

# Statistical moments for rough surface scatter from two-way parabolic integral equation at low grazing angles

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## ABSTRACT

The moments of a plane wave scattered at low grazing angles from a one-dimensional perfectly reflecting rough surface are considered. The mean intensity and autocorrelation of the scattered field and the corresponding angular spectrum are obtained to second order in surface height. The derivations are based on an operator expansion of the extended (two-way) parabolic integral equation solution. The resulting operator series describes successively higher-order surface interactions between forward and backward going components. The expressions derived may be regarded as backscatter corrections to those obtained via the standard (one-way) parabolic integral equation method.

## KEYWORDS

Rough surface scattering; two-way parabolic equation; second moment; angular spectrum

## 1. Introduction

Analytical treatment of wave scattering statistics from rough surfaces remains a difficult challenge (1–7), especially for incidence at low grazing angles (8–12), where multiple scattering is inherent even for slight roughness. Away from grazing angles methods such as small slope approximation, Kirchhoff approximation, perturbation theory, and the smoothing method are powerful and well-established (3, 5, 9, 13). At near-grazing incidence, however, single scattering and purely local approximations break down.

Further progress can be made under the assumption that forward scattering predominates. The full free space Green's function can then be approximated by a one-way parabolic equation Green's function (14), and the usual Helmholtz integral equations replaced by the (one-way) parabolic equations (PIE). In 2D problems this PE Green's function takes closed form (by contrast to the full Green's function - the reverse is true in 3D), and this approach has proved highly adaptable in numerical simulations (14–17). For small surface heights the amenable form of the Green's function facilitates derivation of analytical solutions (18–20) for the mean field and angular spectrum. Such solutions are valid in the perturbation regime but include some multiple forward-scattered surface interactions.

The major drawback is that backward-going interactions are precluded. In order to overcome this, the one-way PIE method was extended to include backscatter by use of a two-way form of the Green's function (21). This is obtained in a straightforward way by replacing the one-way parabolic Green's function by a form symmetrical in range to within a phase factor. This system was then treated by applying left-right operator splitting, and truncating the resulting series. Operator series solutions such as this and the method of ordered multiple interactions (22–28) have proved versatile in both two and three dimensions, These approaches typically proceed by expanding the surface fields about the dominant 'forward-going' component.

In order to obtain theoretical solutions we impose the additional restriction to the perturbation regime of small surface height  $\sigma$ , within which analytical expressions for the mean field and autocorrelation function are obtained. This extends the corresponding results(18, 19) derived under the standard PIE. The approach there was first to obtain the scattered field to second order in  $\sigma$  at the mean surface plane, and find the far-field under the assumption that propagation outwards from the surface is governed by the full Helmholtz equation. That allowed for 'direct' backscatter but precluded any component due to coherent addition of reversible paths(30–32), because interactions at the surface were allowed only to take place in the forward direction. The two-way formulation presented here removes this restriction (and in principle allows separation of the forward and backward going interactions to various orders, although this aspect is not explored in detail here). In particular this method produces a correction term, whose statistics can be obtained in the perturbation regime. All results presented are for Dirichlet boundary condition.

The paper is organised as follows: The parabolic integral equation method and its recently-presented two-way extension are summarised in §2, and the series solution is explained. The main results are in §3 where analytical expressions for the statistics under the extended method are derived, including mean field and angular spectrum.

## 2. Extended parabolic integral equation method

The standard parabolic integral equation method and its two-way extension have been described elsewhere (14–16), and (21). We summarize these here, focusing mainly on the extended form and its solution by means of operator series.

Consider a scalar time-harmonic wave field  $p$  scattered from a one-dimensional rough surface  $h(x)$  with Dirichlet boundary condition. (Equivalently,  $p$  is a  $TE$  polarised electromagnetic wave and  $h$  is a perfectly conducting corrugated surface whose generator is in the plane of incidence.) The field has wavenumber  $k$  and is governed by the wave equation  $(\nabla^2 + k^2)p = 0$ . The coordinate axes are  $x$  and  $z$  where  $x$  is the horizontal (directed to the right) and  $z$  is the vertical (directed out of the medium). Angles of incidence and scatter are assumed to be small with respect to the  $x$ -direction. It will be assumed that the surface is statistically stationary to second order, i.e. its mean and autocorrelation function are translationally invariant. We choose coordinates so that  $h(x)$  has mean zero. The autocorrelation function  $\langle h(x)h(x+\xi) \rangle$  is denoted by  $\rho(\xi)$ , and we assume that  $\rho(\xi) \rightarrow 0$  at large separations  $\xi$ . (The angled brackets here denote the ensemble average.) Then  $\sigma^2 \equiv \rho(0)$  is the variance of surface height, so that the surface roughness is of order  $O(\sigma)$ . We define the reduced wave to be the slowly-varying component  $\psi(x, z) = p(x, z) \exp(-ikx)$ . Reduced forms of the incident and scattered components  $\psi_i$  and  $\psi_s$  are defined similarly, so that  $\psi = \psi_i + \psi_s$ .

Under the assumption of small angles of scatter Thorsos (14) replaced the full free space Green's function by a one-way (right-going) parabolic form  $G_p$ , allowing the usual Helmholtz or Stratton-CHu integral equations for rough surface scatter to be replaced by the parabolic integral equation method (14, 15).

We can modify these equations to account for backscattering, retaining the assumption of small angles with respect to the mean surface plane. To do this, we replace  $G_p$  by its symmetrical analogue  $G$  obtained again by applying the small angle approximation to the full free space Green's function. We thus obtain

$$G(x, z; x', z') \begin{cases} = \alpha \sqrt{\frac{1}{x-x'}} \exp\left[\frac{ik(z-z')^2}{2(x-x')}\right], & x' < x \\ = \alpha \sqrt{\frac{1}{x'-x}} \exp\left[\frac{ik(z-z')^2}{2(x'-x)}\right] \exp[2ik(x'-x)] & x' \geq x \end{cases} \quad (1)$$

The factor  $\exp[-2ik(x'-x)]$  arises for  $x' \geq x$  because we are solving for the reduced wave  $\psi$ .

Applying this Green's function to the reduced wave  $\psi$  we obtain

$$\psi_s(x, z) = \int_0^\infty G(\mathbf{r}, \mathbf{r}') \frac{\partial \psi(\mathbf{r}')}{\partial z} dx' \quad (2)$$

where  $\mathbf{r} = (x, z)$ ,  $\mathbf{r}' = (x', h(x'))$ . This represents the modified or extended analogue of the standard PIE effectively containing a backscatter correction. The key difference is that in PIE the upper limit of integration is  $x$ , so that only values to the left contribute to the integral in that case. Taking the limit of (7) as  $z \rightarrow h(x)$  yields an integral equation relating the incident field to the scattered field at the surface:

$$\psi_i(x, h(x)) = - \int_0^\infty G(\mathbf{r}_s, \mathbf{r}') \frac{\partial \psi(\mathbf{r}')}{\partial z} dx' \quad (3)$$

where now  $\mathbf{r}_s = (x, h(x))$ ,  $\mathbf{r}' = (x', h(x'))$  both lie on the surface. (Note that the addition of a correction to the parabolic equation is along the lines proposed by Thorsos(14).) Equations (7), (8) can be written in operator notation:

$$\psi_s(x, z) = -(L + R) \frac{\partial \psi}{\partial z} \quad (4)$$

$$\psi_i(x, h(x)) = (L + R) \frac{\partial \psi}{\partial z} \quad (5)$$

where  $L, R$  are defined by

$$Lf(x, z) = \int_0^x G(\mathbf{r}, \mathbf{r}') f(x') dx', \quad Rf(x, z) = \int_x^\infty G(\mathbf{r}, \mathbf{r}') f(x') dx'$$

and  $\mathbf{r} = (x, z)$ ,  $\mathbf{r}' = (x', h(x'))$ . These integral operators and their inverses are Volterra, or 'one-sided' in an obvious sense.

## 2.1. Operator series solution

Integral equation (5) has formal solution

$$\frac{\partial\psi}{\partial z} = (L + R)^{-1}\psi_i \quad (6)$$

which formally can be expanded in a series

$$\frac{\partial\psi}{\partial z} = \left[ L^{-1} - L^{-1}RL^{-1} + (L^{-1}R)^2 L^{-1} - \dots \right] \psi_i \quad (7)$$

Under the assumption that  $R$  is small (in a sense already required implicitly for the standard PE) the series (7) is convergent; the series can then be truncated after finitely many terms. By ‘small’ we mean that  $R\phi/|\phi|$  is small for all terms  $\phi$  in the series. It can be shown heuristically that this assumption is indeed justified at low grazing angles, since the kernel of  $R$  oscillates rapidly especially at small wavelengths. It is nevertheless difficult to give this a precise range of validity, and we will not attempt to do so here.

Solution for the field can be obtained by truncating the series (7) and substituting into the integral (2). The first term  $L^{-1}\psi_i$  in series (7) corresponds to the solution for  $\partial\psi/\partial z$  under the standard PIE method (e.g. (15)). Denote this first approximation by  $\tilde{\psi}$ , i.e.

$$\frac{\partial\psi}{\partial z} \cong \tilde{\psi} = L^{-1}\psi_i. \quad (8)$$

Note however that the integral (2) using the two-way Green’s function allows for outgoing components scattered to the left, unlike its PIE analogue, so even this lowest order truncation gives backscatter. This can be considered the *direct backscatter* component.

Truncation of (7) at the second term gives:

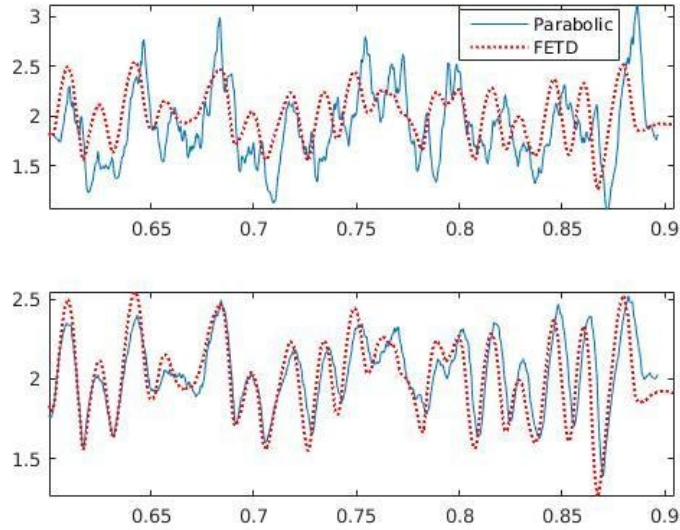
$$\frac{\partial\psi}{\partial z} \cong \tilde{\psi} + C \quad (9)$$

where  $C$  is a correction term,

$$C = L^{-1}RL^{-1}\psi_i. \quad (10)$$

Note that this is the lowest-order truncation consistent with reversible ray paths. The above expression will be used below to obtain some statistical measure of the backscattered component in the perturbation regime of small surface height.

As an illustration of the method, Figure 1 shows comparisons between the first and second iterations of the operator series with ‘exact’ computations from (24), restricted to the rough portion of the surface. The computations were carried out using a Finite Element Time Domain solution. These examples are due to scattering from a fairly jagged surface having a power-law autocorrelation, which for the purposes of the calculation was embedded as a section of an longer flat surface.



**Figure 1.** Comparison between parabolic integral equation (—) and finite element time domain calculations (...). The upper graph shows the first iteration and the lower graph shows the second.

### 3. Perturbation solution and statistics of backscatter

#### 3.1. Perturbation solution

The mean field and higher moments based on the standard parabolic equation approximation were obtained elsewhere(18, 19) to second order in surface height for pure forward scattering. In this section the statistics of the backscatter correction (eq. (9)) due to the two-way PIE method will be derived.

Suppose that a reduced plane wave  $\psi_i^\theta = \exp(ik[xS + z \cos \theta])$  is incident on the rough surface at an angle  $\theta$  measured from the normal, where  $S = \sin \theta - 1$ . We first summarize the perturbational calculation used to obtain the scattered field statistics previously. Suppose that a plane  $z = z_1$ , say, can be chosen ‘close’ to every point on the surface. The scattered field is obtained to second order in surface height along this plane, for a given incident plane wave, and the statistics are found from this. For convenience we may set  $z_1 = 0$ . An expression is thus found for the scattered field

$$\psi_s(x, 0) = -\psi_i^\theta(x, h) - h \left[ \frac{\partial \psi}{\partial z} - \frac{\partial \psi_i}{\partial z} \right] - \frac{1}{2} h^2 \frac{\partial^2 \psi_i(x, 0)}{\partial z^2} + O(\sigma^3). \quad (11)$$

The only term here which is not known *a priori* is  $\partial \psi / \partial z$ . The standard PIE solution  $\tilde{\psi}$  for  $\partial \psi / \partial z$  is given (18, 19) to second order in  $\sigma$  by:

$$\frac{\partial \psi}{\partial z} \cong \tilde{\psi} = -\frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{\psi_i^\theta(x', h(x'))}{\alpha \sqrt{x - x'}} dx'. \quad (12)$$

This arises from (8) by substitution of the flat surface form of  $L$  (see (15) below).

Denote by  $\tilde{\psi}_s$  the approximation to  $\psi_s$  obtained by substituting (12) in (11), so that

$$\begin{aligned} \tilde{\psi}_s(x, 0) \equiv & -\psi_i^\theta(x, h) + h \left[ \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{\psi_i^\theta(x', h(x'))}{\alpha\sqrt{x-x'}} dx' + \frac{\partial\psi_i^\theta(x, h)}{\partial z} \right] \\ & - \frac{h^2}{2} \frac{\partial^2\psi_i^\theta(x, 0)}{\partial z^2}. \end{aligned} \quad (13)$$

We now wish to calculate the backscatter correction to this expression due to the replacement of  $\partial\psi/\partial z$  in (11) by the corrected two-way PE solution ( $\tilde{\psi}+C$ ) (equations (9), (10)). We therefore repeat the above derivation replacing (8) by (9), to obtain

$$\psi_s(x, 0) = \tilde{\psi}_s(x, 0) + h(x)C(x). \quad (14)$$

Since the correction term  $C$  appears here with a factor  $h$ , it is necessary to evaluate it only to order  $O(\sigma)$ .

Expanding  $L$  and  $R$  (eqs. (4)-(5)) in surface height  $h(x)$ , it is seen that  $L = L_0 + O(\sigma^2)$ ,  $R = R_0 + O(\sigma^2)$ , where  $L_0, R_0$  denote the deterministic (i.e. flat surface) forms of the operators  $L$  and  $R$  respectively:

$$L_0(f) = \alpha \int_0^x \frac{f(x')}{\sqrt{x-x'}} dx', \quad R_0(f) = \alpha \int_x^\infty \frac{f(x')}{\sqrt{x'-x}} dx' \quad (15)$$

In evaluating  $C$  (eq. (10)) to order  $O(\sigma)$  we may thus ignore fluctuating parts of the operators, and replace  $L, R$  by  $L_0, R_0$  respectively. We can therefore write

$$C = L_0^{-1}R_0\tilde{\psi} + O(\sigma^2). \quad (16)$$

An expression of the form  $f = L_0^{-1}g$  is Abel's integral equation, which has the well-known solution(29)

$$g(x) = \frac{1}{\alpha\pi} \frac{d}{dx} \int_0^x \frac{1}{\sqrt{x-y}} f(y) dy.$$

Now to first order in  $h$ ,  $\tilde{\psi}$  in (21) is given(18) by

$$\tilde{\psi}(r) \cong -\pi \left[ D_\theta(r) + \frac{dI(r)}{dr} \right]. \quad (17)$$

where, for large  $r$ ,  $D$  takes the form (see eq. (10) of (18))

$$D_\theta(r) \sim -2ik\pi\sqrt{2-2\sin\theta}e^{ikSr} \quad (18)$$

and  $I$  is an integral

$$I(r) = \int_0^r ikh(r') \cos\theta \frac{e^{ikSr'}}{\alpha\sqrt{r-r'}} dr'. \quad (19)$$

Therefore  $D$  and  $dI/dr$  are  $O(1)$  and  $O(h)$  respectively, so that in eq. (16)  $C$  becomes

$$C(x) = \frac{1}{\alpha^2 \pi} \frac{d}{dx} \left[ \int_0^x \frac{1}{\sqrt{x-y}} \int_y^\infty \frac{\exp(ikr)}{\sqrt{y-r}} \tilde{\psi}(r) dr dy \right]. \quad (20)$$

To second order in surface height the scattered field  $\psi_s(x, 0)$  at the mean surface is therefore described by eq. (14), with  $C$  given by (20).

### 3.2. Mean field

The effect of the correction term  $C$  on the scattered field statistics can now be examined. We first find the mean field  $\langle \psi_s(x, z) \rangle$ . It is sufficient to obtain this quantity on the mean surface plane  $z = 0$ , using equation (14), i.e.

$$\langle \psi_s(x, 0) \rangle = \langle \tilde{\psi}_s(x) \rangle + \langle h(x)C(x) \rangle.$$

The solution for  $\langle \tilde{\psi}_s \rangle$  has been obtained previously (18), and we can restrict attention to finding the correction  $\langle hC \rangle$  to this. Denote the correlation  $\langle h(X)C(x) \rangle$  by  $\mathcal{E}$  for any  $X, x$ , i.e.

$$\mathcal{E}(X, x) = \langle h(X)C(x) \rangle.$$

Consider first the function  $\langle h\tilde{\psi} \rangle$ . Since  $\langle hD_\theta \rangle$  vanishes, eq. (17) gives

$$\langle h\tilde{\psi} \rangle = -\pi \langle h \frac{\partial I}{\partial x} \rangle. \quad (21)$$

Now from eq. (20)

$$\mathcal{E}(X, x) = \left\langle \frac{h(X)}{\alpha^2 \pi} \frac{d}{dx} \int_0^x \frac{1}{\sqrt{x-y}} \int_y^\infty \frac{\exp(ikr)}{\sqrt{y-r}} \tilde{\psi}(r) dr dy \right\rangle.$$

The term  $h(X)$  can be taken under the integral signs as part of the operand of  $d/dx$ . The order of integration and averaging can then be reversed so that, by (26),

$$\mathcal{E}(X, x) = -\frac{1}{\alpha^2} \left[ \frac{d}{dx} \int_0^x \frac{1}{\sqrt{x-y}} \int_y^\infty \frac{e^{ikr}}{\sqrt{y-r}} \left\langle h(X) \frac{\partial I(r)}{\partial r} \right\rangle dr dy \right]. \quad (22)$$

Consider the term  $\langle h(X)dI/dr \rangle$  in the inner integrand. By (19),

$$\begin{aligned} \left\langle h(X) \frac{\partial I(r)}{\partial r} \right\rangle &= \left\langle h(X) \frac{d}{dr} \int_0^r ikh(r') \cos \theta \frac{e^{ikSr'}}{\alpha \sqrt{r-r'}} dr' \right\rangle \\ &= ik \cos \theta \frac{d}{dr} \left[ \int_0^r \frac{e^{ikSr'}}{\alpha \sqrt{r-r'}} \rho(X-r') dr' \right] \end{aligned} \quad (23)$$

This may be substituted into (22) to give an analytical expression for the correlation  $\langle h(X)C(x) \rangle$ . We can simplify this expression by evaluating the derivatives explicitly.

The term  $\rho(X - r')$  is independent of  $r$ , so writing

$$\frac{e^{ikSr'}}{\alpha\sqrt{r-r'}} = f(r, r') \quad (24)$$

the expression (23) becomes

$$\begin{aligned} \left\langle h(X) \frac{\partial I(r)}{\partial r} \right\rangle &= ik \cos \theta \frac{d}{dr} \left[ \int_0^r f(r, r') \rho(X - r') dr' \right] \\ &= ik \cos \theta \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[ \int_0^{r+\epsilon} f(r + \epsilon, r') \rho(X - r') dr' - \int_0^r f(r, r') \rho(X - r') dr' \right] \\ &= ik \cos \theta \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [K_1 + K_2 - K_3] \end{aligned} \quad (25)$$

where

$$\begin{aligned} K_1 &= \int_0^\epsilon f(r + \epsilon, r') \rho(X - r') dr' \\ K_2 &= \int_\epsilon^{r+\epsilon} f(r + \epsilon, r') \rho(X - r') dr' \\ K_3 &= \int_0^r f(r, r') \rho(X - r') dr' \end{aligned} \quad (26)$$

Examining these three integrals in detail, the first gives

$$\begin{aligned} \frac{1}{\epsilon} K_1 &= \frac{1}{\epsilon} \int_0^\epsilon f(r + \epsilon, r') \rho(X - r') dr' \cong \frac{1}{\epsilon} \rho(X) \int_0^\epsilon \frac{1}{\sqrt{r + \epsilon - r'}} dr' \\ &= \frac{2}{\alpha\epsilon} \rho(X) [\sqrt{r + \epsilon} - \sqrt{r}] \\ &\cong \frac{\rho(X)}{\alpha\sqrt{r}} \end{aligned} \quad (27)$$

using a Taylor expansion in  $\epsilon$ . Changing variables,  $K_2$  in can be written

$$\int_\epsilon^{r+\epsilon} f(r + \epsilon, r') \rho(X - r') dr' = \int_0^r f(r + \epsilon, r'' + \epsilon) \rho(X - r'' - \epsilon) dr''. \quad (28)$$

Now

$$f(r + \epsilon, r'' + \epsilon) = \frac{e^{ikS(r'' + \epsilon)}}{\alpha\sqrt{r - r''}} = e^{ikS\epsilon} f(r, r'')$$

so from (28)

$$\int_\epsilon^{r+\epsilon} f(r + \epsilon, r') \rho(X - r') dr' = \int_0^r e^{ikS\epsilon} f(r, r') \rho(X - r' - \epsilon) dr'. \quad (29)$$



Thus the difference  $K_2 - K_3$  in (25) becomes

$$\begin{aligned} & \int_0^r f(r, r') \left[ e^{ikS\epsilon} \rho(X - r' - \epsilon) - \rho(X - r') \right] dr' \\ & \cong \int_0^r f(r, r') \epsilon \left[ ikS\rho(X - r') - \frac{d\rho(X - r')}{dX} \right] dr' \end{aligned} \quad (30)$$

where  $\rho$ , which may be assumed to be differentiable, has been expanded to leading order in  $\epsilon$ . Substituting (27) and (30) in (23), we obtain

$$\left\langle h(X) \frac{\partial I(r)}{\partial r} \right\rangle = \frac{ik}{\alpha} \cos \theta \left\{ \frac{\rho(X)}{\sqrt{r}} + \int_0^r \frac{e^{ikSr'}}{\sqrt{r-r'}} \left[ ikS\rho(X - r') - \frac{d\rho(X - r')}{dX} \right] dr' \right\}. \quad (31)$$

This removes the derivative with respect to  $x$  in (22), and indeed for several important autocorrelation functions eq. (31) can be written in closed form. The term  $\rho(X)/\sqrt{r}$  is an artifact of the finite lower bound of integration and can be dropped, as we can assume the range variable  $X$  to be large. Equation (22) therefore becomes

$$\mathcal{E}(X, x) \equiv \langle h(x)C(x) \rangle = -\frac{ik}{\alpha^3} \cos \theta \times$$

$$\left[ \frac{d}{dx} \int_0^x \frac{1}{\sqrt{x-y}} \int_y^\infty \frac{e^{ikr}}{\sqrt{y-r}} \int_0^r \frac{e^{ikSr'}}{\sqrt{r-r'}} R(X, r') dr' dr dy \right] \quad (32)$$

where

$$R(X, r') = ikS\rho(X - r') - \frac{d\rho(X - r')}{dX}. \quad (33)$$

The derivative with respect to  $x$  in (32) can be evaluated similarly, and after further manipulation (see Appendix) the required expression can be written, setting  $X = x$ ,

$$\langle h(x)C(x) \rangle = -\frac{ik}{\alpha^3} \cos \theta \times$$

$$\begin{aligned} & \left[ \frac{2}{\sqrt{x}} \int_y^\infty \frac{e^{ikr}}{\sqrt{y-r}} \int_0^r \frac{e^{ikSr'}}{\sqrt{r-r'}} R(X, r') dr' dr \right. \\ & \left. - \int_0^x \frac{1}{\sqrt{x-y}} \int_y^\infty \frac{e^{ikr}}{\sqrt{y-r}} \int_0^r \frac{e^{ikSr'}}{\sqrt{r-r'}} \mathcal{F}(X, r') dr' dr dx \right]_{X=x} \end{aligned} \quad (34)$$

where

$$\mathcal{F} = \left\{ (1 + ik \sin \theta) R(X, r') + \frac{dR}{dr'} \right\}. \quad (35)$$

### 3.3. Autocorrelation and angular spectrum

The main quantity of interest is the angular spectrum of intensity, which may be defined as the Fourier transform of the autocorrelation function (i.e. the second moment) of the scattered field. This remains essentially unchanged with distance from the surface, so that we may again concentrate on obtaining the form on the mean surface plane,  $z = 0$ .

Denote the second moment

$$m_2(x, y) = \langle \psi_s(x, 0) \psi_s^*(y, 0) \rangle$$

where \* indicates the complex conjugate, and denote its approximation using the standard parabolic equation method by

$$\tilde{m}_2(x, y) \equiv \langle \tilde{\psi}_s(x, 0) \tilde{\psi}_s^*(y, 0) \rangle.$$

The perturbational solution of  $\tilde{m}_2$  was obtained in (19). It is relatively straightforward to express  $m_2$ , to second order in surface height under the present two-way PIE method, as the sum of  $\tilde{m}_2$  and correction terms. These additional terms, which are expected to be small, represent the ‘indirect’ contribution to the backscatter, i.e. involving backward-going surface interactions. From (14) we have

$$\psi_s(x) \psi_s^*(y) = \tilde{\psi}_s(x) \tilde{\psi}_s^*(y) + \tilde{\psi}_s(x) h(y) C^*(y) + \tilde{\psi}_s^*(y) h(x) C(x) + h(x) h(y) C(x) C^*(y). \quad (36)$$

We can write  $\tilde{\psi}_s$  and  $C$  to zero and first order in surface height,

$$\tilde{\psi}_s = \psi_0 + \psi_1 + O(\sigma^2)$$

where (18)

$$\begin{aligned} \psi_0(x) &= -e^{ikSx} \\ \psi_1(x) &= -2ikh(x) \sqrt{2 - 2 \sin \theta} e^{ikSx} \equiv h(x) D_\theta(x), \end{aligned} \quad (37)$$

and

$$C = C_0 + C_1 \quad (38)$$

where

$$\begin{aligned} C_0 &= -\pi L_0^{-1} R_0 D_\theta \\ C_1 &= -\pi L_0^{-1} R_0 \frac{dI}{dx}. \end{aligned}$$

Therefore to  $O(\sigma^2)$  the second moment can be written

$$\begin{aligned} m_2(x, y) &= \tilde{m}_2(x, y) + \psi_0(x) \langle h(y) C_1^*(y) \rangle + \langle \psi_1(x) h(y) \rangle C_0^*(y) \\ &+ \psi_0^*(y) \langle h(x) C_1(x) \rangle + \langle \psi_1^*(y) h(x) \rangle C_0(x) + \rho(x - y) C_0(x) C_0^*(y). \end{aligned} \quad (39)$$

Since  $\mathcal{E} = \langle hC \rangle = \langle hC_1 \rangle$ , equation (39) can be expressed as

$$m_2(x, y) = \tilde{m}_2(x, y) + \psi_0(x)\mathcal{E}^*(y) + \psi_0^*(y)\mathcal{E}(x) + \rho(x-y)C_0(x)C_0^*(y) + \langle \psi_1(x)h(y) \rangle C_0^*(y) + \langle \psi_1^*(y)h(x) \rangle C_0(x). \quad (40)$$

In this equation, only the last two terms remain to be determined. From (37),  $\langle \psi_1(x)h(y) \rangle$  is just

$$\langle \psi_1(x)h(y) \rangle = \rho(x-y)D_\theta \quad (41)$$

and similarly for  $\langle \psi_1^*(x)h(y) \rangle$  so that (40) becomes

$$m_2(x, y) = \tilde{m}_2(x, y) + \psi_0(x)\mathcal{E}^*(y) + \psi_0^*(y)\mathcal{E}(x) + \rho(\xi) \left[ C_0(x)C_0^*(y) + D_\theta(x)C_0^*(y) + D_\theta^*(y)C_0(x) \right] \quad (42)$$

where  $\xi = x - y$ .

#### 4. Conclusions

We have considered the scattering of a plane-wave incident upon a slightly rough surface at a low grazing angle  $\mu$ , by combining operator series expansion with the two-way parabolic integral equation. The second moment and angular distribution of intensity have been found to second order in surface height, and their dependence upon incident angle and the surface autocorrelation function has been shown.

This approach allows calculation of backscatter due to a scalar wave impinging on a rough surface at low grazing angles. The solution is written in terms of a series of Volterra operators which correspond to multiple scattering resulting from increasing orders of surface interaction. Truncation at the first term gives the leading forward- and back-scattered components; higher-order multiple scattering are available from subsequent terms. With the additional assumption of small surface heights, analytical solutions have then been obtained, to second order in height, for the mean field and its autocorrelation. These provide backscatter corrections to the solutions given in the purely forward-scattered case (18, 19) with the potential for further insight into the role of different orders of multiple scattering. (Small height perturbation theory derived directly from Helmholtz equation has of course been well established for many years and yields particularly simple single scattering results. The results here are from a different perspective; the first term already includes 'multiple-forward-scattering', and subsequent terms incorporate back- and forward-scatter contributions systematically at higher orders.)

Parabolic equation methods remain very widely used for long-range propagation at low grazing angles. The tractable form of the Green's function together with the series decomposition provide computational efficiency and the means to extend existing PE methods to include backscatter.

We should not overstate the advantages. Whilst accounting for some multiple surface interactions the analytical results are nevertheless derived under small roughness assumptions. Furthermore the computational benefits are reduced when considering fully 3-dimensional problems or when truly wide-angle scatter is important. In such

cases a more fruitful approach may be to replicate the analysis based on left-right splitting for the full Green's function equations.

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## Appendix

We can write the expression (29) as

$$\mathcal{E}(X, x) = -\frac{ik}{\alpha^3} \cos \theta \frac{d}{dx} \int_0^x g(x, y) H(X, y) dy \quad (1)$$

where

$$g(x, y) = \frac{1}{\sqrt{x-y}}, \quad (2)$$

$$H(X, y) = \int_y^\infty \frac{e^{ikr}}{\sqrt{y-r}} \int_0^r \frac{e^{ikSr'}}{\sqrt{r-r'}} R(X, r') dr' dr, \quad (3)$$

and  $R$  is given by (30). Differentiation with respect to  $x$  is carried out as for the  $r$ -derivative (equations (22)-(28)): The  $x$ -derivative is thus expressed as a limit of a finite difference, and the integral split into three parts,

$$\mathcal{E}(X, x) = -\frac{ik}{\alpha^3} \cos \theta \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [L_1 + L_2 - L_3] \quad (4)$$

where

$$\begin{aligned} L_1 &= \int_0^\epsilon g(x + \epsilon, y) H(X, y) dy \\ L_2 &= \int_\epsilon^{x+\epsilon} g(x + \epsilon, y) H(X, y) dy \\ L_3 &= \int_0^x g(x, y) H(X, y) dy \end{aligned}$$

We thereby obtain

$$\frac{d}{dx} \int_0^x g(x, y) H(X, y) dy = \frac{2}{\sqrt{x}} H(X, y) + \int_0^x g(x, y) \left[ \frac{dH(X, y)}{dy} - H(X, y) \right] dy. \quad (5)$$

The term  $dH/dy$  is then

$$\frac{dH(X, y)}{dy} = \frac{d}{dy} \int_y^\infty a(y, r) J(X, r) dr \quad (6)$$

where

$$a(y, r) = \frac{e^{ikr}}{\sqrt{y-r}}, \quad (7)$$

$$J(X, r) = \int_0^r \frac{e^{ikSr'}}{\sqrt{r-r'}} R(X, r') dr' \quad (8)$$

Treating the derivative as before gives

$$\frac{dH}{dy} = - \int_y^\infty a(y, r) \left( \frac{dJ(X, r)}{dr} + ikJ(X, r) \right) dr. \quad (9)$$

Finally,

$$\frac{dJ}{dr} = \frac{d}{dr} \int_0^r \frac{e^{ikSr'}}{\sqrt{r-r'}} R(X, r') dr' \quad (10)$$

from which we similarly get

$$\frac{dJ}{dr} = \frac{R(X, 0)}{\sqrt{r}} + \int_0^r \frac{e^{ikSr'}}{\sqrt{r-r'}} \left\{ ikSR(X, r') + \frac{dR}{dr'} \right\} dr'. \quad (11)$$

As before (see (29)) the expression  $R(X, 0)$  vanishes for large  $X$  and can be dropped. Successively substituting (9), (11), (3) and (5) into (1), we eventually obtain

$$\langle h(X)C(x) \rangle = -\frac{ik}{\alpha^3} \cos \theta \times \quad (12)$$

$$\left[ \frac{2}{\sqrt{x}} \int_y^\infty \frac{e^{ikr}}{\sqrt{y-r}} \int_0^r \frac{e^{ikSr'}}{\sqrt{r-r'}} R(X, r') dr' dr \right. \quad (13)$$

$$\left. - \int_0^x \frac{1}{\sqrt{x-y}} \int_y^\infty \frac{e^{ikr}}{\sqrt{y-r}} \int_0^r \frac{e^{ikSr'}}{\sqrt{r-r'}} \mathcal{F}(X, r') dr' dr dx \right] \quad (14)$$

where

$$\mathcal{F} = \left\{ (1 + ik[1 + S])R(X, r') + \frac{dR}{dr'} \right\}. \quad (15)$$

In this expression,  $R$  is given by (30), so that

$$\frac{dR}{dr'} = ikS \frac{d\rho(X - r')}{dr'} - \frac{d^2\rho(X - r')}{dx^2}. \quad (16)$$

It is clear then that the correction term introduces a higher-order dependence on the correlation function.