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THE STRATEGY OF CONQUEST

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This paper develops a theoretical framework for the study of war and conquest. The analysis highlights the role of three factors - the technology of war, resources, and contiguity network - in shaping the dynamics of appropriation and the formation of empires. The analysis reveals that the world of many small kingdoms is characterized by incessant fighting. After an initial phase of uncertain and gradual growth, the expansion of the winning kingdom speeds up, and it grows rapidly through contiguous expansion. The size of the empire is limited by the connectivity of the network. These results provide a parsimonious account of the growth of major empires.

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Abstract

This paper develops a theoretical framework for the study of war and conquest. The analysis highlights the role of three factors – the technology of war, resources, and contiguity network – in shaping the dynamics of appropriation and the formation of empires. The analysis reveals that the world of many small kingdoms is characterized by incessant fighting. After an initial phase of uncertain and gradual growth, the expansion of the winning kingdom speeds up, and it grows rapidly through contiguous expansion. The size of the empire is limited by the connectivity of the network. These results provide a parsimonious account of the growth of major empires.

Keywords Resources, Networks, Contests, Empire.

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1 Introduction

The history of the world .... is an imperial history, the history of empires. Empires were systems of influence or rule where ethnic, cultural or ecological boundaries were overlapped or ignored. Their ubiquitous presence arose from the fact that .... the endowments needed to build strong states were very unequally distributed. Against the cultural attraction, or physical force, of an imperial state, resistance was hard, unless reinforced by geographical remoteness or unusual cohesion. (Darwin [2007]; page 491)

A recurring theme in history is that the presence of small kingdoms is accompanied by bloody conflict; rulers fight each other incessantly, small parcels of land are exchanged, treasures are plundered, and capture of human beings is common. However, once a ruler acquires a large advantage relative to his neighbours he then quickly goes on to take them over, one after the other, and to create an empire.[1] This record of war and conquest leads us to ask: what are the circumstances under which rulers will choose to fight, what is the optimal timing of attack – now or later, when will the resource advantage of a ruler translate rapidly into domination over neighbours, and what are the limits to the size of the empire? The goal of this paper is to develop a theoretical framework to address these questions.

We consider a set of 'kingdoms'. Every kingdom is endowed with resources and controlled by a ruler. Rulers desire to expand territory and acquire more resources. The ruler can wage a war on neighboring kingdoms. The winner of a war takes control of the loser's resources and his kingdom; the loser is eliminated. The probability of winning a war depends on the resources of the combatants and on the technology of war that is defined by a *contest success function*.[2] As the winning ruler expands his domain, he may be able to access and attack new kingdoms. The neighborhood structure between kingdoms is reflected in a *contiguity network*. We model the interaction between rulers as a dynamic game and study its (Markov Perfect) equilibria.

We start by establishing that there exists a pure strategy Markov Perfect equilibrium and the equilibrium payoffs are unique. This sets the stage for a study of how the main parameters

[1] Classical studies on the formation of empire include Polybius [2010], Tacitus [2009] and Khaldun [1989]. Starting with Gibbon [1776], there is a long tradition of modern work on empires, see e.g., Braudel [1995], Darwin [2007], Elliott [2006], Lewis [2010], Morris and Scheidel [2009], and Thapar [1997, 2002]. Mathematical models of the evolution of empire include Levine and Modica [2013] and Turchin [2007].

[2] Classical writers on war and more recent research both point to the decisive role of the army size and financial resources in securing victory, see e.g., Lewis [2010], Tzu [2008], Clausewitz [1993] and Howard [2009].

– resources, the contiguity network, and the contest function – affect the dynamics of war and peace.

Consider two rulers A and B, with resources r_A and r_B, and suppose $r_A > r_B$. When they fight, the expected payoff of A is given by $(x_A + x_B)p(x_A, x_B)$, where $p(x_A, x_B)$ is the contest success function that defines the probability of winning for ruler A. The contest success function is said to be *rich rewarding* if fighting is profitable for A (and unprofitable for B), i.e., $(x_A + x_B)p(x_A, x_B) > x_A$. The technology is said to be *poor rewarding*, otherwise. The technology shapes the optimal timing and the target of attack. When the technology is rich rewarding *no-waiting* is optimal: attacking the two rivals in sequence is preferable to attacking the merged kingdom. In the poor rewarding setting, waiting is optimal: attacking the larger kingdom formed after two rivals have fought is best. Moreover, with a rich (poor) rewarding technology it is optimal for a ruler to attack opponents in increasing (decreasing) order of resources. Equipped with these results, we turn to the study of equilibrium dynamics.

Our main result, Theorem 1 shows that, with a rich rewarding technology, in any configuration with three or more kingdoms, *all* rulers find it optimal to attack a neighbour. Thus, we are in a world with incessant warfare, the violence only stops when all opposition is eliminated. When the network is connected, all opposition is eliminated only with the hegemony of a single ruler.[3] The arguments underlying this result are fairly general. We start by defining a *strong* ruler: this is a ruler who has a 'full attacking sequence' (involving all other opponents), such that at each point he is stronger than the opponent. Clearly, at any point in time, the richest ruler is a strong ruler. It follows from the rich rewarding property that, if everyone else is peaceful, then such a strong ruler has a strict incentive to fight every other ruler. Next consider the case when other rulers may also wish to attack: does the strong ruler still have an incentive to implement a fully attacking sequence? Given the no-waiting property identified above, it then follows that the strong ruler has a dominant strategy: a full attacking sequence. So, there is always at least one ruler who wishes to fight to the finish. Anticipating this, and given the no-waiting property, every ruler, no matter how poor, has an incentive to fight a neighbour. Thus, in a connected network, in equilibrium, eventually there will be only one ruler left.

Turning to the role of resources and networks in shaping the prospects of individual rulers, for ease of exposition, in this part, we restrict attention to the well known Tullock Contest Function: the probability of ruler A winning is $p(x_A, x_B) = x_A^\gamma/(x_A^\gamma + x_B^\gamma)$, for some $\gamma \in \mathbb{R}_+$.

[3] A network is connected if there is a path between any two kingdoms.

It can be shown that the function is rich rewarding if $\gamma > 1$ and poor rewarding if $\gamma < 1$ (and rulers are indifferent between war and peace if $\gamma = 1$). When γ is large, the probability of a weak ruler becoming a hegemon becomes negligible. Within the set of strong rulers, those who have 'exclusive' access to weak kingdoms that have a significantly greater probability of becoming the hegemon (relative to their strong rivals). We show that the dynamics of appropriation have powerful redistribution effects: in particular, they tend to take resources away from the richest kingdoms and the poorest kingdoms and toward the middle resource kingdoms.

We next take up poor rewarding contest success functions. A poor ruler gains from fighting a rich rival; however, waiting is better: so the poorer ruler would prefer to wait and allow for opponents to become large before engaging in a fight. This creates the possibility of peace. To make progress we divide the analysis into two parts. To start, consider resource distributions with a single rich ruler: if this ruler is sufficiently rich then his kingdom becomes an 'irresistible' prize; all other rulers have a strict incentive to fight to acquire the rich kingdom. So peace cannot be sustained and the outcome is hegemony. Next, consider the case where no ruler is very rich. Here we show that perpetual peace and a phase of war followed by peace may be sustained in equilibrium. The key to sustaining peace is the threat of imminent war. The equilibrium has the following structure: no ruler wishes to fight a single fight because, once this fight is undertaken, all rulers have an incentives to fight till the finish. It is this latter phase of war that makes war today unattractive. The analysis highlights the role of resource inequality: across a range of networks and contest success functions, peace is more likely when resources are more similar. And, we show that the dynamics of appropriation in the poor rewarding setting are 'equalizing': the ex-post distribution of resources is less unequal as compared to the starting resource distribution.

In the basic model, all the resources are taken over by the winner and the winner can implement a full attacking sequence (while all other rulers remain passive). Section 5 shows how our methods of analysis can be applied after we relax these features. The dynamics are now considerably richer and they yield new insights. The introduction of gradual capture of parts of an empire leads to dynamics in which conflict among rulers can be protracted and involves the exchange of small parcels of territories. However, once a ruler succeeds in expanding his territory, he gets more secure against being fully defeated. This motivates fighting, and the growing size of a ruler in turn speeds up the emergence of a hegemon. As a result, in the gradual expansion model, hegemony is the outcome both for rich and for poor rewarding contest success functions. We then consider an extension of the model in which a rulers choose short attack sequences only: this accommodates the idea that rival rulers can become active once a ruler begins an attack sequence. The incentives to wage war remain strong in this setting and hegemony is still the norm.[4]

To summarize, our analysis suggests that starting in a situation with multiple kingdoms, the dynamics are characterized by incessant fighting. After an initial phase of uncertain and gradual growth, the pace of expansion of a 'kingdom' speeds up, and it grows rapidly through contiguous expansion. This expansion, and consequently the size of the empire, is limited by the connectivity of the network.

These predictions are consistent with episodes of imperial expansion in world history, e.g., the First Chinese Empire, the growth of the Roman Empire, Cyrus forming the first Persian Empire, Alexander's campaigns leading to the Greek Empire in Asia, Chandragupta setting up the Mauryan Empire in India, and the creation of the first Arab Empire. Section 6 closes the paper by presenting a case study of the rise of the First Chinese Empire. This discussion maps the principal theoretical insights on to the Warring States Period, the reforms of Qin leading to major resource augmentation, the monotonic sequence of attack, and the very rapid expansion of territory leading to the emergence of the first Chinese Empire.

We now place our paper in the context of the literature and clarify its contributions. Our paper studies the dynamics of war and peace and the formation of empires; related work includes Hirshleifer [1995], Jordan [2006], Krainin and Wiseman [2016], Levine and Modica [2013, 2016], and Piccione and Rubinstein [2007]. A number of aspects of our framework set it apart from existing work: we develop a non-cooperative and dynamic game with far-sighted players, we consider general contest functions, and there is a network structure which shapes the sequence of attack strategies and the scale of empires. Our analysis introduces new concepts – rich/poor rewarding contest success functions and strong/weak rulers. They enable us to address a range of very different questions, such as the timing and monotonicity of optimal attack strategies, and how the prospects of individual rulers depend on the network and on the nature of the contest success function. Finally, we provide a mapping from our results onto the history of empire. Along all these dimensions we go beyond the existing work.

The theoretical framework combines elements from the literature on contests, on resource wars, and on networks. We now discuss the relationship between our paper and these literatures.

[4]We have also studied a number of other factors: ties in a war, the guns vs butter trade-off, resurrection of defeated rulers, and asymmetric contest success functions. The details of these extensions are available from the authors.

There is a large literature on contests, for surveys see Konrad [2009] and Garfinkel and Skaperdas [2012]. We consider a general model of multi-player contests inspired by the axiomatic work of Skaperdas [1996].[5] In recent work, Konrad and Kovenock [2009], Groh, Moldovanu, Sela, and Sunde [2012], and Anbarcı, Cingiz, and Ismail [2018] study multi-player sequential contests. In these papers the contest takes the form of an all-pay auction. The interest is in how individual heterogeneity and the sequential contest structure determine aggregate efforts and winning probabilities. By contrast, in our model, we abstract away from effort so that we can study the dynamics of conflict with general contest success functions and networks. To the best of our knowledge, the results on rich/poor rewarding contest success functions and strong/weak rulers, and the mapping from these results to imperial history, are novel in the context of this literature.[6]

The role of resources in shaping violent conflict is an active field of study, see e.g., Acemoglu, Golosov, Tsyvinski, and Yared [2012], Caselli, Morelli, and Rohner [2015], and Novta [2016]. This literature provides evidence for appropriation of resources as a major motivation for war. The theoretical work is mostly limited to two players or to symmetric models; for an overview of the theory, see Baliga and Sjöström [2012]. Our paper contributes to this literature by studying the cumulative dynamics of appropriation and the expansion of territory within a contiguity network, and by linking these dynamics to major episodes of world history.

Finally, our paper is a contribution to the recent literature on conflict and networks, see e.g., Franke and Öztürk [2015], Hiller [2017], Kovenock and Roberson [2012], Huremović [2015], Jackson and Nei [2015], and König, Rohner, Thoenig, and Zilibotti [2017]. For an overview see Dziubiński, Goyal, and Vigier [2016]. Our paper advances this literature on two fronts: one, the dynamics of appropriation in inter-connected conflict and two, how these dynamics are decisively shaped by the contiguity network, the resources, and the contest success function.

The rest of the paper is organized as follows. Section 2 presents the basic model. Section 3 studies the incentives to fight and the optimal timing of attack. Section 4 presents the results on equilibrium dynamics. Section 7 concludes. The Appendix contains the proofs of the main results, while the Online Appendix discusses a number of extensions of the basic model.

[5] For an early study of optimal strategy of attack in a three player game, see Shubik [1954]. Olszewski and Siegel [2016] study static contests with a large numbers of players.

[6] In our model, a rich rewarding contest success function provides a rationale for waging a sequence of wars due to the compounding of spoils of war. This bears some resemblance to the earlier work of Garfinkel and Skaperdas [2000] and McBride and Skaperdas [2014] who study incentives for war in settings where rewards extend through time. In their model, war today is attractive as it facilitates expansion tomorrow.

2 The Model

We study a dynamic game in which rulers seek to maximize the resources they control by waging war and capturing new territories. There are three building blocks in our model: the interconnected 'kingdoms', the resource endowment for every kingdom, and the contest success function.

Let $V = \{1, 2, \ldots, n\}$, where $n \geq 2$ is the set of vertices. Every vertex $v \in V$ is endowed with resources, $r_v \in \mathbb{R}_{++}$. The vertices are connected in a network, represented by an undirected graph $G = \langle V, E \rangle$, where $E = \{uv : u, v \in V,\ u \neq v\}$ is the set of edges (or links) in G. A network G is said to be connected if there is a path between any two vertices. For expositional simplicity, we restrict attention to (undirected) connected networks. Our insights extend in a natural way to directed networks.

A link between two vertices signifies 'access'. Access may reflect physical contiguity. But, in principle, it goes beyond geography: we do not restrict attention to planar graphs.[7] So our model allows for 'virtual' links, i.e., links made possible by advances in military and transport technology.

Every vertex $v \in V$ is owned by one ruler. At the beginning, there are $N = \{1, 2, \ldots, n\}$ rulers. Let $\mathbb{o} : V \to N$ denote the ownership function. The resources of ruler $i \in N$ under $\mathbb{o}$, are given by

$$R_i(\mathbb{o}) = \sum_{v \in \mathbb{o}^{-1}(i)} r_v \tag{1}$$

The network together with the ownership configuration induces a neighbor relation between the rulers: two rulers $i, j \in N$ are *neighbors* in network $G = \langle V, E \rangle$ if there exists $u \in V$, owned by i, and $v \in V$, owned by j, such that $uv \in E$. Figure 1 illustrates vertices, resource endowments, and connections; vertices controlled by the same ruler share a common colour. The light line between vertices represents the interconnections, the dotted lines encircling vertices owned by the same ruler indicate the ownership configuration, and the thick lines between vertices reflect the induced neighborhood relation between rulers.

When two rulers fight, the probability of winning is specified by a *contest success function*. Following Skaperdas [1996], we consider symmetric contest success functions with no ties. Given two rulers, A and B, with resources $x_A \in \mathbb{R}_{++}$ and $x_B \in \mathbb{R}_{++}$, respectively, $p(x_A, x_B)$ is the probability that A wins the conflict and $p(x_B, x_A)$ is the probability that B wins the

[7] A graph is planar if it can be embedded in a plane, i.e. drawn in a plane in such a way that the edges intersect at their endpoints only. An example of a graph that is not planar is a clique with 5 nodes.

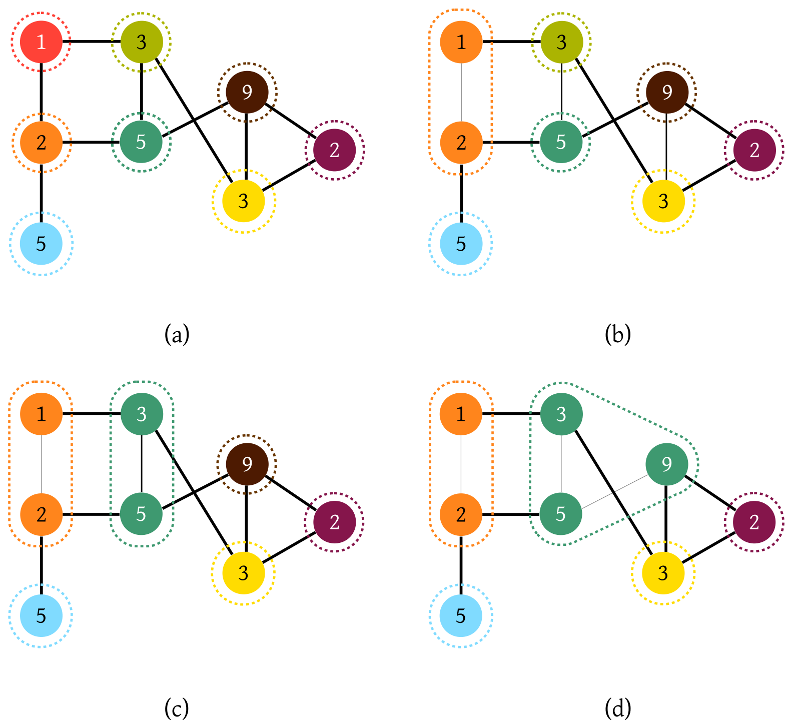


Figure 1: Neighboring Rulers

conflict.

The game takes place in discrete time: rounds are numbered $t = 1, 2, 3 \ldots$. At the start of a round, each of the rulers is picked with equal probability. The chosen ruler, (say) i, chooses either to be peaceful or to attack one of his neighbors. If a ruler attacks a rival, he does so with all his current resources. If he chooses peace, one of the remaining rulers is picked, and asked to choose between war and peace, and so forth. If no ruler chooses war, the game ends. If the attacker loses, the round ends. Otherwise, the attacker is allowed to attack neighbors until he loses, chooses to stop, or there are no neighbors left to attack.[8] When two rulers i and j fight, the winner takes over the entire kingdom of the loser (and also inherits the boundaries, and hence the connections). This dynamic is illustrated in Figure 1: the orange kingdom wins the war with the red kingdom and expands. This expansion brings it in contact with new neighbors, the light and dark green kingdoms. The game ends when all rulers choose to be

[8] De Jong, Ghiglino, and Goyal [2014] introduced a model of conflict with resources and a network: the key difference is that conflict is imposed exogenously. Links are picked at random and rulers *must* fight. By contrast, in the present paper, the choice of waging a war or being at peace is the central object of study.

peaceful (the case of a single surviving ruler is a special case, as there is no opponent left to attack). Observe that, given these rules, the game ends after at most $n-1$ rounds. It may of course end earlier: this happens if all the rulers choose peace at a round.

The configuration of kingdoms and rulers – who is a neighbor of whom – is (potentially) evolving over time. Given a set of vertices $U \subseteq V$, $G[U] = \langle U, \{vu \in E : v, u \in U\}\rangle$ is the subgraph of G restricted to vertices in U and links between them. The set of valid ownership configurations, given graph G, is denoted by

$$\mathbb{O} = \{\mathbb{o} \in N^V : \text{ for all } i \in N,\, G[\mathbb{o}^{-1}(i)] \text{ is connected}\}. \tag{2}$$

As the graph is fixed, for simplicity, we omit it as an argument.

A *state* is a pair $(\mathbb{o}, P)$, where $P \subseteq N$, is the set of rulers who were picked prior to i and chose peace at $\mathbb{o}$. Ruler i, picked at state $(\mathbb{o}, P) \in \mathbb{O} \times 2^{N\setminus\{i\}}$, chooses a sequence of rulers to attack. A sequence σ is *feasible* at $\mathbb{o}$ in graph G if either σ is empty, or if $\sigma = j_1, \ldots, j_k$ and for all $1 \leq l < k$, $j_l \notin \{i, j_1, \ldots, j_{l-1}\}$ and j_l is a neighbor of one of the rulers from $\{i, j_1, \ldots, j_{l-1}\}$ under $\mathbb{o}$ in G. A sequence σ is *attacking* if it is non-empty. Let N^* denote the set of all finite sequences over N (including the empty sequence). A *strategy* of ruler i is a function $s_i : \mathbb{O} \times 2^{N\setminus\{i\}} \to N^*$ such that for every ownership configuration, $\mathbb{o} \in \mathbb{O}$, and every set of rulers, $P \subseteq N \setminus \{i\}$, $s_i(\mathbb{o}, P)$ is feasible at $\mathbb{o}$ in G.[9] Given ruler $i \in N$ and graph G, the set of strategies of i is denoted by S_i; $\boldsymbol{S} = \prod_{i\in N} S_i$ denotes the set of strategy profiles.

The probability that ruler 1 with resources R_1 wins a sequence of conflicts with rulers with resources $R_2, \ldots, R_m$, accumulating the resources of the losing opponents at each step of the sequence is

$$p_{\text{seq}}(R_1, \ldots, R_m) = \prod_{k=2}^{m} p\left(\sum_{j=1}^{k-1} R_j, R_k\right). \tag{3}$$

Given $\mathbb{o}$, a set of rulers, P, and a strategy profile $\boldsymbol{s} = (s_1, s_2, \ldots, s_n) \in \boldsymbol{S}$, the probability that the game ends at $\mathbb{o}'$, is given by $F(\mathbb{o}' \mid \boldsymbol{s}, \mathbb{o}, P)$. We shall sometimes refer to a final ownership configuration as an *outcome*. The expected payoff to ruler i from strategy profile $\boldsymbol{s} \in \boldsymbol{S}$ at state $(\mathbb{o}, P)$ is:

[9] Observe that the only feasible sequence for rulers who do not own any vertices, and for the ruler who owns all vertices, is the empty sequence.

$$\Pi_i(\boldsymbol{s} \mid \mathbb{o}, P) = \sum_{\mathbb{o}' \in \mathbb{O}} F(\mathbb{o}' \mid \boldsymbol{s}, \mathbb{o}, P) R_i(\mathbb{o}'). \tag{4}$$

Every ruler seeks to maximize his expected payoff. The goals of rulers have been studied extensively; for classical discussions see Hobbes [1651], Machiavelli [1992], and for more recent work see Jackson and Morelli [2007].[10]

A strategy profile $\boldsymbol{s} \in \boldsymbol{S}$ is a Markov perfect *equilibrium* of the game if and only if, for every ruler $i \in N$, every strategy $s_i' \in S_i$, and every state, $(\mathbb{o}, P) \in \mathbb{O} \times 2^{N \setminus \{i\}}$, $\Pi_i(\boldsymbol{s} \mid \mathbb{o}, P) \geq \Pi_i((s_i', \boldsymbol{s}_{-i}) \mid \mathbb{o}, P)$. Standard arguments can be employed to establish:

Proposition 1. *Fix a connected graph G. For any symmetric contest success function, p, and any resource endowment, $\boldsymbol{r} \in \mathbb{R}_{++}^V$, there exists an equilibrium and all equilibria are payoff equivalent.*

The proof is presented in the Appendix.

3 The Incentives to Fight

This section introduces a general class of contest success functions and presents general results on incentives to fight for the two and three ruler setting. The notions of rich and poor rewarding contest success functions are introduced and a characterization is presented in terms of standard properties such as increasing and decreasing returns. The interest then turns to the timing and order of optimal attacks: conditions on the contest success functions are obtained under which rulers prefer to wait/not wait to attack.

In general, a contest success function is function $q : \mathbb{R}_{++}^2 \to [0,1]^2$. Following Skaperdas [1996]), we consider three axioms for contest success functions, together with an additional, fourth axioms, that substitutes independence of irrelevant alternatives axiom for the case of bilateral contests.[11]

[10] We assume that ruler's utility is linear in resources. Risk-averse and risk-loving preferences can easily be accommodated. Suppose utility is given by $u(x)$, with $u(0) = 0$, $u' > 0$ and $u'' < 0$. This means that $u(x+y) < u(x) + u(y)$. Expected payoff to x *vs* y can be written as:

$$p(x,y)u(x+y) = p(x,y)(u(x) + u(y))(1 - d(x,y))$$

where $d(x,y) = 1 - u(x+y)/(u(x) + u(y))$. So $0 < d(x,y) < 1$: in other words, risk-aversion creates a 'cost' of conflict.

[11] Skaperdas [1996] proposes five axioms for contest success functions, the first three of them correspond to axioms **A1-3**, the fourth, consistency axiom, is always satisfied in the case of two bilateral contests, and the

A1 For all $(x_1, x_2) \in \mathbb{R}^2_{++}$, $q_1(x_1, x_2) + q_2(x_1, x_2) = 1$,

A2 For all $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, $q_i(x_i, x_j)$ is increasing in x_i and decreasing in x_j,

A3 For all $(x_1, x_2) \in \mathbb{R}^2_{++}$, $q_1(x_1, x_2) = q_2(x_2, x_1)$,

A4 For all $i \in \{1, 2\}$ and $(x_1, x_2, x_3) \in \mathbb{R}^3$, $q_i(x_1, x_2)q_i(x_2, x_3)q_i(x_3, x_1) = (1 - q_i(x_1, x_2))(1 - q_i(x_2, x_3))(1 - q_i(x_3, x_1))$.

By axiom **A3**, the contest success function is symmetric and can be represented by function $p : \mathbb{R}^2_{++} \to [0, 1]$, where $q_1(x_1, x_2) = p(x_1, x_2)$ and $q_2(x_1, x_2) = p(x_2, x_1)$. Using the additional axiom, **A4**, the proof of Skaperdas [1996] extends to show that a bilateral contest success function satisfying axioms **A1-4** necessarily takes the form

$$p(x, y) = \frac{f(x)}{f(x) + f(y)}. \tag{5}$$

with an increasing, positive, function $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$.[12] The study of contests remains a very active field of study; see Fu and Pan [2015] for a a recent contribution and for references to the literature.

Recall that $(x + y)p(x, y)$ is the expected payoff of a ruler with resources x who fights an opponent with resources y. We shall say that the contest success function, p, is *rich rewarding* if for all $x, y \in \mathbb{R}_{++}$ with $x > y$,

$$(x + y)p(x, y) > x \tag{6}$$

Similarly, we shall say that p is *poor rewarding* if for all $x, y \in \mathbb{R}_{++}$ with $x < y$,

$$(x + y)p(x, y) > x \tag{7}$$

A rich rewarding contest success function gives the richer side an incentive to fight, while poor rewarding one gives the poorer side an incentive to fight. We characterize rich and poor rewarding contest success functions in terms of standard properties of the function f. We also examine the timing of optimal attack: whether to attack now or to wait and attack later. A contest success function, p, is said to have the *no-waiting* property if for all $x, y, z \in \mathbb{R}_{++}$, $p(x, y)p(x + y, z) > p(x, y + z)$. It is said to have the *waiting* property if for all $x, y, z \in \mathbb{R}_{++}$,

fifth axiom, independence of irrelevant alternatives, applies to contests with at least three participants.

[12] In addition, f is unique up to positive multiplicative transformations.

$p(x,y)p(x+y,z) < p(x,y+z)$. With contest success functions having the no-waiting property, it is profitable for a ruler to attack the other two rulers in a sequence rather than wait to fight the merged kingdom. The converse is true in the case of contest success functions that exhibit the waiting property. Rich/poor rewarding and the timing of attacks are intimately related. Turning to the optimal order of attack, a contest success function has the *poor-first* property if the expected payoffs of attacking the poor ruler followed by the rich ruler are larger, i.e., for all $x, y, z \in \mathbb{R}_{++}$, with $y < z$, $p(x,y)p(x+y,z) > p(x,z)p(x+z,y)$. A contest success function has the *rich-first* property if the converse holds. Define

$$h(s,t) = \frac{f(t)f(s+t)}{f(s+t) - f(s) - f(t)}.$$

Proposition 2. *Consider a contest success function, p, that satisfies* (5)*. The function p is rich rewarding if and only if f exhibits increasing returns to scale; it is poor rewarding if and only if f exhibits decreasing returns to scale. In addition:*

1. *Timing of attack: If p is rich rewarding then it has the no-waiting property, while if p is poor rewarding then it has the waiting property.*

2. *Order of attack: p has the poor-first (rich-first) property if and only if $h(s,t)$ is strictly increasing (decreasing) in $t \in \mathbb{R}_{++}$, for all $s \in \mathbb{R}_{++}$.*

The proof is presented in the Appendix. The argument for the first part proceeds as follows. Suppose that $x > y$. If f exhibits increasing returns then $f(x)/(f(x) + f(y)) > x/(x + y)$. Multiplying both sides by $x + y$ now yields the desired implication. On the other hand, if the stronger side gains in expectation, then it must be that $(x + y)f(x)/(f(x) + f(y)) > x$. Rewriting and rearranging this gives us the inequality $f(x)/(f(x) + f(y)) > x/(x + y)$, which requires that f exhibits increasing returns. A similar line of reasoning applies to the poor rewarding case. The argument for the second part proceeds as follows. In the case of timing of attack, we begin by showing that the no-waiting property is equivalent to f being super-additive. The next step demonstrates that super-additivity is a weaker property than increasing returns to scale, and that concludes the proof. In the case of order of attack, rewriting of the poor-first property derives the required expression.

We note that the optimal order of attack result can be generalized to cover n opponents: if all opponents are neighbours, then the order of attack is monotonically increasing (decreasing) in the resources of opponents if $h(x,y)$ is increasing (decreasing) in y for all x (this result is stated and proved in Appendix).[13]

We illustrate the scope of these results through a consideration of the widely studied *Tullock* contest success function.[14]

$$p(x, y) = \frac{x^\gamma}{x^\gamma + y^\gamma},$$

where $\gamma > 0$. Hence, $f(x) = x^\gamma$. If $\gamma > 1$ then f has increasing returns to scale. From Proposition 2 it follows that the contest success function is rich rewarding and has the no-waiting property. On the other hand, if $\gamma < 1$, then f exhibits diminishing returns to scale. It is therefore poor rewarding and the ruler would prefer to wait. Finally, observe that $(x + y)p(x, y) = x$, for all $x, y \in \mathbb{R}_{++}$ if $\gamma = 1$. So the contest success function is *reward neutral*; it is also *timing neutral* (as for all $x, y, z \in \mathbb{R}_{++}$, $p(x, y)p(x + y, z) = p(x + y, z)$). Lastly, in the case of $\gamma > 1$, $h(s, t)$ is increasing in t for all s. Hence in this case the contest success function has the poor first property. On the other hand, in the case of $\gamma < 1$, $h(s, t)$ is decreasing in t for all s and the contest success function has rich first property. Since $h(s, t)$ remains constant in t for all s, if $\gamma = 1$, so the contest success function is order neutral in this case. To summarize:

Corollary 1. *The Tullock contest success function is rich rewarding, has the no-waiting and poor first properties if $\gamma > 1$; it is poor rewarding, has the waiting and rich first properties if $\gamma < 1$. It is reward, timing, and order neutral if $\gamma = 1$.*

The condition with regard to order of attack generalizes to larger sequences. Hence, in the case of the Tullock Contest Function, it yields a clean implication: if $\gamma > 1$ then the optimal attack strategy prescribes attacking rivals in increasing order of resources; the converse holds if $\gamma < 1$. These results set the stage for the study of $n \geq 3$ rulers located in a connected network.[15]

[13] The qualification 'if all opponents are neighbors' is important. If some opponents are not neighbors then it may be optimal to attack a richer neighbor in preference to a poor neighbor, so as to reach other poorer opponents first. Here is an example. Suppose G is a line network with 4 rulers, a, b, c, and d, each controlling one vertex (in that order). Suppose that resources of ruler a are $x \in (0, 2)$. The resources of b, c and d are respectively 2, 2.01 and 1. Assume Tullock contest success function with $f(x) = x^2$. If $x < 1.83$ then the optimal full attacking sequence of ruler b is (a, c, d): so it prescribes attacking the weakest neighbor first. On the other hand, if $x > 1.84$ then the optimal full attacking sequence is (c, d, a): it is better to first attack a stronger neighbor, c, to get access to weak d, and only then attack a.

[14] The Online Appendix presents a discussion of the Hirshleifer Difference Contest Function.

[15] The literature has tended to assume $\gamma \leq 1$. This is because of concerns about the existence of an equilibrium in models where resources are costly. In our setting, the ruler chooses whether to fight or not and in this setting the existence of equilibrium does not depend on the value of γ.

We have not been able to locate clear empirical evidence on the nature of contest success functions. The key of resources has been noted in the context of the formation of the first Chinese Empire and the expansion of the Roman Empire (Lewis [2010], Polybius [2010]). For more recent times, Clausewitz [1993], drawing inspiration from the Napoleonic wars in Europe, argued that superiority in numbers was fundamental: an army twice as large as its opponent almost never lost the battle (not even against a great general like Bonaparte). Howard [2009] likewise argues that army size was critical factor in the victory of Germany over France in the Franco-Prussian Wars. These observations are consistent with a probability of winning that is responsive to army size and resources. In what follows, we therefore present equilibrium analysis for both rich and poor rewarding contest success functions.[16]

4 Conquest and Empire

This section studies equilibrium dynamics of war and peace and the formation of empires. The analysis for the rich rewarding case is reasonably complete: we show that equilibrium is characterized by incessant warfare and that the outcome is hegemony. The connectivity of the network defines the limits of the hegemony. The concepts of strong and weak rulers – that reflect resources and network architecture – play a key role in this analysis. The analysis of poor rewarding contest functions is more partial because the dynamics are considerably more complicated: we show that perpetual peace, perpetual war (and hegemony), and a phase of war followed by peace can all arise in equilibrium. Greater equality in initial resources makes peace more likely.

Given ownership configuration $\obar$, the set of *active* rulers at $\obar$ is

$$\mathrm{Act}(\obar) = \{i \in N : \varnothing \subsetneq \obar^{-1}(i) \subsetneq V\}.$$

An ordering of the elements of the set $\mathrm{Act}(\obar) \setminus \{i\}$, σ, such that the sequence σ is feasible for i in G under $\obar$ is called a *full attacking sequence* (or f.a.s). Figure 2 illustrates such a sequence(for the orange kingdom).

We are now ready to state our first main result on equilibrium dynamics.

[16]To get a sense of the numbers, consider the Tullock Contest Function and suppose one army is twice the size of the other army. With an exponent $\gamma = 2$, the probability of winning for the larger army is 0.8, and with an exponent $\gamma = 4$ it is (approx) 0.95. On the other hand, with $\gamma = 0.5$ the probability of winning is 0.6, and with $\gamma = 0$ the probability is 0.5.

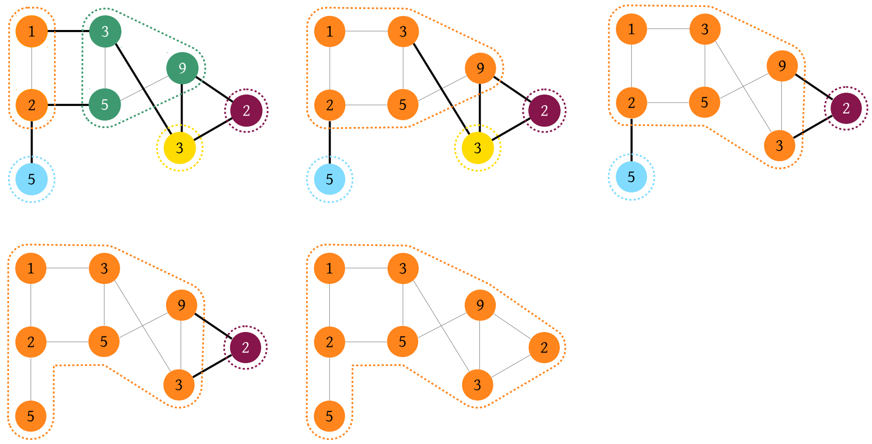


Figure 2: Full Attacking Sequence

Theorem 1. *Consider a rich rewarding contest success function that satisfies* (5)*. Suppose G is a connected network and let $\boldsymbol{r} \in \mathbb{R}^V_{++}$ be a generic resource profile. In equilibrium, every active ruler chooses to attack a neighbor if $|A(\mathbb{o})| \geq 3$, and at least one of the active rulers attacks his opponent if $|A(\mathbb{o})| = 2$. The outcome is hegemony and the probability of becoming a hegemon is unique for every ruler.*

The proof is presented in the Appendix. The result offers an account of the dynamics of conflict in a rich rewarding setting when rulers are driven by a desire to maximize resources under their control. It predicts incessant fighting, preemptive attacks, and long attacking sequences. It is worth drawing attention to the generality of this result: it holds for all rich rewarding contest functions, for any connected network, and for generic resources.

We discuss the arguments underlying the theorem. A ruler is said to be *strong* if he has an attacking sequence $\sigma = i_1, \ldots, i_k$, where for all $l \in \{1, \ldots, k\}$,

$$\sum_{j=0}^{l-1} R_{i_j}(\mathbb{o}) > R_{i_l}(\mathbb{o}).$$

In other words, at every step in the attacking sequence, the ruler has more resources than the next opponent. The set of *strong* rulers at ownership configuration $\mathbb{o}$ is

$$\mathrm{S}(\mathbb{o}) = \{i \in \mathrm{Act}(\mathbb{o}) : i \text{ has a strong f.a.s. } \sigma \text{ at } \mathbb{o}\}.$$

A ruler who is not *strong* is said to be *weak*. Note that (generically) in any state, the ruler with the most resources is strong, while the ruler with the least resources is weak. Thus both sets are non-empty in every network and for (generic) resource profiles.

The first step is to show that, assuming that all other rulers choose peace in all states, it is optimal for a strong ruler to choose a full attacking sequence. This is true because the contest success function is rich rewarding and so a strong ruler has a full attacking sequence that increases his resources in expectation, at every step, along the sequence. The second step extends the argument to cover opponents who choose war. If opponents are active then the no-waiting property (from Proposition 2) tells us that it is even more attractive to not give them an opportunity to move. For a strong ruler it is therefore a dominant strategy to use an optimal full attacking sequence. The final step in the proof covers non-strong rulers to establish that with 3 or more active rulers, it is optimal for *every* ruler to choose a full attacking sequence. Observe that we have already shown that every non-strong ruler knows that he will be facing an attack sooner or later. This means that waiting can only mean that the opposition will become (larger and) richer. The no-waiting property then tells us that every ruler must attack as soon as possible. If there are only two active rulers then the richer ruler has a strict incentive to attack the poorer opponent (this follows from the definition of the rich rewarding contest function).

We now examine the role of the contiguity network and resources more closely. For expositional simplicity, we focus on the Tullock contest success function. Notice that, due to timing and order neutrality, there are no interesting network effects when $\gamma = 1$: equilibrium expected resources of any ruler remain equal to his initial resources. When γ is large it is never optimal to attack a richer ruler if other options are available. The optimal strategy for a strong ruler must involve attacking a poorer ruler at every stage in the attack sequence. Such a sequence is clearly not available for a weak ruler: the probability of a weak ruler becoming a hegemon converges to zero, as γ grows.

Given the initial ownership configuration Φ_0, a γ, and resources $\boldsymbol{r}$, let $\mathrm{Prob}_i(\boldsymbol{r}, \gamma \mid \Phi_0)$ be the equilibrium probability of ruler i becoming the hegemon. Define

$$\mathrm{Prob}_i^*(\boldsymbol{r} \mid \Phi_0) = \mathrm{Prob}_i(\boldsymbol{r}, \lim_{\gamma \to +\infty} \gamma \mid \Phi_0).$$

Proposition 3. *Suppose the contest success function is Tullock, the network G is connected, and the resources $\boldsymbol{r} \in \mathbb{R}^n_{++}$ are generic. The probability of a weak player becoming a hegemon becomes negligible as γ grows. Specifically,*

$$\mathrm{Prob}_i^*(\boldsymbol{r} \mid \Phi_0) \begin{cases} \geq \frac{1}{|\mathrm{Act}(\Phi)|}, & \textit{if } i \in \mathrm{S}(\Phi_0) \\ = 0, & \textit{otherwise.} \end{cases}$$

The proof is presented in the Appendix.

Whether a ruler is strong or weak depends both on the distribution of resources and on the position of the ruler in the contiguity network. In Figure 3 we represent strong rulers in red and weak rulers in yellow. It is helpful to define the boundary of a set of vertices $U \subseteq V$ in G is

$$B_G(U) = \{v \in V \setminus U : \text{ there exists } u \in U \text{ s.t. } uv \in E\}$$

A set of vertices, U, is *weak* if $G[U]$ is connected, $B_G(U) \neq \varnothing$, and for all $v \in B_G(U)$, $r_v > \sum_{u \in U} r_u$. A weak set of nodes is surrounded by a boundary, constituted of nodes, each of whom is endowed with more resources than the sum of resources of vertices within the set. Weak sets are illustrated in Figure 3. It is easy to see that, for any initial state ϕ, a ruler is weak if his vertex belongs to a weak set and, otherwise, the ruler is strong.

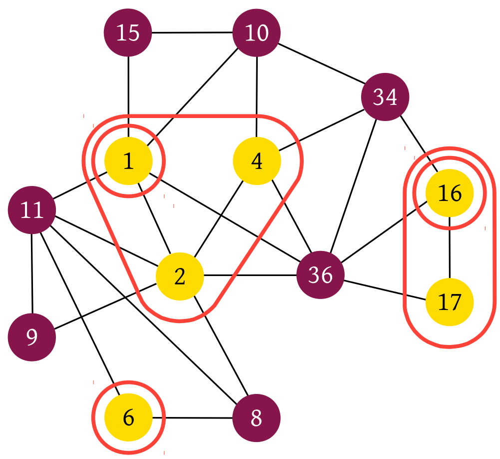


Figure 3: Weak rulers (surrounded by thick lines) and strong rulers

Proposition 3 covers the case of large γ. We now turn to examples to show that the distinction between strong and weak rulers is central to the study of dynamics more generally, across rich rewarding γ. Consider three networks with 10 nodes: the clique network (with 45 links), a connected network with 27 links and a tree network (with 9 links). The resources endowments at the nodes are 2, 3, 6, 11, 13, 15, 16, 18, 21, and 23, respectively. The strong rulers are presented in purple, while the weak rulers are presented in yellow. These networks and resource endowments are presented in Figure 4. Observe that as we delete links from clique to obtain the network with 27 links, the number of weak rulers increases strictly (from 2 to 3) and the same happens as we go move from network with 27 links to the network with 9 links (the number goes up from 3 to 4).

We compute the equilibrium payoffs in these examples;[17] the results are summarized in Figure 5.[18] The key point to note is that, even for $\gamma = 8$, the long run prospects of a ruler are essentially determined by whether he is strong or weak. Further study of examples that span a range of different values of γ reveal that this pattern is reinforced when we increase γ.

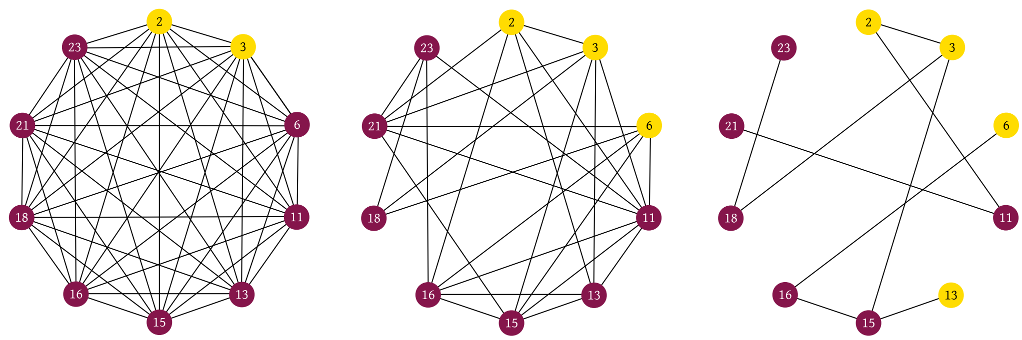


Figure 4: Examples of Networks

Given the importance of strong and weak rulers, we briefly comment on how changes in resources and links affect the set of strong and weak rulers. Given a resource profile, observe that adding links to a network offers all rulers potentially more sequences of attack. This means that a ruler who was weak may now have a strong sequence. Adding links (weakly) therefore expands the set of strong rulers. The number of strong rulers is maximized in the complete network and it is minimized when the strongest ruler is at the center of a star network. Given a network and a resource configuration, an increase in resources of a ruler either maintains his status or switches him from weak to strong. Observe that an increase in resources of a ruler may well lead to another ruler becoming weak. From Proposition 3 we can infer that additional resources for one ruler can make a big difference to his and others' long term prospects.

The discussion so far has focused on the difference between strong and weak rulers. We now argue that the network structure also shapes the relative prospects of different strong rulers. Consider an example with two strong rulers. Suppose the two rulers are 1 and 2, and they own vertices v_1 and v_2, respectively. The set of the remaining vertices, $V \setminus \{v_1, v_2\}$, can

[17] We would like to stress that all the computational examples in the paper are obtained by means of numerical calculations of equilibrium strategies and payoffs and not by simulations. This allows us to obtain much more accurate results.

[18] In the figures we present the relation between initial end expected equilibrium resources using scatter diagrams and we present the distribution of resources using Lorenz curves. For any $x \in [0, 100]$, a Lorenz curve represents the fraction of the total resources owned by poorest $x\%$ of the rulers.

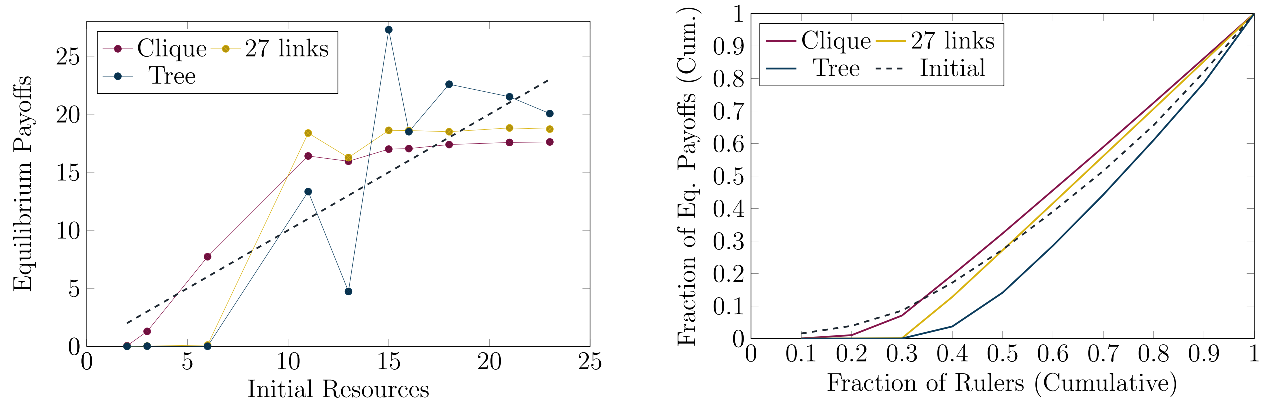


Figure 5: Equilibrium Payoffs and Lorenz Curves: $\gamma = 8$.

be partitioned into three sets: the set of nodes reachable from v_2 via v_1 only, denoted by U_1, the set of nodes reachable from v_1 via v_2 only, denoted by U_2, and the remaining nodes, U_{12} (c.f. Figure 6).

To see the effects of the networks structure easily, suppose that γ is large and that $r_{v_1} + R_{U_1} > r_{v_2} + R_{U_2} + R_{U_{12}}$. This ensures that ruler 1 remains strong as long as he is active. Ruler 2, on the other hand, becomes weak if ruler 1 accumulates enough resources from the set U_1. The probability of ruler 1 becoming a hegemon is approximately $1/2 + q$, where q is the probability that ruler 2 is picked to move before ruler 1 is picked to move *and* he is weak when that happens. Thus q is the probability that 1 conquers sufficiently many nodes before ruler 2 is picked to move. To fix ideas, suppose that 1 needs to acquire all the nodes in U_1 to become uniquely strong. Suppose $|U_1| = k$. If $G[U_1]$ is a fully disconnected network then q is approximately equal to $k!/(k+2)! = 1/((k+1)(k+2))$. If, on the other hand, $G[U_1]$ is a clique with $k-1$ strong rulers then q is approximately equal to $(k-1)(k+1)!/(2(k+2)!) = (k-1)/(2(k+2))$. As k gets large, the probability that ruler 1 becomes the hegemon converges to $1/2$ in the former case, and to 1 in the latter case.

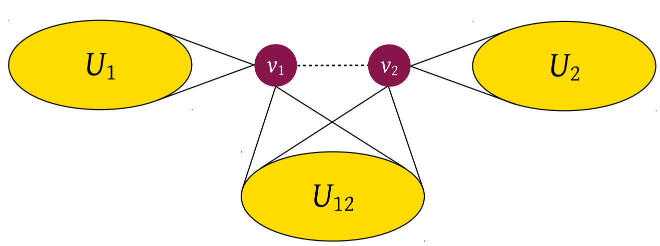


Figure 6: Partitioning of a Graph with Two Strong Rulers

4.1 Poor Rewarding Contest Success Functions

We begin by recalling that in the poor rewarding setting, every bilateral conflict is profitable to the poorer of the two opponents. However, the poor rewarding property also implies that rulers have a preference to wait before they fight. These two considerations suggest that the dynamics can be complicated. We are especially interested in the possibility of peace.

We start with noting that in equilibrium, at every ownership configuration, there is either peace or fight, regardless of the order in which the rulers are picked to move. Formally, given a strategy profile, $\boldsymbol{s}$, an ownership configuration $\mathbb{o} \in \mathbb{O}$ is *peaceful* under $\boldsymbol{s}$, if for all $i \in N$ and all $P \in 2^{N \setminus \{i\}}$, $s_i(\mathbb{o}, P)$ is the empty sequence. An ownership configuration $\mathbb{o} \in \mathbb{O}$ is *conflictual* under $\boldsymbol{s}$ if for every sequence $i_1, \ldots, i_n$ of rulers from N there exists $k \in \{1, \ldots, n\}$ such that $s_{i_k}(\mathbb{o}, \{i_1, \ldots, i_{k-1}\})$ is not empty. In other words, regardless of the order in which the rulers are picked to move at $\mathbb{o}$, one of the rulers chooses an attacking sequence.

By the observation above, the possibility of peace means that, in equilibrium, there exist ownership configurations, with two or more active rulers, at which all the rulers prefer staying peaceful to choosing fight. To make progress we divide the analysis into two parts: first, we characterize situations where peace is impossible, and second, we turn to situations where peace may be sustainable. We say that there is perpetual peace in a given strategy profile, if the initial state is peaceful. We say that there is war followed by peace in a given strategy profile if the initial state is not peaceful and no equilibrium outcome is hegemony.

Proposition 4. *Consider a generic poor rewarding contest success function that satisfies* (5)*.*

1. *For any connected network, G, and any generic resource endowment, $\boldsymbol{r} \in \mathbb{R}^V_{++}$, every ownership configuration $\mathbb{o} \in \mathbb{O}$ is either peaceful or conflictual in equilibrium.*

2. *For any connected network, G, any node $v \in V$, and any resource endowment of the other nodes, $\boldsymbol{r}_{-v}$, there exists a resource level $\tilde{r}_v$ such that for all $r_v > \tilde{r}_v$, there is fight till hegemony in equilibrium under resource endowment $(r_v, \boldsymbol{r}_{-v})$.*

3. *For any $n \geq 4$, there exists a network and a generic resource profile such that there is perpetual peace in equilibrium. Similarly, there exists a network and a generic resource profile such that there is war followed by peace in equilibrium.*

The proof of this result is presented in the Appendix. The result should be seen as a possibility result: it illustrates the rich range of outcomes possible under the poor rewarding contest success function. A comparison of Theorem 1 with Proposition 4 reveals contrasting optimal strategies (full attacking sequence versus no fighting) and outcomes (hegemony versus multiple kingdoms) and highlights the key role of the contest success function in shaping conflict dynamics. The hegemony result relies on quite different arguments than the hegemony result under rich rewarding contest success function. In the poor rewarding case, the existence of a sufficiently rich ruler motivates other rulers to fight. However, due to the waiting property, these rulers may choose to fight only if others do not. This is in contrast to the rich rewarding case, where each ruler chooses fight whenever he is given a chance. The peace and war followed by peace outcomes rely on the idea of fear of conflict escalation. We propose a network and a (generic) resource profile for which, whenever any ruler chooses to fight, there will be fight till hegemony in the following states and the ruler who started the conflict will be involved in all the following conflicts. The resource endowments are such that it is never profitable for any ruler to be involved in fight till hegemony starting from the initial state. The main challenge is to show that such a resource endowment exists, for general n.

We next examine how networks, resources, and the contest success function affect the prospects of peace. We consider Tullock contest success function with two values of γ: 0.05 (low) and 0.8 (high) and networks with 10 nodes (as in the rich rewarding case). In addition we consider eight ranges of resources: $[45, 55]$, $[40, 60]$, $[35, 65]$, $[30, 70]$, $[25, 75]$, $[20, 80]$, $[15, 85]$, $[10, 90]$. For each triple of γ, number of links, k, and resource range, $[a, b]$, we pick 1000 random samples of connected networks of k links with resources drawn uniformly from the set (of $10,000$ evenly spaced values from) $[a, b]$. Figure 7 presents the frequencies of samples exhibiting peace in the first round as a function of the resource range. It suggests that peace is more likely when resources are drawn from a smaller range: this is true for both high and low values of γ and true also across a wide range of networks. Taking together, Proposition 4 and our examples show that *resource equality is conducive for peace.*

This section concludes with an observation on equilibrium payoffs. We take up the same three networks as in the rich rewarding case (from Figure 4) and we fix the Tullock parameter γ to be equal to 0.05. Figure 8 presents the equilibrium payoffs and the Lorenz curves for the three networks and the initial resources. It is clear that, when γ is very small, the equilibrium dynamics are powerfully equalizing. A comparison with Figure 5 also reveals the big difference between the rich and poor rewarding setting: the poorer kingdoms gain significantly in the latter setting, and this is reflected at the aggregate level via the Lorenz curves.

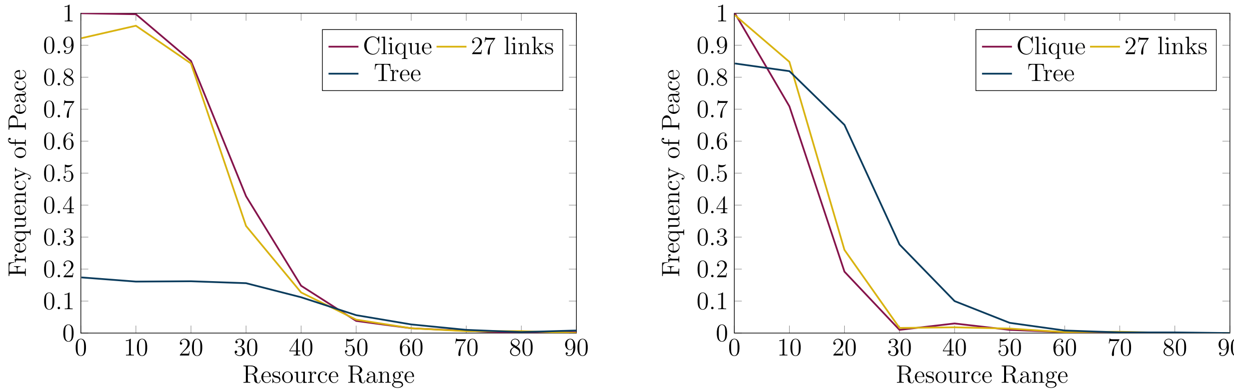


Figure 7: Frequency of Peace: $\gamma = 0.05$ (left), $\gamma = 0.8$ (right).

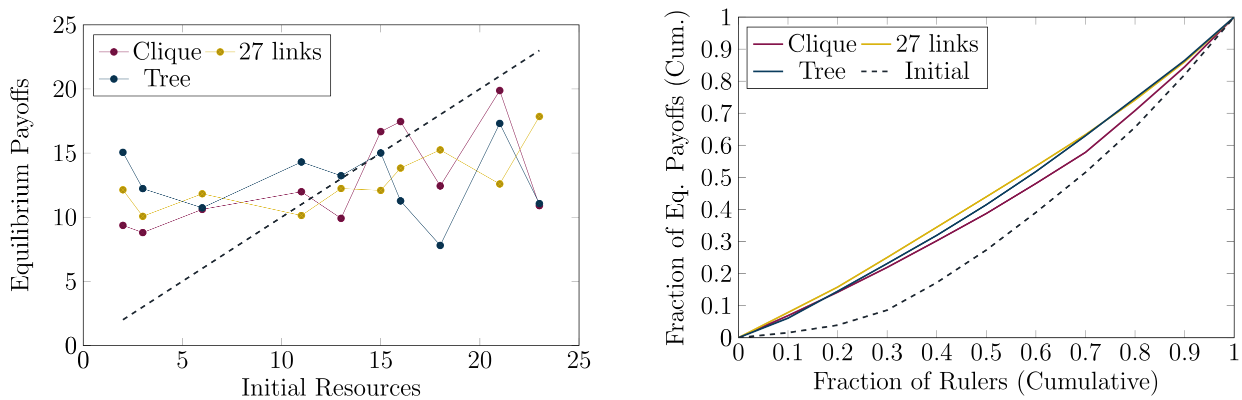


Figure 8: Equilibrium Payoffs and Lorenz Curves: $\gamma = 0.05$.

5 Extensions

In the basic model, in a war there is always a winner and a loser, the loser is eliminated, and all his resources are taken over by the winner. Moreover, the victor can employ his augmented resources to execute a long sequence of attacks against rivals, while they remain passive. In this section we relax these features of the model. The dynamics are now richer and this allows us to develop new insights. A general point that emerges is that the incentives to wage war remain strong. Allowing for gradual expansion reveals that conflict among rulers with small resources can be protracted and involves the exchange of small parcels of territories. However, once a ruler succeeds in expanding his territory, he gets more secure against being fully defeated, he gets stronger, and he reduces the number of fights needed to become a hegemon. This can lead rapidly to the emergence of a hegemony. These extensions suggest that the central results on expansion and hegemony continue to obtain in more realistic models.

5.1 Gradual Conquest

This section studies a model in which a winning ruler acquires constituent parts of the losing ruler's kingdom. As the losing ruler is not eliminated after one defeat, this creates the possibility that he may retaliate and recover the lost territories. The dynamics are now much richer: parcels of land may exchange hands for long periods of time. Moreover, as expansion is gradual, waiting to fight a larger ruler is no longer as attractive as it was in the basic model. The elimination of a ruler calls for a sequence of defeats that corresponds to the size of the ruler. We show that dynamics must converge to an absorbing state and that the absorbing state will exhibit hegemony, for all contest success functions satisfying properties A1-A3.

The model of conquest is modified as follows: A ruler picked to move chooses peace or chooses war against a neighbour. In case of war, he chooses a *link* between his own kingdom and the neighbouring kingdom. This link determines the launching node as well as the attacked node. The battle involves all the resources of the two rivals. If the attacker wins, he gains the attacked node and if he looses he looses the node from which the attack is launched. The picked ruler is allowed to attack neighbours until he looses all his nodes, or he chooses to stop, or there are no more neighbours left to attack. We now define the game formally.

As in the basic model, the game proceeds in rounds. Within a round, a state is given by the owner configuration and the set of rulers who have already been picked to fight. Ruler i, picked to move at state $(\mathbb{o}, P) \in \mathbb{O} \times 2^{N \setminus \{i\}}$, chooses a plan of conquest $\sigma : \mathbb{O} \to E \cup \{\varepsilon\}$. A plan of conquest is feasible if, for each $\mathbb{o} \in \mathbb{O}$, either $\sigma(\mathbb{o}) = \varepsilon$ (which means that i chooses to stop attacking) or $\sigma(\mathbb{o}) = uv$ such that $uv \in E(\mathbb{o})$, $\mathbb{o}(u) = i$ and $\mathbb{o}(v) = j \neq i$ (which means that i attacks vertex v of ruler j launching the attack from his own vertex, u). We call attack sequence *fully attacking* if for all $\mathbb{o}' \in \mathbb{O}$ with $|\mathrm{Act}(\mathbb{o}')| \geq 2$, $\sigma(\mathbb{o}') \neq \varepsilon$. A strategy of ruler i is a function $s_i : \mathbb{O} \times 2^{N \setminus \{i\}} \to (E \cup \{\varepsilon\})^{\mathbb{O}}$ such that for every ownership configuration, $\mathbb{o} \in \mathbb{O}$, and every set of rulers, $P \subseteq N \setminus \{i\}$, $s_i(\mathbb{o}, P)$ is a feasible plan of conquest for i at $\mathbb{o}$ in G. Given ruler i and graph G, the set of strategies of i is denoted by S_i and $\boldsymbol{S} = \prod_{i \in N} S_i$ denotes the set of strategy profiles.

It is possible to show that any strategy profile determines an absorbing Markov chain (this is done in the Online Appendix, Lemma 1). Given a strategy profile, $\boldsymbol{s}$, the ownership configurations at the absorbing states are called the outcomes of the game. For any ownership configuration, $\mathbb{o} \in \mathbb{O}$, set of rulers, $P \subseteq N$, a strategy profile $\boldsymbol{s} \in \boldsymbol{S}$, the probability that the outcome of the game is $\mathbb{o}'$, $F(\mathbb{o}' \mid \boldsymbol{s}, \mathbb{o}, P)$, is well defined. Starting at state at state

$(\mathbb{o}, P) \in \mathbb{O} \times 2^{N\setminus\{i\}}$, the expected payoff to ruler $i \in N$ from strategy profile $\boldsymbol{s} \in \boldsymbol{S}$ is

$$\Pi_i(\boldsymbol{s} \mid \mathbb{o}, P) = \sum_{\mathbb{o}' \in \mathbb{O}} F(\mathbb{o}' \mid \boldsymbol{s}, \mathbb{o}, P) R_i(\mathbb{o}').$$

Every ruler seeks to maximize his expected payoff. We consider (Markov Perfect) equilibrium of the game.

We establish the following result.

Proposition 5. *Let p be a contest success function satisfying axioms **A1**–**A3**.*

- *Consider any connected network with two active rulers. The equilibrium outcome is hegemony.*
- *Consider the complete network with three or more active rulers. The equilibrium outcome is hegemony.*

The proof is presented in the Online Appendix.

If there are only two active rulers then at least one of them has a strictly profitable – an improving – fully attacking plan of conquest. This is because a fully attacking plan of conquest for one ruler can be used by the other ruler as well (by replacing the roles of the attacked node and the node from which the attack is launched). Since a fully attacking plan of conquest leads to only two outcomes – hegemony of one of rulers – one of the rulers must find it improving.

Next consider three or more active rulers in the complete network. We establish that, at every state, at least one active ruler has an improving fully attacking plan of conquest. The claim is true for 2 rulers, from above. Suppose that the claim is true for $k-1$ active rulers. Consider a configuration with k rulers. In this configuration, we can remove all the nodes owned by one of the rulers, say i. The remaining network is also complete – a clique – with territories of other rulers unchanged. From the induction basis it follows that there is a ruler, say j, in this residual graph who has an improving plan of conquest. Either j finds it beneficial to continue and fully conquer ruler i, in the original network. In this case we are done. If this is not true then it must be true that i has an improving fully attacking plan of conquest. Observe that by assumption it is unprofitable for j to attack i after having captured all the other $k-2$ rulers. So it must be the case that i finds it profitable to attack j, when there are only two rulers i and j. The final step is to note that the payoffs to i from carrying out a fully attacking plan of conquest at the original ownership configuration are even larger, because he would then face weaker opponents than j. This follows from the assumption that the probability of winning is falling in the resources of the opponent (implied by assumptions **A1**–**A3**).

5.2 Short Attack Sequences

In the basic model, a ruler is allowed to choose a full attacking sequence of attacks. In particular, all other rivals remain passive, while this ruler executes this sequence. In this extension, we allow for rivals to have more opportunity to react and the goal of this section is to examine if our results are robust to this generalization.

We consider a variant of our model where rulers, when picked to move, can either choose peace or choose a sequence of attack of length 1 only, and then a new mover is drawn. A strategy of a ruler i is a function $s_i : \mathbb{O} \times 2^{N \setminus \{i\}} \to N \cup \{\varepsilon\}$ such that for every ownership configuration, $\mathbb{o} \in \mathbb{O}$, and every set of rulers, $P \subseteq N \setminus \{i\}$, $s_i(\mathbb{o}, P)$ is feasible at $\mathbb{o}$ in G, that is either $s_i(\mathbb{o}, P)$ is empty or $s_i(\mathbb{o}, P)$ consists of a neighbor of i under $\mathbb{o}$ in G. As the problem is especially relevant under the no-waiting property, in this discussion, we restrict attention to rich rewarding contest success functions. Notice that the proof of Proposition 1 can be adjusted in a straightforward way and so Proposition 1 is valid for the short-attack variant of the model. In particular, equilibrium existence and payoff equivalence of equilibria hold in this model as well.

First, we take up the setting with a unique strong ruler. This situation arises naturally if one ruler controls more than half of the resources. But the condition is significantly more general. Given any network, G, recall that a maximal set of nodes such that any two distinct nodes in the set are reachable from each other by a path in G is called a component in G. The set of all components of G is denoted by $\mathcal{C}(G)$. In addition, given a set of nodes, $U \subseteq V$, $G - U = G[V \setminus U]$ denotes the graph obtained by removing the nodes in U and all their links from G. A connected graph G with resource endowment $\boldsymbol{r}$ has a unique strong node if and only if there exists a node $v \in V$ such that for every component $C \in \mathcal{C}(G - \{v\})$, $r_v > R_C$.

Proposition 6. *Consider a rich rewarding contest success function that satisfies* (5)*. Suppose the network G is connected and a (generic) resource profile $\boldsymbol{r} \in \mathbb{R}^n_{++}$ is such that there is exactly one strong node. In equilibrium, at every ownership configuration, $\mathbb{o}$, at least one ruler attacks his neighbor. So the outcome is hegemony and the probability of becoming a hegemon is unique for every ruler.*

The proof is presented in the Online Appendix. The first observation is that if there is a unique strong ruler under some ownership configuration, then in every ownership configuration that follows in the course of the game, there is also a unique strong ruler. This is because no weak ruler can become strong, unless he fights and beats a strong ruler (in which case he becomes the unique strong ruler). Given this observation, we now show that at any state, for any strategy profile of the other rulers, the unique strong ruler increases his resources in expectation using the 'optimal attacking' strategy. We proceed by induction. For two rulers, which is the base step, the claim clearly holds. Assume now that the claim holds for k rulers. We show that the result holds for $k+1$ active rulers. This is because, due to the rich-rewarding contest success function, any fight between the strong ruler and any other ruler increases his resources in expectation and then, by the induction hypothesis, the expected resources at the end of the game are even higher. If any other two rulers fight, then the resources of the strong ruler remain unchanged in any following state and then, by the induction hypothesis, they increase. Thus, in any equilibrium there cannot be peace, because, at any ownership configuration, the strong ruler prefers to attack one of his neighbours over remaining peaceful.

We now examine the case of multiple strong rulers, with the help of examples. Consider the same three networks (with corresponding resources) as in the basic model (c.f. Figure 4) and assume that $\gamma = 8$. Equilibrium payoffs and Lorenz curves for the results are presented in Figure 9. By way of illustration, the figure contains also the corresponding outcomes for the basic model. In all the examples every ruler chooses to fight in every state and so the outcome is hegemony. As in the case of the basic model, the expected resources of the weak rulers are close to 0. There is, however, much greater variation across the strong rulers. Their equilibrium payoffs are more affected by the initial resource distribution, as compared to the basic model. The 'richest' ruler gains most from the dynamics and has much higher expected payoffs. The Lorenz curves confirm this point: the one-step dynamics lead to greater inequality than the dynamics in the basic model.

Next, we study the frequency of peace. Suppose again that $n = 10$ nodes. We run numerical calculations for $\gamma \in \{2, 4, 8, 16, 32\}$, number of links, $k \in \{9, 18, 27, 36, 45\}$, and resource ranges $[45, 55]$, $[40, 60]$, $[35, 65]$, $[30, 70]$, $[25, 75]$, $[20, 80]$, $[15, 85]$, and $[10, 90]$. For each combination of the three parameters we have drawn 1000 random samples, ensuring that there are at least two strong rulers. In each case we observe that there is fight till hegemony in equilibrium. Moreover, at every state (on and off the equilibrium path) all rulers chose fight. Taken together, Proposition 6 and these examples suggest that incessant warfare and the emergence of hegemony are robust features of the dynamics of appropriation in the rich

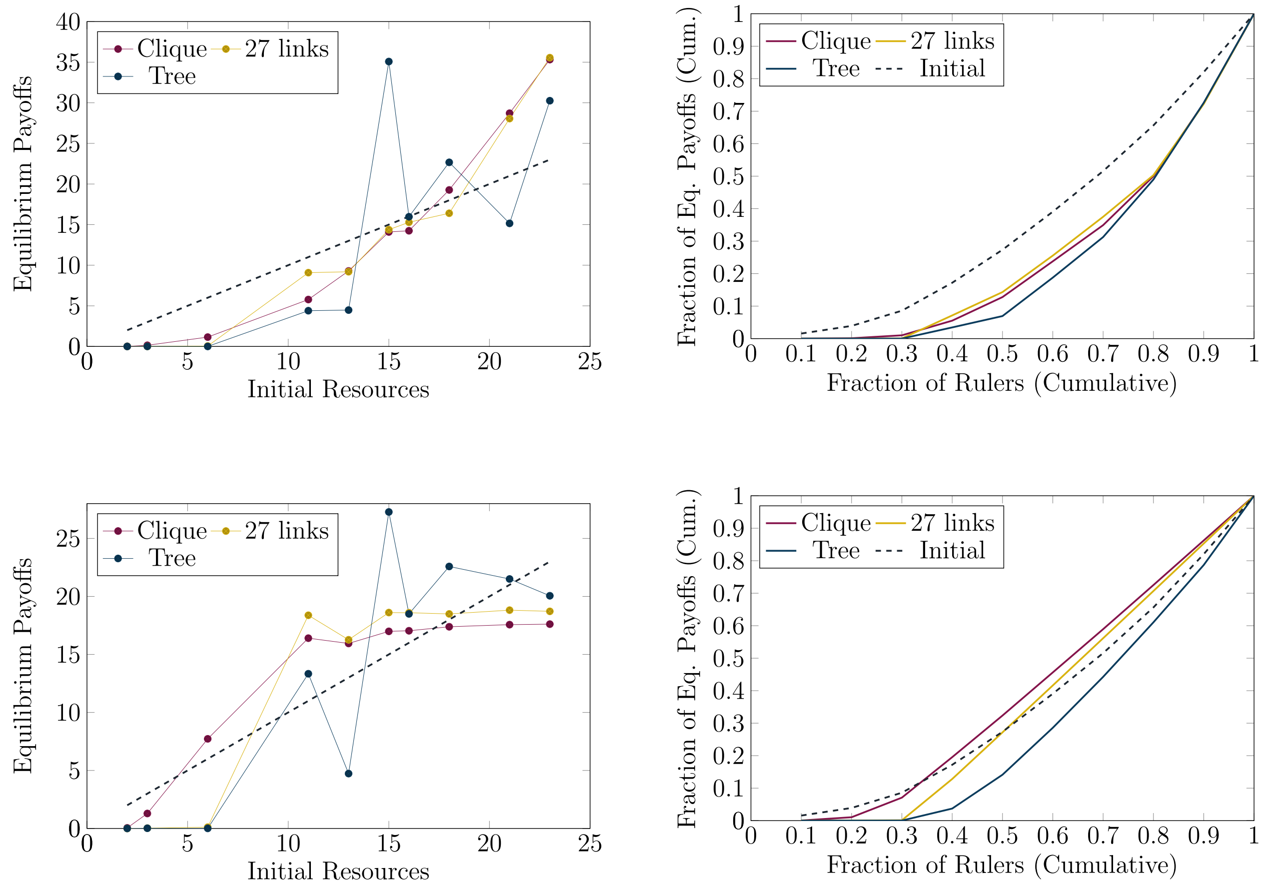


Figure 9: Equilibrium Outcomes $\gamma = 8$: short attacks model (top) and basic model (bottom).

rewarding setting.

6 Theory and History

The analysis suggests that the dynamics will exhibit incessant warfare. Once a ruler becomes dominant relative to his neighbours – either due to superior resource endowments or due to institutional or technological innovations – he will more easily expand his territory. Subsequent wars become decisive, and the speed of the expansion gathers pace. The size of empires is limited by the connectivity of the network. We now relate these prediction in relation to the developments leading to the creation of the First Chinese empire.

The First Chinese Empire in 221 BC: The discussion draws heavily on Lewis [2010] and Overy [2010]. In China, the years between 475 BC and 221 BC were characterized by almost uninterrupted warfare between seven major states which is referred to as the *Warring States Period*. The seven major kingdoms were Qin (located in the far west) the three Jins (located in the center on the Shanxi plateau; Han south along the Yellow River, Wei located in the middle, Zhao the northernmost of the three), Qi (centred on the Shandong Peninsula), Chu (with its core territory around the valleys of the Han River), and Yan (centered on modern-day Beijing). Initially, wars led to changes in the power of the different dynasties but all the kingdoms survived. However, from 320 BC to 221 BC, there was a major consolidation and by 221 BC, the Qin defeated all the other kingdoms and unified the entire area under one ruler, Qin Shi Huang. Figure 11 illustrates the dynamics and Figure 12 summarizes it.

We now relate, in some detail, our theoretical model the process of the emergence of this first empire. The *first* observation is that over a period stretching several hundred years, there was incessant warfare. The *second* observation is that in the first part of this period, from 475 BC to 360 BC, the armies were relatively small and the wars did not lead to the elimination of the major rulers. The *third* observation concerns the changes from 360 BC onward. The period after 360 BC witnessed major reforms of the Qin minister, Shuang Yang. After these reforms and the accompanying technological developments, the scale and violence in a war changed dramatically: now elimination of the losing ruler and conquest of his kingdom became much more likely, especially in a war between the Qin and one of the other warring states.

> ... the rise of Qin to dominance and its ultimate success in creating a unified empire depended on two major developments. First, under Shang Yang it achieved the most systematic version of the reforms that characterized the Warring States. These reforms entailed the registration and mobilization of all adult males for military service and the payment of taxes. While all Warring States were organized for war, Qin was unique in its extension of this pattern to every level of society, and in the manner in which every aspect of administration was devoted to mobilizing and provisioning its forces for conquest. (Lewis, 2010; page 38-39).

These reforms meant that the ruler had the resources – both in terms of army size and in terms of tax revenue – to wage large scale wars. Equipped with such a large army the Qin ruler was able to implement a long attacking sequence: in 230 BC, Qin conquered Han, the weakest of the Seven Warring States. In 225 BC, Qin conquered Wei, followed in 223 BC by the conquest of Chu.[19] Qin conquered Zhao and Yan in 222 BC. Finally, in 221 BC, Qin

[19]The size of the army was crucial in this contest: the first Qin invasion was a failure, when 200,000 Qin

turned its attention to the last surviving Warring State opponent: the Qi. In the face of the great threat Qi surrendered.

Our *fourth* observation concerns the order of attack: in line with the predictions of Proposition 2 for a rich rewarding technology, there was a tendency to attack the weaker states first before the stronger ones. Han, the weakest of the seven, was the first to fall. Qin's policy of attacking the nearby states and befriending the faraway states was partly determined by proximity and partly driven by the fact that Han and Wei were relatively weak, while Qi and Chu had the most resources. Yan was also a weak state and was the object of attack by Zhao and Qi. The table below provides the estimated size of armies during the late Warring States Period. The data is taken from Zhao and Xie [1988] (pages 18-19).[20]

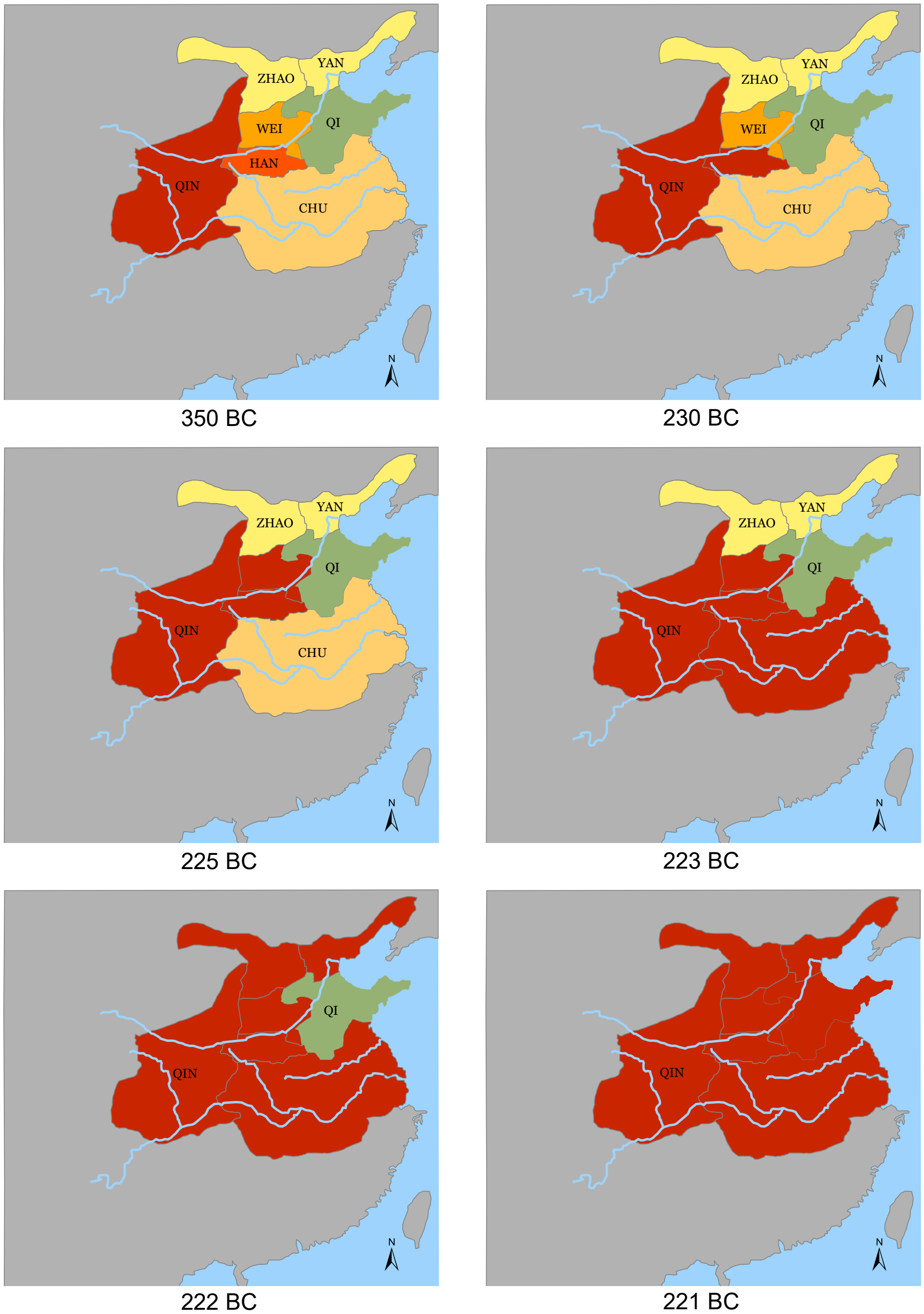
Kingdom	Size of Army	End Year
Qin	800,000	-
Chu	800,000	BC 223
Qi	600,000	BC 221
Zhao	500,000	BC 222
Wei	400,000	BC 225
Han	300,000	BC 230
Yan	300,000	BC 222

Table 1: Chinese Kingdoms: Army Size and End Year

Our *final* observation concerns the frontiers of the empire: the Qin empire was bounded by forests in the South, deserts and the Tibetan Plateau on the West, wasteland in the North and the Pacific Ocean in the East. These physical features, especially in the South, the West and the East, presented a physical constraint on further expansion. It is then possible to interpret China as a distinct 'component' of the world network, somewhat isolated from other parts of the world. The first Chinese Empire was a hegemon that was 'limited' by the connectivity of the physical contiguity network.

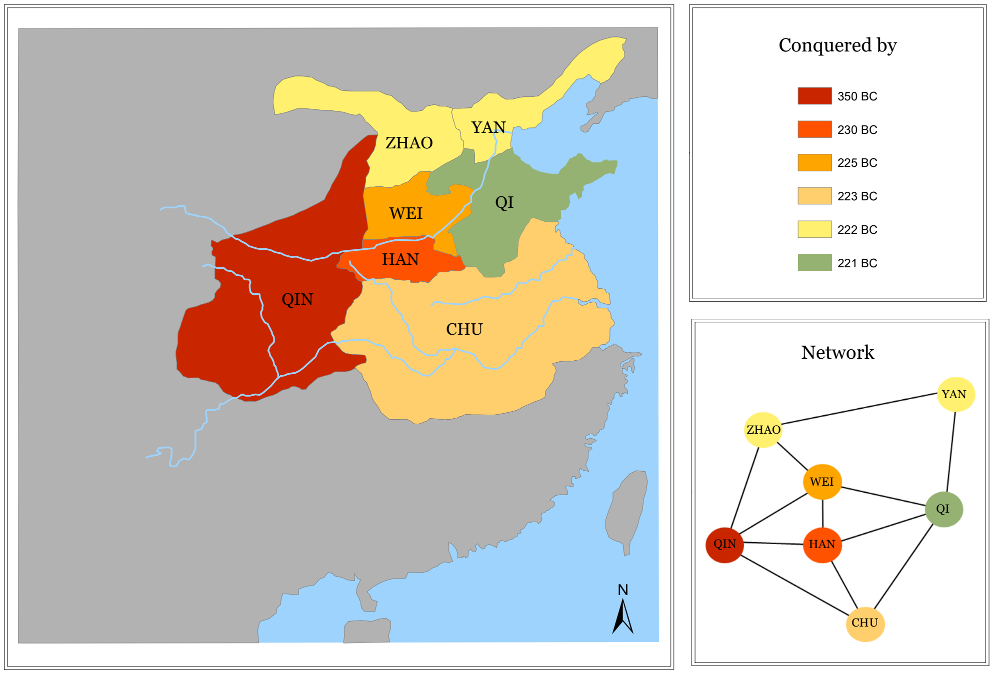
troops were defeated by a much larger Chu army with around 500,000 troops. The following year, Qin mounted a second invasion with 600,000 men and they defeated the Chu state. At their peak, the combined armies of Chu and Qin are estimated to have been in excess of a million soldiers.

[20] We thank Sng Tuan Hwee for providing us this data on army size in Early China.



Source: Overy, Richard (2010). Times Complete History of the World (8th Edition).

Figure 10: The First Chinese Empire: Dynamics



Source: Overy, Richard (2010). Times Complete History of the World (8th Edition).

Figure 11: The First Chinese Empire: Summary

7 Concluding Remarks

This paper develops a theoretical framework for the study of the incentives to wage war to conquer territory and resources. Our innovation is that we locate the dynamics of appropriation within a contiguity network. The analysis develops a number of results on the interplay between the technology of war, the resources of rulers, and contiguity, that illuminate the process of the formation of empires. In a setting where the contest functions are rich rewarding, starting from a situation with multiple kingdoms, the dynamics are characterized by incessant fighting. After an initial phase of uncertain and gradual growth, the pace of expansion of a 'kingdom' speeds up, and it grows rapidly through contiguous expansion. This expansion, and consequently the size of the empire, is limited by the connectivity of the network. These results provide a parsimonious account of the growth of major empires. We illustrate this through the case study of the First Chinese Empire.

The paper highlights the importance of network connections and the contest success functions. Rulers can alter both through strategic investments. In future work it would be interesting to incorporate these choices within a general framework. The model examines external constraints on the extent of the empire: it is clear that institutional arrangements play a role in defining the limits of empire.[21] This offers another avenue for further work.

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[21] For an early discussion of the role of institutions in empire building, see Polybius [2010].

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APPENDIX: PROOFS

EQUILIBRIUM EXISTENCE AND PAYOFF UNIQUENESS

Proof of Proposition 1. We start with introducing a natural partial order of precedence on the set of ownership configurations, $\mathbb{O}$, and on the set of states, $\mathbb{O} \times 2^N$. Given any two ownership configurations, $\mathbb{o} \in \mathbb{O}$ and $\mathbb{o}' \in \mathbb{O}$, $\mathbb{o} \sqsubseteq \mathbb{o}'$ if and only if for all $v \in V$, either $\mathbb{o}(v) = \mathbb{o}'(v)$ or $\mathbb{o}(v) \neq \mathbb{o}'(u)$, for all $u \in V$. Informally, if $\mathbb{o}$ and $\mathbb{o}'$ are ownership configurations such that $\mathbb{o}'$ is obtained from $\mathbb{o}$ by some rulers expanding their territories, then $\mathbb{o} \sqsubseteq \mathbb{o}'$. Given any two states, $(\mathbb{o}, P) \in \mathbb{O} \times 2^N$ and $(\mathbb{o}', P') \in \mathbb{O} \times 2^N$, $(\mathbb{o}, P) \preceq (\mathbb{o}', P')$ if and only if either $\mathbb{o} \sqsubseteq \mathbb{o}'$ or $\mathbb{o} = \mathbb{o}'$ and $P \subseteq P'$. Informally, if state $(\mathbb{o}, P)$ precedes state $(\mathbb{o}', P')$ in the course of the game, then $(\mathbb{o}, P) \preceq (\mathbb{o}', P')$. We will also use $\sqsubset$ and $\prec$ to denote the strict orders associated with the respective partial orders, defined above. Given an ownership configuration, $\mathbb{o} \in \mathbb{O}$, let $Succ(\mathbb{o}) = \{\mathbb{o}' \in \mathbb{O} : \mathbb{o} \sqsubset \mathbb{o}')\}$ be the set of all ownership configurations that $\mathbb{o}$ precedes. Let $\overline{Succ}(\mathbb{o}) = Succ(\mathbb{o}) \cup \{\mathbb{o}\}$. Similarly, given a state $(\mathbb{o}, P) \in \mathbb{O} \times 2^N$, let

$Succ(\mathbb{o}, P) = \{(\mathbb{o}', P') \in \mathbb{O} \times 2^N : (\mathbb{o}, P) \prec (\mathbb{o}', P')\}$ be the set of all states that $(\mathbb{o}, P)$ precedes, and let $\overline{Succ}(\mathbb{o}, P) = Succ(own, P) \cup \{(own, P)\}$.

Since $\mathbb{O}$ and $\mathbb{O} \times 2^N$ are finite, there exist maximal elements of $\sqsubseteq$ and $\preceq$. Take any strategy profile, $\boldsymbol{s}$, defined recursively on $\mathbb{O} \times 2^N$ starting from the maximal elements of $\preceq$ as follows. If $(\mathbb{o}, P)$ is such that $\mathbb{o}$ is maximal according to $\sqsubseteq$ (i.e. there is only one active ruler at $\mathbb{o}$) then, for all $i \in N$, $s_i(\mathbb{o}, P) = \varepsilon$ (the unique feasible sequence of i at $\mathbb{o}$). Otherwise, let $s_i(\mathbb{o}, P)$ be any sequence that maximises i's expected payoff given the continuation payoff determined by $\boldsymbol{s}$ defined on the states in $Succ(\mathbb{o}, P)$. Clearly any such strategy profile is well defined and is a Markov perfect equilibrium of the game. Moreover, given the Markov perfection requirement and since at each state there are no simultaneous moves (only one player is picked to make a choice), every Markov equilibrium is a strategy profile of the form defined above.

We now turn to showing payoff equivalence of equilibria. Take any two Markov perfect equilibria of the game, $\boldsymbol{s}$ and $\boldsymbol{s}'$, and suppose that they are not payoff equivalent. Let $(\mathbb{o}, P) \in \mathbb{O} \times 2^N$ be a maximal state, according to $\preceq$, such that there exists a ruler $i \in N \setminus P$ with $\Pi_i(\boldsymbol{s} \mid \mathbb{o}, P) \neq \Pi_i(\boldsymbol{s}' \mid \mathbb{o}, P)$. Suppose that $\Pi_i(\boldsymbol{s} \mid \mathbb{o}, P) > \Pi_i(\boldsymbol{s}' \mid \mathbb{o}, P)$ (the arguments for the inverse inequality are symmetric and omitted). Then i could strictly improve his payoff under $\boldsymbol{s}'$ by choosing a strategy s_i'' different to s_i' at state $(\mathbb{o}, P)$ only: $s_i''(\mathbb{o}, P) = s_i(\mathbb{o}, P)$. Since $(\mathbb{o}, P)$ is a maximal state, according to $\preceq$, for which $\Pi_i(\boldsymbol{s} \mid \mathbb{o}, P) \neq \Pi_i(\boldsymbol{s}' \mid \mathbb{o}, P)$, so for all states in $Succ(\mathbb{o}, P)$, $\boldsymbol{s}$ and $\boldsymbol{s}'$ yield the same payoff to i and the payoff to i at $(\mathbb{o}, P)$ depends on his resources at $(\mathbb{o}, P)$ and on his payoff at these states only. Thus $\Pi_i((\boldsymbol{s}'_{-i}, s_i'') \mid \mathbb{o}, P) = \Pi_i(\boldsymbol{s} \mid \mathbb{o}, P) > \Pi_i(\boldsymbol{s}' \mid \mathbb{o}, P)$, a contradiction with the assumption that $\boldsymbol{s}'$ is a Markov perfect equilibrium of the game. Hence for all $i \in N$ and $(\mathbb{o}, P) \in \mathbb{O} \times 2^{N \setminus \{i\}}$, $\boldsymbol{s}$ and $\boldsymbol{s}'$ must yield the same payoff to i. $\square$

INCENTIVES TO FIGHT

Proof of Proposition 2. We start with the rich rewarding case. The proof for the poor rewarding case is similar and omitted. Let p be a contest success function satisfying (5). Suppose that p is rich rewarding and take any $x, y \in \mathbb{R}_{++}$ such that $x > y$. Rewriting $(x+y)p(x,y) > x$, it is equivalent to $\frac{1}{1+\frac{f(y)}{f(x)}} > \frac{1}{1+\frac{y}{x}}$. Further, this is equivalent to $f(x)/x > f(y)/y$. Hence rich rewarding property is equivalent to $f(x)/x$ being strictly on $\mathbb{R}_{++}$, that is to f exhibiting increasing returns to scale.

We next turn to the timing results and consider the no-waiting case. The proof for the waiting case is similar and omitted. Let p be a contest success function satisfying (5). Suppose that p has the no-waiting property. Then, for any $x, y, z \in \mathbb{R}_{++}$,

$$\frac{f(x)}{f(x)+f(y)}\frac{f(x+y)}{f(x+y)+f(z)} > \frac{f(x)}{f(x)+f(y+z)}.$$

In particular, the inequality holds for $z = x$ so

$$\frac{f(x)}{f(x)+f(y)}\frac{f(x+y)}{f(x+y)+f(x)} > \frac{f(x)}{f(x)+f(x+y)}.$$

holds for any $x, y \in \mathbb{R}_{++}$. Dividing both sides by $f(x)$ and multiplying them by the denominators we get

$$f(x+y)(f(x)+f(x+y)) > (f(x)+f(y))(f(x+y)+f(x)).$$

Dividing both sides by $f(x)+f(x+y)$ yields

$$f(x+y) > f(x)+f(y).$$

Thus f is super-additive.

Next suppose that f is super-additive. Then, for any $y, z \in \mathbb{R}_{++}$,

$$f(y+z) > f(y)+f(z).$$

Multiplying both sides of the inequality above by $f(x+y)$, for any $x, y, z \in \mathbb{R}_{++}$,

$$f(x+y)f(y+z) > f(x+y)(f(y)+f(z)).$$

Moreover,

$$f(x+y)f(y+z) > f(x+y)f(y)+f(x+y)f(z) > f(x+y)f(y)+(f(x)+f(y))f(z).$$

Adding $f(x)f(x+y)$ to both sides we get

$$f(x+y)\left(f(x)+f(y+z)\right) > \left(f(x)+f(y)\right)\left(f(x+y)+f(z)\right).$$

This can be rewritten as

$$\frac{1}{f(x)+f(y)}\frac{f(x+y)}{f(x+y)+f(z)} > \frac{1}{f(x)+f(y+z)}.$$

Multiplying both sides by $f(x)$ we get

$$\frac{f(x)}{f(x)+f(y)}\frac{f(x+y)}{f(x+y)+f(z)} > \frac{f(x)}{f(x)+f(y+z)}.$$

To complete the argument, we show that increasing returns to scale imply super-additivity and that decreasing returns to scale imply sub-additivity. Suppose that f has increasing returns to scale. So $f(x)/x$ is strictly increasing on $\mathbb{R}_{++}$. For any $x, y \in \mathbb{R}_{++}$,

$$xf(x+y) > (x+y)f(x) \text{ and } yf(x+y) > (x+y)f(y).$$

Adding the two inequalities and dividing both sides by $x+y$ we get $f(x+y) > f(x)+f(y)$, that is f is strictly super-additive. The arguments for decreasing returns are similar and omitted.

By what was shown above, rich rewarding property of p implies that f exhibits increasing returns to scale which, in turn, implies that f is super-additive and, further, that p has the no-waiting property. By similar argument, poor rewarding property implies the waiting property.

Finally we turn to the order of attack result. We provide the proof for the poor-first case. The arguments for the rich-first case are similar and omitted. Let $x, y, z \in \mathbb{R}_{++}$ with $y > z$ and suppose that $p(x,y)p(x+y,z) > p(x,z)p(x+z,y)$. This may be rewritten as:

$$\frac{f(x)}{f(x)+f(y)}\frac{f(x+y)}{f(x+y)+f(z)} > \frac{f(x)}{f(x)+f(z)}\frac{f(x+z)}{f(x+z)+f(y)}$$

Dividing both sides by $f(x)$ and multiplying them by the denominators, we get

$$f(x+y)(f(x)+f(z))(f(x+z)+f(y)) > f(x+z)(f(x)+f(y))(f(x+y)+f(z))$$

This is equivalent to

$$f(y)f(x+y)(f(x)+f(z)) + f(x)f(x+z)f(x+y) + f(z)f(x+z)f(x+y) >$$
$$f(z)f(x+z)(f(x)+f(y)) + f(x)f(x+y)f(x+z) + f(y)f(x+y)f(x+z)$$

Subtracting $f(x)f(x+z)f(x+y) + f(z)f(x+z)f(x+y) + f(y)f(x+y)f(x+z)$ from both sides this is equivalent to

$$f(y)f(x+y)(f(x)+f(z)) - f(y)f(x+y)f(x+z) >$$
$$f(z)f(x+z)(f(x)+f(y)) - f(z)f(x+z)f(x+y)$$

Reorganizing and multiplying both sides by -1, this is equivalent to

$$f(z)f(x+z)(f(x+y)-(f(x)+f(y))) > f(y)f(x+y)(f(x+z)-(f(x)+f(z))).$$

Dividing both sides by $(f(x+y)-(f(x)+f(y)))(f(x+z)-(f(x)+f(z)))$, this is equivalent to

$$\frac{f(z)f(x+z)}{f(x+z)-f(x)-f(z)} > \frac{f(y)f(x+y)}{f(x+y)-f(x)-f(y)}.$$

This completes the proof. □

The timing part of Proposition 2 can be generalized to arbitrary sequences of fights: with poor-first property attacking opponents in increasing order with respect to their resources is optimal, while with rich-first property attacking them in the reversed order is optimal. This is stated in the corollary below.

Corollary 2. *Let $m \geq 3$ and $x_0, x_1, \ldots, x_m \in \mathbb{R}_{++}$, be such that $x_1 < \ldots < x_m$. Then, for any permutation $\pi : \{1, \ldots, m\} \to \{1, \ldots, m\}$,*

$$p_{\mathrm{seq}}\left(x_0, x_{\pi(1)}, \ldots, x_{\pi(m)}\right) \leq \begin{cases} p_{\mathrm{seq}}(x_0, x_1, \ldots, x_m), & \text{if } p \text{ has poor-first property,} \\ p_{\mathrm{seq}}(x_0, x_m, \ldots, x_1), & \text{if } p \text{ has rich-first property.} \end{cases}$$

with equality only if the permutations on both sides are the same.

Proof. We provide the proof for the poor-first property. The proof for the rich-first property is similar and omitted. Assume p has poor-first property. Let $\pi : \{1, \ldots, m\} \to \{1, \ldots, m\}$ be a permutation of $\{1, \ldots, m\}$. A pair of indices $(i, j) \in \{1, \ldots, m\}$ such that $i < j$ and $\pi(i) > \pi(j)$

is called an *inverse* of π. We will show that for any permutation π of $\{1,\ldots,m\}$ with at least one inverse there exists a permutation π' of $\{1,\ldots,m\}$ with less inverses that yields higher p_{seq}: $p_{\text{seq}}(x_0,x_{\pi(1)},\ldots,x_{\pi(m)}) < p_{\text{seq}}(x_0,x_{\pi'(1)},\ldots,x_{\pi'(m)})$. Since the identity is the unique permutation of $\{1,\ldots,m\}$ with no inverses, this implies the proposition. Throughout the proof, given a permutation π and $j \in \{1,\ldots,m\}$ we will use $X_{\pi(j)}$ to denote $\sum_{l=1}^{j} x_{\pi(l)}$.

So take any permutation π on $\{1,\ldots,m\}$ with at least one inverse, (i,j). Then there exists $i \le k < j$ such that $(k,k+1)$ is also an inverse of π. Let π' be a permutation of $\{1,\ldots,m\}$ obtained from π by exchanging $\pi(k)$ and $\pi(k+1)$, i.e. $\pi'(k)=\pi(k+1)$, $\pi'(k+1)=\pi(k)$, and $\pi'(l)=\pi(l)$ for $l \in \{1,\ldots,m\}\setminus\{k,k+1\}$. There is at least one inverse less in π' than in π. Moreover, $p_{\text{seq}}\left(x_0,x_{\pi'(1)},\ldots,x_{\pi'(m)}\right) = p_{\text{seq}}\left(x_0,x_{\pi'(1)},\ldots,x_{\pi'(k-1)}\right)\cdot p\left(X_{\pi'(k-1)},x_{\pi'(k)}\right)\cdot p\left(X_{\pi'(k)},x_{\pi'(k+1)}\right)\cdot p_{\text{seq}}\left(X_{\pi'(k+1)},x_{\pi'(k+2)},\ldots,x_{\pi'(m)}\right) = p_{\text{seq}}\left(x_0,x_{\pi(1)},\ldots,x_{\pi(k-1)}\right)\cdot p\left(x_{\pi'(k-1)},x_{\pi'(k)}\right)\cdot p\left(X_{\pi'(k)},x_{\pi'(k+1)}\right)\cdot p_{\text{seq}}\left(x_{\pi(k+1)},x_{\pi(k+2)},\ldots,x_{\pi(m)}\right)$. By poor-first property, this is greater than $p_{\text{seq}}\left(x_0,x_{\pi(1)},\ldots,x_{\pi(k-1)}\right)\cdot p\left(x_{\pi(k-1)},x_{\pi(k)}\right)\cdot p\left(X_{\pi(k)},x_{\pi(k+1)}\right)\cdot p_{\text{seq}}\left(X_{\pi(k+1)},x_{\pi(k+2)},\ldots,x_{\pi(m)}\right) = p_{\text{seq}}\left(x_0,x_{\pi(1)},\ldots,x_{\pi(m)}\right)$. This completes the proof. □

CONQUEST AND EMPIRE

We start by noting that the waiting and no-waiting properties extend to sequences of arbitrary length. Formally, let $m \ge 3$, $x_1,\ldots,x_m \in \mathbb{R}_{++}$, and $1 \le i < j \le m$ such that $i \ne 1$ or $j \ne m$. If p has the no-waiting property, then

$$p_{\text{seq}}(x_1,\ldots,x_{i-1},x_i,\ldots,x_j,x_{j+1},\ldots,x_m) > p_{\text{seq}}\left(x_1,\ldots,x_{i-1},\sum_{l=i}^{j} x_l,x_{j+1},\ldots,x_m\right) \tag{8}$$

Proof of Theorem 1. The proof proceeds in three steps.

Step 1: Fix some state ϕ with $|\text{Act}(\phi)| \ge 2$. For a strong ruler i, the optimal full attacking sequence maximizes his payoffs across all attacking sequences. Moreover, in generic case, it is a unique maximizer.

Let ϕ be a state with $|\text{Act}(\phi)| = m \ge 2$. Take an active ruler $j_0 \in \text{Act}(\phi)$ with maximal amount of resources $R_{j_0}(\phi)$. For generic resource values, such a ruler is unique. Pick a full attacking sequence $j_1,\ldots,j_{m-1}$ consisting of rulers in $\text{Act}(\phi)\setminus\{j_0\}$ that is feasible for j_0 in G under ϕ (clearly such a sequence exists because G is connected). Since j_0 has maximal amount of resources so, for all $1 \leq k \leq m-1$, we have

$$\sum_{l=0}^{k-1} R_{j_l}(\phi) \geq R_{j_k}(\phi). \tag{9}$$

The expected payoff to ruler j_0 from the attacking sequence is

$$\begin{aligned} \pi_{j_0}(\phi \mid j_1, \ldots, j_{m-1}) &= \left(\sum_{l=0}^{m-1} R_{j_l}(\phi) \right) \prod_{k=1}^{m-1} p \left(\sum_{l=0}^{k-1} R_{j_l}(\phi), R_{j_k}(\phi) \right) \\ &= R_{j_0}(\phi) \prod_{k=1}^{m-1} p \left(\sum_{l=0}^{k-1} R_{j_l}(\phi), R_{j_k}(\phi) \right) \left(\frac{\sum_{l=0}^{k} R_{j_l}(\phi)}{\sum_{l=0}^{k-1} R_{j_l}(\phi)} \right). \end{aligned} \tag{10}$$

Since p is rich rewarding, so

$$p \left(\sum_{l=0}^{k-1} R_{j_l}(\phi), R_{j_k}(\phi) \right) \left(\frac{\sum_{l=0}^{k} R_{j_l}(\phi)}{\sum_{l=0}^{k-1} R_{j_l}(\phi)} \right) \geq 1, \tag{11}$$

with equality only if $k = 1$ and $R_{j_0}(\phi) = R_{j_1}(\phi)$.

At every step in the sequence, the expected resources are growing. So, for generic resource values, there is a full attacking sequence that dominates any partial attacking sequence. By definition, the optimal full attacking sequence maximizes payoffs across all attack sequences.

The first step has a powerful implication: in any state with 2 or more active rulers there is at least one ruler who has a strict incentive to attack, given that other rulers do not attack. Hence, in equilibrium, there must exist a hegemon.

In the dynamic game, in principle, a strong ruler may prefer to wait and allow others to move and then attack later. The next step shows that an optimal full attacking sequence dominates all such waiting strategies.

Step 2: Fix some state ϕ with $|\mathrm{Act}(\phi)| \geq 2$ and a set or rulers, P. For any ruler $i \in N \setminus P$ strong at ϕ, an optimal full attacking sequence is a dominant choice at (ϕ, P). Moreover, the choice is strictly dominant if $|\mathrm{Act}(\phi)| \geq 3$.

Fix some state ϕ. Let $\sigma_i(\phi)$ be the optimal sequence of ruler i at ϕ, assuming that the game ends after i executes the sequence (successfully or not). In other words, $\sigma_i(\phi)$ is the myopic optimal sequence of ruler i at ϕ. Notice that this sequence is independent of the set of rulers who chose peace prior to i's move at a round at the state ϕ. Let $\bar{\pi}_i(\phi) = \pi_i(\phi \mid \sigma_i(\phi))$ denote the optimal myopic payoff ruler i can attain at ϕ.

Claim. *The optimal myopic payoff is the highest that ruler i can hope to attain, i.e., for any state, ϕ, and any set of rulers, $P \subseteq N \setminus \{i\}$, $\bar{\pi}_i(\phi) \geq \Pi_i(\boldsymbol{s} \mid \phi, P)$ for any feasible strategy profile $\boldsymbol{s}$. Moreover, if i is strong and there are at least three active rulers, then the inequality is strict.*

The proof is by induction on the number of active rulers. For the induction basis, we show that the claim holds for 2 active rulers. If i is the richer ruler then, from the rich rewarding property, his myopic optimal strategy is to attack. It is also clear that attacking yields strictly higher payoffs if the other ruler does not attack, and weakly higher payoffs if the other ruler does attack. If i is the poorer ruler then not attacking is the optimal myopic strategy. In case the richer ruler attacks, the expected payoff to i is less due to the rich rewarding property. That completes the argument for 2 active rulers.

For the induction step, suppose that the claim holds for all $y \leq X$, where $X \geq 2$, active rulers: we will show that it also holds for $X + 1$ active rulers. Given state ϕ', set of rulers, P', and strategy profile, $\boldsymbol{s}'$, we will use $\text{Atck}(\boldsymbol{s}', \phi', P')$ to denote the set of rulers choosing attack at (ϕ', P') under $\boldsymbol{s}'$, i.e. $\text{Atck}(\boldsymbol{s}', \phi', P') = \{j \in N \setminus P' : s'_j(\phi', P') \neq \varepsilon\}$.[22] Fix some state ϕ with $X + 1$ active rulers and a set of rulers, P such that $\text{Act}(\phi) \setminus P \neq \varnothing$. Take an active ruler $i \in \text{Act}(\phi) \setminus P$ and any strategy profile $\boldsymbol{s}$. If for all $P' \subseteq N$ such that $P \subseteq P'$, $\text{Atck}(\boldsymbol{s}, \phi, P') = \varnothing$, i.e. all players choose peace following P at ϕ, then the claim follows, because $\sigma_i(\phi)$ is at least as good as the empty sequence at ϕ:

$$\bar{\pi}_i(\phi) \geq \pi_i\left(\phi \mid \sigma_i(\phi)\right) \geq \pi_i\left(\phi \mid \varepsilon\right) = \Pi_i(\boldsymbol{s} \mid \phi, P). \tag{12}$$

Moreover, by Step 1, the inequality is strict if i is strong.

For the remaining part of the argument assume that there exists $P' \subseteq N$ with $P \subseteq P'$ such that $\text{Atck}(\boldsymbol{s}, \phi, P') \neq \varnothing$. We will establish that $\bar{\pi}_i(\phi) \geq \Pi_i(\boldsymbol{s} \mid \phi, P)$. Given a set of rulers P' and ruler $j_0 \in N \setminus P'$ such that $s_{j_0}(\phi, P') \neq \varepsilon$, let $\Pi_i(\boldsymbol{s} \mid \phi, P', j_0)$ denote the expected payoff to ruler i from strategy profile $\boldsymbol{s}$ conditional on ruler j_0 being selected to move at (ϕ, P') and

[22] Throughout the proofs we use the standard notation, ε, to denote empty sequences.

$q(j_0, P' \mid \mathbb{o}, \boldsymbol{s})$ denote the probability that j_0 is picked after P' at $\mathbb{o}$ under $\boldsymbol{s}$. Then

$$\begin{aligned}\Pi_i(\boldsymbol{s} \mid \mathbb{o}, P) = & \sum_{P' \subseteq N \text{ s.t. } P \subseteq P'} \sum_{j_0 \in \mathrm{Atck}(\boldsymbol{s}, \mathbb{o}, P')} q(j_0, P' \mid \mathbb{o}, \boldsymbol{s}) \Pi_i \left(\boldsymbol{s} \mid \mathbb{o}, P', j_0\right) + \\ & \left(1 - \sum_{P' \subseteq N \text{ s.t. } P \subseteq P'} \sum_{j_0 \in \mathrm{Atck}(\boldsymbol{s}, \mathbb{o}, P')} q(j_0, P' \mid \mathbb{o}, \boldsymbol{s})\right) \pi_i \left(\mathbb{o} \mid \varepsilon\right) \end{aligned} \tag{13}$$

As we established above, $\bar{\pi}_i(\mathbb{o}) \geq \pi_i(\mathbb{o} \mid \varepsilon)$, with strict inequality if i is strong. Thus to show the claim, it is enough to show that

$$\bar{\pi}_i(\mathbb{o}) \geq \Pi_i \left(\boldsymbol{s} \mid \mathbb{o}, P', j_0\right), \tag{14}$$

for each $P' \subseteq N$ with $P \subseteq P'$ and each attacking ruler $j_0 \in \mathrm{Atck}(\boldsymbol{s}, \mathbb{o}, P')$, with strict inequality for at least one P' and $j_0 \in \mathrm{Atck}(\boldsymbol{s}, \mathbb{o}, P')$ in the case of i being strong.

So take any set of rulers, $P' \subseteq N$ with $P \subseteq P'$ and any ruler $j_0 \in \mathrm{Atck}(\boldsymbol{s}, \mathbb{o}, P')$. Three cases are possible:

(i). $j_0 \neq i$ and i is not in the attacking sequence $s_{j_0}(\mathbb{o})$ of j_0,

(ii). $j_0 \neq i$ and i is in the attacking sequence $s_{j_0}(\mathbb{o})$ of j_0,

(iii). $j_0 = i$.

Case (i). Ruler j_0 is different to i and does not have i in his attacking sequence $s_{j_0}(\mathbb{o})$. Let $F(\mathbb{o}' \mid \boldsymbol{s}, \mathbb{o}, P', j_0)$ be the probability of reaching ownership state $\mathbb{o}'$ in the next round from state $\mathbb{o}$ under strategy profile $\boldsymbol{s}$ when j_0 is selected to move after P' (and executes attacking sequence $s_{j_0}(\mathbb{o}, P')$). Then

$$\Pi_i \left(\boldsymbol{s} \mid \mathbb{o}, P', j_0\right) = \sum_{\mathbb{o}' \in \mathbb{O}} F\left(\mathbb{o}' \mid \boldsymbol{s}, \mathbb{o}, P', j_0\right) \Pi_i(\boldsymbol{s} \mid \mathbb{o}'). \tag{15}$$

To show (14) it is enough to show that

$$\bar{\pi}_i(\mathbb{o}) \geq \Pi_i(\boldsymbol{s} \mid \mathbb{o}') = \Pi_i(\boldsymbol{s} \mid \mathbb{o}', \varnothing), \tag{16}$$

for each state $\mathbb{o}'$ that can be reached in the next round with positive probability from $\mathbb{o}$ when j_0 plays the attacking sequence $s_{j_0}(\mathbb{o}, P')$ after P' at $\mathbb{o}$. We will show that the inequality is strict when i is strong.

Ownership state $\mathbb{o}'$ is reached after at least one fight and so has at most X active rulers. Hence, by the induction hypothesis, $\bar{\pi}_i(\mathbb{o}') \geq \Pi_i(\boldsymbol{s} \mid \mathbb{o}', \varnothing)$, and so to show (16) it is enough to show that

$$\bar{\pi}_i(\mathbb{o}) \geq \bar{\pi}_i(\mathbb{o}'). \tag{17}$$

Take an optimal myopic sequence, $\sigma_i(\mathbb{o}')$, of i at $\mathbb{o}'$. There are two sub-cases to be considered.

(a) Sequence $\sigma_i(\mathbb{o}')$ does not contain the rulers in the sequence of fights that leads to $\mathbb{o}'$. This means, in particular, that $\sigma_i(\mathbb{o}')$ is not a full attacking sequence. Hence, by Step 1, i is not strong.

Since $\sigma_i(\mathbb{o}')$ does not contain the rulers in the sequence of fights that leads to $\mathbb{o}'$, it can be executed at state $\mathbb{o}$. By optimality of $\sigma_i(\mathbb{o})$ at $\mathbb{o}$

$$\bar{\pi}_i(\mathbb{o}) = \pi_i\left(\mathbb{o} \mid \sigma_i(\mathbb{o})\right) \geq \pi_i\left(\mathbb{o} \mid \sigma_i(\mathbb{o}')\right) = \pi_i\left(\mathbb{o}' \mid \sigma_i(\mathbb{o}')\right) = \bar{\pi}_i(\mathbb{o}'). \tag{18}$$

(b) Sequence $\sigma_i(\mathbb{o}')$ contains at least one ruler in the sequence of fights that leads to $\mathbb{o}'$. This is true, in particular, when i is strong because, by Step 1, $\sigma_i(\mathbb{o}')$ must be a full attacking sequence then.

Since $\sigma_i(\mathbb{o}')$ contains at least one ruler in the sequence of fights that leads to $\mathbb{o}'$, so $\sigma_i(\mathbb{o}') = \sigma_i^1(\mathbb{o}'), k, \sigma_i^2(\mathbb{o}')$, where k is the ruler who won the sequence of fights leading to $\mathbb{o}'$. We can construct a sequence $\sigma' = \sigma_i^1 \tau \sigma_i^2$ that is feasible for i at $\mathbb{o}$, with τ being a sequence of rulers involved in the sequence of fights leading to $\mathbb{o}'$. By point 1 of Proposition 2 p has the no-waiting property. As we observed prior to the proof of the theorem, the no-waiting property extends to sequences of fights of arbitrary length – (8). Given this observation, σ' yields a strictly higher payoff than $\sigma_i(\mathbb{o}')$. By construction, $\sigma_i(\mathbb{o})$ is an optimal myopic strategy for i at $\mathbb{o}$ and so payoff dominates σ' at $\mathbb{o}$. Hence

$$\bar{\pi}_i(\mathbb{o}) = \pi_i\left(\mathbb{o} \mid \sigma_i(\mathbb{o})\right) \geq \pi_i(\mathbb{o} \mid \sigma') > \pi_i\left(\mathbb{o}' \mid \sigma_i(\mathbb{o}')\right) = \bar{\pi}_i(\mathbb{o}'). \tag{19}$$

Hence (17) and, consequently, (16) hold with strict inequality.

Case (ii). Ruler j_0 is different to i and has i in his attacking sequence $s_{j_0}(\mathbb{o})$. Let $s_{j_0}(\mathbb{o}) = j_1, \ldots, j_m$ be the sequence selected by j_0 at $\mathbb{o}$ under strategy s_{j_0}. Then $i = j_k$ for some $1 \leq k \leq m$. Given $l \in \{1, \ldots, m\}$, let $\mathbb{o}^l$ be the state reached after j_0 looses the l'th fight in the sequence. The expected payoff to i from $\boldsymbol{s}$ at $\mathbb{o}$ given that j_0 is selected to move after set

P' or rulers is equal to

$$\Pi_i\left(\boldsymbol{s} \mid \mathbb{O}, P', j_0\right) = \sum_{l=1}^{k-1} F\left(\mathbb{O}^l \mid \boldsymbol{s}, \mathbb{O}, P', j_0\right) \Pi_i\left(\boldsymbol{s} \mid \mathbb{O}^l\right) + \\ \left(1 - \sum_{l=1}^{k-1} F\left(\mathbb{O}^l \mid \boldsymbol{s}, \mathbb{O}, P', j_0\right)\right) p\left(r_i(\mathbb{O}), \sum_{l=0}^{k-1} r_{j_l}(\mathbb{O})\right) \Pi_i\left(\boldsymbol{s} \mid \mathbb{O}^k\right), \tag{20}$$

where $j_1, \ldots, j_{k-1}$ are the rulers attacked by j_0 prior to attacking i.

Hence to show (14) it is enough to show that (16) holds for all $\mathbb{O}' = \mathbb{O}^l$, $l \in \{1, \ldots, k-1\}$, reachable after a sequence of fights of j_0 in which j_0 looses before facing i, and that

$$\bar{\pi}_i(\mathbb{O}) \geq p\left(r_i(\mathbb{O}), \sum_{l=0}^{k-1} r_{j_l}(\mathbb{O})\right) \Pi_i\left(\boldsymbol{s} \mid \mathbb{O}^k\right). \tag{21}$$

holds for $\mathbb{O}^k$, reachable by a sequence of fights of j_0 in which i is attacked by j_0 and wins. (16) is shown by the same arguments as in point (ii) above. In particular, the inequality in (16) is strict when i is strong. For (21), let τ be a sequence of rulers $\{j_0, \ldots, j_{k-1}\}$ feasible to i at $\mathbb{O}$ (clearly such a sequence exists). Then sequence $\sigma' = \tau\sigma_i(\mathbb{O}^k)$, consisting of τ and an optimal myopic sequence of i at $\mathbb{O}^k$, is feasible for i at $\mathbb{O}$. By the no-waiting property and its generalization, (8), τ yields at least the same payoff to i as the sequence of fights that leads to $\mathbb{O}'$ (the inequality is strict, unless $k = 1$). Combining this with the induction hypothesis we get

$$\bar{\pi}_i(\mathbb{O}) \geq \pi_i(\mathbb{O} \mid \tau\sigma_i(\mathbb{O}^k)) \geq p\left(R_i(\mathbb{O}), \sum_{l=0}^{k-1} R_{j_l}(\mathbb{O})\right) \pi_i\left(\mathbb{O}^k \mid \sigma_i\left(\mathbb{O}^k\right)\right) \\ \geq p\left(R_i(\mathbb{O}), \sum_{l=0}^{k-1} R_{j_l}(\mathbb{O})\right) \Pi_i\left(\boldsymbol{s} \mid \mathbb{O}^k\right), \tag{22}$$

with strict inequality, unless $k = 1$.

Case (iii). Ruler i is picked to move at $\mathbb{O}$ after P'. The strategy chosen by i under strategy profile $\boldsymbol{s}$ at $(\mathbb{O}, P')$ is $s_i(\mathbb{O}, P')$. Let $\mathbb{O}'$ be the state that is reached if i wins all the attacks in sequence $s_i(\mathbb{O}, P')$. Then sequence $\sigma' = s_i(\mathbb{O}, P')\sigma_i(\mathbb{O}')$, consisting of $s_i(\mathbb{O}, P')$ and an optimal myopic sequence of i at $\mathbb{O}'$, is feasible for i at $\mathbb{O}$. State $\mathbb{O}'$ is reached after at least one fight and has at most X active rulers. By the induction hypothesis, $\bar{\pi}_i(\mathbb{O}') \geq \Pi_i(\boldsymbol{s} \mid \mathbb{O}')$ and it follows that

$$\bar{\pi}_i(\mathbb{o}) \geq \pi_i\left(\mathbb{o} \mid s_i(\mathbb{o})\sigma_i(\mathbb{o}')\right) \geq \pi_i\left(\mathbb{o}' \mid \sigma_i(\mathbb{o}')\right) = \bar{\pi}_i(\mathbb{o}') \geq \Pi_i(\boldsymbol{s} \mid \mathbb{o}'). \tag{23}$$

The inequality is strict unless the sequence $s_i(\mathbb{o})\sigma_i(\mathbb{o}')$ is the same as the optimal myopic sequence of i at $\mathbb{o}$.

To complete the proof of the claim, we argue that $\bar{\pi}_i(\mathbb{o}) > \Pi_i(\boldsymbol{s} \mid \mathbb{o}, P)$ if i is strong and there are at least 3 active rulers at $\mathbb{o}$. As we established above, if i is strong then (14) holds with equality in two cases only: $j_0 = i$ and $s_i(\mathbb{o}, P)$ is the optimal myopic sequence of i at $\mathbb{o}$, or $j_0 = j \neq i$, j_0 attacks i first under $s_{j_0}(\mathbb{o}, P)$ and j_0 is the first ruler to be attacked by i under his optimal myopic sequence of attacks. Generically the second case is possible for at most one ruler other then i. Hence with at least three active rulers there is at least one for which the inequality in (14) is strict. This completes the proof of the claim.

From Step 1, we know that in any state $\mathbb{o}$, there exists a strong ruler for whom the full attacking sequence is the optimal stand alone strategy and it is optimal for him to choose it after any set of rulers P at $\mathbb{o}$. It now follows from the claim above that for this strong ruler the optimal full attacking sequence dominates all other strategies, and the domination is strict if there are at least three active rulers at $\mathbb{o}$. The final step in the proof takes up non-strong rulers. We show that faced with rulers such that at every state at least one of them attacks, every ruler will find it profitable to choose an optimal full attacking sequence.

Step 3: *Let $i \in N$ be a ruler, $\tilde{\boldsymbol{s}}$ be a strategy profile such that for every state $\mathbb{o}$ and for every permutation of N, $j_1, \ldots, j_n$, there exists $k \in \{1, \ldots, n\}$ such that $j_k \neq i$ and $\tilde{s}_{j_k}(\mathbb{o}, \{j_1, \ldots, j_{k-1}\}) \neq \varepsilon$. Let s_i be a best response of i to $\tilde{\boldsymbol{s}}_{-i}$. Then for every state $\mathbb{o}$ such that $i \in \mathrm{Act}(\mathbb{o})$ and $|\mathrm{Act}(\mathbb{o})| \geq 3$, and for every set of rulers, $P \subseteq N \setminus \{i\}$ such that $\mathrm{Atck}(\tilde{\boldsymbol{s}}, \mathbb{o}, P) \setminus \{i\} \neq \varnothing$, $s_i(\mathbb{o}, P)$ is an optimal full attacking sequence of i at $\mathbb{o}$.*

Let $i \in N$ be a ruler and let $\tilde{\boldsymbol{s}}_{-i}$ be a strategy profile of the other rulers, as stated above. The assumption means that at every state $\mathbb{o}$, for any draw of rulers, with probability 1 a ruler other than i would choose attack if i would not. Let s_i be a strategy such that at every state $\mathbb{o}$ where ruler i is active and there are at least three active rulers, and for every $P \subseteq N \setminus \{i\}$ with $\mathrm{Atck}(\tilde{\boldsymbol{s}}, \mathbb{o}, P) \setminus \{i\} \neq \varnothing$, $s_i(\mathbb{o}, P)$ is an optimal full attacking sequence for i. We show that for any other strategy, s_i', of ruler i, every state $\mathbb{o} \in \mathbb{O}$ with $|\mathrm{Act}(\mathbb{o})| \geq 2$, and every set of rulers $P \subseteq N \setminus \{i\}$ such that $\mathrm{Atck}(\tilde{\boldsymbol{s}}, \mathbb{o}, P) \setminus \{i\} \neq \varnothing$,

$$\Pi_i\left((s_i, \tilde{\boldsymbol{s}}_{-i}) \mid \mathbb{o}, P\right) \geq \Pi_i((s_i', \tilde{\boldsymbol{s}}_{-i}) \mid \mathbb{o}, P)), \tag{24}$$

with strict inequality when $|\mathrm{Act}(\phi)| \geq 3$. Notice that, by Step 2, the claim holds if i is strong at ϕ. For the remaining part of the proof we will consider rulers who are not strong at the given states.

The argument is by induction on the number of active rulers. As it proceeds along lines similar to Step 2, it is omitted. $\square$

Proof of Proposition 3. A sequence $\sigma \in \mathbb{R}^*$ is *strong* if either $\sigma = \varepsilon$ or $\sigma = x_0, \ldots, x_m$ and for all $k \in \{1, \ldots, m\}$, $\sum_{j=0}^{k-1} x_j > x_k$. A sequence $\sigma \in \mathbb{R}^*$ is *weak* if it is not strong.

Let $p(x, y \mid \gamma) = \frac{x^\gamma}{x^\gamma + y^\gamma}$. Since

$$\frac{\partial p}{\partial \gamma} = \left(\frac{x^\gamma y^\gamma}{x^\gamma + y^\gamma} \right) (\ln(x) - \ln(y))$$

and

$$\lim_{\gamma \to +\infty} \frac{x^\gamma}{x^\gamma + y^\gamma} = \lim_{\gamma \to +\infty} \frac{1}{1 + \left(\frac{y}{x}\right)^\gamma} = \begin{cases} 1, & \text{if } x > y \\ 0, & \text{if } x < y. \end{cases}$$

so for $x > y$, $p(x, y \mid \gamma)$ is increasing and converges to 1 when $\gamma \to +\infty$, and for $x < y$, $p(x, y \mid \gamma)$ is decreasing and converges to 0 when $\gamma \to +\infty$. In addition, for any strong sequence σ, $p_{\mathrm{seq}}(\sigma \mid \gamma)$ is increasing when γ is increasing. This is because for all $k \in \{1, \ldots, m\}$, $\sum_{j=0}^{k-1} x_j > x_k$, and so $\lim_{\gamma \to +\infty} \prod_{k=1}^m p\left(\sum_{j=0}^{k-1} x_j, x_k \middle| \gamma\right) = 1$ and $\prod_{k=1}^m p\left(\sum_{j=0}^{k-1} x_j, x_k \middle| \gamma\right)$ is increasing when γ is increasing. On the other hand, for any weak $\sigma = x_0, \ldots, x_m$, $\lim_{\gamma \to +\infty} p_{\mathrm{seq}}(\sigma \mid \gamma) = 0$. This is because there exists $k \in \{1, \ldots, m\}$ such that $\sum_{j=0}^{k-1} x_j < x_k$ and for any such k, $\lim_{\gamma \to +\infty} p\left(\sum_{j=0}^{k-1} x_j, x_k \middle| \gamma\right) = 0$. Since for all other $k \in \{1, \ldots, m\}$, $p\left(\sum_{j=0}^{k-1} x_j, x_k \middle| \gamma\right) \leq 1$ so $\lim_{\gamma \to +\infty} \prod_{k=1}^m p\left(\sum_{j=0}^{k-1} x_j, x_k \middle| \gamma\right) = 0$. Consequently, for any non-empty sequence $\sigma = x_0, \ldots, x_m$,

$$\lim_{\gamma \to +\infty} p_{\mathrm{seq}}(\sigma \mid \gamma) = \begin{cases} 1, & \text{if } \sigma \text{ is strong} \\ 0, & \text{if } \sigma \text{ is weak.} \end{cases}$$

The claim on probability of hegemony for strong rulers now follows. $\square$

ONLINE APPENDIX

POOR REWARDING CONTEST SUCCESS FUNCTIONS

Proof of Proposition 4. **Part 1:** The argument presented here is true for general contest functions. Take any ownership configuration, $\mathbb{o} \in \mathbb{O}$, and any active ruler, $i \in \mathrm{Act}(\mathbb{o})$. Given a state $(\mathbb{o}, P) \in \mathbb{O} \times 2^{N \setminus \{i\}}$, an attacking sequence, σ, is an optimal attacking sequence if it maximises the payoff of i at $(\mathbb{o}, P)$ across all attacking sequences that are feasible to i at $\mathbb{o}$ and given the continuation payoffs determined by $\boldsymbol{s}$ on the states in $Succ(\mathbb{o}, P)$. Notice that if a sequence is an optimal attacking sequence for i at $(\mathbb{o}, P)$, then it is an optimal attacking sequence of i at $(\mathbb{o}, P')$, for any $P' \subseteq N$. Thus its optimality depends on the ownership configuration and the expected payoff determined by $\boldsymbol{s}$ on ownership configurations $\mathbb{o}' \in Succ(\mathbb{o})$, only. Clearly, at every state $(\mathbb{o}, P) \in \mathbb{O} \times 2^{N \setminus \{i\}}$ an expected payoff maximising ruler chooses between the empty sequence (peace) and an optimal attacking sequence at $\mathbb{o}$. Given ownership configuration $\mathbb{o}$, let $E(\boldsymbol{s}, \mathbb{o})$ be the set of rulers, active at $\mathbb{o}$, for whom an optimal attacking sequence at $\mathbb{o}$ yields higher payoff than the empty sequence. It is easy to see that if $E(\boldsymbol{s}, \mathbb{o}) = \varnothing$ and $\boldsymbol{s}$ is an equilibrium, then $s_i(\mathbb{o}, P) = \varepsilon$, for all $i \in N$ and $P \in 2^{N \setminus \{i\}}$. On the other hand, suppose that $E(\boldsymbol{s}, \mathbb{o}) \neq \varnothing$ and take any sequence $i_1, \ldots, i_n$ of rulers from N. Let i_k be the last ruler from $E(\boldsymbol{s}, \mathbb{o})$ in the sequence. Generically, no ruler is indifferent between peace and an optimal attacking sequence. Hence, if $\boldsymbol{s}$ is an equilibrium then, for every $l > k$, $s_{i_l}(\{i_1, \ldots, i_{l-1}\})$ is the empty sequence and, consequently, $s_{i_k}(\{i_1, \ldots, i_{k-1}\})$ is an optimal attacking sequence of i_k at $\mathbb{o}$ under the continuation of $\boldsymbol{s}$. Hence if $E(\boldsymbol{s}, \mathbb{o}) \neq \varnothing$ then $\mathbb{o}$ is conflictual under $\boldsymbol{s}$.

Part 2: Let p be a poor rewarding contest success function satisfying (5). Then $p(x, y) = f(x)/(f(x) + f(y))$ and, by Propoistion 2, $f(x)/x$ is decreasing. Since $f(x)/x$ is decreasing and positive on $\mathbb{R}_{++}$ so $\lim_{x \to +\infty} f(x)/x$ exists and is finite. Let $\lim_{x \to +\infty} f(x)/x = L$.

Consider a sequence of fights where a ruler with $x \in \mathbb{R}_{++}$ resources first fights a ruler with $y \in \mathbb{R}_{++}$ resources and then fights with $m \geq 1$ rulers with resources $z_1, \ldots, z_m \in \mathbb{R}_{++}$. The expected payoff to the rulers with x resources from such a sequence of fights is is equal to

$$\begin{aligned}
\pi(x, y, z_1, \ldots, z_m) &= p_{\mathrm{seq}}(x, y, z_1, \ldots, z_m)(x + y + z_1 + \ldots + z_m) \\
&= x \cdot \frac{f(x)}{f(x) + f(y)} \cdot \frac{x + y}{x} \cdot \prod_{i=1}^{m} \left(\frac{f\left(x + y + \sum_{j=1}^{i-1} z_j\right)}{f\left(x + y + \sum_{j=1}^{i-1} z_j\right) + f(z_i)} \cdot \frac{x + y + \sum_{j=1}^{i} z_j}{x + y + \sum_{j=1}^{i-1} z_j} \right).
\end{aligned}$$

We will show that for sufficiently large y, $\pi(x, y, z_1, \ldots, z_m) > x$. We consider two cases separately: $L > 0$ and $L = 0$.

Suppose first that $L > 0$. Notice that

$$\lim_{y \to +\infty} \frac{f(x)}{f(x) + f(y)} \frac{x + y}{x} = \lim_{y \to +\infty} \frac{\frac{f(x)}{x}}{\frac{f(x)}{y} + \frac{f(y)}{y}} \left(\frac{x}{y} + 1 \right) = \frac{\frac{f(x)}{x}}{L} > 1.$$

Similarly

$$\lim_{y \to +\infty} \frac{f\left(x + y + \sum_{j=1}^{i-1} z_j\right)}{f\left(x + y + \sum_{j=1}^{i-1} z_j\right) + f(z_i)} \cdot \frac{x + y + \sum_{j=1}^{i} z_j}{x + y + \sum_{j=1}^{i-1} z_j} = \frac{L}{L} = 1.$$

Hence $\lim_{y \to +\infty} \pi(x, y, z_1, \ldots, z_m) = t > x$ and so for sufficiently large y, $\pi(x, y, z_1, \ldots, z_m) > x$.

Second, suppose that $L = 0$. After winning the conflict with the ruler with y resources, in every subsequent conflict in the sequence the starting ruler has higher resources than his opponent. Hence the probability of winning each of these conflicts is more than $1/2$. In the event of winning all the conflicts in the sequence, the starting ruler owns at least $x + y + \sum_{j=1}^{m} z_j$ resources. By these observations $\pi(x, y, z_1, \ldots, z_m) \geq \left(\frac{1}{2^m}\right) \left(\frac{f(x)}{f(x)+f(y)}\right) (x + y)$. On the other hand, since $L = 0$ so, for sufficiently large y,

$$\frac{f(y)}{y} + \left(1 - \frac{1}{2^m}\right) \frac{f(x)}{y} < \frac{1}{2^m} \frac{f(x)}{x}.$$

Multiplying both sides by $y/f(x)$ and reorganizing, this is equivalent to

$$\frac{f(y)}{f(x)} + 1 < \frac{1}{2^m} \left(1 + \frac{y}{x}\right).$$

Taking the inverses of both sides and then multiplying both sides by $(x + y)/2^m$, this is equivalent to

$$\left(\frac{1}{2^m}\right) \left(\frac{f(x)}{f(x) + f(y)}\right) (x + y) > x.$$

Hence, for sufficiently large y, $\pi(x, y, z_1, \ldots, z_m) > x$.

Now, let G be a connected network over the set of nodes, V, and let $\boldsymbol{r} \in \mathbb{R}_{++}$ be a resource endowment. Fix any vertex $v \in V$. Take any ownership configuration $\mathbb{o} \in \mathbb{O}$. If there is a ruler who owns all the vertices under $\mathbb{o}$ then we are done. Assume otherwise. There are at least two active rulers under $\obar$, $|\mathrm{Act}(\obar)| \geq 2$. Let i be the ruler owning vertex v, $\obar(v) = i$, and let $j \in \mathrm{Act}(\obar)$ be any active neighbor of i under $\obar$. Let σ be a permutation of $\mathrm{Act}(\obar) \setminus \{j\}$ starting with i. Sequence σ is a full attacking sequence of j at $\obar$. By what we have shown above, if r_v is sufficiently large, then $\Pi(j, \obar; \sigma) > R_j(\obar)$ and so by choosing σ ruler j strictly increases his expected payoff. Since at every ownership configuration $\obar$ with at least two active rulers there exists a ruler who can increase his expected resources by choosing attack, so every equilibrium outcome is hegemony.

Part 3: Let $v \in V$ be a vertex and let G be a star network with centre v. Let p be a Tullock contest success function with $\gamma \in (0, 1)$. Take any $y > 0$. Let the resource vector $\boldsymbol{r}$ be such that $r_u = y$, for each spoke $u \in V \setminus \{v\}$, and $r_v = x$, for the centre. We will show that there exists (a range of values of) x such that there is an equilibrium where each ruler chooses peace in the initial ownership configuration. Similarly, we will show that there exists (a range of values of) x such that there is an equilibrium where each ruler at a spoke chooses a sequence of fights that leads to a ownership configuration with peace (so we have war followed by peace in equilibrium).

The expected payoff from a full attacking sequence of m fights to a ruler owning a spoke in a star over at least $m + 1$ vertices, when each spoke is endowed with y resources and the centre is endowed with x resources, is

$$\varphi(x, y, m) = (x + my)p(y, x)\prod_{i=1}^{m-1} p(x + iy, y) = (x + my)\left(\frac{y^\gamma}{x^\gamma + y^\gamma}\right)\prod_{i=1}^{m-1}\left(\frac{(x + iy)^\gamma}{(x + iy)^\gamma + y^\gamma}\right)$$

The key to the constructions of resource endowments enabling equilibria described above is the following claim:

Claim. *For all $m \geq 2$, $\gamma \in [0, 1)$, and $y > 0$, there exists a unique $x^*_m = x^*_m(y, \gamma) > y$, such that*

$$\varphi(x, y, m)\begin{cases} < y & \text{if } x \in (y, x^*_m), \\ = y & \text{if } x = x^*_m, \\ > y & \text{if } x > x^*_m. \end{cases} \tag{25}$$

Moreover, $x^*_{m+1}(y, \gamma) > x^*_m(y, \gamma)$.

Before proving the claim, we provide the construction of resource endowments. Taking any $x \in (\max(y, x^*_{n-2} - y), x^*_{n-1})$ guarantees that no ruler has incentives to engage in a full attacking sequence (and the interval is non-empty, as $x^*_{n-2} > y$, for $n \geq 4$). Moreover, after at least one fight, every ruler at a spoke has incentives to fight if no other ruler fights, as a full attacking sequence yields him expected payoff higher than y. Thus any ruler deviating from peaceful strategy profile leads to fight till hegemony, which is not profitable for the deviating ruler. Therefore there is an equilibrium where all rulers choose peace in the initial ownership configuration. Similarly, taking any $x \in (\max(0, x^*_{n-3} - 2y), x^*_{n-2} - y)$ guarantees that after one fight by a spoke, an ownership configuration with resources at the centre as described above is reached. Moreover, at such a state, no ruler has incentives to engage in a full attacking sequence. Thus there is an equilibrium where (1) in the initial state each ruler owning a spoke chooses to attack the centre and the ruler owning the centre chooses peace, (2) in the state with $n-1$ vertices every vertex chooses peace, and (3) in any state with at most $n-2$ at least one vertex chooses attack. In this equilibrium there is one conflict followed by peace.

Notice that the two constructions given above are generic: analogous argument could be conducted if spokes were endowed with resource sufficiently close to each other and the centre was endowed with resources within a range close to the range given in the construction above.

We now provide the proof of the claim. To this end, we establish four properties of function φ, from which the claim follows. Fix any $\gamma \in [0, 1)$.

First, we show that, for all $x, y \in \mathbb{R}_{++}$ and $m \geq 3$, $\varphi(x, y, m) < \varphi(x, y, m-1)$. Notice that, $y^{1-\gamma} \leq (x+(m-1)y)^{1-\gamma}$. Multiplying both sides by $y^{\gamma}(x+(m-1)y)^{\gamma}$ we get $y(x+(m-1)y)^{\gamma} < y^{\gamma}(x+(m-1)y)$. Reorganizing, we obtain $(x+my)(x+(m-1)y)^{\gamma} < ((x+(m-1)y)^{\gamma} + y^{\gamma})(x+(m-1)y)$. Dividing both sides by the RHS we get $\left(\frac{x+my}{x+(m-1)y}\right)\left(\frac{(x+(m-1)y)^{\gamma}}{(x+(m-1)y)^{\gamma}+y^{\gamma}}\right) < 1$. This, together with the fact that $\varphi(x, y, m) = \varphi(x, y, m-1)\left(\frac{(x+(m-1)y)^{\gamma}}{(x+(m-1)y)^{\gamma}+y^{\gamma}}\right)\left(\frac{(x+my)}{(x+(m-1)y)}\right)$ yields $\varphi(x, y, m) < \varphi(x, y, m-1)$.

Second, we show that φ is strictly increasing in x for $x > y$. First derivative of φ with respect to x is

$$\frac{\partial \varphi}{\partial x} = \left(\frac{\gamma y^{\gamma}}{x^{\gamma}+y^{\gamma}}\right)(x+my)\prod_{i=1}^{m-1}\left(\frac{(x+iy)^{\gamma}}{(x+iy)^{\gamma}+y^{\gamma}}\right)$$
$$\left(\left(\frac{1}{\gamma(x+my)}\right) - \left(\frac{x^{\gamma-1}}{x^{\gamma}+y^{\gamma}}\right) + \sum_{j=1}^{m-1}\frac{y^{\gamma}}{(x+jy)((x+jy)^{\gamma}+y^{\gamma})}\right). \quad (26)$$

Since $\gamma \in [0, 1)$ so $(1-\gamma)(x+y) > 0$. Reorganizing we get $x + y + (m-1)\gamma y > \gamma(x+my)$. Dividing both sides by $\gamma(x+y)(x+my)$ we get $\frac{1}{\gamma(x+my)} + \frac{(m-1)y}{(x+y)(x+my)} > \frac{1}{x+y}$ Since $x > y$ and

$\gamma \in [0,1)$ so $(x/y)^{1-\gamma} > 1$ and so $\frac{1}{x+y} > \frac{1}{x+y\left(\frac{x}{y}\right)^{1-\gamma}} = \frac{x^{\gamma-1}}{x^\gamma+y^\gamma}$. Hence

$$\frac{1}{\gamma(x+my)} + \frac{(m-1)y}{(x+y)(x+my)} > \frac{x^{\gamma-1}}{x^\gamma+y^\gamma}. \tag{27}$$

Notice that

$$\begin{aligned}\frac{(m-1)y}{(x+y)(x+my)} &= \left(\frac{1}{x+y}\right) - \left(\frac{1}{x+my}\right) = \sum_{i=1}^{m-1}\left(\frac{1}{x+iy}\right) - \sum_{i=2}^{m}\left(\frac{1}{x+iy}\right)\\ &= \sum_{i=1}^{m-1}\left(\left(\frac{1}{x+iy}\right) - \left(\frac{1}{x+(i+1)y}\right)\right) = \sum_{i=1}^{m-1}\left(\frac{y}{(x+iy)((x+iy)+y)}\right)\end{aligned}$$

Moreover, for $\gamma \in [0,1)$, $x > y$, and $i \geq 1$,

$$\begin{aligned}\frac{y}{(x+iy)((x+iy)+y)} &= \frac{1}{(x+iy)\left(\left(\frac{x}{y}+i\right)+1\right)} < \frac{1}{(x+iy)\left(\left(\frac{x}{y}+i\right)^\gamma+1\right)}\\ &= \frac{y^\gamma}{(x+iy)((x+iy)^\gamma+y^\gamma)}\end{aligned}$$

Thus

$$\frac{(m-1)y}{(x+y)(x+my)} < \sum_{i=1}^{m-1}\left(\frac{y^\gamma}{(x+iy)((x+iy)^\gamma+y^\gamma)}\right)$$

which, together with (27), implies

$$\frac{1}{\gamma(x+my)} + \sum_{i=1}^{m-1}\left(\frac{y^\gamma}{(x+iy)((x+iy)^\gamma+y^\gamma)}\right) > \frac{x^{\gamma-1}}{x^\gamma+y^\gamma}.$$

Therefore, by that and (26), $\partial\varphi/\partial x > 0$ for all $x > y$ and so φ is increasing in x on $(y, +\infty)$.

Third, we show that for all $y \in \mathbb{R}_{++}$ and $m \geq 3$, $\lim_{x\to+\infty}\varphi(x,y,m) = +\infty$. To see that notice that $\lim_{x\to+\infty}\prod_{i=1}^{m-1} p(x+iy,y) = 1$ and $\lim_{x\to+\infty} p(y,x)(x+my) = \left(\frac{y^\gamma}{1+\left(\frac{y}{x}\right)^\gamma}\right)\left(x^{1-\gamma}+m\left(\frac{y}{x^\gamma}\right)\right) = +\infty$, and so the property follows.

Fourth, we show that $\varphi(y,y,m) < y$. To see that we start with

$$\varphi(y,y,m) = \left(\frac{1}{2}\right)(m+1)y\prod_{i=1}^{m-1}\left(\frac{(i+1)^\gamma}{(i+1)^\gamma+1}\right) = \left(\frac{1}{2}\right)(m+1)y\prod_{i=2}^{m}\left(\frac{i^\gamma}{i^\gamma+1}\right).$$

Since $\frac{i^\gamma}{i^\gamma+1} = 1 - \left(\frac{1}{i^\gamma+1}\right)$, $\gamma \in [0,1)$, $i \geq 1$, so $i^\gamma/(i^\gamma+1)$ is increasing in γ. Hence $\varphi(y,y,n) < \left(\frac{1}{2}\right)(m+1)y\prod_{i=2}^{m}\left(\frac{i}{i+1}\right) = \left(\frac{1}{2}\right)y\left(\frac{n!}{n!}\right)2 = y$.

By the four properties of φ, established above, for all $m \geq 2$, $\gamma \in [0,1)$, and $y > 0$, there exists a unique $x_m^* = x_m^*(y,\gamma) > y$, such that (25) holds. Moreover, since for all $x, y \in \mathbb{R}_{++}$ and $m \geq 3$, $\varphi(x,y,m) < \varphi(x,y,m-1)$, and since φ is increasing in x for $x > y$, $x_{m+1}^*(y,\gamma) > x_m^*(y,\gamma)$. This completes the proof. □

HIRSHLEIFER'S CONTEST SUCCESS FUNCTION

Another widely used contest success function, along the Tullock contest success function, is the so called difference form proposed by Hirshleifer [1989]:

$$p(x,y) = \frac{\exp(\gamma x)}{\exp(\gamma x) + \exp(\gamma y)}, \tag{28}$$

where $\gamma > 0$. Thus $f(x) = \exp(\gamma x)$ and it is easy to check that $f(x)/x$ is increasing on interval $(0, 1/\gamma)$ and decreasing on $(1/\gamma, +\infty)$. Thus the function maintains the poor rewarding and, consequently, the waiting properties on the interval $(0, 1/\gamma)$ and maintains the rich rewarding and, consequently, the no-waiting properties on the interval $(1/\gamma, +\infty)$. Hence if the minimal resources in the network at the initial ownership configuration are greater than $1/\gamma$, all the results obtained for the rich rewarding case would hold for this contest success function as well and if the total resources in the network are less than $1/\gamma$, the results for the poor rewarding case apply.

For the order of fights properties of Hirshleifer's contest success function, notice that

$$h(s,t) = \frac{\exp(\gamma t)\exp(\gamma(s+t))}{\exp(\gamma(s+t) - \exp(\gamma s) - \exp(\gamma t))} = \frac{1}{\exp(-\gamma t) - \exp(-2\gamma t) - \exp(-\gamma(s+t))}.$$

Taking the derivative with respect to t and comparing it to 0 we can see that $h(s,t)$ is decreasing in t when $\exp(-\gamma t) < 1/2 - \exp(-\gamma s)/2$ and is increasing in t when the inequality is reversed. The LHS of the inequality is decreasing in t while the RHS is increasing in s. Moreover, the functions $\exp(-\gamma x)$ and $1/2 - \exp(-\gamma x)/2$ intersect at $x = \ln(3)/\gamma > 1/\gamma$. Thus on the interval $(0, 1/\gamma)$ the contest success function maintains the poor rewarding and the rich first properties and on interval $(\ln(3)/\gamma, +\infty)$ it maintains the rich rewarding and the poor first property.

EXTENSIONS AND ADDITIONAL RESULTS

GRADUAL CONQUEST

We start with defining the Markov chain associated with a given strategy profile $\boldsymbol{s}$. Given an ownership configuration, $\mathbb{o} \in \mathbb{O}$, and two nodes, $\{u, v\} \subseteq V$, such that $\mathbb{o}(u) \neq \mathbb{o}(v)$, we will use $\mathbb{o}[u \to v]$ to denote the ownership configuration obtained from $\mathbb{o}$ by changing the owner of v to the owner of u, that is

$$\mathbb{o}[u \to v](w) = \begin{cases} \mathbb{o}(u), & \text{if } w = v, \\ \mathbb{o}(w), & \text{otherwise.} \end{cases}$$

Let $\mathcal{M}(\boldsymbol{s})$ be defined as follows. The set of states of $\mathcal{M}(\boldsymbol{s})$ is $Z = \{\hat{\mathbb{o}}\} \cup (\mathbb{O} \times P \times N \times \mathbb{O})$, where $\hat{\mathbb{o}} \in \mathbb{O}$ is the initial ownership configuration. The states of the form $(\mathbb{o}, P, i, \mathbb{o}')$ are the states at which ruler i is picked to move at ownership configuration $\mathbb{o}$ after the set of rulers P and reaches ownership configuration $\mathbb{o}'$ executing his plan of conquest $\sigma_i(\mathbb{o}, P)$.

The transition probabilities, $\tau : Z \times Z \to [0, 1]$ are defined as follows:

(a) For all $i \in N$, $\tau(\hat{\mathbb{o}}, (\hat{\mathbb{o}}, \varnothing, i, \hat{\mathbb{o}})) = \frac{1}{|N|}$.

(b) For all $\mathbb{o} \in \mathbb{O}$, $i \in N$, $P \subseteq N \setminus \{i\}$, and $j \in N \setminus (P \cup \{i\})$, if $\sigma_i(\mathbb{o}) = \varepsilon$ then $\tau((\mathbb{o}, P, i, \mathbb{o}), (\mathbb{o}, P \cup \{i\}, j, \mathbb{o})) = \frac{1}{|N|-|P|-1}$.

(c) For all $\mathbb{o} \in \mathbb{O}$, $i \in N$, if $\sigma_i(\mathbb{o}) = \varepsilon$ then $\tau((\mathbb{o}, N \setminus \{i\}, i, \mathbb{o}), (\mathbb{o}, N \setminus \{i\}, i, \mathbb{o})) = 1$.

(d) For all $\mathbb{o} \in \mathbb{O}$, $i \in N$, $P \subseteq N \setminus \{i\}$, if $\sigma_i(\mathbb{o}) = uv$ then

$$\tau((\mathbb{o}, P, i, \mathbb{o}'), (\mathbb{o}, P, i, \mathbb{o}'')) = \begin{cases} p(R_{\mathbb{o}'(u)}(\mathbb{o}), R_{\mathbb{o}'(v)}(\mathbb{o}')), & \text{if } \mathbb{o}'' = \mathbb{o}'[u \to v], \\ 1 - p(R_{\mathbb{o}'(u)}(\mathbb{o}'), R_{\mathbb{o}'(v)}(\mathbb{o}')), & \text{if } \mathbb{o}'' = \mathbb{o}'[v \to u], \\ 0, & \text{otherwise.} \end{cases}$$

(e) For all $\mathbb{o} \in \mathbb{O}$, $i \in N$, $P \subseteq N \setminus \{i\}$, $\mathbb{o}' \in \mathbb{O} \setminus \{\mathbb{o}\}$, and $j \in N$, if $\sigma_i(\mathbb{o}) = \varepsilon$ then $\tau((\mathbb{o}, P, i, \mathbb{o}'), (\mathbb{o}', \varnothing, j, \mathbb{o}')) = \frac{1}{|N|}$.

(f) In all the remaining cases of $z', z'' \in Z$, $\tau(z', z'') = 0$.

Transition (a) corresponds to the draw of the first ruler to move at the initial state. Transition (b) corresponds to drawing the next ruler to move after a ruler chooses peace. Transition (c) corresponds to reaching peace after the last drawn ruler chooses peace. Transition (d) corresponds to transitions during execution of a plan of conquest by a ruler. Transition (e) corresponds to drawing the first ruler to move after a ruler finished executing his plan of conquest.

First, we establish that the Markov chain $\mathcal{M}(\boldsymbol{s})$ is absorbing for any strategy profile $\boldsymbol{s} \in \boldsymbol{S}$. The absorbing states are the states of the form $(\mathbb{o}, N \setminus \{i\}, i, \mathbb{o})$ with $\sigma_i(\mathbb{o}) = \varepsilon$.

Lemma 1. *Fix a connected graph G. For any strategy profile $\boldsymbol{s} \in \boldsymbol{S}$, the Markov chain $\mathcal{M}(\boldsymbol{s})$ is absorbing and the set of absorbing states is*

$$\mathrm{Abs}_{\boldsymbol{s}} = \{(\mathbb{o}, N \setminus \{i\}, i, \mathbb{o}) : \mathbb{o} \in \mathbb{O},\ i \in N,\ \sigma_i(\mathbb{o}, N \setminus \{i\}) = \varepsilon\}\,.$$

Proof. To prove the lemma we show, for any state $z \in Z$, that z is either transient or absorbing and that z is absorbing if and only if $z = (\mathbb{o}, N \setminus \{i\}, i, \mathbb{o})$ for some $\mathbb{o} \in \mathbb{O}$ and $i \in N$ with $\sigma_i(\mathbb{o}, N \setminus \{i\}) = \varepsilon$. Clearly $z = \hat{\mathbb{o}}$ is transient.

For the remaining part of the proof we consider states $z \in Z \setminus \{\hat{\mathbb{o}}\}$. Given an ownership configuration $\mathbb{o}$, let

$$B(\mathbb{o}) = \{uv \in G : \mathbb{o}(u) \neq \mathbb{o}(v)\}$$

be the set of links connecting nodes belonging to different rulers under $\mathbb{o}$.

We will show that any $z = (\mathbb{o}, P, i, \mathbb{o}')$ is either transient of absorbing and that it is absorbing if and only if it is an element of $\mathrm{Abs}_{\boldsymbol{s}}$. The remaining part of the proof is by induction on $|B(\mathbb{o}')|$. For the induction basis, suppose that $|B(\mathbb{o}')| = 0$. Since G is connected, this implies that $\mathbb{o}'$ is hegemony. Hence the only valid strategy for every ruler j is such that $\sigma_j(\mathbb{o}') = \varepsilon$. If $\mathbb{o}' = \mathbb{o}$ and $P = N \setminus \{i\}$ then z is absorbing, by (c). If $\mathbb{o}' = \mathbb{o}$ and $|P| < |N| - 1$ then, by one or more applications of (b), with probability 1, z reaches an absorbing state $z' = (\mathbb{o}, P', j, \mathbb{o})$ with $P' = N \setminus \{j\}$. Lastly, if $\mathbb{o}' \neq \mathbb{o}$ then, by (e), z transits to a state $z' = (\mathbb{o}', \varnothing, j, \mathbb{o}')$ and z' is transient by what was shown above. Hence z is transient in this case as well.

For the induction step, let $|B(\mathbb{o}')| = m > 1$ and suppose that the induction hypothesis holds for any state of the form $(\tilde{\mathbb{o}}, \tilde{P}, k, \tilde{\mathbb{o}}')$ with $|B(\tilde{\mathbb{o}}')| < m$. If $\sigma_i(\mathbb{o}') = uv$ (and either $\mathbb{o}' = \mathbb{o}$ or $\mathbb{o}' \neq \mathbb{o}$) then, by (d), z transits to state $z' = (\mathbb{o}, P, i, \mathbb{o}'[u \to v])$ with probability greater than 0 or two a state $z'' = (\mathbb{o}, P, i, \mathbb{o}'[v \to u])$ with probability greater than 0. Since

$B(\mathbb{o}'[u \to v]) = B(\mathbb{o}'[v \to u]) = B(\mathbb{o}') \setminus \{uv\}$ so, by the induction hypothesis, z' is either absorbing or transient and z'' is either absorbing or transient. Therefore z is transient. If $\sigma_i(\mathbb{o}') = \varepsilon$, $\mathbb{o}' = \mathbb{o}$, and $P = N \setminus \{i\}$, then z is absorbing, by (c). If $\sigma_i(\mathbb{o}') = \varepsilon$, $\mathbb{o}' = \mathbb{o}$, and $|P| < |N| - 1$, then, by one or more applications of (b), with probability 1, z reaches a state $z' = (\mathbb{o}, P', j, \mathbb{o})$ with $P \subseteq P'$ and such that either $\sigma_j(\mathbb{o}) \neq \varepsilon$ or $P = N \setminus \{j\}$. As we just shown, z' is either absorbing or transient. Hence z is transient. Lastly, if $\sigma_i(\mathbb{o}') = \varepsilon$ and $\mathbb{o}' \neq \mathbb{o}$ then, by (e), z transits to a state $z' = (\mathbb{o}', \varnothing, j, \mathbb{o}')$ and, by what was shown above, z' is transient. Hence z is transient as well. This completes the proof. $\square$

Given two states, $z, z' \in Z$, let $A_{\boldsymbol{s}}(z, z')$ be the absorption probability of z' in $\mathcal{M}(\boldsymbol{s})$ starting at z. The probability that the game ends at an ownership configuration $\mathbb{o}' \in \mathbb{O}$, $F(\mathbb{o}' \mid \boldsymbol{s}, \mathbb{o}, P)$, starting at an ownership configuration, $\mathbb{o} \in \mathbb{O}$ and a set of rulers, $P \subseteq N$, under a strategy profile $\boldsymbol{s} \in \boldsymbol{S}$, is defined as follows:

$$F(\mathbb{o}' \mid \boldsymbol{s}, \mathbb{o}, P) = \frac{1}{|N| - |P|} \sum_{i \in N \setminus P} \sum_{(\mathbb{o}', N \setminus \{j\}, j, \mathbb{o}') \in \mathrm{Abs}_{\boldsymbol{s}}} A_{\boldsymbol{s}}((\mathbb{o}, P, i, \mathbb{o}), (\mathbb{o}', N \setminus \{j\}, j, \mathbb{o}')) \qquad (29)$$

Now we state and prove a more general version of Proposition 5 that covers any connected graphs in the case of three or more active rulers. One can immediately see that the result implies Proposition 5.

Proposition 7. *Let p be a contest success function satisfying axioms **A1**–**A3**. For any equilibrium $\boldsymbol{s}$ and for any state $(\mathbb{o}, P) \in \mathbb{O} \times 2^N$ at which all active rulers choose peace under $\boldsymbol{s}$, either $|\mathrm{Act}(\mathbb{o})| = 1$ or $|\mathrm{Act}(\mathbb{o})| \geq 3$ and there exists $i \in \mathrm{Act}(\mathbb{o})$ such that $G[\mathbb{o}^{-1}(i)]$ is not connected.*

Before we provide proof of Proposition 7 we need the following auxiliary lemma. The following notation will be convenient. Given a set of nodes $X \subseteq V$ let $G - X = G[V \setminus X]$ denote the graph obtained from G by removing from it all nodes from X and all the links ending at nodes in X.

Lemma 2. *Let $H \in \mathcal{G}$ be a connected graph and let $\mathbb{o} \in \mathbb{O}(H)$ be an ownership configuration such that $\mathrm{Act}(\mathbb{o}) \neq \varnothing$ and for all $i \in \mathrm{Act}(\mathbb{o})$, $H[\mathbb{o}^{-1}(i)]$ is connected. There exists $i \in \mathrm{Act}(\mathbb{o})$ such that $H - \mathbb{o}^{-1}(i)$ is connected.*

Proof. Let Q be the neighbourhood graph over the set of active rulers determined by $\mathbb{o}$ and H as follows: $V(Q) = \mathrm{Act}(\mathbb{o})$ and $E(Q) = \{ij : \{i, j\} \subseteq \mathrm{Act}(\mathbb{o})$ and there exist $uv \in$

$E(H)$ such that $i = \mathbb{o}(u)$ and $j = \mathbb{o}(v)\}$. Since H is connected so Q is connected as well. Hence Q has a spanning tree and for any leaf i of any spanning tree of Q, $Q - \{i\}$ is connected. Since in addition, for every $j \in \mathrm{Act}(\mathbb{o})$, $H[\mathbb{o}^{-1}(j)]$ is connected so $H - \mathbb{o}^{-1}(i)$ is connected as well. □

Now we are ready to prove Proposition 7.

Proof of Proposition 7. Let $\mathcal{G}(V)$ be the set of graphs that can be formed over V and let $\bar{\mathcal{G}} = \bigcup_{U \subseteq V} \mathcal{G}(V)$ be the set of all graphs that can be formed over V or any of its subsets.

Given an ownership configuration, $\mathbb{o} \in \mathbb{O}(H)$, on graph H and a set of nodes $U \subseteq V$, we will use $\mathbb{o}_{-U} = \mathbb{o}|_{V \setminus U}$ to denote the ownership configuration $\mathbb{o}$ restricted to the nodes in $V \setminus U$. Notice that if $\mathbb{o} \in \mathbb{O}(H)$ then $\mathbb{o}_{-U} \in \mathbb{O}(H - U)$.

The proof proceeds in two steps. In the first step we show, for any connected graph $H \in \mathcal{G}$ and any ownership configuration, $\mathbb{o} \in \mathbb{O}(H)$, with at least two active rulers, $|\mathrm{Act}(\mathbb{o})| \geq 2$, and such that either $|\mathrm{Act}(\mathbb{o})| = 2$ or for all $i \in \mathrm{Act}(\mathbb{o})$, $H[\mathbb{o}^{-1}(i)]$ is connected, that there exists an active ruler $i \in \mathrm{Act}(\mathbb{o})$ who has a fully attacking plan of conquest that increases his resources in expectation. In the second step we use this fact to conclude that at every such state there exists an active ruler who prefers fight to peace.

Proof of the first claim is by induction on the number of nodes in H, $|V(H)|$. Notice that the minimum number of nodes needed to have at least two active rulers is 2. For the induction basis, take any connected graph $H \in \mathcal{G}$ with $|V(H)| = 2$ nodes and any ownership configuration, $\mathbb{o} \in \mathbb{O}(H)$ with $|\mathrm{Act}(\mathbb{o})| = 2$ active nodes. Let i and j denote the two different active rulers under $\mathbb{o}$. Since $p(R_i(\mathbb{o}), R_j(\mathbb{o})) = 1 - p(R_i(\mathbb{o}), R_j(\mathbb{o}))$ so either $p(R_i(\mathbb{o}), R_j(\mathbb{o}))(R_i(\mathbb{o}) + R_j(\mathbb{o})) \geq R_i(\mathbb{o})$ or $p(R_j(\mathbb{o}), R_i(\mathbb{o}))(R_i(\mathbb{o}) + R_j(\mathbb{o})) \geq R_j(\mathbb{o})$, with inequality being strict if and only if either $R_i(\mathbb{o}) = R_j(\mathbb{o})$ or $p(x, y) = x/(x + y)$. Since both cases are non-generic, the inequalities are generically strict and attacking the other ruler is an improving fully attacking plan of conquest for one of the rulers.

For the induction step, take any connected graph $H \in \mathcal{G}$ with $|V(H)| > 2$ and any any ownership configuration , $\mathbb{o} \in \mathbb{O}(H)$, with at least two active rulers, $|\mathrm{Act}(\mathbb{o})| \geq 2$, and such that for all $i \in \mathrm{Act}(\mathbb{o})$, $H[\mathbb{o}^{-1}(i)]$ is connected. We consider two cases separately: (i) $|\mathrm{Act}(\mathbb{o})| = 2$ and (ii) $|\mathrm{Act}(\mathbb{o})| \geq 3$. For case (i), suppose that $|\mathrm{Act}(\mathbb{o})| = 2$. Let σ be a fully attacking plan of conquest for a ruler $i \in \mathrm{Act}(\mathbb{o})$ and let j denote the other ruler in $\mathrm{Act}(\mathbb{o})$. Since σ is fully attacking so starting from a state $z = (\mathbb{o}, P, i, \mathbb{o})$, it will lead to a state $(\mathbb{o}, P, i, \mathbb{o}')$ with an ownership configuration $\mathbb{o}'$ such that $|\mathrm{Act}(\mathbb{o}')| = 1$. Since there is only one ruler at this state, execution of σ ends there and, regardless of the strategy of the other ruler, the game will end at an ownership configuration $\mathbb{o}'$. There are two such ownership configurations: one, $\mathbb{i}$, where i owns all the nodes, and the other one, $\mathbb{j}$, where j owns all the nodes. Let $Q_\sigma(\mathbb{o}, \mathbb{j})$ be the probability of reaching the state $(\mathbb{o}, P, i, \mathbb{j})$ from z and $Q_\sigma(\mathbb{o}, \mathbb{i})$ be the probability of reaching the state $(\mathbb{o}, P, i, \mathbb{i})$ from z. Then $Q_\sigma(\mathbb{o}, \mathbb{j}) = 1 - Q_\sigma(\mathbb{o}, \mathbb{i})$. The expected payoff to ruler i from plan σ at $\mathbb{o}$ is $Q_\sigma(\mathbb{o}, \mathbb{i})(R_i(\mathbb{o}) + R_j(\mathbb{o}))$ and the expected payoff to ruler j from plan σ at $\mathbb{o}$ is $(1 - Q_\sigma(\mathbb{o}, \mathbb{i}))(R_i(\mathbb{o}) + R_j(\mathbb{o}))$. Hence either $Q_\sigma(\mathbb{o}, \mathbb{i})(R_i(\mathbb{o}) + R_j(\mathbb{o})) \geq R_i(\mathbb{o})$ or $(1 - Q_\sigma(\mathbb{o}, \mathbb{i}))(R_i(\mathbb{o}) + R_j(\mathbb{o})) \geq R_j(\mathbb{o})$. Generically these inequalities are strict. Let σ' be a plan of conquest such that, for $\mathbb{o}' \in \mathbb{O}(H)$,

$$\sigma'(\mathbb{o}') = \begin{cases} \sigma(\mathbb{o}'), & \text{if } \mathrm{Act}(\mathbb{o}') = \{i, j\}, \\ \varepsilon, & \text{otherwise.} \end{cases}$$

Notice that σ' is a valid plan of conquest for ruler j. Intuitively, at every ownership configuration $\mathbb{o}' \in \mathbb{O}(H)$, if $\sigma(\mathbb{o}')$ prescribes i attacking a node v of j from node u then $\sigma'(\mathbb{o}')$ prescribes j attacking node u of i from v.

Notice that the sets of ownership configurations reachable from $\mathbb{o}$ in under σ and under σ' are the same and that the transition probabilities are the for these reachable states under the two plans of conquest. Hence $Q_{\sigma'}(\mathbb{o}, \mathbb{i}) = Q_\sigma(\mathbb{o}, \mathbb{i})$ and $Q_{\sigma'}(\mathbb{o}, \mathbb{j}) = Q_\sigma(\mathbb{o}, \mathbb{j})$. Thus either σ results in i increasing his resources in expectation at $\mathbb{o}$, or σ' results in j increasing his resources in expectation at $\mathbb{o}$. Therefore there exists a player who has an improving fully attacking plan of conquest.

For case (ii), suppose that $|\mathrm{Act}(\mathbb{o})| \geq 3$ and suppose that the induction hypothesis holds for any connected graph $H' \in \mathcal{G}$ with $|V(H')| < |V(H)|$ nodes. By Lemma 2, there exists an active ruler $i \in \mathrm{Act}(\mathbb{o})$ such that $H_{-i} = H - \mathbb{o}^{-1}(i)$ is connected. Let $\mathbb{o}_{-i} = \mathbb{o}_{-\mathbb{o}^{-1}(i)}$ be the ownership configuration $\mathbb{o}$ restricted to the nodes owned by rulers other than i. By the induction hypothesis, there exists a ruler $j \in \mathrm{Act}(\mathbb{o}_{-i})$ with an improving fully attacking plan of conquest, σ_{-i}, at $\mathbb{o}_{-i}$ on H_{-i}. Let $\tilde{\mathbb{o}}$ be an ownership configuration on H such that, for $u \in V(H)$,

$$\tilde{\mathbb{o}}(u) = \begin{cases} i, & \text{if } \mathbb{o}(u) = i \\ j, & \text{otherwise,} \end{cases}$$

so all the nodes owned by i under $\mathbb{o}$ are owned by i under $\tilde{\mathbb{o}}$ and all the remaining nodes are owned by j under $\tilde{\mathbb{o}}$. By point (i), either i or j has an improving fully attacking plan of conquest at $\tilde{\mathbb{o}}$ on H. Suppose that j is such a ruler and let $\tilde{\sigma}$ be an improving fully attacking plan of conquest of j at $\tilde{\mathbb{o}}$ on H. Define plan of conquest, σ', as follows (for $\mathbb{o}' \in \mathbb{O}(H)$):

$$\sigma'(\mathbb{o}') = \begin{cases} \sigma_{-i}(\mathbb{o}'), & \text{if } \mathbb{o}'^{-1}(i) = \mathbb{o}^{-1}(i) \text{ and } \mathbb{o}' \neq \tilde{\mathbb{o}}, \\ \tilde{\sigma}(\mathbb{o}'), & \text{otherwise.} \end{cases}$$

Informally, following plan σ', ruler j first conquers all the nodes owned by rulers other than i, following plan σ_{-i}, and then fights with i, following plan $\tilde{\sigma}$. We will show that σ' is an improving fully attacking plan of conquest at $\mathbb{o}'$. By its definition, it is a fully attacking plan of conquest at $\mathbb{o}$. Since σ' is fully attacking, it stops either at an ownership configuration $\mathbb{j}$, where j owns all the nodes, or at an ownership configuration where j owns no nodes. In addition, before reaching $\mathbb{j}$ from $\mathbb{o}$, σ' must first lead to $\tilde{\mathbb{o}}$ from $\mathbb{o}$ and then to $\mathbb{j}$ from $\tilde{\mathbb{o}}$. The probability of σ' leading to $\mathbb{j}$ is $Q_{\sigma'}(\mathbb{o}, \mathbb{j}) = Q_{\sigma_{-i}}(\mathbb{o}_{-i}, \tilde{\mathbb{o}}_{-i}) Q_{\tilde{\sigma}}(\tilde{\mathbb{o}}, \mathbb{j})$. The expected payoff to j from σ' at $\mathbb{o}$ is

$$\begin{aligned} Q_{\sigma'}(\mathbb{o}, \mathbb{j}) R_j(\mathbb{j}) &= Q_{\sigma_{-i}}(\mathbb{o}_{-i}, \tilde{\mathbb{o}}_{-i}) Q_{\tilde{\sigma}}(\tilde{\mathbb{o}}, \mathbb{j}) R_j(\mathbb{j}) \\ &= R_j(\mathbb{o}) Q_{\sigma_{-i}}(\mathbb{o}_{-i}, \tilde{\mathbb{o}}_{-i}) \frac{R_j(\tilde{\mathbb{o}}_{-i})}{R_j(\mathbb{o}_{-i})} Q_{\tilde{\sigma}}(\tilde{\mathbb{o}}, \mathbb{j}) \frac{R_j(\mathbb{j})}{R_j(\tilde{\mathbb{o}})} > R_j(\mathbb{o}), \end{aligned}$$

because $R_j(\mathbb{o}) = R_j(\mathbb{o}_{-i})$, $Q_{\sigma_{-i}}(\mathbb{o}_{-i}, \tilde{\mathbb{o}}_{-i}) R_j(\tilde{\mathbb{o}}_{-i}) > R_j(\mathbb{o}_{-i})$ (as σ_{-i} is improving at $\mathbb{o}_{-i}$) and $Q_{\tilde{\sigma}}(\tilde{\mathbb{o}}, \mathbb{j}) R_j(\mathbb{j}) > R_j(\tilde{\mathbb{o}})$ (as $\tilde{\sigma}$ is improving at $\tilde{\mathbb{o}}$). Thus σ' is improving at $\mathbb{o}$.

Suppose now that i has an improving fully attacking plan of conquest at $\tilde{\mathbb{o}}$ on H. Take any such a plan and denote it by $\tilde{\sigma}$. Given an ownership configuration, $\mathbb{o}'$, let $d\mathbb{o}'$ be the ownership configuration where each owner of a node other than i is replaced by j, that is

$$d\mathbb{o}'(v) = \begin{cases} \mathbb{o}'(v), & \text{if } \mathbb{o}'(v) = i \\ j, & \text{otherwise.} \end{cases}$$

Define a plan of conquest, σ', at $\mathbb{o}$ as follows (for $\mathbb{o}' \in \mathbb{O}(H)$): $\sigma'(\mathbb{o}') = \tilde{\sigma}(d\mathbb{o}')$. Following σ', i chooses links over which the subsequent attacks are launched according to the set of nodes he owns only and makes this decision according to $\tilde{\sigma}$. Since $\tilde{\sigma}$ is fully attacking, σ' is fully attacking as well. The probability of winning each conflict is higher under σ', starting at $\mathbb{o}$, than under $\tilde{\sigma}$, starting at $\tilde{\mathbb{o}}$. This is because at every link at which an attack is performed the resources of i are the same but the resources of the opponent are the same or smaller, for i faces an opponent who owns at most all the nodes that i does not own under σ' starting at $\mathbb{o}$, while he faces an opponent j who owns all the nodes that i does not own under $\tilde{\sigma}$ at $\tilde{\mathbb{o}}$. By axioms **A1**–**A3**, the probability of winning a bilateral conquest is increasing when resources of the opponent are decreasing. Hence the probability of reaching an ownership configuration $\mathbb{i}$, where i owns all the nodes, is at least as high under σ' starting at $\mathbb{o}$ as under $\tilde{\sigma}$ starting at $\tilde{\mathbb{o}}$. Since $\tilde{\sigma}$ is improving at $\tilde{\mathbb{o}}$ for i so is σ' at $\mathbb{o}$. This concludes proof of the first step.

For the second step, let $\boldsymbol{s}$ be an equilibrium strategy profile and let $(\mathbb{o}, P)$ be a state with $\mathrm{Act}(\mathbb{o}) \geq 2$ and, in the case of $\mathrm{Act}(\mathbb{o}) \geq 3$, with $G[\mathbb{o}^{-1}(i)]$ being connected for every $i \in \mathrm{Act}(\mathbb{o})$. By the first step, there exists $i \in \mathrm{Act}(\mathbb{o})$ who has an improving fully attacking plan of conquest at $\mathbb{o}$. Let $I \subseteq \mathrm{Act}(\mathbb{o})$ be the set of all such rulers. Then ruler $i \in I$, at every state $(\mathbb{o}, P)$ such that $I \setminus \{i\} \subseteq P$, prefers to choose his improving fully attacking plan of conquest to choosing ε. □

SHORT ATTACK SEQUENCES

Proof of Proposition 6. Throughout the proof we use the precedence relations on ownership configurations and states, as well as the sets $Succ$ and $\overline{Succ}$ introduced in proof of Proposition 1.

Notice that if $i \in \mathrm{Act}(\mathbb{o})$ is the unique strong ruler at $\mathbb{o}$, then for all $\mathbb{o}' \in Succ(\mathbb{o})$ there is exactly one strong ruler in $\mathrm{Act}(\mathbb{o}')$ and if $i \in \mathrm{Act}(\mathbb{o}')$ then i is strong. This is because no weak ruler has a strong full attacking sequence and therefore no such ruler can become strong, unless he wins a conflict with a strong ruler (in which case he replaces the unique strong ruler in the subsequent state).

Given a ruler $i \in N$ and an ownership configuration, $\mathbb{o}$, a strategy s_i of i is an *attacking strategy at* $\mathbb{o}$ if, for every ownership configuration $\mathbb{o}' \in \overline{Succ}(\mathbb{o})$ such that $i \in \mathrm{Act}(\mathbb{o})$ and $|\mathrm{Act}(\mathbb{o})| \geq 2$, and every set of rulers $P \in 2^{N \setminus \{i\}}$, $s_i(\mathbb{o}', P) \neq \varepsilon$. Thus, at state $\mathbb{o}$ and at any state following $\mathbb{o}$ in the course of the game, i never chooses to stay peaceful under s_i, unless he is not active or is the unique active ruler.

Given a ruler $i \in N$, an ownership configuration, $\mathbb{o}$, and a strategy profile of the other ruler, $\boldsymbol{s}_{-i}$, we define an attacking strategy s_i that is a *best attacking response of* i *to* $\boldsymbol{s}_{-i}$ *at* $\mathbb{o}$. The strategy is defined recursively on the set of states $\overline{Succ}(\mathbb{o}, \varnothing)$, starting from the maximal elements under $\preceq$. If $(\mathbb{o}', P)$ is such that $\mathbb{o}'$ is maximal according to $\sqsubseteq$ in $\overline{Succ}(\mathbb{o})$ then, for all $i \in N$, $s_i(\mathbb{o}', P) = \varepsilon$ (the unique feasible choice of i at $\mathbb{o}$). Otherwise, let $s_i(\mathbb{o}, P)$ be any neighboring ruler attacking whom maximises i's expected payoff across all neighbors of i at $\mathbb{o}$, given the continuation payoff determined by $\boldsymbol{s} = (s_i, \boldsymbol{s}_{-i})$ defined on states in $Succ(\mathbb{o}, P)$.

Notice that if j is such a ruler at (o, P) then, for any $P \in 2^{N\setminus\{i\}}$, attacking j maximises i's expected payoff across all neighbors of i. Moreover, generically, such a neighbor is unique.

Now we are ready to give main part of the proof. First we show, for any strategy profile, $\boldsymbol{s}$, any ownership configuration, $\text{o} \in \mathbb{O}$, any ruler $i \in N$, and any set of rulers $P \subseteq N \setminus \{i\}$, that if i is the unique strong ruler at o then any best attacking response, s_i^*, of i to $\boldsymbol{s}_{-i}$ at o yields i an expected payoff greater than $R_i(\text{o})$.

The proof is by induction on the number of active rulers at o. For the induction basis, suppose that $|\text{Act}(\text{o})| = 2$ and that i is the single strong ruler at o. Let j be the other active ruler. Since p is rich rewarding and i is strong, the other active ruler is weak and attacking him increases i's payoff in expectation. Thus the claim holds.

For the induction step, take any $2 < m \leq n$ suppose that the claim holds for any ownership configuration o with $|\text{Act}(\text{o})| < m$ active rulers. Take any ownership configuration, $\text{o} \in \mathbb{O}$, with a unique strong ruler, $i \in \text{Act}(\text{o})$. Notice that since s_i^* is an attacking strategy, so $s_i^*(\text{o}, P) \neq \varepsilon$, for all $P \in 2^{N\setminus\{i\}}$. Hence, with probability 1, a ruler choosing attack will be selected at o. Thus the strategy profile $\tilde{\boldsymbol{s}} = (s_i^*, \boldsymbol{s})$ determines a probability distribution $Q(\cdot \mid \tilde{\boldsymbol{s}}, \text{o})$ on the set $A(\text{o}) = \{(j,k) \in \text{Act}(\text{o}) \times \text{Act}(\text{o}) : j \neq k\}$ where, given $(j,k) \in A(\text{o})$, $Q(j,k \mid \tilde{\boldsymbol{s}}, \text{o})$ is the probability that ruler j attacks ruler k at o. Given two rulers, $j, k \in \text{Act}(\text{o})$, active at o let $\text{o}[j \to k]$ denote the ownership configuration resulting from j wining a conflict with k. The expected payoff to i at o, $\Pi_i(\tilde{\boldsymbol{s}} \mid \text{o})$, is equal to

$$\begin{aligned}
\Pi_i(\tilde{\boldsymbol{s}} \mid \text{o}) = \sum_{\substack{(j,k)\in A(\text{o})\\ j\neq i, k\neq i}} & Q(j,k \mid \tilde{\boldsymbol{s}}, \text{o})\Big(p(R_j(\text{o}), R_k(\text{o}))\Pi_i(\tilde{\boldsymbol{s}} \mid \text{o}[j \to k]) + \\
& \qquad\qquad p(R_k(\text{o}), R_j(\text{o}))\Pi_i(\tilde{\boldsymbol{s}} \mid \text{o}[k \to j])\Big) + \\
\sum_{(j,i)\in A(\text{o})} & Q(j,i \mid \tilde{\boldsymbol{s}}, \text{o}) p(R_i(\text{o}), R_j(\text{o}))\Pi_i(\tilde{\boldsymbol{s}} \mid \text{o}[i \to j]) + \\
& Q(i, s_i^*(\text{o}, P) \mid \tilde{\boldsymbol{s}}, \text{o}) p(R_i(\text{o}), R_{s_i^*(\text{o},P)}(\text{o}))\Pi_i(\tilde{\boldsymbol{s}} \mid \text{o}[i \to s_i^*(\text{o}, P)])
\end{aligned}$$

By the observation at the beginning of the proof, i remains a unique strong ruler at each ownership configuration $\text{o}[j,k]$ with $(j,k) \in A(\text{o})$ such that $j \neq i$ and $k \neq i$. Similarly, i remains a unique strong ruler at each ownership configuration $\text{o}[i,j]$ with $j \in A(\text{o})$. Thus, by the induction hypothesis, for all $(j,k) \in A(\text{o})$, $\Pi_i(\tilde{\boldsymbol{s}} \mid \text{o}[j \to k]) > R_i(\text{o}[j \to k])$. In the case of $j \neq i$ and $k \neq i$, $R_i(\text{o}[j \to k]) = R_i(\text{o})$. In the case of $k = i$, $p(R_i(\text{o}), R_j(\text{o}))\Pi_i(\tilde{\boldsymbol{s}} \mid \text{o}[i \to j]) >$

$p(R_i(\mathbb{o}), R_j(\mathbb{o}))R_i(\mathbb{o}[j \to k]) = p(R_i(\mathbb{o}), R_j(\mathbb{o}))(R_i(\mathbb{o}) + R_j(\mathbb{o})) > R_i(\mathbb{o})$, as $R_i(\mathbb{o}) > R_j(\mathbb{o})$ and i is rich rewarding. Hence $\Pi_i(\tilde{\boldsymbol{s}} \mid \mathbb{o}) > R_i(\mathbb{o})$.

Now, suppose that there is a unique strong node in G under resource endowment $\boldsymbol{r}$. Then there is a unique strong ruler at the ownership configuration. Take any equilibrium $\boldsymbol{s}$ of the game. By the observation above, there is a unique strong ruler at every ownership configuration $\mathbb{o} \in \mathbb{O}$. In addition, point 1 of Proposition 4 extends immediately to the short sequence of attack (the proof does not make any assumptions about the sequences that the rulers choose). Hence in the short sequence model, like in the basic model, every ownership configuration is either peaceful or conflictful under $\boldsymbol{s}$. Take any peaceful ownership configuration $\mathbb{o}$. It must be that there is a unique active ruler at $\mathbb{o}$ as otherwise, by what was shown above, if no other active ruler attacks his neighbor, the unique strong ruler attacks one of his neighbors. Hence there is fight till hegemony under $\boldsymbol{s}$. By generic uniqueness of equilibrium payoffs, the probability of becoming a hegemon is generically unique. □