



**UNIVERSITY OF LEEDS**

This is a repository copy of *On the origin of dual Lax pairs and their r-matrix structure*.

White Rose Research Online URL for this paper:  
<http://eprints.whiterose.ac.uk/117931/>

Version: Accepted Version

---

**Article:**

Avan, J and Caudrelier, V (2017) On the origin of dual Lax pairs and their r-matrix structure. *Journal of Geometry and Physics*, 120. pp. 106-128. ISSN 0393-0440

<https://doi.org/10.1016/j.geomphys.2017.05.010>

---

© 2017 Elsevier B.V. This manuscript version is made available under the CC-BY-NC-ND 4.0 license <http://creativecommons.org/licenses/by-nc-nd/4.0/>

**Reuse**

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

**Takedown**

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing [eprints@whiterose.ac.uk](mailto:eprints@whiterose.ac.uk) including the URL of the record and the reason for the withdrawal request.



[eprints@whiterose.ac.uk](mailto:eprints@whiterose.ac.uk)  
<https://eprints.whiterose.ac.uk/>

## Accepted Manuscript

On the origin of dual Lax pairs and their  $r$ -matrix structure

Jean Avan, Vincent Caudrelier

PII: S0393-0440(17)30132-8

DOI: <http://dx.doi.org/10.1016/j.geomphys.2017.05.010>

Reference: GEOPHY 3001

To appear in: *Journal of Geometry and Physics*

Received date: 8 May 2017

Accepted date: 8 May 2017

Please cite this article as: J. Avan, V. Caudrelier, On the origin of dual Lax pairs and their  $r$ -matrix structure, *Journal of Geometry and Physics* (2017), <http://dx.doi.org/10.1016/j.geomphys.2017.05.010>

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



# On the origin of dual Lax pairs and their $r$ -matrix structure

Jean Avan<sup>a</sup>, Vincent Caudrelier<sup>b</sup>

<sup>a</sup>Laboratoire de Physique Théorique et Modélisation (CNRS UMR 8089),  
Université de Cergy-Pontoise, F-95302 Cergy-Pontoise, France

<sup>b</sup>School of Mathematics, University of Leeds,  
LS2 9JT Leeds, United Kingdom

## Abstract

We establish the algebraic origin of the following observations made previously by the authors and coworkers: (i) A given integrable PDE in  $1 + 1$  dimensions within the Zakharov-Shabat scheme related to a Lax pair can be cast in two distinct, dual Hamiltonian formulations; (ii) Associated to each formulation is a Poisson bracket and a phase space (which are not compatible in the sense of Magri); (iii) Each matrix in the Lax pair satisfies a linear Poisson algebra a la Sklyanin characterized by the *same* classical  $r$  matrix. We develop the general concept of dual Lax pairs and dual Hamiltonian formulation of an integrable field theory. We elucidate the origin of the common  $r$ -matrix structure by tracing it back to a single Lie-Poisson bracket on a suitable coadjoint orbit of the loop algebra  $\mathfrak{sl}(2, \mathbb{C}) \otimes \mathbb{C}(\lambda, \lambda^{-1})$ . The results are illustrated with the examples of the nonlinear Schrödinger and Gerdjikov-Ivanov hierarchies.

*This paper is dedicated to the memory of Anjan Kundu*

## 1 Introduction

We propose in this paper an elucidation of the mechanisms underlying the appearance of certain Hamiltonian structures in integrable hierarchies of  $1 + 1$ -dimensional PDEs. We consider integrable PDEs obtained in the well-known Lax pair formulation as a zero-curvature condition for a 2-dimensional connection  $\{d/dx - L(\lambda), d/dt - M(\lambda)\}$ , realizing the so-called Ablowitz-Kaup-Newell-Segur or AKNS scheme [1, 2, 3]. Our motivation stems from our previous papers [4, 5] where the original question of the Hamiltonian formulation of integrable defects in a  $1 + 1$  field theory raised in [6] led to the construction of two Poisson brackets acting on different phase spaces for the same PDE. The two Poisson structures acted respectively on each of the two connection matrices  $L, M$  in the Lax representation of the PDE, and surprisingly endowed both matrices with a Poisson algebra structure parametrized by *the same* classical  $r$ -matrix. This aspect was investigated more systematically in [7]. This situation differs fundamentally from the Poisson bracket hierarchy à la Magri [8] where mutually compatible Poisson structures act on the same phase space.

Such a result may be viewed as a coincidence but we prove here that it in fact originates in a systematic construction of  $1 + 1$  dimensional Lax pairs for integrable field theories, from a single initial  $0 + 1$  dynamical system living on a coadjoint orbit of a loop algebra, suitably identified with the dual of the positive-index half of a loop algebra (in our case based on the Lie algebra  $sl(2, \mathbb{C})$ ) as exposed by Flaschka, Newell and Ratiu (FNR) [9]. This paper describes the complete consistent construction of the  $r$ -matrix properties of such Lax pairs. The above mentioned duality of Poisson structures and Lax representations for the same integrable PDE, together with the occurrence of the same  $r$  matrix structure in both representations, will thus be accounted for.

Note that the terminology of “duality” that what we used in [5, 7] and also in the present paper is not to be confused with that of e.g. [10, 11, 12]. It is to be understood as a notion similar to the world–volume duality in string theory, exchanging as it does the role of the space/time coordinates of the underlying space time; whereas dualities such as are found in [10, 11, 12] are target-space dualities exchanging the role of dynamical variables  $p, q$ .

The structure of the paper follows closely the construction procedure. In Section 2 we recall the basis of the coadjoint orbit construction [9] starting from the canonical matrix  $L(\lambda)$  defined on a coadjoint orbit of the loop algebra of Laurent series with coefficients in  $sl(2, \mathbb{C})$  picked to be the negative power series. The complex parameter  $\lambda$  defining the evaluation representations of the loop algebra and its expression in formal power series, is identified as “spectral parameter”. The standard Adler-Kostant-Symes construction endows the coadjoint orbit with two compatible Poisson structures and associated  $r$ -matrix structures. We describe the integrability properties for the Hamiltonian evolutions triggered by suitable invariant traces, and the associated isospectral evolutions taking the generic form  $\frac{d}{dt_n} L = [L^{(n)}, L]$  where  $L^{(n)} \equiv P^+ \lambda^n L$ .

We then proceed as follows:

### Step 1 (Section 3.1) The constraint equations

Choosing one specific time variable  $t_n$  one restricts the dynamical variables in  $L$  to obey the  $n$ -th time evolution. Implementation of this restriction constrains the dynamical variables in  $L$  to obey specific algebraic and differential relations encoded in a map  $\Psi_n$ . We thus define a restricted phase space, this time however of  $t_n$ -valued fields, and a projected and  $\Psi$ -restricted Lax matrix  $V_n^{(n)}(\lambda, t_n) \equiv \Psi_n L^{(n)}$ . It naturally acquires a canonical ultralocal  $r$ -matrix structure inherited from the  $r$ -matrix of  $L$ .

### Step 2 (section 3.2) Integrability after constraint

We compare:

- The result of applying a given constraint  $\Psi_n$  and extending the dynamical variables to being  $t_n$ -dependent fields, onto every other FNR operator  $L^{(k)}$ ,  $k \neq n$ .
- The result of computing the operator describing the  $t_k$  time evolution for the  $k$ -th Hamiltonian deduced from the expansion of the monodromy matrix of the differential operator  $\partial_{t_n} - V_n^{(n)}(\lambda, t_n)$ .

The resulting matrices are identical. Hence the complete hierarchy of commuting FNR flows on the coadjoint orbit restricts consistently under any constraint procedure of Step 1, to the hierarchy structure of the time Lax operators  $L^{(k)}$  deduced from the monodromy of the specific “space” Lax operator  $L^{(n)}$  picked in Step 1.

### Step 3 (section 3.3) Duality

We prove that the zero curvature conditions obtained from the respective two distinct choices of an AKNS scheme in step 1 and 2 *i.e.* either first  $t_n$  and then  $t_k$  or first  $t_k$  and then  $t_n$  are identical. As a consequence, the corresponding  $1 + 1$  integrable field theory exhibits two “dual” Lax pair representations, characterized by different phase spaces, same  $r$ -matrix structure, and the exchange of  $x = t_n$  and  $t = t_k$  as “space” vs “time” variables in the AKNS formulation. The duality, pointed out and investigated in [7] from a covariant field theory point of view, is therefore identified in a purely algebraic framework as a general feature directly originating in the constraint procedure leading from  $0 + 1$  integrable theories on a coadjoint orbit of infinite dimension to  $1 + 1$  integrable theories.

The above steps are captured in the diagram below which explains graphically the duality of Hamiltonian formulation of a given integrable PDE. The important and nontrivial results are shown in a box and correspond to Sections 3.1, 3.2 and 3.3 respectively.

For a fixed  $n \geq 1$ , Step 1 and 2 go all the way down from the FNR picture and its associated  $R$ -operator formulation (the distinction between  $r$ -matrix and  $R$  operator formulation [13] will be clear in the text) to the AKNS formulation and its associated  $r$ -matrix. The third step brings in the right-hand-side of the diagram, corresponding to a parallel but different choice of initial time in the FNR picture. The commutativity of the diagram expresses the notion of dual Hamiltonian formulation of an integrable PDE.

We illustrate our results in Section 4, using the NLS equation as a first example to make contact with previous [7]. We also provide an example never treated before from this dual point of view: the Gerdjikov-Ivanov equation [14]. This is all the more interesting as this integrable

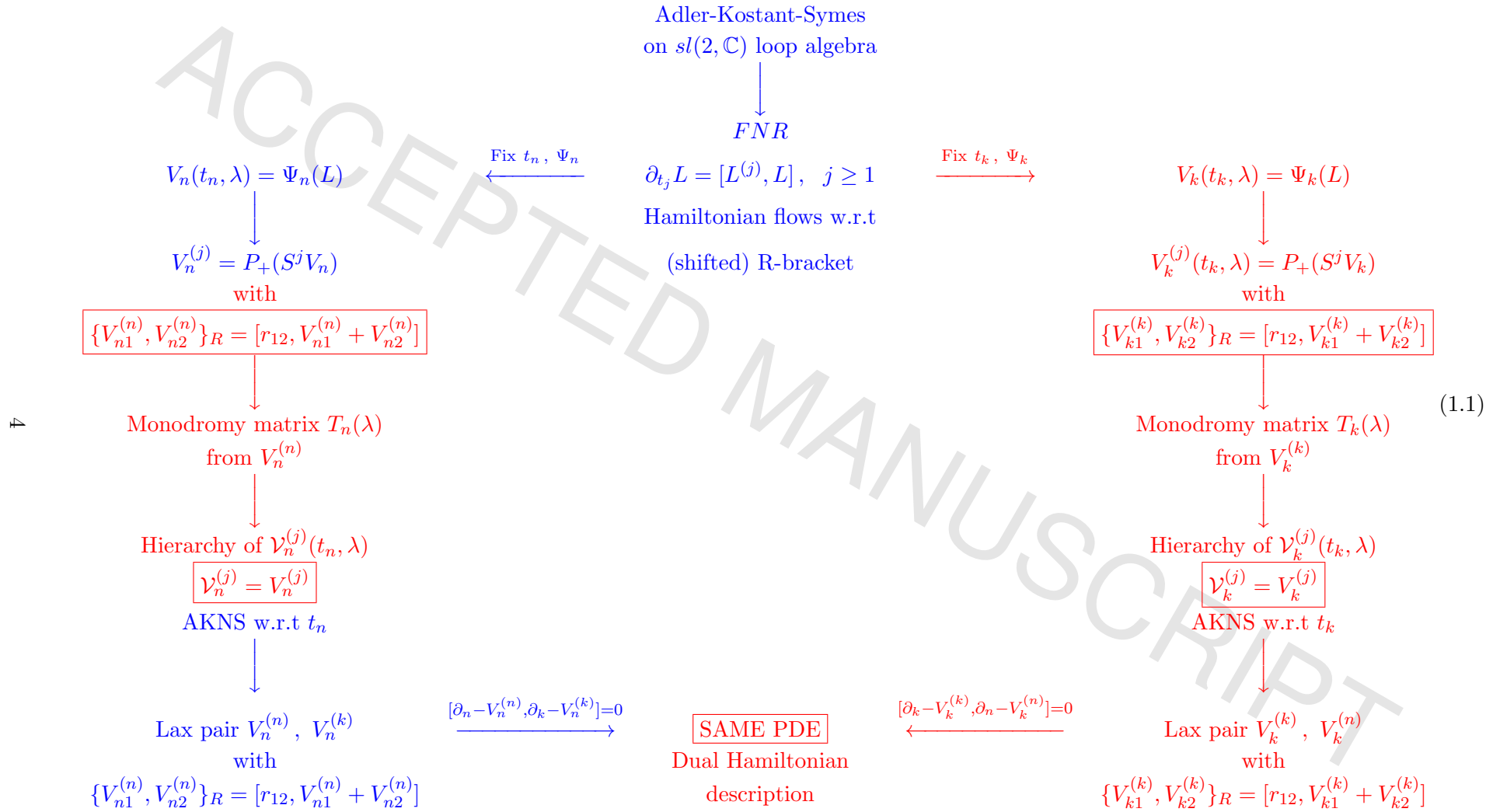
PDE is not part of the traditional AKNS hierarchy based on  $V_1^{(1)}$  but rather, is related to the Kaup-Newell hierarchy [15]. We end up with some open questions and connections with other issues in classical integrable systems.

Let us round up this introduction by putting our current results in perspective with respect to the state of the art. So far and to the best of our knowledge, only parts of one of the two sides of this diagram have been used as the basis of the theory of integrable PDEs/classical field theories. Although it is impossible to pinpoint faithfully the major results in the huge available literature (see references in e.g. [16, 17] for a partial overview) on such a simple linear diagram, let us point out some of the main historical developments and how they relate to the present construction. The traditional AKNS picture corresponds to picking e.g. the left part of the diagram, say with  $n = 1$  and  $t_1 = x$ . The literature is then essentially split into two main streams:

- The top (left) half of the diagram (in blue) in which the classical  $r$ -matrix approach is not required and not dealt with. This part of the literature is formulated in the coadjoint orbit picture [9]. Most studies on Hamiltonian and multi-Hamiltonian structures involve formal (differential) algebra based on the loop algebra of  $\mathfrak{sl}(2, \mathbb{C})$  (and some higher rank generalizations) together with the Adler-Kostant-Symes (AKS) construction. See e.g. Chapter 10 in [17].

- The bottom (left) part of the diagram (in blue) in which the full power of the classical  $r$ -matrix to describe Hamiltonian properties of an integrable hierarchy is used. In this approach, the building ingredients are the Lax matrix  $L$ , i.e. the matrix associated to the “space part” of the flat connection in the auxiliary problem and its classical  $r$ -matrix structure. These data allow one to deduce the hierarchy of higher time-shift or Lax-partner matrices from expansion in the spectral parameter of the monodromy matrix of the space-shift differential operator  $d/dx - L(\lambda, x)$ . The knowledge of the analytical properties is therefore crucial in this approach (see e.g. [16]). The higher Lax-partner matrices are by-products of the construction and they are not known to exhibit specific Poisson properties w.r.t to the only Poisson brackets ( $r$ -matrix for  $L$ ) used in this picture. See e.g. Part I, Chapter 3 in [16].

In addition to this short summary of the research in that area is the paper [18] (see also [19]). The results provide a Hamiltonian description of the hierarchies of integrable equations based on a Lax matrix of the type  $V_n^{(n)}$  in our notations, for arbitrary  $n \geq 1$ . The author’s starting point is at the AKNS level and he derives the corresponding FNR equation for a fixed but arbitrary  $t_n$  (denoted  $x$  in the paper). So the results could be summarised by a set of arrows going from the bottom to the top of the left part of our diagram. Concerning the notion of duality that we develop here, and which is absent from [18], it worth mentioning a major set of works not easily fitted in our diagram. It deals with the topic of Hamiltonian properties of stationary manifolds [20] in an integrable hierarchy. A geometric treatment of this question is provided in [18]. In [21, 22], the idea to swap the roles of the two independent variables is used extensively to derive the (multi)-Hamiltonian structure of integrable stationary equations and is traced back to [23]. The  $r$ -matrix formalism is not used there and one ultimately works with one independent variable, and hence ODEs, by reduction to time-independent fields.



## 2 Algebraic construction of the hierarchy: FNR construction

We first recall the construction of the AKNS hierarchy along the lines of [9]. The power of the procedure is to set on equal footing all time flows within a well-defined hierarchy from the start. It also provides by construction a Hamiltonian formulation of the hierarchy, and the related integrability properties, built in by the application of the Adler-Kostant-Symes procedure [24, 25, 26] to the  $\mathfrak{sl}(2, \mathbb{C})$  loop algebra endowed with a second Poisson bracket structure compatible with the canonical one.

The procedure goes as follows. Let  $\mathcal{L}$  be the Lie algebra of Laurent series  $X : \lambda \mapsto X(\lambda)$  with coefficients in  $\mathfrak{sl}(2, \mathbb{C})$  *i.e.*

$$X(\lambda) = \sum_{j=-\infty}^N X_j \lambda^j, \quad X_j \in \mathfrak{sl}(2, \mathbb{C}); \quad \text{for some integer } N, \quad (2.1)$$

with the standard Lie bracket

$$[X, Y](\lambda) = \sum_k \sum_{i+j=k} [X_i, Y_j] \lambda^k. \quad (2.2)$$

There is a natural decomposition of  $\mathcal{L}$  into Lie subalgebras  $\mathcal{L} = \mathcal{L}_- \oplus \mathcal{L}_+$  where

$$\mathcal{L}_- = \left\{ \sum_{j=-\infty}^{-1} X_j \lambda^j \right\}, \quad \mathcal{L}_+ = \left\{ \sum_{j=0}^{\infty} X_j \lambda^j; \quad X_j = 0 \quad \forall j > N \quad \text{for some integer } N \geq 0 \right\}. \quad (2.3)$$

This yields two projectors  $P_+$  and  $P_-$  and we define the  $R$  operator as

$$R = P_+ - P_-. \quad (2.4)$$

It is well-known that this operator satisfies the modified classical Yang-Baxter equation and allows one to define a second Lie bracket  $[\cdot, \cdot]_R$  on  $\mathcal{L}$  (see e.g. [27])

$$[X, Y]_R = \frac{1}{2}[(RX), Y] + \frac{1}{2}[X, (RY)]. \quad (2.5)$$

Using the following ad-invariant nondegenerate symmetric bilinear form on  $\mathcal{L}$ , for all  $X, Y \in \mathcal{L}$ ,

$$(X, Y) = \sum_{i+j=-1} \text{Tr}(X_i, Y_j) \equiv \text{Res}_\lambda \text{Tr}(X(\lambda)Y(\lambda)), \quad (2.6)$$

where  $\text{Tr}$  in the first equality is the Killing form on  $\mathfrak{sl}(2, \mathbb{C})$ , the two Lie brackets allow to define Lie-Poisson brackets on  $C^\infty(\mathcal{L})$  by setting, for all  $F, G \in C^\infty(\mathcal{L})$  and  $X \in \mathcal{L}$ ,

$$\{F, G\}(X) = (X, [\nabla F(X), \nabla G(X)]), \quad (2.7)$$

$$\{F, G\}_R(X) = (X, [\nabla F(X), \nabla G(X)]_R). \quad (2.8)$$

where  $\nabla$  denotes as usual the gradient.



We denote the second Poisson bracket as the  $R$ -bracket. The main statement of FNR [9] is that all equations of the AKNS hierarchy can be encapsulated into a single set of mutually commuting Hamiltonian time flows:

$$\partial_{t_k} Q = [Q^{(k)}, Q], \quad k = 1, 2, \dots, \quad (2.9)$$

with  $Q \in \mathcal{L}^0 \equiv \left\{ \sum_{j=-\infty}^0 X_j \lambda^j \right\}$  and

$$Q^{(k)} = P_+(S^k Q). \quad (2.10)$$

The shift operator  $S^k$  on  $\mathcal{L}$  is defined for all  $k \in \mathbb{Z}$  by

$$S^k(X)(\lambda) = \lambda^k X(\lambda). \quad (2.11)$$

The associated mutually Poisson-commuting Hamiltonians turn out to be the dynamical coefficients in the  $\lambda$  power expansion of  $\text{Tr}(L(\lambda)^2)$ . These objects are indeed identified as:

$$\phi_k(L) = -\frac{1}{2}(S^k(L), L), \quad k \in \mathbb{Z}. \quad (2.12)$$

This is a consequence of the Adler-Kostant-Symes scheme applied to this specific splitting once the time-evolution operator is identified by the formula (2.10). We now recall the main steps leading to the FNR result using the systematic reformulation in terms of  $R$ -operator proposed by Semenov-Tian-Shanski [13] and adapted to our setting.

**Proposition 2.1** [13] *Let  $I$  be a finite or countable set and let  $C_n$ ,  $n \in I$  be a collection of Casimir functions for the Lie-Poisson bracket  $\{ , \}$  on  $\mathcal{L}$ . Then*

1. *The Casimir functions are in involution with respect to the  $R$ -bracket.*
2. *The commuting Hamiltonian flows generated by  $C_n$  with respect to  $\{ , \}_R$  take the Lax form*

$$\partial_{t_n} L = -\frac{1}{2}[(R\nabla C_n(L)), L], \quad \forall L \in \mathcal{L}. \quad (2.13)$$

The functions  $\phi_k$  in (2.12) provide a family of Casimirs for the Lie-Poisson bracket on  $\mathcal{L}$ . Applying the proposition, we obtain

**Proposition 2.2** *For any  $k \geq 1$ , the equation*

$$\partial_{t_k} Q = [Q^{(k)}, Q] \quad (2.14)$$

*is the flow generated by the function  $\phi_k$  with respect to  $\{ , \}_R$ , restricted to  $\mathcal{L}^0_-$ .*

Proof: Note that, consistently with (2.12)

$$\nabla \phi_k(X) = -S^k(X), \quad \forall X \in \mathcal{L}. \quad (2.15)$$

Let  $F \in C^\infty(\mathcal{L})$  then

$$\partial_{t_k} F(X) = \{\phi_k, F\}_R(X) = -\frac{1}{2} \left( X, [(R(S^k(X))), \nabla F(X)] \right). \quad (2.16)$$

Note that  $[S^k(X), X] = 0$ , and  $\partial_{t_k} F(X) = (\partial_{t_k} X, \nabla F(X))$  hence we deduce

$$\partial_{t_k} X = [P_+(S^k(X), X)]. \quad (2.17)$$

It remains to show that this equation on  $\mathcal{L}$  can be restricted to  $\mathcal{L}_-^0$ . For  $X \in \mathcal{L}$ , denote  $X = X_+ + X_-$  the unique decomposition of  $X$  where<sup>1</sup>  $X_- \in \mathcal{L}_-^0$ . Then (2.17) is equivalent to

$$\partial_{t_k} (X_+ + X_-) = -[P_-(S^k(X_-), X_+) - [P_-(S^k(X_-), X_-)], \quad (2.18)$$

which in turn implies

$$\partial_{t_k} X_+ = -[P_-(S^k(X_-), X_+)]_+. \quad (2.19)$$

We can thus consistently set  $X_+ = 0$ . Inserting back in (2.18), we obtain the reduced equation to  $\mathcal{L}_-^0$  as

$$\partial_{t_k} X_- = -[P_-(S^k(X_-), X_-)] = [P_+(S^k(X_-), X_-)]. \quad (2.20)$$

yielding (2.14) with  $Q = X_-$ . ■

It is important to note that (2.14) is a **nonlinear** equation in  $Q$ . It is different in substance from the starting point in strictly-speaking AKNS-type integrable hierarchies consisting of defining resolvents  $R$  associated to an operator  $\partial_{t_k} - V^{(k)}$  where  $V^{(k)}$  is a **given** polynomial of degree  $k$  in  $\lambda$  with coefficients in a given Lie algebra ( $\mathfrak{sl}(2, \mathbb{C})$  for the AKNS hierarchy), *i.e.*,

$$[\partial_{t_k} - V^{(k)}, R] = 0. \quad (2.21)$$

This is the approach developed to a great level of generality for instance in [28] (using a formal algebraic formulation). Eq (2.21) is **linear** in  $R$ . This allows one to develop a full theory of resolvents using the convenient notion of (formal) dressing. It actually lies at the core of the dressing transformation procedure for Zakharov-Shabat type Lax representations, such as described in e.g. [29].

Our purpose now is to make contact with this level of the theory that we shall denote as AKNS formalism from now on, to distinguish it from the FNR formalism discussed so far. In a nutshell, we want to transfer the Hamiltonian and  $R$ -operator content of FNR down to AKNS. To some extent, the Hamiltonian content is already addressed in [9]. We now translate the language of the  $R$  operator used so far into the more well-known classical  $r$ -matrix formalism that is widely used at the AKNS level of the theory. We first need to obtain the central FNR equation (2.14) in a slightly more general manner. We use the notion of *intertwining operator* as defined in [30].

---

<sup>1</sup>We draw the reader's attention to the fact that  $X_\pm \neq P_\pm(X)$  as the decomposition is performed along different subalgebras of  $\mathcal{L}$ .

**Definition 2.3** A linear operator  $A : \mathcal{L} \rightarrow \mathcal{L}$  is called *intertwining* if

$$A[X, Y] = [AX, Y] = [X, AY], \quad \forall X, Y \in \mathcal{L}. \quad (2.22)$$

**Proposition 2.4** [30]

If  $R$  is a solution of the modified classical Yang-Baxter equation and  $A$  is an intertwining operator then  $\tilde{R} = R \circ A$  is also a solution of the modified classical Yang-Baxter equation.

As a consequence, one can consistently define a Lie bracket  $[\cdot, \cdot]_{\tilde{R}}$  on  $\mathcal{L}$  and hence the corresponding  $\tilde{R}$ -bracket. It is easy to check that the shift operator  $S^k$  is an intertwining operator on  $\mathcal{L}$  for all  $k \in \mathbb{Z}$ .

**Definition 2.5** The family of Poisson brackets on  $\mathcal{L}$  associated to  $R \circ S^k$ ,  $k \in \mathbb{Z}$  is denoted by  $\{ \cdot, \cdot \}_k$ .

We can now revisit Proposition 2.2 and state

**Proposition 2.6** Let  $k \geq 1$  be fixed. The equation

$$\partial_{t_k} Q = [Q^{(k)}, Q] \quad (2.23)$$

is the flow generated by the function  $\phi_m$  in (2.12) with respect to  $\{ \cdot, \cdot \}_{k-m}$  for all  $m \in \mathbb{Z}$ , restricted to  $\mathcal{L}_-^0$ .

Proof: It suffices to note that for all  $m, n \in \mathbb{Z}$ , using (2.16),

$$\{\phi_m, F\}_n(L) = \{\phi_{m+n}, F\}_R(L), \quad \forall L \in \mathcal{L}, \quad \forall F \in C^\infty(\mathcal{L}). \quad (2.24)$$

■

We have the following useful lemma

**Lemma 2.7** Let  $k \in \mathbb{Z}$ ,

$$\{F \circ S^k, G \circ S^k\}_{-k}(L) = \{F, G\}_R(S^k L), \quad \forall L \in \mathcal{L}, \quad \forall F, G \in C^\infty(\mathcal{L}). \quad (2.25)$$

Now we make the connection with the classical  $r$ -matrix formalism. It will be convenient to use the so-called auxiliary space notation in the rest of the paper. For instance, for any  $X \in \mathcal{L}$ , we write<sup>2</sup>

$$X_1 = X \otimes \mathbb{I}, \quad X_2 = \mathbb{I} \otimes X. \quad (2.26)$$

Let us extend the bilinear form (2.6) to the tensor product  $\mathcal{L} \otimes \mathcal{L}$  by setting

$$(X \otimes Y, Z \otimes W) = (X, Z)(Y, W) \equiv \text{Res}_{\lambda, \mu} \text{Tr}(X(\lambda)Z(\lambda))\text{Tr}(Y(\mu)W(\mu)), \quad (2.27)$$

for all  $X, Y, Z, W \in \mathcal{L}$ . Another notation we will use to remember the dependence on  $\lambda, \mu$  is

$$\text{Res}_{\lambda, \mu}(X(\lambda) \otimes Y(\mu), Z(\lambda) \otimes W(\mu)). \quad (2.28)$$

<sup>2</sup>Note that we are rather loose in the definition as the object  $\mathbb{I}$  is not properly defined. Doing so would make the paper even longer. We prefer to take the view that this is a well-established notation for an appropriate identity map which should not lead to any confusion.

**Definition 2.8** *The classical  $r$ -matrix is the element of  $\mathcal{L} \otimes \mathcal{L}$  defined by (again with  $X, Y$  any two elements of  $\mathcal{L}$ ):*

$$(RX, Y) = (r, Y \otimes X) = \text{Res}_{\lambda, \mu}(r(\lambda, \mu), Y(\lambda) \otimes X(\mu)), \quad \forall X, Y \in \mathcal{L}. \quad (2.29)$$

**Definition 2.9** *Let  $L \in \mathcal{L}$  and  $k \in \mathbb{Z}$ . The element of  $\mathcal{L} \otimes \mathcal{L}$  denoted by  $\{L_1(\lambda), L_2(\mu)\}_k$  is defined by*

$$\text{Res}_{\lambda, \mu}(\{L_1(\lambda), L_2(\mu)\}_k, Y(\lambda) \otimes X(\mu)) = (L, [X, Y]_k), \quad \forall X, Y \in \mathcal{L}. \quad (2.30)$$

Here and in the following discussions the notation  $X_i$  denotes the element of the (possibly multiple but at least double) tensor power of  $\mathcal{L}$  where  $X$  is positioned as the  $i$ -th factor and all other tensorial factors are taken to be 1.

With these two definitions, we make contact between the  $k$ -bracket defined in (2.5) and the celebrated Sklyanin formula [31]

**Proposition 2.10** *The following formula holds*

$$\{L_1(\lambda), L_2(\mu)\}_k = -\frac{1}{2}[r_k(\lambda, \mu), L(\lambda) \otimes \mathbb{I}] + \frac{1}{2}[\Pi r_k(\mu, \lambda), \mathbb{I} \otimes L(\mu)], \quad (2.31)$$

where  $\Pi$  is the permutation operator on two copies of  $\mathcal{L}$ :  $\Pi(X \otimes Y) = Y \otimes X$  and

$$r_k(\lambda, \mu) = -2 \frac{\mu^k}{\lambda - \mu} t, \quad t = \Pi^{sl(2, \mathbb{C})} \quad (\text{Casimir}). \quad (2.32)$$

In particular,

$$\{L_1(\lambda), L_2(\mu)\}_R = \{L_1(\lambda), L_2(\mu)\}_0 = \left[ \frac{\Pi}{\lambda - \mu}, L_1(\lambda) + L_2(\mu) \right] \quad (\text{Sklyanin formula}). \quad (2.33)$$

It is useful here to make this form of Poisson brackets more explicit. Using a basis  $\{e^a \lambda^i\}$  for  $\mathcal{L}$  where  $\{e^a\}$  is a basis for  $\mathfrak{sl}(2, \mathbb{C})$ , write

$$\{L_1(\lambda), L_2(\mu)\}_k = (\{L_1(\lambda), L_2(\mu)\}_k)_{ij}^{ab} e^a \lambda^i \otimes e^b \mu^j \quad (\text{summation implied over repeated indices}), \quad (2.34)$$

and let us use the convenient notation

$$\{L_i^a, L_j^b\}_k \equiv (\{L_1(\lambda), L_2(\mu)\}_k)_{ij}^{ab}. \quad (2.35)$$

The following component form of the  $k$ -bracket is useful

$$\{L_i^a, L_j^b\}_k = \epsilon_{ij}^k (K^{-1})_{ac} C_b^{cd} L_{i+j+1-k}^d, \quad (2.36)$$

where  $C_f^{cd}$  are the structure constants of  $\mathfrak{sl}(2, \mathbb{C})$ ,  $K$  is the matrix with entry  $K_{ab} = \text{Tr}(e^a e^b)$  and

$$\epsilon_{ij}^k = \begin{cases} -1, & i, j < k, \\ 1, & i, j \geq k, \\ 0 & \text{otherwise.} \end{cases} \quad (2.37)$$

### 3 From FNR to AKNS: transfer of the integrability structure

#### 3.1 The map $\Psi_k$ : From r-matrix structure of FNR to r-matrix structure of AKNS

Let  $k \geq 1$  be fixed in all this section. Our goal is to show that the  $r$ -matrix structure used to express the FNR equation (2.14) is the origin of the  $r$ -matrix structure of the AKNS realization of the object  $L^{(k)} = P_+(S^k L)$ . In view of Lemma 2.7, this suggests that we should use the bracket  $\{, \}_{-k}$  as a starting point. This is where our reformulation of the FNR flows in Proposition 2.6 becomes useful.

The following proposition is well-known in the literature (see e.g. [9]). We formulate it in the form that motivates our subsequent analysis.

**Proposition 3.1** *Fix a time variable  $t_k$ . For  $L \in \mathcal{L}_-^0$ ,*

$$L = \sum_{j=0}^{\infty} \ell_j \lambda^{-j}, \quad \ell_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}, \quad (3.1)$$

*Consider the flow equation*

$$\frac{d}{dt_k} L = [L^{(k)}, L], \quad (3.2)$$

*with periodic boundary conditions in  $t_k$  and  $\ell_0 = \sigma_3$  i.e.  $a_0 = 1, b_0 = 0 = c_0$ . Then, one can view the elements  $a_j, b_j, c_j$  as periodic functions of  $t_k$  and*

1.  $b_j, c_j, j = 1, \dots, k$  are not constrained.
2.  $b_j, c_j, j > k$  are polynomials respectively denoted  $P_k^{b_j}$  and  $P_k^{c_j}$  in  $b_n^{(\ell)}, c_n^{(\ell)}, n = 1, \dots, k$  where  $u_n^{(\ell)}$  is the  $\ell$ -th  $t_k$  derivative of  $u_n, u = b, c$ .
3.  $a_j, j \geq 1$  are polynomials denoted  $Q_k^{a_j}$  in  $b_n^{(\ell)}, c_n^{(\ell)}, n = 1, \dots, k$ . For  $j = 0 \dots, k, a_j$  is in fact a polynomial in  $b_n^{(0)}, c_n^{(0)}, n = 0, \dots, k$  only, no derivatives appear.

The last point on the form of  $a_j, j = 0 \dots, k$  has a translation in Hamiltonian terms that we will exploit below. Denote now  $V_k$  the solution of the equation thus obtained. We want to introduce a map  $\Psi_k$  that conveniently captures the solution procedure of the previous proposition and which assigns the solution  $V_k$  to  $L$ ,

$$\Psi_k(L) = V_k. \quad (3.3)$$

$V_k$  is now a function of  $t_k$  (and of course a power series in  $\lambda^{-1}$ ). We may sometimes write explicitly  $V_k(\lambda, t_k)$  to emphasize this point. The point is that after imposing the flow equation of Proposition 3.1, the variables  $a_j, b_j, c_j$ , which are free,  $t_k$ -independent coordinates on the coadjoint orbit, become  $t_k$ -dependent functions which are all expressible in terms of a finite number of functions  $b_j(t_k), c_j(t_k), j = 1, \dots, k$  and their  $t_k$  derivatives. The map  $\Psi_k$  assigns to each coadjoint orbit variable precisely its expression after resolution of the constraints imposed

by the flow equation (3.2) with the chosen condition on  $\ell_0$ . To formalise this, we need to define an algebra where  $V_k$  lives.

**Definition 3.2**

1. Let  $\mathcal{B}$  be the algebra over  $\mathbb{C}$  of polynomials in  $a_i, b_i, c_i$ ,  $i = 0, 1, 2, \dots$  regarded as smooth complex-valued functions of the variable  $t_k$ .
2. Let  $\mathcal{B}_n$  be the quotient of  $\mathcal{B}$  by the ideal  $\mathcal{I}_n$  generated by  $\{a_i(t_k), b_i(t_k), c_i(t_k)\}_{i=n+1}^\infty$ .  $\mathcal{B}_n$  is identified with the associative algebra generated by  $\{a_i(t_k), b_i(t_k), c_i(t_k)\}_{i=0}^n$ .
3. Let  $\mathcal{B}_n^0$  be the algebra over  $\mathbb{C}$  of polynomials in  $\{b_i(t_k), c_i(t_k)\}_{i=0}^n$ .
4. Let  $d\mathcal{B}_n^0$  be the algebra over  $\mathbb{C}$  of polynomials in  $\{b_i^{(\ell)}(t_k), c_i^{(\ell)}(t_k)\}_{i,\ell=(0,0)}^{(n,\infty)}$  where

$$b_i^{(0)}(t_k) = b_i(t_k), \quad c_i^{(0)}(t_k) = c_i(t_k), \quad (3.4)$$

$$\partial_k u_i^{(\ell)}(t_k) = u_i^{(\ell+1)}(t_k), \quad i = 0, \dots, n, \quad \ell \geq 0, \quad u = b, c. \quad (3.5)$$

**Definition 3.3** The map  $\Psi_k$  from  $\mathcal{L}_-^0$  to  $\mathcal{L}_{d\mathcal{B}_k}$  is defined in two steps: first by giving its action on the elements  $\{a_i, b_i, c_i\}_{i=0}^\infty$  of  $L \in \mathcal{L}_-^0$  as

$$a_0 \mapsto 1, \quad b_0 \mapsto 0, \quad c_0 \mapsto 0 \quad (3.6)$$

$$(b_i, c_i) \mapsto (b_i(t_k), c_i(t_k)), \quad i = 1, \dots, k \quad (3.7)$$

$$a_i \mapsto Q_k^{a_i}(b_1(t_k), c_1(t_k), \dots, b_{i-1}(t_k), c_{i-1}(t_k)), \quad i = 1, \dots, k \quad (3.8)$$

$$b_i \mapsto P_k^{b_i}(b_1^{(\ell)}(t_k), c_1^{(\ell)}(t_k), \dots, b_{i-1}^{(\ell)}(t_k), c_{i-1}^{(\ell)}(t_k)), \quad i \geq k+1 \quad (3.9)$$

$$c_i \mapsto P_k^{c_i}(b_1^{(\ell)}(t_k), c_1^{(\ell)}(t_k), \dots, b_{i-1}^{(\ell)}(t_k), c_{i-1}^{(\ell)}(t_k)), \quad i \geq k+1 \quad (3.10)$$

$$a_i \mapsto Q_k^{a_i}(b_1^{(\ell)}(t_k), c_1^{(\ell)}(t_k), \dots, b_{i-1}^{(\ell)}(t_k), c_{i-1}^{(\ell)}(t_k)), \quad i \geq k+1 \quad (3.11)$$

and second, by extending it componentwise to a morphism of algebras from  $\mathcal{L}_-^0$  to  $\mathcal{L}_{d\mathcal{B}_k}$ , the algebra of power series in  $\lambda^{-1}$  with coefficients in  $\mathfrak{sl}(2, d\mathcal{B}_k^0)$ .  $P_k^{b_i}$ ,  $P_k^{c_i}$  and  $Q_k^{a_i}$  are the polynomials determined by the solution procedure of Proposition 3.1.

**Remark 3.4** A geometric realization of the map  $\Psi_k$  restricted to  $\{a_i, b_i, c_i\}_{i=0}^k$  is given in Proposition 2 of [18] (there  $k = N$  and  $t_N \equiv x$ ).

**Remark 3.5** The motivation for the definition of the various algebras in Definition 3.2 can be seen as follows. If we write  $V_k(\lambda, t_k) = \Psi_k(L)$  as

$$V_k(\lambda, t_k) = \sum_{j=0}^{\infty} V_{k,j}(t_k) \lambda^{-j}, \quad (3.12)$$

then we see from Definition 3.3 that for  $j > k$ ,  $V_{k,j}(t_k)$  belongs to  $\mathfrak{sl}(2, d\mathcal{B}_k^0)$  and for  $j = 0, \dots, k$ , it belongs to  $\mathfrak{sl}(2, \mathcal{B}_k^0)$  (a subalgebra of  $\mathfrak{sl}(2, d\mathcal{B}_k^0)$ ). We will only be interested in the Poisson properties of the generalized Lax matrix  $V_k^{(k)}$  (see next Definition). The latter only involves  $V_{k,j}(t_k)$ ,  $j = 0, \dots, k$  and hence, we will be concerned with a certain Poisson structure on  $\mathcal{B}_k^0$ . The latter is most easily derived from a natural Poisson structure on  $\mathcal{B}$  and its subalgebra  $\mathcal{B}_k$ .

**Definition 3.6** For  $n \geq 0$ , we define the (generalized) Lax matrices as

$$V_k^{(n)}(\lambda, t_k) = P_+(S^n(\Psi_k(L))). \quad (3.13)$$

For  $k = 1$ , one recovers the well known Lax matrices of the ANKS hierarchy, with  $b_1 = q$  and  $c_1 = r$ , the two fundamental fields in that hierarchy (more details in the Examples section below).

The map  $\Psi_k$  is now used to transfer the  $r$ -matrix structure of the FNR theory over elements  $L \in \mathcal{L}_-^0$  to the Poisson structure of the Lax matrix  $V_k^{(k)}$ . This requires endowing  $\mathcal{B}$  and  $\mathcal{B}_n$  with natural Poisson structures such that the map  $L \mapsto V_k^{(k)}(\lambda, t_k)$  enjoys nice Poisson properties. To this end, note that the associative algebra  $\mathcal{A}$  over  $\mathbb{C}$  of polynomials in the symbols  $a_i, b_i, c_i$ ,  $i = 0, 1, 2, \dots$  inherits a Poisson structure from  $\{, \}_{-k}$  on  $\mathcal{L}_-^0$  by viewing it as a subspace of  $C^\infty(\mathcal{L}_-^0)$  (polynomials in those symbols are particular smooth functions in the entries of elements of  $\mathcal{L}_-^0$ ). In other words, it is enough to specify the Poisson brackets of  $a_i, b_i, c_i$  and the latter are obtained via the identifications

$$\ell_i^+ \mapsto b_i, \quad \ell_i^- \mapsto c_i, \quad \ell_i^3 \mapsto a_i, \quad (3.14)$$

where we write an element  $L$  of  $\mathcal{L}_-^0$  as

$$L = \sum_{j=0}^{\infty} \ell_j \lambda^{-j}, \quad \ell_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} = \ell_j^+ \sigma_+ + \ell_j^- \sigma_- + \ell_j^3 \sigma_3, \quad (3.15)$$

and we have made a choice of basis in  $\mathfrak{sl}(2, \mathbb{C})$  corresponding to the three matrices

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.16)$$

Using (2.36), we get:

$$\{b_m, c_n\}_{-k} = -2\tilde{\epsilon}_{mn}^k a_{m+n-k-1}, \quad \{b_m, a_n\}_{-k} = \tilde{\epsilon}_{mn}^k b_{m+n-k-1}, \quad \{c_m, a_n\}_{-k} = -\tilde{\epsilon}_{mn}^k c_{m+n-k-1}, \quad (3.17)$$

where

$$\tilde{\epsilon}_{mn}^k = \begin{cases} 1, & m, n > k, \\ -1, & m, n \leq k, \\ 0 & \text{otherwise.} \end{cases} \quad (3.18)$$

The Poisson bracket on  $\mathcal{A}$  is extended componentwise to Laurent series with coefficients in  $\mathfrak{sl}(2, \mathcal{A})$ .  $\mathcal{L}_-^0$  is naturally embedded in the subalgebra  $\mathcal{L}_{\mathcal{A}}^0$  of power series in  $\lambda^{-1}$  with coefficients in  $\mathfrak{sl}(2, \mathcal{A})$ .

Similarly,  $\mathcal{B}$  can be viewed as the subalgebra in  $C^\infty(\mathfrak{S})$  where  $\mathfrak{S}$  is the space of smooth periodic maps from  $S^1$  to  $\mathcal{L}_-^0$ . Therefore, the passage from a Poisson structure on  $\mathcal{A}$  to one on  $\mathcal{B}$  uses a well-known procedure of central extension to incorporate the extra variable  $t_k$  and the corresponding derivation  $\partial_k$ . The details of this construction, involving the so-called double loop algebra, can be found for instance in Lecture 7 of [27] for the case of the  $R$ -bracket. The generalization to the  $k$ -bracket was performed in [30] and this is the result we use here. We keep the same notation  $\{ , \}_k$  for the bracket resulting from the central extension construction.

We first extend the bilinear form to elements of  $\mathfrak{S}$  as

$$(X, Y) = \text{Res}_\lambda \int \text{Tr}(X(\lambda, t_k)Y(\lambda, t_k)) dt_k. \quad (3.19)$$

One then introduces the following 2-cocycle on  $\mathfrak{S}$

$$\omega(X, Y) = \int \text{Tr}(X(\lambda, t_k)\partial_k Y(\lambda, t_k)) dt_k \quad (3.20)$$

and the corresponding central extension defined as the Lie algebra of pairs  $(X, p)$  where  $X \in \mathfrak{S}$  and  $p \in \mathbb{C}(\lambda, \lambda^{-1})$  endowed with the Lie bracket

$$[(X, p), (Y, q)] = ([X, Y], \omega(X, Y)). \quad (3.21)$$

The natural extension of the bilinear form on  $\mathfrak{S}$  to its central extension is

$$\langle (X, p), (Y, q) \rangle = (X, Y) + \text{Res}_\lambda p(\lambda)q(\lambda). \quad (3.22)$$

We now use a well-known result:

**Proposition 3.7** *If  $R$  satisfies the modified classical Yang-Baxter equation and  $\omega$  is a 2-cocycle with respect to the Lie bracket  $[ , ]$  then  $\omega_R$  defined by*

$$\omega_R(X, Y) = \frac{1}{2}\omega(RX, Y) + \frac{1}{2}\omega(X, RY) \quad (3.23)$$

*is a 2-cocycle with respect to the Lie bracket  $[ , ]_R$ .*

This allows one to define a central extension of  $\mathfrak{S}$  with respect to  $\omega_R$  as the Lie algebra of pairs  $(X, p)$  equipped with the following bracket

$$[(X, p), (Y, q)]_R = ([X, Y]_R, \omega_R(X, Y)). \quad (3.24)$$

It can be viewed consistently as the  $R$ -bracket associated to  $[ , ]$  in (3.21) if we define the action of  $R$  on  $(X, p)$  as

$$R(X, p) = (RX, p). \quad (3.25)$$



This construction applies to any solution of the classical Yang-Baxter equation. In particular, it applies to  $R \circ S^k$ . We need to specify the action of the intertwining operator  $S^k$  on  $(X, p)$  as

$$S^k(X, p) = (S^k X, s^k p), \quad (s^k p)(\lambda) = \lambda^k p(\lambda). \quad (3.26)$$

A simple but important observation is that the Lie bracket (3.21) does not depend on  $p$  or  $q$ . This is what allows one to define a Poisson bracket on smooth functions on  $\mathfrak{G}$  starting from the standard construction of a Lie-Poisson bracket for functions on the central extension of  $\mathfrak{G}$ . More precisely, we define, for  $k \in \mathbb{Z}$ ,

$$\{F, G\}_k^p(L) = \langle (L, p), [(\nabla F(L), 0), (\nabla G(L), 0)]_{R \circ S^k} \rangle, \quad \forall L \in \mathfrak{G}, \quad \forall F, G \in C^\infty(\mathfrak{G}), \quad (3.27)$$

where the gradient of  $F \in C^\infty(\mathfrak{G})$  is calculated with respect to (3.19) now. The following proposition gives explicit formulas for this bracket that are useful in our subsequent analysis.

**Proposition 3.8** *We have*

$$\{F, G\}_n^p(L) = -\frac{1}{2} ([RS^n \nabla F(L), L] - S^n R[\nabla F(L), L] + (pRS^n - S^n Rp)\partial_k \nabla F(L)), \nabla G(L)) \quad (3.28)$$

where one uses the bilinear form (3.19) now, and the generalization of (2.36) to  $\mathfrak{G}$  reads [30], with our conventions,

$$\{L_i^a(t_k), L_j^b(\tau_k)\}_k^p = \epsilon_{ij}^k (K^{-1})_{ac} C_b^{cd} L_{i+j+1-k}^d(t_k) \delta(t_k - \tau_k) - \epsilon_{ij}^k (K^{-1})_{ab} \delta_{i+j+1-k, s} p_s \delta'(t_k - \tau_k), \quad (3.29)$$

where  $p = p_s \lambda^s$  and we have denoted

$$\{L_i^a(t_k), L_j^b(\tau_k)\}_k^p \equiv \{f_i^a(t_k), f_j^b(\tau_k)\}_k^p \quad (3.30)$$

where  $f_i^a(t_k)$  is the coordinate function  $f_i^a(t_k) : L \mapsto L_i^a(t_k)$  if  $L(\lambda, t_k) = \sum_{i=-\infty}^0 L_i^a(t_k) e^a \lambda^i$ .

Proof: The first equation is obtained by noting that

$$\langle (Y, p), [(X, 0), (Z, 0)] \rangle = -([X, Y] + p \partial_k X, Z), \quad \forall X, Y, Z \in \mathfrak{G}, \quad (3.31)$$

and

$$(RS^n X, Y) = -(X, S^n RY), \quad \forall n \in \mathbb{Z}, \quad \forall X, Y \in \mathfrak{G}. \quad (3.32)$$

The coordinate expression is obtained by direct calculation from the first equation using the equality

$$\nabla f_i^a(t_k)(L) = (K^{-1})_{ac} e^c \lambda^{-i-1} \delta_{t_k}, \quad \delta_{t_k}(\theta) = \delta(t_k - \theta) \quad (\text{Dirac distribution}). \quad (3.33)$$

**Remark:** the appearance of  $K^{-1}$  compared to [30] is due to our implicit choice of identifying the loop algebra and its dual from the beginning using the ad-invariant bilinear form  $(\cdot, \cdot)$ .

The bracket  $\{ \cdot, \cdot \}_k^p$  enjoys the following property which generalizes Lemma 2.7.

**Lemma 3.9** For all  $k \in \mathbb{Z}$  and  $p \in \mathbb{C}(\lambda)$ ,

$$\{F, G\}_k^{s^k p}(S^k L) = \{F \circ S^k, G \circ S^k\}_R^p(L) \quad \forall L \in \mathfrak{G} \quad \forall F, G \in C^\infty(\mathfrak{G}) \quad (3.34)$$

**Notation:** in practice, we will use only functions  $p$  of the form  $p(\lambda) = \lambda^r$  for a given  $r \in \mathbb{Z}$  so we will denote the corresponding  $k$ -bracket by  $\{, \}_k^r$ . We have a family of Poisson brackets that are labeled by two integers  $k$  and  $r$ . To simplify even further the notations, we will simply write  $\{, \}_k$  for  $\{, \}_k^k$ , *i.e.* when  $r = k$ .

**Proposition 3.10**  $\mathcal{B}_k$  is a Poisson subalgebra of  $\mathcal{B}$  equipped with the Poisson bracket  $\{, \}_{-k}$ .

Proof: We need to check that the ideal  $\mathcal{I}_k$  is also a Poisson ideal for  $\{, \}_{-k}$ . Recall that

$$\begin{pmatrix} a_j(t_k) & b_j(t_k) \\ c_j(t_k) & -a_j(t_k) \end{pmatrix} = \ell_j(t_k) = L_{-j}(t_k), \quad j \geq 0, \quad (3.35)$$

so

$$\{\ell_i^a(t_k), \ell_j^b(\tau_k)\}_{-k} = \epsilon_{-i, -j}^{-k}(K^{-1})_{ac} C_b^{cd} \ell_{i+j-1-k}^d(t_k) \delta(t_k - \tau_k) - \epsilon_{-i, -j}^{-k}(K^{-1})_{ab} \delta_{i+j-1-k, k} \delta'(t_k - \tau_k). \quad (3.36)$$

Hence, for  $0 \leq i \leq k$  and  $j \geq k+1$  we get

$$\{\ell_i^a(t_k), \ell_j^b(\tau_k)\}_{-k} = 0 \in \mathcal{I}_k, \quad (3.37)$$

and for  $i, j \geq k+1$ ,

$$\{\ell_i^a(t_k), \ell_j^b(\tau_k)\}_{-k} = -(K^{-1})_{ac} C_b^{cd} \ell_{i+j-1-k}^d(t_k) \delta(t_k - \tau_k) \in \mathcal{I}_k \quad (3.38)$$

where the last claim follows from  $i, j \geq k+1 \Rightarrow i+j-k-1 \geq k+1$ .  $\blacksquare$

In practice, this means that we can restrict  $\{, \}_{-k}$  to the Poisson subspace of elements of the form

$$\Lambda^{(k)}(t_k) = \sum_{j=0}^k \begin{pmatrix} a_j(t_k) & b_j(t_k) \\ c_j(t_k) & -a_j(t_k) \end{pmatrix} \lambda^{-j} \equiv \sum_{j=0}^k \ell_j(t_k) \lambda^{-j} = S^{-k} P_+(S^k(L(t_k))) \quad (3.39)$$

$\Lambda^{(k)}(t_k)$  is thus simply a truncation of  $L(t_k)$  to its first  $k+1$  terms.

As can be seen from (3.6)-(3.8), for any  $n$  such that  $0 \leq n \leq k$ , the map  $\Psi_k$  restricted to  $\mathcal{B}_n$ , acts trivially on  $\{b_i, c_i\}_{i=0}^n$  and substitutes for  $a_i$  certain polynomials in  $b_j, c_j, j < i$ . On each element  $\Lambda^{(n)}(t_k) = \sum_{j=0}^n \ell_j(t_k) \lambda^{-j}$  we can define the analog of the Hamiltonian functions (2.12) of the FNR theory which we also denote by  $\phi_j$ ,

$$\phi_j(\Lambda^{(n)}(t_k)) = -\frac{1}{2} \text{Res}_\lambda \lambda^j \text{Tr}(\Lambda^{(n)}(t_k))^2. \quad (3.40)$$

It turns out that the action of  $\Psi_k$  on  $a_i$  coincides precisely to fixing  $\phi_j(\Lambda^{(n)}(t_k))$ ,  $j = 0, \dots, n-1$  to constant values. This is a well-known consequence of solving (3.2). It is most often stated as the fact that in the equation for  $a_j$  in (3.2)

$$\partial_k a_j = P(a_i, b_i, c_i) \quad (3.41)$$

the right-hand-side is always a total  $t_k$  derivative. Therefore one can always integrate the equation up to a constant of integration. In the particular case where  $j = 1, \dots, n$ ,  $n \leq k$ , the constant of integration is half the constant value of  $\phi_{j-1}$ . One of the crucial aspects of our construction is that, from the point of view of the Poisson structure  $\{ , \}_{-k}$ , this procedure of reducing the phase space of the theory by fixing the value of the functions  $\phi_i$  is consistent and the Poisson structure then projects down to the reduced phase space. We have

**Proposition 3.11** *The functions  $\phi_j$ ,  $j = 0, \dots, n-1$  are Casimir functions on  $\mathcal{B}_n$  equipped with  $\{ , \}_{-n}$  for any  $n \leq k$ .*

It is interesting to give two proofs of this fundamental statement using respectively the  $r$ -matrix and  $R$  operator formalism.

Proof: ( $r$ -matrix)

This proof runs along the following steps:

Consider first the Poisson bracket structure  $\{ , \}_{-n}$  without  $\delta'$  terms.

**Step 1:** The functions  $\phi_j$  indexed from 0 to  $n-1$  are identified with the  $n$  first powers of the Trace of  $L^2(t_k)$  (from 0 to  $-n+1$ ) since the truncation of  $L$  to  $\Lambda$  does not modify the  $n$  first terms of the squared trace. They are thus trivially also the  $n$  first powers (from  $n+1$  to  $2n$ ) of the Trace of  $(\lambda^n L)^2(t_k)$

**Step 2:** The Poisson bracket structure  $\{ , \}_{-n}$  evaluated on  $(\lambda^n L)(t_k) \equiv L^n$  (not to be confused with its projected image  $L^{(n)}$ ) is again parametrized by the skew-symmetric canonical  $r$ -matrix. We now incorporate the distribution  $\delta(x-y)$  into the definition of  $r$  :

$$\{L_1^n(\lambda, x), L_2^n(\mu, y)\}_{-n} = -\frac{1}{2}[r(\lambda, \mu), L^n(\lambda, x) \otimes \mathbb{I}] + \frac{1}{2}[r(\mu, \lambda), \mathbb{I} \otimes L^n(\mu, y)], \quad (3.42)$$

where again  $\Pi$  is the permutation operator on two copies of  $\mathcal{L}$ :  $\Pi(X \otimes Y) = Y \otimes X$  and

$$r(\lambda, \mu) = -2\frac{1}{\lambda - \mu}t \delta(x - y), \quad t = \Pi^{sl(2, \mathbb{C})} \quad (3.43)$$

This now identifies  $\{ , \}_{-n}$  acting on  $(\lambda^n L)(t_k) \equiv L^n \equiv S^n L(t_k)$  with the original Poisson bracket  $\{ , \}_R$ .

**Step 3:** Compute now the Poisson bracket of Trace of  $(L^n)^2(\lambda, t_k)$  with  $L^n(\mu)$ . By cyclicity of trace and properties of the  $\Pi$  operator it yields  $\frac{[L^n(\lambda), L^n(\mu)]}{\lambda - \mu}$ . Counting now available powers of  $\lambda$  indicates that no power higher than  $n$  occur in this expression whereas the Poisson bracket of the  $n$  higher  $\lambda$  powers of  $(\lambda^n L)^2(t_k)$  with  $L^n(\mu)$  should yield powers of  $\lambda$  running from  $n$  to

$2n$ . This Poisson bracket is therefore null and the statement is proven. The functions  $\phi_j$  are actually even Casimirs on  $\mathcal{B}$ .

Consider now the contribution of the  $\delta'$  terms. They do not contribute to the Poisson brackets of the  $n$  first terms of the matrix  $L$  due to the term  $\delta_{i+j-1-n,n}$  in the centrally extended Poisson structure (3.36). The statement that the functions  $\phi_j$ ,  $j = 0, \dots, n-1$  be Casimir functions thus holds at least when Poisson-bracketed with elements of  $\mathcal{B}_n$ . It actually holds indeed for  $\mathcal{B}$  as seen now from the second proof: ■

Proof: ( $R$ -operator) Given that

$$\nabla \phi_j(L(t_k)) = -S^j(L)\delta_{t_k}, \quad (3.44)$$

a direct calculation from (3.28) yields,

$$\{\phi_j, G\}_{-n}(L(t_k)) = -\frac{1}{2} \left( ([RS^{j-n}L, L] + S^{-n}R[S^jL, L])\delta_{t_k}, \nabla G(L) \right) \quad (3.45)$$

$$+ \frac{1}{2} \left( (pRS^{-n} - S^{-n}Rp)S^jL\delta'_{t_k}, \nabla G(L) \right). \quad (3.46)$$

Remembering that  $j < n$ , we have  $RS^{j-n}L = -S^{j-n}L$ . Hence the first line is zero since  $[S^mL, L] = 0$  for all  $m \in \mathbb{Z}$ . The second line is also seen to be zero by noting that, since  $p(\lambda) = \lambda^{-n}$  (see the remark on notations after Lemma 3.9), we have for all  $X \in \mathfrak{S}$ ,

$$(pRS^{-n} - S^{-n}Rp)X = (S^{-n}RS^{-n} - S^{-n}RS^{-n})X = 0, \quad p(\lambda) = \lambda^{-n}. \quad (3.47)$$

We have actually shown more than announced in the Proposition since we see that  $\phi_j$ ,  $j = 0, \dots, n-1$  is in fact a Casimir on the whole algebra  $\mathcal{B}$  equipped with  $\{, \}_{-n}$ . By restriction, it is also true on the subalgebra  $\mathcal{B}_n$ . ■

The functions  $\phi_j$  being Casimirs, we can fix them to a definite value and obtain reduced Poisson manifolds as the level sets of  $\phi_j$ ,  $j = 1, \dots, n-1$ . This proposition is particularly useful when  $n = k$  since in that case we know that  $\mathcal{B}_k$  is a Poisson subalgebra of  $\mathcal{B}$ . Combined with the previous observation which allows us to further restrict the Poisson bracket to the submanifold parametrized by  $\phi_j(\Lambda^{(n)}(t_k)) = cst$ ,  $j = 0, \dots, k-1$ , this means that the elements of the form

$$S^{-k}P_+(S^k(\Psi_k(L))) = \sum_{j=0}^k \begin{pmatrix} Q_k^{a_j} & b_j(t_k) \\ c_j(t_k) & -Q_k^{a_j} \end{pmatrix} \lambda^{-j} \quad (3.48)$$

form a Poisson submanifold of  $\mathcal{L}_{\mathcal{B}_k^0}$  equipped with  $\{, \}_{-k}$ , where  $Q_k^{a_j}$  is the polynomial substituted to  $a_j$  by  $\Psi_k$  (cf Definition 3.3). We are now ready to state the main theorem of this section. We use the auxiliary space notation (2.26) e.g.

$$V_{k,1}^{(k)}(\lambda, t_k) = V_k^{(k)}(\lambda, t_k) \otimes \mathbb{1}. \quad (3.49)$$

**Theorem 3.12** *The map  $P_+ \circ S^k \circ \Psi_k : L \mapsto V_k^{(k)}(\lambda, t_k)$  is a Poisson map from  $(\mathcal{L}_-, \{ , \}_{-k})$  to  $(\mathcal{L}_{\mathcal{B}_k^0}, \{ , \}_R)$ . As a consequence, the Lax matrix  $V_k^{(k)}(\lambda, t_k)$  satisfies the following ultralocal Poisson algebra*

$$\{V_{k,1}^{(k)}(\lambda, t_k), V_{k,2}^{(k)}(\mu, \tau_k)\}_R = \delta(t_k - \tau_k)[r(\lambda, \mu), V_{k,1}^{(k)}(\lambda, t_k) + V_{k,2}^{(k)}(\mu, \tau_k)] \quad (3.50)$$

where  $r(\lambda, \mu)$  is the characteristic  $\mathfrak{sl}(2, \mathbb{C})$  classical  $r$ -matrix of the AKNS hierarchy. The underlying labels 1, 2 in  $V_k$  take the same meaning of “position in the tensor square of  $\mathcal{L}$ ”, as described in e.g. (2.30).

Proof:

To prove the first part of the statement, note that the map  $\Phi_k \equiv S^{-k} \circ P_+ \circ S^k \circ \Psi_k$  is a Poisson map with respect to  $\{ , \}_{-k}$ . This is a consequence of Propositions 3.10, 3.11 and the discussion following them. Indeed, Proposition 3.10 states that the quotient map  $\mathcal{B} \rightarrow \mathcal{B}/\mathcal{I}_k \cong \mathcal{B}_k$  is a Poisson map. This implies that the map  $\mathcal{L}_-^0 \rightarrow \mathcal{L}_{\mathcal{A}_k}$  which acts as  $L \mapsto S^{-k} P_+ S^k(L) \equiv \Lambda^{(k)}$  is also a Poisson map<sup>3</sup>. Now,

$$S^{-k} P_+ S^k \Psi_k(L) = \Psi_k S^{-k} P_+ S^k(L). \quad (3.51)$$

In the right-hand side,  $\Psi_k$  acts as the identity on  $b_j, c_j, j = 1, \dots, k$  and exactly as the fixing of the Casimirs  $\phi_j, j = 0, \dots, k-1$  to constants (say zero), on  $a_p, p = 1, \dots, k$  with the effect of replacing  $a_p$  by the polynomial  $Q_k^{a_p}$ . Proposition 3.11 thus ensures that it is a Poisson map from  $\mathcal{L}_{\mathcal{A}_k}$  to  $\mathcal{L}_{\mathcal{B}_k^0}$ <sup>4</sup>. Putting everything together yields the claim on  $\Phi_k$ . To conclude, it suffices to note that  $P_+ \circ S^k \circ \Psi_k = S^k \circ \Phi_k$  and that  $S^k$  maps the bracket  $\{ , \}_{-k}$  to the  $R$ -bracket according to Lemma 3.9.

The second part of the statement is a consequence of the Poisson map property combined with the classical  $r$ -matrix representation of the  $k$ -bracket discussed in Proposition 2.10, properly extended to functions of  $t_k$  according to the central extension construction explained above (see also [30]). For the  $R$ -bracket, the latter reads generically

$$\{L_1(\lambda, t_k), L_2(\mu, \tau_k)\}_R = \delta(t_k - \tau_k) \left[ \frac{\Pi}{\lambda - \mu}, L_1(\lambda) + L_2(\mu) \right]. \quad (3.52)$$

The equality

$$\{F \circ S^k \circ \Phi_k, G \circ S^k \circ \Phi_k\}_{-k}(L(t_k)) = \{F, G\}_R(S^k \circ \Phi_k(L(t_k))) \quad (3.53)$$

ensures that we can consistently restrict (3.52) to  $S^k \circ \Phi_k(L(t_k)) = V_k^{(k)}(\lambda, t_k)$  to obtain (3.50) as claimed. ■

<sup>3</sup>Note that in this intermediate step, we work with the algebra  $\mathcal{A}_k$  defined from  $\mathcal{A}$  in the same way as  $\mathcal{B}_k$  is defined from  $\mathcal{B}$ . This is because the map  $\Psi_k$  has not been applied yet so the elements  $a_i, b_i, c_i$  are not yet viewed as functions of  $t_k$ . In practice, this simply means that we work with the Poisson structure based on the  $r$ -matrix before central extension. This has no bearings on the arguments which only depend on the algebraic structure of the  $r$ -matrix and not on the details of the central extension, as the reader can check.

<sup>4</sup>See previous footnote.

**Remark:** this theorem proves in full generality the conjecture made in our previous paper [7] where we had explicitly established (3.50) for  $k = 1, 2, 3$ . We had used a completely different approach based on a Lagrangian and ideas of covariant field theory and presented arguments in favour of the claim that (3.50) holds beyond  $k = 3$ .

### 3.2 From FNR flows to AKNS Lax operators

To establish consistency of the  $\Psi_k$  constraint procedure with integrability structures we now consider the construction of the time-flow operators in the AKNS hierarchy, as derived from traces of the monodromy of the Lax ‘space’ operator, and compare it with the application of the constraint onto the time-flow matrices in the FNR procedure. We will see that both objects are identical. More precisely, the generalized Lax matrices  $V_k^{(n)}(\lambda, t_k)$ ,  $n \in \mathbb{N}$  obtained by applying  $\Psi_k$  in the FNR picture should coincide with the matrices  $\mathcal{V}_k^{(n)}(\lambda, t_k)$  constructed from the monodromy of  $V_k^{(k)}(\lambda, t_k)$  at the AKNS level and which generate the  $t_n$  flow of  $V_k^{(k)}(\lambda, t_k)$  with respect to  $\{ , \}_R$ . The proof runs in three steps.

#### First step:

The commutativity of the flows at the FNR level imply the (weak) zero curvature condition

$$[\partial_n L^{(k)} - \partial_k L^{(n)} + [L^{(k)}, L^{(n)}], L] = 0 \quad (3.54)$$

for any pair of times  $t_k, t_n$ , where we recall that  $L^{(n)} = P_+(S^n L)$ . In fact, one has the stronger result

**Lemma 3.13** *The FNR equations*

$$L_{t_k} = [L^{(k)}, L] = \frac{1}{2}[RS^k L, L], \quad k \geq 1, \quad (3.55)$$

imply the (strong) zero curvature condition:

$$\partial_n L^{(k)} - \partial_k L^{(n)} + [L^{(k)}, L^{(n)}] = 0. \quad (3.56)$$

We propose again two distinct proofs of this Lemma.

Proof: The proof relies on the following facts:

1.  $R = P_+ - P_-$  satisfies the modified classical Yang-Baxter equation

$$[RX, RY] - 2R[X, Y]_R = -[X, Y], \quad (3.57)$$

2.  $S^j$  is an intertwining operator,

3.  $R$  and  $S^j$  commute with all the differential operators  $\partial_{t_k}$ ,  $k \geq 1$ .

Now we have on the one hand

$$\begin{aligned}
 (RS^n L)_{t_k} - (RS^k L)_{t_n} + \frac{1}{2}[RS^n L, RS^k L] &= \frac{1}{2}RS^n[RS^k L, L] - \frac{1}{2}RS^k[RS^n L, L] + \frac{1}{2}[RS^n L, RS^k L] \\
 &= R[S^k L, S^n L]_R - \frac{1}{2}[RS^k L, RS^n L] \\
 &= \frac{1}{2}[S^k L, S^n L] \\
 &= 0
 \end{aligned}$$

and on the other hand, using  $R = 2P_+ - I$ ,

$$\begin{aligned}
 (RS^n L)_{t_k} - (RS^k L)_{t_n} + \frac{1}{2}[RS^n L, RS^k L] &= 2 \left( L_{t_k}^{(n)} - L_{t_n}^{(k)} + [L_{t_k}^{(n)}, L_{t_n}^{(k)}] \right) \\
 &\quad - S^n \left( L_{t_k} - [L^{(k)}, L] \right) \\
 &\quad + S^k \left( L_{t_n} - [L^{(n)}, L] \right) \\
 &= 2 \left( L_{t_k}^{(n)} - L_{t_n}^{(k)} + [L_{t_k}^{(n)}, L_{t_n}^{(k)}] \right). \quad (3.58)
 \end{aligned}$$

A more compact but less rigorous proof is given as follows:

Proof: Equation (3.54) implies that  $\partial_n L^{(k)} - \partial_k L^{(n)} + [L^{(k)}, L^{(n)}]$  is a matrix commuting with  $L$ . Hence it must reduce in general to a polynomial in  $L$  hereafter denoted  $P(L)$ . However  $\partial_n L^{(k)} - \partial_k L^{(n)} + [L^{(k)}, L^{(n)}]$  is at most of order  $n+k$  as a polynomial in  $\lambda$  whilst any polynomial in  $L$  must exhibit all finite powers of  $\lambda$ . Moreover  $P(L)$  cannot be reduced to the identity matrix since  $\partial_n L^{(k)} - \partial_k L^{(n)} + [L^{(k)}, L^{(n)}]$  is traceless. This leaves us with  $P(L) = 0$ , which is (3.56). ■

### Second step:

We fix a time  $t_k$  and assume that one has solved the FNR equation with respect to  $t_k$  first, producing the map  $\Psi_k$ . The application of  $\Psi_k$  to (3.56) yields the AKNS zero-curvature condition

$$\partial_n V_k^{(k)}(\lambda, t_k) - \partial_k V_k^{(n)}(\lambda, t_k) + [V_k^{(k)}(\lambda, t_k), V_k^{(n)}(\lambda, t_k)] = 0, \quad (3.59)$$

for any other time  $t_n$ , where Lax matrices should now be viewed as functions of  $t_n$  as well. The collection of these equations can be cast into a single equation using the following generating function

$$V_k(\lambda, \mu, t_k) = \sum_{n=1}^{\infty} \mu^{-n} V_k^{(n-1)}(\lambda, t_k) = \sum_{n=1}^{\infty} \mu^{-n} P_+(S^{n-1}(\Psi_k(L(t_k)))) = \sum_{n=1}^{\infty} \mu^{-n} P_+(\lambda^{n-1} V_k(\lambda, t_k)) \quad (3.60)$$

and the differential operator  $D = \sum_{n=1}^{\infty} \mu^{-n} \partial_{n-1}$ ,

$$DV_k^{(k)}(\lambda, t_k) - \partial_k V_k(\lambda, \mu, t_k) + [V_k^{(k)}(\lambda, t_k), V_k(\lambda, \mu, t_k)] = 0. \quad (3.61)$$

On the other hand, at the AKNS level, the following generalization of a classic argument (see e.g. [16] for an  $r$ -matrix derivation or [19] for a geometric treatment) was proved in [7]. Denote  $\xi = t_k$  and assume  $\xi \in [-M, M]$  (with periodic boundary conditions). Let  $T(\xi_1, \xi_2, \lambda)$  be the finite-interval monodromy of the auxiliary problem

$$\partial_\xi \Psi(\lambda, \xi) = V_k^{(k)}(\lambda, \xi) \Psi(\lambda, \xi) \quad (3.62)$$

normalised to  $T(\xi, \xi, \lambda) = \mathbb{I}$ . Introduce the following decomposition, understood as a series in  $\lambda^{-1}$ ,

$$T(\xi_1, \xi_2, \lambda) = (\mathbb{I} + W(\xi_1, \lambda)) e^{Z(\xi_1, \xi_2, \lambda)} (\mathbb{I} + W(\xi_2, \lambda))^{-1} \quad (3.63)$$

where  $W$  is off-diagonal and  $Z$  is proportional to  $\sigma_3$  and admits the expansion

$$Z(\xi_1, \xi_2, \lambda) = \sum_{n=0}^k Z^{(-n)}(\xi_1, \xi_2) \lambda^n + \sum_{n=1}^{\infty} Z^{(n)}(\xi_1, \xi_2) \lambda^{-n}. \quad (3.64)$$

Define

$$H_k^{(n-1)} = \frac{1}{2} \text{Tr}(\sigma_3 Z^{(n)}(-M, M)), \quad n \geq 1. \quad (3.65)$$

The functions  $H_k^{(n)}$  are thus commuting Hamiltonians which respectively generate the  $t_n$  flows with respect to  $\{, \}_R$ . We have

**Proposition 3.14** [7] *The Hamiltonian flow equation*

$$\partial_n V_k^{(k)}(\lambda, t_k) = \{V_k^{(k)}(\lambda, t_k), H_k^{(n)}\}_R \quad (3.66)$$

is equivalent to the zero curvature equation

$$\partial_n V_k^{(k)}(\lambda, t_k) - \partial_k \mathcal{V}_k^{(n)}(\lambda, t_k) + [V_k^{(k)}(\lambda, t_k), \mathcal{V}_k^{(n)}(\lambda, t_k)] = 0 \quad (3.67)$$

where  $\mathcal{V}_k^{(n)}(\lambda, t_k)$  is obtained from the generating function

$$\mathcal{V}_k(\lambda, \mu, \xi) = \frac{-1}{\lambda - \mu} (\mathbb{I} + W(\xi, \mu)) \sigma_3 (\mathbb{I} + W(\xi, \mu))^{-1} \quad (3.68)$$

via the expansion

$$\mathcal{V}_k(\lambda, \mu, \xi) = \sum_{n=0}^{\infty} \frac{\mathcal{V}_k^{(n)}(\lambda, \xi)}{\mu^{n+1}}. \quad (3.69)$$

**Third step:**

We now have two candidate generating functions for what is expected to be the same hierarchy of Lax matrices. This is consistent if they are equal. Indeed we prove:

**Theorem 3.15** *The two generating functions  $V_k(\lambda, \mu, t_k)$  and  $\mathcal{V}_k(\lambda, \mu, t_k)$ , obtained either directly from the FNR level by application of the map  $\Psi_k$ ; or in a self-contained manner at the AKNS level as explained above; are equal, and therefore:*

$$V_k^{(n)}(\lambda, t_k) = \mathcal{V}_k^{(n)}(\lambda, t_k), \quad n \geq 0. \quad (3.70)$$



Proof: On the one hand,

$$V_k(\lambda, \mu, t_k) = \sum_{n=1}^{\infty} \mu^{-n} P_+(\lambda^{n-1} V_k(\lambda, t_k)) \quad (3.71)$$

$$= \sum_{n=0}^{\infty} \mu^{-n-1} P_+(\lambda^n \sum_{j=0}^{\infty} V_{k,j}(t_k) \lambda^{-j}) \quad (3.72)$$

$$= \frac{1}{\mu} \sum_{n=0}^{\infty} \sum_{\ell=0}^n \mu^{\ell-n} \frac{\lambda^\ell}{\mu^\ell} V_{k,n-\ell}(t_k) \quad (3.73)$$

$$= \frac{1}{\mu} \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\mu^\ell} \sum_{n=\ell}^{\infty} \mu^{\ell-n} V_{k,n-\ell}(t_k) \quad (3.74)$$

$$= \frac{-1}{\lambda - \mu} \sum_{p=0}^{\infty} \mu^{-p} V_{k,p}(t_k) \quad (3.75)$$

$$= \frac{-1}{\lambda - \mu} V_k(\mu, t_k) \quad (3.76)$$

and by definition of  $V_k(\mu, t_k)$ , we have

$$\partial_k V_k(\mu, t_k) = [V_k^{(k)}(\mu, t_k), V_k(\mu, t_k)]. \quad (3.77)$$

On the other hand

$$\mathcal{V}_k(\lambda, \mu, t_k) = \frac{-1}{\lambda - \mu} \mathcal{V}_k(\mu, t_k). \quad (3.78)$$

where

$$\mathcal{V}_k(\mu, t_k) = (\mathbb{I} + W(t_k, \mu)) \sigma_3 (\mathbb{I} + W(t_k, \mu))^{-1}. \quad (3.79)$$

Hence,

$$\partial_k \mathcal{V}_k(\mu, t_k) = [\partial_k W(\mu, t_k) (\mathbb{I} + W(\mu, t_k))^{-1}, \mathcal{V}_k(\mu, t_k)] \quad (3.80)$$

Now, we use the fact that  $T(\xi_1, \xi_2, \lambda)$  in (3.63) is a solution of (3.62) for  $\xi_1 = \xi = t_k$  and  $\xi_2$  arbitrary. We obtain, dropping  $(\mu, t_k)$  for conciseness,

$$\partial_k W(\mathbb{I} + W)^{-1} = V_k^{(k)} - (\mathbb{I} + W) \partial_k Z (\mathbb{I} + W)^{-1}. \quad (3.81)$$

Inserting in (3.80) and remembering that  $Z$  is proportional to  $\sigma_3$ , so that the contribution  $(\mathbb{I} + W) [\partial_k Z, \sigma_3] (\mathbb{I} + W)^{-1}$  is zero, we are left with

$$\partial_k \mathcal{V}_k(\mu, t_k) = [V_k^{(k)}(\mu, t_k), \mathcal{V}_k(\mu, t_k)] \quad (3.82)$$

which is the same equation as for  $V_k(\mu, t_k)$ . Solutions  $\Upsilon$  of this equation, known as resolvents, have been studied extensively (see e.g. [28], Chap. 9 and 10). They are known to form a two-dimensional vector space spanned by  $\Upsilon^\alpha = \phi E_{\alpha\alpha} \phi^{-1}$ ,  $\alpha = 1, 2$  where  $\phi$  is a so-called dressing factor realising a gauge transformation from  $V_k^{(k)}$  to a diagonal matrix  $D(\mu)$  and  $E_{\alpha\alpha}$  is the

matrix with the only non zero entry equal to 1 at position  $(\alpha, \alpha)$ . The dressing  $\phi$  can be chosen in the form  $\mathbb{I} + W$  as we have here. Then a resolvent is completely characterised by fixing its constant coefficient (in front of  $\mu^0$ ). Here both  $\mathcal{V}_k(\mu, t_k)$  and  $V_k(\mu, t_k)$  have constant coefficient equal to  $\sigma_3$ . Therefore, they are both equal to

$$(\mathbb{I} + W(t_k, \mu)) \sigma_3 (\mathbb{I} + W(t_k, \mu))^{-1} \quad (3.83)$$

as required.  $\blacksquare$

**Remark:** To obtain this equality, we have chosen to set all the constants of integration in the map  $\Psi_k$  to zero so that diagonal elements  $a_j$ ,  $j \geq 1$  contain no constant term. This is the standard ANKS normalisation.

### 3.3 Zero curvature representation and dual Hamiltonian formulation of an integrable PDE

To complete our diagram on Figure 1, we must show that if we pick two times  $t_k$  and  $t_n$ , applying first  $\Psi_k$  and obtaining the zero curvature associated to  $t_n$  or, applying first  $\Psi_n$  and obtaining the zero curvature associated to  $t_k$ , yields the same set of PDEs for the same variables. Without loss of generality, fix  $1 \leq n < k$ . We begin with a simple but convenient observation.

**Lemma 3.16** *Considering the matrices  $\ell_j$  as functions of both  $t_n$  and  $t_k$ , applying  $\Psi_n$  to obtain the generalized Lax matrices  $V_n^{(j)}$  and imposing the zero curvature condition*

$$\partial_k V_n^{(n)} - \partial_n V_n^{(k)} + [V_n^{(n)}, V_n^{(k)}] = 0 \quad (3.84)$$

is equivalent to imposing the following set of simultaneous equations

$$\ell_0 = \sigma_3, \quad \partial_n \ell_p = \sum_{j=0}^n [\ell_j, \ell_{p+n-j}], \quad p \geq 1, \quad (3.85)$$

$$\partial_k L^{(n)} - \partial_n L^{(k)} + [L^{(n)}, L^{(k)}] = 0, \quad L^{(j)} = \sum_{m=0}^j \ell_m \lambda^{j-m}. \quad (3.86)$$

Proof: Eq (3.85) is just the component form of the FNR equation (3.2) of Proposition 3.1 which gives rise to the map  $\Psi_n$ . Hence, under (3.85), the zero curvature condition (3.86) is equivalent to (3.84).  $\blacksquare$

We are now in a position to prove

**Proposition 3.17** *Let  $1 \leq n < k$ . The set of equations*

$$\ell_0 = \sigma_3, \quad \partial_n \ell_p = \sum_{j=0}^n [\ell_j, \ell_{p+n-j}], \quad p \geq 1, \quad (3.87)$$

$$\partial_k L^{(n)} - \partial_n L^{(k)} + [L^{(n)}, L^{(k)}] = 0. \quad (3.88)$$

gives rise to the same set of PDEs for the variable  $b_j, c_j, j=1, \dots, n$  as the set of equations

$$\ell_0 = \sigma_3, \quad \partial_k \ell_p = \sum_{j=0}^k [\ell_j, \ell_{p+k-j}], \quad p \geq 1, \quad (3.89)$$

$$\partial_n L^{(k)} - \partial_k L^{(n)} + [L^{(k)}, L^{(n)}] = 0. \quad (3.90)$$

The common set of  $2n$  PDEs for the  $2n$  variables  $b_j, c_j, j=1, \dots, n$  reads

$$\partial_k V_{\sharp, p} - \partial_n V_{\sharp, k+p-n} + \sum_{j=0}^{n-p} [V_{\sharp, n-j}, V_{\sharp, k+p+j-n}] = 0, \quad p = 1, \dots, n \quad (3.91)$$

where  $V_{\sharp}^{(j)} = \sum_{m=0}^j V_{\sharp, m} \lambda^{j-m}$  and  $V_{\sharp}^{(j)}$  is either  $V_k^{(j)}$  or  $V_n^{(j)}$ ,  $j = n, k$ .

Proof: The discussion of the map  $\Psi_k$  for any  $k \geq 1$  in Section 3.1 has established that it constrains the variable  $b_j, c_j$  for  $j > k$  and  $a_j$  for all  $j \geq 1$  in a specific manner. Hence, since  $k > n$  here, it is not obvious a priori that (3.89)-(3.90) constrains the variables  $b_j, c_j$  for  $j = n+1, \dots, k$  in the same way as (3.87)-(3.88) does. The same remark goes for  $a_j, j \geq 1$ . That they do so is the key to obtain the complete equivalence of the two descriptions of (3.91). Let us spell out (3.87)-(3.88),

$$\begin{cases} \ell_0 = \sigma_3, \quad \partial_n \ell_p = \sum_{j=0}^n [\ell_j, \ell_{p+n-j}], \quad p \geq 1, \\ \partial_k \ell_p - \partial_n \ell_{k+p-n} + \sum_{j=0}^{n-p} [\ell_{n-j}, \ell_{k+p+j-n}] = 0, \quad p = 1, \dots, n, \\ \partial_n \ell_p = \sum_{j=0}^n [\ell_{n-j}, \ell_{p+j}], \quad p = 1, \dots, k-n. \end{cases} \quad (3.92)$$

Of course, the third equation coming from the zero curvature is redundant here because of the first equation which is the FNR equation w.r.t.  $t_n$ . Upon solving the latter, *i.e.* applying  $\Psi_n$ , we are therefore left with (3.91) with  $V_n^{(j)}$ ,  $j = n, k$  with all their entries being polynomials in  $b_j, c_j, j = 1, \dots, n$  and their derivatives w.r.t.  $t_n$ .

Consider now (3.89)-(3.90), equivalent to,

$$\begin{cases} \ell_0 = \sigma_3, \quad \partial_k \ell_p = \sum_{j=0}^k [\ell_j, \ell_{p+k-j}], \quad p \geq 1, \\ \partial_k \ell_p - \partial_n \ell_{k+p-n} + \sum_{j=0}^{n-p} [\ell_{n-j}, \ell_{k+p+j-n}] = 0, \quad p = 1, \dots, n, \\ \partial_n \ell_p = \sum_{j=0}^n [\ell_{n-j}, \ell_{p+j}], \quad p = 1, \dots, k-n. \end{cases} \quad (3.93)$$

We see that the first and third equation yield different informations. For the  $k$  relevant matrices

$$\ell_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}, \quad j = 1, \dots, k \quad (3.94)$$

the first equation does not constrain  $b_j, c_j$   $j = 1, \dots, k$ . It also gives expressions for  $a_j$  in terms of  $b_j, c_j$   $j = 1, \dots, k$  and *their derivatives w.r.t.  $t_k$* . So the situation is a priori very different from the first route. However, the third equation provides one part of the equations with respect to  $t_n$  that we need. The required remaining equations w.r.t.  $t_n$  are obtained by noting that the combination of the first equation for  $p = 1, \dots, n$  in (3.93) and the second equation yield

$$\partial_n \ell_p = \sum_{j=0}^n [\ell_{n-j}, \ell_{p+j}], \quad p = k - n + 1, \dots, k. \quad (3.95)$$

Together with the third equation, this yields a total of  $k$  equations with respect to  $t_n$

$$\partial_n \ell_p = \sum_{j=0}^n [\ell_{n-j}, \ell_{p+j}], \quad p = 1, \dots, k. \quad (3.96)$$

To complete the argument, we must show that this truncated system of equations constrains  $b_j, c_j$ ,  $j = n + 1, \dots, k$  and  $a_p$ ,  $p = 1, \dots, k$  in exactly the same way as the analogous non truncated system in (3.92) which defines the map  $\Psi_n$ .

Consider first the  $b, c$  variables. Since  $\ell_0 = \sigma_3$ , for  $p = 1, \dots, k - n$ , the truncated system determines recursively  $b_j, c_j$  for  $j = n + 1, \dots, k$  in terms of  $a_m^{(\ell)}, b_m^{(\ell)}, c_m^{(\ell)}$ ,  $m \leq n$ , exactly as  $\Psi_n$  does.

Consider now the  $a$  variables. We must show that  $a_p$ ,  $p = 1, \dots, n$  takes the same form as under  $\Psi_n$ . Upon examination of (3.96), we see that  $a_p$ ,  $p = 1, \dots, k$  determined by the truncated system does not depend on the entries of  $\ell_m$  for  $m > n$ . Therefore, the truncated system can be embedded in the nontruncated system without altering the expression for  $a_p$ ,  $p = 1, \dots, k$ . We can apply the argument in the non truncated system that yields the expressions of  $a_p$  as polynomials in  $a_m, b_m, c_m$ ,  $m < p$  (meaning that the right hand side of (3.87) projected onto the diagonal is an exact  $t_n$  derivative). When restricted to  $a_p$ ,  $p = 1, \dots, k$ , this yields the same expression as one would obtain via  $\Psi_n$ .

We have now shown that the two systems (3.92) and (3.93) yield the same generalized matrices  $V_k^{(j)}$  and  $V_n^{(j)}$ ,  $j = n, k$ . In the first case, both are obtained by straightforward application of  $\Psi_n$ . In the second case, the missing equations with respect to  $t_n$  are provided by the zero curvature condition itself, combined with the FNR equations with respect to  $t_k$ . ■

Combining Proposition 3.17 with Proposition 3.14 and Theorem 3.15 of the previous section, we can now state the main theorem of this section.

**Theorem 3.18** *Let  $n, k \geq 1$  be given. The set of PDEs obtained from  $\Psi_n$  or  $\Psi_k$ , together with the corresponding zero curvature condition, has two equivalent Hamiltonian formulations as*

$$\partial_k V_n^{(n)} = \{V_n^{(n)}, H_n^{(k)}\}_R^{t_n} \quad (3.97)$$

and

$$\partial_n V_k^{(k)} = \{V_k^{(k)}, H_k^{(n)}\}_R^{t_k} \quad (3.98)$$

where the two different Poisson brackets  $\{, \}_R^{t_n}$  and  $\{, \}_R^{t_k}$  are the brackets derived from  $\Psi_n$  and  $\Psi_k$  respectively.  $V_n^{(n)}$  (resp.  $V_k^{(k)}$ ) satisfies the ultralocal Poisson algebra (3.50) with respect to  $\{, \}_R^{t_n}$  (resp.  $\{, \}_R^{t_k}$ ). The Hamiltonian  $H_n^{(k)}$  (resp.  $H_k^{(n)}$ ) is constructed as in (3.62)-(3.65) (swapping the roles of  $t_n$  and  $t_k$  appropriately).

This theorem is the generalization to an arbitrary pair  $t_n, t_k$  of the notion of dual Hamiltonian formulation that was introduced in [7], based on explicit constructions for the pairs  $n = 1, k = 2$  and  $n = 1, k = 3$  with a special reduction ( $b_1 = c_1^*$ ) corresponding to the NLS equation and the complex modified KdV. In both cases, the variable  $t_1$  was called  $x$  and the corresponding Poisson bracket  $\{, \}_R^{t_1}$  was called the equal-time bracket  $\{, \}_S$ . The variables  $t_2$  and  $t_3$  were taken in turn as the time  $t$  and the corresponding Poisson bracket  $\{, \}_R^{t_k}$  was called the equal-space bracket  $\{, \}_T^{(k)}$ . The present approach puts all the times  $t_n$  and  $t_k$  on equal footing from the start.

## 4 Examples

### 4.1 Recovering known results: the NLS case

The NLS equation

$$iq_t + q_{xx} - 2\epsilon|q|^2q = 0 \quad \epsilon = \pm 1, \quad (4.1)$$

is well-known to follow from the FNR/AKNS scheme described previously by setting  $n = 1, k = 2$ . Solving the FNR equations w.r.t.  $t_1$  and applying the zero curvature equation involving  $t_2$  yields the following PDEs

$$\begin{cases} \partial_{t_2} b_1 - \frac{1}{2} \partial_{t_1}^2 b_1 + b_1^2 c_1 = 0, \\ \partial_{t_2} c_1 + \frac{1}{2} \partial_{t_1}^2 c_1 - c_1^2 b_1 = 0. \end{cases} \quad (4.2)$$

Applying the reduction

$$b_1 = q, \quad c_1 = \epsilon q^*, \quad (4.3)$$

and setting  $t_1 = x$  and  $t_2 = 2it$  gives NLS. There is no need to work only within this reduction so we will keep using  $b_1$  and  $c_1$  in the following, thus generalizing the results of [5, 7] which dealt only with the particular reduction (4.3).

Here the two Lax matrices are, in the AKNS normalization,

$$V_1^{(1)} = \begin{pmatrix} \lambda & b_1 \\ c_1 & -\lambda \end{pmatrix}, \quad V_1^{(2)} = \begin{pmatrix} \lambda^2 - \frac{1}{2} b_1 c_1 & \lambda b_1 + \frac{1}{2} \partial_{t_1} b_1 \\ \lambda c_1 - \frac{1}{2} \partial_{t_1} c_1 & -\lambda^2 + \frac{1}{2} b_1 c_1 \end{pmatrix}. \quad (4.4)$$

The Poisson bracket  $\{, \}_R^{t_1}$  used for the Hamiltonian description of (non reduced) NLS is the famous one for which  $b_1$  and  $c_1$  are canonically conjugate variables. It yields the well-known

ultralocal Poisson algebra (3.50) of Theorem 3.12 satisfied by  $V_1^{(1)}$ . In [7], it was denoted  $\{ , \}_S$ . The Hamiltonian  $H_1^{(2)}$  obtained from (3.65) reads

$$H_1^{(2)} = \frac{1}{16} \int_{-M}^M (b_1 \partial_{t_1}^2 c_1 + b_1 \partial_{t_1}^2 c_1 - 2b_1^2 c_1^2) dt_1 \quad (4.5)$$

and the flow equation (3.97) boils down to

$$\partial_{t_2} b_1 = \{b_1, H_1^{(2)}\}_R^{t_1} = \frac{1}{2} \partial_{t_1}^2 b_1 - b_1^2 c_1, \quad (4.6)$$

$$\partial_{t_2} c_1 = \{c_1, H_1^{(2)}\}_R^{t_1} = -\frac{1}{2} \partial_{t_1}^2 c_1 + c_1^2 b_1 \quad (4.7)$$

thus reproducing (4.2) as required. To compute the Poisson bracket of  $b_1, c_1$  with the Hamiltonian  $H_1^{(2)}$ , we use the fact that Poisson algebra (3.50) is equivalent to

$$\{b_1(t_1), c_1(\tau_1)\}_R^{t_1} = 4\delta(t_1 - \tau_1), \quad (4.8)$$

and all other Poisson brackets being zero.

The dual Hamiltonian approach to (non reduced) NLS corresponds to swapping the role of  $n$  and  $k$ . This yields the following two Lax matrices

$$V_2^{(1)} = \begin{pmatrix} \lambda & b_1 \\ c_1 & -\lambda \end{pmatrix}, \quad V_2^{(2)} = \begin{pmatrix} \lambda^2 - \frac{1}{2} b_1 c_1 & \lambda b_1 + b_2 \\ \lambda c_1 + c_2 & -\lambda^2 + \frac{1}{2} b_1 c_1 \end{pmatrix}. \quad (4.9)$$

As we expected, we work with four fields now in this dual picture. The zero curvature equation yields the following PDEs

$$\begin{cases} \partial_{t_1} b_1 - 2b_2 = 0, \\ \partial_{t_1} c_1 + 2c_2 = 0, \\ \partial_{t_2} b_1 - \partial_{t_1} b_2 + b_1^2 c_1 = 0, \\ \partial_{t_2} c_1 - \partial_{t_1} c_2 - c_1^2 b_1 = 0. \end{cases} \quad (4.10)$$

We see on this example the procedure explained in full generality in Proposition 3.17. The first two equations produced by the zero curvature equation define the extra fields  $b_2, c_2$  in terms of  $b_1, c_1$ . They are the FNR equations of the map  $\Psi_1$  that are missing when we go over to the dual picture and apply the map  $\Psi_2$ . It is obvious that under the first two equations, the last two are the same as (4.2). In [7], we obtained a clear physical interpretation of the extra variables  $b_2, c_2$  appearing in the dual formulation of NLS: they were the canonically conjugate momenta of  $b_1 = q$  and  $c_1 = q^*$  under the covariant Legendre transformation with respect to  $t_1 = x$ . In the present work, they simply appear as additional coordinates in the reduced phase space whose construction from the infinite coadjoint orbit description of FNR we detailed in the previous section.

The dual Hamiltonian formulation of (non reduced) NLS is now obtained using the Poisson bracket  $\{ , \}_R^{t_2}$ , the Hamiltonian

$$H_2^{(1)} = \frac{1}{2} \int_{-M}^M \left( b_2 c_2 + \frac{1}{4} (c_1 \partial_{t_2} b_1 - b_1 \partial_{t_2} c_1 + b_1^2 c_1^2) \right) dt_2, \quad (4.11)$$

and the flow equation (3.98) to obtain

$$\partial_{t_1} b_1 = \{b_1, H_2^{(1)}\}_R^{t_2} = 2b_2, \quad (4.12)$$

$$\partial_{t_1} c_1 = \{c_1, H_2^{(1)}\}_R^{t_2} = -2c_2, \quad (4.13)$$

$$\partial_{t_1} b_2 = \{b_2, H_2^{(1)}\}_R^{t_2} = \partial_{t_2} b_1 + b_1^2 c_1, \quad (4.14)$$

$$\partial_{t_1} c_2 = \{c_2, H_2^{(1)}\}_R^{t_2} = \partial_{t_2} c_1 - c_1^2 b_1, \quad (4.15)$$

as required. The Poisson bracket  $\{, \}_R^{t_2}$  captures the ultralocal Poisson algebra (3.50) of Theorem 3.12 for  $V_2^{(2)}$  now. In [7], it was denoted  $\{, \}_T^{(2)}$  and called the equal-space Poisson bracket. Here, it is equivalent to the following relations on the fields

$$\{b_1(t_2), c_2(\tau_2)\}_R^{t_2} = 4\delta(t_2 - \tau_2), \quad \{c_1(t_2), b_2(\tau_2)\}_R^{t_2} = -4\delta(t_2 - \tau_2), \quad (4.16)$$

and all other Poisson brackets being zero. Under the special reduction (4.3), the previous results reproduce those originally derived in [5, 7].

The same analysis can be performed with  $n = 1$  and  $k = 3$ . The results generalize those found in [7] for the complex modified KdV equation

$$q_t + q_{xxx} - 6\epsilon|q|^2 q_x = 0. \quad (4.17)$$

which is recovered by applying the reduction (4.3) and setting  $t_1 = x$  and  $t_3 = -4t$ . The two Hamiltonians

$$H_1^{(3)} = \frac{1}{32} \int_{-M}^M (c_1 \partial_{t_1}^3 b_1 - b_1 \partial_{t_1}^3 c_1 + 3b_1 c_1 (b_1 \partial_{t_1} c_1 - c_1 \partial_{t_1} b_1)) dt_1 \quad (4.18)$$

and

$$H_3^{(1)} = \frac{1}{8} \int_{-M}^M (c_1 \partial_{t_3} b_1 - b_1 \partial_{t_3} c_1 + 2b_1 c_1 (b_1 c_2 + b_2 c_1) + 4b_3 c_2 + 4b_2 c_3) dt_3 \quad (4.19)$$

produce the PDEs corresponding to the zero curvature for  $(V_1^{(1)}, V_1^{(3)})$  and  $(V_3^{(1)}, V_3^{(3)})$  using (3.97) and (3.98) respectively. Both systems boil down to

$$\partial_{t_3} b_1 - \frac{1}{4} \partial_{t_1}^3 b_1 + \frac{3}{2} b_1 c_1 \partial_{t_1} b_1 = 0, \quad (4.20)$$

$$\partial_{t_3} c_1 - \frac{1}{4} \partial_{t_1}^3 c_1 + \frac{3}{2} b_1 c_1 \partial_{t_1} c_1 = 0, \quad (4.21)$$

upon elimination of  $b_2, c_2, b_3, c_3$  in the dual picture based on  $(V_3^{(1)}, V_3^{(3)})$ . The Poisson bracket on the fields are

$$\{b_1(t_1), c_1(\tau_1)\} = 4\delta(t_1 - \tau_1), \quad (4.22)$$

as before, and

$$\{b_3(t_3), c_3(\tau_3)\}_R^{t_3} = -2b_1(t_3)c_1(\tau_3)\delta(t_3 - \tau_3), \quad \{b_3(t_3), c_1(\tau_3)\}_R^{t_3} = 4\delta(t_3 - \tau_3), \quad (4.23)$$

$$\{b_1(t_3), c_3(\tau_3)\}_R^{t_3} = 4\delta(t_3 - \tau_3), \quad \{b_2(t_3), c_2(\tau_3)\}_R^{t_3} = 4\delta(t_3 - \tau_3), \quad (4.24)$$

in the dual picture. Note that  $b_3$  and  $c_3$  are not canonical variables but rather obey a quadratic Poisson bracket relation. This is a rather general feature: one cannot expect in general that the variables  $b_j, c_j$  inherited from the coadjoint orbit description of the FNR scheme are canonical variables for the associated Poisson bracket that we constructed. Finding such variables is a difficult task in general. However, in Section 7 of [9], the authors provide a set of variables (their equation (40)) that provide conjugate variables in their setting. It would be interesting to cast this aspect of their results into our classical  $r$ -matrix approach but that will not be done in this paper.

#### 4.2 A new example: Gerdjikov-Ivanov-type hierarchy

We now apply our scheme to  $n = 2$  and  $k = 4$ . This gives rise to equations that generalize those studied in [14] which are related to a wealth of NLS-type equations: derivative NLS equation of the Kaup-Newell hierarchy [15], the Chen-Lee-Liu equation [33], the WKI equation [32]) via gauge transformations [34]. This shows that the Gerdjikov-Ivanov equation can be embedded in a hierarchy whose classical  $r$  matrix structure is of the rational type used all along this paper, a feature that is new to the best of our knowledge. Of course, it is known that some reductions (like the ones we will apply below) and gauge transformations do not preserve the  $r$  matrix structure. This example also illustrates a point that we mention in the next section and that deserves a more careful analysis, going beyond the scope of the present paper: thanks to our dual approach, one can “travel” in a multidimensional lattice of Lax pairs and make connections between hierarchies that are traditionally thought of as distinct. One ends up with multiple multiHamiltonians hierarchies.

The Lax pair corresponding to  $t_2$  and  $t_4$  reads

$$V_2^{(2)} = \begin{pmatrix} \lambda^2 - \frac{1}{2}b_1c_1 & \lambda b_1 + b_2 \\ \lambda c_1 + c_2 & -\lambda^2 + \frac{1}{2}b_1c_1 \end{pmatrix}, \quad (4.25)$$

$$V_2^{(4)} = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & -\alpha(\lambda) \end{pmatrix}, \quad (4.26)$$

with

$$\alpha(\lambda) = \lambda^4 - \frac{1}{2}b_1c_1\lambda^2 - \frac{1}{2}(b_2c_1 + b_1c_2)\lambda + \frac{1}{8}(2b_1\partial_{t_2}c_1 - 2\partial_{t_2}b_1c_1 - 4b_2c_2 - b_1^2c_1^2) \quad (4.27)$$

$$\beta(\lambda) = b_1\lambda^3 + b_2\lambda^2 + \frac{1}{2}\partial_{t_2}b_1\lambda + \frac{1}{2}(\partial_{t_2}b_2 - c_2b_1^2 - b_2c_1b_1), \quad (4.28)$$

$$\gamma(\lambda) = c_1\lambda^3 + c_2\lambda^2 - \frac{1}{2}\partial_{t_2}c_1\lambda - \frac{1}{2}(\partial_{t_2}c_2 + b_2c_1^2 + b_1c_2c_1). \quad (4.29)$$



The corresponding PDEs read

$$\partial_{t_4} b_1 - \frac{1}{2} \partial_{t_2}^2 b_1 + b_2^2 c_1 + 2b_2 c_2 b_1 - \frac{1}{2} b_1^2 \partial_{t_2} c_1 + \frac{1}{4} c_1^3 b_1^2 = 0, \quad (4.30)$$

$$\partial_{t_4} c_1 + \frac{1}{2} \partial_{t_2}^2 c_1 - c_2^2 b_1 - 2c_2 b_2 c_1 - \frac{1}{2} c_1^2 \partial_{t_2} b_1 - \frac{1}{4} b_1^3 c_1^2 = 0, \quad (4.31)$$

$$\partial_{t_4} b_2 - \frac{1}{2} \partial_{t_2}^2 b_2 + b_2^2 c_2 + (b_1 c_2 + b_2 c_1) \partial_{t_2} b_1 + \frac{1}{2} b_1^2 \partial_{t_2} c_2 + \frac{3}{2} b_1^2 c_1^2 b_2 + \frac{1}{2} b_1^3 c_1 c_2 = 0, \quad (4.32)$$

$$\partial_{t_4} c_2 + \frac{1}{2} \partial_{t_2}^2 c_2 - c_2^2 b_2 + (b_1 c_2 + b_2 c_1) \partial_{t_2} c_1 + \frac{1}{2} c_1^2 \partial_{t_2} b_2 - \frac{3}{2} b_1^2 c_1^2 c_2 - \frac{1}{2} c_1^3 b_1 b_2 = 0. \quad (4.33)$$

In the special case  $b_2 = c_2 = 0$ , this reduces to

$$\partial_{t_4} b_1 - \frac{1}{2} \partial_{t_2}^2 b_1 - \frac{1}{2} b_1^2 \partial_{t_2} c_1 + \frac{1}{4} b_1^3 c_1^2 = 0, \quad (4.34)$$

$$\partial_{t_4} c_1 + \frac{1}{2} \partial_{t_2}^2 c_1 - \frac{1}{2} c_1^2 \partial_{t_2} b_1 - \frac{1}{4} c_1^3 b_1^2 = 0. \quad (4.35)$$

The further reduction  $b_1 = q$ ,  $c_1 = i\epsilon q^*$ ,  $\epsilon = \pm 1$  and the choice  $t_2 = x$ ,  $t_4 = 2it$  yields the Gerdjikov-Ivanov equation

$$iq_t + q_{xx} + i\epsilon q^2 q_x^* + \frac{1}{2} |q|^4 q = 0. \quad (4.36)$$

As before it can be checked that the system of PDEs (4.30) is of Hamiltonian type with Hamiltonian

$$H_2^{(4)} = \frac{1}{16} \int_{-M}^M (b_2 \partial_{t_2}^2 c_1 + c_2 \partial_{t_2}^2 b_1 + c_1 \partial_{t_2}^2 b_2 + b_1 \partial_{t_2}^2 c_2) dt_2 \quad (4.37)$$

$$+ \frac{1}{32} \int_{-M}^M (b_1 c_1 (b_2 \partial_{t_2} c_1 - c_2 \partial_{t_2} b_1) + 3b_1^2 c_2 \partial_{t_2} c_1 - 3b_2 c_1^2 \partial_{t_2} b_1) dt_2 \quad (4.38)$$

$$+ \frac{1}{32} \int_{-M}^M (-2b_1^3 c_1^2 c_2 - 2b_1^2 b_2 c_1^3 - 8b_1 b_2 c_2^2 - 8c_1 c_2 b_2^2) dt_2 \quad (4.39)$$

and Poisson brackets between the fields as

$$\{b_1(t_2), c_2(\tau_2)\}_R^{t_2} = 4\delta(t_2 - \tau_2), \quad \{c_1(t_2), b_2(\tau_2)\}_R^{t_2} = -4\delta(t_2 - \tau_2), \quad (4.40)$$

all other Poisson brackets being zero. We see that these are the same Poisson brackets as in the NLS case in the dual formulation. This is of course consistent with the fact that the latter is also based on  $V_2^{(2)}$ . In fact, NLS in the dual formulation is nothing but the first level in the same hierarchy as the present one: one simply picks the  $t_1$  flow with associated Lax matrix  $V_2^{(1)}$  instead of the  $t_4$  flow used here. Roughly speaking, we can “travel” from the standard NLS hierarchy based on  $V_1^{(1)}$  to the present different hierarchy based on  $V_2^{(2)}$  via the dual formulation of NLS which precisely shifts the emphasis from  $V_1^{(1)}$  to  $V_2^{(2)}$ . Our general results show that the  $r$ -matrix structure follows in the process.

The details of the dual formulation of the system (4.30) are not very illuminating in themselves and only confirm the consistency of our general picture. The PDEs (4.30) are recovered from the zero curvature based on the Lax pair  $(V_4^{(2)}, V_4^{(4)})$  and from the Hamiltonian flow (3.98) with respect to  $\{ , \}_R^{t_4}$  using the Hamiltonian  $H_4^{(2)}$ .

## 5 Discussion and conclusion

### 5.1 Outlook

The puzzling observation that the two Lax matrices involved in a given zero curvature condition each satisfy the same ultralocal Poisson algebra (albeit with respect to very different and not compatible Poisson brackets) was the main motivation for the present paper. In the process of understanding this, we have now uncovered the general features of the diagram shown in the introduction. The appearance of the same  $r$ -matrix structure in the Poisson brackets of the two Lax matrices is traced back to the R operator formulation of the FNR scheme. In addition, we made no use of the Lagrangian formalism here and the associated cumbersome covariant Legendre transformations and Dirac procedure, but have only used algebraic tools.

Of immediate attention is now the question of generalizing the results of this paper to other types of classical  $r$ -matrix contexts: trigonometric  $r$ -matrix and non-ultralocal  $r$ -matrix. It is not clear how our approach can be extended to those cases since the specific pole structure of the  $r$ -matrix seems to play a crucial role in several demonstrations. However, at least the trigonometric extensions have in many cases suitable interpretations in terms of coadjoint orbit constructions, which gives us some hope regarding a partial extension to this case. The elliptic case seems much more tricky.

### 5.2 Covariance and connection with results on dressing transformations

Combining the results of the present paper (purely algebraic) with the previous results of [7] (purely field theoretic), we are naturally led to suggest that the classical  $r$ -matrix has a fundamental covariant nature that has not been investigated so far. Since its discovery, it has always been tied to a traditional Hamiltonian approach which singles out one time variable. Our paper represents a first step at elucidating the covariant nature of the classical  $r$  matrix. To our knowledge, the most advanced theory of integrable systems from a covariant field theory point of view has been achieved in Chapter 19 of [28]. However, the classical  $r$ -matrix is totally absent there. In our opinion, these two important topics deserve to be fused into a consistent integrable covariant field theory.

A possible direction of investigation towards understanding this  $r$ -matrix “universality” uncovered in this paper may be suggested by general features of classical integrable systems, namely the structures of the associated dressing transformations. Let us indeed summarize precisely our results and compare them with some known facts on dressing transformations of integrable PDEs.

We have established that for any choice of a continuous variable  $x \equiv t_n$  amongst the hierarchy of time variables in the FNR formalism the linear Poisson structure of the constrained Lax matrix ( $x$ -component of the Lax connection) and therefore the subsequent quadratic Poisson structure of the associated monodromy matrix were described by the  $r$  matrix  $r_{12} = \frac{\Pi_{sl(2,C)}}{\lambda - \mu}$ .

It is in particular true of the two Lax matrices in a pair of dual Lax representations such as were discussed in Section 3.3, yielding two zero-curvature representations of the same nonlinear integrable PDE.

We argue that this universality of the  $r$  matrix structure in a dual pair could have been proposed ab initio from arguments based on the general properties of the group of dressing transformations acting on the set of solutions to the PDE (possibly endowed with a manifold structure if a moduli space structure can be defined).

The relevant general results on dressing transformations are formulated as follows:

Assuming that some integrable PDE is described in a AKNS-ZS framework by a zero-curvature condition of a connection  $\{d/dx - L(\lambda), d/dt - M(\lambda)\}$  depending on one complex spectral parameter  $\lambda$ , it was proved that the dressing group was:

1. Isomorphic as a group to the direct product of groups of germs of Lie algebra-valued analytic functions in  $\lambda$  at each (simple) pole in  $\lambda$  of the Lax pair [29].
2. Endowed with a Lie-Poisson structure directly obtained from the Lie-Poisson structure of the monodromy matrix identified as a generator of the dressing group [17, 35], and consistent with the algebraic group structure.
3. In addition, the pole structure of the classical Lax matrix is determined by the precise structure of the Adler-Kostant-Symes split, defining intrinsically the integrability structure, and yielding directly the  $R$ -operator, hence the  $r$  matrix.

Given that the dressing group acts on the (moduli) space of the solutions to the PDE, it is expected to be only sensitive (as a group) to the Lagrangian aspect and not to any Hamiltonian or zero-curvature representation of the PDE. Hence both its own algebraic structure and associated Lie-Poisson structure are a priori intrinsic. At the same time the algebraic structure and Lie-Poisson structure are technically obtained [29] from the poles of the Lax matrix which themselves determine the poles of the  $r$ -matrix by the duality procedure between  $r$ -matrix and  $R$ -operator, as discussed in Point 3. Finally, the Lie-Poisson structure of the dressing group allows one to turn the action of the dressing group on the space of solutions into a Poisson action [35], implying that the Poisson structure (whatever it is) underlying the integrable PDE must exhibit some compatibility property with respect to the Lie-Poisson structure of the dressing group.

This strongly suggests that whichever choice of Hamiltonian structure and associated Lax representation is picked for a PDE (within some limits to be investigated more precisely, in particular regarding the Lie algebra structure involved in the construction), we should expect the  $r$ -matrix structure to always be consistent with the fundamental algebraic features, described in Points 1 and 2, of the dressing transformation group. Hence it should be independent of this particular choice; as is indeed the case for the “dual” representations.

To summarize this informal discussion, it is not surprising *a posteriori* to observe the same  $r$ -matrix structure for the two Lax matrices involved in the description of a given integrable PDE, if one relates it to the dressing transformation group of that PDE since the latter is not sensitive to the distinction between an  $x$  or  $t$  variable.

### 5.3 Multiple hierarchies

Having established the pattern to get any “space” Lax operator from a chosen time-evolution in a coadjoint orbit formulation, we can pose the problem of connecting the construction of two non-compatible Poisson brackets associated with each term of a Lax pair, to the well-established theory of hierarchies of Poisson structures a la Magri attached to a given Lax matrix endowed with its Poisson structure. Let us describe the issue more precisely.

Start from an initial dual Lax pair compactly denoted  $(L, M)$ . From the monodromy of each of our space Lax operator  $L$  or  $M$ , together with its Poisson structure and  $r$ -matrix, we generate a hierarchy of mutually commuting time evolutions. They have been shown here to be also obtained by applying the relevant constraint-extension map  $\Psi$  to the corresponding operator in the coadjoint orbit framework.

Consider now a pair made of the initial space Lax operator  $L$  and one of its time Lax operator  $M^{(k)}$  in the hierarchy. Given our results the duality property still holds, hence there exists a Poisson structure under which  $M^{(k)}$  becomes a “space” operator and  $L$  a “time” operator. We may then use the monodromy of  $M^{(k)}$  to compute a hierarchy of new time operators now denoted  $L^{(k,s)}$ . Proceeding similarly with the initial “time” operator  $M$  of the dual pair now taken as “space” operator, we can build from its monodromy a hierarchy of new operators  $L^{(s')}$ , any of which may then be taken by duality as “space” operator to generate the operators  $M^{(s',k')}$ . Repeating the process in this way, we should be able to build a multilabel structure of Lax-type operators.

The beginning of this construction was undertaken in the previous paper [7]. A key question is the following: is this multilabel structure a “flat” lattice? In other words, can one identify operators  $L^{(k,s)}$  and  $M^{(s',k')}$  for instance (provided  $s = s', k = k'$  or maybe some weaker condition)? More generally, does the previous process close after a finite number of steps? This already seemed not to be the case in the simplest situations. In fact as we have seen the identification of operators obtained by monodromy construction versus operators obtained by constraint-extensions is valid on-shell as is clear from the formulation of Proposition 3.17. Only at the level of equations can we establish the equivalence. This point thus clearly requires a very careful examination, which is left for future work.

## Acknowledgements

We thank L-C Li for his useful comments on a draft of this paper. V.C. acknowledges the hospitality of the University of Cergy-Pontoise and l’Institut des Etudes Avancées for supporting his visit at the LPTM where most of this research was carried out. V.C. is indebted to A. Fordy for stimulating discussions and for drawing his attention to the theory of stationary manifolds and the references [21, 22]. We are also indebted to M. Semenov-Tian-Shanski for reading the first version of our preprint and pointing out references [18, 19] that helped us put our results into perspective.

## References

- [1] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, *Method for solving the Korteweg-de Vries equation*, Phys. Rev. Lett. **19** (1967), 1095.
- [2] V.E. Zakharov, A.B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Sov. Phys. JETP **34** (1972), 62.
- [3] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, *The inverse scattering transform - Fourier analysis for nonlinear problems*, Stud. Appl. Math. **53** (1974), 249.
- [4] V. Caudrelier, *Multisymplectic approach to integrable defects in the sine-Gordon model*, J. Phys. **A48** (2015), 195203.
- [5] V. Caudrelier, A. Kundu, *A multisymplectic approach to defects in integrable classical field theory*, JHEP **02** (2015), 088.
- [6] V. Caudrelier, *On a systematic approach to defects in classical integrable field theories*, IJGMMP **5** (2008), 1085.
- [7] J. Avan, V. Caudrelier, A. Doikou, A. Kundu, *Lagrangian and Hamiltonian structures in an integrable hierarchy and space-time duality*, Nucl. Phys. **B902** (2016), 415.
- [8] F. Magri, *A simple model of the integrable Hamiltonian equations*, J. Math. Phys. **19** (1978), 1156.
- [9] H. Flaschka, A.C. Newell, T. Ratiu, *Kac-Moody Lie algebras and soliton equations II.*, Physica **9D** (1983), 300.
- [10] S. N. M. Ruijsenaars, *Action-angle maps and scattering theory for some finite-dimensional integrable systems: I. The pure soliton case*, Comm. Math. Phys. **115** (1988), 127.
- [11] L. Feher, C. Klimcik *Poisson-Lie interpretation of trigonometric Ruijsenaars duality*, Comm. Math. Phys. **301** (2011), 55.
- [12] L. Feher, C. Klimcik *Self-duality of the compactified Ruijsenaars-Schneider system from quasi-Hamiltonian reduction*, Nucl. Phys. B **860** (2012), 464.
- [13] M. A. Semenov-Tyan-Shanskii, *What is a classical r-matrix?*, Funct. Anal. Appl. **17** (1983), 259.
- [14] V.S. Gerdjikov, M.I. Ivanov, *The quadratic bundle of general form and the nonlinear evolution equations*, Joint Inst. Nucl. Res. (1982) E2-82-595.
- [15] D.J. Kaup, A.C. Newell, *An exact solution for a derivative nonlinear Schrödinger equation*, J. Math. Phys. **19** (1978), 798.

- [16] L.D. Faddeev, L.A. Takhtadjan, *Hamiltonian Methods in the Theory of Solitons*, Springer-Verlag 1987.
- [17] O. Babelon, D. Bernard, M. Talon, *Introduction to classical integrable systems*, Cambridge University Press, 2003.
- [18] A.G. Reiman, *A unified Hamiltonian system on polynomial bundles, and the structure of stationary problems*, J. Math. Sci. **30** (1985), 2319 (translated from Zap. Nauchn. Sem. POMI **131** (1983), 118).
- [19] P.P. Kulish, A.G. Reiman, *Hamiltonian structure of polynomial bundles*, J. Math. Sci. **28** (1985), 505 (translated from Zap. Nauchn. Sem. LOMI **123** (1983), 67).
- [20] O.I. Bogoyavlenskii, S.P. Novikov, *The relationship between Hamiltonian formalisms of stationary and nonstationary problems*, Func. Anal Appl., **10**, (1976), 8.
- [21] A.P. Fordy, S.D. Harris, *Hamiltonian flows on stationary manifolds*, In A.S. Fokas and I.M. Gelfand, editors, Methods and Applications of Analysis, **4** (1997), 212.
- [22] A.P. Fordy, S.D. Harris, *Hamiltonian structures in stationary manifold co-ordinates*, In A.S. Fokas and I.M. Gelfand, editors, Algebraic Aspects of Integrable Systems, Dedicated to the memory of Irene Dorfman, Birkhauser, **26**, (1997), 103.
- [23] M. Antonowicz, M. Blaszak, *On a non-standard Hamiltonian description of NLEE*. In: Nonlinear Evolution Equations and Dynamical Systems (Eds. S. CARillo and O. Ragnisco), Springer-Verlag, Berlin, 1990, p152.
- [24] M. Adler, *On a trace functional for formal pseudodifferential operators and symplectic structure of the Korteweg de Vries equation*, Invent. Math.**50** (1979), p. 219
- [25] B. Kostant, *The solution to the generalized Toda lattice and representation theory* Adv. Math. **34** (1979), p. 195
- [26] W.W. Symes, *Systems of Toda type, inverse spectral problems and representation theory*, Invent. Math. **59** (1980), p. 13
- [27] M. Semenov-Tian-Shansky, *Integrable Systems: the r-matrix Approach*, RIMS-1650 (2008), Kyoto University.
- [28] L.A. Dickey, *Soliton Equations and Hamiltonian Systems*, World Scientific, 2nd Edition, 2003.
- [29] J. Avan, M. Bellon, *Infinite-dimensional transformation groups for the two-dimensional principal chiral models* Phys . Lett. **B 213** (1988), 459
- [30] A.G. Reyman, M.A. Semenov-Tian-Shansky, *Compatible Poisson structures for Lax equations: a r-matrix approach*, Phys. Lett. A **130** (1988), 456.

- [31] E.K. Sklyanin, *Method of the inverse scattering problem and quantum nonlinear Schrödinger equation*, Dokl. Acad. Nauk SSSR **244**, No. 6 (1978), 1337.
- [32] M. Wadati, K. Konno, Y-H. Ichikawa, *A Generalization of Inverse Scattering Method*, J. Phys. Soc. Jpn. **46** (1979), 1965.
- [33] H.H. Chen, Y.C. Lee, C.S. Liu, *Integrability of Nonlinear Hamiltonian Systems by Inverse Scattering Method*, Phys. Scr. **20** (1979), 490.
- [34] A. Kundu, *LandauLifshitz and higher-order nonlinear systems gauge generated from nonlinear Schrödinger-type equations*, J. Math. Phys. **25** (1984), 3433.
- [35] M. A. Semenov-Tyan-Shanskii, *Dressing transformations and Poisson group actions*, Publ. RIMS (Kyoto) **21** (1985), 1237.