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ABELIAN HURWITZ-HODGE INTEGRALS

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ABSTRACT. Hodge classes on the moduli space of admissible covers with monodromy group G are associated to irreducible representations of G. We evaluate all linear Hodge integrals over moduli spaces of admissible covers with abelian monodromy in terms of multiplication in an associated wreath group algebra. In case G is cyclic and the representation is faithful, the evaluation is in terms of double Hurwitz numbers. In case G is trivial, the formula specializes to the well-known result of Ekedahl-Lando-Shapiro-Vainshtein for linear Hodge integrals over the moduli space of curves in terms of single Hurwitz numbers.

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0. INTRODUCTION

0.1. **Moduli of covers.** Let $\mathcal{M}_{g,n}$ be the moduli space of nonsingular, connected, genus g curves over $\mathbb{C}$ with n distinct points. Let G be a finite group. Given an element $[C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n}$, we will consider principal G-bundles,

$$
\begin{array}{ccc}
G \longrightarrow & & P \\
 & & \downarrow \pi \\
 & & C \setminus \{p_1, \ldots, p_n\}
\end{array} \quad , \qquad (1)
$$

over the punctured curve. Denote the G-action on the fibers of π by

$$\tau : G \times P \to P.$$

The monodromy defined by a positively oriented loop around the i^{th} puncture determines a conjugacy class $\gamma_i \in \text{Conj}(G)$. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be the n-tuple of monodromies. The moduli space

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of covers $\mathcal{A}_{g,\gamma}(G)$ parameterizes G-bundles (1) with the prescribed monodromy conditions. There is a canonical morphism

$$\epsilon : \mathcal{A}_{g,\gamma}(G) \to \mathcal{M}_{g,n}$$

obtained from the base of the G-bundle. Both $\mathcal{A}_{g,\gamma}(G)$ and $\mathcal{M}_{g,n}$ are nonsingular Deligne-Mumford stacks.

A compactification $\mathcal{A}_{g,\gamma}(G) \subset \overline{\mathcal{A}}_{g,\gamma}(G)$ by *admissible covers* was introduced by Harris and Mumford in [15]. An admissible cover

$$[\pi, \tau] \in \overline{\mathcal{A}}_{g,\gamma}(G)$$

is a degree $|G|$ finite map of complete curves

$$\pi : D \to (C, p_1, \ldots, p_n)$$

together with a G-action

$$\tau : G \times D \to D$$

on the fibers of π satisfying the following properties:

(i) D is a possibly disconnected nodal curve,
(ii) $[C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n}$ is a stable curve,
(iii) π maps the nonsingular points to nonsingular points and nodes to nodes,

$$\pi(D^{ns}) \subset C^{ns}, \quad \pi(D^{sing}) \subset C^{sing},$$

(iv) $[\pi, \tau]$ restricts to a principal G-bundle over the punctured nonsingular locus

$$\pi^{open} : D^{open} \to C^{ns} \setminus \{p_1, \ldots, p_n\}$$

with monodromy γ,
(v) distinct branches of a node $\eta \in D^{sing}$ map to distinct branches of $\pi(\eta) \in C^{sing}$ with equal ramification orders over $\pi(\eta)$,
(vi) the monodromies of the G-bundle π^{open} determined by the two branches of C at $\eta \in C^{sing}$ lie in opposite conjugacy classes.

Harris and Mumford originally considered only symmetric group Σ_d monodromy, but the natural setting for the construction is for all finite G.

An admissible cover may be alternatively viewed as a principal G-bundle over the stack quotient[1] $[D/G]$ inducing a stable map to the classifying space

$$f : [D/G] \to \mathcal{B}G. \tag{2}$$

Then, $\overline{\mathcal{A}}_{g,\gamma}(G)$ is simply a moduli space of stable maps [2, 5] [2],

$$\overline{\mathcal{A}}_{g,\gamma}(G) \stackrel{\sim}{=} \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G).$$

The deformation theory of stable maps endows $\overline{\mathcal{A}}_{g,\gamma}(G)$ with a canonical nonsingular Deligne-Mumford stack structure. We take the stable maps perspective here.

There are two flavors of such stable map theories. If the base C is required to be connected as above, we write $\overline{\mathcal{M}}^{\circ}_{g,\gamma}(\mathcal{B}G)$. If disconnected bases C are allowed, we write $\overline{\mathcal{M}}^{\bullet}_{g,\gamma}(\mathcal{B}G)$. In the

[1] $[D/G]$ differs from C only by possible stack structure at the markings p_i and the nodes. In both cases, the order of the isotropy group is the order of the local monodromy in G.

[2] We do not trivialize the marked gerbes on the domain in the definition of $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$.

disconnected case, the genus g may be negative. If the superscript is omitted, the connected case is assumed.

Our results are restricted to abelian groups G. Here, Conj(G) is the set of elements of G. Of course, the cyclic groups $\mathbb{Z}_a$ will play the most important role. In case G is trivial, there is no extra monodromy data, and the moduli space of maps $\overline{\mathcal{M}}_{g,(0,\ldots,0)}(\mathcal{B}\mathbb{Z}_1)$ specializes to $\overline{\mathcal{M}}_{g,n}$.

0.2. **Hodge integrals.** Let R be an irreducible $\mathbb{C}$-representation of G. If G is abelian, R is a character

$$\phi^R : G \to \mathbb{C}^*.$$

By associating to each map $[f] \in \overline{\mathcal{M}}_{g,\gamma}(G)$ presented as (2) above the R-summand of the G-representation $H^0(D, \omega_D)$, we obtain a vector bundle

$$\mathbb{E}^R \to \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \ .$$

The rank of $\mathbb{E}^R$ is locally constant and determined by the orbifold Riemann-Roch formula discussed in Section 1. The *Hodge classes* on $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ are Chern classes of $\mathbb{E}^R$,

$$\lambda_i^R = c_i(\mathbb{E}^R) \in H^{2i}(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G), \mathbb{Q}).$$

The i^{th} cotangent line bundle L_i on the moduli space of curves has fiber

$$L_i|_{(C,p_1,\ldots,p_n)} = T^*_{p_i}(C).$$

Descendent classes on $\overline{\mathcal{M}}_{g,n}$ are defined by

$$\psi_i = c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Descendent classes $\bar{\psi}_i$ on the space of stable maps are defined by pull-back via the morphism

$$\epsilon : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,n}$$

to the moduli space of curves,

$$\bar{\psi}_i = \epsilon^*(\psi_i) \in H^2(\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G), \mathbb{Q}).$$

The *Hodge integrals* over $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ are the top intersection products of the classes $\{\lambda_i^R\}_{R \in Irr(G)}$ and $\{\bar{\psi}_j\}_{1 \le j \le n}$. Linear Hodge integrals are of the form

$$\int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)} \lambda_i^R \cdot \prod_{j=1}^{n} \bar{\psi}_j^{m_j}.$$

The term *Hurwitz-Hodge integral* was used in [3] to emphasize the role of the covering spaces.

0.3. **Hurwitz numbers.** Let g be a genus and let ν and μ be two (unordered) partitions of $d \geq 1$. Let $\ell(\nu)$ and $\ell(\mu)$ denote the lengths of the respective partitions. A Hurwitz cover of $\mathbb{P}^1$ of genus g with ramifications ν and μ over $0, \infty \in \mathbb{P}^1$ is a morphism

$$\pi : C \to \mathbb{P}^1$$

satisfying the following properties:

(i) C is a nonsingular, connected, genus g curve,
(ii) the divisors $\pi^{-1}(0), \pi^{-1}(\infty) \subset C$ have profiles equal to the partitions ν and μ respectively,
(iii) the map π is simply ramified over $\mathbb{C}^* = \mathbb{P}^1 \setminus \{0, \infty\}$.

By condition (ii), the degree of π must be d. Two covers

$$\pi : C \to \mathbb{P}^1, \ \pi' : C' \to \mathbb{P}^1$$

are isomorphic if there exists an isomorphism of curves $\phi : C \to C'$ satisfying $\pi' \circ \phi = \pi$. Each cover π has an naturally associated automorphism group $\mathrm{Aut}(\pi)$.

By the Riemann-Hurwitz formula, the number of simple ramification points of π over $\mathbb{C}^*$ is

$$r_g(\nu, \mu) = 2g - 2 + \ell(\nu) + \ell(\mu).$$

Let $U_r \subset \mathbb{C}^*$ be a fixed set of $r_g(\nu,\mu)$ distinct points. The set of $r_g(\nu,\mu)^{th}$ roots of unity is the standard choice. The *double Hurwitz number* $H_g(\nu,\mu)$ is a weighted count of the distinct Hurwitz covers π of genus g with ramifications ν and μ over $0, \infty \in \mathbb{P}^1$ and simple ramification over U_r. Each such cover is weighted by $1/|\mathrm{Aut}(\pi)|$. The count $H_g(\nu,\mu)$ does not depend upon the location of the points of U_r.

There are two flavors of Hurwitz numbers. The connected case defined above will be denoted $H^\circ_g(\nu,\mu)$. If C is allowed to be disconnected, the Hurwitz count is denoted $H^\bullet_g(\nu,\mu)$. Again, the absence of a superscript indicates the connected theory.

Disconnected Hurwitz numbers are easily expressed as products in the center $\mathcal{Z}\Sigma_d$ of the group algebra of Σ_d,

$$H^\bullet_g(\nu,\mu) = \frac{1}{d!}\left(C_\nu T^{r_g(\nu,\mu)} C_\mu\right)_{[\mathrm{Id}]}. \tag{3}$$

Here, C_ν and C_μ are the sums in the group algebra of all elements of Σ_d with cycle types ν and μ respectively, and T is the sum of all transpositions. The subscript denotes the coefficient of the identity [Id].

Multiplication in $\mathcal{Z}\Sigma_d$ is diagonalized by the representation basis. Hurwitz numbers can be written as sums over characters of Σ_d and conveniently expressed as matrix elements in the infinite wedge representation. The latter formalism naturally connects Hurwitz numbers to integrable systems [20, 21, 24].

0.4. **Formula for $\mathbb{Z}_a$.** The formula for linear Hodge integrals is simplest in case the monodromy group is $\mathbb{Z}_a$ and the representation U is given by

$$\phi^U : \mathbb{Z}_a \to \mathbb{C}^*, \quad \phi^U(1) = e^{\frac{2\pi i}{a}}.$$

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a vector[3] of *nontrivial* elements of $\mathbb{Z}_a$,

$$\gamma_i \in \{1, \ldots, a-1\}.$$

Let μ be a partition of $d \geq 1$ with parts μ_j and length ℓ,

$$\sum_{j=1}^{\ell} \mu_j = d.$$

Let $\gamma - \mu$ denote the vector of elements of $\mathbb{Z}_a$ defined by

$$\gamma - \mu = (\gamma_1, \ldots, \gamma_n, -\mu_1, \ldots, -\mu_\ell).$$

[3] The length n may be taken to be 0 in which case $\gamma = \emptyset$.

While the parts of μ are unordered, an ordering is chosen for $\gamma - \mu$. The vector $\gamma - \mu$ may contain trivial parts. We will consider Hodge integrals over the moduli space $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$.

For nonemptiness, the parity condition

$$d - \sum_{i=1}^{n} \gamma_i = 0 \mod a \tag{4}$$

is required. non-negativity,

$$d - \sum_{i=1}^{n} \gamma_i \geq 0,$$

and boundedness,

$$\forall i \neq j, \ \ \gamma_i + \gamma_j \leq a$$

will also be imposed. If $\gamma = \emptyset$, non-negativity and boundedness are satisfied.

An automorphism of a partition is an element of the permutation group preserving equal parts. Let $|\mathrm{Aut}(\gamma)|$ and $|\mathrm{Aut}(\mu)|$ denote the orders of the automorphism groups.[4] Let γ_+ be the partition of d determined by adjoining $\frac{d-\sum_{i=1}^{n}\gamma_i}{a}$ parts of size a,

$$\gamma_+ = (\gamma_1, \ldots, \gamma_n, a, \ldots, a).$$

A calculation shows

$$r_g(\gamma_+, \mu) = 2g - 2 + n + \ell + \frac{d}{a} - \sum_{i=1}^{n} \frac{\gamma_i}{a}.$$

Let the monodromy group $\mathbb{Z}_a$ and representation ϕ^U be specified as above. Our main result for linear $\mathbb{Z}_a$-Hodge integrals is the following formula.

Theorem 1. *Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be nontrivial monodromies in $\mathbb{Z}_a$ satisfying the parity, non-negativity, and boundedness conditions with respect to the partition μ. Then,*

$$H_g(\gamma_+, \mu) =$$

$$\frac{r_g(\gamma_+,\mu)!}{|\mathrm{Aut}(\gamma)|\,|\mathrm{Aut}(\mu)|} a^{1-g-\sum_{i=1}^{n}\frac{\gamma_i}{a}+\sum_{j=1}^{\ell}\left\langle\frac{\mu_j}{a}\right\rangle} \prod_{j=1}^{\ell} \frac{\mu_j^{\left\lfloor\frac{\mu_j}{a}\right\rfloor}}{\left\lfloor\frac{\mu_j}{a}\right\rfloor!} \int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty}(-a)^i\lambda_i^U}{\prod_{j=1}^{\ell}(1-\mu_j\bar{\psi}_j)}\ .$$

The integer and fractional parts of a rational number are denoted in the above formula by

$$q = \lfloor q \rfloor + \langle q \rangle\,, \ \ q \in \mathbb{Q}.$$

The cotangent lines in the denominator on the far right are associated to the stack points of the stable map domain corresponding to the parts of μ.

Theorem 1 is proven by virtual localization on the moduli space of stable maps to the stack $\mathbb{P}^1[a]$ with $\mathbb{Z}_a$-structure at 0 following the arguments of [9, 12]. The space of stable maps to $\mathbb{P}^1[a]$ is discussed in Section 1, and the proof is given in Section 2. The formula is easily seen to determine *all* linear $\mathbb{Z}_a$-Hodge integrals with respect to U in terms of double Hurwitz numbers. In fact, the set of evaluations with $\gamma = \emptyset$ is sufficient. Conversely, every double Hurwitz number is realized for a sufficiently large.

[4]Here, γ is considered as a partition by forgetting the ordering of the elements.

For the disconnected formula, we assume $\gamma = \emptyset$ and the parity condition $d = 0 \pmod a$.[5] Then, Theorem 1 holds in exactly the same form,

$$H_g^\bullet(\emptyset_+, \mu) = \frac{r_g(\emptyset_+, \mu)!}{|\mathrm{Aut}(\mu)|} a^{1-g+\sum_{j=1}^{\ell} \langle \frac{\mu_j}{a} \rangle} \prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor !} \int_{\overline{\mathcal{M}}^\bullet_{g,-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)} . \tag{5}$$

The ELSV formula [6] for linear Hodge integrals on the moduli space of curves arises from the $a = 1$ specialization of Theorem 1,

$$H_g(\mu) = \frac{(2g-2+d+\ell)!}{|\mathrm{Aut}(\mu)|} \prod_{j=1}^{\ell} \frac{\mu_j^{\mu_j}}{\mu_j!} \int_{\overline{\mathcal{M}}_{g,\ell}} \frac{\sum_{i=0}^{g} (-1)^i \lambda_i}{\prod_{j=1}^{\ell} (1 - \mu_j \psi_j)} .$$

For $a = 1$, we must have $\gamma = \emptyset$.

The conditions γ allow for greater freedom in the $a > 1$ case. For example, the proof of Theorem 1 yields a remarkable vanishing property. The monodromy conditions γ satisfy negativity if

$$d - \sum_{i=1}^{n} \gamma_i < 0$$

and strong negativity if

$$d - n - \frac{d - \sum_{i=1}^{n} \gamma_i}{a} < 0.$$

Strong negativity is easily seen to imply negativity.

Theorem 2. *Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be nontrivial monodromies in $\mathbb{Z}_a$ satisfying the parity condition with respect to the partition μ. In addition, let γ satisfy at least one of the following two conditions:*

(i) negativity and boundedness, or

(ii) strong negativity.

Then, a vanishing results for Hurwitz-Hodge integrals holds:

$$\int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)} = 0.$$

A few examples of Theorems 1 and 2 where alternative approaches to the integrals are available are presented in Section 3.

0.5. **Abelian G.** Since any faithful representation R of $\mathbb{Z}_a$ differs from U by an automorphism of $\mathbb{Z}_a$, Theorem 1 determines linear Hodge integrals with respect to R. Representations of $\mathbb{Z}_a$ with kernels require an additional analysis.

Let G be an abelian group with group law written additively. Consider an irreducible representation R,

$$\phi^R : G \to \mathbb{C}^*,$$

[5] If $\gamma \neq \emptyset$, the non-negativity condition may satisfied globally but be violated on connected components.

with associated exact sequence

$$0 \to K \to G \xrightarrow{\phi^R} \mathrm{Im}(\phi^R) \cong \mathbb{Z}_a \to 0. \tag{6}$$

The homomorphism ϕ^R induces a canonical morphism

$$\rho : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a).$$

The morphism ρ satisfies

$$\rho^*(\lambda_i^U) = \lambda_i^R$$

and has the same degree over each component of $\overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$. Therefore, linear Hodge integrals with respect to R can be calculated by multiplying the formula of Theorem 1 by the degree of ρ.

In Section 4, the solution for arbitrary G and R is cast in a more appealing way. When

$$\phi^R(\gamma) = -\mu \in \mathbb{Z}_a,$$

Hodge integrals of the form

$$\int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)} \frac{\sum_{i=0}^{\infty}(-a)^i \lambda_i^R}{\prod_{j=1}^{\ell}(1-\mu_j\bar{\psi}_j)}$$

are expressed in terms of Hurwitz numbers for K_d, the wreath product of K with the symmetric group Σ_d. Since the infinite wedge formalism for Σ_d extends to a Fock space formalism for the wreath product K_d, there is again a connection to integrable systems [25].

Conjugacy classes in K_d are indexed by Conj(K)-weighted partitions of d,

$$\overline{\mu} = \{(\mu_1,\kappa_1),\ldots,(\mu_{\ell(\mu)},\kappa_{\ell(\mu)})\}.$$

Here, μ is a partition of d with parts μ_j, the weights $\kappa_i \in$ Conj(K) are conjugacy classes in K, and $\overline{\mu}$ is an unordered set of pairs. Let Aut$(\overline{\mu})$ denote the automorphism group of $\overline{\mu}$. Let $C_{\overline{\mu}} \in \mathcal{Z}K_d$ be the element of the group algebra associated to the conjugacy class $\overline{\mu}$. The transposition element $T \in \mathcal{Z}K_d$ is associated to conjugacy class of K_d indexed by

$$\overline{\tau} = \{(2,0),(1,0),\ldots,(1,0)\}$$

where all the Conj(K)-weights are 0.

The wreath product K_d has a forgetful map to Σ_d which sends elements of cycle type $\overline{\mu}$ to elements of type μ. The *K_d-Hurwitz number* $H_{g,K}(\overline{\nu},\overline{\mu})$ counts the degree $d|K|$-fold covers of $\mathbb{P}^1$ with monodromy in K_d given by $\overline{\nu}$ and $\overline{\mu}$ at $0,\infty \in \mathbb{P}^1$ and $\overline{\tau}$ at all the points of

$$U_{r_g(\nu,\mu)} \subset \mathbb{P}^1.$$

Since $K \subset K_d$ is contained in the center, any such cover has a canonical K-action which defines a K-bundle over a punctured Hurwitz cover counted by $H_g(\nu,\mu)$. The connectivity requirement we place on covers counted by $H_{g,K}(\overline{\nu},\overline{\mu})$ is *not* that the $d|K|$-fold cover is connected, but only that the associated Hurwitz d-fold cover is connected. Similarly, g is the genus of the d-fold cover.

The natural extension of formula (3) for disconnected Hurwitz covers for the wreath product K_d is

$$H^{\bullet}_{g,K}(\overline{\nu},\overline{\mu}) = \frac{1}{|K_d|}\left(C_{\overline{\nu}} T^{r_g(\nu,\mu)} C_{\overline{\mu}}\right)_{[\mathrm{Id}]},$$

where the product on the right takes place in the group algebra of K_d.

Select an element $x \in G$ with $\phi^R(x) = 1$. Let $k = ax \in K$. Denote by $-\overline{\mu}$ the $\ell(\mu)$-tuple of elements of G defined by:

$$-\overline{\mu} = (\kappa_1 - \mu_1 x, \kappa_2 - \mu_2 x, \ldots, \kappa_{\ell(\mu)} - \mu_{\ell(\mu)} x).$$

Although the parts of $\overline{\mu}$ are unordered, an ordering is chosen for $-\overline{\mu}$. The parity condition is now

$$\sum_{j=1}^{\ell} \kappa_j - \mu_j x = 0 \in G.$$

Denote by $\emptyset_+(k)$ the conjugacy class given by

$$\emptyset_+(k) = \{\underbrace{(a, -k), \ldots, (a, -k)}_{d/a \text{ times}}\}.$$

Theorem 3. *For weighted-partitions $\overline{\mu}$ satisfying the parity condition,*

$$H_{g,K}(\emptyset_+(k), \overline{\mu}) = \frac{r_g(\emptyset_+, \mu)!}{|\mathrm{Aut}(\overline{\mu})|} a^{1-g+\sum_{j=1}^{\ell} \langle \frac{\mu_j}{a} \rangle} \prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor !} \int_{\overline{\mathcal{M}}_{g,-\overline{\mu}}(\mathcal{B}G)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^R}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)} .$$

Theorem 3 determines all linear Hurwitz-Hodge integrals for G and holds in exactly the same form for the disconnected theories $H^{\bullet}_{g,K}(\emptyset_+(k), \overline{\mu})$ and $\overline{\mathcal{M}}^{\bullet}_{g,-\overline{\mu}}(\mathcal{B}G)$.

0.6. **Future directions.** The ELSV formula has two immediate applications in Gromov-Witten theory. The first is the determination of descendent integrals over $\overline{\mathcal{M}}_{g,n}$ via asymptotics to remove the Hodge classes [18, 21]. The second is the exact evaluation of the vertex integrals in the localization formula for $\mathbb{P}^1$ in [22, 23]. The latter requires the Hodge classes.

Since $\epsilon : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,n}$ is a finite map, a geometric approach to the descendent integrals is not strictly necessary [16]. However, for the calculation of the Gromov-Witten theory of target curves with orbifold structure [17], Theorem 3 is essential. The results may be viewed as a first step for orbifolds along the successful line of exact Hodge integral formulas which have culminated in the topological and equivariant vertices in ordinary Gromov-Witten theory.

Hurwitz-Hodge integrals can be viewed as pairings of tautological classes

$$\epsilon_*(\lambda_i^R) \in H^{2i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

against the descendents ψ_i. Given an action

$$\alpha : G \times \{1, \ldots, k\} \to \{1, \ldots, k\}$$

on a set with k elements, there is a second map to the moduli space of curves. Let

$$\mathcal{C} \to \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G), \;\; \mathcal{D} \to \mathcal{C}$$

be the universal domain curve and the universal G-bundle respectively. A second universal curve

$$\mathcal{D}^{\alpha} = \mathcal{D} \times_G \{1, \ldots, k\} \to \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$$

is obtained by the mixing construction. We obtain

$$\epsilon^{\alpha} : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g^{\alpha},n^{\alpha}},$$

where g^α and n^α are the genus and the number of distinguished sections[6] of the universal curve $\mathcal{D}^\alpha$. Two questions immediately arise:

(i) Do the classes $\epsilon^\alpha_*(\lambda^R_i)$ lie in the tautological ring of $\overline{\mathcal{M}}_{g^\alpha,n^\alpha}$?
(ii) Do the pairings of $\epsilon^\alpha_*(\lambda^R_i)$ against the descendents of $\overline{\mathcal{M}}_{g^\alpha,n^\alpha}$ admit simple evaluations?

The answer to (i) is known [11] to be false for $g = 1$, but may be true for $g = 0$. See [8] for positive results related to (i) for the standard action of the symmetric group Σ_k in the $g = 0$ case.

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1. Stable relative maps

1.1. **Definitions.** For $a \geq 1$, let $\mathbb{P}^1[a]$ be the projective line with a single stack point of order a at 0. Let

$$\langle \zeta_a \rangle \subset \mathbb{C}^*, \quad \zeta_a = e^{\frac{2\pi i}{a}}$$

be the group of a^{th}-roots of unity. Locally at 0, $\mathbb{P}^1[a]$ is the quotient stack $\mathbb{C}/\langle \zeta_a \rangle$. Alternatively, $\mathbb{P}^1[a]$ is the a^{th}-root stack of $\mathbb{P}^1$ along the divisor 0.

Let $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$ be the stack of stable relative maps to $(\mathbb{P}^1[a], \infty)$ where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a vector of nontrivial elements

$$1 \leq \gamma_i \leq a - 1, \ \ \gamma_i \in \mathbb{Z}_a,$$

and μ is a partition of $d \geq 1$ with parts μ_j and length ℓ. The moduli space parametrizes maps

$$[\ f : (C, p_1, \ldots, p_n) \to \mathbb{P}^1[a]\ ] \in \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$$

for which

(i) the domain C is a nodal curve of genus g with stack structure at p_i determined by γ_i,
(ii) relative conditions over $\infty \in \mathbb{P}^1[a]$ are given by the partition μ.

The isotropy group of $p_i \in C$ is the subgroup of $\mathbb{Z}_a$ generated by γ_i. Let a_i denote the order of γ_i. The domain C, called a *twisted curve*, may have additional stack structure at the nodes, see [2].

We recall the Riemann-Roch formula for twisted curves.[7] Let C be a twisted curve whose nonsingular stack points are $p_1, ..., p_n$ with cyclic isotropy groups $I_1, \ldots, I_n$. The group I_i is identified with the a_i^{th}-roots of unity via the action on $T_{p_i}C$,

$$I_i \xrightarrow{\sim} \langle \zeta_{a_i} \rangle \subset \mathbb{C}^*, \ \ \zeta_{a_i} = e^{\frac{2\pi i}{a_i}}.$$

[6]We suppress the ordering issues here.
[7]See Theorem 7.2.1 of [1] for precisely our situation.

Let E be a locally free sheaf over the stack C. Then, I_i acts on the restriction $E|_{p_i}$. Let

$$E|_{p_i} = \bigoplus_{0\le s\le a_i-1} V_s^{\oplus e_s}$$

be the direct sum decomposition, where V_s is the irreducible representation of $\mathbb{Z}_{a_i}$ associated to the character

$$\phi^s : I_i \to \mathbb{C}^*, \quad \phi^s(\zeta_{a_i}) = \zeta_{a_i}^s.$$

The *age* of E at p_i is defined by

$$\text{age}_{p_i}(E) = \sum_{0\le s\le a_i-1} e_s \frac{s}{a_i} \, .$$

The Riemann-Roch formula for twisted curves is given as follows:

$$\chi(C,E) = \text{rk}(E)(1-g) + \text{deg}(E) - \sum_{i=1}^{n} \text{age}_{p_i}(E). \tag{7}$$

The virtual dimension of $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)$ is calculated by the Riemann-Roch formula (7). Let

$$[\ f : (C, p_1, \ldots, p_n) \to \mathbb{P}^1[a]\ ] \in \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu).$$

Certainly, $\deg\left(f^*T_{\mathbb{P}^1[a]}(-\infty)\right) = d/a$. By the quotient presentation of $\mathbb{P}^1[a]$, the character of $f^*T_{0,\mathbb{P}^1[a]}$ at p_i is

$$\zeta_{a_i} \mapsto \zeta_{a_i}^{\frac{\gamma_i a_i}{a}} = \zeta_a^{\gamma_i}.$$

Therefore, $\text{age}_{p_i}\left(f^*T_{\mathbb{P}^1[a]}(-\infty)\right) = \frac{\gamma_i}{a}$ and

$$\begin{aligned}\text{vdim}\, \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu) &= 3g-3+n+\ell+\chi(C, f^*T_{\mathbb{P}^1[a]}(-\infty)) \\ &= 3g-3+n+\ell+1-g+\frac{d}{a}-\sum_{i=1}^{n}\frac{\gamma_i}{a} \\ &= 2g-2+n+\ell+\frac{d}{a}-\sum_{i=1}^{n}\frac{\gamma_i}{a}\ .\end{aligned}$$

To simplify notation, let r denote the above virtual dimension. Since r must be an integer, $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)$ is empty unless the parity condition $d = \sum_{i=1}^n \gamma_i \pmod{a}$ holds.

1.2. **Hurwitz numbers.** We now impose the non-negativity condition,

$$d - \sum_{i=1}^{n} \gamma_i \ge 0.$$

Let $H_{g,a}(\gamma,\mu)$ denote the weighted count of degree d representable maps from nonsingular, connected, genus g twisted curves with stack points of type γ to $\mathbb{P}^1[a]$ with profile μ over ∞ and simple ramification over r fixed points in $\mathbb{P}^1[a] \setminus \{0,\infty\}$.

Lemma 1. *$H_{g,a}(\gamma,\mu)$ is well-defined and equal to* $|\text{Aut}(\gamma)| \cdot H_g(\gamma_+,\mu)$.

Given a stack map $[f : C \to \mathbb{P}^1[a]] \in \mathcal{M}_{g,\gamma}(\mathbb{P}^1[a], \mu)$ satisfying the simple ramification condition over the r points, the associated coarse map

$$f^c : C^c \to \mathbb{P}^1$$

is a usual Hurwitz covering counted by $H_g(\gamma_+, \mu)$. The representability condition implies the point p_i has ramification profile γ_i over 0 for the coarse map. Conversely, we have the following result.

Lemma 2. *Let C^c be a nonsingular curve and let $f^c : C^c \to \mathbb{P}^1$ be a nonconstant map. Then, there is a unique (up to isomorphism) twisted curve $(C, p_1, \ldots, p_m)$ and a representable morphism $f : C \to \mathbb{P}^1[a]$ whose induced map between coarse curves is f^c.*

Proof. Since the natural map $\mathbb{P}^1[a] \to \mathbb{P}^1$ is an isomorphism over $\mathbb{P}^1[a] \setminus [0/\mathbb{Z}_a]$, we may consider the composite

$$C^c \setminus (f^c)^{-1}(0) \xrightarrow{f^c} \mathbb{P}^1 \setminus \{0\} \xrightarrow{\sim} \mathbb{P}^1[a] \setminus \{[0/\mathbb{Z}_a]\} \subset \mathbb{P}^1[a].$$

The Lemma follows by applying Lemma 7.2.6 of [2]. □

To proceed, we need to identify the ramification profile of f^c over 0. Since $\mathbb{P}^1[a]$ is a root stack, we may use classification results on maps to root stacks proven in [4]. According to Theorem 3.3.6 of [4], maps considered in our stack Hurwitz problem are in bijective correspondence with maps $f^c : C^c \to \mathbb{P}^1$ from a coarse curve C^c satisfying

$$(f^c)^*[0] = \sum_{i=1}^{n} \gamma_i[\bar{p}_i] + aD, \tag{8}$$

where $\bar{p}_1, ..., \bar{p}_n \in C^c$ are distinct points and $D \subset C^c$ is a divisor consisting of $\frac{d-\sum_{i=1}^n \gamma_i}{a}$ additional distinct points.

The proof of Lemma 1 is complete. The factor $|\text{Aut}(\gamma)|$ occurs since the stack points of C are labelled while the corresponding ramification points on the Hurwitz covers enumerated by $H_g(\gamma_+, \mu)$ are not. □

1.3. **Branch maps.** There exists a basic branch morphism for stable maps,

$$\text{br} : \overline{\mathcal{M}}_g(\mathbb{P}^1, \mu) \to \text{Sym}^{2g-2+d+\ell}(\mathbb{P}^1),$$

constructed in [9]. By composing with the coarsening map, we obtain

$$\text{br} : \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu) \to \text{Sym}^{2g-2+d+\ell}(\mathbb{P}^1).$$

To proceed, we impose the boundedness condition,

$$\forall i \neq j, \ \ \gamma_i + \gamma_j \leq a.$$

Lemma 3. *If the parity, non-negativity, and boundedness conditions are satisfied,*

$$\text{Im(br)} \subset \left(d - n - \frac{d - \sum_{i=1}^n \gamma_i}{a} \right) [0] + \text{Sym}^r(\mathbb{P}^1) \subset \text{Sym}^{2g-2+d+\ell}(\mathbb{P}^1).$$

Proof. Let $f : C \to \mathbb{P}^1[a]$ be a Hurwitz cover counted by $H_{g,a}(\gamma, \mu)$. The expression

$$E = d - n - \frac{d - \sum_{i=1}^n \gamma_i}{a}$$

is the order of $[0]$ in br$([f])$. The claim of the Lemma is simply that the minimum order of $[0]$ in br(f) is achieved at such Hurwitz covers f.

The proof requires checking all possible degenerations of f over 0. If the stack points $p_1, \ldots, p_n$ do not bubble off the domain, the claim follows easily as in the coarse case. We leave the details to the reader.

A more interesting calculus is encountered if a subset of stack points $p_1, \ldots, p_l$ bubbles off the domain together over $[0/\mathbb{Z}_a] \in \mathbb{P}^1[a]$. We do the analysis for a single bubble. We can assume the bubble is of genus 0 since higher genus increases the branching order. The multi-bubble calculation is identical.

The genus 0 bubble is attached to the rest of the curve in m stack points of type

$$\delta_1, \ldots, \delta_m \in \mathbb{Z}_a, \ \ 1 \leq \delta_j \leq a$$

on the noncollapsed side. The parity condition

$$\sum_{i=1}^{l} \gamma_i - \sum_{j=1}^{m} \delta_j = ka \tag{9}$$

must be satisfied with $k \in \mathbb{Z}$.

The branch contribution over 0 of the bubbled map is at least

$$E' = \sum_{i=l+1}^{n} (\gamma_i - 1) + \sum_{j=1}^{m} (\delta_j - 1) + 2m - 2 + \frac{d - \sum_{i=l+1}^{n} \gamma_i - \sum_{j=1}^{m} \delta_j}{a} (a-1).$$

All the terms on the right are obtained from the ramifications on the noncollapsed side except for the $2m$ from the m nodes of the bubble and the -2 from the bubble itself, see [9]. Rewriting using the parity condition (9), we find

$$E' = E + l + m - 2 - k.$$

By connectedness and bubble stability, we have

$$m \geq 1, \ \ l + m \geq 3.$$

If $k \leq 0$, we conclude $E' > E$. If $k \geq 0$, then $k \leq l - 2$ by the boundedness condition and the positivity of δ_1. Again, $E' > E$. □

By Lemma 3, we may view the branch map with restricted image,

$$\text{br}_0 : \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu) \to \text{Sym}^r(\mathbb{P}^1).$$

The proof of Lemma 3 shows the maps $f : C \to \mathbb{P}^1[a]$ satisfying $[0] \notin \text{br}_0(f)$ have no contraction over 0 and coarse profile exactly γ_+. The usual nonsingularity and Bertini arguments [9] then imply the following result.

Lemma 4. *If the parity, non-negativity, and boundedness conditions are satisfied,*

$$H_{g,a}(\gamma, \mu) = \int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \text{br}_0^*(H^r),$$

where $H \in H^2(\text{Sym}^r(\mathbb{P}^1), \mathbb{Q})$ *is the hyperplane class.*

2. LOCALIZATION

2.1. **Fixed loci.** The standard $\mathbb{C}^*$-action on $\mathbb{P}^1$, defined by $\xi \cdot [z_0, z_1] = [z_0, \xi z_1]$, lifts canonically to $\mathbb{C}^*$-actions on $\mathbb{P}^1[a]$ and $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$. We will evaluate the integral

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathrm{br}_0^*(H^r) \tag{10}$$

by virtual localization for relative maps [10, 13] following [9, 12]. We assume the parity, non-negativity, and boundedness conditions.

The first step is to define a lift of the $\mathbb{C}^*$-action to the integrand. Certainly the $\mathbb{C}^*$-action lifts canonically to $\mathrm{Sym}^r(\mathbb{P}^1)$. A lift of H^r can be defined by choosing the $\mathbb{C}^*$-fixed point $r[0] \in \mathrm{Sym}^r(\mathbb{P}^1)$. The tangent weights at $[0/\mathbb{Z}_a], \infty \in \mathbb{P}^1[a]$ are $\frac{t}{a}$ and $-t$ respectively. The equivariant Euler class of the normal bundle to $r[0]$ in $\mathrm{Sym}^r(\mathbb{P}^1)$ has weight $r!t^r$.

The second step is to identify the $\mathbb{C}^*$-fixed locus $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)^{\mathbb{C}^*} \subset \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$. The components of the $\mathbb{C}^*$-fixed locus lie over the $r+1$ points of $\mathrm{Sym}^r(\mathbb{P}^1)^{\mathbb{C}^*}$. By our lifting of H^r, we need only consider

$$\overline{\mathcal{M}}_0^{\mathbb{C}^*} = \overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)^{\mathbb{C}^*} \cap \mathrm{br}_0^{-1}(r[0]).$$

Because of the strong restriction on the branching, the maps

$$[f : C \to \mathbb{P}^1[a]] \in \overline{\mathcal{M}}_0^{\mathbb{C}^*}$$

have a very simple structure:

(i) $C = C_0 \cup \coprod_{j=1}^{\ell} C_j$,

(ii) $f|_{C_0}$ is a constant map from a genus g curve to $[0/\mathbb{Z}_a] \in \mathbb{P}^1[a]$,

(iii) the coarse map $f^c|_{C_j} : C_j^c \to \mathbb{P}^1$ is a $\mathbb{C}^*$-fixed Galois cover of degree μ_j for $j > 0$,

(iv) C_0 meets C_j at a node q_j.

The stack structure at $q_j \in C_j$ is easily determined using the relationship between stack Hurwitz covers of $\mathbb{P}^1[a]$ and ordinary Hurwitz covers of $\mathbb{P}^1$ discussed in Section 1.2. The stack structure at $q_j \in C_j$ is of type $\mu_j \in \mathbb{Z}_a$. The stack structure at $q_j \in C_0$ where C_j is attached is of the *opposite* type $-\mu_j \in \mathbb{Z}_a$. The map

$$f|_{C_0} : (C, p_1, \ldots, p_n, q_1, \ldots, q_\ell) \to [0/\mathbb{Z}_a]$$

is an element of $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$.

The $\mathbb{C}^*$-fixed locus may be identified with a quotient of a fibered product,

$$\overline{\mathcal{M}}_0^{\mathbb{C}^*} \cong \left(\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \times_{(\bar{I}\mathcal{B}\mathbb{Z}_a^\ell)} P_1 \times ... \times P_\ell\right)_{/\mathrm{Aut}(\mu)},$$

where $\bar{I}\mathcal{B}\mathbb{Z}_a$ is the rigidified inertia stack of $\mathcal{B}\mathbb{Z}_a$ and P_j is the moduli stack of $\mathbb{C}^*$-fixed Galois covers of degree μ_j. By the standard multiplicity obtained from gluing stack $\mathbb{Z}_a$-bundles, the projection

$$\overline{\mathcal{M}}_0^{\mathbb{C}^*} \to \left(\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \times P_1 \times ... \times P_\ell\right)_{/\mathrm{Aut}(\mu)} \tag{11}$$

has degree $\prod_{j=1}^{\ell} \frac{a}{b_j}$ where b_j is the order of $\mu_j \in \mathbb{Z}_a$.

Fortunately, the residue integral over $\overline{\mathcal{M}}_0^{\mathbb{C}^*}$ in the virtual localization formula for (10) is pulled-back via (11). Instead of integrating over $\overline{\mathcal{M}}_0^{\mathbb{C}^*}$, we will integrate over

$$\widetilde{\mathcal{M}}_0^{\mathbb{C}^*} = \overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a) \times P_1 \times ... \times P_\ell$$

and multiply by

$$\frac{1}{|\mathrm{Aut}(\mu)|} \prod_{j=1}^{\ell} \frac{a}{b_j}.$$

2.2. **Virtual normal bundle.** The virtual localization formula for (10) with our choice of equivariant lifts takes the following form:

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathrm{br}_0^*(H^r) = \frac{1}{|\mathrm{Aut}(\mu)|} \prod_{j=1}^{\ell} \frac{a}{b_j} \int_{\widetilde{\mathcal{M}}_0^{\mathbb{C}^*}} \frac{r!\, t^r}{e(\mathrm{Norm}^{vir})}. \tag{12}$$

The equivariant Euler class of the virtual normal bundle is

$$\frac{1}{e(\mathrm{Norm}^{vir})} = \frac{e(H^1(C, f^*T_{\mathbb{P}^1[a]}(-\infty)))}{e(H^0(C, f^*T_{\mathbb{P}^1[a]}(-\infty)))} \frac{1}{\prod_{j=1}^{\ell} e(N_j)}, \tag{13}$$

see [10]. The last product is over the nodes of C, and N_j is the equivariant line bundle associated to the smoothing of q_j. The terms in (13) are computed via the normalization sequence of the domain C. The various contributions over the components $C_0, C_1, \ldots, C_\ell$ are computed separately.

First consider the collapsed component C_0. The space $H^0(C_0, f|_{C_0}^* T_{\mathbb{P}^1[a]}(-\infty))$ is identified with the subspace of $T_{\mathbb{P}^1[a]}(-\infty)|_{[0/\mathbb{Z}_a]}$ consisting of vectors invariant under the action of the image of the monodromy representation $\pi_1^{orb}(C_0) \to \mathbb{Z}_a$. Therefore, H^0 vanishes unless the monodromy representation is trivial, in which case H^0 is 1-dimensional with weight $\frac{t}{a}$.

The trivial monodromy representation $\pi_1^{orb}(C_0) \to \mathbb{Z}_a$ is possible only if

$$\gamma = \emptyset \quad \text{and} \quad \forall j,\ \mu_j = 0 \mod a\,.$$

Even then, the locus with trivial monodromy is just a component[8] of $\overline{\mathcal{M}}_{g,(0,\ldots,0)}(\mathcal{B}\mathbb{Z}_a)$. The trivial monodromy representation locus will play a slightly special role throughout the calculation. But, in the final formula, no different treatment is required.

The space $H^1(C_0, f|_{C_0}^* T_{\mathbb{P}^1[a]}(-\infty))$ yields the vector bundle

$$\mathbb{B} = (\mathbb{E}^U)^\vee$$

over $\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$ whose rank may be calculated by the orbifold Riemann-Roch formula. Over the component of the fixed locus where the monodromy representation $\pi_1^{orb}(C_0) \to \mathbb{Z}_a$ is trivial, the rank of $\mathbb{B}$ is g. Otherwise, the rank is

$$r_{\mathbb{B}} = g - 1 + \sum_{i=1}^{n} \frac{\gamma_i}{a} + \sum_{\mu_j \neq 0 \bmod a} \left(1 - \left\langle \frac{\mu_j}{a} \right\rangle\right). \tag{14}$$

[8] If $g > 0$, there will typically be other components as well.

The $H^1 - H^0$ contribution from the collapsed component to the localization formula is

$$\sum_{i=0}^{r_{\mathbb{B}}} \left(\frac{t}{a}\right)^{r_{\mathbb{B}}-i} c_i(\mathbb{B}) = \sum_{i=0}^{r_{\mathbb{B}}} \left(\frac{t}{a}\right)^{r_{\mathbb{B}}-i} (-1)^i \lambda_i^U. \tag{15}$$

For the component where the monodromy representation is trivial, an additional factor of $\frac{a}{t}$ must be inserted in (15).

Next consider the $H^1 - H^0$ contribution from the $\mathbb{C}^*$-fixed Galois covers. Since

$$\deg(f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty)) = \frac{\mu_j}{a},$$

we have

$$H^k(C_j, f|_{C_j}^* T_{\mathbb{P}^1[a]}(-\infty)) = H^k\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\left(\left\lfloor \frac{\mu_j}{a} \right\rfloor\right)\right).$$

The H^0 weights are

$$\frac{t}{\mu_j}, 2\frac{t}{\mu_j}, ..., \left\lfloor \frac{\mu_j}{a} \right\rfloor \frac{t}{\mu_j},$$

where the weight 0 is omitted.[9] The group H^1 vanishes. The $H^1 - H^0$ contribution is

$$t^{-\lfloor \frac{\mu_j}{a} \rfloor} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor !}.$$

Finally, consider the $H^1 - H^0$ contribution from the nodal point q_j. If $\mu_j \neq 0 \pmod a$, then q_j is a stack point and

$$H^0(q_j, f^* T_{\mathbb{P}^1[a]}(-\infty)|_{q_j}) = 0$$

as there is no invariant section. If $\mu_j = 0 \pmod a$ then $H^0(q_j, f^* T_{\mathbb{P}^1[a]}(-\infty)|_{q_j})$ is 1-dimensional and contributes a factor $\frac{t}{a}$. Certainly, H^1 vanishes here for dimension reasons.

The contribution from smoothing the node q_j is the tensor product of the tangent lines of the two branches incident to q_j,

$$e(N_j) = \frac{1}{b_j}\left(-\bar{\psi}_j + \frac{t}{\mu_j}\right).$$

After putting the component calculations together in (13), we obtain the following expression for for $1/e(\text{Norm}^{vir})$:

$$\left(\sum_{i=0}^{r_{\mathbb{B}}} \left(\frac{t}{a}\right)^{r_{\mathbb{B}}-i} (-1)^i \lambda_i^U\right) \cdot \prod_{j=1}^{\ell} \left(t^{-\lfloor \frac{\mu_j}{a} \rfloor} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor !} \frac{1}{\frac{1}{b_j}\left(-\bar{\psi}_j + \frac{t}{\mu_j}\right)}\right) \cdot \prod_{j=1}^{\ell} \left(\frac{t}{a}\right)^{\delta_{0,\langle \frac{\mu_j}{a} \rangle}}.$$

Regrouping of terms yields

$$\frac{\prod_{j=1}^{\ell} b_j \mu_j}{a^{r_{\mathbb{B}} + \sum_{j=1}^{\ell} \delta_{0,\langle \frac{\mu_j}{a} \rangle}}} \left(\prod_{j=1}^{\ell} \frac{\mu_j^{\lfloor \frac{\mu_j}{a} \rfloor}}{\lfloor \frac{\mu_j}{a} \rfloor !}\right) \left(\sum_{i=0}^{r_{\mathbb{B}}} t^{r_{\mathbb{B}}-i} (-a)^i \lambda_i^U\right) \cdot t^{-\sum_{j=1}^{\ell} \lfloor \frac{\mu_j}{a} \rfloor} \prod_{j=1}^{\ell} \frac{t^{\delta_{0,\langle \frac{\mu_j}{a} \rangle}}}{(t - \mu_j \bar{\psi}_j)}. \tag{16}$$

For the component with trivial monodromy representation, a factor of $\frac{a}{t}$ must be inserted in the formulas for $1/e(\text{Norm}^{vir})$.

[9]The 0 weight is from reparameterization of the domain C_j and is not in the virtual normal bundle.

2.3. **Proof of Theorem 1.** Putting the calculations of Section 2.2 together and passing to the non-equivariant limit, we obtain the following evaluation

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathrm{br}_0^*(H^r) = \frac{r!}{|\mathrm{Aut}(\mu)|} \frac{a^\ell}{a^{r_{\mathbb{B}}+\sum_{j=1}^\ell \delta_{0,\left\langle\frac{\mu_j}{a}\right\rangle}}} \prod_{j=1}^\ell \frac{\mu_j^{\left\lfloor\frac{\mu_j}{a}\right\rfloor}}{\left\lfloor\frac{\mu_i}{a}\right\rfloor!} \int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^\infty (-a)^i \lambda_i^U}{\prod_{j=1}^\ell (1-\mu_j\bar{\psi}_j)}.$$

On the right side, we have included the fundamental class factors

$$\prod_{j=1}^\ell \frac{1}{\mu_j}$$

of the moduli spaces P_j. For the component with trivial monodromy representation, a factor of a must be inserted in the formula.

We can simplify the integral evaluation by using the calculation (14) of $r_{\mathbb{B}}$,

$$\begin{aligned} & r_{\mathbb{B}} + \sum_{i=1}^\ell \delta_{0,\left\langle\frac{\mu_j}{a}\right\rangle} - \ell \\ &= g-1+\sum_{i=1}^n \frac{\gamma_i}{a} + \sum_{\mu_j\neq 0 \bmod a} \left(1-\left\langle\frac{\mu_j}{a}\right\rangle\right) + \left(\sum_{\mu_j=0 \bmod a} 1\right) - \ell \\ &= g-1+\sum_{i=1}^n \frac{\gamma_i}{a} - \sum_{j=1}^\ell \left\langle\frac{\mu_j}{a}\right\rangle. \end{aligned}$$

The above calculation is not valid for the component with trivial monodromy since $r_{\mathbb{B}} = g$ not $g-1$. The discrepancy is exactly fixed by the extra factor a required for the trivial monodromy case. We conclude

$$(17) \quad \int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathrm{br}_0^*(H^r) = \frac{r!}{|\mathrm{Aut}(\mu)|} a^{1-g-\sum_{i=1}^n \frac{\gamma_i}{a}+\sum_{j=1}^\ell \left\langle\frac{\mu_j}{a}\right\rangle} \prod_{j=1}^\ell \frac{\mu_j^{\left\lfloor\frac{\mu_j}{a}\right\rfloor}}{\left\lfloor\frac{\mu_i}{a}\right\rfloor!} \int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^\infty (-a)^i \lambda_i^U}{\prod_{j=1}^\ell (1-\mu_j\bar{\psi}_j)}.$$

holds uniformly. Theorem 1 is then obtained from Lemmas 1 and 4. □

In degenerate cases, unstable integrals may appear on the right side of the formula in Theorem 1. The unstable integrals come in two forms and are defined by the localization contributions:

$$\int_{\overline{\mathcal{M}}_{0,(0)}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i\geq 0}(-a)^i \lambda_i^U}{(1-x\bar{\psi}_1)} = \frac{1}{a}\cdot\frac{1}{x^2},$$

$$\int_{\overline{\mathcal{M}}_{0,(m,-m)}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i\geq 0}(-a)^i \lambda_i^U}{(1-x\bar{\psi}_1)(1-y\bar{\psi}_2)} = \frac{1}{a}\cdot\frac{1}{x+y}.$$

With the above definitions, Theorem 1 holds in all cases.

The disconnected formula (5) follows easily from the connected case by the usual combinatorics of distributing ramification points to the components of Hurwitz covers.

2.4. **Proof of Theorem 2.** Suppose γ satisfies the parity and strong negativity condition with respect to μ. Since

$$\delta = d - n - \frac{d - \sum_{i=1}^{n} \gamma_i}{a} < 0,$$

the virtual dimension r of $\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a], \mu)$ is greater than $2g - 2 + d + \ell$. As a consequence, we immediately obtain the vanishing

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathrm{br}^*(H^r) = 0 \tag{18}$$

since $H^r = 0 \in H^*(\mathrm{Sym}^{2g-2+d+\ell}(\mathbb{P}^1), \mathbb{Q})$.

We may nevertheless calculate (18) by localization with the lift

$$H^r = (2g - 2 + d + \ell)[0] \cdot t^{-\delta}$$

which does *not* vanish equivariantly. The analysis is identical to the calculations of Sections 2.1-2.3. We find the integral (18) is proportional (with nonzero factor) to

$$\int_{\overline{\mathcal{M}}_{g,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{\sum_{i=0}^{\infty} (-a)^i \lambda_i^U}{\prod_{j=1}^{\ell} (1 - \mu_j \bar{\psi}_j)},$$

and therefore conclude the vanishing.

Assume now strong negativity does not hold, but γ satisfies the parity, negativity, and boundedness condition. By the proof of Lemma 3, using the boundedness condition, the maps

$$f : C \to \mathbb{P}^1[a]$$

which satisfy $[0] \notin \mathrm{br}_0(f)$ have no contraction over 0 and coarse profile determined by γ. By the negativity condition, no such maps exists. Hence, $[0]$ is always in $\mathrm{br}_0(f)$. Therefore,

$$\int_{[\overline{\mathcal{M}}_{g,\gamma}(\mathbb{P}^1[a],\mu)]^{vir}} \mathrm{br}_0^*(H^r) = 0$$

and we conclude as above.□

3. Examples

3.1. $\mathbb{Z}_2$ **example.** The Hodge bundle $\mathbb{E}^U$ has a very simple interpretation in the $\mathbb{Z}_2$ case. Let

$$\mathcal{C} \to \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_2), \ \ \mathcal{D} \to \mathcal{C}$$

be the universal domain curve and the universal $\mathbb{Z}_2$-bundle. Let

$$\epsilon : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_2) \to \overline{\mathcal{M}}_g, \ \ \tilde{\epsilon} : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}\mathbb{Z}_2) \to \overline{\mathcal{M}}_{g-1+\frac{n}{2}}$$

be the maps to moduli obtained from $\mathcal{C}$ and $\mathcal{D}$ respectively. The exact sequence

$$0 \to \epsilon^*(\mathbb{E}_g) \to \tilde{\epsilon}^*(\mathbb{E}_{g-1+\frac{n}{2}}) \to \mathbb{E}^U \to 0.$$

exhibits $\mathbb{E}^U$ as the K-theoretic difference of the pulled-back Hodge bundles. If $g = 0$, then the situation[10] is even simpler,

$$\mathbb{E}^U \cong \tilde{\epsilon}^*(\mathbb{E}_{g-1+\frac{n}{2}}). \tag{19}$$

[10]The map ϵ is not well-defined here for stability reasons.

Consider the case of Theorem 1 where $g = 0$, $\gamma = (1, 1)$, and $\mu = (1, 1)$. The statement is

$$H_0((1,1),(1,1)) = \frac{2}{2!2!} 2^1 \int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \frac{1 - 2\lambda_1^U}{(1-\bar{\psi}_1)(1-\bar{\psi}_2)}.$$

The double Hurwitz number on the left is $\frac{1}{2}$. Expansion of the right side yields:

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \frac{1 - 2\lambda_1^U}{(1-\bar{\psi}_1)(1-\bar{\psi}_2)} &= \frac{1}{2} \int_{\overline{\mathcal{M}}_{0,4}} \frac{1}{(1-\psi_1)(1-\psi_2)} - 2 \int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \lambda_1^U \\ &= 1 - 2 \int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \lambda_1^U. \end{aligned}$$

To evaluate the last integral, we note the map

$$\tilde{\epsilon} : \overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2) \to \overline{\mathcal{M}}_{1,1},$$

where the first branch point is selected for the marking on the elliptic curve, is of degree 6. Moreover, λ_1^U is the pull-back of λ_1 under $\tilde{\epsilon}$ by (19). Hence,

$$1 - 2 \int_{\overline{\mathcal{M}}_{0,(1,1,1,1)}(\mathcal{B}\mathbb{Z}_2)} \lambda_1^U = 1 - 2 \cdot 6 \cdot \frac{1}{24} = \frac{1}{2}.$$

3.2. **Vanishing example.** The simplest example of the vanishing of Theorem 2 occurs for $\mathbb{Z}_2$. Let $g = 0$,

$$\gamma = (\underbrace{1, \ldots, 1}_{n})$$

and $\mu = (1)$. By the parity condition, n must be odd. Boundedness holds. For the negativity condition, we require $n \geq 2$. By Theorem 2 (i),

$$\int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \frac{\sum_{i\geq 0}(-2)^i \lambda_i^U}{1 - \bar{\psi}_1}$$

vanishes for all odd $n \geq 3$.

We now use the identification of λ_i^U with the Chern classes of the Hodge bundle $\tilde{\epsilon}^*(\mathbb{E}_{\frac{n-1}{2}})$ whose fiber over

$$f : [D/\mathbb{Z}_2] \to \mathcal{B}\mathbb{Z}_2$$

is simply given by the space of differential forms on the genus $\frac{n-1}{2}$ curve D. The Chern roots of $\tilde{\epsilon}^*(\mathbb{E}_{\frac{n-1}{2}})$ can be identified by the vanishing sequence at a Weierstrass point of D. The Weierstrass point can be chosen to lie above the marking corresponding to the single part of μ. The Chern roots of $\tilde{\epsilon}^*(\mathbb{E}_{\frac{n-1}{2}})$ are then $L, 3L, \ldots, (n-2)L$ where L is the Chern class of the cotangent line of the Weierstrass point. The class L on $\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)$ is $\frac{1}{2}\bar{\psi}_1$. Expanding the Chern roots, we find

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \frac{\sum_{i\geq 0}(-2)^i \lambda_i^U}{1 - \bar{\psi}_1} &= \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \frac{\prod_{i=1}^{\frac{n-1}{2}} (1 - (2i-1)\bar{\psi}_1)}{(1-\bar{\psi}_1)} \\ &= \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_2)} \prod_{i=2}^{\frac{n-1}{2}} (1 - (2i-1)\bar{\psi}_1) \\ &= 0, \end{aligned}$$

where the last integral vanishes for dimension reasons.

3.3. $\mathbb{Z}_\infty$ **example.** An interesting feature of Theorem 1 is the possibility of studying the behavior for large a. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ determine a partition of d,

$$d = \sum_{i=1}^{n} \gamma_i.$$

Let $\mu = (d)$ consist of a single part. For $a > d$, the rank of the Hodge bundle

$$\mathbb{E}^U \to \overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)$$

is 0 by (14). Since the parity, non-negativity, and boundedness conditions hold for $a > d$, we may apply Theorem 1 to conclude

$$\begin{aligned} H_0(\gamma, (d)) &= \frac{(n-1)!}{|\mathrm{Aut}(\gamma)|}\, a \int_{\overline{\mathcal{M}}_{0,\gamma-\mu}(\mathcal{B}\mathbb{Z}_a)} \frac{1}{1-d\bar{\psi}_1} \\ &= \frac{(n-1)!}{|\mathrm{Aut}(\gamma)|}\, d^{n-2}, \end{aligned}$$

which is a well-known formula for genus 0 double Hurwitz numbers.

3.4. **1-point series.** If $\mu = (d)$ consists of a single part, the entire generating series for double Hurwitz numbers has been computed [11] in [14]:

$$\sum_{g\geq 0} t^{2g}(-1)^g H_g(\nu,(d)) = \frac{r!\, d^{r-1}}{|\mathrm{Aut}(\nu)|} \prod_{k\geq 1} \left(\frac{\sin(kt/2)}{kt/2}\right)^{m_k(\nu)-\delta_{k,1}}, \tag{20}$$

where $r = r_g(\nu, (d))$ and $m_k(\nu)$ is the number of times k appears as a part of ν. Single part double Hurwitz numbers are considerably simpler because such covers are automatically connected and the only characters with nonzero evaluation on the d-cycle are exterior powers of the standard $(d-1)$-dimensional representation.

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a vector of nontrivial elements of $\mathbb{Z}_a$ satisfying the boundedness condition. We will consider degrees d for which the parity and non-negativity conditions are satisfied. Then,

$$d - \sum_{i=1}^{n} \gamma_i = ab$$

for an integer $b \geq 0$. Consider the generating series

$$F_\gamma(t,z) = \sum_{g=0}^{\infty} \sum_{l=-\infty}^{g} t^{2g} z^l \int_{\overline{\mathcal{M}}_{g,\gamma-(d)}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_0^{2g-2+\ell(\gamma)+l} \lambda_{g-l}^U$$

where $\bar{\psi}_0$ is the class corresponding to the point with monodromy $-d$.

[11]We write Theorem 3.1 of [14] in terms of $\sin$ instead of $\sinh$ and divide by $|\mathrm{Aut}(\nu)|$ since we do not mark ramifications in our definition of Hurwitz numbers.

The double Hurwitz number formula of Theorem 1 is

$$\begin{aligned} H_g(\gamma_+,(d)) &= \frac{r!}{|\mathrm{Aut}(\gamma)|} a^{1-g-\sum_{i=1}^n \frac{\gamma_i}{a}+\left\langle\frac{d}{a}\right\rangle} \frac{d^{\left\lfloor\frac{d}{a}\right\rfloor}}{\left\lfloor\frac{d}{a}\right\rfloor!} \sum_{l=-\infty}^{g} d^{r-b-1+l}(-a)^{g-l} \int_{\overline{\mathcal{M}}_{g,\gamma-(d)}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_0^{r-b-1+l}\lambda_{g-l}^U \\ &= (-1)^g \frac{ad^{r-1}r!\left(\frac{d}{a}\right)^{\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor}}{|\mathrm{Aut}(\gamma)|\left(b+\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor\right)!} \sum_{l=-\infty}^{g}\left(\frac{-d}{a}\right)^l \int_{\overline{\mathcal{M}}_{g,\gamma-(d)}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_0^{r-b-1+l}\lambda_{g-l}^U \end{aligned}$$

or, equivalently,

$$\sum_{g\geq 0}(-1)^g t^{2g} H_g(\gamma_+,(d)) = \frac{ad^{r-1}r!}{|\mathrm{Aut}(\gamma)|\left(b+\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor\right)!}\left(\frac{d}{a}\right)^{\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor} F_\gamma(t,-d/a)$$

where $r = r_g(\gamma_+,(d))$. After combining with (20), we obtain

$$F_\gamma(t,-d/a) = \frac{1}{a}\frac{\left(b+\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor\right)!}{b!}\left(\frac{a}{d}\right)^{\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor}\prod_{k\geq 1}\left(\frac{\sin(kt/2)}{kt/2}\right)^{m_k(\gamma_+)-\delta_{k,1}}. \tag{21}$$

for $b \geq 0$.

Theorem 4. $F_\gamma(t,z)$ *equals*

$$\frac{1}{a}\frac{\left(-z-\sum\frac{\gamma_i}{a}+\sum\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor\right)!}{\left(-z-\sum\frac{\gamma_i}{a}\right)!}(-z)^{-\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor}\left(\frac{\sin(at/2)}{at/2}\right)^{-z-\sum\frac{\gamma_i}{a}}\prod_{k\geq 1}\left(\frac{\sin(kt/2)}{kt/2}\right)^{m_k(\gamma)-\delta_{k,1}}.$$

Proof. Using the standard polynomial expansion

$$\frac{\left(-z-\sum\frac{\gamma_i}{a}+\sum\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor\right)!}{\left(-z-\sum\frac{\gamma_i}{a}\right)!} = \left(-z-\sum\frac{\gamma_i}{a}+\sum\left\lfloor\frac{\sum\gamma_i}{a}\right\rfloor\right)\cdots\left(-z-\sum\frac{\gamma_i}{a}+1\right),$$

we see the t^{2g} coefficients of both sides of Theorem 4 are Laurent polynomials in z. Equation (21) shows Theorem 4 holds for all evaluations of the form $z=-d/a$ where

$$d-\sum_{i=1}^n \gamma_i = ab$$

and b is a non-negative integer. Since there are infinitely many such evaluations, the coefficient Laurent polynomials in z must be equal for all t^{2g}. □

If we specialize Theorem 4 to the case where $\gamma=\emptyset$, we obtain

$$\frac{1}{a}+\sum_{g>0}\sum_{l=0}^{g} t^{2g}z^l \int_{\overline{\mathcal{M}}_{g,1}(\mathcal{B}\mathbb{Z}_a)} \bar{\psi}_1^{2g-2+l}\lambda_{g-l}^U = \frac{1}{a}\left(\frac{at/2}{\sin(at/2)}\right)^z \frac{t/2}{\sin(t/2)} \tag{22}$$

If $\gamma=\emptyset$ and $a=1$ we recover

$$1+\sum_{g>0}\sum_{l=0}^{g} t^{2g}z^l \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2+l}\lambda_{g-l} = \left(\frac{t/2}{\sin(t/2)}\right)^{z+1} \tag{23}$$

first calculated in [7].

In (22), the term λ_g^U vanishes for dimensional reasons except over the trivial monodromy component, where it agrees with the usual λ_g. Indeed, setting $z = 0$ in (22) yields

$$\frac{1}{a} + \sum_{g>0} t^{2g} \int_{\overline{\mathcal{M}}_{g,1}(\mathcal{B}\mathbb{Z}_a)} \psi_1^{2g-2} \lambda_g^U = \frac{1}{a} \frac{t/2}{\sin(t/2)}$$

which is the expected contribution from (23) with a factor of $1/a$ coming from the automorphisms.

4. Abelian groups

4.1. **Pull-back.** For an abelian group G and irreducible representation R, recall the sequence (6),

$$0 \to K \to G \xrightarrow{\phi^R} \mathrm{Im}(\phi^R) \cong \mathbb{Z}_a \to 0.$$

By construction $R \cong \phi^{R*}(U)$. The homomorphism ϕ^R induces a canonical map

$$\rho : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$$

by sending a principal G-bundle to its quotient by K.

Lemma 5. $\mathbb{E}^R \cong \rho^*(\mathbb{E}^U)$.

Proof. Recall $\mathbb{E} \to \overline{\mathcal{M}}_{g,n}(\mathcal{B}H)$ is the bundle whose fiber over

$$[f] : [D/H] \to \mathcal{B}H \in \overline{\mathcal{M}}_{g,n}(\mathcal{B}H)$$

is $H^0(D, \omega_D)$. The latter can be understood as the space of 1-forms α on the normalization $\tilde{D}$ of D with possible simple poles with opposite residues at the two preimages of each node q_i.

Let $\tilde{\rho}$ be the map between the universal principal G- and $\mathbb{Z}_a$-curves that induces ρ. We obtain

$$d\tilde{\rho} : \rho^*(\mathbb{E}) \to \mathbb{E}$$

by pulling-back differential forms. An easy verification shows $\tilde{\rho}$ is well-defined even at points in the moduli space $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ for which the G-curve is nodal.

The map $d\tilde{\rho}$ is injective on each fiber since the pull-back of a nonzero differential form by a finite surjective map is nonzero. Certainly $d\tilde{\rho}$ carries the subbundle $\rho^*(\mathbb{E}^U)$ to the subbundle $\mathbb{E}^R$. These bundles have the same dimension by the Riemann-Roch formula for twisted curves. Hence, $d\tilde{\rho}$ is an isomorphism. □

The map ρ does not preserve the isotropy groups at the marked points. However, since the classes $\bar{\psi}_i$ are pulled-back from $\overline{\mathcal{M}}_{g,n}$,

$$\rho^*(\bar{\psi}) = \bar{\psi}.$$

By Lemma 5, we concluded the integrand in Theorem 3 is exactly the integrand of Theorem 1 pulled-back via ρ.

4.2. **Degree.** The degree of ρ is determined by the following result.

Lemma 6. *We have*

$$\deg(\rho) = \begin{cases} 0 & \sum_i \gamma_i \neq 0 \\ |K|^{2g-1} & \sum_i \gamma_i = 0 \end{cases} .$$

Proof. Consider a nonsingular curve $[C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n}$. Let

$$\Gamma = \pi_1(C \setminus \{p_1, \ldots, p_n\}) = \left\langle \Gamma_i, A_j, B_j \middle| \prod_{i=1}^{n} \Gamma_i \prod_{j=1}^{g} [A_j, B_j] \right\rangle,$$

where Γ_i is a loop around p_i and the loops A_j, B_j are the standard generators of $\pi_1(C)$.

The elements of $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ lying above $[C, p_1, \ldots p_n]$ are in bijective correspondence with the homomorphisms[12] $\varphi : \Gamma \to G$ with

$$\varphi(\Gamma_i) = \gamma_i. \tag{24}$$

Since G is abelian, $\varphi([A_j, B_j]) = 0$. Hence, the parity condition

$$\sum_{i=1}^{n} \gamma_i = 0 \tag{25}$$

must be satisfied for $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ to be nonempty.

If the parity condition holds, then the images of A_j and B_j are completely unconstrained. There are $|G|^{2g}$ homomorphisms ϕ satisfying (24). Stated in terms of homomorphisms, the map ρ corresponds to composing $\varphi : \Gamma \to G$ with $\phi^R : G \to \mathbb{Z}_a$. Since there are $|K|$ elements of G in the preimage of any element of $\mathbb{Z}_a$, there are $|K|^{2g}$ elements in a generic fiber of ρ. Since G is abelian, a cover in $\mathcal{M}_{g,\gamma}(\mathcal{B}G)$ has automorphism group G. A cover in the image of ρ only has automorphism group $\mathbb{Z}_a$. Thus, the degree of ρ is $|K|^{2g-1}$. □

Although $\overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$ may have several components, Lemma 6 implies the degree of ρ is the same over each component. In the nonabelian case, the situation is much more complicated. For example, let η be the conjugacy class of a 3-cycle in Σ_3, let

$$s : \Sigma_3 \to \mathbb{Z}_2$$

be the sign representation, and let

$$\rho : \overline{\mathcal{M}}_{1,\eta}(\mathcal{B}\Sigma_3) \to \overline{\mathcal{M}}_{1,0}(\mathcal{B}\mathbb{Z}_2)$$

be the map induced by s. The space $\overline{\mathcal{M}}_{1,0}(\mathcal{B}\mathbb{Z}_2)$ consists of two components: one with trivial monodromy, and one with nontrivial monodromy. There are covers in $\overline{\mathcal{M}}_{1,\eta}(\mathcal{B}\Sigma_3)$ lying above the nontrivial monodromy component. If $t_1 \neq t_2 \in \Sigma_3$ are two transpositions, then $[t_1, t_2]$ is a 3-cycle. On the other hand, there are no elements of $\overline{\mathcal{M}}_{1,\eta}(\mathcal{B}\Sigma_3)$ lying above the trivial monodromy component. All the monodromy in such a cover would lie in the abelian group $\mathbb{Z}_3 = \ker(s)$, and there are no such covers with nontrivial monodromy about the one marked point by (25). As the formula in Theorem 1 considers all components of $\overline{\mathcal{M}}_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$ at once, a more nuanced approach would be required to understand Hurwitz-Hodge integrals for nonabelian groups, even for 1-dimensional representations.

[12] Composition in Γ is written multiplicatively while composition in G is additive.

In the disconnected case $\rho : \overline{\mathcal{M}}^\bullet_{g,\gamma}(\mathcal{B}G) \to \overline{\mathcal{M}}^\bullet_{g,\phi^R(\gamma)}(\mathcal{B}\mathbb{Z}_a)$, Lemma 6 has a few minor complications:

(i) The monodromy condition $\sum_i \gamma_i = 0 \in G$ cannot be checked globally, but must be verified separately on each domain component.
(ii) The number of components matters. For disconnected curves with h components, each of which satisfies the monodromy requirements, the degree of ρ is $|K|^{2g-2+h}$.

When ρ is nonzero, the degree $|K|^{2g-2+h}$ is independent of G and the monodromy conditions (25). The only role these conditions play is to determine when the degree is nonzero.

4.3. **Wreath Hurwitz numbers.** The wreath product K_d is defined by

$$K_d = \{(k,\sigma) \mid k = (k_1,\dots,k_d) \in K^d, \sigma \in \Sigma_d\},$$
$$(k,\sigma)(k',\sigma') = (k + \sigma(k'), \sigma\sigma').$$

Conjugacy classes of K_d are determined by their cycle types [19]. Since K is abelian, for each m-cycle $(i_1 i_2 \cdots i_m)$ of σ, the element $k_{i_m} + k_{i_{m-1}} + \cdots + k_{i_1}$ is well-defined. The resulting Conj(K)-wieghted partition of d is the called the *cycle type* of (k,σ). Two elements of K_d are conjugate exactly when they have the same cycle type.

We index the conjugacy classes of K_d by Conj(K)-weighted partitions of d. Let

$$\overline{\nu} = \{(\nu_1,\iota_1),\dots,(\nu_{\ell(\nu)},\iota_{\ell(\mu)})\},$$
$$\overline{\mu} = \{(\mu_1,\kappa_1),\dots,(\mu_{\ell(\mu)},\kappa_{\ell(\mu)})\}$$

be two such partitions. Let ν^* be the partition with parts ν_j with a partial labelling given by ι_j. Then

$$\text{Aut}(\nu^*) = \text{Aut}(\overline{\nu}).$$

The Hurwitz number $H_g(\nu^*,\mu^*)$ counts cover with the additional labelling data,

$$H_g(\nu^*,\mu^*) = \frac{|\text{Aut}(\nu)|}{|\text{Aut}(\nu^*)|}\frac{|\text{Aut}(\mu)|}{|\text{Aut}(\mu^*)|} H_g(\nu,\mu).$$

Lemma 7. *$H_{g,K}(\overline{\nu},\overline{\mu})$ is the count of the covers $\pi : C \to \mathbb{P}^1$ enumerated by $H_g(\nu^*,\mu^*)$ with multiplicity m_π. The multiplicity m_π is the automorphism-weighted count of principal K-bundles on $C \setminus \pi^{-1}(\{0,\infty\})$ with monodromy ι_i at $p_i \in \pi^{-1}(0)$ and κ_j at $q_j \in \pi^{-1}(\infty)$.*

Proof. Let $\pi' : D \to \mathbb{P}^1$ be a cover counted by $H_{g,K}(\overline{\nu},\overline{\mu})$. By definition, π' is a $d|K|$-fold cover of $\mathbb{P}^1$ with monodromies $\overline{\nu},\overline{\mu}$ and $\overline{\tau}$ over $0,\infty$ and the points of U_r respectively.

Each such cover has an associated cover $\pi : C \to \mathbb{P}^1$ counted by $H_g(\nu^*,\mu^*)$. Algebraically, the cover is obtained by the forgetful map from $K_d \to \Sigma_d$. Geometrically, the cover is obtained by taking the quotient of D by the diagonal subgroup $K \subset K_d$. There is a natural map $f : D \to C$. Away from the preimages of $0,\infty$ and U_r, the map f is a principal K-bundle.

Consider the point $p_i \in \pi^{-1}(0)$ corresponding to a cycle ν_i which is labelled with $\iota_i \in K$. A small loop winding once around p_i on C has an image that winds ν_i times around 0. But we know that the monodromy for $\pi' : D \to \mathbb{P}^1$ around 0 is given by $\overline{\nu}$. By the definition of the cycle type, the monodromy of f around p_i is ι_i. An identical argument shows the monodromy at q_i over ∞ is κ_i and the monodromy around all preimages of a point in U_r is zero.

The above process is reversible. We start with a d-fold cover $\pi' : C \to \mathbb{P}^1$ counted by $H_g(\nu^*, \mu^*)$ and a principal K-bundle $f : D \to C$ with monodromy ι_i around p_i and κ_i around q_i. Then, the composition $\pi = \pi' \circ f$ is a cover counted by $H_{g,K}(\overline{\nu}, \overline{\mu})$. □

In other words, if $\rho' : \overline{\mathcal{M}}_{g,\iota\cup\kappa}(\mathcal{B}K) \to \overline{\mathcal{M}}_{g,\ell(\lambda)+\ell(\mu)}$ is the natural map, then

$$H_{g,K}(\overline{\nu}, \overline{\mu}) = \deg(\rho')H_g(\nu^*, \mu^*).$$

4.4. **Proof of Theorem 3.** By Lemma 5, we can compute the integral in Theorem 3 by computing the analogous Hurwitz-Hodge integral (appearing in Theorem 1) over $\overline{\mathcal{M}}_{g,-\mu}(\mathcal{B}\mathbb{Z}_a)$ and multiplying by the degree of

$$\rho : \overline{\mathcal{M}}_{g,-\overline{\mu}}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,-\mu}(\mathcal{B}\mathbb{Z}_a).$$

On the other hand, by Lemma 7, we can calculate $H_{g,K}(\emptyset_+(k), \overline{\mu})$ by computing $H_g(\emptyset_+, \mu)$, multiplying by the degree of

$$\rho' : \overline{\mathcal{M}}_{g,(-k)^{d/a}\cup\kappa}(\mathcal{B}K) \to \overline{\mathcal{M}}_{g,d/a+\ell(\mu)},$$

and correcting for the difference in the sizes of the automorphism groups Aut(μ) and

$$\text{Aut}(\overline{\mu}) = \text{Aut}(\mu^*).$$

Thus, to deduce Theorem 3 from Theorem 1, we need only check that the degrees of ρ and ρ' agree. By Lemma 6, the degrees agree when nonzero. The last step is to check the parity condition (25) is the same for ρ and ρ'. For ρ, the parity condition is

$$0 = \sum_{j=1}^{\ell}(-\overline{\mu})_j = \sum_{j=1}^{\ell}(\kappa_j - \mu_j x) = \sum_{j=1}^{\ell}\kappa_j - dx.$$

For ρ', the parity condition is

$$0 = -\frac{d}{a}k + \sum_{j=1}^{\ell}\kappa_j.$$

Since $ax = k$, the conditions are equivalent. □

As in the faithful case, unstable integrals may appear on the right side of the formula in Theorem 3. These unstable terms are defined in a completely analogous manner, and extend Theorem 3 to all contributions:

$$\int_{\overline{\mathcal{M}}_{0,(0)}(\mathcal{B}G)} \frac{\sum_{i\geq 0}(-a)^i\lambda_i^R}{(1-x\bar{\psi}_1)} = \frac{1}{|G|}\cdot\frac{1}{x^2},$$

$$\int_{\overline{\mathcal{M}}_{0,(m,-m)}(\mathcal{B}G)} \frac{\sum_{i\geq 0}(-a)^i\lambda_i^R}{(1-x\bar{\psi}_1)(1-y\bar{\psi}_2)} = \frac{1}{|G|}\cdot\frac{1}{x+y}.$$

Alternatively, using a theory of stable maps relative to a stack divisor[13] at ∞, Theorem 3 could be proven in a manner closely parallel to the proof of Theorem 1.

[13] We avoid the foundational discussion of this theory.

REFERENCES

[1] D. Abramovich, T. Graber, A. Vistoli, *Gromov-Witten theory for Deligne-Mumford stacks*, math/0603151.
[2] D. Abramovich and A. Vistoli, *Compactifying the space of stable maps*, JAMS **15** (2002), 27–75.
[3] J. Bryan, T. Graber, and R. Pandharipande, *The orbifold quantum cohomology of* $\mathbb{C}^2/\mathbb{Z}_3$ *and Hurwitz-Hodge integrals*, J. Alg. Geom. **17** (2008), 1 – 28.
[4] C. Cadman, *Using stacks to impose tangency conditions on curves*, Amer. J. Math. **129** (2007), 405–427.
[5] W. Chen and Y. Ruan, *Orbifold Gromov-Witten theory* in *Orbifolds in mathematics and physics (Madison, WI, 2001)*, 25–85, Contemp. Math. **310** (2002).
[6] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. 146 (2001), 297-327.
[7] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), 173–199.
[8] C. Faber and R. Pandharipande, *Relative maps and tautological classes*, JEMS **7** (2005), 13–49.
[9] B. Fantechi, R. Pandharipande, *Stable maps and branch divisors*, Compositio Math. **130** (2002), 345–364.
[10] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), 487–518.
[11] T. Graber and R. Pandharipande, *Constructions of nontautological classes on moduli spaces of curves*, Michigan Math J **51** (2003), 93–109.
[12] T. Graber and R. Vakil, *Hodge integrals and Hurwitz numbers via virtual localization*, Compositio Math. **135** (2003), 25–36.
[13] T. Graber and R. Vakil, *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, Duke Math. J. **130** (2005), 1–37.
[14] I.P. Goulden, D.M. Jackson, and R. Vakil, *Towards the geometry of double Hurwitz numbers*, Advances in Mathematics **198** (2005), 43–92.
[15] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), 23–88.
[16] T. Jarvis and T. Kimura, *Orbifold quantum cohomology of the classifying space of a group* in *Orbifolds in mathematics and physics (Madison, WI, 2001)*, 123–134, Contemp. Math. **310** (2002).
[17] P. Johnson, in preparation.
[18] M. Kazarian and S. Lando, *An algebro-geometric proof of Witten's conjecture*, JAMS **20** (2007), 1079 – 1089.
[19] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford University Press, 1995.
[20] A. Okounkov, *Toda equations for Hurwitz numbers*, Math. Res. Letters **7** (2000), 447-453.
[21] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and matrix models*, math/0102017.
[22] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and completed cycles*, Ann. of Math **163** (2006), 517 – 560.
[23] A. Okounkov and R. Pandharipande, *The equivariant Gromov-Witten theory of* $\mathbf{P}^1$, Ann. of Math **163** (2006), 561 – 605.
[24] R. Pandharipande, *The Toda equation and the Gromov-Witten theory of the Riemann sphere*, Lett. Math. Phys. **53** (2000), 59 – 74.
[25] Z. Qin and W. Wang, *Hilbert schemes of points on the minimal resolution and soliton equation* in *Lie algebras, vertex operator algebras and their applications*, Y.-Z. Huang and K. Misra (eds.), 435–462, Contemp. Math. **442** (2007).

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