.55
10	.97	.4397E+00	.1518	.4765E+00	.0589	8.4	.11	.4372E+00	.1581	-.6	.33	.1000E+01	.0000	127.4	.55
20	.30	.8847E+00	1.1987	.8834E+00	1.1924	-.1	.55	.8829E+00	1.1897	-.2	1.59	.8804E+00	-1.1770	-.5	2.25
20	.50	.7826E+00	.7810	.7766E+00	.7608	-.8	.55	.7730E+00	.7488	-1.2	1.54	.7760E+00	-.7588	-.8	2.36
20	.70	.6643E+00	.4243	.6537E+00	.3953	-1.6	.55	.6410E+00	.3611	-3.5	1.54	.6582E+00	-.4075	-.9	2.41
20	.90	.5196E+00	.0490	.5255E+00	.0641	1.2	.49	.4911E+00	.0222	-5.5	1.59	.5157E+00	-.0395	-.7	2.36
20	.97	.4480E+00	.1306	.4836E+00	.0410	7.9	.55	.4308E+00	.1744	-3.9	1.59	.1000E+01	.0000	123.2	2.36
30	.30	.9466E+00	1.6130	.9464E+00	1.6111	.0	1.87	.9462E+00	1.6090	.0	4.18	.9442E+00	-1.5909	-.3	5.82
30	.50	.8612E+00	1.0858	.8557E+00	1.0613	-.6	1.86	.8532E+00	1.0504	-.9	4.23	.8566E+00	-1.0653	-.5	6.09
30	.70	.7429E+00	.6524	.7265E+00	.6022	-2.2	1.87	.7144E+00	.5663	-3.8	4.23	.7374E+00	-.6352	-.8	6.15
30	.90	.5790E+00	.1992	.5721E+00	.1818	-1.2	1.87	.5322E+00	.0809	-8.1	4.29	.5732E+00	-.1846	-1.0	6.09
30	.97	.4910E+00	.0225	.5195E+00	.0488	5.8	1.87	.4562E+00	.1100	-7.1	4.28	.1000E+01	.0000	103.7	6.21
40	.30	.9840E+00	2.1455	.9847E+00	2.1626	.1	4.67	.9847E+00	2.1624	.1	8.84	.9831E+00	-2.1236	-.1	12.25
40	.50	.9321E+00	1.4918	.9302E+00	1.4776	-.2	4.67	.9295E+00	1.4719	-.3	8.90	.9294E+00	-1.4714	-.3	12.75
40	.70	.8349E+00	.9736	.8183E+00	.9088	-2.0	4.67	.8101E+00	.8781	-3.0	8.90	.8299E+00	-.9539	-.6	12.85
40	.90	.6679E+00	.4341	.6460E+00	.3745	-3.3	4.67	.6011E+00	.2563	-10.0	8.95	.6609E+00	-.4150	-1.0	12.85
40	.97	.5644E+00	.1622	.5818E+00	.2066	3.1	4.67	.5143E+00	.0359	-8.9	8.95	.1000E+01	.0000	77.2	12.85
50	.30	.9823E+00	2.1027	.9831E+00	2.1225	.1	9.77	.9830E+00	2.1209	.1	16.48	.9810E+00	-2.0746	-.1	22.90
50	.50	.9238E+00	1.4311	.9213E+00	1.4139	-.3	9.83	.9197E+00	1.4029	-.4	16.48	.9199E+00	-1.4047	-.4	23.67
50	.70	.8159E+00	.8999	.7961E+00	.8277	-2.4	9.78	.7835E+00	.7841	-4.0	16.48	.8093E+00	-.8752	-.8	23.94
50	.90	.6362E+00	.3482	.6125E+00	.2858	-3.7	9.83	.5597E+00	.1501	-12.0	16.59	.6276E+00	-.3254	-1.4	23.83
50	.97	.5285E+00	.0714	.5470E+00	.1181	3.5	9.77	.4706E+00	.0737	-10.9	16.59	.1000E+01	.0000	89.2	23.72

Table 4.1. Equicorrelated series systems, $\beta < 1.5$.

N	c	Exact value		Method 1				Method 2				Interpolation				T-M Method			
		Prob.	R.I.	Prob.	R.I.	Err %	C.T.	Prob.	R.I.	Err %	C.T.	Prob.	R.I.	Err %	C.T.	Prob.	R.I.	Err %	C.T.
10	.30	.4603E-04	3.9106	.4612E-04	3.9101	.2	.11	.4603E-04	3.9106	.0	.33	.4603E-04	3.9106	.0	.33	.4603E-04	3.9106	.0	.44
10	.50	.4512E-04	3.9154	.4609E-04	3.9103	2.1	.11	.4504E-04	3.9159	-.2	.33	.4504E-04	3.9159	-.2	.32	.4504E-04	3.9159	-.2	.49
10	.70	.4095E-04	3.9388	.4603E-04	3.9106	12.4	.11	.3987E-04	3.9452	-2.6	.33	.4135E-04	3.9364	1.0	.33	.3997E-04	3.9446	-2.4	.55
10	.90	.2870E-04	4.0233	.4583E-04	3.9117	59.7	.11	.2656E-04	4.0415	-7.5	.33	.2957E-04	4.0162	3.0	.33	.2754E-04	4.0329	-4.0	.55
10	.97	.2112E-04	4.0949	.4563E-04	3.9127	116.0	.11	.2050E-04	4.1018	-3.0	.33	.2358E-04	4.0693	11.6	.33	.2091E-04	4.0972	-1.0	.55
20	.30	.1135E-06	5.1758	.1135E-06	5.1758	.0	.55	.1135E-06	5.1758	.0	1.54	.1135E-06	5.1758	.0	1.59	.1135E-06	5.1758	.0	2.14
20	.50	.1130E-06	5.1766	.1135E-06	5.1758	.4	.55	.1130E-06	5.1766	.0	1.59	.1130E-06	5.1766	.0	1.54	.1130E-06	5.1766	.0	2.14
20	.70	.1068E-06	5.1872	.1135E-06	5.1758	6.3	.49	.1049E-06	5.1905	-1.8	1.59	.1075E-06	5.1859	.7	1.60	.1049E-06	5.1905	-1.8	2.31
20	.90	.7120E-07	5.2621	.1134E-06	5.1759	59.3	.55	.6059E-07	5.2917	-14.9	1.59	.7051E-07	5.2639	-1.0	1.59	.6390E-07	5.2820	-10.2	2.42
20	.97	.4295E-07	5.3543	.1133E-06	5.1760	163.9	.55	.3606E-07	5.3858	-16.0	1.59	.4608E-07	5.3415	7.3	1.59	.3996E-07	5.3673	-7.0	2.47
30	.30	.1312E-04	4.2038	.1315E-04	4.2034	.2	1.81	.1312E-04	4.2038	.0	4.17	.1312E-04	4.2038	.0	4.23	.1312E-04	4.2038	.0	5.77
30	.50	.1278E-04	4.2099	.1314E-04	4.2034	2.9	1.81	.1271E-04	4.2110	-.5	4.17	.1271E-04	4.2110	-.5	4.23	.1271E-04	4.2110	-.5	5.82
30	.70	.1096E-04	4.2445	.1314E-04	4.2035	19.9	1.87	.1005E-04	4.2638	-8.3	4.23	.1102E-04	4.2431	.6	4.29	.1012E-04	4.2622	-7.6	6.15
30	.90	.5992E-05	4.3779	.1313E-04	4.2037	119.1	1.87	.4347E-05	4.4473	-27.5	4.23	.5788E-05	4.3854	-3.4	4.22	.4978E-05	4.4181	-16.9	6.37
30	.97	.3279E-05	4.5076	.1312E-04	4.2038	300.2	1.87	.2377E-05	4.5754	-27.5	4.29	.3528E-05	4.4920	7.6	4.23	.2893E-05	4.5340	-11.8	6.43
40	.30	.9175E-01	1.3301	.9420E-01	1.3153	2.7	4.67	.8851E-01	1.3500	-3.5	8.90	.8851E-01	1.3500	-3.5	8.95	.8802E-01	1.3530	-4.1	12.75
40	.50	.6883E-01	1.4846	.7611E-01	1.4317	10.6	4.66	.5866E-01	1.5661	-14.8	8.95	.5866E-01	1.5661	-14.8	8.95	.6100E-01	1.5464	-11.4	12.97
40	.70	.4563E-01	1.6887	.6017E-01	1.5533	31.9	4.67	.3176E-01	1.8555	-30.4	8.95	.4055E-01	1.7443	-11.1	8.96	.3817E-01	1.7724	-16.4	13.24
40	.90	.2253E-01	2.0041	.4716E-01	1.6731	109.3	4.67	.1431E-01	2.1888	-36.5	9.01	.2065E-01	2.0406	-8.4	8.95	.1938E-01	2.0667	-14.0	13.23
40	.97	.1354E-01	2.2103	.4329E-01	1.7137	219.7	4.67	.9885E-02	2.3307	-27.0	8.96	.1489E-01	2.1729	10.0	8.95	.1000E+01	.0000	*****	13.01
50	.30	.2845E-03	3.4460	.2880E-03	3.4427	1.2	9.84	.2842E-03	3.4463	-.1	16.48	.2842E-03	3.4463	-.1	16.59	.2842E-03	3.4463	-.1	22.68
50	.50	.2596E-03	3.4706	.2866E-03	3.4440	10.4	9.78	.2495E-03	3.4813	-3.9	16.48	.2495E-03	3.4813	-3.9	16.53	.2498E-03	3.4810	-3.8	23.56
50	.70	.1925E-03	3.5501	.2847E-03	3.4458	47.9	9.83	.1465E-03	3.6214	-23.9	16.47	.1924E-03	3.5502	.0	16.53	.1529E-03	3.6103	-20.6	24.33
50	.90	.8458E-04	3.7611	.2822E-03	3.4481	233.7	9.84	.4534E-04	3.9143	-46.4	16.59	.7928E-04	3.7773	-6.3	16.53	.6048E-04	3.8442	-28.5	24.38
50	.97	.4081E-04	3.9396	.2812E-03	3.4491	589.1	9.77	.2402E-04	4.0650	-41.1	16.59	.4534E-04	3.9143	11.1	16.53	*********	*******	*****	*****

Table 4.2. Equicorrelated series systems, $\beta > 1.5$

N	Dir. Simulation Pr/R.I.	C.T.	Bounds L.B.	U.B.	C.T.	Method 1 Pr/R.I.	C.T.	Method 2 Pr/R.I.	C.T.	Interpolation Pr/R.I.	C.T.	M-T estimate Pr/R.I.	C.T.
10	.4428E-04	31.5	.3507E-04	.3799E-04	.22	.4200E-04	.06	.3604E-04	.33	.3741E-04	.33	.3571E-04	.50
	3.9200					3.9327		3.9693		3.9604		3.9715	
10	.1101E-04	28.8	.1093E-04	.1190E-04	.22	.1477E-04	.06	.1126E-04	.33	.1203E-04	.33	.1114E-04	.49
	4.2434					4.1769		4.2384		4.2234		4.2407	
10	.2313E-02	28.2	.2396E-02	.2684E-02	.22	.3067E-02	.05	.2512E-02	.33	.2637E-02	.33	.2482E-02	.54
	2.8319					2.7405		2.8054		2.7898		2.8093	
20	.7726E-02	131.8	.5265E-02	.9856E-02	1.70	.1093E-01	.54	.6680E-02	1.60	.7637E-02	1.59	.6475E-02	2.42
	2.4216					2.2929		2.4740		2.4258		2.4852	
20	.6094E-02	128.6	.4630E-02	.8217E-02	1.75	.9539E-02	.50	.5851E-02	1.60	.6701E-02	1.59	.5691E-02	2.47
	2.5066					2.3440		2.5210		2.4729		2.5307	
20	.2670E-02	124.0	.2261E-02	.3150E-02	1.76	.3723E-02	.56	.2608E-02	1.60	.2882E-02	1.59	.2540E-02	2.36
	2.7857					2.6762		2.7934		2.7610		2.8020	
30	.8846E-06	324.4	.9361E-06	.1035E-05	6.92	.1038E-05	1.86	.9492E-06	4.23	.9796E-06	4.29	.9444E-06	5.82
	4.7781					4.7460		4.7639		4.7576		4.7650	
30	.8471E-02	349.8	.5806E-02	.1404E-01	6.87	.1203E-01	1.82	.7664E-02	4.28	.8802E-02	4.28	.7212E-02	6.31
	2.3879					2.2561		2.4246		2.3738		2.4465	
30	.7124E-02	327.2	.5368E-02	.1182E-01	6.86	.1036E-01	1.81	.6624E-02	4.28	.7636E-02	4.23	.6217E-02	6.32
	2.4510					2.3132		2.4770		2.4259		2.4996	
40	.1178E-01	727.3	.6143E-02	.1616E-01	19.67	.1374E-01	4.67	.8502E-02	9.01	.1015E-01	9.01	.7971E-02	13.07
	2.2643					2.2045		2.3866		2.3207		2.4103	
40	.6701E-05	564.5	.7018E-05	.8674E-05	19.77	.8777E-05	4.71	.7315E-05	8.95	.7879E-05	9.06	.7227E-05	12.31
	4.3534					4.2939		4.3342		4.3178		4.3369	
40	.8445E-02	677.4	.5186E-02	.1551E-01	19.61	.1312E-01	4.73	.7122E-02	9.00	.8881E-02	9.00	.6625E-02	13.23
	2.3891					2.2227		2.4510		2.3705		2.4770	
50	.2652E-03	1052.3	.1829E-03	.3565E-03	45.76	.3632E-03	9.83	.2195E-03	16.64	.2697E-03	16.70	.2041E-03	23.72
	3.4649					3.3794		3.5155		3.4604		3.5348	
50	.1240E-04	968.3	.1081E-04	.1574E-04	45.87	.1677E-04	9.94	.1209E-04	16.59	.1397E-04	16.70	.1174E-04	23.12
	4.2166					4.1480		4.2223		4.1896		4.2289	
50	.1010E-01	1095.3	.5610E-02	.2041E-01	45.70	.1695E-01	9.83	.8287E-02	16.65	.1104E-01	16.64	.7480E-02	24.55
	2.3227					2.1214		2.3960		2.2889		2.4333	

Table 4.3. General series systems. $\beta > 1.5$.

N	c	Exact value		Method 1				Method 2				M-T Method			
		Prob.	R.I.	Prob.	R.I.	Err %	C.T.	Prob.	R.I.	Err %	C.T.	Prob.	R.I.	Err %	C.T.
10	.30	.1666E-02	2.9354	.1628E-02	2.9425	-2.3	.11	.1621E-02	2.9438	-2.7	.33	.1594E-02	2.9491	-4.3	.44
10	.50	.9732E-02	2.3365	.9725E-02	2.3368	-.1	.11	.9632E-02	2.3404	-1.0	.33	.9365E-02	2.3509	-3.8	.44
10	.70	.3078E-01	1.8695	.3203E-01	1.8517	4.1	.05	.3178E-01	1.8553	3.3	.33	.3055E-01	1.8728	-.7	.49
10	.90	.7757E-01	1.4216	.7989E-01	1.4058	3.0	.11	.8302E-01	1.3850	7.0	.33	.7969E-01	1.4072	2.7	.49
10	.97	.1088E+00	1.2331	.1026E+00	1.2666	-5.6	.06	.1129E+00	1.2111	3.8	.33	.1105E+00	1.2239	1.6	.55
20	.30	.7490E-04	3.7914	.6662E-04	3.8204	-11.0	.55	.6627E-04	3.8217	-11.5	1.59	.5912E-04	3.8497	-21.1	2.15
20	.50	.1586E-02	2.9506	.1489E-02	2.9699	-6.1	.55	.1473E-02	2.9734	-7.1	1.54	.1321E-02	3.0065	-16.7	2.14
20	.70	.9658E-02	2.3394	.9954E-02	2.3281	3.1	.55	.9845E-02	2.3322	1.9	1.59	.9017E-02	2.3649	-6.6	2.31
20	.90	.3770E-01	1.7780	.4043E-01	1.7458	7.2	.49	.4207E-01	1.7272	11.6	1.59	.3853E-01	1.7679	2.2	2.42
20	.97	.6143E-01	1.5428	.5936E-01	1.5602	-3.4	.55	.6537E-01	1.5112	6.4	1.59	.6219E-01	1.5367	1.2	2.41
30	.30	.3084E-04	4.0063	.2583E-04	4.0480	-16.2	1.87	.2571E-04	4.0490	-16.6	4.17	.2123E-04	4.0937	-31.2	5.60
30	.50	.1037E-02	3.0794	.9416E-03	3.1081	-9.2	1.87	.9341E-03	3.1104	-9.9	4.18	.8143E-03	3.1507	-21.5	5.66
30	.70	.7678E-02	2.4239	.7818E-02	2.4173	1.8	1.82	.7758E-02	2.4201	1.1	4.23	.7210E-02	2.4466	-6.1	5.93
30	.90	.3279E-01	1.8413	.3572E-01	1.8027	8.9	1.81	.3651E-01	1.7927	11.4	4.23	.3415E-01	1.8230	4.2	6.16
30	.97	.5469E-01	1.6010	.5466E-01	1.6013	-.1	1.87	.5942E-01	1.5596	8.7	4.29	.5582E-01	1.5909	2.1	6.15
40	.30	.5226E-04	3.8799	.4302E-04	3.9269	-17.7	4.67	.4285E-04	3.9279	-18.0	8.95	.3780E-04	3.9579	-27.7	11.98
40	.50	.1609E-02	2.9461	.1467E-02	2.9746	-8.8	4.67	.1459E-02	2.9763	-9.3	8.90	.1351E-02	2.9998	-16.1	12.03
40	.70	.1106E-01	2.2883	.1140E-01	2.2767	3.1	4.67	.1137E-01	2.2779	2.8	8.95	.1092E-01	2.2932	-1.3	12.52
40	.90	.4455E-01	1.7001	.4843E-01	1.6602	8.7	4.73	.5056E-01	1.6394	13.5	8.95	.4685E-01	1.6761	5.2	12.69
40	.97	.7337E-01	1.4512	.7157E-01	1.4642	-2.4	4.67	.7982E-01	1.4063	8.8	8.95	.7550E-01	1.4360	2.9	12.79
50	.30	.1459E-04	4.1798	.1092E-04	4.2451	-25.1	9.84	.1088E-04	4.2460	-25.4	16.48	.8927E-05	4.2902	-38.8	22.51
50	.50	.8024E-03	3.1550	.6971E-03	3.1958	-13.1	9.83	.6942E-03	3.1971	-13.5	16.48	.6265E-03	3.2265	-21.9	22.57
50	.70	.7291E-02	2.4426	.7427E-02	2.4359	1.9	9.83	.7448E-02	2.4349	2.2	16.53	.7173E-02	2.4485	-1.6	23.24
50	.90	.3466E-01	1.8163	.3775E-01	1.7774	8.9	9.83	.3949E-01	1.7566	13.9	16.53	.3671E-01	1.7902	5.9	23.62
50	.97	.5883E-01	1.5647	.5775E-01	1.5740	-1.8	9.78	.6327E-01	1.5279	7.6	16.64	.5987E-01	1.5558	1.8	23.62

Table 4.4. Equicorrelated parallel systems, $\beta < 1.5$.

N	c	Exact value		Method 1				Method 2				T-M Method			
		Prob.	R.I.	Prob.	R.I.	Err %	C.T.	Prob.	R.I.	Err %	C.T.	Prob.	R.I.	Err %	C.T.
10	.30	.5211E-08	5.7237	.4736E-08	5.7399	-9.1	.11	.4740E-08	5.7398	-9.0	.33	.3710E-08	5.7812	-28.8	.44
10	.50	.1188E-05	4.7185	.1095E-05	4.7351	-7.9	.06	.1090E-05	4.7359	-8.2	.33	.7956E-06	4.7994	-33.0	.49
10	.70	.2931E-04	4.0183	.2882E-04	4.0223	-1.7	.11	.2847E-04	4.0252	-2.9	.33	.2238E-04	4.0814	-23.6	.50
10	.90	.2902E-03	3.4406	.3000E-03	3.4316	3.4	.11	.3016E-03	3.4302	3.9	.32	*********	.0000	*****	.06
10	.97	.5634E-03	3.2568	.5487E-03	3.2643	-2.6	.06	.5708E-03	3.2531	1.3	.33	*********	.0000	*****	.05
20	.30	.1514E-10	6.6452	.1115E-10	6.6901	-26.4	.50	.1118E-10	6.6897	-26.1	1.54	.3856E-11	6.8438	-74.5	2.19
20	.50	.4687E-07	5.3385	.3608E-07	5.3857	-23.0	.55	.3606E-07	5.3858	-23.1	1.54	.1252E-07	5.5729	-73.3	2.20
20	.70	.3881E-05	4.4716	.3404E-05	4.4996	-12.3	.55	.3358E-05	4.5025	-13.5	1.59	.1788E-05	4.6347	-53.9	2.25
20	.90	.8729E-04	3.7532	.9291E-04	3.7375	6.4	.55	.9090E-04	3.7431	4.1	1.54	*********	.0000	*****	.11
20	.97	.2491E-03	3.4817	.2612E-03	3.4690	4.8	.55	.2673E-03	3.4628	7.3	1.59	*********	.0000	*****	.11
30	.30	.2563E-09	6.2152	.1978E-09	6.2558	-22.8	1.87	.1978E-09	6.2557	-22.8	4.23	.5517E-10	6.4521	-78.5	5.66
30	.50	.3785E-06	4.9461	.3229E-06	4.9770	-14.7	1.86	.3229E-06	4.9770	-14.7	4.18	.1121E-06	5.1781	-70.4	5.77
30	.70	.1805E-04	4.1312	.1728E-04	4.1411	-4.2	1.87	.1720E-04	4.1422	-4.7	4.23	.1035E-04	4.2572	-42.6	5.77
30	.90	.2297E-03	3.5034	.2402E-03	3.4914	4.6	1.87	.2385E-03	3.4934	3.8	4.23	*********	.0000	*****	.27
30	.97	.4926E-03	3.2947	.4931E-03	3.2944	.1	1.86	.5050E-03	3.2877	2.5	4.23	*********	.0000	*****	.22
40	.30	.2055E-10	6.6000	.1343E-10	6.6628	-34.7	4.67	.1345E-10	6.6627	-34.6	8.90	.1944E-11	6.9412	-90.5	12.03
40	.50	.9665E-07	5.2057	.7262E-07	5.2585	-24.9	4.67	.7261E-07	5.2585	-24.9	8.90	.1670E-07	5.5226	-82.7	12.14
40	.70	.8134E-05	4.3108	.7255E-05	4.3360	-10.8	4.67	.7197E-05	4.3377	-11.5	8.96	.3813E-05	4.4754	-53.1	12.25
40	.90	.1574E-03	3.6028	.1673E-03	3.5869	6.3	4.67	.1640E-03	3.5921	4.2	8.95	*********	.0000	*****	.38
40	.97	.4056E-03	3.3489	.4253E-03	3.3358	4.9	4.67	.4308E-03	3.3322	6.2	9.01	*********	.0000	*****	.39
50	.30	.1059E-10	6.6976	.4912E-11	6.8090	-53.6	9.84	.4902E-11	6.8093	-53.7	16.42	.4678E-12	7.1397	-95.6	22.46
50	.50	.1041E-06	5.1918	.6516E-07	5.2784	-37.4	9.84	.6469E-07	5.2797	-37.9	16.48	.1079E-07	5.5989	-89.6	22.68
50	.70	.1262E-04	4.2126	.1101E-04	4.2434	-12.8	9.83	.1091E-04	4.2454	-13.6	16.53	.4286E-05	4.4504	-66.0	22.80
50	.90	.3115E-03	3.4214	.3354E-03	3.4013	7.7	9.83	.3456E-03	3.3930	11.0	16.53	*********	.0000	*****	1.16
50	.97	.7820E-03	3.1625	.7587E-03	3.1713	-3.0	9.83	.8216E-03	3.1481	5.1	16.64	*********	.0000	*****	.66

Table 4.5. Equicorrelated parallel systems, $\beta > 1.5$.

N	c	Estimate	Exact value	Error %	CPU (sec)
10	.30	.0000000828	.0000000812	2.04	5.00
10	.70	.0000000827	.0000000783	5.62	5.16
10	.90	.0000000814	.0000000593	37.39	5.39
10	.97	.0000000802	.0000000406	97.32	5.50
20	.30	.0093803064	.0088112171	6.46	47.62
20	.70	.0053704273	.0057554262	-6.69	64.98
20	.90	.0030503857	.0032444067	-5.98	75.69
20	.97	.0021670129	.0021653750	.08	79.53
40	.30	.0072928602	.0067792989	7.58	415.51
40	.70	.0057361021	.0040867685	40.36	488.72
40	.90	.0022264505	.0020162515	10.43	628.57
40	.97	.0012634876	.0011552814	9.37	645.43

Table. 4.6 Trisections method. Series systems
Beta > 1.5

CHAPTER V

SYSTEM RELIABILITY BOUNDS

5.0 INTRODUCTION

An alternative to direct integration of the joint probability density over the failure region, is the bounding techniques. Instead of formulating the probability of failure as a multiple integral, it may be viewed as the probability of occurrence of a union of a discrete set of events . Each event F_i corresponds to a failure mode, expressed in terms of a limit-state function (or a safety margin)

$$g_i(x) \leq 0 \qquad (5.1)$$

and the probability of its occurrence is defined as

$$P(F_i) = P(\ g_i(x) \leq 0) \qquad (5.2)$$

The overall probability of failure is formulated as a series system probability

$$p_f = P(\ \bigcup_{i}^{n} F_i\ ) \qquad (5.3)$$

The evaluation of p_f is replaced by the search for an upper and lower bound of its exact value, which are close enough to each other for practical purposes.

The exact expression of the probability of union of events, by inclusion-exclusion theorem, is

$$P(\bigcup_{i=1}^{n}) = \sum_{i=1}^{n} P(F_i) - \sum_{j<i} P(F_i \cap F_j) + \sum_{k<j<i} P(F_i \cap F_j \cap F_k)$$
$$- \sum_{l<k<j<i} P(F_i \cap F_j \cap F_k \cap F_l) \ldots + (-1)^{n-1} P(F_1 \cap F_2 \cap \ldots \cap F_n) \qquad (5.4)$$

The bounds can be formulated from this expression by truncation at some m^{th} order. It can be shown that truncation after an even order sum (i.e. a sum of intersections of even number of events) would result in a lower bound. Conversely, a truncation after an odd order sum would yield an upper bound. This assumes that the probability of intersection of the events, up to the mth order are known, or can be evaluated at an acceptable cost, which is rarely the case. A computationally viable truncation can be made only up to the second or third order at most. Because partial truncations up to the second order can produce better bounds than full truncations of third order, much of the effort on bounds improvements has been devoted to second order bounds, and the techniques of partial truncations. By m^{th} order full truncation, it is understood here that any intersection up to the mth order is included and none is discarded. The converse is true for partial truncations.

In the following are reviewed different formulations of the first, second and third order bounds, and their possible improvements by suitable ordering of the events.

5.1 FIRST-ORDER BOUNDS

The earliest known bounds, and the simplest, are due to Boole [1854], and are referred to as Boolean bounds. The probability of individual events only are involved in these bounds

$$\underset{i=1}{\overset{n}{Max}} P(F_i) \leq P(\bigcup_{i=1}^{n} F_i) \leq \sum_{i=1}^{n} P(F_i) \qquad (5.5)$$

The lower bound coincides with the exact result if the events are fully correlated. If the events are fully independent (or mutually exclusive), then the upper bound provides the exact result. As the failure modes in structural systems are highly correlated, the gap between the bounds is often too wide to provide an acceptable estimate of the probability.

5.2 SECOND-ORDER BOUNDS

The first lower bound to include bisections (pairwise intersection between events) is due to Bonferroni [1936]. It consists of the straightforward truncation of the right hand side expression of equation 5.4 after the second-order term over all subscripts $i < j$, in an unselective way. The upper bound is the same as the first-order Boolean bound. This gives

$$P(F_1) + \sum_{i=2}^{n} [P(F_i) - \sum_{j=1}^{i-1} P(F_i \cap F_j)] \leq P(\bigcup_{i=1}^{n} F_i) \leq \sum_{i=1}^{n} P(F_i) \qquad (5.6)$$

5.2.1 Improvements on the lower bound

The lower bound can be 'too low' if some of the terms in the left hand side of Equation 5.6 become negative. To overcome such problem and to improve the bounds, several attempts have been made during the fifties and the sixties [Chung & Erdos 1952; Whittle 1959; Gallot 1966; Dawson & Sankoff 1967]. But in all these attempts the effect of the events labelling seems to be overlooked. Kounias [1968] put forward a formulation, with a truncation across the first and second-order terms

$$P_f \geq MAX_J [\sum_{i \in J} P_i - \sum_{i < j, \ i,j \in J} P_{ij}] \qquad (5.7)$$

where

$$J \subseteq \{1, 2, \ldots, n\} \qquad (5.8)$$

This is the first formulation to take account of the numbering, despite the fact that this does not appear explicitly in the formula, as will be shown later. At first sight, this formulation looks too combinatorial to be of any practical use, as the identification of the optimal subset(s) J_0 among all subsets J seemed to be a formidable task. In a later work, Kounias et al. [1976] formulated the second-order lower bound as an objective function of a linear programming problem as follows

$$Maximise \qquad b_0 + \sum_{1 \leq i \leq n} b_i P_i - \sum_{1 \leq i \leq j \leq n} b_{ij} P_{ij} \qquad (5.9a)$$

Subject to :

$$b_0 + \sum_{i \in J_r} b_i - \sum_{i<j\,(i,j \in J_r)} b_{ij} \leq 1 \qquad b_0 \leq 0 \qquad (5.9b)$$

where $J_r \subset \{1,2,\ldots,n\}$ and r the number of elements in J_r.

The number of constraints is equal to 2^n (number of all subsets of a set of size n), n being the number of events. This approach becomes rapidly too expensive with increase in n, for the identification of all possible solutions. Nevertheless, improved lower bounds can be obtained from the above formulation. The aim, although less ambitious, is to find points of high objective function value, inside or on the boundaries of the feasible domain defined by the inequalities 5.9b. Each feasible point, which will be referred to as a lower bound point (LBP) in the sequel, yields a class of lower bounds, which can be optimised over all permutations of J_n. This suboptimisation is merely a relabelling of the events for the given LBP. For the particular case where

$$b_i = a \quad if \ i \in J_r \ , \quad b_i = 0 \ otherwise$$
$$b_{ij} = b \quad if \ i,j \in J_r \ , \quad b_{i,j} = 0 \ otherwise \qquad (5.10)$$

this renumbering is limited to selecting the r events contributing to the lower bound, which is independent of the ordering within J_r. In [Kounias, 1976], some vertices of the feasible domain are given, and correspond more or less to the case described by equation 5.7.

It is important to mention here that almost all second-order

lower bound formulations published by different authors before or after the publication of [Kounias, 1976], are particular solutions to the Kounias linear programming problem. The so called Ditlevsen lower bound (DLB) [1979b] in structural reliability

$$P_f \leq P_1 + \sum_{i=2}^{n} MAX[(P_i - \sum_{j=1}^{i-1} P_{ij}), 0] \qquad (5.11)$$

assumes some ordering. Its wide use in structural reliability is believed to be due to its simplicity. Ditlevsen used indicator functions to derive this bound. The search for a better feasible points within the linear programming framework has been inhibited, because of the lack on one hand of a general computationally viable method of feasibility check, and on the other hand of an optimal ordering algorithm to make the best use of any LBP. The latter problem is tackled for the LBP corresponding to DLB, in the following subsection where an optimal ordering method is proposed. But the feasibility check is still a challenging topic. It is believed that the identification of lower bound points of straightforward feasibility check, as did Kounias, is the most computationally viable way of lower bound improvement. In the following example with four events and with three LBP are given for illustrative purpose. The probability matrix is

$$\begin{pmatrix} .424 & .360 & .272 & .360 \\ .360 & .408 & .208 & .296 \\ .272 & .208 & .416 & .272 \\ .360 & .296 & .272 & .544 \end{pmatrix} \qquad (5.12)$$

The following are three LBP and their corresponding lower bound values

$$
\begin{pmatrix}
4/5 & & & \\
3/5 & 4/5 & & \\
1/5 & 1/5 & 2/5 & \\
2/5 & 2/5 & 1/5 & 3/5
\end{pmatrix}
\begin{pmatrix}
6/7 & & & \\
5/7 & 6/7 & & \\
1/7 & 1/7 & 2/7 & \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & & & \\
1 & 1 & & \\
1 & 1 & 1 & \\
0 & 0 & 0 & 0
\end{pmatrix}
\qquad (5.13)
$$

$$
[\quad 0.5304 \qquad\qquad 0.5063 \qquad\qquad 0.4080 \quad]
$$

It can be easily seen that any single permutation in the order of the events would give different lower bound value, as none of the above LBP 's is found by solving the system 5.9, but are arbitrarily selected feasible points. This confirms that among all identified LBP's an optimal ordering is to be found for each one of them and to select the highest bound.

Theoretically, any LBP is a potential optimal lower bound with the right ordering. However, the family of Bonferroni-type LBP's , i.e feasible points with only 1's and 0's, are vertices of the feasible domain and are therefore potential global optimal solutions to the linear programming system 5.9. The LBP of the form

$$
\begin{pmatrix}
1 & & & & \\
1 & 1 & & & \\
1 & 1 & 1 & & \\
. & . & . & . & . \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\qquad (5.14)
$$

can be made (sub)optimal by putting the events of most contribution at the top of the events list, leading to an optimal form of DLB, or to the selection of the optimal subset J_0 in the first Kounias LB of Equation 5.7. This means that the order in which the elements of J_0 appear in the events list is immaterial for the value of the bound. For all other numberings, DLB yields a smaller value than Kounias's. This can be made clear in a different way as follows:

Let J_D be the subset of events with positive contribution (i.e. with positive outer sum terms in Equation 5.6), for some arbitrary numbering. Then for every event of J_D , it can be easily seen that

$$P_i - \sum_{j=1}^{i-1} P_{ij} \leq P_i - \sum_{j=1, j \in J_0}^{i-1} P_{ij} \qquad (5.15)$$

and that

$$J_D \subseteq J_0$$

i.e. any event, with positive contribution in Bonferroni LB, is a necessarily part of Kounias's LB optimal subset J_0 and its contribution for the latter is at least equal to that of the former if not higher. On the other hand any member of J_0 does not necessarily make a positive contribution to Bonferroni LB or Ditlevsen LB, as more bisections are substracted (involving events not in J_0).

The optimal subset J_0

$$J_0 = \{i_1, i_2, \ldots \ldots, i_k\} \qquad k \leq n \qquad (5.16)$$

can be found using the following method. Its suboptimality is shown in the light of Kounias linear programming formulation. The events are represented by a graph where the vertices represent the events probabilities and the arcs represent the intersection probabilities. A subgraph is built up from the whole graph of events, such that at each iteration, the new vertex i added and corresponds to the maximal value of the quantity

$$P_i - \sum_{j=1}^{i-1} P_{ij} \qquad (5.17)$$

where the j's correspond to the events already selected. The process is repeated until the quantity above is negative. This algorithm has the following features:

• the events are ordered according to their contribution in the new numbering,

• the optimal contribution is selected at each stage. At the lth stage, l events have been so far renumbered such that any single substitution by another non renumbered event would reduce the current value of the bound. The only way of improving the bound is by <u>adding</u> new events with positive contribution. The algorithms is stopped if there is no more events with positive contribution to be added. In linear programming terms, the LBP represented in Equation 5.14, which corresponds to the numbering which put the l selected events first on the list, is built up by increasing the value of l new coordinates from 0 to 1. If the feasible domain is restricted to its intersection with the unit cube $[0,1]^n$, at

each stage *l*, the current LBP is a vertex of this reduced domain. This amounts to moving from one vertex to another by adding one event at a time, ensuring the best improvement of the bound, which corresponds to shifting the objective hyperplane parallel to itself in the direction to its gradient. At the last stage the optimal vertex remains the only intersection of the objective hyperplane passing through the current vertex with the feasible domain. The analogy and similarity with the simplex algorithm is a striking feature of the algorithm as the vertex-to-vertex path would be identical if the feasible domain is reduced as above. It follows that with this algorithm, the best possible subset J_0 of Kounias LB is identified.

Illustrative examples

a. Example 1

The example quoted above (Equation 5.12), is optimally reordered, by selecting the events with positive contribution as follows

- First selected event : $max_i\ P_i\ \rightarrow\ P_4$

- Second event : $max_i\ (P_i\text{-}P_{4i})\ \rightarrow\ P_3$

- Third event : $max_i\ (P_i\ -\ P_{3i}\ -\ P_{4i})\ <\ 0$

After the second selected event, the remaining others have no contribution to the lower bound, which yields

$$J_0\ -\ \{\ 3,\ 4\ \}$$

and the suboptimal lower bound is

$$P_L = P_3 + P_4 - P_{34} = .688$$

which is a significant improvement on the lower bound (.472) that would be obtained by direct application of DLB expression to the initial ordering.

b. *Example 2* Ten events with the following probability matrix (the figures correspond to 10^4 times the probabilities):

$$
\begin{bmatrix}
1.55 & & & & & & & & & \\
.29 & 4.02 & & & & & & & & \\
.12 & .46 & .65 & & & & & & & \\
.14 & .05 & .04 & .47 & & & & & & \\
.15 & .13 & .12 & .04 & 1.07 & & & & & \\
.31 & .49 & .35 & .14 & .80 & 4.28 & & & & \\
.21 & 1.33 & .47 & .03 & .87 & 2.53 & 17.52 & & & \\
.02 & .16 & .08 & .02 & .11 & .46 & .44 & .67 & & \\
.03 & .10 & .11 & .01 & .04 & .31 & .27 & .07 & 1.27 & \\
.04 & .34 & .12 & .00 & .12 & .21 & .49 & .09 & .05 & .72
\end{bmatrix}
$$

With the above ordering, it can easily be checked that the lower bound is $2.082 \cdot 10^{-3}$. With the reordering only six events have positive contribution, yielding the following probability submatrix:

$$
\begin{bmatrix}
17.52 & & & & & \\
1.33 & 4.02 & & & & \\
2.53 & .49 & 4.28 & & & \\
.21 & .29 & .31 & 1.55 & & \\
.27 & .10 & .31 & .03 & 1.27 & \\
.03 & .05 & .14 & .14 & .01 & .47
\end{bmatrix}
$$

and a lower bound of $2.287 \cdot 10^{-3}$.

The remaining question on the second-order lower bound is that which one of the two families of LBP's discussed above, can

produce the best lower bound ? Numerical experience, although limited, tends to suggest that the Bonferroni-type LBP's (0's and 1's) are the overall optimal solutions. This could be the case for some class of practical problems, but it is not easy to prove. In theory at least, no LBP can be ruled out as a potential global optimum. From the practical point of view, the optimal numbering is much costly with LBP with non-zero coefficients less than one. The above algorithm could be used to improve the bound, but the overall optimal ordering needs more involved method, as after the first event with negative contribution is found, the possibility of positive contribution in the next rows of the LBP cannot be ruled out. The number of possible combinations for the remaining events may be too high for an exhaustive search.

5.2.2 Improvements on the upper bound

The first known improvement on the upper bound (UB) is due to Kounias [1968]

$$P(\bigcup_{i=1}^{n} F_i) \le \sum_{i=1}^{n} P_i - \max_{1 \le k \le n} \sum_{i=1, i \ne k}^{n} P_{ki} \qquad (5.18)$$

Using Moses and Kinser [1967] expression of the probability, as a linear combination of the conditional probabilities of the survival for the first i-1 events given that the ith fails, Vanmarcke [1971] derived the following UB

$$P_f \le P(F_1) + \sum_{i=2}^{n} \min_{k=1}^{i-1} P(F_k^c \cap F_i) \qquad (5.19)$$

where F_k^c is the kth survival event. This UB can be easily written as

$$P_f \leq \sum_{i=1}^{n} P_i - \sum_{i=1}^{n} \max_{k=1}^{i-1} P_{ki} \qquad (5.20)$$

although it has not been presented in this form by Vanmarcke. This UB has been also derived by Ditlevsen using indicator functions [Ditlevsen, 1979b].

Hunter[1976] derived the simpler formulation

$$P_f \leq \sum_{i=1}^{n} P_i - \sum_{i=1}^{n} P_{ki} \qquad (5.21)$$

for some arbitrary $k < i$. This UB is not only dependent on the events ordering, but also on the choice of k. Obviously, for a given ordering, the choice of k would lead to the use of Vanmarcke formulation (Equation 5.19). The best ordering is the one which maximises the quantity

$$\sum_{i=1}^{n} P_{ik} \qquad (5.22)$$

Hunter used graph theory to derive the best ordering. A graph is constructed in the same way as in section (5.2.1). A connected subgraph is called <u>spanning tree</u> if it has (n-1) branches of which at lest one is incident to each of the n nodes. Hunter showed that the bisections to be included in the upper bound should form a spanning tree of all nodes. The

selection of the best ordering becomes a selection of the *best spanning tree*. Hunter made use of Kruskal algorithm [Kruskal 1956] to obtain the best spanning tree as follows :

The bisections P_{ij} are allocated to the tree in decreasing order, such that the node *i* is part of a previously selected branch, and *j* not yet selected, or vice-versa. The process is repeated until all n nodes are used up. This algorithm, despite its simplicity, is very powerful; it selects the ordering that spreads the highest bisections among the rows in the subdiagonal part of the probability matrix.

Hunter contribution can be summarised as an implicit ordering optimisation for Vanmarcke upper bound.

In [Kounias, 1976], the linear programming approach for the lower bound is also adopted for the upper bound. The upper bound linear programming system is

$$minimise : c_0 + \sum_{i=1}^{n} c_i P_i - \sum_{1 \leq i < j \leq n} c_{ij} P_{ij} \tag{5.23a}$$

subject to

$$0 \leq c_0$$

$$1 \leq c_0 + \sum_{i \in J_r} c_i - \sum_{i < j, i, j \in J_r} c_{ij} \quad \text{for all} \quad J_r \in J_n$$

$$\tag{5.23b}$$

Here again the previous UB's are feasible solutions (UBP) of the above system. A catalogue of a few UBP's is given in [Kounias, 1976]. The identification of vertices 'by inspection' is much more difficult than for the lower bound

case, which reduces the chances of improving the value of the bound.

5.3 THIRD-ORDER BOUNDS

Using indicator functions, Ramachandran [1984,1985] derived third-order bounds. The lower bound is

$$P_f \geq RL^3 = P_1 + P_2 - P_{12} + \sum_{i=3}^{n} \max\{ [P_i - \sum_{j=1}^{i-1} P_{ij} + \max\sum P_{ilk}], 0 \}$$

(5.24)

where l and k are arbitrary choices in $\{1,2,\ldots,i-1\}$, such that the corresponding branches form a spanning tree of all vertices $1,2,\ldots,i-1$. The upper bound is

$$P_f \leq RU^3 = P_1 + P_2 - P_{12} + \sum_{i=3}^{n} \{P_i - \max_{k,l<i} [P_{il} + P_{ik} - P_{ikl}]\}$$

(5.25)

As the trisections evaluation is very costly and makes the use of third-order bounds less attractive, Ramachandran proposed a nonlinear lower bound to the trisections for the special case of events defined by linear functions. This bound is

$$P(F_i \cap F_j \cap F_k) \geq \frac{P(F_i \cap F_j) \cdot P(F_i \cap F_k)}{P(F_i)}$$

(5.26)

if $\rho_{jk} > \rho_{ij}\rho_{ik}$, where ρ_{ij} is the correlation between linear functions corresponding to the events i and j. This normal trivariate bound is used for the third-order bound by replacing the third-order terms by the expression

$$\max\{ \ \frac{P_{ij}P_{ik}}{P_i} \ , \ \frac{P_{ij}P_{jk}}{P_j} \ , \ \frac{P_{ik}P_{jk}}{P_k} \ \} \tag{5.27}$$

which satisfies the correlation condition above.

5.4 HIGHER ORDER BOUNDS

Hohenbichler and Rackwitz [1983], assumed that intersections are known up to some order $l<n$, and derived a general formulation of the bound in terms of those intersections. Let J_r be the subset $\{ \ j_1,j_2,\ldots,j_r \ \}$, with $r<l$, it has been shown that an upper bound is

$$P(\bigcup_{i=1}^{n} F_i) \le O^1 - O^1(F_1,F_2,\ldots,F_n)$$

$$= P(F_1) + \sum_{i=2}^{n} \{P(F_i) - \max_{J_r,r<l} [P(\bigcup_{j \in J_r} (F_i \cap F_j))]\} \tag{5.27}$$

By considering for each i successively, the $i-1$ G_j events

$$\{ \ G_j = F_i \cap F_j / j = 1,2,\ldots,i-1 \ \} \tag{5.29}$$

and applying the above upper bound to each one, a lower bound is easily derived

$$P(\bigcup_{i=1}^{n} F_i) \geq L^l - P(F_1) + \sum_{i=2}^{n} \max\{P(F_i) - O^{l-1}(G_1, G_2, \ldots, G_i), 0\}$$

(5.30)

It is not difficult now to see that application of the above bounds to the case $l=1$ and $l=2$ yields respectively the bounds derived earlier. The third-order bounds, after a change of notation, come as follows

$$L^3 = P_1 + P_2 - P_{12} + \sum_{i=3}^{n} \max\{P_i - P_{i1} - \sum_{j=2}^{i-1} [P_{ji} - \max_{k<j} P_{jik}], 0\}$$

(5.31)

and

$$O^3 = P_1 + P_2 - P_{12} + \sum_{i=3}^{n} \{P_i - \max_{j,k;k<j<i} [P_{ij} + P_{ik} - P_{ijk}]\}$$ (5.32)

where P_{ijk} is the intersection of the events i, j, k.

5.5 BOUNDING BY CONDITIONING

Another type of strategy for bounding has been adopted by Rackwitz [1978] and Ditlevsen [1982] and is outlined below. Rackwitz approach is based on the following theorem given by Sidak[1968]:

" if $\{\rho_{ij}\}$ and $\{\gamma_{ij}\}$ are two correlation matrices such that :

$$\rho_{ij} \leq \gamma_{ij}$$

then :

$$\phi_n(\beta,\gamma) \leq \phi_n(\beta,\rho)$$

and the reverse (i.e. $\geq$) is true."

If the events can be represented in terms of linear safety margins of the form

$$Z_i = v_i U + \sqrt{1-v_i^2}\, U_i \, , \qquad i=1,2,\ldots,n \qquad (5.33)$$

where U and U_i are independent standard normal variates, and $|v_i| \leq 1$, it follows that the correlation between events is

$$\rho_{ij} = v_i v_j$$

Then the probability of failure can be expressed as [Dunnett and Sobel,1955]

$$p_f = \int_{-\infty}^{+\infty} [1 - \prod_{i=1}^{n} \phi(\frac{\beta_i - v_i u}{\sqrt{1-v_i^2}})] \, \varphi(u) \, du \qquad (5.34)$$

The probability would be exact if one can find the values of v_i's yielding the exact correlation matrix. Unfortunately, such values are the solutions of an overdetermined nonlinear system of $n(n-1)/2$ equations and n unknowns, for which solutions do not always exist. However, one can always find values such that the conditions of Sidak's theorem are satisfied, thus yielding a lower bound for the case ($v_i v_j \leq \rho_{ij}$) and an upper bound for ($v_i v_j \geq \rho_{ij}$), provided that the condition of non-negative definiteness is met. The attractiveness of the method comes from the fact that it brings down a multiple integral to a single dimensional one. But the hard bone of this approach is to find values of v_i's that reduces the gap

between the bounds, bearing in mind that different sets may be needed for the two bounds. There is no known viable technique for such a purpose. Moreover, numerical experience showed that the bounds obtained are normally wide, and a search for a better v_i's is likely to outweigh the single dimensional integral feature in term of efficiency.

In a slightly different approach, Ditlevsen assumes the variates U_i correlated and seeks the v_i's such that their correlation is minimised [Ditlevsen 1979b, 1982]. This amounts to replacing the original set of linear margins by another of less correlated ones, in order to reduce the gap between the bounds. This method has many drawbacks, which has been indicated by Ramachandran and Baker [1984]. The most important is the computing time which is about five times higher than the methods discussed above. Using bounds on the bivariate integrals may have negative effect on the gap between the bounds. Moreover, the single dimensional integral over the variable u may be very sensitive to the integration step. A further point which has not been raised is the evaluation of the constants v_i's which ensures low correlations between the equivalent safety margins; this may be a difficult and costly optimisation problem.

CHAPTER VI

OPTIMISATION OF MULTIEXTREMAL FUNCTIONS

6.0 INTRODUCTION

This chapter presents a concise review of the best known techniques used in solving unconstrained nonconvex optimisation problems, and examines the relevance of these techniques to the generation of critical load-resistance directions in structural reliability. As the aim of these methods is to find the global optimum, any other local optimum found on the way comes only as a by-product, they cannot be directly used to solve a problem in which a selection of the 'best' local optima is to be found. Therefore any method used has to be modified to suit this objective in the most efficient way. Existing methods are examined first and an algorithm adapted to the selective optimisation problem in structural reliability is presented in the last section.

Finding the global optimum to multiextremal functions has been and is still a big challenge in optimisation research. So far there is no method which can claim to solve the problem efficiently despite the restrictive assumptions often made on the objective function. The most restrictive methods are of the deterministic type with finite convergence, but with exponential increase in computational cost with the dimension of the problem. Less restrictive are the probabilistic

are the two major classes of global optimisation methods, but some methods include certain random elements within a deterministic framework to enhance their performance or to widen their range of applicability.

The problem is formalised, for a maximisation, as follows. Given the objective function f

$$f : x \ \text{------------} \rightarrow \ f(x) \in S \qquad \qquad (6.1)$$

where S is a compact subset of $\mathbf{R}^n$, find x^*, y^* such that

$$y^* = f(x^*) \geq f(x), \forall x \in S \qquad \qquad (6.2)$$

Unconstrained optimisation would mean that S is the maximal subset of $\mathbf{R}^n$ for which real function values for f exist. In most practical situations the so called unconstrained problems have the domain S as a box with simple bounds on the variables. The methods described below make the same assumption but the problem is still referred to as unconstrained one.

6.1 DETERMINISTIC METHODS

6.1.1 Space covering techniques

These methods attempt to cover the whole domain S by an exhaustive search. They are limited to Lipschitzian functions, for which there is a constant K such that

for which there is a constant K such that

$$| f(x_1) - f(x_2) | \leq K | x_1 - x_2 | \quad \forall x_1, x_2 \in S \qquad (6.3)$$

i.e. there is a bound K on the rate of change of the objective function. Moreover this bound is assumed to be known. In practice, the existence of such a constant is not always evident, and its value can be hard if not impossible to find. This is the first drawback of these methods.

The first known method of this type was the grid method of Fisher [1935] which covers S simply by identical hypercubes. A better approach is suggested by Evtushenko [1971] in which a sequence of points is generated until the domain S is covered by a set of hyperspheres centred at these points, or hypercubes inscribed in them. Under the same assumption, Shubert [1972] developed an algorithm which ,although impractical for multidimensional functions [Dixon 1975, Archetti 1984], turns out to be very efficient for single dimensional problems. It could also be used in a sequential scheme within the multidimensional case [Schoen, 1982].

Other domain-partitioning algorithms developed recently [Mayne & Polak 1984, Meewella & Mayne 1989] for Lipschitz continuous functions are claimed to perform well. Using interval arithmetic, Hansen [1980] proposed an algorithm for functions that are rational and their first and second derivative are also rational, but not necessarily Lipschitzian.

Another type of method referred to as the "regions of attraction" [Treccani et al. 1975, Corles 1975] assumes a twice continuous differentiability for the objective function. The region of attraction for a local optimum is defined as the connected subset of S from each point of which a steepest search leads to this local optimum. The domain is scanned for the hills (or the basins) corresponding to each local optimum via the search for saddle points. The neighbourhoods of the saddle points found are explored for starting points leading to new local optima.

6.1.2 Trajectory techniques

Less sensitive to the dimensionality of the problem are the trajectory methods. Trajectories of constant gradient direction are built up through the solution of a differential equation; local optima, or good starting point for local search are generated in the process. The original algorithm due to Branin [1971], laid down the path to the development and the critical study of this approach [Treccani 1975, Gomulka 1975, Hardy 1975, Griewank 1981].

6.1.3 Tunnelling method

The basic idea of this method is to generate a point with objective function value better than the local optimum last found, by solving an equation involving all the previously

found local optima. But the solution of this equation, and specially deciding if there are other roots or not, could be more difficult than the original global optimisation. The method was first published by Levy & Montalvo [1977].

6.2 STOCHASTIC METHODS

One major problem in global optimisation is the possibility of bounding the error on the optimal solution. With deterministic algorithms, this bounding can only be achieved for a restricted type of functions, of a very moderate dimensionality. To avoid this difficulty, the stochastic approach seeks an asymptotic guarantee of getting the true global optimum rather than an absolute one. This is not a serious drawback if there is no other efficient way of handling the problem. The basic idea behind the stochastic methods is that, by sampling points from some continuous distribution over the domain S, the probability of hitting any given point or its neighbourhood tends to 1 as the sample size tends to infinity [Brooks 1958; Rubinstein & Weissman 1977]. The choice of a sampling distribution, the inclusion of a local search, and the termination rules are the basic criteria behind the classification of the various stochastic methods. A few methods are outlined in the following sections.

6.2.1 Pure random search

The pure random search (or nonadaptive random search) is the

simplest method. A random sample of N points is generated from some distribution in S, and the function is evaluated at these points. The best function value is then selected as the solution of the problem. Brooks [1958] and Anderssen [1972] studied such a method and showed its asymptotic convergence for the uniform distribution. Rubinstein and Weissman [1977] studied the general case of a continuous distribution. It is obvious that the choice of non uniform distribution assumes some prior knowledge of the location of the optimal solution within S.

Let A be a neighbourhood of the optimal solution x^* and α such that

$$\alpha = \frac{m(A)}{m(S)} \tag{6.4}$$

where m is the Lebesgue measure on S (or simply volume). If a uniform sample $\{x_i\}$ of size N is drawn from S, the probability that at least one point from the sample falls within A is

$$P = 1 - (1 - \alpha)^N \tag{6.5}$$

it follows that the minimum sample size to ensure the probability level P, for a given α is

$$N^* = \frac{\log(1-P)}{\log(1-\alpha)} \tag{6.6}$$

Without prior assumptions on the objective function f, this minimum sample size does not guarantee that the optimum value of the sample falls inside A. This is a very weak point of this method as the objective value might be 'close' to the optimum value, with completely wrong location. Moreover, the probability, that the error on the optimal objective value is lower than some prefixed value, is not known. Another problem, linked to the prior knowledge about the objective function, is the acceptability of the global optimum estimate. This problem has been discussed by Anderssen [1972] who introduced the idea of hypothesis testing, by which a minimum confidence level for the value and location of the optimum should be ensured, using all available information about the physical problem.

A straightforward improvement on the crude uniform random search is the 'multistart' algorithm, where a local optimisation is carried out at every sample point. The best local optimum is selected as the estimate of the global optimum . Due to the computing cost of the local search, severe limitation on the sample size is often unavoidable. To overcome this problem, a more sophisticated algorithm, combining random sampling and a cluster analysis, has been developed (see section II.3 for more details).

6.2.2 Adaptive random search

While in the previous method the points are sampled

independently from each other, the adaptive random search generates a sequence of points with improving objective function values, and an update of the sampling distribution at each iteration. This approach has been investigated by many authors, and a bibliographic coverage can be found in [Rubinstein 1986].

Matyas [1965] appears to be the first to have developed an adaptive random search for unimodal functions and to prove its convergence in probability. Matyas's algorithm generates a sequence $\{x_k\}$ of improving objective function value, taking account of the previous observations at each stage.

Baba et al. [1977] re-examined Matyas's convergence theorem and generalised it to multiextremal functions. A comprehensive study of the adaptive random search techniques is due to Solis & Wets [1981]. They generalised the concept of adaptive search to fit most of the previously known algorithms. The sequence $\{x_k\}$ is such that

$$x_{k+1} = T(x_k, y_k)$$

(6.7)

where y_k is generated from some distribution h_k updated at each stage k. Two basic conditions are shown to ensure the convergence in probability :

- The mapping T and the sampling distribution h_k must ensure the improvability of the objective function. It can be a random or deterministic search for the local optimum.

- The distribution h_k should not ignore systematically any subset A of S.

The above convergence is shown under the condition of measurability of both S and the objective function.

It can be seen from this general formulation that the two main ingredients of any recipe for global optimisation, using an adaptive random search, are the sampling distribution h_k and the mapping T. The behaviour of any derived algorithm depends very much on the combined effect of these two ingredients.

6.2.3 Multistart methods

The simplest method is the crude multistart referred to at the end of section II.1. But without prior knowledge on the number of local optima, or the 'topography' of the function, two difficult questions arise :

- Is the estimated optimum 'close enough' to the true global one ?
- How can we ovoid converging to the same local optimum several times, as many points can be sampled from the same region of attraction ?

In stochastic methods, with asymptotic guarantee of convergence, the first question can only have a probabilistic answer formulated in stopping rules of the sequential

sampling. More details on stopping rules are given in section II.5 in the general framework of termination criteria of random search techniques.

The second question can be tackled if one can identify the regions of attraction of all local optima. This means to find contours of 'basins' and 'hills' of the objective function, which is more difficult. An alternative to exact identification, which can serve the same purpose, is to group the sample points into subsets of points belonging to the same basin (or hill). By doing so, we can virtually ensure that only one local search is carried out in each basin (hill) from which at least one point is sampled. This technique is well known in statistics, and referred to as *cluster analysis* [Everitt 1974; Hartigan 1975; Anderberg 1973].

Application of cluster techniques in conjunction with random sampling and good stopping rules, has produced relatively efficient algorithms for global optimisation.

6.2.3.1 Clustering techniques and global optimisation

Cluster Analysis is the term used to refer to techniques which seek to divide a set of N objects or elements into subsets according to contiguity or some similarity criterion. It is widely used in statistics for data analysis. Clustering techniques are as various as their users, but the ones

considered in the following are those which have been used as a part of some global optimisation algorithms. Becker & Lago [1970] are reported to have put forward the idea of clustering the points around the local optima in order to avoid unnecessary function evaluations and local searches, resulting in a substantial savings in the computational cost. From a uniformly drawn sample, regions of high density are created around the local optima by removing a prefixed proportion of the sample points with lowest (highest) function values. This is known as the *sample reduction technique*. In the ideal case, reduction would create a clearly distinct sub-agglomerations of sample points leading to the same local optima. But in practice, very often these sub-agglomerations correspond to the connected components of some level set *L(y)* defined as

$$L(y) = \{ x \in S \mid g(x) \leq y \} \qquad (6.8)$$

Torn [1976] devised and implemented a clustering-based algorithm which increases the density of sample points in the hills (basins) by undergoing one or a few steepest ascent (descent) steps from every point of the sample. This operation is known as *sample concentration*. This is further improved by sample reduction [Torn, 1978]. This sample concentration would theoretically help to overcome the problem left by simple reduction, by creating low density 'corridors' between contiguous hills (basins), making their identification easier. However, this comes in a conflict with the clustering procedure which assumes and exploits the

uniformity of the sample distribution.

To identify the clusters, Torn used a simple version of the *density clustering technique*, a natural way of grouping points in a metric space, which identifies areas of high density. He considered the clusters as hyperspheres of stepwise increasing volume from a starting seed point. Assuming that the local optima are the best seed points, he approximated them by the sample points with the best function values. The seed points are selected among the best points not yet clustered. The main advantage of this method is the fact that it does not need the computation of the whole similarity matrix for growing the cluster (i.e. for allocating a point to a given cluster only this point and the seed point are considered). The cluster is grown by increasing the radius of the hyperspheres until the density within the cluster becomes higher than the original uniform density over the whole sample. But the approximation of the clusters' shape, which represents the regions of attraction, by hyperspheres is rather crude and can only fit a very limited class of functions. Nevertheless, Torn reported some success in handling a series of test functions. Gomulka [1978] reported her experience with Torn's algorithm and found it 'consistently reliable'. But one drawback which has not been pointed out before, is the effect of the boundaries of *S* on the peripheral clusters' densities. A spherical cluster not entirely contained in *S* would have its density underestimated. However, the negative effect decreases as the overall average density increases. This problem can be completely eliminated for the particular case where *S* is a

subset of the boundaries of a convex subset of $\mathbf{R}^n$. This idea is further developed in section 3.

Boender et al [1982] suggested an improvement on the shape of the clusters ; hyperellipsoids are used instead of hyperspheres through second order approximation of the objective function at the local optima (seed points). The range of applicability although slightly wider, is still very narrow. In the same reference, further improvement on the clusters' shape, is sought, based on the distance between any sample point and its closest neighbour. This is called *single linkage clustering* which can produce clusters of virtually any shape. In this approach, a cluster consists originally of a single point, other points are added if their distance to any point already in that cluster is below some critical threshold. Two clusters are fused, if the distance of any point from the first to any point from the second is below the threshold, which is updated as the overall sample size grows.

The single linkage approach is used for global optimisation by defining the distance between two points x_1 and x_2, in the neighbourhood of a local optimum x^* chosen as a seed point, as

$$d(x_1,x_2) = \sqrt{(x_1 - x_2)' H(x^*) (x_1 - x_2)} \qquad (6.9$$

where H is the hessian of the function. Points are allocated to a cluster until the smallest distance from any point inside this cluster to any other point outside it exceeds some threshold level. This threshold criterion is dictated by the

distribution of the sample ; it should be uniform within each cluster. With the metric defined above, the average volume v of the neighbourhood of each sample point containing no other sample point is approximated by

$$v = \frac{d^n \pi^{n/2}}{\Gamma(n/2 + 1)|H(x^*)|^{1/2}} \qquad (6.10)$$

where d is the average distance to the nearest sample point. The probability of having only one point out of N sample points within this volume is

$$\alpha = (1 - \frac{v}{m(S)})^{N-1} \qquad (6.11)$$

If d is taken as the threshold level, it can be estimated by

$$d = [\frac{\Gamma(1+n/2)\sqrt{|H(x^*)|}\,m(S)}{\pi^{n/2}} (1-\alpha^{1/(N-1)})]^{1/n} \qquad (6.12)$$

The experience reported on the density and the single linkage methods shows that both perform equally well, except for the case where the shape of the clusters is far from ellipsoidal where the density clustering tends to terminate the clustering prematurely. Both methods seem to be better than Torn's algorithm and many other existing methods, in terms of number of function evaluations.

Rinnoy Kan and Timmer [1984] suggest the following critical distance for the single linkage approach

$$d = \pi^{-1/2} [\Gamma(1+n/2) \, m(S) \, \sigma \frac{\log(N)}{N}]^{1/n} \qquad (6.13)$$

using the normal euclidian norm, and setting the probability α to

$$\alpha = (1 - \frac{\sigma \ln(N)}{N})^N \qquad (6.14)$$

They also show that by *single linkage*,

- if $\sigma > 2$, the probability that a local search is applied at the kth cycle tends to 0 with increasing k.

- if the parameter σ exceeds 4, the total number of local searches is finite with probability 1, for infinite sample size.

- In every connected component of a given level set $L(y)$ in which a point is sampled, a local optimum will be found within a finite number of iterations.

Rinnoy Kan and Timmer [1984, 1987] also suggested another approach referred to as *The Mode Analysis*. This is in fact a simplified adaptation of the clustering method of Wishart [1969], known under the same name. The domain S is first partitioned into small cells (hypercubes). After sample reduction, the cells are classified into full cells and empty ones. The cells with more than half the expected number of sample points are considered full, otherwise they are empty. Then a single linkage clustering is carried out on the cells as elements, the clusters being the subsets of S corresponding

to a connected subsets of full cells.

One common drawback, in both the single linkage and the mode analysis, is that each cluster correspond to the connected components of the level set defined by the sample reduction, which can cover more than one region of attraction, and yet only one is identified. The problem may be even more serious due to an inherent weakness of the single linkage method, known as *chaining*. Here some disjoint contiguous components of the level set are put in the same cluster, suggesting a spurious connection. This increases substantially the number of overlooked regions of attraction hit by the reduced sample. This comes on top of the effect of the sample reduction where the discarded points of 'bad' function values might belong to the region of attraction of the global optimum.

This led Rinnoy Kan and Timmer [1984, 1987] to introduce the so-called *Multi-level methods*. The procedure is the same as for the two previous techniques, except that the local optimisation is started from every sample point without a neighbour of better function value within some *critical distance*. This results in a sequence of monotonically increasing (decreasing) sample points of the objective function value, from which a local search is applied. If the critical distance of equation 6.13 is used, then the asymptotic results concerning the local searches remain valid. Moreover, if σ is positive and x is an arbitrary sample point, then the probability that a local search is initiated from x, would tend to 0 with increasing number of sampling/clustering

cycles *k*.

The mode analysis method is further extended to the *multi-level mode analysis* (MLMA), in the same way as the multi-level single linkage. The value of each full cell is the best function value among its sample points. A local search is started from the best point of each cell which has no neighbour of a higher value.

Numerical experiments [Rinnoy Kan and Timmer 1984, 1987] show that multi-level single linkage method gives promising results compared to other clustering-based methods (SL,MA,MLMA). Compared to other methods described in previous sections, its performance seems to be the best both in terms of the number of function evaluations and unit standard time. But it has been recognised that any fair comparison should take account of the fact that, on one hand some parameters proper to each method have to be optimised (critical distance parameter σ, reduction percentage, sample size per iteration...), and on the other hand the stopping rules must be the same.

6.2.3.2 Termination criteria

All global optimisation algorithms using random sampling comprise a global probabilistic part and/or a local deterministic part. The latter part performs the numerical approximation of the global optimum candidate and the former

controls the sampling and the *decisions* for making a local search and most importantly whether its outcome is the global solution or not, in order to terminate or to carry on searching. Deciding when to stop sampling and accept the best solution found so far as the solution, is the critical part of this class of algorithms. Much of the recent work done on global optimisation has been devoted to devising "good stopping rules". Two different ways of handling the problem have been considered ; in the first one, the function values of the sampled points are seen as realisations of a random variable for which the distribution is approximated; in the second one, the number of regions of attraction is estimated as that of the cells of a multinomial distribution.

6.2.3.2.1 Function level set probability termination

This class of termination techniques aims at estimating the probability of finding a <u>better local optimum</u>, than the ones found so far. The search is terminated if this probability is less than some prefixed value. This gives no information on the error on the true global optimum, and just indicates the chance of any improvement of the function value. Consider the following function

$$\xi(y) = \frac{m(L(y))}{m(S)} \tag{6.15}$$

where $L(y)$ is the level set as defined previously. The global minimum can be defined as

$$y^* = \{ \, y \mid \xi(y) = 0 \, \} \qquad\qquad (6.16)$$

and a ε-approximation of y^* is

$$y_\varepsilon^* = \max \{ \, y \mid \xi(y) \leq \varepsilon \, \} \qquad\qquad (6.17)$$

It can be easily seen that the function defined by equation 6.15 can be interpreted as a probability distribution of $f(X)$, if X is uniformly distributed over S, and the ε-approximation is simply its quantile of order ε.

6.2.3.2.2 Regions of attraction numbering termination

This approach was put forward first by Zielinski [1981] in order to find the number of regions of attraction and their relative sizes, and later extended by Boender & Rinnoy Kan [1983].

The domain S of multimodal functions can be partitioned according to the regions of attraction R_k of the local maxima (minima) x_k^*. Let ω_k be the relative share of the kth region

$$\omega_k = \frac{m(R_k)}{m(S)} \qquad\qquad (6.18)$$

The number of local optima is l. If a uniform sample of size N is taken from S, and $n_1, n_2, \ldots, n_l$ are the shares of the regions from the sample, then

$$n_1 + n_2 + \ldots n_l - N$$

The $\{ n_1, n_2, \ldots, n_l \}$ set is a realisation of the random set $\{ N_1, N_2, \ldots, N_l \}$ from a multinomial distribution. Because the number of cells is not known, the maximum likelihood estimate of l is shown to be infinite for all possible outcomes of n_j [Boender & Rinnoy Kan, 1983].

If the parameters l and $\{ \omega_1, \omega_2, \ldots, \omega_l \}$ are assumed also to be random variables, with a specified *prior distributions*, *posterior distributions* can be found using the sample outcome, through Bayes theorem. The prior distributions are such that L is uniform over the set of positive integers, and $\{ \omega_1, \omega_2, \ldots, \omega_l \}$ jointly uniform over the $(l-1)$-dimensional unit simplex. Under such conditions, it can be shown that [Boender & Rinnoy Kan, 1983] :

i) the marginal probability that $(L=l \mid l \geq w)$ is

$$\frac{(l-1)! \, l! \, (N-1)! \, (N-2)!}{(N+l-1)! \, (l-w)! \, w! \, (w-1)! \, (N-w-2)!} \tag{6.19}$$

ii) the posterior expected value of L is

$$\frac{w(N-1)}{(N-w-2)} \qquad for \quad N \geq w+3 \tag{6.20}$$

and its variance is

$$\frac{w(w+1)(N-1)(N-2)}{(N-w-2)^2(N-w-3)} \qquad for \quad N \geq w+4 \tag{6.21}$$

iii) the transition probability that a new trial will result in a new local optimum is

$$\frac{w(w+1)}{N(N-1)} \qquad for \quad N \geq w+2 \qquad\qquad (6.22)$$

iv) given a pair (w,N), and $c = c_1/c_2$, where c_1 is the cost of premature termination assumed to be proportional to the relative error of the posterior expectation, and c_2 the cost of each trial assumed constant, then the optimal decision is to terminate sampling if

$$N \geq N'(c) = \lceil 6 + \frac{8}{c} + \sqrt{\frac{80}{c} - 13} \rceil \qquad\qquad (6.23)$$

6.3 MULTIEXTREMAL OPTIMISATION OVER A HYPERSPHERE

6.3.1 Method selection

In order to have a useful comparison between different global optimisation methods, each approach will be appraised for its suitability for identifying the stochastically dominant directions in the load-resistance space.

The function to be optimised in this study corresponds to the limit-state surface of a framed structure with random resistances and subject to random loads. The integration of the probability of failure is carried out by the simulation

using a directional importance sampling approach. The directions of most importance are in the neighbourhood of local maxima of the function. But the number of local maxima is too high to allow them to be set as a target in their totality; moreover only a few of them, with the highest function values, are of practical relevance to importance sampling. Therefore, the objective would be to find a subset of local maxima of function value higher than some threshold. In this respect, the goal is different from that of the conventional global optimisation which seeks only the global maximum as the end result and ignores the other optima. It is also different from the multiextremal function analysis which tries to find all local optima. Therefore any method used in this particular case should be some appropriate blend of the methods used for solving the two situations referred to above.

Among the deterministic methods, most of the space covering techniques are developed for a particular class of functions such as rationals, lipshitzians, and twice differentiable. As limit state functions are not given in a closed form, these methods are not suitable. Moreover, nearly all these methods are time-exponential and can be unacceptably expensive for large dimensionality.

The trajectory techniques seem to be unaffected by the dimensionality, however the convergence for all cases is still questionable.

The tunnelling method generates a sequence of improving local

optima intended to converge to the global one, in such a way that many relevant 'peaks' may be overlooked. More importantly, it generates a sub-problem more difficult to handle than the original one.

Generally speaking, the range of applicability of the deterministic methods in their present state of development, is too narrow to be of direct use in the present study. Therefore it is believed that only the stochastic methods can ensure some useful solution at an acceptable cost in structural reliability analysis.

The pure random search techniques (PRS) cannot be directly used in this case for obvious reasons. Adaptive random search (ARS) as described by Solis and Wets, using local search for the mapping T (see section II.2), can be modified to keep record of any relevant local optimum found on the way, instead of generating an improving sequence only. It may appear that our problem can be solved by this method, but the number of local searches would be unacceptably high. Therefore, even modified ARS cannot be used.

The Multistart (MS) approach remains the most promising alternative; by its structure, it generates local optima and can be modified to meet closely our purpose. Of course, the crude MS is not suitable for lack of efficiency. The Multistart/clustering technique is most suitable; it keeps record of all (relevant) optima found on the way in a systematic manner, and the sample is arranged in order to give

out the maximum information about the function and its shape; it is flexible and can lend itself easily to adaptive modifications; a selective scheme of the local optima can be efficiently included avoiding wasted local searches for non relevant optima.

Within the multistart/clustering scheme, the choice is still wide. Many decisions, concerning the different phases of any potential algorithm, are yet to be made. In the following are discussed the different alternatives for each phase. These phases are Sample modifications, Clustering, and Stopping rules.

6.3.2 Sample modifications

The crucial point in this phase is to decide whether reduction and/or concentration should be made, and by how much the sample should be reduced or how many steepest steps are to be done for concentration.

After sampling and function evaluations, the points are theoretically evenly distributed and no sub-groups of important size can be singled out. Agglomerations of points of 'good' function values can be distinguishable only if they are surrounded by areas of density significantly below average. This can be achieved either by removing low function value sample points or by moving the points a few steps toward the corresponding local maxima in the steepest

ascent.

As we are not interested in all local optima, concentration of the sample would create clusters corresponding to unwanted optima (i.e. with function values lower than a prefixed threshold) and therefore concentration of the sample cannot be selective. Furthermore, concentration leads to at least n function evaluations per sample point, which would increase substantially the overhead cost of the clustering. Consequently, concentration of the sample is not suitable and very expensive.

Reduction of the sample is much more attractive in the present situation. Firstly, its cost is very marginal. Secondly, the choice of the threshold of the function value for the relevancy of local optima, would automatically sieve off the unwanted ones and highlight well-separated clusters. In existing methods, reduction is done by taking off from the sample some percent γ. The value of γ seems to be chosen arbitrarily without reference to the type of objective function, and so would be the threshold level of the unwanted local optima. It is thought that the choice should be done the other way around, by fixing the function level limit first according to the dispersion of the function values in the sample. Information on this dispersion can be obtained from the statistical analysis of the function values (mean μ_g, standard deviation σ_g). An appropriate form would be

$$\mu_g + \rho\sigma_g$$

where ρ is a chosen positive coefficient. This coefficient may be updated at later stages of sampling in order to meet some clustering conditions, such as the number of clusters.

6.3.3 Clustering procedure

The clustering methods currently used in global optimisation are as follows

-Density clustering (DC)

-Single linkage clustering (SLC)

-Mode Analysis (MA)

-Multi-level single linkage (MLSL)

-Multi-level mode analysis (MLMA)

The DC techniques are of limited use as the shape of the 'hills' cannot always be approximated by hyperspheres or even by hyperellipsoids. The MA and MLMA techniques suffer the same drawback as the deterministic domain-partitioning methods in which the number of cells increases exponentially with the dimension. Moreover, the values given to the cells (the best sample point of the cell) can be misleading unless their size is made very small. The SL approach has a good performance, but does not make full use of the information the sample can provide which result in unnecessary local searches; this can be significantly improved by the MLSL method. This is in agreement with the experience reported by several authors which seems to show that the best choice is the MLSL.

One crucial point in single linkage clustering is the value of the critical distance which determines allocation of points to the clusters. The euclidian norm is not suitable as a clustering criterion. As our domain is the surface of the unit hypersphere, it is more convenient to use polar coordinates. The distance between any two points is better represented by the angle between them

$$\psi(x,y) = |\arccos(x,y)| \qquad (6.24)$$

This choice is very convenient for deriving the critical distance from the distribution of the distance from a given point to its closest neighbour in a uniformly drawn sample of a given size N. This distribution can be obtained using order statistics. Let $\{X_1, X_2, \ldots, X_N\}$ be a set of uniform random vectors from S and x a given point in S. The angle between x and any realisation x_i of X_i is

$$\psi_i(x_i,x) = |\arccos(x_i,x)| \qquad (6.25)$$

Assume that the above angles are labelled such that

$$\psi_1 \leq \psi_2 \leq \ldots \leq \psi_N \qquad (6.26)$$

Let H_r be the distribution of the rth order statistic

$$H_r(\psi) = Pr(\psi_{(r)} \leq \psi)$$
$$= Pr(\text{ at least } r \text{ of the } \psi_i \leq \psi)$$
$$= \sum_{i=r}^{N} \binom{N}{i} P_\Psi(\psi) [1-P_\Psi(\psi)]^{N-i} \qquad (6.27)$$

where P_Ψ is the distribution of Ψ. For the particular case of r=1, we have the distribution of the closest neighbour

$$H_1(\psi) = Prob(\psi_1 \leq \psi) = 1 - [1 - P_\Psi(\psi)]^N \qquad (6.28)$$

The density distribution can be easily derived from the cumulative distribution H_1

$$h_1(\psi) = \frac{dH_1(\psi)}{d\psi} = N \cdot p_\Psi(\psi)[1 - P_\Psi(\psi)]^{(N-1)} \qquad (6.29)$$

where p_Ψ is the density distribution of Ψ, which can be easily seen as the ratio

$$p_\Psi(\psi) = \frac{m(s_\psi)}{m(S)} \qquad (6.30)$$

where s_ψ is the set of points of S such that the angle between them / and x is Ψ; it is the surface of an n-1 hypersphere of radius $\sin(\Psi)$. The density is therefore given by

$$p_\Psi(\psi) = \frac{\Gamma(n/2)(\sin(\psi))^{n-2}}{\Gamma((n-1)/2)\sqrt{\pi}} \qquad (6.31)$$

and the cumulative distribution

$$P_\Psi(\psi) = \int_0^\psi p_\Psi(t)\,dt = \frac{\Gamma(n/2)}{\Gamma((n-1)/2)\sqrt{\pi}} \int_0^\psi (\sin(t))^{n-2}dt \qquad (6.32)$$

If

$$I_n = \int_0^\psi \sin^n(t)\,dt \qquad (6.33)$$

then it can be easily shown that

$$I_n = \frac{n-1}{n} I_{n-2} - \frac{\sin^n(t)\cos(t)}{n} \qquad (6.34)$$

which allows the evaluation of the distribution numerically. The expected value of Ψ_1 can then be formulated as

$$E(\psi_1) = \int_0^\pi \psi\, h_1(\psi)\, d\psi = N \int_0^\pi \psi p_\Psi(\psi)\, [1 - P_\Psi(\psi)]^{N-1} d\psi \qquad (6.35)$$

which on integration by parts gives

$$E[\psi_1] = \int_0^\pi [1 - P_\Psi(\psi)]^N d\psi \qquad (6.36)$$

and this can be evaluated numerically.

For a given dimension n and sample size N, the shape of the density h_1 is slightly skew to the left, with both tails having very small probability content, particularly the upper one. The density h_1 and the corresponding cumulative distribution H_1 are represented in Fig 6.1 through Fig 6.4 for different values of n and N. A 'pseudo-upper bound' on the unsampled spherical portion can be evaluated, with some confidence level, and used as a critical distance for clustering. This critical distance and its corresponding probability distribution are a probabilistic measure of the space covered by the sample. It can be easily seen that the probability of non exceedance η becomes very close to 1, shortly after the modal value of the density. Let ψ_η be the

corresponding angle value.

For different values of n and N, the range of variation of ψ_η is relatively wide. Fig 6.5, represents the curve of n versus ψ_η for different dimensions, and sample sizes, with η = .9999. It shows that ψ_η varies almost linearly with the dimension n, specially for high values of n. For a given η, the angle ψ_η can be easily shown to approach 0 as the sample size tends to infinity. It follows that if x^* is a local optimum, its closest sample point will be within an angle less than ψ_η with probability η.

The value of η can be either constant throughout the iteration process, or a function ε of the current sample size. A possible choice of this function is

$$\varepsilon = 1 - \alpha \qquad (6.37)$$

where α is given by equation 6.14. The critical angle becomes therefore a function of σ. The choice of a value for σ is not necessarily the same as for the case of objective functions defined over a convex domain, with an euclidian critical distance. Numerical experience seems to suggest that for values exceeding 1 and a reasonable sample size, dominant regions are overlooked and only a fraction of the local optima is identified. A value of $\sigma=0.5$ seems to be a sensible choice. However, a thorough parametric study for the class of function under investigation is necessary for an optimal choice of σ.

The clustering procedure may be summarized as follows

i) The points of the reduced sample are relabelled in decreasing objective function values.

ii) The best of the remaining unclustered points is selected as a seed point of a new cluster. All unclustered points within a critical distance of a point already in the cluster are allocated to it, and a record is kept of the points which have been already used as a starting search point, or fall within a critical distance of a previously found local optimum.

Virtually any new portion of Ω_n of radius less or equal to $\sin(\psi_\eta)$ is identified and recorded so that one local search at most is made from it.

The clustering procedure generates a sequence of sets of locally improving seed points converging (with an increasing probability η) towards the local optima. It follows that the repeated sampling/reduction/clustering scheme without a deterministic local search constitutes an approximate multiextremal optimisation algorithm on its own.

The method can be made efficient if one tries to make the most of the particularities of this original problem, in order to reduce the sample size. This is possible if the function is confined to some connected and regular portion of the unit sphere (orthant, circular portion,...), from which direct uniform sampling is possible. Its area is also easy to

evaluate. This reduces the sampling space to a fraction of the *n-sphere*. A typical case is the one, where some of the variables are either positive or negative. If *k* is the number of sign restriction-free variables, this fraction can be easily shown to be

$$\Omega = \frac{\Omega_n}{2^{n-k}} \qquad (6.38)$$

where Ω_n is the measure of the *n-sphere* surface. This 2^{n-k}-fold reduction allows a dramatic reduction in the sample size. But with this reduced sample space, the reformulation of the density h_1 and distribution H_1 is highly complicated, because the nice feature of rotational symmetry is now lost. Therefore, some way of using h_1 and H_1 as formulated for the unit sphere, and sampling only from the reduced space, is necessary. This can be achieved by considering the sampled set, from the reduced space, as a part of a fictitious and much bigger sample that would be sampled from the whole sphere. The fictitious sample size can be estimated from the real one by extrapolation using equation 6.38, and used to evaluate the critical angle ψ_η.

6.3.4 Local search

The efficiency of the clustering procedure in identifying the sample points closest to the local optima, reduces considerably the local search cost. If the objective function is smooth enough and does not present very sharp peaks,

efficient conventional local optimisation routines can be used. However, in practical problems, the objective function can often present local ill-conditioning, or can be piece-wise differentiable. The deterministic local search cannot be carried out at an acceptable cost. In this case, one has to accept the seed points as approximations to the local optima, and try to improve them by increasing the number of sampling/clustering cycles.

Because the objective function might present the ill-conditioning behaviour in some parts of the feasible domain and be smooth in others, a sensible method is to incorporate a local search routine with a test for smoothness (by checking the gradient for example), so that the local optimum is approximated by the cluster seed point whenever the test is negative.

/

6.3.5 Stopping rules

This is the Achilles' heel for stochastic multiextremal optimisation problems. It would be too optimistic to expect to find *good stopping rules*.

The criterion for termination based on the probability of finding a better local optimum described in subsection II.5.1 is clearly not suitable here. Many relevant local optima can be systematically overlooked, simply because the global optimum has been incidently found at an early iteration.

The alternative, based on the bayesian estimation of the regions of attraction corresponding to the local optima, is much more suitable. For well behaved functions with a very moderate number of optima, this latter approach could be used in a straightforward way. But in our case, we cannot adopt these assumptions, and at the same time our scheme is a selective optima seeking one. Two possibilities can be considered if this method is to be used :

- to derive analytical results equivalent to those in section II.5.2 corresponding to the selective case. For example, one may want to evaluate the posterior expected number of local optima with function value above some threshold level.

- to modify the function in such a way that makes the results of II.5.2 directly usable.

f

The region of attraction in II.5.2 is defined as a cell of relative size θ_i independent of its optimal function value. Hence the theoretical framework does not seem to favour the first possibility. The second one is much more manageable. As a matter of fact, in the sample reduction and the clustering procedure described in section III.2 above, is embedded the solution of the 'function modification'. If we consider the set D of points of S with function value less than the threshold for the local optima, this set contains all regions of attraction of the unwanted optima. Moreover, the points of this set which belong to the regions of useful optima are not used either for clustering or for local search, and are

treated in the same way as any other point of *D*. This suggests the idea of considering the set *D* as another cell of the multinomial distribution, i.e. a particular cluster with special clustering rule; allocation of a point to *D* is done if its function value is less than the threshold level. Here the whole sample is used and the reduction is simply dumping the 'bad' points in *D*. This is merely replacing the original function by a fictitious one in which the number of regions of attraction exceeds the number of useful local optima by one.

All regions of attraction or cells, of the fictitious function described above are taken into consideration in a nonselective way; therefore all analytical results of II.5.2 are valid and can be used directly, bearing in mind that the number of local optima is the number of cells less one.

The optimal decision formulated in paragraph (iv) of II.5.2 is expressed in terms of the ratio *c* of the sampling cost to the premature termination cost. Its value depends very much on the physical nature of the problem, the subsequent use of the local optima and the subjective judgement of the user. In the case of structural reliability the local optima are subsequently used either as modal sampling points for directional simulation, or as linearization points for a first order approximation, or both. The impact of missing some relevant local optima concerns the accuracy of the probability of failure. There is no general method for assessing this impact, which is case dependent. Even if this impact can be estimated or bounded, it has to be converted into 'cost'. This

conversion can be very subjective and may depend on the investigator. The uncertainty of the value of the loss ratio c would be therefore too high to decide on a termination rule.

This suggests the search for an approach independent of the loss ratio c, which can couple the use of a mixed criterion based on pure bayesian rules, and some special feature of the particular problem under investigation, as follows:

- The use of the transition probability given by the expression 6.22. This expression quantifies the chance of finding a new local maximum for the next sample. Or more conveniently, the probability of not finding a new local maximum

$$1 - \frac{w(w+1)}{N(N-1)} \qquad (6.39)$$

could be used. For k consecutive samples, the expression

$$1 - \prod_{i=1}^{k} [1 - \frac{w(w+1)}{(N+i)(N+i-1)}] \qquad (6.40)$$

gives the probability of finding at least one new local maximum above the current threshold function level, after k new samples. It may be used as a termination test by comparison to some prefixed maximum confidence level. As the expression 6.40 is dependent on k, its choice should be carefully made by the investigator depending on the size of the problem.

- The current average value of the function f is an unbiased estimate of the integral of f over the domain S. The fluctuations of such an average are assessed through the current estimate of the variance (or more conveniently the coefficient of variation), which decreases as the sampling proceeds. With uniform sampling and objective functions with no sharp peaks, a small value of the variance can be interpreted as that the sample has scanned almost all important regions. A stabilisation in the average value of the objective function can therefore be a good indicator for termination. This of course assumes that the shape of the objective function does not present sharp peaks which can be easily missed out by the sample.

If the function is such that the value of its integral can be evaluated by some approximation from the knowledge of the local optima, a possible criterion is to compare the average of f to such an approximation. The ratio of the two estimates of the integral should lie within some prefixed bracket linked to the accuracy required by the investigator. This is precisely the case for the problem of identifying dominant failure modes in structural reliability. More details are given in chapter 7.

The coupling of the two criteria described above is believed to be more relevant than an expected posterior loss based on a hypothetical value of the loss ratio. The threshold levels for both criteria have a direct physical meaning for the investigator and are much easier to fix values for.

6.3.6 Algorithm summary

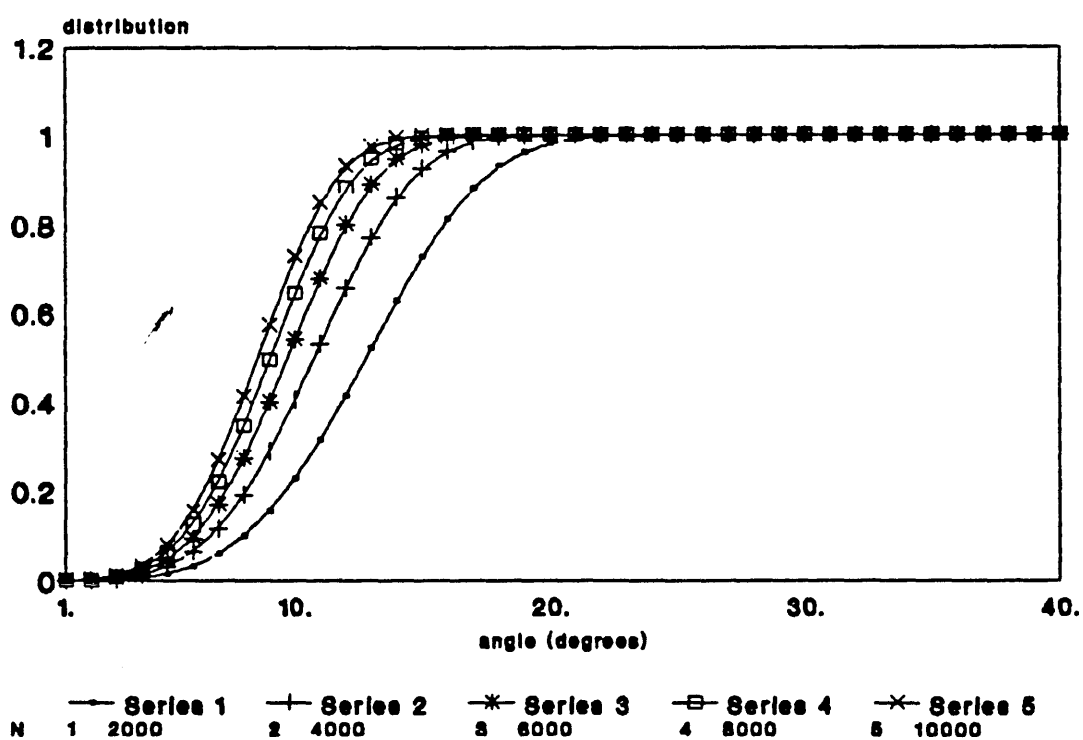In the following is a summary of an algorithm proposed for the use in structural reliability

Step 1. Evaluate the critical angle. Set $N=N^*$

Step 2. Sample N^* points and evaluate their function values, and the threshold for reduction.

Step 3. Relabel the points in decreasing order of their function values.

Step 4. Reduce the sample according to the current threshold.

Step 5. Clustering procedure of the reduced sample.

Step 6. Local search from the best point of each sub-cluster, if not already used for such purpose, or if it does not fall within a critical distance of some local optimum.

Step 7. Termination test : for bayesian rule of equation 6.40, and integral approximation. If positive, go to step 9.

Step 8. Draw new sample of size k and evaluate their function values, and add them to the current set. Set $N=N+k$, and update the critical angle. Go to step 3.

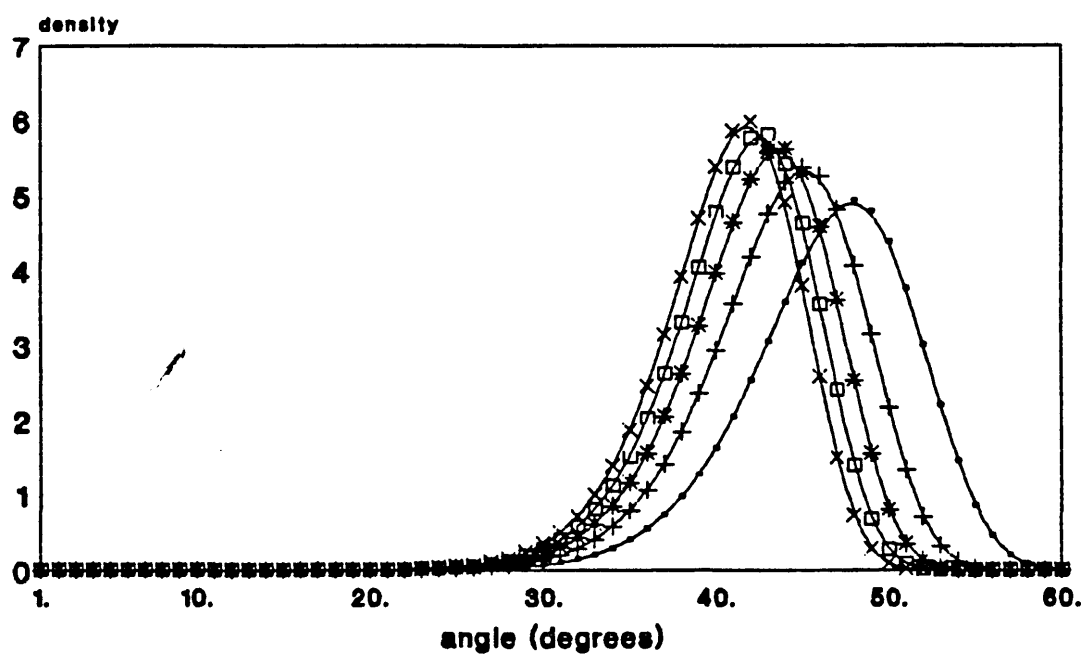Step 9. End

Fig 6.1. Closest point from uniform
sample. Density
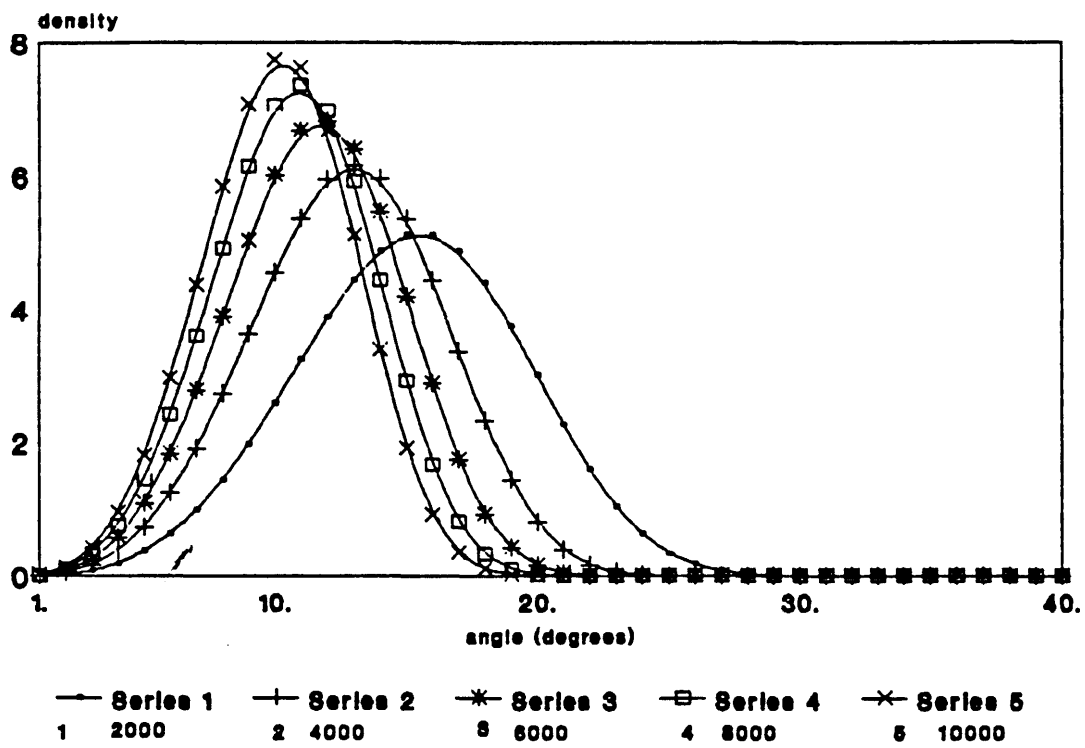
Fig 6.2. Closest point from uniform sample. Cumulative distribution

distribution
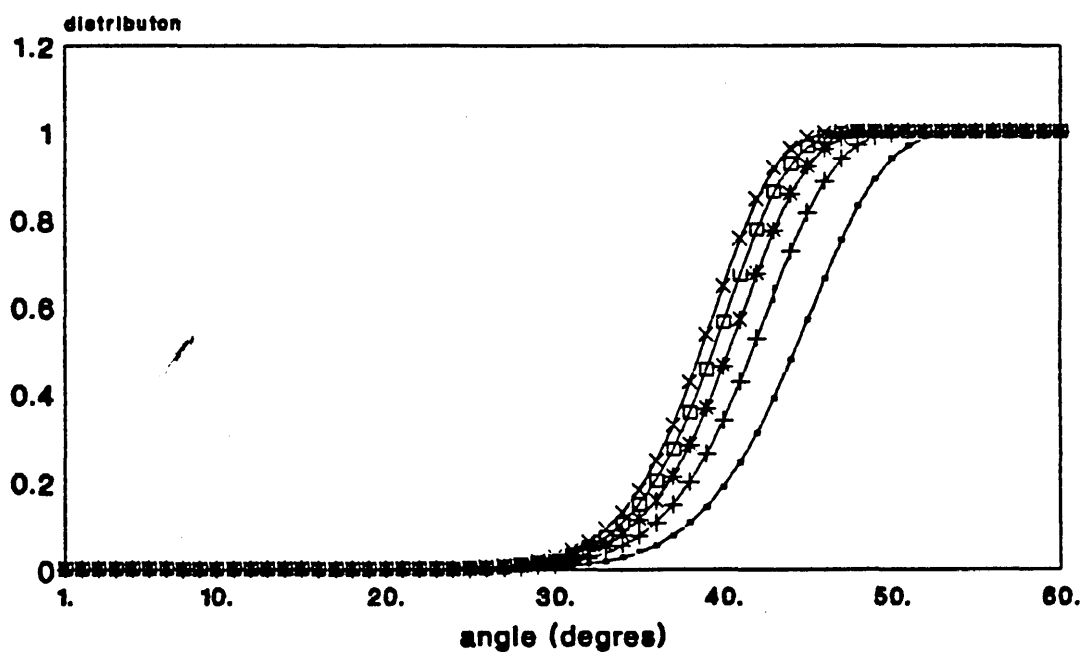
angle (degrees)

	Series 1	Series 2	Series 3	Series 4	Series 5
N	1 2000	2 4000	3 6000	4 8000	5 10000

Dimension : 5

Fig 6.3 Closest point from uniform sample. Density

density

angle (degrees)

	Series 1		Series 2		Series 3		Series 4		Series 5
N	1	2000	2	4000	3	6000	4	8000	5	10000

Dimension :15

Fig 6.4. Closest point from uniform sample. Cumulative distribution

Fig 6.5. Closest point with probability
.9999 for various sample sizes
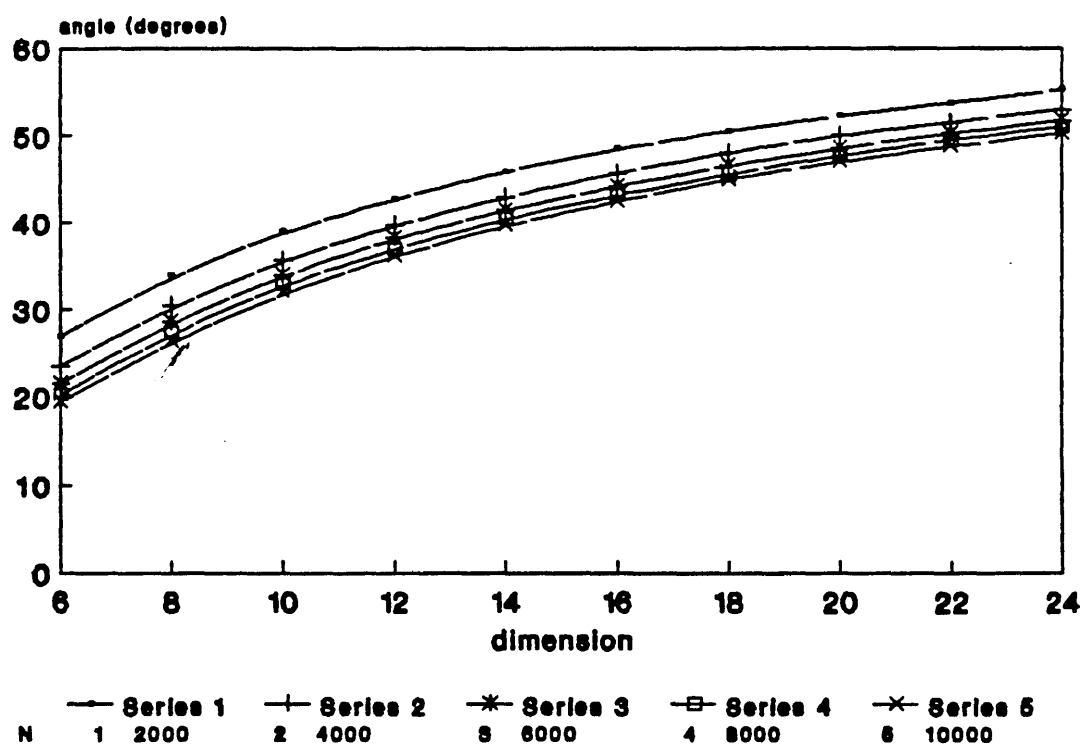
STRUCTURAL RELIABILITY BY SIMULATION

The wide use of Monte Carlo (MC) methods in structural reliability is relatively recent. However, the idea of MC integration is not new in structural reliability; Warner and Kabaila [1968], have applied it to reinforced concrete columns, with some 'selective sampling'; this is in fact a stratified sampling. But the lack of interest in the method in the seventies has delayed its development, while the FORM and SORM methods received much attention. It has been now widely accepted that MC techniques, are not merely a fantasy approach with uncertain outcome, but a serious alternative for estimating the reliability of large structures which otherwise cannot be handled with conventional methods. This began in the early eighties with the paper published by Shinozuka [1983] and Harbitz [1983] where the latter has applied importance sampling technique to an 8-dimensional fatigue problem

7.1 HIT-OR-MISS MONTE CARLO

7.1.1 Crude hit-or-miss

The probability integral is formulated as follows

$$p_f = \int_{\mathscr{F}} f_X(x)\, dx = \int_{\mathbb{R}^n} I[G(x)] \cdot f_X(x)\, dx \qquad (7.1)$$

where $I[.]$ is an indicator function which takes the value of one if x is in F and 0 otherwise, and $G(x)$ represents the limit-state function. The probability p_f is formulated now as an expected value of $I[G(x)]$ which is distributed according to f_x. An unbiased estimator of $\hat{p}_f$ is

$$\hat{p}_f = \frac{1}{N} \sum_{i=1}^{N} I[G(x_i)] \qquad (7.2)$$

where the x_i's are sampled from f_x.

As the reliability of structures is very high, p_f is very small and for a method with inherent poor efficiency, a very large sample is needed to achieve an acceptable confidence level. The variance can be estimated as

$$Var(\hat{p}_f) = \frac{\hat{p}_f(1-\hat{p}_f)}{N} \qquad (7.3)$$

7.1.2 Improved Hit-or-miss

7.1.2.1 Reduced sample space

An improvement to the MC integration of the probability of failure is by the partition of the region suggested by Shreider [1966] in a paper by Harbitz [1986]. It has been applied to the case of standardised uncorrelated normal variables. Let x^* be the point on the limit-state surface with

smallest distance to the origin (i.e. the so-called design point in the FORM analysis), and refer to this distance as β. Every point within the hypersphere with radius β centred at the origin is obviously a safe point. To make good use of such information , a β-sphere truncated distribution ϕ_{tr} is defined as follows

$$\varphi_{\beta,tr}(x) = \frac{\varphi_n(x)}{(1-\chi_n^2(\beta^2))} \quad if \quad \|x\| \leq \beta$$

$$= 0 \quad if \quad \|x\| \leq \beta \tag{7.4}$$

where χ_n^2 is the chi-square distribution of degree n. Thus sampling from $\phi_{\beta,tr}$ instead of ϕ_n would exclude the β-sphere from the sampling space allowing more points to be sampled outside the safe domain for a given total sample size, resulting in estimating a conditional probability of an order of magnitude higher than the original one. In other words, the so-called crossing rate becomes significant even for a small sample. The resulting variance reduction of the new estimator $\hat{p}_{fr}$ can be shown to be

$$Var(\hat{p}_{fr}) \leq (1 - \chi_n^2(\beta)) Var(\hat{p}_f) \tag{7.5}$$

Csenki [1988] attempts to extend this idea, by excluding from the sampling domain a much larger hypersphere. He suggested the use of a hypersphere of radius R with largest probability content, that can be inscribed in the safe domain, for a given centre point a, using classical optimisation techniques. Its probability content is equal to the distribution function of the non-central chi-square distribution with n degrees of

freedom and non-centrality parameter $\|a\|^2$, evaluated at $r=R$.

7.1.2.2 Importance sampling

If some distribution h_Y entirely different from the original distribution is used for sampling, equation 7.1 becomes

$$p_f - \int_{\mathbb{R}^n} I[G(x)]\, \frac{f_X(x)}{h_Y(x)}\, h_Y(x)\, dx \qquad (7.6)$$

and p_f is now the expected value of $I[G(x)].f_X(x)/h_Y(x)$. Two major unknowns are to be found for a good sampling distribution: its density function and its modal point(s).

A natural choice of h_Y is a function with the mode at the point of maximum likelihood of the failure domain, this point lies on the boundaries of F. Structures fail in many different modes, leading to many local 'checking' points; however the case of a single failure mode is first considered so that any ideas developed for this case can form the basis for handling the multifailure case.

If the maximum likelihood point (MLP) is assumed to be unique, it can in principle be found by using conventional optimisation algorithms, but one should bear in mind that this involves implicit limit-state function evaluations at points which are not part of the sample. Those function values cannot be accumulated with the sampled ones as they are

deterministic, and therefore constitute an overhead cost to be taken into account when comparing with uniform sampling or any other method. Any possibility of using those points by estimating their densities would have beneficial effect on the method's efficiency.

The choice of an importance density function is mainly a user's decision based on his prior knowledge of the integrand function and the limit-state surface in the vicinity of the assumed 'checking point'. But without such prior knowledge, a common choice of the importance sampling density function is the multinormal density with diagonal covariance matrix. The covariance matrix may be adjusted to take advantage of any appropriate information available. For the case of concave limit-state function with jointly gaussian distributed variables, a better choice is a half-space-truncated standard gaussian density, defined on the half-space bounded by the tangent-hyperplane to the MLP. For other specific cases, appropriate sampling distributions may be worked out to minimise the variance.

In an attempt to develop a general method, with minimum prior information specially concerning the 'checking' point, Melchers[1988c] proposed a search based importance sampling scheme in which the sampling process provides usable integrand function evaluations together with an optimisation path for finding the maximum likelihood point (MLP). In other words, the sampling distribution has a sequentially changing mode, theoretically converging towards the MLP. This is merely the

adaptive random search method used in global optimisation [Matyas 1965; Baba 1977; Wets & Solis 1981] referred to in chapter 6, in which the sample function values are accumulated to provide the probability of failure estimates.

Melchers defined the 'sampling efficiency' as the rate of failure points within the sample, and pointed out that it can measure the closeness to the limit-state surface, and that a minimum 'sampling efficiency' of 0.5 is necessary, but not sufficient, for convergence.

Concerning the variance update, initial values for standard deviations are chosen relatively large and are sequentially reduced as one gets 'closer' to the MLP. Generally the sampling variance should not exceed the original variance of the basic variables. Melchers reported good results for ratios within the interval [1,2], lower values may overestimate p_f and both higher and lower ones reduce the convergence rate. He proposed the following formula for the standard deviation update

$$(4 \mid \eta_{m-1} - 0.5 \mid + 1) \sigma_{X_i} \qquad (7.7)$$

where σ_{X_i} is the original standard deviation of the variable X_i and η_m is the sampling efficiency recorded for the mth group of k sample points.

The effectiveness of importance sampling by the above approach can be very limited, because it is carried out right from the

beginning without prior scanning for the regions of high probability density regions. One needs to know that, at some stage, sampling is happening in the close vicinity of the true MLP, with an acceptable level of confidence; otherwise one cannot ensure that sampling is targeted to genuine important regions, as it is meant to be.

For multiple failure modes cases, each mode has its own local 'checking' point. Not all of them are to be considered for deriving the sampling distribution, but only the most significant ones. The sampling distribution is chosen as a multimodal density with local modes at the selected 'checking' points. A convenient choice [Melchers 1987] is

$$h_Y(x) = \sum_{i=1}^{M} a_i h_{iY}(x) \qquad (7.8)$$

where each h_{iY} is a sampling distribution corresponding to the ith failure mode defined in the same way as in the single mode case described above. The coefficients a_i are weighting constants summing up to one and chosen proportionally to the failure probability associated with each considered failure mode.

The most crucial problem is finding these 'checking' points. For the same reasons discussed in the single failure mode case, Melchers extended his approach to multiple failure limit-state. The starting points are either chosen randomly or pre-assigned, but nothing is said about their number M.

This is a very important point as a multimodal sampling function with too many modes would tend to a uniform one and the importance sampling becomes almost spurious one. Moreover, if some randomly selected starting points are close enough to each other they would converge to the same modal area causing an overlapping effect among the modal parts of the partial densities h_{iy}. The starting points are, therefore, better selected in a more refined way to avoid such problems. An approach to deal with such problems is proposed in the next section (7.3), based on a multiextremal optimisation technique presented in Chapter 6, applied to a directional sampling integration.

The search based importance sampling approach is a double-task algorithm which tries to handle optimisation and integration simultaneously. But in order to have an efficient integration, the convergence of the optimisation part needs to be as fast as possible so that the importance sampling can be effective for the majority of the samples. In that respect, the method has room for improvement, preferably without too many wasted function evaluations.

For the case where the 'checking' points are assumed to be known, Fu and Moses [1987] proposed an approach which they refer to as the 'weighted general normal sampling distribution' method. It is basically similar to the method of Melchers [1984], except that the sampling covariance matrix is identical to that of the original distribution (and not simply diagonal) and the sample size allocated to each individual

partial density is chosen in advance to be in relative proportion with the original densities at the 'checking' points. The weights a_i are also chosen proportional to the original densities, whereas in Melchers' method they are proportional the reliability indices of each safety margin.

Fu and Moses reported results slightly better than those of Melchers [1984], but the difference seems to be marginal compared to the overhead due the use of correlated sampling. Somewhat similar ideas have been published by Schueller and Stix [1987].

7.2 Directional simulation

7.2.1 Uniform sample-mean

7.2.1.1 General formulation

The use of MC integration in structural reliability has been confined during the first half of the eighties to the hit-or-miss approach with a considerable effort of improvement by the methods described above. Despite these improvements, the sample size needed is still very high due to the high reliability level of the structures. A move towards a sample-mean approach becomes a necessary alternative. This move has been triggered by the relationship between the standard normal distribution and the chi-square distribution: the sum of the squares of n standard normal variables is

chi-square distributed, independently of the direction of the vector formed by this n variables. This gives the idea of radial integration with polar coordinates, instead of cartesian ones, for the multinormal integral. This idea was first put forward by Deak [1980]. But its use in structural reliability has to wait until 1985 when Ditlevsen, Oleson and Hasofer [1985] applied it to a load combination problem.

A full use of this approach for the evaluation of the probability of failure, assuming gaussian distribution for all variables, is proposed by Bjerager [1988] and Ditlevsen and Bjerager [1989].

From equation 3.15 (chapter 3), one can define the conditional probability of failure given some proportioning of the random variable, $P(A)$, as a function of such proportioning

$$P(A) = \int_{R(A)}^{\infty} f_X(tA)\, t^{(n-1)} dt \qquad (7.9)$$

If A is distributed according to a distribution h_A defined over the unit hypersphere, this distribution must correspond to that of the normalised form (unit vector) of a random vector Y distributed according to h_Y over R^n. The expression for h_A is therefore

$$h_A(\alpha) = \int_0^{\infty} h_Y(t\alpha)\, t^{n-1} dt \qquad (7.10)$$

and outcomes of *A* are obtained by normalising to unity outcomes of *Y*. Then the probability of failure can be expressed as an expected value as follows

$$p_f - E_A \left[\frac{P(A)}{\int_0^\infty h_Y(tA) \, t^{n-1} dt} \right] \qquad (7.11)$$

It can be clearly seen that the efficiency of the approach is directly dependent on the computational viability of the one-dimensional radial integrations. Any further development needs an explicit formulation of the sampling distribution h_Y, and its choice.

Three cases are possible :

   a. the sampling distribution is chosen identical to the distribution of the original variables: then the directional density distribution of equation 7.10 has to be evaluated numerically if it cannot be obtained in an analytical closed form. The sampling of directions is not uniform, but this non-uniformity is not necessarily beneficial like the importance sampling distribution chosen in relation to the limit-state function, because it is completely independent of the safe region boundaries. However, for highly correlated variables and limit-state surface shape close to a spherical one, sampling from the original density is more likely to target the regions of high density in the failure set.

   b. A standard normal distribution is chosen for the sampling distribution h_Y centred at the origin : then the

directional density h_A is uniformly distributed on the unit hypersphere and has the constant value of

$$h_A(\alpha) = \frac{1}{\Omega_n} = \frac{\Gamma(n/2)}{2\pi^{n/2}} \qquad (7.12)$$

This case is the typical crude sample-mean MC applied to the function defined by equation 7.9. Then the probability of failure becomes

$$p_f = \mathbf{E_A} \left[\frac{2\pi^{n/2}}{\Gamma(n/2)} \int_{r(A)}^{\infty} f_X(tA)\, t^{n-1} dt \right] \qquad (7.13)$$

The particular case of f_X being standard gaussian itself was studied by Bjerager [1988]. The availability of the radial integral of the standard normal distribution ϕ_n in a closed form, or at least in a recursive form, increases significantly the efficiency of directional simulation methods. The probability of failure takes the simple form

$$p_f = \mathbf{E_A} [\, 1 - \chi_n^2 (r(A)^2) \,] \qquad (7.14)$$

This is simply the expression of the multinormal integral in the polar coordinates space. An estimation of the probability of failure is obtained by sampling N directions α_i from h_A and averaging

$$\hat{p}_f = \frac{1}{N} \sum_{i=1}^{N} [\, 1 - \chi_n^2 (r(\alpha_i)^2) \,] \qquad (7.15)$$

and the variance is estimated by

$$var(\hat{p}_f) = \frac{1}{N(N-1)} \sum_{i=1}^{N} [1 - \chi_n^2(r(\alpha_i)^2) - \hat{p}_f]^2 \qquad (7.16)$$

Because of the closed form of the radial integral in the case of standard normal variables, directional simulation has been first used in normal space. This assumes that, if the original variables are not normal, a transformation T is made to a standard normal space. Sampling is then made from the transformed space, and the transformed limit-state equation is solved for $r(\alpha)$. However, the solution of the limit-state equation becomes more difficult and needs an expensive iterative algorithm. The new limit-state function is

$$\zeta[r(\alpha)] = G[T^{-1}(r(\alpha))] \qquad (7.17)$$

As a result any possible simplicity in the solution of the original function G, such as linearity or a closed form expression, becomes almost useless. Moreover, if importance sampling is to be carried out, the search for important regions is made costly by the solution of equation 7.17.

c. Sampling distributions are in principle different from the original variables' distributions, but the choice takes account of the relation between the safe domain boundaries and this latter distribution, such that the fluctuations of the radial integral of f_x in the failure domain are minimised. This is called the *directional importance sampling*, in which

the sampling space is the surface of the *n*-dimensional unit hyperesphere. The different aspects of directional importance sampling and some possible choices of sampling distributions are discussed in the next section.

7.2.1.1 Probability integration over the failure surface

If the original distribution f_X is not normal, its radial integral in equation 7.9 has to be carried out numerically, or by approximation techniques. Errors on this radial integral, from different sample outcomes, would not have any harmful effect if their expected value can be shown to be zero (integration in presence of noise, [Rubinstein 1981]), and if their variance is not too high. In deterministic numerical approximation methods, although the variance is usually very small, the expected value has often some systematic error inherent to the approximation technique used. The use of Monte Carlo simulation in this single dimension integral would not be efficient due to the high variance even with an unbiased estimator with zero mean. Although the numerical experience has shown that errors on the radial integral, using methods such as Gauss-Legendre, are negligible, the computing time depends on the accuracy required (i.e the number of Gauss points). Therefore an analytical closed form of the radial integral would be very beneficial both in computing cost and accuracy.

An approach which provides a closed form for the integrand,

under some restrictions, is to use Green's theorem to transform an integral over a given domain (say volume) into an integral over its boundaries (surface). Similar ideas has been used by Shinozuka [1983] to reduce a trivariate normal integral to a bivariate normal integral, and a bivariate normal integral to a single normal one.

Let V_f be the vector whose components are the conditional cumulative distributions $F_i(X)$ of each variable, i.e

$$V_f - (F_1(x), F_2(x), \ldots, F_n(x))$$ (7.18)

and

$$F_i(x) - \int_{-\infty}^{x_i} f_X(x_1, x_2, \ldots, t_i, \ldots, x_n) dt_i$$ (7.19)

It can be easily seen that

$$div(V_f) - n.f_X(x)$$ (7.20)

To apply Green's theorem, a closed surface boundary of the failure domain needs to be defined. In theory, the failure domain is usually unbounded, but because the values of the marginal distributions tend to zero as the distance to the origin tends to infinity, one can easily construct a bounded domain with the failure surface as the inner boundary and the surface of some hypersphere, centred at the origin with arbitrarily large radius, as a fictitious outer boundary (See Fig. 7.1). It follows, by Green's theorem, that

$$p_f - \frac{1}{n} \int_{\mathcal{T}} div(V_f)\, dx - \frac{1}{n} \int_{\partial\mathcal{T} \cup B} V_f \cdot \eta\, dS \qquad (7.21)$$

where η is the outer unit normal vector to the failure surface. The radius of the outer sphere can be chosen large enough to make V_f arbitrarily small. The probability of failure then becomes

$$p_f - \frac{1}{n} \int_{\partial\mathcal{T}} V_f \cdot \eta\, dS \qquad (7.22)$$

In polar coordinates, the surface element dS can be easily shown to be

$$dS - \frac{r(A)^{n-1}}{A \cdot \eta(A)}\, dA \qquad (7.23)$$

where $r(A)$ is the distance from the origin to the failure surface in the direction A . This yields

$$p_f - \frac{1}{n} \int_{\Omega_n} \frac{V_f \cdot \eta(A)}{A \cdot \eta(A)}\, r(A)^{n-1} dA \qquad (7.24)$$

If A is assumed to be a random unit vector distributed according to some density h_A, derived as in equation 7.10, then p_f can be expressed as an expectation

$$p_f - \frac{1}{n} E_A \left[\frac{V_f \cdot \eta(A)}{A \cdot \eta(A)\, h_A(A)}\, r(A)^{n-1} \right] \qquad (7.25)$$

This assumes that the analytical integration of $V_f(\alpha)$ is possible and the normalised gradient $\eta(\alpha)$ is easy to evaluate. An obvious case, when the conditional distribution can be written in closed form, is when the original variables are independent. Another obvious case for easy evaluation of the gradient is when the failure surface is piece-wise linear, which corresponds to the rigid-plastic assumptions in structural failures.

7.2.2 Directional importance sampling

Generally speaking, the derivation of the sampling distribution in integral estimation is a case-dependent problem, as the shape of the sampling density should be as close as possible to that of the integrand. In most of the cases for which the MC method is considered to be the last resort, it is very difficult if not impossible to find a tractable function of similar shape to the integrand. However, the numerical experience in importance sampling has shown that substantial improvement on the accuracy can be obtained even with functions of approximate shapes. Some convenient 'pass key' sampling distributions are described in the following.

7.2.2.1 Some typical sampling distributions

Most of the common sampling distributions are related to the normal distribution, the main reason is its convenience and ease of use.

a. the standard normal distribution centred at the mean point (Fig. 7.2): it results in uniform directional sampling and its formulation is given in subsection 2.2.3.1.

b. the normal distribution centred at some point x^*, usually in the 'best direction', i.e. the maximum radial integral or as close as possible to it (Fig. 7.3).

The sampling density corresponding to a direction A is

$$h_A(\alpha) = \int_0^\infty \varphi_n(tA-x^*)\, t^{n-1}dt$$

$$= \frac{\varphi_n(x^*)}{\sqrt{2\pi}\; \varphi(x^{*'}A)} \int_0^\infty \exp[-(t-x^{*'}A)^2/2]\, t^{n-1}dt \qquad (7.26)$$

The last integral in this formula can be evaluated by a recursive procedure. Let S_n be such an integral

$$S_n = \int_0^\infty \exp[-(t-x^{*'}A)^2/2]\, t^{n-1}dt \qquad (7.27)$$

It can then be shown that

$$S_{n+1} = nS_{n-1} + (x^{*'}A)\,S_n \qquad (7.28)$$

if one assumes that $S_{-1}=0$ and $S_0=(2\pi)\phi(-x^{*'}A)$

c. the *U-truncated standard normal distribution* (Fig 7.4): If the limit-state surface lies on a half-space, then the curvatures at the checking point are all positive or negative. Let U be a convex set the boundaries of which coincide in the

neighbourhood of this checking point with the limit-state surface and the curvature has the same sign at this point. This set can theoretically be used to construct a U-truncated standard normal distribution. If U is such that its probability content can be exactly evaluated and it can be easily sampled from, this distribution is very convenient as the radial integral can be expressed in term of the chi-square distribution. A typical set is the set whose cross-sections normal to the direction of x^*, is an *(n-1)-sphere;* this includes the half-space bounded by the hyperplane normal to η and at distance β to the origin, and all circular paraboloids tangent to it at x^*. Let p_U be the probability content of U and $I_U[.]$ the indicator function of U. The expression for the truncated density is

$$\varphi_{n,tr}(x,U,\beta) = I_U[x] \frac{\varphi_n(x)}{p_U} \qquad (7.29)$$

This simplifies, for the hyperplane normal to η , to

$$\varphi_{n,tr}(x,\eta,\beta) = I[\eta'.\alpha] \frac{\varphi_n(x)}{\Phi(-\beta)} \qquad (7.30)$$

The directional sampling density can then be shown to be

$$h_A(\alpha) = \sqrt{2\pi} \frac{\Gamma(n/2)2^{n/2-1}}{\Phi(-\beta)} [1 - \chi_n 2(\frac{\beta^2}{(\eta'.\alpha)^2})].I[\eta'.\alpha]$$

$$(7.31)$$

Samples from a *U-truncated* distribution are obtained as follows

$$\alpha = \frac{u + (v-u'\eta)\eta}{\|u + (v-u'\eta)\eta\|} \tag{7.32}$$

where v is sampled from a β-truncated normal distribution, and u is an outcome from a *truncated* multinormal standardised distribution to the *n-sphere* of same diameter as that of the *(n-1)-sphere* cross-section normal to η at distance v to the origin.

d. the simplex-truncated standard normal distribution (Fig. 7.5). This is a generalisation of the previous case. The sampling domain is bounded by a set of m (not exceeding n) hyperplanes of normal vectors η_i and distances to the origin β_i, and is defined as follows :

$$\varphi_{tr}(x, N'N) = \frac{\varphi_n(x)}{\Phi(-B, N'N)} I[N'x-B] \tag{7.33}$$

where

$$N = \{\eta_1, \eta_2, \ldots, \eta_m\} \tag{7.34}$$

and

$$B = \{\beta_1, \beta_2, \ldots, \beta_m\} \tag{7.35}$$

The truncated density is

$$h_A(\alpha) = \sqrt{2\pi} \frac{\Gamma(n/2) 2^{n/2-1}}{\Phi_m(-B, N'N)} [(1 - \chi_n^2 (\min_i \frac{\beta_i^2}{(\eta'_i\alpha)^2})] . I[N'\alpha] \tag{7.36}$$

Outcome from this distribution can be obtained by

normalising standard gaussian outcomes from the simplex defined by the *m* hyperplanes. A sampling procedure based on conditioning is described by Bjerager [1988].

This distribution is suitable for multiple checking points problems, or rather multiple 'checking directions'. The β_i's should be chosen to reflect the importance of the 'checking directions', using the radial transformation described in Chapter 3.

$$1 - \chi_n^2(\beta_i^2) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_{r(\alpha_i^*)}^{\infty} f_X(t\alpha_i^*) t^{n-1} dt \qquad (7.37)$$

where α_i^* is the ith 'checking direction'. Each hyperplane generated in this way can be considered as a local 'equivalent gaussian approximating hyperplane' and the probability content of the simplex is a FORM approximation to the exact probability.

This can be explained by the fact that the sampling density and the integrand coincide in the most important directions, and are expected to have close values in neighbouring directions.

The probability content of the simplex can be estimated using the methods described in Chapter 4, by sequential conditioning and hyperplane representation, or if the number of β-points is small enough, bounding techniques of Chapter 5 can give acceptable bounds.

Another point concerning this sampling distribution is the number of β-points that can be considered, which cannot exceed the dimension of the problem. This means that, in some cases, important regions may be systematically overlooked.

e. The class of mixed standard normal and gaussian-derived distribution

$$h_Y(y) - w.\varphi_n(y) + (1-w)\theta_Y(y) \tag{7.38}$$

where w is the mixing probability and θ_Y can be any of the previously described distributions.

Sampling from the standard gaussian (uniform directions) is carried out with probability w and sampling from the θ_Y distribution (importance sampling) is done with probability $1-w$. The use of mixed distribution is recommended when the identification of the important regions is believed to be uncertain. The uniform fraction of the distribution would ensure that unidentified important regions are sampled from.

f.The class of distributions formed by mixing several gaussian-derived distributions

The basic principle is the same as in equation 7.8 used by Melchers in hit-or-miss importance sampling. Each gaussian-derived distribution corresponds to an important region or a dominant failure mode.

This mixed distribution is much more flexible than the simplex-truncated one, as there is no need to approximate the probability content of the simplex-domain, and there is no restriction on the number of the β-points or dominant failure modes, although some restriction may be needed for other specific reasons. Moreover, sampling from the simplex is much more involved and needs solving linear systems for each sampled direction [Bjerager 1988]

7.2.2.2 Updating sampling density:

The shape of the sampling densities used are seldom similar to that of the integrand, as the optimal parameters values are not known prior to sampling. Good starting values for those parameters may be guessed by the investigator, but he may want to improve this values as the sampling proceeds and update the density accordingly. Among the parameters concerned are the modal points (or rather modal directions), the covariance matrix and the mixing parameters.

7.2.2.2.1 Modal points updating :

For the modal points updating, the adaptive random search for the maximum of the integrand, as discussed in section 7.1.2.2 for the hit-or-miss importance sampling, may be used. The only difference is in the objective function which is now the radial integral instead of the point density. The limitations and drawbacks of the method discussed in section 7.1.2.2, may be reduced by

- a new sampling strategy such that most use is made of information available at each stage of sampling, and to locate the maximum possible number of important regions.

- inclusion of bayesian estimations of confidence level for the number and/or the important regions.

This second point would have an informative role, as bayesian rules could be used to know how likely the current approximate modal points are close to the exact ones, and the chance of any important region not being represented by a modal point. This could help to assess how effective importance sampling is as the sampling proceeds, and to estimate how many more points are yet to be sampled to meet a given confidence level on a given accuracy target.

The approach suggested by Melchers [1988c], even improved by bayesian rules, will still suffer in the following two situations concerning the starting points

- with 'bad' starting points, the sampling may get trapped in non-important regions for a significant fraction of the whole sample, and thus may result in underestimating the probability. This needs some sampling variance tuning, linked to the rate of success in improving these points.

- with a 'very good' starting point (i.e. with high integrand value), any other important region not already represented, with local maximum slightly lesser than the initial ones, have a very low chance of been selected as a modal area, no matter how significant is its contribution. Even the selection of other more important

regions is very dependent on the initial sampling variances.

It follows that the method would be very sensitive to the initial conditions of modal points and variances. A significant part of the sample may be used up before importance sampling becomes effective, leading to biased estimates.

A new sampling strategy should be conceived in order to reduce the risk of those shortcomings, or at least to decide how much should be spent a priori for 'bad importance sampling' initially in order to identify the maximum number of significantly important regions. The sampling process can be carried out in two phases:

>   i. An optimisation phase; in which a stochastic multiextremal optimisation (see Chapter 6) of the integrand is undertaken using the first phase sample points, resulting in the identification of a set of local maxima. The points are sampled from some appropriate distribution and their corresponding integrand values are accumulated for use in probability estimation.

>   ii. An importance sampling phase; using the local maxima found in the first phase as modal points, a new sampling distribution is constructed, and an effective importance sampling from such distribution takes over the initial sampling. The optimisation phase can still be carried

out, if it is believed that more important regions could be discovered, or if the locations of the already found modal points have not been obtained by conventional and accurate methods. This can be done by allowing a uniform fraction in the current sampling distribution so that the scanning for more important regions can continue during the importance sampling phase. This uniform phase also avoids biased estimates, if the shape of the non-uniform part of the sampling distribution is not close enough to the shape of the objective function.

The details of the method and a corresponding algorithm are given in section 7.3.

7.2.2.2.2 Variance with samples from different densities:

Given the outcomes of m different samples drawn from different densities and their sample averages $\{P_1, \ldots, P_n\}$, and a set of weighting coefficients $\{w_1, \ldots, w_m\}$, an unbiased estimate of the probability is

$$p_f = \sum_{i=1}^{m} w_i N_i P_i \qquad (7.39)$$

where the N_i's are the sample sizes and the weighting coefficients w_i are such that

$$\sum_{i=1}^{m} w_i N_i = 1 \qquad (7.40)$$

In [Ditlevsen & Bjerager, 1989], a formulation of the overall variance is given based on the assumption that the w_i's are deterministic. It is shown that an unbiased estimate of the variance is

$$\hat{Var}[\hat{p}_f] = \frac{\sum_{i=1}^{m} [\frac{w_i^2}{1-2w_i} \sum_{k=1}^{N_i} (p_{ik}-\hat{p}_f)^2]}{1 + \sum_{i=1}^{m} \frac{N_i w_i^2}{(1-2w_i)}} \qquad (7.41)$$

which is the generalisation of the case $w_i=1/N$. Theoretically, if the m variances are known, optimal weighting coefficients are inversely proportional to these variances. But the use of variance estimates corresponding to different densities, would yield different weights for different sets of samples, which would make the weights random rather than deterministic, thus contradicting the basic assumption which led to equation 7.41. It seems that more advanced methods are necessary in order to find the optimal combination of the outcomes of different sampling densities, which minimises the overall variance.

7.2.2.3 Parametric sensitivity analysis

In the probability integration discussed so far, it is assumed that the distribution parameters are known with an acceptable accuracy, and are supplied as data. In a single reliability analysis, of a structure with known geometric properties, one

might need to measure the effect of a possible uncertainty on the assumed value of some particular parameter. More important is the case when some parameters are assumed unknown and are to be optimised in order to either minimize the probability or to meet some prefixed target probability. This is often the case of the parameters depending directly on the cross sectional properties (such as the resistance mean values, which are considered as design variables in reliability-based design problems). A convenient way of measuring the sensitivity of the probability to parameter variations is to evaluate the gradient with respect to those parameters. As the evaluation of the probability is already a costly process, one would surely expect that evaluation of its gradient to be even more expensive. In the following, is a derivation of the analytical expression of this gradient, and a finite-difference evaluation using the sample information already available for the probability evaluation.

A change in a given parameter might induce a change in three different functions:

  a. the original distribution f_x, a typical parameter is the coefficient of variation of a given variable.

  b. the limit-state equation, A change in the mean values results in a shifting of the limit-state surface and possibly in a change also in the shape.

  c. the sampling distribution, in the importance sampling case, where the sampling density is derived from the limit-state surface.

Let I_1 and I_2 be respectively the integrals in equations 7.9 and 7.10. The general formulation of the derivative of the probability of failure, expressed as a directional expected value, with respect to a given parameter ζ, is therefore

$$\frac{\partial p_f}{\partial \zeta} = E_A \left[\frac{\frac{\partial I_1}{\partial \zeta} I_2 - I_1 \frac{\partial I_2}{\partial \zeta}}{(I_2)^2} \right] \tag{7.42}$$

The derivative of I_1 can be expressed as follows

$$\frac{\partial I_1}{\partial \zeta} = \int_{r(\alpha)}^{\infty} \frac{\partial f_X(t\alpha)}{\partial \zeta} t^{n-1} dt - \frac{\partial r(\alpha)}{\partial \zeta} f_X(r(\alpha)) r^{n-1} \tag{7.43}$$

while the derivative of I_2 is simply

$$\frac{\partial I_2}{\partial \zeta} = \int_{0}^{\infty} \frac{\partial h_Y(t\alpha)}{\partial \zeta} t^{n-1} dt \tag{7.44}$$

The derivative of $r(A)$ with respect to ζ can then be shown to be

$$\frac{\partial r(A)}{\partial \zeta} = \frac{1}{\nabla G(A, \zeta) . A} \frac{-\partial G(r(A) A, \zeta)}{\partial \zeta} \tag{7.45}$$

The derivative of the probability of failure with respect to the parameter ζ is therefore

$$\frac{\partial p_f}{\partial \zeta} = E_A \left[\frac{1}{I_2} \left(\int_{r(A)}^{\infty} \frac{\partial f_X(tA)}{\partial \zeta} t^{n-1} dt + \frac{1}{A'.\nabla(A,\zeta)} \frac{\partial G(r(A) A, \zeta)}{\partial \zeta} f_X(r(A)A) \right) \right]$$
$$- E_A \left[\frac{I_1}{I_2^2} \int_{0}^{\infty} \frac{\partial h_Y(tA)}{\partial \zeta} t^{n-1} dt \right] \tag{7.46}$$

The expression above for the evaluation of the sensitivity factors, suggests the use of MC estimation by directional sampling. A straightforward idea is to use the same sample directions as for the probability. But one should bear in mind that a given function and its derivative are rarely of similar shape. Therefore, unless it is uniform, a sampling distribution tailored for a given function would not necessarily be good for its derivative and may yield a very poor estimate. It follows that, if importance sampling is performed for the probability, a new importance sampling distribution should be devised for the sensitivity factors using new sample directions. This necessitates repetition of the same process of importance sampling integration from scratch for every parameter considered. Moreover, for nonlinear limit-state functions not known in a closed form, the gradient evaluation can be very expensive; a finite difference involving a number of structural analysis for each simulation is often necessary.

The discussion in the above paragraph shows that an efficient use of the derivative in sensitivity factors estimation is very costly, and any approach that can make use of the original sample would be far more viable. If one considers the probability as a parametric function of ζ, and if a good sampling distribution is found for a given value of ζ, it can be used without significant loss of accuracy for $\zeta+d\zeta$, where $d\zeta$ is a small finite change in ζ. This would yield a good

finite difference approximation to the derivative at ζ. Moreover, the use of the same sample sequence for both estimations leads to a significant reduction in the variance of the final estimate. This is the well known *correlated sampling* method for variance reduction [Rubinstein 1981]. Although it needs two probability evaluations, it is not difficult to see that the method needs less calculations than the use of the derivative expression 7.46.

Let p_{f1} and p_{f2} be the probabilities corresponding to the parameter values ζ and $\zeta + \Delta\zeta$ respectively , where $\Delta\zeta$ is a finite change in ζ. An estimate of the rate of change in probability per unit of ζ is

$$\frac{\Delta \hat{p}_f}{\Delta \zeta} = \frac{\hat{p}_{f2} - \hat{p}_{f1}}{\Delta \zeta} \qquad (7.47)$$

The variance of Δp_f is

$$Var[\Delta p_f] = Var[\hat{p}_{f1}] + Var[\hat{p}_{f2}] - 2Cov[\hat{p}_{f1}, \hat{p}_{f2}] \qquad (7.48)$$

This variance is minimal, if the covariance between the estimates is maximal. Due to the small change in ζ, and the use of the same sequence of samples for the estimate of both probabilities, a high positive correlation between them is ensured. It follows that a good importance sampling distribution for the probability estimate can also ensure good estimate for the sensitivity factors. The burden of evaluating a new sampling distribution is removed along with the cost of new samples, if the two probability estimates are carried out

simultaneously.

7.3 A STOCHASTIC SEARCH BASED DIMS ALGORITHM

7.3.0 Introduction

In this section the algorithm described in chapter 6 is adapted in order to be incorporated in an importance sampling integration scheme, with the task of finding the dominant directions to be used as modal points for the sampling distribution. As presented in Chapter 6, the algorithm assumes that no prior knowledge is available about the objective function , except that it is or can be defined over a unit hypersphere.

For the problem under investigation, there is always some information available either on the shape of the feasible domain S or the objective function or both, and can be used to improve or derive some variant of the algorithm. This information can affect the clustering procedure.

It follows that this algorithm can be modified to suit the particularities of the structure and its assumed failure model. The case of linear limit-state is examined with gaussian variables and an efficient variant of the algorithm is presented.

7.3.1 Linear limit-state and gaussian variables

For a structural system with e members where the failure is defined as the failure of k members, the failure mode is identified by this k-subset. If the joint distribution is normal or closer to normal, it is possible in this case to optimise the number of local searches such that no local optimum is found more than once. Moreover, if the failure surface corresponding to any failure mode can be expressed in a closed analytical form, then the local search can be carried out with one single constraint at a time.

A typical case, which is described in the following, is that of a rigid-plastic model for structural failures. The linearity of the limit-state is exploited to make the algorithm of Chapter 6 more efficient. As the gradient of the limit-state is piece-wise constant, two points with the same gradient can be assumed to belong to the same region of attraction, which suggests another criterion for clustering the sample points according to the failure mechanisms. The clustering procedure would result in the identification of a set of hyperplanes. A failure mechanism is identified by the set of rotations at the potential hinge locations. The values of these rotations are given by the reduced cost matrix of the linear programming tableau, which identifies the hinge locations for each simulated direction. This means that one needs to store this matrix for every sample direction with objective value above the threshold. The storage requirement would be very high and the cost of handling the comparison between mechanisms may become prohibitive as the structure becomes larger. As the mechanism can be identified through the

indices of the hinge locations, no storage of the rotation values is needed. This reduces the storage requirements dramatically, by storing only the indices of the hinged sections. Despite this, the storage requirement and its corresponding computing time remain very high for large sample sizes and large structures. To overcome this problem, the following approach is suggested.

Let I_{ph} be the set of indices of potential hinge locations and n_h their number

$$I_{ph} = \{ 1,2,\ldots,n_h \} \qquad (7.49)$$

A kinematically admissible mechanism m is identified by the subset I_m of its k hinge indices

$$I_m = \{ i_1,i_2,\ldots,i_k \} \qquad (7.50)$$

and it can be shown that the quantity

$$K = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_k} \qquad (7.51)$$

corresponds to a unique subset of I_{ph}, i.e. two different subsets yield two different values of K. It follows that storing K is equivalent to storing I_{ph}. But the storage space is theoretically k times smaller, so that with a single column integer array one can keep record of all mechanisms identified as the sampling proceeds. In other words, K is a 'finger-print' code of the mechanism. However, as the value of

K grows exponentially with that of n_h , it might exceed the maximum integer representable by the machine. Let s be the maximal integer such that 2^s is representable. The mechanism code number K can be factored as polynomial expression of 2^s as follows

$$K = \sum_{l=0}^{L} a_l (2^s)^l \qquad (7.52)$$

where the coefficient a_l is

$$a_l = \sum_{sl \le i_r \le s(l+1)} 2^{(i_r - sl)} \qquad (7.53)$$

the indices i_r are those of the current mechanism and L the degree of the polynomial factorisation. The storage, which now concerns the above coefficients, needs an integer array of $l+1$ columns. For example if $s=30$, and the maximal hinge index is 90, each mechanism would need four integer storage, this is still more manageable and less expensive than storing all hinge indices.

The coupling of this coding procedure with the multilevel single linkage procedure ensures that each local search result in a new local optimum (i.e. origin projection point of a significant failure mode, not outside the safe domain).

The local search consists of successive evaluations of the origin projections of the limit-state hyperplanes, and plastic analysis until the origin projection falls on the safe set

boundaries. An immediate advantage of this new clustering procedure, is that a local search started from different points of the same failure mode lead to the same local optimum.

7.3.2 Non-gaussian variables

The clustering procedure cannot be linked to the failure modes, even for linear limit-state, and should be linked to the critical angle defined in Chapter 6. For a piece-wise linear limit-state function, the local optimisation is carried out with one linear constraint at a time, which is updated by a structural analysis, if the local optimum found falls outside the safe domain.

7.3.3 Algorithm summary

The basic steps in the modified algorithm are as follows

> **Step 1.** Sample N^* points and evaluate their integrand values, their mechanism code, the corresponding mean value and standard deviation. Evaluate the critical angle. Set $N=N^*$
>
> **Step 2.** Relabel the points in decreasing order of their function values.
>
> **Step 3.** Reduce the sample according to the current threshold.
>
> **Step 4.** Perform the clustering procedure of the reduced sample according to the critical angle or the mechanism

code.

Step 5. Perform a local search from the best point of each cluster, if not already used for such purpose, or if it does not fall within critical distance of some local optimum.

Step 6. Termination test; use the bayesian rule of equation 6.49, and integral approximation. If the test is positive, go to step 8.

Step 7. Draw a new sample of size k, evaluate the function values and mechanism codes. Set $N=N+k$. Update the critical angle and go to step 2.

Step 8. Construct a set of hyperplanes normal to each of the optimal directions found previously, and at distance β to the origin obtained from equation 3.18.

Step 9. Start the importance sampling phase, using the new sampling density based on the previously found hyperplanes.

7.3.4 Illustrative examples

A single-storey and a double-storey single bay frames shown in Fig. 7.6 and Fig. 7.7 are analyzed. The distributions considered for the plastic moments and the loads respectively are Normal/Normal, lognormal /lognormal and lognormal/Gumbel. The parameters of the variables are given in Tables (7.1) and (7.2). The results are given in Tables (7.3) and (7.4) with the sample sizes inside parentheses. The initial sample size for all cases is 200, and the number of sample points per iteration is 100. For comparison purposes, the coefficient of

variation of the simulations outcomes, as random quantities, is given instead of the variance.

The results show clearly the variance reduction effect of the directional importance sampling. For non-normal variables, it can be also seen that the number of local searches initiated is moderate compared to that of the identified important regions.

The sensitivity analysis by correlated sampling is illustrated, by the effect on the probability on a small change in the mean plastic moments, in the last column of Table (7.3).

Concerning the computing time, one is interested in assessing the extra cost due to clustering/local search and non-uniform sampling, compared to a uniform sampling run with the same overall sample size. For the case of lognormal/lognormal distribution and single storey frame for example, a 7000 sample size run, of which 6000 uniform and 1000 non-uniform, took 863 seconds (on an IBM-compatible microcomputer), and a uniform 7000 sample size run, with no clustering or optimisation, took 750 seconds, i.e. an extra cost of 15%. The former analysis gave a probability estimate of $0.96 \cdot 10^{-5}$ with a C.O.V of 0.03. In order to evaluate the efficiency of this directional simulation with stochastic optimisation method, this analysis is repeated with 50000 uniform samples. A uniform 50000 sample size run gave a probability estimate of $0.97 \cdot 10^{-5}$ and a coefficient of variation of 0.04.

	Normal		Lognormal		Gumbel	
	μ (Kn)	C.O.V	μ (Kn)	C.O.V	μ (Kn)	C.O.V
M_1	100.0	.10	110.0	.10	–	–
M_2	150.0	.10	160.0	.10	–	–
H	10.1	.20	10.1	.20	15.0	.30
V_1	53.8	.15	180.0	.15	50.0	.20

Table 7.1. Distribution parameters. Single storey frame.

	Normal or Lognormal		Gumbel	
	μ (Kn)	C.O.V	μ (Kn)	C.O.V
M_1	110.0	.15	–	–
M_2	110.0	.15	–	–
M_3	275.0	.15	–	–
H	16.0	.25	30.	.30
V_1	180.	.15	60.	.20
V_2	90.	.25	40.	.20

Table 7.2. Distribution parameters. Double storey frame.

	Uniform phase		Importance sampling phase		Num. of local sear-ches/ Num. of iden. reg.	Sensi. with respect to mean plastic moment (μ_1 and μ_2)
Distrib. (Resist./ Loads)	$p_f \times 10^5$	COV	$p_f \times 10^5$	COV		
Norm/Norm	1.17 (3200)	0.10	1.16 (1000)	.003	2/2	-.103 -.177
Logn/Logn	.92 (6000)	0.13	.96 (1000)	.03	13/7	-.108 -.210
Logn/Gumb	1.10 (6000)	0.14	.90 (1000)	.03	14/7	-.079 -.060

Table 7.3. Probabilities of failure for a single storey frame.

	Uniform phase		Importance sampling phase		Num. of local search -es.	Num. of identi. regions
Distrib. (Resist./ Loads)	$p_f \times 10^4$	COV	$p_f \times 10^4$	COV		
Norm/Norm	22.3 (6000)	.05	25.0 (1000)	.01	10	10
Logn/Logn	16.0 (4800)	.10	15.0 (1000)	.03	69	39
Logn/Gumb	1.8 (5000)	.03	1.8 (1000)	.03	31	20

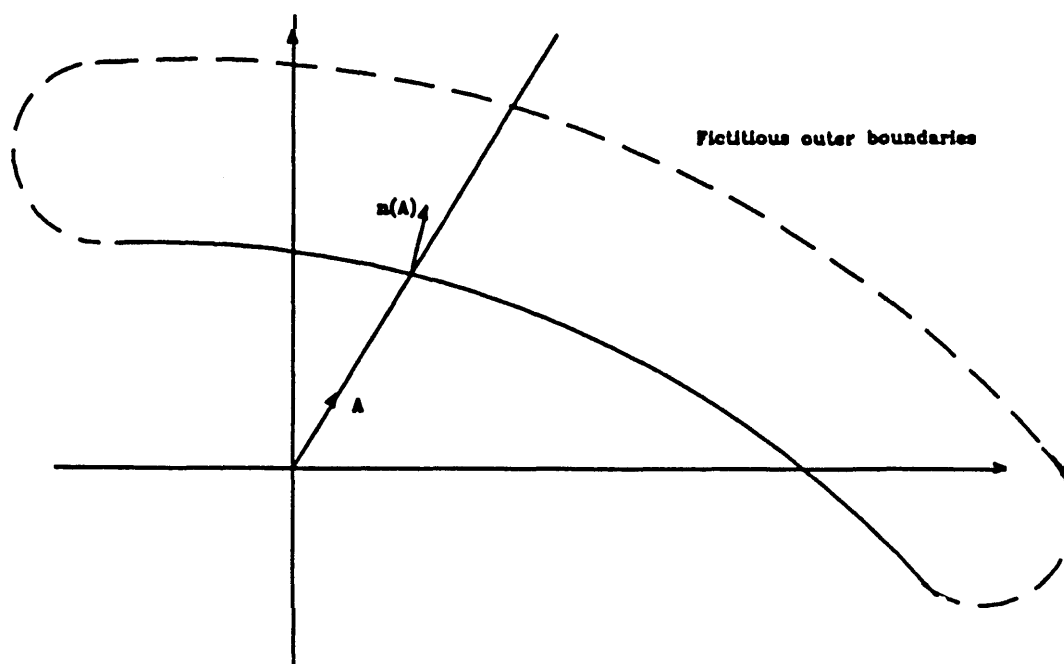**Table 7.4. Probabilities of failure for a double storey frame.**

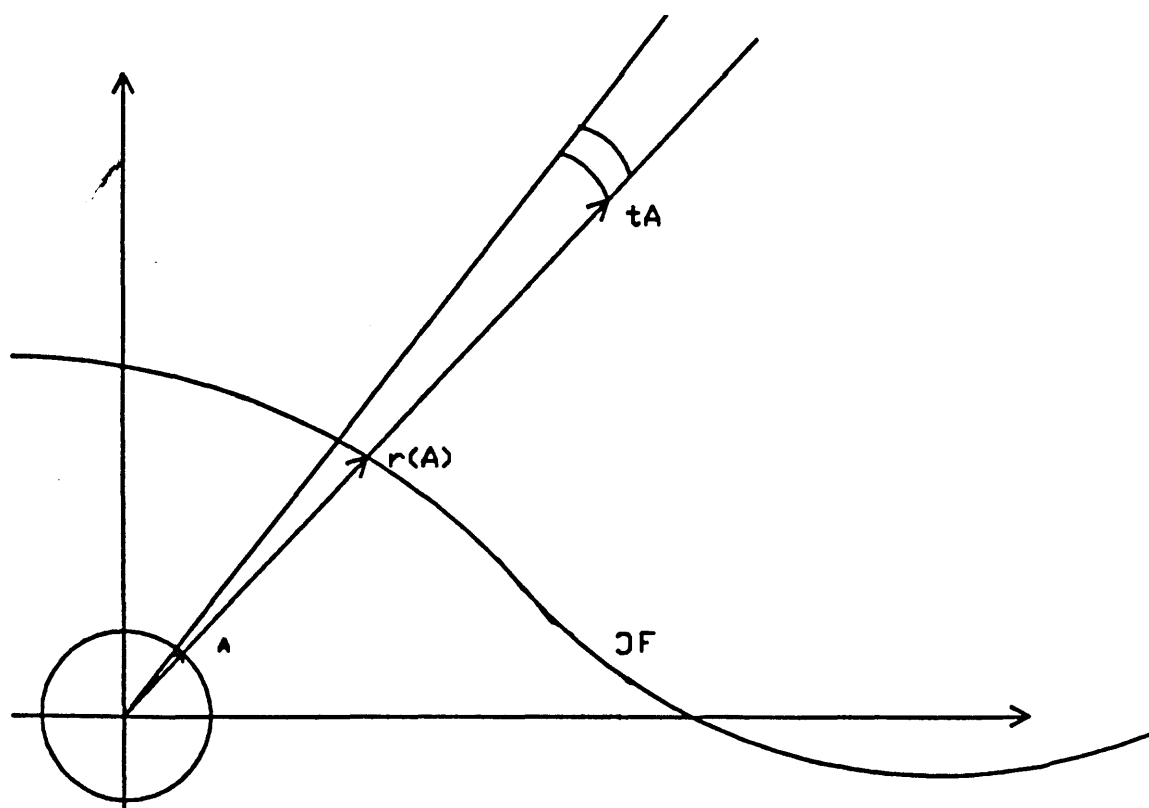**Fig. 7.1 Integration boundaries for Green's theorem formulation**
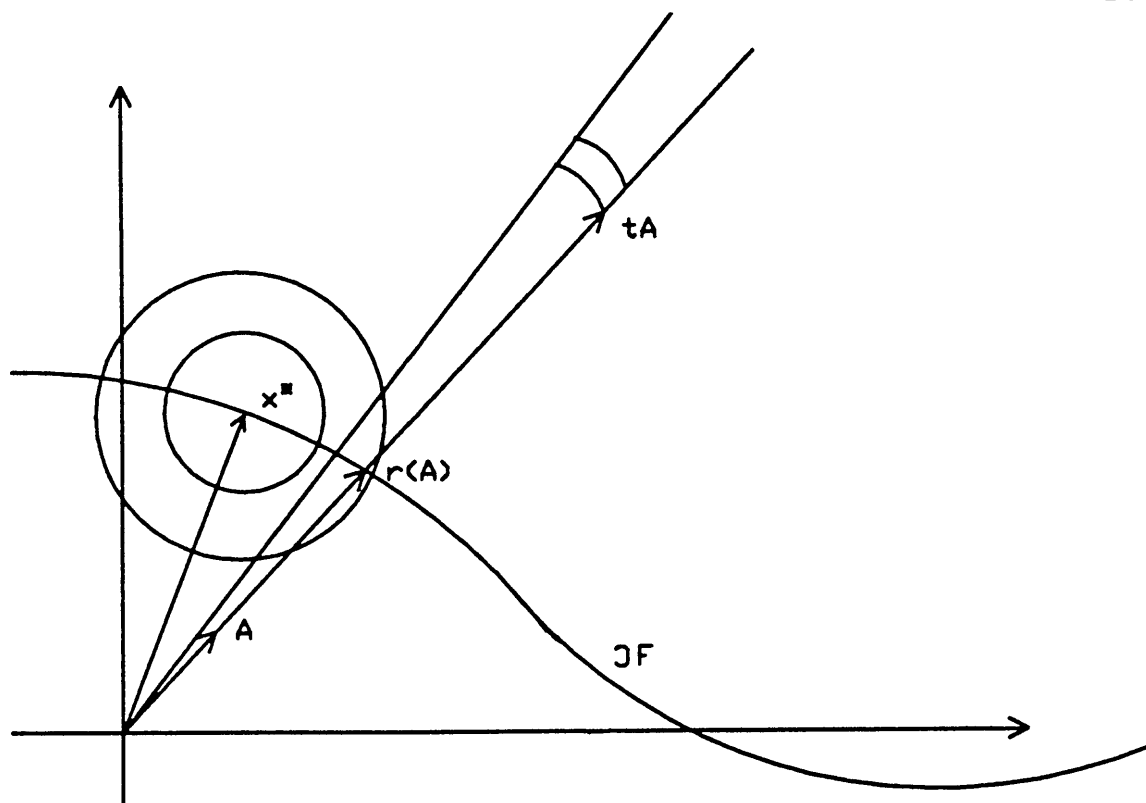

Fig 7.2 Uniform directional sampling
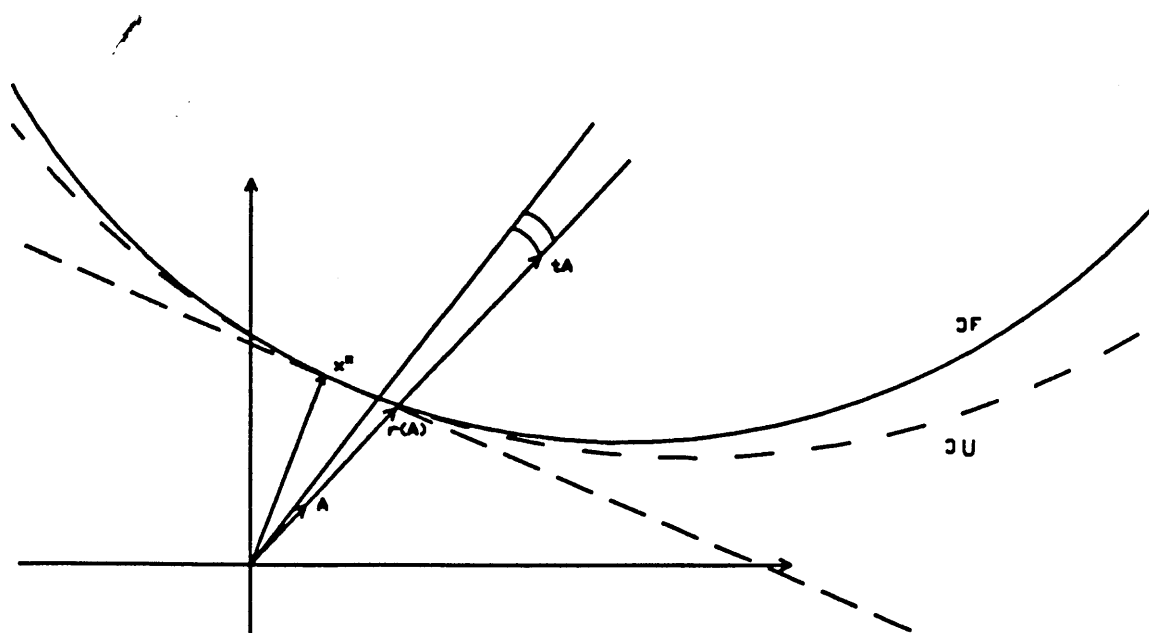
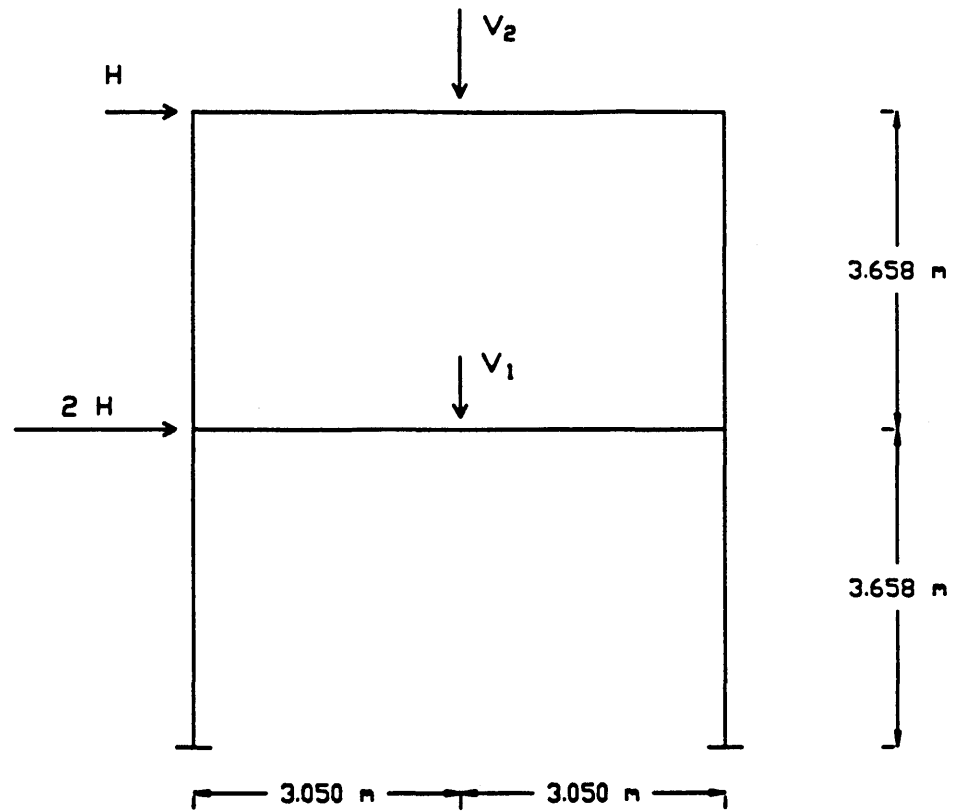**Fig. 7.3 Gaussian directional sampling**

Fig. 7.4 U-truncated gaussian sampling distribution

Fig. 7.5 Simplex-truncated gaussian sampling distribution

Fig. 7.6 A single storey frame.

Fig. 7.7 A double storey frame.

CHAPTER VIII

RELIABILITY-BASED STRUCTURAL OPTIMISATION

8.0 INTRODUCTION

A comprehensive reliability-based optimisation is a complex decision-making operation, that would take account of all aspects such as technical and scientific models, functionality, economy, as well as the human and social factors. However, not all aspects are directly quantifiable and therefore their incorporation into the optimisation model is too difficult if not impossible, and usually many decisions have to be taken only through experience and intuitive judgement of the designer. Despite the recent advancements in artificial intelligence, it cannot yet provide an operational tool for an overall optimisation including non measurable factors such as functional or human. This is why most of the work carried out so far deals with quantifiable aspects mainly concerning the cost through the material quantity utilised. By far the most studied aspect is that of sizing the cross sections, followed by that of the cross sectional shape and/or overall shape (i.e geometric nodal layout).

The concept of structural optimisation with the reliability as constraint can be traced back to Forsell as early as 1924. However, proper interest in the subject started in the 1950's

[Johnson 1953; Ferry Borges 1954; Freudenthal 1956; Paez and Torroja 1959]. The need for lighter structures in industry increased significantly that interest in the sixties, among the pioneering works in that respect were those of Hilton and Feign [1960], and Kalaba [1962], followed by contributions from Switzky [1964], Kinser [1966], Khachaturian and Heider [1966], Moses and Kinser [1967], Benjamin [1968] and Cornell [1969a].

During the seventies, the horizons of the reliability-based structural optimisation become much wider to include multicriteria cases (serviceability and ultimate limit-states) [Parimi and Cohn 1978] and sensitivity analysis for changes in the criteria, or any of the non-design random variable parameters (such as loading variables), type of distributions ... [Moses 1970; Moses and Stevenson 1970].

During the last decade, the concept of reliability-based design becomes more and more accepted among the structural engineering community and applications embrace a wide range of structures, particularly industry-related ones such as off-shore platforms and pressure vessels. Among the recent contributors are Murotsu et al. [1984], Frangopol [1985a], Moses and Feng [1986], Sorensen [1986], Frangopol and Fu [1989]. Thoft-Christensen and Sorensen [1987] discussed an integrated reliability-based optimal design concept in which the meaning of the total cost is extended to include the future cost of inspection and repair. Soltani and Corotis [1988] considered the failure cost as linear combinations of

the modal failures costs weighted by their corresponding probabilities.

8.1 PROBLEM FORMULATIONS

8.1.1 Mathematical programming formulation

The problem of reliability-based structural optimisation can be formulated in many different ways. Two main categories are a) cost minimisation with probability constraint (s), and b) probability minimisation with cost constraint and/or probability constraint. According to the number of limit-states involved, the problem is referred to as a single criterion or multicriteria optimisation. The single criterion case involves only one limit- state of failure either the full collapse or unserviceability. If the objective is to minimise the cost, the optimisation problem can be set as follows

$$\begin{aligned} & \textit{Minimise} \qquad C_t \\ & \textit{subject to:} \qquad p_f \leq p_0 \end{aligned} \qquad (8.1)$$

where p_0 is the allowable limit for the system probability of failure. A multicriteria formulation is

$$\begin{aligned} & \textit{Minimise} \qquad C_t \\ & \textit{Subject to:} \qquad p_f^{\,u} \leq p_0^{\,u} \\ & \qquad\qquad\qquad p_f^{\,s} \leq p_0^{\,s} \end{aligned} \qquad (8.2)$$

where the superscripts u and s refer to the ultimate and serviceability limit-state.

The system probability of failure can be set as objective to be minimised, for a given cost as follows

$$\begin{aligned} &\textit{Minimise} &&p_f \\ &\textit{Subject to}: &&C_t - C_0 \end{aligned} \qquad (8.3)$$

A different formulation considers constraints on the probabilities of failure of the individual members of the structure (potential hinge locations, bars for trusses) instead of the system failure probability, as follows

$$\begin{aligned} &\textit{Minimise} &&C_t \\ &\textit{Subject to}: &&p_{f_i} \leq p_{0_i} \qquad i=1,2,\ldots,n \end{aligned} \qquad (8.4)$$

This formulation has been used by many authors mainly because the probability evaluation is much easier and less costly. It may be also argued that the component reliability formulation is more flexible in allowing the risk levelling throughout the structure, something which cannot be controlled through the overall system reliability [Mahadevan and Haldar 1989] . Moreover, the interaction of different types of internal forces is much easier to take account of by considering components reliabilities. Furthermore, if the system reliability is not considered, the chance constrained program can be easily transformed into a deterministic mathematical programming one.

However, a comprehensive and rational design should include both system and components reliabilities, which can be carried out only at a very high computational cost. If both ultimate and serviceability limit-states are to considered (multicriteria optimisation), the problem becomes even more costly.

8.1.2 Failure probability function as a constraint

The expression for a reliability function depends directly on the method used for its approximation, which determines both the efficiency of the method and the confidence level in the optimal solution with respect to the exact failure probability. First order second moment method was very common in the seventies, with the Hasofer-Lind reliability index as a reliability measure (of the component or system). By the late 1970's and early 1980's, the so-called advanced first-order reliability methods based on normal tail approximation led to first order estimate of the probability of failure. In the mean time, the crude Monte-Carlo and bounding techniques were also used. Improvements of the failure probability estimates were sought through the so-called second order reliability method (SORM) [Breitung 1984]. Many of the methods used in reliability-based optimisation were mixed methods where FORM or SORM and simulation techniques are coupled in order to provide the best possible estimate of the failure probability or the reliability index.

An important step in setting up the optimisation program is to decide values of the allowable limit for the probabilities of failure (or reliability indices) . Although the limit for the system failure probability may be considered, up to some extent, as a matter of decision for the client, setting limit(s) corresponding to the components is more an engineering problem than a client's requirement. Setting a uniform target reliability, for all components, is not believed to be a good basis for a rational design. The consequences of a failure of a given member depends on its location within the structure on one hand, and on the limit-state considered on the other. It follows that each component or group of components must have its own target reliability for full collapse and for unserviceability. Moreover, the set of the components target reliabilities must be consistent within itself and with the system target reliability. This does not seem to have attracted the attention of the researchers in the subject, and the matter has been left to the subjective judgement and the experience of the designer. Although the designer's intuition and experience are necessary, a scientific basis for the allowable failure probabilities will be always desirable.

8.1.3 Cost function as an objective (or a constraint)

The cost function may be defined as the total of the initial construction cost and the cost of the damage due to the failure of the structure

$$C_t = C_i + C_f p_f \qquad (8.5)$$

The second term C_f is very difficult to estimate, as it means assigning cost values to human life, social consequences or similar damage. A credible estimate should based on the expertise of economists and jurists, and not left to the structural engineer to decide. As long as no precise rules are not laid down in the official codes, account of the failure cost in the optimum design will remain an academic topic.

The first term C_i is mainly due to the material weight of the structure, and can be expressed as function of the cross section sizes for framed structures. Generally, the initial cost function is taken as the structural volume

$$W = \sum_{1}^{n} l_i A_i \qquad (8.6)$$

where l_i is the length of the ith member and A_i its cross-section area. This expression can be directly used for trusses. For framed structures, the limit-state analysis is carried out in terms of the plastic moments (or the moments of inertia.). The cross sectional area and the plastic moment are found to be related by the formula

$$A = \kappa M_p^{\gamma} \qquad (8.7)$$

where κ and γ are constants depending on the material and the cross-sectional shape. This makes the objective function nonlinear and adds a great deal of difficulty to the problem. Based on a suggestion by Save and Massonnet [1965], many

authors use a linear expression of the plastic moments as substitute to the structural weight

$$W = \sum_{i}^{n} l_i M_{p_i} \qquad (8.8)$$

which assumes that sizing the plastic moments is almost equivalent to sizing the cross sectional areas.

The interaction between initial cost and failure cost has been addressed by Moses [1977]. Soltani and Corotis [1988] proposed a multi-objective formulation with both costs as objectives. The curve of c_i versus p_f is a monotonically decreasing function, while the failure cost increases with the failure probability. It follows that the total cost , as function of the failure probability, can present a minimum value for some value of p_f, which can be found by unconstrained minimisation. This would yield an optimum value for failure probability, and remove the problem of selecting an allowable value for it. However, due to the difficulty mentioned earlier of setting values to c_f , the unconstrained total cost minimisation has been avoided for lack of confidence in its outcome.

8.2 SOLUTION METHODS

In the following is a review of the methods that are most used in reliability-based structural optimisation. They are by no

means the only ones that can be used among the multitude of non-linear programming methods. From the recorded experience in the subject, it is very difficult to decide which non-linear programming algorithm is most efficient mainly because different authors adopt different definitions for failure and different methods for their probability estimation.

The problem can be classified as *'chance constrained programming'* one, if the components probabilities only are considered, and *'joint-chance constrained programming'* for the case of system reliability. One would expect the problem to be treated as one of stochastic programming. However, most of the methods used are conventional nonlinear programming algorithms.

A commonly used approach is Zoutendijk's feasible directions method [Moses 1969; Vanderplaats and Moses 1973; Frangopol 1984, 1985a]. Starting from a feasible point x_0, a new feasible point of a better objective value x is found by a line-search in a direction s such that

$$x = x_0 + \alpha s \qquad \alpha > 0 \qquad\qquad (8.9)$$

where s and α are evaluated such that the improvement in the objective function is maximal (usable direction), without leaving the safe set (feasible direction). This means that the direction s should not lie outside the cone defined by the tangent hyperplanes to the objective hypersurface and the active constraints at x_0. An optimal choice of s is found by

solving the linear program

$$\text{Maximise} \qquad \gamma$$

$$\text{subject to :} \qquad s^T.\nabla p_f + \xi\gamma \leq 0$$

$$s^T.\nabla W + \gamma \leq 0 \qquad (8.10)$$

$$-1 \leq s_i \leq 1$$

where ξ is a fixed constant controlling the extent to which s is projected away from the safe set hypersurface and it is known as the *push-off factor*. The main advantage of this algorithm is that it operates only in the feasible region, generating an improving sequence of feasible points, so that even if it fails to converge to the optimum solution, the designer is left with a better design. Concerning the speed of convergence, the LP of equation 8.10 ensures that the shortest path toward the optimum solution is followed. Considering the line-search along s as very expensive for large design applications, Belsare and Arora [1983] proposed instead (for deterministic design) the evaluation of bounds on the optimum cost function value, and a systematic search to be carried out in the design space between these bounds. The bounds gap is successively reduced until the optimum solution is found. This idea sounds very attractive, however finding a lower bound for the optimum solution seem to very difficult, and no practical approach was suggested by the above authors. Showing that for a given cost value, no feasible design is possible, is not an obvious task.

Another popular method is the *barrier function method*, also known as the *interior-point unconstrained minimisation*

technique [Frangopol 1984]. Its use in structural design in general, and reliability-based design in particular, is reassuring in a sense that it generates improving points within the safe set. The constrained problem is transformed into an unconstrained one of the form

$$\zeta(x,r) = W(x) + r\varpi(p_f) \qquad r > 0 \qquad (8.11)$$

where ϖ is a positive function of the failure probability (constraint) on the interior of the feasible domain (usually logarithmic or inverse function). A main weakness of the method is that for small values of r, the problem becomes ill-conditioned and difficult to solve. This sensitivity to r can be significantly reduced by using the so-called *extended barrier functions* [Cassis and Schmit 1976], [Prasad and Haftka 1979], [Prasad 1980].

A similar approach to the barrier function method is the *penalty function method*, also known as the *exterior-point unconstrained minimisation technique*. A penalty term, function of the constraint(s), is added to the objective so that a high cost is associated with the constraint violation, resulting in an unconstrained problem. Its use in optimum design has been inhibited by the fact that the optimum solution is approached from the unsafe set. Its has the same ill-conditioning problems as the barrier method, but this can be overcome by using the *augmented lagrangian method*. The penalty function is then added to the lagrangian function instead of the objective function.

A different class of methods is based on linearization techniques. A straightforward one is the *recursive linear programming method*, used by Morutsu et al. [1984]. It consists of solving a sequence of linear programs obtained by linearization of the objective and the constraints at the optimum solution of the previous LP. As the method converges toward a vertex of the safe set, it can only be used if the components reliabilities are considered, where their number is usually higher than that of the design variables. A slightly more sophisticated approach is the Kelly's *cutting plane method* (Kelly, J.E. 1960). Polyhedral envelopes to the constraints are built up by adding linearizations of the constraints at each iteration. As the reliability-based optimisation is not necessarily a convex problem, the cutting plane method is not recommended because safe regions containing the global optimum may be cut off at an intermediate stage. Moreover, the sequence of points generated falls mainly in the failure set.

One might think of using the *recursive quadratic programming method* in reliability-based design. It consists of solving the stationarity conditions of the lagrangian function using Newton methods. This assumes the evaluation of the Hessian matrix, which is in reliability-based optimisation is a very costly process, especially if the system reliability is considered. The use of quasi-Newton algorithms needs the approximation of the Hessian matrix via recursive update formulae [Broyden 1967; Fletchers and Powell 1963]. However, despite the low computational cost, due to the inherent low

accuracy in system reliability estimation and very low failure probabilities in structural engineering, Hessian estimation may be of doubtful validity.

Gradient projections techniques could also be used in reliability-based optimisation. As they consist of a steepest descent along the active constraints surface, using gradients information, it is almost equivalent to the feasible directions method used with zero push-off factor. The sequence of points generated falls outside the safe set, and their restoration involves an additional cost, often carried out in the non-improving direction of the objective.

Studies for linear cost function [Frangopol 1984 and Moses 1969], show that the shape of the isoprobability contours in the design space is such that the locus of the minima of the failure probability for constant cost presents very smooth curvatures, almost linear. It follows that once one point is found on this locus, a good approximation can be obtained by moving along the objective's gradient direction toward the boundaries of the feasible domain. The new point can be either accepted or used as a starting point for the next iteration. The optimisation of the failure probability, at constant cost, can therefore be a key step for a fast reliability-based design, for linear or very smooth cost function. It can be easily transformed into an unconstrained program with one less dimension. A more extensive study on the shapes of the isoprobability contours would be very useful in the assessment of the efficiency of such approach.

8.3 SENSITIVITY ANALYSIS

Due to the lack of sufficient data, the statistical parameters used in reliability analysis are never known exactly. They are rather estimated from records over a limited period of time or from finite number of measurements, due to cost limitations. Moreover, the mathematical models used, both probabilistic and structural, are idealisations of the true physical and stochastic behaviour of the structures and this is on top of the human factors which results in errors of analysis and judgement. The resulting uncertainties have a direct influence on the level of confidence in the optimum design. A sensitivity analysis would show the impact of those uncertainties on the optimum design and optimum cost. This would help to set up a minimum level of accuracy for the non-design parameters involved, and to identify the critical ones which have most influence on the overall design. Improvements in the efficiency of reliability-based optimisation techniques can be devised through the knowledge of the effect of the various parameters.

The early known sensitivity study is due to Moses [1970], were the cost sensitivity to change in statistical parameters, idealisation model and analysis errors are addressed. Sensitivity of the optimum design of a frame structure, to the allowable failure probability, the loads coefficients of variation and even the nature of the distribution functions, has been also studied by Moses and Stevenson [1970]. The

effect of the allowable failure probability has been also addressed by Parimi and Cohn [1978], while a sensitivity to human errors study has been carried out by Nowak [1983]. Using Vanderplaats approach [1984] to sensitivity in deterministic design, Frangopol [1985b] investigated the case of rigid-plastic structures and particularly the effect of the correlation between the resistance variables. The local effect of a change in a given parameter ζ, in the vicinity of the optimum solution, is evaluated by treating ζ as a new design variable and performing a search for the optimal feasible direction with respect to the objective function improvement.

8.4 CONCLUSIONS

The validity of the results and conclusions of optimisation and sensitivity studies mentioned above depends very much on the level of confidence in the reliability analysis approach adopted. If one excludes the cases where the Hasover-Lind reliability index measures the reliability, in most cases either Cornell bounds or the so-called Ditlevsen bounds has been used, for simple portal frames where the failure modes are identifiable by inspection and their number is very small. The gap between the bounds is small enough to give an acceptable estimate of the failure probability. However, this is not the case for large structural systems, where the bounds are significantly different from the exact failure probability. The real optimum design may be very different from the estimated one and so would be its sensitivity to the

non-design parameters variations. A more accurate methods for reliability analysis should be used for large structural systems, if the conclusions drawn from the above studies are to be valid.

The problem of failure probability minimisation, for a given cost, should be given more attention, especially the case of linear cost function. Efficient methods for solving such problem may be very helpful in solving the cost optimisation program.

The possible multiextremal character of the reliability-based optimisation has not been addressed explicitly, except that some authors recommend to repeat the optimisation process from different starting points [Frangopol 1984]. The non-convexity of the problem has been shown by Sorensen and Thoft-Christensen [1986]. The global optimisation approach (see Chapter 6) to the *reliability-based optimisation* would ensure a high confidence level in the optimum design. However, due to the high computational cost, a special and efficient global optimisation technique may be necessary. This remains a challenging topic for future research.

CHAPTER IX

CONCLUSIONS AND RECOMMENDATIONS

9.1 GENERAL CONCLUSIONS

The diverse aspects of structural reliability under time-invariant loads has been addressed, using various optimisation techniques. These aspects cover the variables' transformations and limit state surface approximations on one hand, and system reliability estimation on the other.

For structures or structural systems with polyhedral failure domains in a normal space, good estimation of the failure probability as a multinormal integral is possible. A new formulation for equivalent gaussian safety margins representation presented in this thesis appears to be fast and relatively accurate. The method uses the second moment theory but the correlation matrix is evaluated directly. Qualitative interpretation of the method is made clear by the study of the skewness of the distribution of non gaussian safety margins. Although the accuracy is not uniform over the whole probability space, it gives fairly acceptable results even for extreme cases. The case of highly reliable series systems very frequently met in structural reliability analysis, is made particularly accurate by a suitable interpolation method.

The bounds on the reliability of discrete systems has been

thoroughly investigated. Second-order bounds seem to be cost effective, as the evaluation of higher order intersections is very expensive even with fast computers. The gap between the bounds increases with number and the correlation between the safety margins, and is very dependent on the ordering of the events (safety margins). The research studies on the ordering by the author have shown that great improvements can be achieved by a proper indexation of the events, at a marginal cost. An algorithm has been developed for the ordering of the events to get optimal lower bound. This algorithm has been tested on a large number of examples and has been found to be very efficient.

For events representable by gaussian safety margins, both bounding techniques and direct estimation of the multinormal integral (chapter 4) can be used. For small number of safety margins, both methods seem to provide good accuracy; the choice should therefore be based on the computing time which seem to favour the bounding methods. As the number of safety margins grows, the gap between the bounds increases significantly, and the computing time grows much faster than that needed for direct estimation; theses estimates fall within the bounds and can be obtained at a much lower cost.

The Rosenblatt transformation has been used generally by reliability theorists to map a non-normal space into a normal one. The cost of this transformation, if the conditional probability densities cannot be integrated analytically, could be very high. An alternative approach, which preserves the

directions of the original variables, is proposed. This new approach is applicable to highly reliable systems, which is the case for structures. It is independent of the variables' numbering and allows direct evaluation of the sensitivity factors (with respect to the locally closest points to the origin) which are the same in both x and u-spaces.

The use of Monte-Carlo methods in failure probability estimation has been fully investigated. The hit-and-miss method is particularly inaccurate and needs unacceptably large samples for the estimate to be of any significance. Substantial improvements can be obtained by reducing the sampling space or by importance sampling. However significantly improved results can be obtained by directional sampling with polar coordinates. The accuracy can be further improved with directional importance sampling, provided a good sampling distribution is found. A single figure for the coefficient of variation of the estimate can be easily obtained with some gaussian-derived sampling distributions. For non normal variables, sampling directly from the original space is more interesting than sampling from the transformed normal space. Within the directional importance sampling context of probability estimation, the sensitivity of the estimate to changes in the distribution parameters is better carried out by finite difference rather than direct integration of the analytical expression of the derivative with respect to such parameters.

In order to have a good approximation of the limit state

surface, or to derive a good sampling distribution, identification of the stochastically dominant failure modes or regions is necessary. This is the most difficult and most costly part of any reliability analysis procedure, due to the high non-convexity of the problem. In normal space, the problem can be formulated as a linear program or a nonlinear one. Investigations on the multiple objective linear programming method for limit state generation, have shown that the method is not very useful for practical structures, as it is voraciously time consuming. The applicability of all these methods is limited to plastic structures, and do not guarantee the selection of all dominant modes. Heuristic methods (Branch and bound method, Beta-unzipping methods) can be applied to structures with various material properties (ductile, brittle, strainhardening,...), but can also fail to select all significant modes.

In non normal space, the discretisation into failure modes is no longer relevant for identifying the stochastically dominant parts of the failure domain, and more direct non-convex optimisation techniques are necessary. A review of the multiextremal optimisation methods suggests that deterministic approaches are not very suitable for such non-convex problems. Among the stochastic optimisation techniques reviewed, multistart clustering methods are the most suitable for our purpose. The so-called *multi level single linkage approach*, devised by Rinnoy Kan and Timmer [1984] and others for global optimisation over compact domains, is extended to functions defined over the boundaries of star-shaped domains. The

resulting new formulation turns out to be efficient in identifying the neighbourhoods of significant local maxima of objective functions, defined as the radial integral over the failure part in a given direction. This is achieved with a minimum number of local searches. Coupling of this optimisation method with probability integration by directional simulation in a two-phase approach, resulted in an efficient package for reliability analyses. The first phase scans the failure domain for dominant parts and provides an initial estimate. In the second part, information gained from the first phase is used to derive a suitable sampling distribution and a proper importance sampling is performed. The interesting feature of the procedure is that the extra cost for stochastic optimisation constitutes only a fraction of the overall cost. The result of the stochastic optimisation phase can be used as input for FORM or SORM estimation of the probability. Care must be taken, when using them, as high curvatures at the β-points may give inaccurate results.

Reliability-based optimisation of structural systems has been critically reviewed. Numerous work has been carried out by many authors during the last two decades using different conventional convex optimisation algorithms. The performances of these algorithms are very dependent on the reliability analysis approach adopted. The two aspects of reliability analysis, i.e. dominant failure modes/regions identifications and probability estimation, are tackled in different ways by different authors. Because of this any comparative assessment has only limited validity.

9.2 RECOMMENDATIONS FOR FURTHER RESEARCH

The direct estimation of the multinormal integral for series systems (chapter 4) comprises an interpolation formula. There is still room for improving the estimate by refining this formula, through more extensive numerical experiments on equicorrelated safety margins. Derivation of analytical expressions or numerical procedures for bounding the error on the estimate remains a challenging topic.

Concerning the system reliability bounds, the linear programming formulation of Kounias and Marin [1976] is worth further exploration aiming at generating more extreme points. A challenging topic is to prove whether extreme points, other than those with 0's and 1's, can yield better bounds.

Despite the numerous efforts made by many authors in devising techniques for dominant failure modes/regions identification, existing methods are not very efficient and more research in this direction is necessary. The method developed in chapter 6 and applied in chapter 7, can be significantly improved, if

- methods of optimising the choice of the critical angle on which the clustering procedures is based, can be developed.

- A 'good stopping rules' can be derived for the optimisation process.

Methods of constructing good sampling distributions, which

imitate closely the radial integral function, would improve dramatically the accuracy of the estimates obtained by directional importance sampling. A great deal of progress is expected to be made in that respect in the very near future. The change of sampling distribution during the sampling process needs an optimal weighting for combining the different estimates resulting from each sampling distribution, such that the overall variance is minimised. Development of highly involved statistical techniques for such purpose is an open topic for future research.

Finally, in reliability-based optimisation, more attention should be given to minimising the probability of failure for a fixed cost, for the case of linear cost function. Its optimal solution may lead directly to optimal probabilities for different cost values. The reliability-based optimisation should be addressed as a proper multiextremal problem and special methods should be devised for such purpose.

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