Computations in monotone Floer theory

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Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. Chapters 2 and 3 of this dissertation are my own work and contain nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. Chapter 4 is based on the work done in collaboration with Renato Vianna, whereto each of us contributed equally.

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Floer theory is a rich collection of tools for studying symplectic manifolds and their Lagrangian submanifolds with the help of holomorphic curves. Its origins lie in estimating the numbers of equilibria in Hamiltonian dynamics, and more recently it has become a major component of the Homological Mirror Symmetry conjecture. This work presents several new computations in Floer theory which combine the use of geometric symmetries, naturally arising in various contexts, with advanced algebraic structures related to Floer theory, like the string maps and the Fukaya category.

The three main directions of our study are: the Floer cohomology for a pair of commuting symplectomorphisms; the Fukaya algebra of a Lagrangian submanifold invariant under a circle action; and rigidity properties of non-monotone Lagrangian submanifolds based on the use of low-area versions of the string maps.

In each of the three mentioned setups we provide concrete applications of our general results to the study of symplectic manifolds. For example, we prove that Dehn twists in most projective hypersurfaces have infinite order in the symplectic mapping class group; prove that the real projective space split-generates the Fukaya category of the complex projective space and therefore must intersect any other Lagrangian submanifold that is non-trivial in that Fukaya category; and we exhibit a continuous family of Lagrangian tori in the complex projective plane that cannot be made disjoint from the standard Clifford torus by a Hamiltonian isotopy.

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It has been a delightful experience working alongside the other symplectic topologists who were at Cambridge while I was doing my PhD—in particular Yankı Lekili and Andreas Ott,—and the ones who are still here at the time this thesis is being written: Georgios Dimitroglou Rizell, Jack Smith and Renato Vianna. They are great company.

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On a much wider scale, my parents Ivan and Lyudmila, and my girlfriend Ksenia have been making my life happy by their love.

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Chapter 1

Overview

This work belongs to the field of symplectic topology, whose objects of study are symplectic manifolds and their Lagrangian submanifolds. The main tool to study them—or rather a wide collection of far-reaching tools—is called Floer theory. There is an abundance of notions and algebraic structures involved in Floer theory, but their idea always lies in defining symplectic invariants by counting various (pseudo-)holomorphic curves, possibly with boundary and punctures, with respect to an almost complex structure J that *tames* the given symplectic form ω on a manifold. Generally, computing holomorphic curves is an extremely hard problem where new calculations are always valuable as they can provide insights into the properties of symplectic manifolds, or related subjects like Homological Mirror Symmetry.

Apart from this introduction, the present work is composed of three self-contained chapters based, respectively, on the author's paper [109], author's preprint [108], and author's joint work with Renato Vianna [110]. Each chapter develops a separate thread of general theorems and applications; their unifying feature is that they perform or use new computations of Floer-theoretic invariants in setups that exhibit different sorts of geometric symmetry, used in a crucial way.

Each of the three main chapters (that is, Chapters 2–4) has its own introductory section that explains the more specialised background and motivation relevant for the chapter. In this overview, we present a condensed summary of the most important concepts in Floer theory that will be used later, and highlight some of our main results from Chapters 2–4.

Floer cohomology for symplectomorphisms

Classical mechanical systems have the property that their phase space is a symplectic manifold (X,ω), and the system itself is given by the Hamiltonian flow of a function on X, for example the energy. This function is itself called a Hamiltonian, and in general may be time-dependent. A natural question is to give a lower estimate on the number of equilibria that the time-1 map of this flow can have. Throughout this work, our symplectic manifolds will be assumed to be compact, in which case the Arnold conjecture states that the number of fixed points of a time-1 Hamiltonian flow (provided the points are non-degenerate) cannot be lower than $\dim H^*(X;\mathbb{R})$. Classically, Floer theory was used to prove the Arnold conjecture conjecture for *monotone* and, more generally, weakly monotone symplectic manifolds; only such manifolds will be considered here. Recall that (X,ω) is called monotone if ω and $c_1(X)$ are positively proportional in $H^2(X;\mathbb{R})$, where $c_1(X)$ is the first Chern class of (TX,J) for any almost complex structure J taming ω.

The time-1 flow of a time-dependent Hamiltonian function $H\colon \mathbb{R}\times X \to \mathbb{R}$ is called a Hamiltonian symplectomorphism. It is indeed a symplectomorphism in the sense that it preserves ω, but not all symplectomorphisms of X have to be Hamiltonian. The symplectic mapping class group $Symp(X)/Ham(X)$ is defined to be the quotient of the space of all symplectomorphisms of X by Hamiltonian ones. When $\pi_1(X)$ is trivial, $Symp(X)/Ham(X)$ is a discrete group which is the symplectic counterpart of the smooth mapping class group studied in algebraic topology. The above question can be addressed to any symplectic mapping class, taking the following form: given a symplectomorphism $f\colon X\to X$, we wish to find a lower bound on the number of fixed points that any other symplectomorphism Hamiltonian isotopic to f must have. In this case, there is no conclusive answer like the Arnold conjecture, but the tool for addressing the problem stays the same; it is called Floer cohomology. A lot of foundational work is required to give its complete definition and prove the Arnold conjecture; the common references are [41–43, 91, 89, 81, 83, 93]; the excellent and well known book [73] also covers this material.

If (X,ω) is a (compact, weakly monotone) symplectic manifold and $f\colon X\to X$ is a symplectomorphism, one can define a $\mathbb{Z}/2$-graded vector space $HF^*(f)$, called the Floer cohomology of f. It categorifies the topological Lefschetz number $L(f)\in\mathbb{Z}$ in the sense that its Euler characteristic satisfies $\chi(HF^*(f)) = L(f)$. A Hamiltonian isotopy between two symplectomorphisms f,f' produces a canonical isomorphism $HF^*(f)\to HF^*(f')$. In particular, the rank of $HF^*(f)$ is an invariant of f up to Hamiltonian isotopy, which, unlike the Lefschetz number, is no longer a smooth isotopy invariant. This rank is the lower bound on the number of non-degenerate fixed points for symplectomorphisms that are Hamiltonian isotopic to f. It is known that $HF^*(\mathrm{Id}) = H^*(X)$ which constitutes the proof of the Arnold conjecture; similarly if ι is a symplectic involution (or a finite-order map) with connected fixed locus $\mathrm{Fix}\,\iota$, then $HF^*(\iota)\cong H^*(\mathrm{Fix}\,\iota)$. For a general symplectomorphism f, computing $HF^*(f)$ is very hard; one of the goals of Chapter 2 is to provide some new computations.

The complex for computing Floer cohomology is generated, as a vector space, by the fixed points of a perturbation of f by a time-1 Hamiltonian flow of an S^1-dependent Hamiltonian $H\colon\ S^1 \times X \to \mathbb{R}$. When $f = \mathrm{Id}$, the generators of the complex are simply the time-1 periodic orbits of H. The differential on this vector space counts cylinders in X solving a PDE which is a first-order perturbation of the equation for being J-holomorphic; this perturbation is determined by H. While usual holomorphic curves satisfy the removal singularity theorem at punctures, the mentioned cylinders are subject to a similar phenomenon saying that they must be asymptotic to time-1 periodic orbits of H at their infinite ends, provided the cylinders have finite energy. Because the periodic orbits are the generators of the Floer complex, counting the cylinders in an appropriate way allows to introduce a differential on this complex, defining $HF^*(f)$. Figure 1.1 summarises this informal description. In Chapter 2 we recall certain details of the definition, including the case when f is a not necessarily trivial symplectic mapping class.

Fig. 1.1 A cylinder asymptotic to two periodic orbits of H; the counts of such cylinders define Floer's differential.

The elliptic relation

Floer cohomology has a lot more structure, and in Chapter 2 we discuss a particular one that has no counterpart in classical topology. If $f, g : X \to X$ are commuting symplectomorphisms, we explain in Chapter 2 that they induce grading-preserving automorphisms

$$f_{\text{floer}} : HF^*(g) \to HF^*(g), \quad g_{\text{floer}} : HF^*(f) \to HF^*(f),$$

called the action of f on the Floer cohomology of g, and vice versa. In Chapter 2 we prove the following theorem proposed by Paul Seidel. It is called the *elliptic relation*.

Theorem A (Theorem 2.1.1)**.** *It always holds that* $STr f_{\text{floer}} = STr g_{\text{floer}}$.

Here STr is the supertrace, defined to be the difference of traces on the 0-graded and 1-graded parts of the vector space. The term "elliptic relation" is suggested by the proof, which employs gluing pseudo-holomorphic cylinders that appear in the definitions of f_{floer} and g_{floer} into pseudo-holomorphic tori, and relating those tori to each other by a change of their conformal structure.

When one of the two commuting symplectomorphisms is the identity, the theorem is nothing more than the Lefschetz fixed point theorem, but when we take g to be a symplectic involution, it produces a new lower bound on the dimension of $HF^*(f)$.

Corollary B (Proposition 2.1.4)**.** *If a symplectomorphism f commutes with a symplectic involution ι, we have* $\dim HF^*(f) \geq |L(f|_{\mathrm{Fix}\,\iota})|$.

There is an obvious forgetful map $Symp(X)/Ham(X) \to \pi_0 Diff X$ from the symplectic mapping class group to the smooth one. The first instances when this map is not surjective have been found by Paul Seidel in his thesis [92]; the examples are the so-called Dehn twists which we recall in Chapter 2. In an exact or a Calabi-Yau manifold (X,ω), Dehn twists are known to have infinite order as elements of $Symp(X)/Ham(X)$ [94]. In the next theorem, we find examples when Dehn twists have infinite order in the non-Calabi-Yau setting, which has been relatively unexplored.

Theorem C (Theorem 2.1.2)**.** *Let n be odd and $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d such that $3 \leq d \leq n$ or $d \geq 2n-3$. Let $L \subset X$ be a vanishing Lagrangian sphere for an algebraic degeneration of X. Then τ_L is a symplectomorphism which has infinite order in the group $Symp(X)/Ham(X)$, although τ_L^{2k} is smoothly isotopic to the identity for some k.*

The idea of proof is to put L and X is a position invariant under a hyperplane reflection, and to deduce using Corollary B the growth of Floer cohomology of a certain composition of iterated Dehn twists. The proof of Theorem C should hold for all degrees $d \geq 3$ if one uses virtual perturbation theory making Floer cohomology well-defined on symplectic manifolds which are not necessarily weakly monotone (the extra conditions on the degree are only used for the well-definedness of Floer theory on X).

Fukaya category and string maps

The results of Chapters 3 and 4 use more advanced algebraic structures related to Floer theory, namely Lagrangian Floer cohomology, Fukaya categories and string maps. We will now give a very brief summary of these notions, providing references to the formal definitions. A Lagrangian submanifold in a $2n$-dimensional symplectic manifold (X,ω) is an n-dimensional submanifold such that $\omega|_L \equiv 0$. Lagrangian submanifolds are very natural as far as holomorphic curves are concerned, because J-holomorphic curves with boundary on Lagrangian submanifolds share many nice features with closed holomorphic curves. Most notably, they come in finite-dimensional moduli spaces, and they satisfy Gromov compactness and gluing theorems.

For technical reasons, classical Floer theory requires to consider only *monotone* Lagrangians; this means that the Maslov class $\mu \in H^2(X,L;\mathbb{R})$ has to be a positive multiple of the symplectic area class $\omega \in H^2(X,L;\mathbb{R})$. Further, L has to be orientable and spin, unless we work with $\mathbb{Z}/2$ coefficients. Given such a Lagrangian submanifold L, one can define the complex $CF^*(L,L)$ to be generated by the time-1 Hamiltonian trajectories from the Lagrangian submanifold L back to itself. This involves a choice of a time-dependent Hamiltonian; the trajectories correspond to the intersections of L with its perturbation by the corresponding time-1 Hamiltonian symplectomorphism. The differential counts strips $[0,1]\times\mathbb{R}$ inside X with both boundary components lying on L as shown in Figure 1.2(a); they satisfy a J-holomorphic equation perturbed by the chosen Hamiltonian and are asymptotic to the trajectories generating the complex $CF^*(L,L)$. The resulting Floer cohomology is denoted by $HF^*(L,L)$. The complete definitions are found in [41, 77] and the book [47]. As before, properly setting up the definitions requires substantial analytic background, in particular to prove the relevant gluing and compactness theorems.

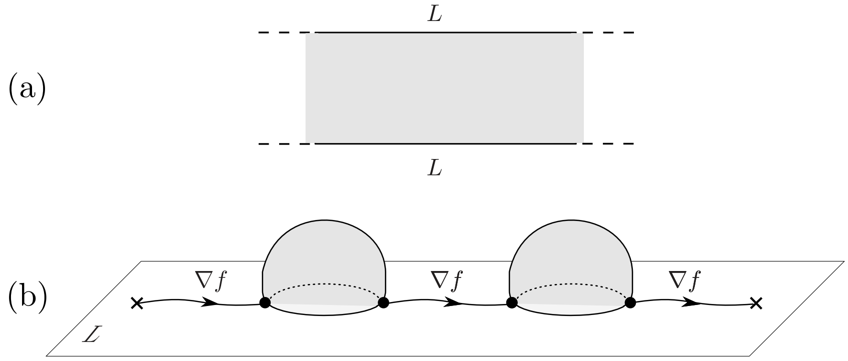


Fig. 1.2 (a): a strip in X with both boundaries on L; the counts of such pseudo-holomorphic strips, asymptotic to Hamiltonian chords from L to itself, define Floer's differential; (b): a pearly trajectory connecting two critical points of f.

When the Hamiltonian perturbation from the definition of $HF^*(L,L)$ tends to zero, the strips converge to *pearly trajectories* shown in Figure 1.2 (b); this convergence is called the adiabatic limit. Counting pearly trajectories instead of holomorphic strips is an alternative way of defining Floer cohomology, this time denoted by $HF^*(L)\cong HF^*(L,L)$. The theory of Floer cohomology using pearly trajectories is developed in [78, 17–19] and is sometimes better suited to explicit computations. Let us spell out some details of this definition. One starts with the complex generated, as a vector space, by the critical points of a Morse function f on L; the same vector space is used for defining Morse cohomology in classical topology. The differential counts holomorphic pearly trajectories that consist of gradient flowlines of f interrupted by J-holomorphic disks with boundary on L; in particular, the

Morse differential is a part of the full pearly differential (corresponding to pearly trajectories with no holomorphic disks). In this setup, the disks are just J-holomorphic; there is no Hamiltonian perturbation of the J-holomorphic equation involved.

Returning to Floer cohomology $HF^*(L,L)$ defined using a Hamiltonian perturbation, it turns out that $HF^*(L,L)$ is an associative algebra. Moreover, the chain complex $CF^*(L,L)$ supports a more refined structure called the A_∞ algebra, defined by counting pseudo-holomorphic disks in X with many boundary punctures shown in Figure 1.3(a). These provide the A_∞ structure maps

$$\mu^k\colon\ CF^*(L,L)^{\otimes k} \to CF^*(L,L)$$

that satisfy a series of quadratic identities called the A_∞ relations. The boundary condition for the disks in Figure 1.3 is the Lagrangian L, and the asymptotic conditions at the boundary punctures roughly speaking correspond to the Hamiltonian chords that generate $CF^*(L,L)$. A disk with two punctures is bi-holomorphic to the strip in Figure 1.2(a), and the μ^1 operation is precisely Floer's differential. One can more generally equip each segment of the boundary of the disk with a separate Lagrangian boundary condition L_i; this is the direction towards defining the Fukaya category of a symplectic manifold. For the definition of the Fukaya category, we refer to the books [96, 47], the papers [87, 103, 20] targeting the monotone setting specifically, and the surveys [11, 105]. An alternative pearly trajectory definition of the Fukaya category (only for exact manifolds) has been carried out in [102]; it uses *pearly trees* and we will use some aspects of this setup (that carry over to monotone manifolds) in Chapter 3.

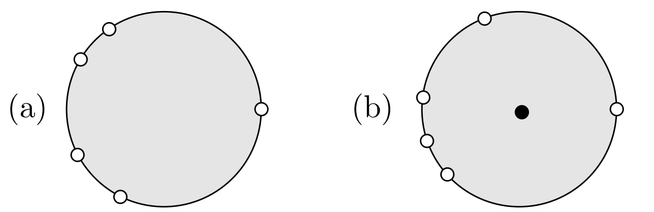


Fig. 1.3 (a): a disk with $k+1$ boundary punctures, among which one puncture is marked as an output and the remaining k ones as inputs; the counts of pseudo-holomorphic disks of this form define the A_∞ structure map $\mu^k\colon\ CF^*(L,L)^{\otimes k} \to CF^*(L,L)$; (b): the same disks with an additional interior marked point, used to define the string maps.

Other important algebraic structures related to Floer theory are the open-closed and the closed-open string maps. First, there are the so-called zeroth-order string maps (or cohomology-level string maps):

$$HF^*(L,L) \xrightarrow{\mathscr{OC}^0} QH^*(X), \quad QH^*(X) \xrightarrow{\mathscr{CO}^0} HF^*(L,L).$$

These maps can further be refined to the maps

$$HH_*(L,L) \xrightarrow{\mathscr{OC}^*} QH^*(X), \quad QH^*(X) \xrightarrow{\mathscr{CO}^*} HH^*(L,L).$$

Here $QH^*(X)$ is the small quantum cohomology of X, and $HH^*(L,L)$, $HH_*(L,L)$ is the Hochschild (co)homology of the Fukaya A_∞ algebra of L. The definition of Hochschild cohomology will be recalled in Chapter 3. For example, $\mathscr{OC}^*$ counts pseudo-holomorphic disks shown in Figure 1.3(b); they are similar to the ones used to define the A_∞ algebra structure, except that now they have an additional interior marked point, which is required to pass through a given homology cycle in X. As a result, for a homology class $a \in QH^*(X)$, we obtain an element

$$\mathscr{CO}^0(a) \in HF^*(L,L)$$

coming from the disks in Figure 1.3(b) with a single boundary puncture, serving as the output, and the maps

$$\mathscr{CO}^k(a)\colon\, CF^*(L,L)^{\otimes k} \to CF^*(L,L)$$

coming from the disks in Figure 1.3(b) with $k+1$ boundary punctures, among which k are inputs. While $\mathscr{CO}^0(a)$ is closed and gives an element of Floer cohomology $HF^*(L,L)$ invariant of all choices, the higher maps $\mathscr{CO}^k(a)$ are not invariant individually. To make an invariant out of them, the maps $\{\mathscr{CO}^k(a)\}_{k\geq 0}$ must be packaged together into a single element $\mathscr{CO}^*(a)$ of the Hochschild cohomology $HH^*(L,L)$ as written above.

In Chapter 4, where we only speak of the string maps on the Floer cohomology level, we shall write $\mathscr{OC}$, $\mathscr{CO}$ instead of $\mathscr{OC}^0$, $\mathscr{CO}^0$. In that chapter, it is more helpful to use the pearly trajectories definition of the maps $\mathscr{OC}^0$, $\mathscr{CO}^0$; these definitions are part of the pearly trajectories package (see the references above). They are more convenient because they compute $\mathscr{OC}^0$, $\mathscr{CO}^0$ from unperturbed J-holomorphic disks with boundary on L.

Circle-invariant Lagrangians and real loci

In Chapter 3 we study the symplectic topology of the real locus L of a toric variety X. Thus L is naturally a Lagrangian submanifold, and building on previous work on the subject by Haug [54], Hyvrier [57], Charette and Cornea [27] we show, for instance, the following.

Theorem D (Proposition 3.1.1)**.** *In characteristic two,* $\mathbb{R}P^n$ *split-generates the Fukaya category of* $\mathbb{C}P^n$*.*

We prove a similar theorem for some other toric varieties including the blowup of $\mathbb{C}P^n$ along a linear k-dimensional subspace, with certain restrictions on n,k. Subsequently, using one of the results of the present work but otherwise a completely different approach, Evans and Lekilı [40] proved a similar split-generation result for any compact toric Fano variety X provided the real locus $L \subset X$ is orientable.

The reader is referred to [96] for the definition of split-generation; let us mention that split-generation is interesting for at least two reasons. First, it implies a purely geometric statement about Lagrangian intersections: it follows that L must non-trivially intersect any other monotone Lagrangian L' which has non-vanishing Floer cohomology and whose so-called obstruction number equals the one of L. For example, $\mathbb{R}P^n \subset \mathbb{C}P^n$ is not displaceable by a Hamiltonian isotopy from the Lagrangian Clifford torus in $\mathbb{C}P^n$, which was earlier proved by Alston and Amorim [7]. Second, split-generation results are usually used in proofs of the Homological Mirror Symmetry conjecture. In this context, the following theorem is also interesting.

Theorem E (Corollary 3.1.2)**.** *The Fukaya* A_∞ *algebra of the Lagrangian* $\mathbb{R}P^{4n+1}$ *inside* $\mathbb{C}P^{4n+1}$ *is not formal in characteristic two.*

Formality means the existence of a quasi-isomorphism between the A_∞ algebra and its associative cohomology algebra, which in our case is the Floer cohomology. This theorem has not appeared in the literature even for the case of $S^1 \subset S^2$, and is remarkable because S^1 is topologically formal in any characteristic.

To prove Theorem D, we use a split-generation criterion originally due to Abouzaid [1], see [2, 87, 99, 103] for the relevant setup: if the closed-open map

$$\mathscr{C}\mathscr{O}^* : QH^*(X)_w \to HH^*(L,L)$$

is injective, then L split-generates the Fukaya category $\mathscr{F}uk(X)_w$. Here $QH^*(X)_w$ is a generalised eigenspace of the quantum multiplication by $c_1(X)$; see Chapter 3. Previous applications of the split-generation criterion, see e.g. [102, 103, 86], were mainly reliant on the injectivity of the cruder map $\mathscr{C}\mathscr{O}^0$, which is no longer injective in the situation of Theorem D. Therefore a computation of some "higher order" terms of the full closed-open map $\mathscr{C}\mathscr{O}^*$ is necessary to establish its injectivity. To do so, in Chapter 3 we prove a theorem that, for a given Lagrangian submanifold L invariant under a loop γ of Hamiltonian symplectomorphisms, computes a higher order term of $\mathscr{C}\mathscr{O}^*$ capturing the homology class of γ-orbits inside L. We should notice that very few direct computations of the closed-open string map $\mathscr{CO}^*$ beyond $\mathscr{CO}^0$ have been made prior to this.

Low-area Floer theory

As we have mentioned, classical Floer theory requires to work with monotone Lagrangian submanifolds. One of its applications addresses the question of displaceability, which is a natural Lagrangian version of the question about estimating the number of fixed points of a symplectomorphism. Given a pair of monotone Lagrangian submanifolds L,K, assume that $HF^*(L,K) \neq 0$. In that case, L and K are non-displaceable, meaning that there is no Hamiltonian symplectomorphism that would map L to a submanifold disjoint from K. In recent years, several new technologies have led to non-displaceability results concerning non-monotone Lagrangians, making it a flourishing area of study.

One of such technologies is called Floer cohomology with bulk deformations [48]. Using it, Fukaya, Oh, Ohta and Ono [49] proved there is a one-parametric family of Lagrangian tori $\hat{T}_a \subset S^2 \times S^2$, $a \in (0,1/2]$, such that each torus $\hat{T}_a$ is non-displaceable from itself by a Hamiltonian isotopy. The tori are invariant under the involution of $S^2 \times S^2$ permuting the factors, so taking the quotient gives a one-parametric family of tori $T_a \subset \mathbb{C}P^2$.

Conjecture F (Conjecture 4.1.1)**.** *For any $a \in (0,1/3]$, there exists no Hamiltonian diffeomorphism of $\mathbb{C}P^2$ that would displace $T_a \subset \mathbb{C}P^2$ from itself.*

The methods of [49] no longer work here because the tori T_a have vanishing Floer cohomology with respect to any (degree two) bulk deformation. To attack the problem, we introduce a new version of Floer theory for non-monotone Lagrangian submanifolds, which we call low-area Floer theory. It is based on the pearly trajectory definition of Floer cohomology but uses holomorphic disks of least area. In Chapter 4 we recall a theorem of Biran and Cornea stating that if, for two monotone Lagrangian submanifolds $L,K \subset X$, the composition of the open-closed and closed-open maps

$$HF^*(L) \xrightarrow{\mathscr{OC}^0} QH^*(X) \xrightarrow{\mathscr{CO}^0} HF^*(K)$$

does not vanish, L and K must have non-empty intersection. This obviously implies that L and K are non-displaceable, by the Hamiltonian invariance of Floer cohomology and string maps.

In Chapter 4 we prove that in favourable cases, the mentioned non-displaceability theorem continues to hold for non-monotone Lagrangian submanifolds if one considers "low-area" versions of $\mathscr{CO}^0$, $\mathscr{OC}^0$ by using holomorphic disks of least area only. Although we could not solve the above conjecture, with the help of low-area Floer theory we prove the following theorem in Chapter 4. This is the first non-displaceability result that would concern a continuous family of Lagrangians in $\mathbb{C}P^2$.

Theorem G (Theorem 4.1.2)**.** *For any $a \in (0, 1/9]$, there exists no Hamiltonian diffeomorphism of $\mathbb{C}P^2$ that would displace $T_a \subset \mathbb{C}P^2$ from the monotone Clifford torus $T_{Cl} \subset \mathbb{C}P^2$.*

The holomorphic Maslov index 2 disks with boundary on the tori T_a, T_{Cl} are known, see the references in Chapter 4. The new idea is in using these disks in a proper way, keeping in mind that one of the Lagrangians in question is not monotone. The eventual argument invokes gluing holomorphic disks into annuli and changing their conformal parameter, similarly to the way it is done in the proofs of Biran-Cornea's theorem and Abouzaid's split-generation criterion for monotone manifolds. In our proof, we use the fact that our annuli have sufficiently low area to rule out unwanted disk bubbling.

Chapter 2

Commuting symplectomorphisms and Dehn twists in divisors

This chapter is based on the author's paper [109].

2.1 Introduction

2.1.1 Overview

Let X be a symplectic manifold and $Symp(X)/Ham(X)$ be the group of all symplectomorphisms of X modulo Hamiltonian isotopy. When X is simply-connected, this group is the same as $\pi_0 Symp(X)$. If one denotes by $\pi_0 Diff(X)$ the smooth mapping class group, there is an obvious forgetful map

$$Symp(X)/Ham(X) \quad \xrightarrow{forgetful} \quad \pi_0 Diff(X).$$

Paul Seidel in his thesis [92] found examples when this map is not injective: if X is any complete intersection of complex dimension 2 other than $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, and $\tau\colon X \to X$ is a certain symplectomorphism called the Dehn twist, then τ^2 is smoothly isotopic to the identity, but not Hamiltonian isotopic to the identity. Later Seidel proved [94] that the kernel of the above map is infinite for some K3 surfaces, again by considering the group generated by a Dehn twist. Using a new technique, we study Dehn twists in certain divisors (the main examples are divisors in Grassmannians) and extend the range of examples when the above forgetful map has infinite kernel.

Suppose X satisfies the so-called W^+ condition, which is slighly stronger than weak monotonicity. We define, for two commuting symplectomorphisms $f, g\colon X \to X$, their actions on Floer cohomology $f_{\mathrm{floer}}\colon HF^*(g) \to HF^*(g)$, $g_{\mathrm{floer}}\colon HF^*(f) \to HF^*(f)$. We then prove a theorem which was proposed by Paul Seidel, cf. [99, Remark 4.1], who suggested it be called the elliptic relation.

Theorem 2.1.1 (Elliptic relation)**.** *If X is a symplectic manifold satisfying the W^+ condition and $f,g\colon X \to X$ are two commuting symplectomorphisms, then*

$$STr(f_{\mathrm{floer}}) \;=\; STr(g_{\mathrm{floer}}) \in \Lambda.$$

Here Λ is the Novikov field. In the rest of the introduction, we explain the elliptic relation, state its Lagrangian version, and consider applications to Dehn twists in divisors. We begin by discussing our results regarding Dehn twists.

2.1.2 Order of Dehn twists in divisors

Let $Gr(k,n)$ be the Grassmannian of k-planes in $\mathbb{C}^n$. Let $\mathscr{O}(d)$ be the line bundle on $Gr(k,n)$ which is the pullback of $\mathscr{O}_{\mathbb{P}^N}(d)$ under the Plücker embedding $Gr(k,n) \subset \mathbb{P}^N$. Consider a smooth divisor $X \subset Gr(k,n)$ in the linear system $|\mathscr{O}(d)| = \mathbb{P}H^0(Gr(k,n),\mathscr{O}(d))$. The results below are interesting even for $Gr(1,n) = \mathbb{P}^{n-1}$, so for simplicity one can take $X \subset \mathbb{P}^{n-1}$ to be a smooth projective hypersurface of degree d throughout this subsection.

For $d \geq 2$, X contains a class of Lagrangian spheres which we call $|\mathscr{O}(d)|$-vanishing Lagrangian spheres, which, briefly, are vanishing cycles for algebraic degenerations of X inside the linear system $|\mathscr{O}(d)|$. To every parametrised Lagrangian sphere $L \subset X$ one associates a symplectomorphism $\tau_L\colon X \to X$ called the Dehn twist around L. (The definitions are given in Section 2.3.) We prove the following.

Theorem 2.1.2. *Let $X \subset Gr(k,n)$ be a smooth divisor in the linear system $|\mathscr{O}(d)|$, and $L \subset X$ be an $|\mathscr{O}(d)|$-vanishing Lagrangian sphere. Suppose*

$$3 \leq d \leq n \qquad \text{or} \qquad d \geq k(n-k)+n-2.$$

Then the Hamiltonian isotopy class of τ_L is an element of infinite order in the group $Symp(X)/Ham(X)$.

When $d=2$ and $k=1$ (X is a projective quadric), τ_L has order 1 or 2 depending on the parity of n [104, Lemma 4.2]. While our proof crucially uses $d \geq 3$, further restrictions on d are only needed to make X satisfy the W^+ condition, so that the “classical” definition of Floer cohomology of symplectomorphisms $X \to X$ applies. There are techniques [51] defining Floer cohomology of symplectomorphisms on arbitrary symplectic manifolds. With their help the proof of Theorem 2.1.2 (and of Theorem 2.1.1) should work for all $d \geq 3$.

Recall the forgetful map $Symp(X)/Ham(X) \to \pi_0 Diff(X)$. If $\dim_{\mathbb{C}} X$ is odd and $d \geq 3$, the image of τ_L has infinite order in $\pi_0 Diff(X)$ by the Picard-Lefschetz formula, so Theorem 2.1.2 becomes trivial. However, when $\dim_{\mathbb{C}} X$ is even, the image of τ_L has finite order in $\pi_0 Diff(X)$ (see Subsection 2.3.4 for details), so Theorem 2.1.2 is really of symplectic nature in this case. When X is Calabi-Yau ($d = n$), Theorem 2.1.2 follows from a grading argument of Paul Seidel [94]. Theorem 2.1.2 is new in all cases when $\dim_{\mathbb{C}} X$ is even and $d \neq n$. For instance, it appears to be new even for the cubic surface $X \subset \mathbb{P}^3$.

Let

$$\Delta \subset \mathbb{P}H^0(Gr(k,n), \mathscr{O}(d))$$

be the discriminant variety parameterising all singular divisors in $|\mathscr{O}(d)|$. Theorem 2.1.2 implies a corollary about the fundamental group of the complement to the discriminant. Fix a divisor $X \in |\mathscr{O}(d)|$. For any family $X_t \subset Gr(k,n)$ of smooth divisors in $|\mathscr{O}(d)|$, $t \in [0,1]$, there is a symplectic parallel transport map, a symplectomorphism $X_0 \to X_1$ which depends up to Hamiltonian isotopy only on the homotopy class of the path X_t relative to its endpoints. Applied to loops, parallel transport gives the symplectic monodromy map

$$\pi_1\left(\mathbb{P}H^0(Gr(k,n), \mathscr{O}(d)) \setminus \Delta\right) \xrightarrow{monodromy} Symp(X)/Ham(X).$$

The discriminant complement contains a distinguished conjugacy class of loops γ called meridian loops. A meridian loop

$$\gamma \subset \mathbb{P}H^0(Gr(k,n), \mathscr{O}(d)) \setminus \Delta$$

is the boundary of a 2-disk in $\mathbb{P}H^0(Gr(n,k), \mathscr{O}(d))$ that intersects Δ transversely once. The image of such a loop under the monodromy map is the Dehn twist τ_L where $L \subset X$ is an $|\mathscr{O}(d)|$-vanishing Lagrangian sphere. Theorem 2.1.2 implies the following.

Corollary 2.1.3. *If $3 \leq d \leq n$ or $d \geq k(n-k)+n-2$, and $\gamma \subset \mathbb{P}H^0(Gr(k,n), \mathscr{O}(d)) \setminus \Delta$ is a meridian loop, then*

$$[\gamma] \in \pi_1\left(\mathbb{P}H^0(Gr(k,n), \mathscr{O}(d)) \setminus \Delta\right) \quad \textit{is an element of infinite order.}$$

Note that $[\gamma] \in H_1\left(\mathbb{P}H^0(Gr(k,n), \mathscr{O}(d)) \setminus \Delta; \mathbb{Z}\right)$ has finite order. For the projective space $Gr(1,n) = \mathbb{P}^{n-1}$, the fundamental group $\pi_1(\mathbb{P}H^0(\mathbb{P}^{n-1}, \mathscr{O}(d)) \setminus \Delta)$ is computed by Lönne in [69] and implies Corollary 2.1.3 for $k = 1$. For $k \neq 1$, the corresponding fundamental group seems not to be studied, but Corollary 2.1.3 should allow a more straightforward proof, suggested to us by Dmitri Panov. Namely, assume $\dim_{\mathbb{C}} X$ is even (otherwise the corollary follows from the fact the Dehn twist has infinite order topologically) and consider the $d:1$ cover of $Gr(k,n)$ branched along X, which now has odd complex dimension. A nodal degeneration of X provides an A_d-degeneration of the cover, and the monodromy around such a degeneration, which is a composition of Dehn twists around a chain of Lagrangian spheres, has infinite order in the smooth mapping class group (which uses the Picard-Lefschetz formula and the fact the spheres are now odd-dimensional). This observation is enough to imply Corollary 2.1.3, bypassing the need to consider the Dehn twist in X itself. However, we decided to keep Corollary 2.1.3 to add an additional context to the main theorems.

We prove analogues of Theorem 2.1.2 and Corollary 2.1.3 for divisors in some very ample line bundles $\mathscr{L} \to Y$, where Y is a Kähler manifold which carries a holomorphic involution with certain properties. The precise statement is postponed to Subsection 2.1.7.

2.1.3 Elliptic relation for commuting symplectomorphisms

To prove Theorem 2.1.2, we use the elliptic relation (Theorem 2.1.1) which we now discuss.

Let X be a symplectic manifold satisfying the W^+ condition explained in Section 2.2; for example, X can be a Kähler manifold which is either Fano, or whose canonical class K_X is sufficiently positive. Given a symplectomorphism $f\colon X \to X$, one defines its Floer cohomology $HF^*(f)$. It is a $\mathbb{Z}_2$-graded vector space, $HF^*(f) = HF^0(f) \oplus HF^1(f)$, over the Novikov field

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i q^{\omega_i} : \ a_i \in \mathbb{C},\ \omega_i \in \mathbb{R},\ \lim_{i\to\infty} \omega_i = +\infty \right\}.$$

For any two commuting symplectomorphisms $f,g\colon X \to X$ we define invertible automorphisms

$$g_{\mathrm{floer}}\colon HF^*(f) \to HF^*(f) \quad \text{and} \quad f_{\mathrm{floer}}\colon HF^*(g) \to HF^*(g).$$

The construction of $HF^*(f)$ uses a time-dependent almost complex structure J and a Hamiltonian H to define a vector space $HF^*(f;J,H)$. This vector space is canonically isomorphic (on the chain level) to $HF^*(gfg^{-1};g^*J,H\circ g)$ by composing all pseudo-holomorphic curves with g. If f,g commute, g_{floer} is the composition of isomorphisms

$$HF^*(f;J,H) \longrightarrow HF^*(gfg^{-1};g^*J,H\circ g) = HF^*(f;g^*J,H\circ g) \longrightarrow HF^*(f;J,H)$$

where the last arrow is the continuation map associated to a homotopy of data from $(g^*J, H\circ g)$ to (J,H).

The automorphisms $f_{\text{floer}}, g_{\text{floer}}$ have zero degree, and one can define their supertrace:

$$STr(g_{\text{floer}}) \coloneqq Tr(g_{\text{floer}}|_{HF^0(f)}) - Tr(g_{\text{floer}}|_{HF^1(f)}) \ \in \ \Lambda.$$

Recall that Theorem 2.1.1 asserts the equality $STr(f_{\text{floer}}) = STr(g_{\text{floer}})$.

Now suppose a symplectomorphism f commutes with a finite-order symplectomorphism ϕ, $\phi^k = \mathrm{Id}$, with fixed locus X^ϕ. Then X^ϕ is a disjoint union of symplectic submanifolds. Using an argument reminiscent of the PSS isomorphism, we show that

$$STr(f_{\text{floer}}\colon\ HF^*(\phi) \to HF^*(\phi)) = L(f|_{X^\phi})\cdot q^0.$$

The right hand side is the topological Lefschetz number

$$L(f|_{X^\phi}) = Tr(f^*|_{H^{even}(X^\phi)} - Tr(f^*|_{H^{odd}(X^\phi)}))$$

where $f^*\colon\ H^*(X^\phi)\to H^*(X^\phi)$ is the classical action on the cohomology of X^ϕ. On the other hand, using that ϕ has finite order, we show that $STr(\phi_{\text{floer}}\colon\ HF^*(f)\to HF^*(f))$ equals $a\cdot q^0$ where $|a| \leq \dim_\Lambda HF^*(f)$. Combining this with the elliptic relation, we obtain the following corollary.

Proposition 2.1.4. *Let X be a symplectic manifold satisfying the W^+ condition, $f,\phi\colon\ X\to X$ two commuting symplectomorphisms and $\phi^k = \mathrm{Id}$. Then*

$$\dim_\Lambda HF^*(f) \geq |L(f|_{X^\phi})|.$$

Remark 2.1.1. The fixed locus X^ϕ is allowed to be disconnected, with components of different dimensions.

Remark 2.1.2. If $f\colon\ X\to X$ is a diffeomorphism with smooth fixed locus X^f, such that $\mathrm{Id}-df(x)|_{N_x\Sigma}$ is non-degenerate on the normal space $N_x\Sigma\subset T_xX$ to any connected component $\Sigma\subset X^f$ for every $x\in\Sigma$, then

$$L(f) = \sum_{\Sigma\subset X^f} \mathrm{sign}\left(\det(\mathrm{Id}-df|_{N_x\Sigma})\right)\cdot\chi(\Sigma).$$

Consequently, if $\phi,\psi\colon\ X\to X$ are finite order symplectomorphisms, we get $L(\phi|_{X^\psi}) = L(\psi|_{X^\phi}) = \chi(X^\phi\cap X^\psi)$, provided the latter intersection is clean. This agrees with the elliptic relation and the topological interpretation of the Floer-homological actions for finite order maps.

Remark 2.1.3. It is possible to give a more straightforward proof of Proposition 2.1.4 which does not appeal to Theorem 2.1.1, but still requires some analysis in the spirit of [96, Lemma 14.11]. See Remark 2.2.7 for more details.

Remark 2.1.4. Theorem 2.1.1 holds when f,g commute only up to Hamiltonian isotopy, and more generally when fg^{-1} is isomorphic to Id in the Donaldson category, whose objects are symplectomorphisms of X and $Hom(f,g) = HF^*(fg^{-1})$; the proofs require only minor modifications. In Proposition 2.1.4, f,g can also be allowed to commute up to Hamiltonian isotopy.

2.1.4 Outline of proof of Theorem 2.1.1

The complete proof of Theorem 2.1.1 with all the necessary definitions is given in Section 2.2. Here we provide a sketch, illustrated by Figure 2.1, and indicate the main technical issue we have to solve.

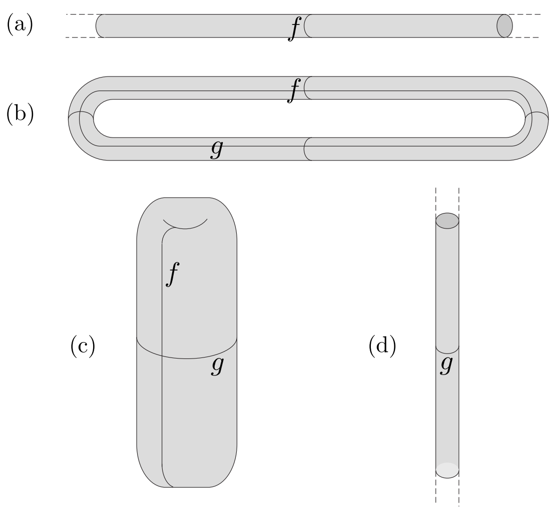


Fig. 2.1 Changing the base of a symplectic fibration in the proof of Theorem 2.1.1.

Let f,g be two commuting symplectomorphisms. By our definition, the supertrace $STr(g_{\text{floer}})$ is computed by counting certain solutions to Floer's continuation equation, or equivalently by counting holomorphic sections of a certain symplectic fibration $E_f \to S^1 \times \mathbb{R}$, see Figure 2.1(a). This fibration has monodromy f along S^1, and an almost complex structure that differs by the action of g over the two ends of the cylinder. We count only those sections whose asymptotics differ by the action of g over the ends of the cylinder. One can therefore glue the fibration, together with the almost complex structure, into a fibration $E_{f,g} \to S^1 \times S^1$.

A gluing theorem in Symplectic Field Theory gives a bijection between holomorphic sections $S^1 \times \mathbb{R} \to E_f$ (with asymptotics as above) and all holomorphic sections $S^1 \times S^1 \to E_{f,g}$ where $S^1 \times S^1$ is endowed with the complex structure which is very "long" in the direction of the second S^1-factor: see Figure 2.1(b). We will refer to this bijection by $(*)$ in the next few paragraphs.

On the other hand, the count of holomorphic sections $S^1 \times S^1 \to E_{f,g}$ does not depend on the chosen complex structure on $S^1 \times S^1$. Take another complex structure on $S^1 \times S^1$ which is "long" in the first S^1-factor instead of the second one, see Figure 2.1(c). The same gluing argument as above $(*)$ implies that the count of holomorphic sections $S^1 \times S^1 \to E_{f,g}$ is equal to the count of holomorphic sections $\mathbb{R} \times S^1 \to E_g$ (with asymptotics different by the action of f over the ends of the cylinder), where $E_g \to \mathbb{R} \times S^1$ is the fibration obtained by cutting $E_{f,g}$ along the first S^1-factor, see Figure 2.1(d). Similarly to what we began with, the latter count of holomorphic sections over $\mathbb{R} \times S^1$ gives $STr(f_{\text{floer}})$.

The key difficulty in upgrading this sketch to a proof is to determine how the bijection $(*)$ behaves with respect to the signs attached to sections over the cylinder (which in general depend on the choice of a "coherent orientation", but are canonical for sections contributing to the supertrace), and signs canonically attached to sections over the torus. The outcome is that $(*)$ multiplies signs by $(-1)^{\deg x}$ where x is a $\pm\infty$ asymptotic periodic orbit of the section over the cylinder. (The $\pm\infty$ asymptotics differ by g and thus have the same degree.) This is Formula (2.27) in Section 2.2. It explains why Theorem 2.1.1 is an equality between supertraces and not usual traces. (We have not found Formula (2.27) elsewhere in the literature. Coherent orientations in SFT are discussed in [37, 24], see especially [24, Corollary 7], but don't seem to give the result we need).

Remark 2.1.5. As the proof uses the torus with different complex structures (i.e. elliptic curves), this justifies the name "elliptic relation". There is some categorical perspective to the elliptic relation, as well: Ben-Zvi and Nadler [14, Theorem 1.2] obtained an equality between the so-called "secondary traces" in a 2-category, which also comes from cutting the torus into pieces in two different ways (however, not into two different cylinders as we do).

2.1.5 Elliptic relation for invariant Lagrangians

Before explaining how the elliptic relation helps to prove Theorem 2.1.2, let us discuss its Lagrangian version. The coefficient field is still Λ. Definitions and sketch proofs are briefly presented in Subsection 2.2.13.

Let X be a connected monotone symplectic manifold (e.g. complex Fano variety), and $L_1, L_2 \subset X$ monotone Lagrangians (e.g. simply connected). Suppose there is a symplectomorphism $\phi\colon X \to X$ such that $\phi(L_1) = L_1$, $\phi(L_2) = L_2$. Under a condition involving spin structures, formulated later as Hypothesis 2.2.17, a version of the open-closed string map provides twisted cohomology classes $[L_1]^\phi \in HF^*(\phi)$, $[L_2]^{\phi^{-1}} \in HF^*(\phi^{-1})$. Consider the quantum product $[L_1]^\phi * [L_2]^{\phi^{-1}} \in QH^*(X)$ and the map $\chi\colon QH^*(X) \to \Lambda$ which is the integration over $[X]$ (sending the volume form to 1 and all elements of $H^{<2n}(X)$, seen as elements of $QH^*(X)$, to 0). Under the assumptions of the next theorem, there is again an action $\phi_{\text{floer}}\colon HF^*(L_1,L_2) \to HF^*(L_1,L_2)$, with Floer cohomology taken over Λ.

Theorem 2.1.5 (Elliptic relation)**.** *Suppose (X,L_1,L_2) are monotone, $\phi\colon X \to X$ is a symplectomorphism, $\phi(L_i) = L_i$. If the base field has* char $\neq 2$*, suppose the L_i are orientable and Hypothesis 2.2.17 is satisfied (e.g. the L_i are simply-connected). Then*

$$STr(\phi_{\text{floer}}) = \chi\left([L_1]^\phi * [L_2]^{\phi^{-1}}\right).$$

If $\phi^k = \mathrm{Id}$ and the fixed loci $L_i^\phi \subset X^\phi$ are smooth and orientable, the q^0-term of the right hand side equals the classical homological intersection $[L_1^\phi]\cdot[L_2^\phi] \in \mathbb{Z}$ inside X^ϕ, where $[L_i^\phi] \in H_{\dim_{\mathbb{R}} X/2}(X;\mathbb{Z})$. On the other hand, eigenvalue decomposition of ϕ_{floer} implies that the left hand side equals $a\cdot q^0$ with $a \in \mathbb{C}$, $|a| \leq \dim_\Lambda HF^*(L_1,L_2)$. The elliptic relation yields the following analogue of Proposition 2.1.4.

Proposition 2.1.6. *Under the assumptions of Theorem 2.1.5, if $\phi^k = \mathrm{Id}$ and the fixed loci L_i^ϕ, X^ϕ are smooth and orientable then*

$$\dim_\Lambda HF^*(L_1,L_2) \geq \left|[L_1^\phi]\cdot[L_2^\phi]\right|.$$

As our Lagrangians are monotone, we can pass from Λ-coefficients to the base field (e.g. $\mathbb{C}$ or $\mathbb{Z}/2\mathbb{Z}$) without changing the dimensions of Floer cohomology [111, Remark 4.4]. So Proposition 2.1.6 gives the same bound on $\dim HF^*(L_1,L_2;\mathbb{C})$ or $\dim HF^*(L_1,L_2;\mathbb{Z}/2\mathbb{Z})$. However, the proof of Proposition 2.1.6 crucially uses Theorem 2.1.5 over Λ, as can be seen from the sketch we presented.

As a simple application of Proposition 2.1.6, consider the hyperplane reflection ι on $\mathbb{CP}^n$ so that for the Lagrangian $\mathbb{R}P^n$ we have $(\mathbb{RP}^n)^\iota = \mathbb{RP}^{n-1}$. Suppose n is odd, so that over $\mathbb{Z}/2$ we have $\chi(\mathbb{R}P^n) = 0$ and $\chi(\mathbb{R}P^{n-1}) = 1$. Then Proposition 2.1.6 implies that $\dim HF^*(\mathbb{RP}^n,\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \geq 1$. This Floer cohomology is actually known to be isomorphic to $H^*(\mathbb{R}P^n;\mathbb{Z}/2)$; see Chapter 3.

In Section 2.6 we provide a more interesting application of Proposition 2.1.6. Namely, we prove that for $L \subset X$ as in Theorem 2.1.2, and if X is in addition Fano and even-dimensional, there is an isomorphism of rings $HF^*(L,L;\mathbb{C}) \cong \mathbb{C}[x]/x^2$. For Lagrangian spheres in the cubic surface, this was proved by Sheridan [103], and after our results had appeared in [109], it was observed by Biran and Membrez [21, Subsection 1.3.2] that for a Lagrangian sphere in a projective hypersurface, which is Fano and of degree at least 3, the isomorphism $HF^*(L,L;\mathbb{C}) \cong \mathbb{C}[x]/x^2$ follows from the known structure of $QH^*(X)$, regardless of the complex dimension of X. Our approach is different: it does not use any knowledge of $QH^*(X)$, and works for hypersurfaces in Grassmannians as well as in some more abstract cases discussed in Section 2.6.

Remark 2.1.6. The action ϕ_{floer} on $HF^*(L_1,L_2)$ (as well the actions in the case of two commuting symplectomorphisms) can be defined using functors coming from Lagrangian correspondences [112, 113]. It is possible that the two versions of the elliptic relation admit a generalisation for Lagrangian correspondences.

2.1.6 Outline of proof of Theorem 2.1.2

We have already mentioned that Theorem 2.1.2 holds for topological reasons when $\dim X$ is odd. Suppose therefore that $\dim_{\mathbb{C}} Gr(k,n)$ is odd, so that $\dim_{\mathbb{C}} X$ is even. The Grassmannian has an involution ι whose fixed locus contains an even-dimensional connected component $\tilde{\Sigma} \subset Gr(k,n)$. For example, when $k=1$ we can take the involution $(x_1 : x_2 : x_3 : x_4 : \ldots : x_n) \mapsto (-x_1 : -x_2 : -x_3 : x_4 : \ldots : x_n)$ and $\tilde{\Sigma} = \mathbb{P}^2(x_1 : x_2 : x_3)$.

The key idea of reducing Theorem 2.1.2 to Proposition 2.1.4 is the following construction performed in Section 2.4. We construct a smooth divisor $X \subset Gr(k,n)$ invariant under ι such that the fixed locus X^ι of the involution $\iota|_X$ contains an odd-dimensional connected component $\Sigma = \tilde{\Sigma} \cap X$. Next, we construct two ι-invariant $|\mathscr{O}(d)|$-vanishing Lagrangian spheres $L_1, L_2 \subset X$ which intersect each other transversely once. Moreover, the fixed loci $L_i^\iota := L_i \cap \Sigma$, $i=1,2$, are Lagrangian spheres in Σ which intersect each other transversely once, see Figure 2. This is where we need $d \geq 3$.

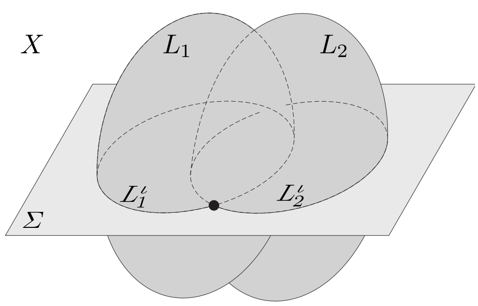


Fig. 2.2 Invariant Lagrangian spheres L_1 and L_2 used in the proof of Theorem 2.1.2.

Theorem 2.1.2 is proved in Section 2.5. Consider the product of iterated Dehn twists $\tau_{L_1}^{2k}\tau_{L_2}^{2k}$. Because L_1, L_2 are ι-invariant, $\tau_{L_1}^{2k}\tau_{L_2}^{2k}$ can be made ι-equivariant. The Lefschetz number of $(\tau_{L_1}^{2k}\tau_{L_2}^{2k})|_\Sigma = \tau_{L_1^\iota}^{2k}\tau_{L_2^\iota}^{2k}$ on Σ is equal to $c-4k^2$, where c is a constant. This follows from the Picard-Lefschetz formula and crucially uses the fact $\dim\Sigma$ is odd. If $\dim\Sigma$ were even, the trace would be independent of k. Consequently by Proposition 2.1.4, $\dim HF^*(\tau_{L_1}^{2k}\tau_{L_2}^{2k})$ grows with k.

Finally we note that L_1, L_2 from our construction can be taken one to another by a symplectomorphism of X. This means τ_{L_1} and τ_{L_2} are conjugate. If $\tau_{L_1}^{2k}$ was Hamiltonian isotopic to Id, then so would be $\tau_{L_2}^{2k}$ and the product $\tau_{L_1}^{2k}\tau_{L_2}^{2k}$. This contradicts the growth of Floer cohomology from above, and proves Theorem 2.1.2 for the specially constructed $|\mathscr{O}(d)|$-vanishing Lagrangian sphere $L_1 \subset X$. If X' is another smooth divisor linearly equivalent to X and $L' \subset X'$ is another $|\mathscr{O}(d)|$-vanishing Lagrangian sphere, Lemma 2.3.8 says there is a symplectomorphism $X \to X'$ taking L to L'. This implies Theorem 2.1.2 in general.

2.1.7 An extension of Theorem 2.1.2

Theorem 2.1.2 is a particular case of the more general, but also more technical theorem which we now state. Let $\mathscr{L}$ be a very ample line bundle over a Kähler manifold Y. It gives an embedding $Y \subset \mathbb{P}^N := \mathbb{P}H^0(Y,\mathscr{L})^*$.

Suppose $\iota\colon Y \to Y$ is a holomorphic involution which lifts to an automorphism of $\mathscr{L}$. The map ι induces a linear involution on $H^0(Y,\mathscr{L})^*$, splitting it into the direct sum of the ± 1 eigenspaces $H^0(Y,\mathscr{L})^*_\pm$. Let $\Pi_\pm \subset \mathbb{P}^N$ be the projectivisations of these eigenspaces. The fixed locus $Y^\iota \subset Y$ of the involution ι is:

$$Y^\iota = (\Pi_+ \sqcup \Pi_-) \cap Y,$$

where the intersection is taken inside $\mathbb{P}^N$. It is automatically smooth, but can have many connected components because the intersections $\Pi_+ \cap Y$, $\Pi_- \cap Y$ may be disconnected.

Theorem 2.1.7. *Under the above notation and assumptions, fix $d \geq 3$ and let $H^0(Y,\mathscr{L}^{\otimes d})_\pm$ denote the ± 1-eigenspace of the involution on $H^0(Y,\mathscr{L}^{\otimes d})$ induced by ι. Further, suppose one of the following:*

(a) d is even, and

Y^ι contains a connected component $\tilde{\Sigma}$ such that $\dim_{\mathbb{C}}\tilde{\Sigma}$ *is even;*

(b) d is odd,

there is a smooth divisor in the linear system $\mathbb{P}H^0(Y,\mathscr{L}^{\otimes d})_+$, and

$\Pi_+ \cap Y$ contains a connected component $\tilde{\Sigma}$ such that $\dim_{\mathbb{C}}\tilde{\Sigma}$ *is even.*

Let $X \subset Y$ be a smooth divisor in the linear system $|\mathscr{L}^{\otimes d}|$ and $L \subset X$ an $|\mathscr{L}^{\otimes d}|$-vanishing Lagrangian sphere. Denote by τ_L the Dehn twist around L, and assume X satisfies the W^+ condition. Then the Hamiltonian isotopy class of τ_L is an element of infinite order in the group $Symp(X)/Ham(X)$. The same is true if we replace symbols $+$ with symbols $-$ in Case (b).

Like Theorem 2.1.2, Theorem 2.1.7 is new when $\dim_{\mathbb{C}} X$ is even and X is not Calabi-Yau.

In Case (a), the existence of a smooth ι-invariant divisor X follows from Bertini's theorem, so it is not included as a condition of the theorem. In Case (b), an invariant divisor can sometimes be found using a strong Bertini theorem [34, Corollary 2.4], which gives the following.

Lemma 2.1.8. *Under the conditions of Theorem 2.1.7, let d be odd. There is a smooth divisor in the linear system $\mathbb{P}H^0(Y,\mathscr{L}^{\otimes d})_{\pm}$ if every connected component of $\Pi_{\mp} \cap Y$ has dimension less than $\frac{1}{2}\dim Y$.*

As in the beginning of the introduction, we have the following corollary.

Corollary 2.1.9. *Under conditions of Theorem 2.1.7, let $\gamma \subset \mathbb{P}H^0(Y,\mathscr{L}^{\otimes d}) \setminus \Delta$ be a meridian loop, defined analogously to one in the paragraph before Corollary 2.1.3. Then*

$$[\gamma] \in \pi_1\left(\mathbb{P}H^0(Y,\mathscr{L}^{\otimes d}) \setminus \Delta\right) \quad \text{is an element of infinite order.}$$

We prove these statements in Section 2.5. We have earlier explained the plan of proof of Theorem 2.1.2; actually we follow this plan to prove the general Theorem 2.1.7 first, and then derive Theorem 2.1.2 from it.

2.1.8 Equivariant transversality approaches

This supplementary subsection is not used in the sequel. Computations of Floer cohomology in the presence of a symplectic involution were discussed by Khovanov and Seidel [61], and Seidel and Smith [101]. Both papers imposed restrictive conditions on the involution which allow one to choose a regular equivariant almost complex structure for computing Floer cohomology.

In [101], it is proved that

$$\dim HF^*(L_1,L_2;\mathbb{Z}/2) \geq \dim HF^*(L_1^\iota,L_2^\iota;\mathbb{Z}/2)$$

when there exists a stable normal trivialisation of the normal bundle to X^ι respecting the L_i. In particular, the Chern classes of this normal bundle should vanish. The right-hand side is

Floer cohomology inside X^ι, where L_i^ι are the fixed loci of L_i and X^ι is the fixed locus of X. Sometimes the right-hand side is easier to compute than the left-hand side (e.g. when all intersection points $L_1^\iota \cap L_2^\iota$ have the same sign). However, the condition on the normal bundle makes this estimate inapplicable to divisors in $Gr(k,n)$.

In a very special case, [61] proves that

$$\dim HF^*(L_1, L_2; \mathbb{Z}/2) = |L_1^\iota \cap L_2^\iota|$$

where the right hand side is the unsigned count of intersection points. The assumption is, roughly, that the fixed locus X^ι has real dimension 2 and $L_1^\iota, L_2^\iota \subset X^\iota$ are curves having minimal intersection in their homotopy class. One could prove a $\mathbb{C}$-version of this equality if the L_i admit ι-equivariant Pin strictures, and apply it to divisors in $\mathbb{P}^{n-1} = Gr(1,n)$, i.e. projective hypersurfaces (thus giving an alternative proof of Theorem 2.1.2 in this case). However, it cannot be applied to divisors in general Grassmannians. When $k > 2$, $Gr(k,n)$ has no holomorphic involution with a connected component of complex dimension 2; this is easy to check because all holomorphic automorphisms $Gr(k,n)$ come from linear ones on $\mathbb{C}^n$, with a single exception when $n = 2k$ [33, Theorem 1.1 (Chow)].

2.2 The elliptic relation

This section proves the elliptic relation for symplectomorphisms (together with its corollary, Proposition 2.1.4) and sketches a proof of the Lagrangian elliptic relation.

2.2.1 Floer cohomology and continuation maps

Definition 2.2.1 (The W^+ condition)**.** A symplectic manifold (X,ω) of dimension $2n$ satisfies the W^+ condition [93], if for every $A \in \pi_2(X)$

$$2 - n \le c_1(A) \le -1 \quad \Longrightarrow \quad \omega(A) \le 0.$$

Let (X,ω) be a compact symplectic manifold satisfying the W^+ condition. Fix a symplectomorphism $f\colon X \to X$. In this subsection we recall the definition of Floer cohomology $HF^*(f)$; see Chapter 1 for the references. Take a family of ω-tame almost complex structures J_s on X, and a family of Hamiltonian functions $H_s\colon X \to \mathbb{R}$, $s \in \mathbb{R}$. They must be f-periodic:

$$H_s = H_{s+1} \circ f, \qquad J_s = f^* J_{s+1}. \tag{2.1}$$

By X_{H_s} we denote the Hamiltonian vector field of H_s, and by $\psi_s \colon X \to X$ the Hamiltonian flow:

$$d\psi_s/ds = X_{H_s} \circ \psi_s, \qquad \psi_0 = \mathrm{Id}. \tag{2.2}$$

The following equation on $u(s,t) \colon \mathbb{R}^2 \to X$ is called Floer's equation:

$$\partial u/\partial t + J_s(u)(\partial u/\partial s - X_{H_s}(u)) = 0. \tag{2.3}$$

This equation comes with the periodicity conditions

$$u(s+1,t) = f(u(s,t)). \tag{2.4}$$

Denote

$$f_H := \psi_1^{-1} \circ f \in Symp(X). \tag{2.5}$$

(The correct notation would be f_{H_s}, but we stick to f_H for brevity). Suppose the fixed points of f_H are isolated and non-degenerate (that is to say, for every $x \in \mathrm{Fix}\, f_H$, $\ker(\mathrm{Id} - df_H(x)) = 0$). Then finite energy solutions to Floer's equation have the following convergence property. There exist points x,y such that

$$\lim_{t\to-\infty} u(s,t) = \psi_s(x), \quad \lim_{t\to+\infty} u(s,t) = \psi_s(y), \quad x,y \in \mathrm{Fix}\, f_H. \tag{2.6}$$

For $x,y \in \mathrm{Fix}\, f_H$, let $\mathscr{M}(x,y;J_s,H_s)$ be the moduli space of all solutions to Floer's equation (2.3) with limits (2.6). For regular J_s,H_s, the moduli space is a manifold which is a disjoint union of the k-dimensional pieces $\mathscr{M}^k(x,y;J_s,H_s)$. They can be oriented in a way consistent with gluings; such orientations are called coherent [44]. There is an $\mathbb{R}$-action on $\mathscr{M}(x,y;J_s,H_s)$, and once a coherent orientation is fixed, $\mathscr{M}^1(x,y;J_s,H_s)/\mathbb{R}$ is a set of signed points.

The Floer complex associated to $(f;J_s,H_s)$ is the Λ-vector space generated by points in $\mathrm{Fix}\, f_H$:

$$CF^*(f;J_s,H_s) := \bigoplus_{x\in \mathrm{Fix}\, f_H} \Lambda\langle x\rangle.$$

The differential on $CF^*(f;J_s,H_s)$ is defined on a generator $x \in \mathrm{Fix}\, f_H$ by:

$$\partial(x) = \sum_{\substack{y\in \mathrm{Fix}\, f_H \\ u\in \mathscr{M}^1(x,y;J_s,H_s)/\mathbb{R}}} \pm q^{\omega(u)} \cdot y. \tag{2.7}$$

Here the signs are those of the points in $\mathcal{M}^1(x,y;J_s,H_s)/\mathbb{R}$, and

$$\omega(u) = \int_{s\in[0,1]} \int_{t\in\mathbb{R}} u^*\omega \, dsdt. \tag{2.8}$$

Suppose J_s, H_s and J'_s, H'_s are two regular choices of almost complex structures and Hamiltonians that satisfy the f-periodicity condition (2.1). Choose a family of ω-tame complex structures $J_{s,t}$ and Hamiltonians $H_{s,t}$, $s,t \in \mathbb{R}$, such that for each t, Condition (2.1) is satisfied and

$$J_{s,t} \equiv J'_s,\ H_{s,t} \equiv H'_s \text{ for } t \text{ near } -\infty, \quad J_{s,t} \equiv J_s,\ H_{s,t} \equiv H_s \text{ for } t \text{ near } +\infty. \tag{2.9}$$

We call $J_{s,t}, H_{s,t}$ a homotopy from J'_s, H'_s to J_s, H_s. Define $\mathcal{M}(x,y;J_{s,t},H_{s,t})$ to be the set of solutions to Floer's continuation equation

$$\partial u/\partial t + J_{s,t}(u)(\partial u/\partial s - X_{H_{s,t}}(u)) = 0 \tag{2.10}$$

with periodicity condition (2.4) and asymptotic conditions:

$$\lim_{t\to-\infty} u(s,t) = \psi_s(x), \quad \lim_{t\to+\infty} u(s,t) = \psi_s(y), \quad x \in \mathrm{Fix}\, f_{H'},\ y \in \mathrm{Fix}\, f_H. \tag{2.11}$$

If $J_{s,t}, H_{s,t}$ are regular, $\mathcal{M}(x,y;J_{s,t},H_{s,t})$ is a manifold. Let $\mathcal{M}^0(x,y;J_{s,t},H_{s,t})$ be its 0-dimensional component, which is a collection of signed points once coherent orientations (consistent with those for J_s, H_s and J'_s, H'_s) are fixed. Define the continuation map $C_{J_{s,t},H_{s,t}}\colon CF^*(f;J'_s,H'_s) \to CF^*(f;J_s,H_s)$ by

$$C_{J_{s,t},H_{s,t}}(x) = \sum_{\substack{y\in\mathrm{Fix}\, f_H \\ u\in\mathcal{M}^0(x,y;J_{s,t},H_{s,t})}} \pm q^{\omega(u)} \cdot y. \tag{2.12}$$

Here $x \in \mathrm{Fix}\, f_{H'}$. For regular $J_{s,t}, H_{s,t}$, it is a chain map inducing an isomorphism on cohomology. So one can actually identify the homologies $HF^*(f;J_s,H_s)$ for all generic J_s, H_s to get a single space $HF^*(f)$. It is called the Floer cohomology of f. It is a $\mathbb{Z}/2$-graded vector space over Λ; we shall recall the grading later.

2.2.2 Commuting symplectomorphisms induce actions on Floer cohomology

As before, let X be a compact symplectic manifold satisfying the W^+ condition. Let $f,g\colon X \to X$ be two commuting symplectomorphisms; we will now define an automorphism $g_{\mathrm{floer}}\colon HF^*(f) \to HF^*(f)$. Pick generic J_s, H_s that satisfy (2.1) to get the complex

$CF^*(f;J_s,H_s)$. Denote

$$J'_s := g^*J_s, \quad H'_s := H_s \circ g. \tag{2.13}$$

This gives us another complex $CF^*(f;J'_s,H'_s)$. Note that $g \circ \psi_1 = \psi'_1$. Let us check that $f_H = f_{H'} \circ g$:

$$f_{H'} \circ g(x) = (\psi'_1)^{-1} fg(x) = (\psi'_1)^{-1} gf(x) = \psi_1^{-1} f(x) = f_H(x).$$

Consequently, g induces a bijection $\operatorname{Fix} f_H \to \operatorname{Fix} f_{H'}$. Extend it by Λ-linearity to

$$g_{\mathrm{push}}\colon\ CF^*(f;J_s,H_s) \to CF^*(f;J'_s,H'_s).$$

Similarly, the composition map $u \mapsto g \circ u$ is an isomorphism

$$\mathscr{M}(x,y;J_s,H_s) \xrightarrow{\cong} \mathscr{M}(g(x),g(y);J'_s,H'_s).$$

So g_{push} is tautologically a chain map inducing an isomorphism on cohomology. Now fix a homotopy $J_{s,t},H_{s,t}$ from J'_s,H'_s to J_s,H_s as in (2.9). Consider the composition

$$CF^*(f;J_s,H_s) \xrightarrow{g_{\mathrm{push}}} CF^*(f;J'_s,H'_s) \xrightarrow{C_{J_{s,t},H_{s,t}}} CF^*(f;J_s,H_s).$$

Definition 2.2.2 (Action on Floer cohomology)**.** We define

$$g_{\mathrm{floer}}\colon\ HF^*(f;J_{s,t},H_{s,t}) \to HF^*(f;J_{s,t},H_{s,t})$$

to be the map induced by the composition of chain maps $C_{J_{s,t},H_{s,t}} \circ g_{\mathrm{push}}$. We will frequently suppress the choice of J_s,H_s and simply write $g_{\mathrm{floer}}\colon\ HF^*(f) \to HF^*(f)$. Also, we will sometimes denote the chain-level map by the same symbol, $g_{\mathrm{floer}} = C_{J_{s,t},H_{s,t}} \circ g_{\mathrm{push}}$.

As a part of this definition, the signs in formula (2.12) for $C_{J_{s,t},H_{s,t}}$ must come from a coherent orientation as explained in Subsection 2.2.8 below. In particular, for any $x \in \operatorname{Fix} f_H$, the sign of an element $u \in \mathscr{M}^0(g(x),x;J_{s,t},H_{s,t})$ is canonical, see Definition 2.2.10, and denoted by $\operatorname{sign}(u)$.

Remark 2.2.1. On the level of cohomology, g_{floer} does not depend on the chosen homotopy $J_{s,t},H_{s,t}$; this follows from the fact that the continuation map $C_{J_{s,t},H_{s,t}}$ does not depend on the choice of homotopy, see e.g. [73, Section 12.1].

Remark 2.2.2 (An analogue in Morse cohomology). A similar construction is known in Morse cohomology [90, 4.2.2]. Suppose $H\colon\ X \to \mathbb{R}$ is a Morse-Smale function on a Riemannian manifold (X,g), and $f\colon\ X \to X$ is a diffeomorphism. Let $C^*(H)$ be the Morse complex of

X generated by points in $\mathrm{Crit}(H)$. Pick homotopies H_t from $H \circ f$ to H, and g_t from f^*g to g, and define $f^*\colon C^*(H) \to C^*(H)$ as follows. Take $x, y \in \mathrm{Crit}(H)$ and let the coefficient of $f^*(x)$ on y be the signed count of flowlines of the gradient $\nabla_{g_t} H_t$ going from $f(x)$ to y. The chain map f^* induces an automorphism of $H^*(X)$ known from elementary topology.

In particular, let us note for future use that the Lefschetz number $L(f)$ can be computed as the sum, over $x \in \mathrm{Crit}(H)$, of $\nabla_{g_t} H_t$-flowlines going from $f(x)$ to x, counted with signs.

Remark 2.2.3 (Relation to Seidel elements). If g is Hamiltonian isotopic to f through symplectomorphisms commuting with f, then one can show $g_{\mathrm{floer}}\colon HF^*(f) \to HF^*(f)$ is the identity. If g is just Hamiltonian isotopic to f, g_{floer} need not be the identity, but can be understood as follows. Take a homotopy g_t, $g_0 = g, g_1 = f$. The path $\gamma_t := g_t^{-1} f g_t$ is actually a loop in $Symp(X)$: $\gamma(0) = \gamma(1) = f$ because $g^{-1} f g = f$. To this path one associates its Seidel element, $S(\gamma) \in QH^*(M;\Lambda)$ [93]. Let $*$ be the quantum multiplication $QH^*(M;\Lambda) \otimes HF^*(f) \to HF^*(f)$. One can check that $g_{\mathrm{floer}}(x) = S(\gamma) * x$ for any $x \in HF^*(f)$. We will not use this observation, so we omit its proof.

2.2.3 Iterations

If f, g commute then f, g^k also commute for any iteration g^k.

Lemma 2.2.3. *The following two automorphisms of $HF^*(f)$ are equal:*

$$(g_{\mathrm{floer}})^k = (g^k)_{\mathrm{floer}}.$$

Proof. We prove the case $k = 2$; the general case is analogous. Take J_s, H_s as in (2.1), J'_s, H'_s pulled by g as in (2.13) and the homotopy $J_{s,t}, H_{s,t}$ as in (2.9). Denote

$$J''_s = g^* J'_s = (g^2)^* J_s, \quad H''_s = H'_s \circ g = H_s \circ g^2.$$

Compare the two compositions given below. The first one induces $(g_{\mathrm{floer}})^2$ on the homological level:

$$CF^*(f;J_s,H_s) \xrightarrow{g_{\mathrm{push}}}$$
$$CF^*(f;J'_s,H'_s) \xrightarrow{C_{J_{s,t},H_{s,t}}} CF^*(f;J_s,H_s) \xrightarrow{g_{\mathrm{push}}} CF^*(f;J'_s,H'_s)$$
$$\xrightarrow{C_{J_{s,t},H_{s,t}}} CF^*(f;J_s,H_s).$$

The second composition gives $(g^2)_{\mathrm{floer}}$, by a gluing theorem for continuation maps:

$$\begin{aligned}&CF^*(f;J_s,H_s)\xrightarrow{g_{\mathrm{push}}}\\&\qquad CF^*(f;J'_s,H'_s)\xrightarrow{g_{\mathrm{push}}}CF^*(f;J''_s,H''_s)\xrightarrow{C_{J'_{s,t},H'_{s,t}}}CF^*(f;J'_s,H'_s)\\&\qquad\qquad\xrightarrow{C_{J_{s,t},H_{s,t}}}CF^*(f;J_s,H_s).\end{aligned}$$

By the definition of $J'_{s,t},H'_{s,t}$ (2.13), g maps Floer solutions (2.10) in $\mathscr{M}(x,y;J_{s,t},H_{s,t})$ to those in $\mathscr{M}(g(x),g(y);J'_{s,t},H'_{s,t})$. This means

$$C_{J'_{s,t},H'_{s,t}}\circ g_{\mathrm{push}}=g_{\mathrm{push}}\circ C_{J_{s,t},H_{s,t}}.$$

This proves Lemma 2.2.3. □

2.2.4 Supertrace

We continue to use notation from Subsection 2.2.1.

Definition 2.2.4 (Grading on Floer's complex)**.** Let $x\in\operatorname{Fix} f_H$. We say $\deg x=0$ if the sign of $\det(\mathrm{Id}-df_H(x))$ is positive and $\deg x=1$ otherwise.

This makes $CF^*(f;J_s,H_s)$ a $\mathbb{Z}_2$-graded vector space over Λ. Floer's differential has degree 1, so the cohomology is also $\mathbb{Z}_2$-graded: $HF^*(f)=HF^0(f)\oplus HF^1(f)$.

Definition 2.2.5 (Supertrace)**.** Let $V=V^0\oplus V^1$ be a $\mathbb{Z}_2$-graded vector space and $\phi\colon V\to V$ an automorphism of zero degree, i.e. $\phi(V^0)\subset V^0$, $\phi(V^1)\subset V^1$. Then $STr(\phi):=Tr(\phi|_{V^0})-Tr(\phi|_{V^1})$.

The automorphism g_{floer} from Definition 2.2.2 has zero degree, so it has well-defined supertrace which is an element of Λ. Supertraces can be computed on the chain level, since all our chain complexes are finite-dimensional. Therefore the following is just a restatement of definitions.

Lemma 2.2.6. *Let X be a symplectic manifold satisfying the W^+ condition and $f,g\colon X\to X$ be two commuting symplectomorphisms. Take J'_s,H'_s as in (2.13) and a homotopy $J_{s,t},H_{s,t}$ from J'_s,H'_s to J_s,H_s as in (2.9). Then*

$$STr(g_{\mathrm{floer}}\colon HF^*(f)\to HF^*(f))=\sum_{\substack{x\in\operatorname{Fix} f_H,\\ u\in\mathscr{M}^0(g(x),x;J_{s,t},H_{s,t})}}(-1)^{\deg x}\cdot\operatorname{sign}(u)\cdot q^{\omega(u)}$$

where $\operatorname{sign}(u)=\pm1$ *is defined in Definition 2.2.10.*

Proof. Pick a generator $x \in \operatorname{Fix} f_H$ of $CF^*(f; J_s, H_s)$. Rewriting the definition of g_{floer} we get:

$$g_{\text{floer}}(x) = \sum_{u \in \mathscr{M}^0(g(x), y; J_{s,t}, H_{s,t})} \pm q^{\omega(u)} \cdot y.$$

When we put $x = y$, the sign $\pm$ is substituted by $\operatorname{sign}(u)$ according to Definition 2.2.2. $\square$

2.2.5 Holomorphic sections

It is useful to reformulate the definition of Floer cohomology using holomorphic sections as in e.g. [97]. If $f\colon X \to X$ is a symplectomorphism, consider the mapping cylinder

$$E_f := \frac{X \times \mathbb{R}^2_{s,t}}{(x,s,t) \sim (f(x), s+1, t)}. \tag{2.14}$$

There is a closed 2-form ω_{E_f} on E_f which comes from $\omega \oplus 0$ on $X \times \mathbb{R}^2$, and a natural fibration $p\colon E_f \to S^1 \times \mathbb{R}$ whose fibres are symplectomorphic to X.

The f-periodicity condition (2.4) on $u\colon \mathbb{R}^2 \to X$ means that it can be seen as a section $u\colon S^1 \times \mathbb{R} \to E_f$. Floer's equation itself (2.3) is equivalent to u being a holomorphic section with respect to the standard complex structure $j_{S^1 \times \mathbb{R}}$ on $S^1 \times \mathbb{R}$ and an almost complex structure $\tilde{J}$ on E_f. In other words, Floer's equation (2.3) becomes:

$$du + \tilde{J} \circ du \circ j_{S^1 \times \mathbb{R}} = 0. \tag{2.15}$$

The almost complex structure $\tilde{J} := \tilde{J}(J_s, H_s)$ is determined by J_s and H_s, see e.g. [73, Section 8.1]. Analogously, if $J_{t,s}, H_{t,s}$ is a continuation homotopy (2.9), the moduli space $\mathscr{M}(x, y; J_{s,t}, H_{s,t})$ consists of sections $u\colon S^1 \times \mathbb{R} \to E_f$ that are holomorphic with respect to $j_{S^1 \times \mathbb{R}}$ and an almost complex structure $\tilde{J}(J_{s,t}, H_{s,t})$ on E_f.

2.2.6 Asymptotic linearised Floer's equation

Let E_f be as in (2.14). We denote

$$T^v E_f = \ker dp$$

the vertical tangent bundle of E_f. The almost complex structures J_s turn $T^v E_f$ into a complex vector bundle. Take a solution $u(s,t)$ to Floer's equation, $u \in \mathscr{M}(x, y; J_s, H_s)$. We regard it as a section $u(s,t)\colon S^1 \times \mathbb{R} \to E_f$ as explained above. The pullback $u^* T^v E_f$ is a complex vector bundle over $S^1 \times \mathbb{R}$. By linearising Floer's equation (2.15), one gets a map

$$D_u\colon\ H^{1,p}(u^*T^vE_f) \to L^p(\Omega^{0,1}(u^*T^vE_f)). \tag{2.16}$$

Here $\Omega^{0,1}(u^*T^vE_f)$ consists of bundle maps $T(S^1 \times \mathbb{R}) \to u^*T^vE_f$ which are complex-antilinear with respect to $\tilde{J}$ and the standard complex structure on $S^1 \times \mathbb{R}$.

We know from (2.6) that u extends to $S^1 \times \{\pm\infty\}$: $u(s,-\infty) = \psi_s(x)$ where ψ_s is the flow (2.2) of X_{H_s}. (The same is true of $t \to +\infty$ and the point y. We will now speak of $t \to -\infty$ only.) Choose a complex trivialisation

$$\Phi_x\colon\ u^*T^vE_f|_{S^1\times\{-\infty\}} \to S^1 \times \mathbb{R}^{2n}. \tag{2.17}$$

We choose a single trivialisation for each point x; this is possible because $u(s,-\infty) = \psi_s(x)$. The operator D_u is asymptotic, as $t \to -\infty$, to the operator

$$L_{A(s)} = \partial/\partial t + J_0\partial/\partial s + A(s)\colon\ H^{1,p}(S^1 \times \mathbb{R}, \mathbb{R}^{2n}) \to L^p(S^1 \times \mathbb{R}, \mathbb{R}^{2n}). \tag{2.18}$$

Here J_0 is the standard complex structure on $\mathbb{R}^{2n}$, and $A(s)$ is a map $S^1 \to Hom(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ taking values in symmetric matrices. It is known that $A(s)$ is determined by J_s, H_s, the point x and the chosen trivialisation $\Phi(x)$. It does not depend on u as long as the $t \to -\infty$ asymptotic of u stays fixed. A reference for these facts is (among others) the thesis of Schwarz [91, Definition 3.1.6, Theorem 3.1.31]. Although that thesis only considers the case $f = \mathrm{Id}$, the proofs of the results we use are valid for any f, as these are general statements about certain Fredholm operators on bundles over S^1 and $S^1 \times \mathbb{R}$.

Lemma 2.2.7 ([91, proof of Lemma 3.1.33])**.** *Consider the operator*

$$J_0\partial/\partial s + A(s)\colon\ C^\infty(S^1, \mathbb{R}^{2n}) \to C^\infty(S^1, \mathbb{R}^{2n}).$$

There is a family of linear maps $\Psi(s)\colon\ [0,1] \to Sp(\mathbb{R}^{2n})$ such that

$$(J_0\partial/\partial s + A(s))\,\Psi(s) = 0, \quad \Psi(0) = \mathrm{Id} \tag{2.19}$$

and $\Psi(1)\colon\ \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ coincides, under the trivialisation Φ_x (2.17), with the differential $df_H(x)$. □

Remark 2.2.4. We identify $S^1 = \mathbb{R}/\mathbb{Z}$ so points of the circle $s = 0$ and $s = 1$ are the same. The statement about $\Psi(1)$ in the lemma above makes sense because $u(0,-\infty) = x$ for some

$x \in \mathrm{Fix}\, f_H$, see (2.6) and (2.2). So $df_H(x)$ acts on $T_xX = u^*T^vE_f|_{(0,-\infty)}$. The trivialisation (2.17) identifies this space with $\mathbb{R}^{2n}$.

Remark 2.2.5. Given $\Psi(s) : [0,1] \to Sp(\mathbb{R}^{2n})$, by solving (2.19) we get

$$A(s) = -J_0(\partial/\partial s\Psi(s))\Psi(s)^{-1} \tag{2.20}$$

with symmetric $A(s)\colon\ [0,1] \to Hom(\mathbb{R}^{2n},\mathbb{R}^{2n})$. Conversely, we can go from $A(s)$ to $\Psi(s)$ by solving (2.19) as an ODE.

Remark 2.2.6. For reader's convenience, we include a correspondence between our notation and that of Schwarz [91], our notation being on the left in each pair: $s \leftrightarrow t$, $t \leftrightarrow s$, $A(s) \leftrightarrow S^\infty(t)$, $\Psi(s) \leftrightarrow \Psi(t)$, and D_u in our notation corresponds to either D_u or DF_h, the latter being the linearisation of Floer's equation at an h which is not necessarily a solution. Equation (2.19) is [91, (3.23)].

2.2.7 An index problem on the torus

The operator $L_{A(s)}$ (2.18) is Fredholm if and only if $\det(\mathrm{Id}-\Psi(1)) = \det(\mathrm{Id}-df_H(x))$ is non-zero. Now, for later use, consider variables (s,t) belonging to the torus $S^1 \times S^1$ instead of the cylinder $S^1 \times \mathbb{R}$. The same formula (2.18) gives the operator

$$L_{A(s)} = \partial/\partial t + J_0\partial/\partial s + A(s)\colon\ C^\infty(S^1 \times S^1,\mathbb{R}^{2n}) \to C^\infty(S^1 \times S^1,\mathbb{R}^{2n})$$

which is now Fredholm of zero index for any family of symmetric matrices $A(s)\colon\ S^1 \to Hom(\mathbb{R}^{2n},\mathbb{R}^{2n})$. For the remainder of this subsection, $L_{A(s)}$ denotes the operator on $S^1 \times S^1$ and not on the cylinder.

Lemma 2.2.8. *Let $A(s)\colon\ S^1 \to Hom(\mathbb{R}^{2n},\mathbb{R}^{2n})$ be a family of symmetric matrices. Suppose $A(s)$ and $\Psi(s)$ satisfy (2.19). Then* $\dim\ker L_{A(s)} = \dim\ker(\mathrm{Id}-\Psi(1))$.

Proof. Any $\xi(s,t) \in \ker L_{A(s)}$ must be independent of t, see [91, Proof of Lemma 3.1.33], so we write $\xi(s,t) \equiv \xi(s)$. The equation on $\xi(s)$ becomes $(J_0\partial/\partial s + A(s))\xi(s) = 0$. This is an ODE whose solutions are of form $\xi(s) = \Psi(s)v$ for some $v \in \mathbb{R}^{2n}$ by (2.19). There are no other solutions by the uniqueness theorem for ODEs, as $v \in \mathbb{R}^{2n}$ sweep out all initial conditions. In addition, our solutions must close up on the circle, meaning $\xi(1) = \xi(0)$, which forces $\Psi(1)v = v$. □

Let $A_0(s), A_1(s)\colon\ S^1 \to Hom(\mathbb{R}^{2n},\mathbb{R}^{2n})$ be two families of symmetric matrices with $L_{A_0(s)}$, $L_{A_1(s)}$ injective. Choose a generic smooth homotopy $A_\tau(s)$ between them, $\tau \in [0,1]$.

Define $\operatorname{sign}(L_{A_0(s)}, L_{A_1(s)}) = (-1)^{\varepsilon}$ where

$$\varepsilon = \sum_{\tau\in[0,1]} \dim_{\mathbb{R}} \ker L_{A_\tau(s)}. \tag{2.21}$$

This sum contains a finite number of non-zero terms as $L_{A_\tau(s)}$ are generically injective, and does not depend modulo 2 on the chosen homotopy.

Lemma 2.2.9. *For $A_0(s)$, $A_1(s)$ as above and $\Psi_0(s), \Psi_1(s)$ satisfying (2.19), we have*

$$\operatorname{sign}(L_{A_0(s)}, L_{A_1(s)}) = \operatorname{sign}\det(\mathrm{Id} - \Psi_0(1)) \cdot \operatorname{sign}\det(\mathrm{Id} - \Psi_1(1)).$$

Proof. For $i = 0, 1$ denote $\tilde{\Psi}_i(s) = e^{s \log \Psi_i(1)}$, $s \in [0;1]$, so that $\tilde{\Psi}_i(0) = \Psi_i(0) = \mathrm{Id}$ and $\tilde{\Psi}_i(1) = \Psi_i(1)$. Let us compute $\tilde{A}_i(s)$ from $\tilde{\Psi}_i(s)$ using (2.20):

$$\tilde{A}_i(s) = -J_0(\partial/\partial s \tilde{\Psi}_i(s))\tilde{\Psi}_i(s)^{-1} = -J_0 \log \Psi_i(1).$$

We see it is a constant s-independent symmetric matrix $\tilde{A}_i(s) \equiv \tilde{A}_i$. Our first claim is that

$$\operatorname{sign}(L_{A_i(s)}, L_{\tilde{A}_i}) = +1. \tag{2.22}$$

Indeed, choose the homotopy $(\Psi_i)_\tau(s) = e^{\tau s \log \Psi_i(1)} e^{(1-\tau)\log \Psi_i(s)}$ from $\Psi_i(s)$ to $\tilde{\Psi}_i(s)$, where $\tau \in [0;1]$, and observe this homotopy has fixed endpoints: we have $(\Psi_i)_\tau(0) = \Psi_i(0)$ for each τ, and also $(\Psi_i)_\tau(1) = \Psi_i(1)$. Passing from $(\Psi_i)_\tau(s)$ to $(A_i)_\tau(s)$ by formula (2.19) we get the linear homotopy $(A_i)_\tau(s) = \tau A_i(s) + (1-\tau)\tilde{A}_i$ from $A_i(s)$ to $\tilde{A}_i$. The corresponding operator $L_{(A_i)_\tau(s)}$ is injective for all τ by Lemma 2.2.8 because we are given $\ker(\mathrm{Id} - \Psi_i(1)) = 0$. This implies that $\operatorname{sign}(L_{A_i(s)}, L_{\tilde{A}_i}) = +1$, as desired.

Let us compute $\operatorname{sign}(L_{\tilde{A}_0}, L_{\tilde{A}_1})$ for two constant matrices $\tilde{A}_i(s) \equiv \tilde{A}_i$, $i = 0, 1$. By linear algebra, one can find a smooth path of matrices $\tilde{A}_\tau$ from $\tilde{A}_0$ to $\tilde{A}_1$ such that $(-1)^{\sum_\tau \dim\ker \tilde{A}_\tau} = \operatorname{sign}\det \tilde{A}_0 \cdot \operatorname{sign}\det \tilde{A}_1$. We will now show that $\dim\ker \tilde{A}_\tau = \dim\ker L_{\tilde{A}_\tau}$ for each τ, and this will immediately imply that

$$\operatorname{sign}(L_{\tilde{A}_0}, L_{\tilde{A}_1}) = \operatorname{sign}\det \tilde{A}_0 \cdot \operatorname{sign}\det \tilde{A}_1. \tag{2.23}$$

For the rest of the paragraph, redenote $\tilde{A}_\tau$ (for some fixed τ) by A: this is an arbitrary symmetric matrix, and if we consider it as an s-independent family and solve (2.20) with respect to Ψ, we get $\Psi(1) = e^{-J_0 A}$. By Lemma 2.2.8, we have $\ker L_A = \ker(\mathrm{Id} - \Psi(1)) = \ker(\mathrm{Id} - e^{-J_0 A})$. The latter equals $\ker A$, as seen by bringing A to the Jordan normal form. So

(2.23) is now justified. Combining all above, we get

$$\begin{aligned}\operatorname{sign}(L_{A_0(s)}, L_{A_1(s)}) &= \operatorname{sign}(L_{A_0(s)}, L_{\tilde{A}_0}) \cdot \operatorname{sign}(L_{\tilde{A}_0}, L_{\tilde{A}_1}) \cdot \operatorname{sign}(L_{\tilde{A}_1}, L_{A_1(s)}) \\ &= \operatorname{sign}(\det \tilde{A}_0) \cdot \operatorname{sign}(\det \tilde{A}_1).\end{aligned}$$

The first equality is true because we can regard the concatenation of three homotopies between the operators appearing in the middle expression as a single homotopy between the eventual endpoints $L_{A_0(s)}$ and $L_{A_1(s)}$; the second equality follows from (2.22) and (2.23). Finally, recall $\tilde{A}_i = -J_0 \log \Psi_i(1)$ and observe that sign $\det \log \Psi_i(1) =$ sign $\det(\mathrm{Id} - \Psi_i(1))$. This completes the proof. □

2.2.8 Signs for the action on Floer cohomology

Let f, g be two commuting symplectomorphisms. We will now complete Definition 2.2.2 of the action $g_{\text{floer}} \colon HF^*(f) \to HF^*(f)$ by specifying the signs appearing there.

Pick regular J_s, H_s to define Floer's complex $CF^*(f; J_s, H_s)$. For each $x \in \operatorname{Fix} f_H$, pick a trivialisation Φ_x (2.17). Then for each x, we get a unique asymptotic linearised operator $L_{A_x(s)}$ (2.18).

Let J'_s, H'_s be pulled back by g (2.13) and $J_{s,t}$, $H_{s,t}$ be a homotopy (2.9). Let $u \in \mathscr{M}^0(g(x), y; J_{s,t}, H_{s,t})$ be a solution to Floer's continuation equation, where $x, y \in \operatorname{Fix} f_H$ so that $g(x) \in \operatorname{Fix} f_{H'}$. Consider the linearisation D_u of Floer's continuation equation at u; its properties are similar to those discussed in Subsection 2.2.6. As $t \to +\infty$, D_u is asymptotic to $L_{A_y(s)}$ because for t close to $+\infty$, $J_{s,t}, H_{s,t}$ are equal to J_s, H_s. On the other hand, as $t \to -\infty$, we can write down D_u in the g-induced trivialisation $\Phi_x \circ dg$ of $u^*TE_f|_{u(-\infty,s)}$. We claim that D_u is asymptotic, as $t \to -\infty$, to $L_{A_x(s)}$. Indeed, the asymptotic operator is determined by the following data: the fixed point $g(x)$, the chosen trivialisation $\Phi_x \circ dg$, and $J_{s,t}, H_{s,t}$ which equal $g^*J_s, H_s \circ g$ for t close to $-\infty$. We see that all this data is pulled back by g from the data x, Φ_x, J_s, H_s which defines the asymptotic linearised operator $A_x(s)$. Clearly, pullback by g does not change the linearised operator at all, so D_u is asymptotic to $L_{A_x(s)}$ as $t \to -\infty$.

The outcome is that the set $\{L_{A_x(s)}\}_{x \in \operatorname{Fix} f_H}$ of asymptotic operators to D_u for $u \in \mathscr{M}(x, y; J_s, H_s)$ (these are solutions to Floer's equations for the differential on $CF^*(f)$, without the second symplectomorphism g involved) is identical to the set of asymptotic operators to D_u for $u \in \mathscr{M}(g(x), y; J_{s,t}, H_{s,t})$ (these are solutions to Floer's continuation equation), provided we use the described trivialisations.

Consequently, the usual definition of coherent orientations [44] on $\mathscr{M}(x, y; J_s, H_s)$ can be applied without any change to orient $\mathscr{M}(g(x), y; J_{s,t}, H_{s,t})$, $x, y \in \operatorname{Fix} f_H$. In Definition 2.2.2, we pick such a coherent orientation on $\mathscr{M}(g(x), y; J_{s,t}, H_{s,t})$. Instead of repeating the complete definition of coherent orientations, we shall only recall a piece relevant to the signs appearing in Lemma 2.2.6 regarding the supertrace of g_{floer}.

Coherent orientations are not unique, but the sign any coherent orientation associates to a point $u \in \mathscr{M}^0(g(x),x;J_{s,t},H_{s,t})$, $x \in \mathrm{Fix}\, f_H$, is canonical. We explain its definition following [44] and [73, Appendix A]. As we have seen, D_u is asymptotic as $t \to \pm\infty$ to the same operator

$$L_{A(s)} = \partial/\partial t + J_0 \partial/\partial s + A(s),$$

where $A(s) = A_x(s)$ in notation of the previous paragraphs. Choose a generic homotopy L_τ from D_u to $L_{A(s)}$, $\tau \in [0,1]$, such that L_τ are Fredholm operators which stay asymptotic to $L_{A(s)}$ as $t \to \pm\infty$.

Definition 2.2.10. For $u \in \mathscr{M}^0(g(x),x;J_{s,t},H_{s,t})$, $x \in \mathrm{Fix}\, f_H$, define $\mathrm{sign}\,(u) = (-1)^{\varepsilon}$ where

$$\varepsilon = \sum_{\tau \in [0,1]} \dim_{\mathbb{R}} \ker L_\tau.$$

Because the operators L_τ have zero index, the sum is well-defined and does not depend modulo 2 on the chosen path. Let us repeat that, as part of Definition 2.2.2, these signs appear in Lemma 2.2.6.

2.2.9 Holomorphic sections over the torus

Subsection 2.2.5 explained that solutions to Floer's equation are holomorphic sections of a fibration $E_f \to S^1 \times \mathbb{R}$, whose monodromy around S^1 equals f. Now, let f, g be two commuting symplectomorphisms of X. In this subsection we define a fibration $p\colon E^{1,R}_{f,g} \to T^{1,R}$ over a 2-torus $T^{1,R}$. The monodromies of this fibration equal f and g around the two basis loops of the torus. After that we recall how to count its holomorphic sections, see [73] for details. We start by defining the torus

$$T^{1,R} := \frac{[0,1] \times [-R,R]}{(s,\{-R\}) \sim (s,\{R\}),\ (\{0\},t) \sim (\{1\},t)}$$

and equipping $T^{1,R}$ with the complex structure $j^{1,R}$ which comes from the standard one on $[0,1] \times \sqrt{-1}[-R,R] \subset \mathbb{C}$. Define

$$E^{1,R}_{f,g} := \frac{X \times [0,1] \times [-R,R]}{(x,s,\{-R\}) \sim (g(x),s,\{R\}),\ (x,\{0\},t) \sim (f(x),\{1\},t)}$$

Here $x \in X$, $s \in [0,1]$, $t \in [-R,R]$. Because $fg = gf$, there is a fibration $p\colon E^{1,R}_{f,g} \to T^{1,R}$ and a fibrewise symplectic closed 2-form $\omega_{E^{1,R}_{f,g}}$ coming from the one on X.

Fix a generic almost complex structure $\tilde{J}$ on $E^{1,R}_{f,g}$ such that $\tilde{J}$ is $\omega^{1,R}_{f,g}$-tame on the fibres and the projection $p\colon E^{1,R}_{f,g} \to T^{1,R}$ is $(\tilde{J}, j^{1,R})$-holomorphic. Let $\mathscr{M}(j^{1,R},\tilde{J})$ be the space of all $(j^{1,R},\tilde{J})$-holomorphic sections $u\colon T^{1,R} \to E^{1,R}_{f,g}$:

$$du + \tilde{J}(u) \circ du \circ j^{1,R} = 0. \tag{2.24}$$

For generic $\tilde{J}$, this moduli space is a smooth manifold that breaks into components of different dimensions. This manifold has a canonical orientation, and in particular its 0-dimensional part $\mathscr{M}^0(j^{1,R},\tilde{J})$ consists of signed points. We will now describe how these signs are defined. Let $u \in \mathscr{M}^0(j^{1,R},\tilde{J})$. Consider the linearised equation (2.24) at u,

$$D_u\colon C^\infty(u^*T^vE^{1,R}_{f,g}) \to \Omega^{0,1}(u^*T^vE^{1,R}_{f,g}).$$

Here $T^vE^{1,R}_{f,g} = \ker dp$ and $u^*T^vE^{1,R}_{f,g}$ is a complex bundle over the torus $T^{1,R}$. Because u has index 0, this bundle has Chern number 0 and hence is trivial; fix its trivialisation. Together with the holomorphic co-ordinates (s,t) on $T^{1,R}$, it induces a trivialisation of $\Omega^{0,1}(u^*TE^{1,R}_{f,g}) = \mathbb{R}^{2n}$. In this trivialisation, D_u is a 0-order perturbation of the Cauchy-Riemann operator:

$$D_u = \partial/\partial t + J_0\partial/\partial s + A(s,t)\colon C^\infty(T^{1,R},\mathbb{R}^{2n}) \to C^\infty(T^{1,R},\mathbb{R}^{2n}) \tag{2.25}$$

where $A(s,t)\colon T^{1,R} \to Hom(\mathbb{R}^{2n},\mathbb{R}^{2n})$. This is the same operator as considered in Subsection 2.2.7, except that now $A(s,t)$ can depend on t as well as on s. The operator D_u is always Fredholm of index 0.

Fix, once and for all, an injective operator of the above form, for example

$$L_{\mathrm{Id}} = \partial/\partial t + J_0\partial/\partial s + \mathrm{Id}.$$

(This one is injective by Lemma 2.2.8, because $\ker(\mathrm{Id} - e^{-J_0}) = 0$.) Find a smooth homotopy of operators L_τ, $\tau \in [0,1]$, from D_u to L_{Id}, by deforming the 0-order part $A(s,t)$ to Id.

Definition 2.2.11 (cf. [73, p. 51 and Appendix A])**.** For $u \in \mathscr{M}^0(j^{1,R},\tilde{J})$, define $\mathrm{sign}(u) := (-1)^\varepsilon$ where

$$\varepsilon = \sum_{\tau\in[0,1]} \dim_{\mathbb{R}} \ker L_\tau.$$

For $u \in \mathscr{M}^0(j^{1,R},\tilde{J})$, denote $\omega(u) := \int_{T^{1,R}} u^*\omega^{1,R}_{E_{f,g}}$. The following is well known.

Proposition 2.2.12.
$$\sharp\mathscr{M}^0(j^{1,R},\tilde{J}) := \sum_{u\in\mathscr{M}^0(j^{1,R},\tilde{J})} \mathrm{sign}\,(u)\cdot q^{\omega(u)}$$
is independent of the complex structure $j^{1,R}$ on the torus and of generic $\tilde{J}$. □

2.2.10 Gluing the fibration over the cylinder to the fibration over the torus

Given a symplectomorphism $f\colon X\to X$, we have constructed a fibration $p\colon E_f\to S^1\times\mathbb{R}$ (2.14); also, given two commuting symplectomorphisms $f,g\colon X\to X$ and a parameter $R\in\mathbb{R}$, we have constructed a fibration $E^{1,R}_{f,g}\to T^{1,R}$. The fibres of both fibrations are symplectomorphic to X. Now, there is a map

$$E_f\supset p^{-1}(S^1\times[-R,R])\to E^{1,R}_{f,g} \tag{2.26}$$

It glues the boundary component $p^{-1}(S^1\times\{R\})$ to the other boundary component $p^{-1}(S^1\times\{-R\})$ via the symplectomorphism $g\colon X\to X$ applied fibrewise along S^1.

Fix regular J_s,H_s (2.1). As in (2.13), set

$$J'_s=g^*J_s,\quad H'_s=H_s\circ g.$$

Choose a homotopy $J_{s,t}$, $H_{s,t}$ (2.9) between J'_s,H'_s and J_s,H_s. This homotopy must be t-independent when t is close to $\pm\infty$; we assume for convenience

$$J_{s,t}\equiv J'_s,\ H_{s,t}\equiv H'_s \text{ for } t\le -R,\quad\text{and}\quad J_{s,t}\equiv J_s,\ H_{s,t}\equiv H_s \text{ for } t\ge R.$$

Finally, let $\tilde{J}:=\tilde{J}(J_{s,t},H_{s,t})$ be the almost complex structure on E_f from Subsection 2.2.5, which has the property that solutions to Floer's continuation equation are precisely $(j_{S^1\times\mathbb{R}},\tilde{J})$-holomorphic sections $S^1\times\mathbb{R}\to E_f$.

By definition, $\tilde{J}|_{p^{-1}(S^1\times\{R\})}$ is the g-pullback of $\tilde{J}|_{p^{-1}(S^1\times\{-R\})}$, which agrees with the gluing (2.26). So $\tilde{J}$ defines a glued almost complex structure $\mathrm{gl}\tilde{J}$ on $E^{1,R}_{f,g}$. Let us recall our notation one more time. $\mathscr{M}(x,y;J_{s,t},H_{s,t})$ consists of holomorphic sections over $S^1\times\mathbb{R}$ which are solutions to Floer's continuation equation (2.10), and $\mathscr{M}(j^{1,R},\mathrm{gl}\tilde{J}(J_{s,t},H_{s,t}))$ consists of holomorphic sections over the torus $T^{1,R}$. We come to an important proposition, of which everything but formula (2.27) is well known.

Proposition 2.2.13. *For each $A>0$ there is $R>0$ such that there is a bijection called the gluing map and denoted by* gl*:*

$$\mathrm{gl}\colon \bigsqcup_{x\in \mathrm{Fix}\, f_H} \mathscr{M}^0(g(x),x;J_{s,t},H_{s,t})^{<A} \xrightarrow{1-1} \mathscr{M}^0(j^{1,R},\mathrm{gl}\tilde{J}(J_{s,t},H_{s,t}))^{<A}.$$

Here the superscripts $*^{<A}$ mean we are taking only those solutions whose ω-area is less than A. The gluing map preserves ω-areas:

$$\int_{S^1\times\mathbb{R}} u^*\omega_{E_f} = \int_{T^{1,R}} \mathrm{gl}(u)^*\omega_{E^{1,R}_{f,g}}$$

and changes the signs from Definitions 2.2.10, 2.2.11 by $(-1)^{\deg x}$:

$$\mathrm{sign}\,(u) = \mathrm{sign}\,(\mathrm{gl}(u))\cdot(-1)^{\deg x} \tag{2.27}$$

Here $u\in\mathscr{M}^0(g(x),x;J_{s,t},H_{s,t})^{<A}$, and $\deg x$ *is defined in Definition 2.2.4.*

Proof. The existence of the bijection gl is well known. The map gl is constructed for the case $f=g=\mathrm{Id}$ in [91], see also [15], and that proof carries over to arbitrary f,g. Alternatively, one can adopt general SFT gluing and compactness theorems [23].

Let $u(s,t)\in\mathscr{M}^0(g(x),x;J_{s,t},H_{s,t})$, cosidered as a section of the fibration as explained above. By a smooth homotopy, this section can be made t-independent for t close to $-\infty$ and $+\infty$. It can further be glued into a smooth section over $T^{1,R}$ by applying (2.26). The smooth section over $T^{1,R}$ we obtain is smoothly homotopic to $\mathrm{gl}(u)$ and hence has the same ω-area as $\mathrm{gl}(u)$: so gluing preserves ω-areas.

Let us explain why gl changes the sign by $(-1)^{\deg x}$. We have illustrated our argument by an informal diagram below; its arrows correspond to homotopies between Fredholm operators, and its labels are the signs determined by the mod 2 count of the dimensions of kernels appearing during the homotopies.

$$\begin{tikzcd}
\substack{\textit{over}\\ S^1\times\mathbb{R}} & D_u \arrow[r, "\mathrm{sign}\,(u)"] \arrow[d, Rightarrow, "\textit{gluing}"'] & \partial/\partial t + J_0\partial/\partial s + A(s) & \\
\substack{\textit{over}\\ T^{1,R}} & D_{\mathrm{gl}(u)} \arrow[r, "\mathrm{sign}\,(u)"] \arrow[rr, bend right, "\mathrm{sign}\,(\mathrm{gl}(u))"'] & \partial/\partial t + J_0\partial/\partial s + A(s) \arrow[r, "(-1)^{\deg x}"] & \partial/\partial t + J_0\partial/\partial s + \mathrm{Id}
\end{tikzcd}$$

Take $u(s,t) \in \mathscr{M}^0(g(x),x;J_{s,t},H_{s,t})$ and consider linearised Floer's operators (2.16) and (2.25):

$$\begin{aligned} D_u: \quad & H^{1,p}(S^1 \times \mathbb{R}, \mathbb{R}^{2n}) \to L^p(S^1 \times \mathbb{R}, \mathbb{R}^{2n}), \\ D_{\mathrm{gl}(u)}: \quad & C^\infty(T^{1,R}, \mathbb{R}^{2n}) \to C^\infty(T^{1,R}, \mathbb{R}^{2n}). \end{aligned}$$

Take a homotopy L_τ from D_u to the operator (2.18) $L_{A(s)} = \partial/\partial t + J_0 \partial/\partial s + A(s)$. By Definition 2.2.10,

$$\operatorname{sign}(u) = (-1)^{\sum_\tau \dim\ker L_\tau} \tag{2.28}$$

Let L^{gl}_τ be a homotopy from $D_{gl(u)}$ to the analogous operator $L_{A(s)} = \partial/\partial t + J_0 \partial/\partial s + A(s)$ over the torus, considered in Subsection 2.2.7. It is well known that

$$\sum_{\tau \in [0,1]} \dim\ker L_\tau \equiv \sum_{\tau \in [0,1]} \ker L^{\mathrm{gl}}_\tau \mod 2. \tag{2.29}$$

(This is a special case of the fact that orientations of moduli spaces of pseudo-holomorphic sections before gluing canonically define orientations on moduli spaces after gluing.)

Take a homotopy $L_{A_\tau(s)}$ from $L_{A(s)}$ to $L_{\mathrm{Id}} = \partial/\partial t + J_0 \partial/\partial s + \mathrm{Id}$. To compute the kernels swept by this homotopy, we will use Lemma 2.2.9. First, let $\Psi(s)$ be the matrix which solves (2.19) with respect to our given $A(s)$, then $\Psi(1) = df_H(X)$ by Lemma 2.2.7. By Definition 2.2.4, sign $\det(\mathrm{Id} - \Psi(1)) = \deg x$. Second, let $\Psi(s)$ instead be the matrix which solves (2.19) with respect to the s-independent matrix $A(s) \equiv \mathrm{Id}$; the solution is e^{-sJ_0}, and for it we obtain sign $\det(\mathrm{Id} - \Psi(1)) = +1$. Now by Lemma 2.2.9,

$$\sum_{\tau \in [0,1]} \dim\ker L_{A_\tau(s)} \equiv \deg x \mod 2. \tag{2.30}$$

The concatenation of homotopies $L_{A_\tau(s)}$ and L^{gl}_τ is a homotopy from $D_{\mathrm{gl}(u)}$ to $\partial/\partial t + J_0 \partial/\partial s +$ Id. So by Definition 2.2.11,

$$\operatorname{sign}(\mathrm{gl}(u)) = (-1)^{\sum_\tau \dim\ker L_{A_\tau(s)}} \cdot (-1)^{\sum_\tau \dim\ker L^{\mathrm{gl}}_\tau}. \tag{2.31}$$

Combining (2.28), (2.29), (2.30), (2.31) we get $\operatorname{sign}(u) = \operatorname{sign}(\mathrm{gl}(u)) \cdot (-1)^{\deg x}$ which completes the proof. □

2.2.11 Proof of the elliptic relation

Proof of Theorem 2.1.1. We only need to compile the previous statements. It suffices to prove that for each $A > 0$, the supertraces are equal up to order q^A: $STr(f_{\mathrm{floer}})/q^A =$

$STr(g_{\text{floer}})/q^A$. By Lemma 2.2.6 and Proposition 2.2.13, for sufficiently large R we have

$$STr(g_{\text{floer}})/q^A = \sum_{\substack{x\in \operatorname{Fix} f_H,\\ u\in \mathscr{M}^0(g(x),x;J_{s,t},H_{s,t})^{<A}}} (-1)^{\deg x}\cdot \operatorname{sign}(u)\cdot q^{\omega(u)} = \sharp \mathscr{M}^0(j^{1,R},\tilde{J}(J_{s,t},H_{s,t}))^{<A}.$$

One can repeat all constructions after swapping f and g to get

$$STr(f_{\text{floer}}\colon\ HF^*(g)\to HF^*(g)) = \sharp \mathscr{M}^0(j^{R,1},\tilde{J}_1)^{<A}.$$

Here $j^{R,1} = j^{1,\frac{1}{R}}$ is another complex structure on the torus (which is “long” in the s-direction, while $j^{1,R}$ is “long” in the t-direction), and $\tilde{J}_1$ some other almost complex structure on the total space. Now Theorem 2.1.1 follows from Proposition 2.2.12. □

2.2.12 Finite order symplectomorphisms

We will now prove two lemmas about the action on Floer cohomology when one of the two commuting symplectomorphisms has finite order, and derive Proposition 2.1.4. The proof of the next lemma is an extension of [55, Lemma 7.1].

Lemma 2.2.14. *Let X be a symplectic manifold satisfying the W^+ condition. Let $g,\phi\colon X\to X$ be two commuting symplectomorphisms. Suppose $\phi^k = \mathrm{Id}$ and the fixed point set X^ϕ is a smooth manifold (maybe disconnected, with components of different dimensions). Then*

$$STr(g_{\text{floer}}\colon\ HF^*(\phi)\to HF^*(\phi)) = L(g|_{X^\phi})\cdot q^0 + \sum_i a_i\cdot q^{\omega_i},\quad \omega_i>0.$$

In other words, $STr(g_{\text{floer}})\in\Lambda$ contains only summands with non-negative powers of q, and the q^0-coefficient is the topological Lefschetz number of $g|_{X^\phi}$. Using the elliptic relation we will later show that the higher order terms $a_i q^{\omega_i}$ actually vanish; this is however a separate argument and we first prove the lemma as stated.

Proof. First we construct a Hamiltonian function on X of special form. Let $U(X^\phi)$ be a ϕ-equivariant tubular neighbourhood of X^ϕ, $p\colon\ U(X^\phi)\to X^\phi$ the projection and *dist* a ϕ-invariant function on $U(X^\phi)$ measuring the distance to X^ϕ in some ϕ-invariant metric. Let H_0 be an arbitrary function on X^ϕ. Define

$$H := H_0\circ p + dist^2.$$

This is a function on $U(X^\phi)$. Extend this function to X in any way and then average it with respect to ϕ (this will not change the function on $U(X^\phi)$). We denote the result by H again.

Note that $H|_{X^\phi} = H_0$ and $Crit(H_0) = Crit(H) \cap X^\phi$. For the rest of the proof, H will be a generic function constructed this way; in particular $H|_{X^\phi}$ is also generic.

Because ϕ has finite order, we can choose a ϕ-invariant compatible almost complex structure J on X which preserves TX^ϕ, and such that $J|_{X^\phi}$ is arbitrary. Since J, H are ϕ-invariant, they satisfy (2.1), with $f = \phi$. Thus Floer's equation (2.3) makes sense for such s-independent data J, H. Denote $J' \equiv g^*J$, $H' \equiv H \circ g$ as in (2.13).

Choose an s-independent homotopy (2.9) H_t, J_t from H' to H (resp. from J' to J). For every t, H_t, J_t must be ϕ-invariant, and as earlier

$$H_t = (H_0)_t \circ p + dist^2 \tag{2.32}$$

on $U(X^\phi)$ where $(H_0)_t = (H_t)|_{X^\phi}$ can be arbitrary. Note that in general, it might not be possible to find s-independent J_t, H_t that would make all solutions of Floer's continuation equation (2.10) regular. However, using [55] we will now argue that some solutions of (2.10) (namely, the gradient flowlines of H_t) are indeed generically regular.

Recall that J_t defines the time-dependent metric $\omega(\cdot, J_t \cdot)$ on X by definition of a compatible almost complex structure. If H is a function on X, its gradient and Hamiltonian vector fields are related by: $\nabla H = JX_H$. So s-independent solutions $u(s,t) \equiv x(t)$ of Floer's continuation equation (2.10) are precisely the $\omega(\cdot, J_t \cdot)$-gradient flowlines of H_t:

$$dx(t)/dt - \nabla H_t = 0.$$

The ϕ-periodicity condition (2.4) now reads $\phi(x(t)) = x(t)$ so we are looking only at gradient flowlines inside X^ϕ. Note that every s-independent solution $u(s,t) \equiv x(t)$ of (2.10) has zero area: $\omega(u) = 0$. Recall that solutions of (2.10) are elements of $\mathscr{M}(x, y; J_t, H_t)$ where $x \in \operatorname{Fix} \phi_{H'}$ and $y \in \operatorname{Fix} \phi_H$. Also note that $\operatorname{Fix} \phi_H = Crit(H|_{X^\phi})$, and similarly $\operatorname{Fix} \phi_{H'} = Crit(H'|_{X^\phi})$.

The following two facts are proved in [55] when H_t, J_t are t-independent and $\phi = \mathrm{Id}$ (that paper is interested in the equations for Floer's differential rather than continuation maps). The proofs are valid in the general case. For example, one can track that the periodicity condition (2.1), which is the only place where ϕ explicitly appears, is not used in the proof of the facts below.

1. For any J_t, H_t as above, an s-independent solution $u(s,t) \equiv x(t)$ of (2.10) is regular, i.e. D_u (2.16) is onto, if and only if the $\omega(\cdot, J_t \cdot)$-gradient flow of H_t is Morse-Smale near X^ϕ [88, Corollary 4.3, Theorem 7.3], compare [55, proof of Theorem 6.1].

2. There is $\varepsilon > 0$ such that every solution $u(s,t)$ of (2.10) with $\omega(u) < \varepsilon$ is s-independent [55, Lemma 7.1].

We claim that the gradient flow of a generic H_t constructed above is Morse-Smale near X^ϕ. Indeed, we can choose $H_t|_{X^\phi}$ freely, so we can make the flow of $H_t|_{X^\phi}$ Morse-Smale. Because H_t is quadratic in the normal direction to X^ϕ (2.32), the stable manifolds of H_t are, near X^ϕ, normal disk bundles over those of $H_t|_{X^\phi}$, and the unstable manifolds of H_t lie in X^ϕ and coincide with those of $H_t|_{X^\phi}$. Consequently, H_t is Morse-Smale near X^ϕ if and only if $H_t|_{X^\phi}$ is Morse-Smale.

By Remark 2.2.2 or [90, 4.2.2],

$$\sum_{\substack{x \in \operatorname{Fix} \phi_H, \\ u \in \mathscr{M}^0(g(x), x; J_t, H_t):\ \omega(u) \leq 0}} (-1)^{\deg x} \cdot \operatorname{sign}(u) \cdot q^{\omega(u)} = L(g|_{X^\phi}) \cdot q^0. \tag{2.33}$$

The left hand side looks exactly like the expression for $STr(g_{\text{floer}})$ from Lemma 2.2.6; however, J_t, H_t need not be regular for all continuation equation solutions, while g_{floer} must be computed using a regular Hamiltonian and almost complex structure. To cure this, we slightly perturb J, H and J_t, H_t by allowing them to depend on s, to get J_s, H_s and $J_{s,t}, H_{s,t}$. For a generic such perturbation, all solutions to (2.10) with respect to $J_{s,t}, H_{s,t}$ become regular. Because s-independent solutions in $\mathscr{M}^0(x,y;J_t,H_t)$ were already regular, they are in 1-1 correspondence (via the continuation map) with some solutions in $\mathscr{M}^0(x,y;J_{s,t},H_{s,t})$ of zero ω-area. By item (2) above, every $u \in \mathscr{M}^0(x,y;J_{s,t},H_{s,t})$ with $\omega(u) < \varepsilon$ actually has zero area and corresponds to an s-independent solution in $\mathscr{M}^0(x,y;J_t,H_t)$. (See [55, proof of Proposition 7.4] for this argument.) In view of (2.33) this means

$$\sum_{\substack{x \in \operatorname{Fix} \phi_H, \\ u \in \mathscr{M}^0(g(x), x; J_{s,t}, H_{s,t}):\ \omega(u) \leq 0}} (-1)^{\deg x} \cdot \operatorname{sign}(u) \cdot q^{\omega(u)} = L(g|_{X^\phi}) \cdot q^0.$$

Lemma 2.2.14 follows from this equality and Lemma 2.2.6. $\square$

Lemma 2.2.15. *Let X be a symplectic manifold satisfying the W^+ condition. Let $g, \phi\colon X \to X$ be two commuting symplectomorphisms. Suppose $\phi^k = \mathrm{Id}$. Then*

$$STr(\phi_{\text{floer}}\colon HF^*(g) \to HF^*(g)) = a \cdot q^0, \quad \textit{where } a \in \mathbb{C} \textit{ and } |a| \leq \dim_\Lambda HF^*(\phi).$$

Proof. By Lemma 2.2.3 $(\phi_{\text{floer}})^k = \mathrm{Id}$, so all eigenvalues of ϕ_{floer} are among the roots of unity $\sqrt[k]{1} \cdot q^0 \in \Lambda$. The signed sum of these eigenvalues gives $STr(\phi_{\text{floer}})$, and Lemma 2.2.15 follows. $\square$

The elliptic relation (Theorem 2.1.1) and Lemma 2.2.15 imply the following corollary.

Corollary 2.2.16. *The terms $a_i \cdot q^{\omega_i}$, $\omega_i > 0$ from Lemma 2.2.14 actually vanish.* □

Proof of Proposition 2.1.4. The proposition follows from Lemma 2.2.14, Lemma 2.2.15 and Theorem 2.1.1. □

Remark 2.2.7. As promised in Remark 2.1.3 we sketch an alternative proof of Proposition 2.1.4 which does not appeal to Theorem 2.1.1. Suppose for simplicity a symplectomorphism $f\colon X \to X$ commutes with a symplectic involution ι and f has non-degenerate isolated fixed points. Note that, for general reasons, $d\iota$ acts by $-\mathrm{Id}$ on the normal bundle to its fixed locus X^ι. To compute $HF^*(f)$, choose the zero Hamiltonian perturbation and an almost complex structure which is ι-invariant at points $x \in \mathrm{Fix}\, f \cap X^\iota$. Then ι_{floer} only counts constant solutions $u(s,t) \equiv x \in \mathrm{Fix}\, f \cap X^\iota$. (Because f has isolated fixed points, the only zero-area solutions are constant, and because $\iota_{\mathrm{floer}}^2 = \mathrm{Id}$, all positive area solutions cancel.) However, the sign associated to a constant solution u is not always positive. The reason is that we must write the linearised Floer's operator D_u in a trivialisation of $u^*T_xX = S^1 \times \mathbb{R} \times T_xX$ which differs by $d\iota(x)$ over the two ends of the cylinder, according to the definition in Subsection 2.2.8. Consider the splitting $T_xX = T_xX^\iota \oplus N_xX^\iota$ into the $+1$ and -1 eigenspaces of $d\iota(x)$. We can choose the constant trivialisation of $u^*T_xX^\iota$ and get the $\mathbb{R}$-independent operator on this subspace, which by definition carries the positive sign. However, we are not allowed to choose the constant trivialisation of $u^*N_xX^\iota$ (instead, an allowed choice is, for example, a rotation from Id to $-\mathrm{Id}$ with parameter t), so D_u will not be the canonical $\mathbb{R}$-invariant operator on N_xX and can carry a nontrivial sign from Definition 2.2.10. We claim that this sign equals sign $\det(\mathrm{Id} - df(x)|_{N_xX^\iota})$. The computation can be essentially be reduced to the index problem considered in Subsection 2.2.7, since D_u can still be chosen independent of the variable s; a related Lagrangian version of this statement is [96, Lemma 14.11]. Once the signs are known, it is easy to see that $STr(\iota_{\mathrm{floer}}) = L(f|_{\mathrm{Fix}\,\iota}) \cdot q^0$:

$$
\begin{aligned}
STr(\iota_{\mathrm{floer}}) &= \sum_{x \in \mathrm{Fix}\, f \cap X^\iota} (-1)^{\deg x} \cdot \mathrm{sign}\, \det(\mathrm{Id} - df(x)|_{N_xX^\iota}) \cdot q^0 \\
&= \sum_{x \in \mathrm{Fix}\, f \cap X^\iota} \mathrm{sign}\, \det(\mathrm{Id} - df(x)|_{T_xX^\iota}) \cdot q^0 \;=\; L(f|_{\mathrm{Fix}\,\iota}) \cdot q^0.
\end{aligned}
$$

The bound $\dim HF^*(f) \geq L(f|_{\mathrm{Fix}\,\iota})$ follows as in Lemma 2.2.15.

2.2.13 Lagrangian elliptic relation

In this subsection, we briefly explain Theorem 2.1.5 and Proposition 2.1.6. Let X be a monotone symplectic manifold, i.e. $[\omega(X)] = \lambda c_1(X)$ as elements of $H^2(X;\mathbb{R})$, $\lambda > 0$. Let

$\phi\colon X \to X$ be a symplectomorphism, and $L_i \subset X$ be two connected monotone Lagrangian submanifolds such that $\phi(L_i) = L_i$.

In order to define the action $\phi_{\text{floer}}\colon HF^*(L_1, L_2) \to HF^*(L_1, L_2)$ over a field of characteristic not equal to two, we must fix the following additional data. First, L_i must be oriented, although ϕ need not preserve the orientations. (In Section 2.6 we use the orientation-reversing case.) Second, the hypothesis below must be satisfied.

Hypothesis 2.2.17. The L_i must be equipped with spin structures S_i together with isomorphisms $\phi^* S_i \to S_i$ if $\phi|_{L_i}$ preserves orientation, and $\phi^* S_i \to \bar{S}_i$ if $\phi|_{L_i}$ reverses orientation. Here $\bar{S}_i$ is the following spin structure on $\bar{L}_i$ (that is, on L_i with the opposite orientation). For simplicity, assume $\dim L_i \geq 3$; then the spin structure S_i is the same as a trivialisation of TL_i over the 1-skeleton of L_i which extends over the 2-skeleton and agrees with the chosen orientation of L_i. We define the spin structure $\bar{S}_i$ to be given by the composition of the trivialisation S_i with a fixed orientation-reversing isomorphism $\mathbb{R}^n \to \mathbb{R}^n$, for example the one which multiplies the first co-ordinate by -1. We note the desired isomorphisms $\phi^* S_i \to S_i$ or $\phi^* S_i \to \bar{S}_i$ always exist if the L_i are simply-connected.

There is a minor difference when $\dim L_i \leq 2$. If one wants to view spin structures on L_i as trivialisations of a bundle over the 1-skeleton, the bundle in question should be a certain stabilisation of TL_i rather than just the tangent bundle [62, Chapter 4]. With this understood, the definition of $\bar{S}_i$ is analogous.

In [96, Section 14], similar data (defined only for an involution ϕ, with an extra condition on the "squares" of the above isomorphisms, but also allowing non-orientable Lagrangians) were called an equivariant Pin structure.

Pick some J_s, H_s defining the Floer cohomology $HF^*(L_1, L_2; J_s, H_s, S_i)$, see Chapter 1 and references therein. We have included the choice of spin structures in our notation. The action ϕ_{floer} is the composition $HF^*(L_1, L_2; J_s, H_s, S_i) \to HF^*(L_1, L_2; \phi^* J_s, H_s \circ \phi, \phi^* S_i) \to HF^*(L_1, L_2; J_s, H_s, S_i)$. Here the first map is the tautological chain-level map that takes all chain generators and Floer's solutions to their ϕ-image; we are using that $\phi L_i = L_i$. The second one is the continuation map. We skip the proof of the next lemma.

Lemma 2.2.18 (cf. [96, Sections (14a) and (14e)])**.** *If* $\phi^k = \mathrm{Id}$ *then* $(\phi_{\text{floer}})^k = \pm\mathrm{Id}$. □

Note that, unlike Lemma 2.2.3 and [96, top of p. 310], we do not necessarily get $(\phi_{\text{floer}})^k = \mathrm{Id}$, but having $(\phi_{\text{floer}})^k = \pm\mathrm{Id}$ is enough for future applications.

Choose J_s, H_s (2.1) to define Floer's complex $CF^*(\phi; J_s, H_s)$. Take the fibration $p\colon E_\phi \to S^1 \times [0, +\infty)$ with monodromy ϕ around the circle as in (2.14), but now over the semi-infinite cylinder $S^1 \times [0, +\infty)$ instead of $S^1 \times \mathbb{R}$. It contains the "boundary condition" manifold

$S^1 \times L \subset p^{-1}(S^1 \times \{0\})$. The symplectic form on X defines a fibrewise symplectic form ω_{E_ϕ} on E_ϕ. Choose a tame almost complex structure $\tilde{J}$ on E_ϕ which, over $S^1 \times [1, +\infty)$, equals $\tilde{J}(J_s, H_s)$ for some J_s, H_s (see Subsection 2.2.5), and in particular is independent of $t \in [1, +\infty)$.

Take $x \in \operatorname{Fix} \phi_H$ (2.5), that is, a generator of $CF^*(\phi; J_s, H_s)$. We define $\mathscr{M}^0(L, x)$ to be the set of all zero index $\tilde{J}$-holomorphic sections $u(s,t)\colon\ S^1 \times [0, +\infty) \to E_\phi$ which are asymptotic, as $t \to +\infty$, to the Hamiltonian trajectory $\psi_s(x)$ (2.2), and satisfy the Lagrangian boundary condition $u(s, 0) \in S^1 \times L$. Then we define

$$[L]^\phi = \sum_{x \in \operatorname{Fix} \phi_H} \sum_{u \in \mathscr{M}^0(L,x)} \pm q^{\omega(u)} \cdot [x] \in HF^*(\phi).$$

Here $[x] \in HF^*(\phi)$ is the cohomology class of the chain generator x, and

$$\omega(u) = \int_{S^1 \times [0,+\infty)} u^* \omega_{E_\phi}.$$

The signs are defined using the chosen spin structures on L_i and coherent orientations for ϕ. One can think of $[L]^\phi$ as the result of applying a "twisted" version of the open-closed string map; when $\phi = \mathrm{Id}$ it coincides with the $\mathscr{C}\mathscr{O}^0$-image of the unit in $HF^*(L, L)$ (see Chapter 1 for a discussion of the string maps).

Next we review the quantum product $HF^*(\phi) \otimes HF^*(\phi^{-1}) \to HF^*(\mathrm{Id}) \cong QH^*(X)$. It counts holomorphic sections of a symplectic fibration over S^2 with three punctures and monodromies $\phi, \phi^{-1}, \mathrm{Id}$ around them. The first two punctures serve as inputs for $HF^*(\phi)$, $HF^*(\phi^{-1})$, and the third puncture is the output, see [73] for details. If one caps the output puncture by a disk, the count of sections over the resulting twice-punctured sphere (see the lower part of Figure 2.3(a)), gives the composition $HF^*(\phi) \otimes HF^*(\phi^{-1}) \to HF^*(\mathrm{Id}) \xrightarrow{\chi} \Lambda$ of the product and the integration map χ (once we identify $HF^*(\mathrm{Id})$ with $QH^*(X)$).

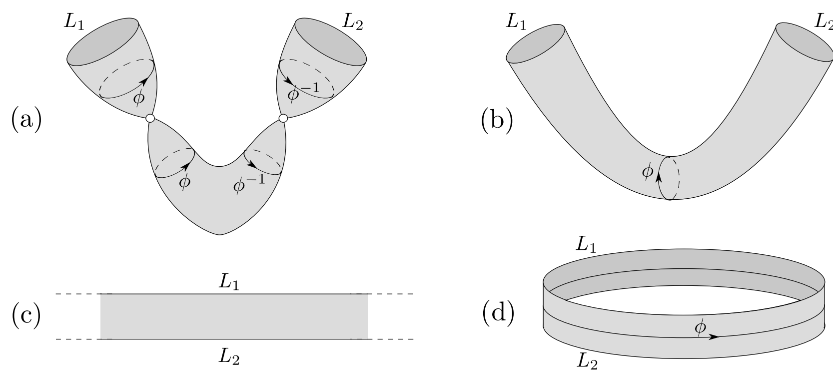


Fig. 2.3 Proving the Lagrangian elliptic relation.

Combining the definitions, $\chi([L_1]^{\phi} * [L_2]^{\phi^{-1}})$ counts holomorphic sections over two cylinders and a twice-punctured sphere which have the same asymptotics over the punctures in two pairs, see Figure 2.3(a). Here the cylinder $S^1 \times [0,+\infty)$ is seen as a once punctured disk. This count equals the number of sections of a glued fibration over an annulus, with monodromy ϕ around the core circle, and Lagrangian conditions $S^1 \times L_1, S^1 \times L_2$ over the boundary of the annulus. The annulus carries a fixed "long" complex structure, see Figure 2.3(b).

On the other hand, $STr(\phi_{\text{floer}})$ counts sections of a trivial fibration over the strip $[0,1]\times\mathbb{R}$ with Lagrangian boundary conditions $\mathbb{R}\times L_i$ and asymptotics differing by ϕ over $t\to\pm\infty$, see Figure 2.3(c). We can glue the fibration over the strip twisting it by ϕ to get a fibration over the annulus which we have already encountered: it carries Lagrangian conditions $S^1\times L_i$ over the boundary and has monodromy ϕ around the core circle, see Figure 2.3(d). By gluing, $STr(\phi_{\text{floer}})$ is equal to the count of holomorphic sections of this fibration, with a fixed ("long", but in the other direction than before) complex structure on the annulus. As the count of sections does not depend on the complex structure on the annulus, we get Theorem 2.1.5. We omit the discussion of signs which was carried out in detail for the case of commuting symplectomorphisms. The signs in present case can be studied by similar arguments if we superficially deform the Lagrangians so that $T_pL_1 = T_pL_2$ for all intersection points $p\in L_1\cap L_2$, keeping these points isolated, and then pick non-degenerate Hamiltonians H_1, H_2 to compute $HF^*(L_1,L_2)$.

Let us now explain Proposition 2.1.6. The most important step is to prove a Lagrangian analogue of Lemma 2.2.14: if ϕ is a map of finite order with fixed locus X^{ϕ} and smooth orientable Lagrangian fixed loci $L_i^{\phi}\subset X^{\phi}$ then

$$\chi([L_1]^{\phi} * [L_2]^{\phi}) = ([L_1^{\phi}]\cdot[L_2^{\phi}])\cdot q^0 + \sum_i a_i\cdot q^{\omega_i}, \quad \omega_i>0. \tag{2.34}$$

Recall that $[L_1^{\phi}]\cdot[L_2^{\phi}]\in\mathbb{Z}$ is the homological intersection of the fixed loci L_1^{ϕ}, L_2^{ϕ} inside X^{ϕ}. (Note that L_i^{ϕ} are automatically isotropic but not necessarily Lagrangian, although we will only use the case when they are Lagrangian. One can get examples of $(\phi_{\text{floer}})^k = -\mathrm{Id}$ in Lemma 2.2.18 when dimensions of L_1^{ϕ}, L_2^{ϕ} are different.)

In order to count sections of the configuration in Figure 2.3(a), we must specify the data $J_{s,t}, H_{s,t}$ over our configuration consisting of two half-cylinders $S^1\times[0,+\infty)$ and a twice-punctured sphere which we will now see as the cylinder $S^1\times\mathbb{R}$. Similarly to Lemma 2.2.14, we choose the data to be of special form, namely independent of the basepoint: $J_{s,t}\equiv J$, $H_{s,t}\equiv H$ (this forces J,H to be ϕ-equivariant). With this data, s-independent ($s\in S^1$) sections become gradient flowlines of the Morse function H inside the fixed locus X^{ϕ}. Rigid sections over $S^1 \times \mathbb{R}$ are constant, while rigid sections over $S^1 \times [0, +\infty)$ are flowlines from L_i to a critical point of H. This way, the count of s-independent rigid configurations in Figure 2.3(a) is $\sum_{x \in Crit^n(H|_{X^\phi})} ([L_1^\phi] \cdot [Stab(x)]) ([L_2^\phi] \cdot [Stab(x)])$ where $Crit^n$ are index n critical points, $n = \frac{1}{2} \dim_{\mathbb{R}} X^\phi$, and $Stab$ are the stable manifolds in X^ϕ. This sum equals the intersection $[L_1^\phi] \cdot [L_2^\phi]$.

Finally, one must argue that these configurations of flowlines are regular, and are the only zero area solutions. (There could be other positive area solutions which are not necessarily regular). This is a variation on the lemmas cited in the proof of Lemma 2.2.14. Then one makes the data J, H regular by allowing them to depend on s, t and argues that the count of zero area solutions (which were already regular) is preserved.

On the other hand, if ϕ is of finite order then $\phi_{\text{floer}}\colon\ HF^*(L_1, L_2) \to HF^*(L_1, L_2)$ is of finite order by Lemma 2.2.18, and the eigenvalues of ϕ_{floer} are among $\sqrt[2k]{1} \cdot q^0$. Consequently, $STr(\phi_{\text{floer}}) = a \cdot q^0$, $|a| \le \dim_\Lambda HF^*(L_1, L_2)$. Now Theorem 2.1.5 and formula (2.34) imply Proposition 2.1.6.

2.3 Vanishing spheres and Dehn twists

Let Y be a Kähler manifold with a Kähler form ω, and $\mathscr{L} \to Y$ a very ample holomorphic line bundle. Let $X \subset Y$ be a smooth divisor in the linear system $|\mathscr{L}|$. In this section we define $|\mathscr{L}|$-vanishing Lagrangian spheres in the symplectic manifold $(X, \omega|_X)$. They exist if the line bundle $\mathscr{L} \to Y$ has zero defect (see below) and are then unique up to symplectomorphism. Throughout this section, we denote by $D \subset \mathbb{C}$ the unit complex disk.

2.3.1 Lefschetz fibrations and vanishing cycles

This subsection reviews well known material, see e.g. [96].

Definition 2.3.1 (Lefschetz fibration with a unique singularity)**.** Suppose E is a smooth manifold, Ω a closed 2-form on E, and $\pi\colon\ E \to D$ is a smooth proper map. The triple (E, Ω, π) is called a Lefschetz fibration with a unique singularity if there is a point $p \in E$ (without loss of generality, we assume $\pi(p) = 0 \in D$), and a neighbourhood $U(p)$ such that:

- π is regular outside $U(p)$, and the restriction of Ω on the regular fibres of π is symplectic;
- there exists a complex structure on $U(p)$ with a holomorphic chart $x_1, \ldots, x_n$ such that

$$\pi(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2;$$

- $\Omega|_{U(p)}$ is Kähler with respect to the above complex structure.

All smooth fibres $E_t := \pi^{-1}(t)$ contain a Lagrangian sphere, uniquely defined up to Hamiltonian isotopy. Let us sketch its construction, as we will refer to it later in the proof of Lemma 2.4.6. Because the smooth fibres E_t are symplectomorphic to each other by parallel transport with respect to the Ω-induced connection on E, it suffices to construct a Lagrangian sphere in E_t for a small $t \in \mathbb{R}_+$. Define $L \subset U(p) \subset E$ by the equation

$$x_1^2 + \ldots + x_n^2 = t, \quad x_i \in \mathbb{R}.$$

Clearly $L \subset E_t$ and is a Lagrangian sphere for $t \in \mathbb{R}_+$ with respect to the standard symplectic structure Ω_{std} on $U(p) \subset \mathbb{C}^n$. However, it is generally not possible to make our form $\Omega|_{U(p)}$ standard by a holomorphic change of co-ordinates preserving π. Instead, we can follow the argument of [95, Lemma 1.6]: there is a function f on $U(p)$ such that $\Omega|_{U(p)} = \Omega_{std} + dd^c f$. We can deform f to 0 in a smaller neighbourhood $U'(p) \subset U(p)$ while leaving f unchanged outside of $U(p)$. Let f_r be such a homotopy and define $\Omega_r := \Omega$ outside of $U(p)$, and $\Omega_r|_{U(p)} := \Omega_{std} + dd^c f_r$. Observe that $\Omega_0 = \Omega$ and $\Omega_1|_{U'(p)} = \Omega_{std}$. For all r, the smooth fibres $(E_t, \Omega_r|_{E_t})$ are symplectic and the cohomology class of $\Omega_r|_{E_t}$ is constant, so by Moser's lemma the smooth fibres are actually symplectomorphic to each other for any r. In particular, the Lagrangian sphere $L \subset (E_t, \Omega_{\text{std}}|_{E_t})$ constructed above can be mapped by this symplectomorphism to a Lagrangian sphere in $(E_t, \Omega|_{E_t})$.

Definition 2.3.2 (Vanishing Lagrangian sphere)**.** A Lagrangian sphere in a smooth fibre E_t is called vanishing for the Lefschetz fibration $E \to D$ if it is Hamiltonian isotopic to the one constructed above.

2.3.2 Defect of a line bundle

Definition 2.3.3 (Defect of a line bundle)**.** Let Y be a complex manifold and $\mathscr{L} \to Y$ a very ample holomorphic line bundle, giving an embedding $Y \subset (\mathbb{P}^N)^*$ where $\mathbb{P}^N = \mathbb{P}H^0(Y, \mathscr{L})$. The discriminant variety $\Delta \subset \mathbb{P}^N$ is the dual variety to Y, parameterising all hyperplanes in $(\mathbb{P}^N)^*$ which are tangent to $Y \subset \mathbb{P}^N$. Equivalently, it parameterises all singular divisors in the linear system $\mathbb{P}H^0(Y, \mathscr{L})$. The defect of $\mathscr{L}$ is the number

$$\operatorname{def} \mathscr{L} = N - 1 - \dim \Delta \geq 0.$$

Line bundles usually have zero defect; for us, it is useful to note the following.

Lemma 2.3.4 ([13, page 532]). *Suppose $\mathscr{L} \to Y$ is a very ample line bundle. If* $\operatorname{def}\mathscr{L} \geq 1$*, there exists a smooth rational curve $l \subset Y$ such that $\mathscr{L} \cdot l = 1$.* □

For completeness, let us sketch a proof. Recall that points in Δ^{reg} correspond to generic hyperplanes $H \subset (\mathbb{P}^N)^*$ which are not transverse to Y. If $\operatorname{def}\mathscr{L} \geq 1$, for such a hyperplane $H \in \Delta^{reg}$ the contact locus $(H \cap Y)^{sing}$ is a linear $\mathbb{P}^{\operatorname{def}\mathscr{L}}$ [107, Theorem 1.18]. Take any line $l \cong \mathbb{P}^1$ in H. Obviously it intersects a generic smooth hyperplane section $\tilde{H} \cap Y$ transversely at a single point, which means $\mathscr{L} \cdot l = 1$.

Corollary 2.3.5. *Suppose $\mathscr{L} \to Y$ is a very ample line bundle. For any $d \geq 2$,* $\operatorname{def}\mathscr{L}^{\otimes d} = 0$. □

2.3.3 $|\mathscr{L}|$-vanishing spheres in divisors

Recall $D \subset \mathbb{C}$ denotes the unit disk.

Definition 2.3.6 (Total space of a family of divisors)**.** Let Y be a Kähler manifold and $\mathscr{L} \to Y$ a very ample line bundle. Take a holomorphic embedding $u\colon D \to \mathbb{P}H^0(Y,\mathscr{L}) = |\mathscr{L}|$, then each point $t \in D$ defines a divisor $X_{u(t)} \subset Y$. We call $\{X_{u(t)}\}_{t\in D}$ a family of divisors. The total space of the family $\{X_{u(t)}\}_{t\in D}$ is

$$E := \{(x,u(t)) : \, x \in X_t, \, t \in D\} \subset Y \times \mathbb{P}H^0(Y,\mathscr{L}).$$

The restriction of the product Kähler form from $Y \times \mathbb{P}H^0(Y,\mathscr{L})$ to E makes E a Kähler manifold. There is a canonical projection $\pi\colon E \to D$ whose fibres are $X_{u(t)}$. In future we shall write $\{X_t\}_{t\in D}$ instead of $\{X_{u(t)}\}_{t\in D}$.

Definition 2.3.7 ($|\mathscr{L}|$-vanishing Lagrangian sphere in a divisor)**.** Let Y be a Kähler manifold and $\mathscr{L} \to Y$ a very ample line bundle with zero defect, and with $\dim \mathbb{P}H^0(Y,\mathscr{L}) \geq 2$. Let $\Delta \subset \mathbb{P}H^0(Y,\mathscr{L})$ be the discriminant variety from Definition 2.3.3. Let $u\colon D \to \mathbb{P}H^0(Y,\mathscr{L})$ be a holomorphic embedding such that $u(0) \in \Delta^{reg}$, $u(t) \notin \Delta$ for $t \neq 0$, and the intersection of $u(D)$ with Δ^{reg} is transverse. Let $\pi\colon E \to D$ be as in Definition 2.3.6.

By [70, 1.8], $\pi\colon E \to D$ is a Lefschetz fibration with a unique singular point over $t = 0$ (in particular, X_0 has a single node). The vanishing sphere $L \subset X_1$ of this fibration is called an $|\mathscr{L}|$-vanishing sphere.

Obviously, every smooth divisor in the linear system $|\mathscr{L}|$ contains an $|\mathscr{L}|$-vanishing sphere, if $\mathscr{L}$ has zero defect. Two different maps $u,u'\colon D \to H^0(Y,\mathscr{L})$ with $u(1) = u'(1)$ can give two $|\mathscr{L}|$-vanishing spheres in X_1 which are not Hamiltonian isotopic and even not homologous to each other, such as in the case of Lemma 2.4.1. However, $|\mathscr{L}|$-vanishing spheres are unique up to symplectomorphism.

Lemma 2.3.8. *Let $\mathscr{L} \to Y$ be a very ample line bundle over a Kähler manifold Y, $\operatorname{def} \mathscr{L} = 0$. Suppose X, X' are two smooth divisors in the linear system $|\mathscr{L}|$ and $L \subset X$, $L' \subset X'$ are two $|\mathscr{L}|$-vanishing Lagrangian spheres. Then there is a symplectomorphism $\psi\colon X \to X'$ such that $\psi(L) = L'$.*

This lemma is probably well known, but we don't have a clear reference for it, so we prove it here. An auxiliary lemma is required.

Lemma 2.3.9. *Let $\pi\colon X \to D \times [0,1]$ be a smooth proper map and Ω a closed 2-form on X. Suppose that for every $s \in [0,1]$, $X_{D;s} := \pi^{-1}(D \times \{s\})$, equipped with the restriction of Ω, is a Lefschetz fibration over D with a unique singularity over $0 \in D$. (In particular, the fibres of π are symplectic.) For $t \in D$, $s \in [0,1]$ denote by $X_{t;s}$ the fibre $\pi^{-1}(\{t\} \times \{s\})$. Let $L_0 \subset X_{1;0}$ (resp. $L_1 \subset X_{1;1}$) be a vanishing sphere of the Lefschetz fibration $X_{D;0}$ (resp. $X_{D;1}$). Then there is a symplectomorphism $\psi\colon X_{1;0} \to X_{1;1}$ such that $\psi(L_0) = L_1$.*

Proof. One can choose a smooth family of Lagrangian spheres $L_s \subset X_{1;s}$ such that L_s is vanishing for the fibration on $X_{D,s}$, and L_0, L_1 are the given spheres. This is easily seen from our definition or from [96, proof of Lemma 16.2].

Fix $s \in [0;1]$ and let $\phi_\varepsilon\colon X_{1;s} \to X_{1;s+\varepsilon}$ be the parallel transport with respect to Ω [96, Section 15a] along the s-direction. Let us look at $\phi_\varepsilon(L_s)$ and $L_{s+\varepsilon}$: these are two Lagrangian spheres in $X_{1;s+\varepsilon}$ which coincide when $\varepsilon = 0$, so they remain sufficiently close to each other for ε small enough, say $|\varepsilon| < \varepsilon(s)$. Being sufficiently close, the two spheres are Hamiltonian isotopic inside $X_{1;s+\varepsilon}$. By composing ϕ_ε with this Hamiltonian isotopy, we get a symplectomorphism $\psi_\varepsilon\colon X_{1;s} \to X_{1;s+\varepsilon}$ taking L_s to $L_{s+\varepsilon}$.

The open cover of $[0,1]$ consisting of the intervals $\{(s-\varepsilon(s), s+\varepsilon(s))\}_{s\in[0;1]}$ admits a finite subcover. We know that for s, s' within a single interval, L_s can be taken to L_s' by a symplectomorphism $X_{1;s} \to X_{1;s'}$; using the finite subcover, we are able to find a finite composition of such maps which is a symplectomorphism $X_{1;0} \to X_{1;1}$ taking L_0 to L_1. □

Proof of Lemma 2.3.8. Let $u, u'\colon D \to \mathbb{P}H^0(Y, \mathscr{L})$ be two holomorphic maps as in Definition 2.3.7, and denote $X = X_{u(1)}$, $X' = X_{u'(1)}$. By Definition 2.3.7, $u(0), u'(0) \in \Delta^{reg}$. Since Δ^{reg} is connected, one can find a path $\alpha(s) \in \Delta^{reg}$ from $u(0)$ to $u'(0)$, $s \in [0,1]$. Next one can find an s-parametric family of holomorphic disks $u_s\colon D \to \mathbb{P}H^0(Y, \mathscr{L})$ such that $u_0 = u$, $u_1 = u'$, $u_s(0) \in \Delta^{reg}$ and $u_s(D)$ intersects Δ^{reg} transversely. Consider the space

$$E := \{(x, u_s(t)) :\ t \in D,\ s \in [0,1],\ x \in X_{u(t)}\} \subset Y \times \mathbb{P}H^0(Y, \mathscr{L}).$$

It carries a closed 2-form which is the restriction of the product Kähler form to Y and $\mathbb{P}H^0(Y,\mathscr{L})$. There is also a canonical projection $E \to D \times [0,1]$. With these data, E satisfies conditions of Lemma 2.3.9. This lemma provides the desired symplectomorphism $\psi\colon X \to X'$ taking an given $|\mathscr{L}|$-vanishing sphere in X to a given one in X'. □

2.3.4 Dehn twists

We recall the definition of Dehn twists from [96, Section (16c)]. First, one defines the Dehn twist as a compactly supported symplectomorphism of T^*S^n. Fix the standard round metric on S^n, and let $|\xi|$ be the norm function on T^*S^n. It is non-smooth at the 0-section; away from the 0-section, its Hamiltonian flow is the normalised geodesic flow. Take a function $b(r)\colon \mathbb{R} \to \mathbb{R}$ with compact support and such that $b(r) - b(-r) = -r$. The Dehn twist $\tau\colon T^*S^n \to T^*S^n$ is the 2π-flow of the Hamiltonian function $b(|\xi|)$. It extends smoothly to the 0-section by the antipodal map, thanks to the special form of $b(r)$. As a result, τ is a compactly supported symplectomorphism of T^*S^n. Its behaviour in T^*S^n is well understood.

Theorem 2.3.10. *1. τ has infinite order in $Symp^c(T^*S^n)/Ham^c(T^*S^n)$, the group of compactly supported symplectomorphisms of T^*S^n modulo compactly-supported symplectic isotopy.*

2. If n is even, τ has finite order in $\pi_0 Diff^c(T^*S^n)$, the group of compactly-supported diffeomorphisms of T^*S^n modulo compactly-supported isotopy [64]. □

When $n = 2$ it is further known that τ generates $\pi_0 Symp^c(T^*S^2) \cong \mathbb{Z}$, and τ^2 is smoothly isotopic to Id in $Diff^c(T^*S^2)$ [97], see also [12, Theorem 1.21].

Next, if $L \subset X$ is a Lagrangian sphere in any symplectic manifold, a neighbourhood of L in X is symplectomorphic to a neighbourhood of the 0-section in T^*S^n. So one can pull back τ using this symplectomorphism and then extend it by the identity to get a map $\tau_L\colon X \to X$. It is a symplectomorphism uniquely defined up to Hamiltonian isotopy (once a parameterisation of L is fixed), supported in a neighbourhood of L.

Definition 2.3.11 (Dehn twist)**.** The symplectomorhism $\tau_L\colon X \to X$ is called the Dehn twist around L.

Lemma 2.3.12 (Picard-Lefschetz formula, [70])**.** *If $\dim_{\mathbb{R}} X = 2n$ and $L \subset X$ is a Lagrangian sphere, then $(\tau_L)_*$ acts by* Id *on $H_i(X)$, $i \neq n$. For any $[A] \in H_n(X)$,*

$$(\tau_L)_*[A] = [A] - \varepsilon \cdot ([L]\cdot[A])[L].$$

Here $\varepsilon = (-1)^{\frac{1}{2}n(n-1)}$. Consequently:

1. *if n is even, then $(\tau_L)_*^2$ acts by* Id *on $H_*(X)$.*

2. *if n is odd and $[L] \in H_n(X;\mathbb{R})$ is non-zero, then $(\tau_L)_*$ is an automorphism of infinite order of $H_*(X)$.* □

Summarising Theorem 2.3.10(2) and Lemma 2.3.12(2), we arrive to the following well known statement.

Corollary 2.3.13. *Let* $\dim X_{\mathbb{R}} = 2n$ *be a compact symplectic manifold and $L \subset X$ a Lagrangian sphere non-zero in $H_n(X;\mathbb{R})$.*

1. *If n is even, τ_L has finite order in $\pi_0 Diff(X)$,*

2. *if n is odd, τ_L has infinite order in $\pi_0 Diff(X)$.* □

The next lemma relates Dehn twists and Lefschetz fibrations, see [96, (15b)] for details.

Lemma 2.3.14 ([95, 96])**.** *Let (E, Ω, π) be a Lefschetz fibration with a unique singularity. Let E_1 be its regular fibre and $L \subset E_1$ a vanishing Lagrangian sphere. Then the Dehn twist $\tau_L \colon E_1 \to E_1$ is Hamiltonian isotopic to the symplectic monodromy map $E_1 \to E_1$ obtained by applying symplectic parallel transport to the fibres E_t along the circle $t \in \partial D$.* □

Remark 2.3.1. Let X be a symplectic manifold and $L \subset X$ a Lagrangian sphere; assume L is non-zero in $H_n(X)$. There are three main previously known cases when τ_L has infinite order in $Symp(X)/Ham(X)$ (if X is non-compact, consider $Symp^c(X)/Ham^c(X)$ instead):

1. $\frac{1}{2}\dim_{\mathbb{R}} X$ is odd, as explained above;

2. X is exact with contact type boundary, and L is exact (Seidel, unpublished);

3. X is Calabi-Yau, and there is another Lagrangian sphere L' intersecting L once transversely [94].

Let $X = Bl_k\mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ in k generic points, $2 \leq k \leq 8$, with the monotone symplectic form, and $L \subset X$ be any Lagrangian sphere. Seidel [97] showed that τ_L has order 2 in $Symp(X)/Ham(X)$ when $k = 2,3,4$ and has order greater than 2 when $k = 5,6,7,8$, but did not prove it was infinite. Note that $X = Bl_6\mathbb{P}^2$ is the cubic surface $X \subset \mathbb{P}^3$, to which Theorem 2.1.2 applies.

2.4 Constructing invariant Lagrangian spheres

The aim of this section is to state and prove Proposition 2.4.2, which will later be used to prove Theorem 2.1.7. We start by stating an essentially known lemma which can be used to prove the simple case of Theorem 2.1.2 when $\dim_{\mathbb{C}} X$ is odd.

Lemma 2.4.1. *Let $\mathscr{L}$ be a very ample line bundle over a Kähler manifold Y. For any $d \geq 3$, every smooth divisor $X \subset Y$ in the linear system $|\mathscr{L}^{\otimes d}|$ contains two $|\mathscr{L}^{\otimes d}|$-vanishing Lagrangian spheres L_1, L_2 that intersect transversely, once.* $\square$

The proposition below should be considered as an equivariant version of Lemma 2.4.1. It will be used to prove the harder case of Theorem 2.1.7 when $\dim_{\mathbb{C}} X$ is even. (When applicable, it in particular provides the conclusion of Lemma 2.4.1 itself. So we will not need to prove Lemma 2.4.1 for our purposes, although the arguments in this section can readily be adopted, in fact simplified, to give such a proof.)

Proposition 2.4.2. *Let $\mathscr{L}$ be a very ample line bundle over a Kähler manifold Y, and let $\iota\colon Y \to Y$ be a holomorphic involution which lifts to an automorphism of $\mathscr{L}$. Fix $d \geq 3$ and let $H^0(Y, \mathscr{L}^{\otimes d})_{\pm}$ denote the ± 1-eigenspace of the involution on $H^0(Y, \mathscr{L}^{\otimes d})$ induced by ι. Let $\Pi_{\pm}$ be as in Theorem 2.1.7. Pick a connected component $\tilde{\Sigma}$ of $Y^{\iota} \subset Y$, $\dim \tilde{\Sigma} \geq 2$. Suppose one of the following:*

(a) d is even;

(b) d is odd, $\tilde{\Sigma} \subset \Pi_{+}$, and there is a smooth divisor in the linear system $\mathbb{P}H^0(Y, \mathscr{L}^{\otimes d})_{+}$.

Then there is a smooth divisor X in the linear system $|\mathscr{L}^{\otimes d}|$ and two $|\mathscr{L}^{\otimes d}|$-vanishing Lagrangian spheres $L_1, L_2 \subset X$ such that:

1. *$\iota(X) = X$, $\Sigma := X \cap \tilde{\Sigma}$ is smooth, $\dim \Sigma = \dim \tilde{\Sigma} - 1$*
2. *$\iota(L_1) = L_1$, $\iota(L_2) = L_2$;*
3. *L_1, L_2 intersect transversely, at a single point which belongs to Σ;*
4. *$L_i^{\iota} = L_i \cap \Sigma$ are Lagrangian spheres in Σ, $i = 1, 2$.*
5. *for $i = 1, 2$ one can choose a symplectomorphism τ_{L_i} of X representing the Hamiltonian isotopy class of the Dehn twist around L_i such that τ_{L_i} commutes with ι, and $\tau_{L_i}|_{X^{\iota}}$ is the Dehn twist around L_i^{ι}.*

The same is true if we replace symbols $+$ with $-$ in Case (b).

2.4.1 A_2 chains of Lagrangian spheres from A_2 fibrations

Definition 2.4.3 (A_2 chain of Lagrangian spheres)**.** Let X be a symplectic manifold. A pair (L_1, L_2) of two Lagrangian spheres in X is called an A_2-chain if L_1 and L_2 intersect at a single point, and the intersection is transverse.

In Section 2.3 we have seen that how to construct Lagrangian spheres as vanishing cycles of Lefschetz fibrations. Similarly, one can get A_2 chains of Lagrangian spheres from fibrations with slightly more complicated singularities.

Definition 2.4.4 (A_2 fibration)**.** Denote by $D \subset \mathbb{C}$ the open unit disk, and by $B_\varepsilon \subset \mathbb{C}$ the open disk of radius ε. Both disks are centered at 0.

Suppose E is a smooth manifold, Ω a closed 2-form on E and $\pi\colon E \to D$ is a smooth map. The triple (E, Ω, π) is called an A_2 fibration if there is a point $p \in E$ (without loss of generality, we assume $\pi(p) = 0 \in D$), and a neighbourhood $U(p)$ such that:

- all but a finite number of fibres of π are regular, and the restriction of Ω is symplectic on them;
- there exists a complex structure on $U(p)$ with a holomorphic chart $x_1, \ldots, x_n$, $x_i \in B_\varepsilon$ such that
$$\pi(x_1, \ldots, x_n) = x_1^2 + \ldots + x_{n-1}^2 + h(x_n),$$
where $h(x_n)$ is holomorphic;
- $h(x_n)$ has at least 3 roots within $B_{\varepsilon/2}$, counted with multiplicities;
- for any $x_n \in B_{\varepsilon/2}$, $\sqrt{h(x_n)} \in B_{\varepsilon/2}$;
- $\Omega|_{U(P)}$ is Kähler with respect to the above complex structure.

Remark 2.4.1. The definition allows π to have singularities outside of $U(p)$. Also, the definition does not require $p\colon E \to D$ to be a proper map, so the smooth fibres E_t need not be symplectomorphic, as we may not be able to integrate the parallel transport vector fields. The generality of this definition is slightly unusual, but it makes no difference to the local construction of A_2 chains of Lagrangian spheres, which is the next thing we discuss.

In order to prove Proposition 2.4.2, we need to introduce A_2 fibrations with involutions.

Definition 2.4.5 (Involutive A_2 fibration)**.** Let (E, Ω, π) be an A_2 fibration. It is called an involutive A_2 fibration with involution $\iota\colon E \to E$ if in the holomorphic chart from Definition 2.4.4 we have in addition:

$$\iota(x_1, \ldots, x_l, x_{l+1}, \ldots, x_n) = (-x_1, \ldots, -x_l, x_{l+1}, \ldots, x_n)$$

for some $l < n$. We denote by E^ι the fixed locus of ι.

Remark 2.4.2. It follows from this definition that $\pi|_{E^\iota}\colon E^\iota \to D$ is also an A_2 fibration. Note that $x \in E^\iota$ is regular for π if and only if it is regular for $\pi|_{E^\iota}$. Indeed, we can decompose $T_xE = T_xE^\iota \oplus N_x$ where N_x is the (-1)-eigenspace of $d\iota(x)$. Since $\pi\iota = \pi$, $N_x \subset \ker d\pi(x)$. So $\mathrm{rk}\, d\pi(x) = \mathrm{rk}\, d\pi(x)|_{T_xE^\iota}$. Consequently, for a regular fibre E_t, the fixed locus E_t^ι is smooth.

The following is a slight refinement of [61, Lemma 6.12].

Lemma 2.4.6. *Let $\pi\colon E \to D$ be an A_2 fibration. Then for every sufficiently small $t \in D$ such that the fibre $E_t := \pi^{-1}(t)$ is smooth, E_t contains an A_2 chain of Lagrangian spheres.*

We will use the following equivariant analogue of this lemma.

Lemma 2.4.7. *Let $\pi\colon E \to D$ be an involutive A_2 fibration with an involution ι. Then for every sufficiently small $t \in D$ such that the fibre $E_t := \pi^{-1}(t)$ is smooth, E_t contains an A_2 chain of Lagrangian spheres (L_1, L_2) which satisfy properties (2)—(5) from Proposition 2.4.2 with $X := E_t$, and Σ the connected component of E_t^ι which is a subset of the connected component of the point p in E^ι.*

Remark 2.4.3. Note that $\dim \Sigma = l - 1$, where l is the number coming from the co-ordinate chart in Definition 2.4.5.

Proof of Lemma 2.4.6. Let $U'(p) \subset U(p)$ be the ball around p given by $|x_i| < \varepsilon/2$, $i = 1, \ldots, n$. As in Subsection 2.3.1, it suffices to assume $\Omega|_{U'(p)}$ is the standard symplectic form in the holomorphic chart $(x_1, \ldots, x_n)$ from Definition 2.4.4.

The condition that E_t is smooth means the equation $h(x_n) = t$ has no multiple roots with $x_n \in B_{\varepsilon/2}$. Therefore by Definition 2.4.4, the equation $h(x_n) = 0$ has at least 3 roots with $x_n \in B_{\varepsilon/2}$. So for sufficiently small t the equation $h(x_n) = t$ also has at least 3 distinct roots with $x_n \in B_{\varepsilon/2}$. Pick three such roots, say $z_1, z_2, z_3 \in B_{\varepsilon/2}$: $h(z_i) = t$. Let $\gamma_{12} \subset B_{\varepsilon/2}$ be a path from z_1 to z_2 whose interior avoids the roots of $h - t$. Define

$$L_1 := \bigsqcup_{z \in \gamma_{12}} \{(x_1, \ldots, x_n) \in B_{\varepsilon/2} \cap \pi^{-1}(t) : \ |x_i| \in \mathbb{R} \cdot \sqrt{-h(z)}\}.$$

This is a smooth Lagrangian sphere in $\pi^{-1}(t)$ with respect to the restriction of the standard symplectic form on $\mathbb{C}^n$ to $\pi^{-1}(t)$. Similarly, let $\gamma_{23} \subset B_{\varepsilon/2} \subset \mathbb{C}$ be a path from z_2 to z_3 and define L_2 by the same formula replacing γ_{12} by γ_{23}. If γ_{12} and γ_{23} are transverse at their common endpoint z_2, then (L_1, L_2) is an A_2 chain of Lagrangian spheres by [61, Lemma 6.12]. Note that L_1, L_2 lie in $U'(p)$ by the fourth condition in Definition 2.4.4. $\square$

Proof of Lemma 2.4.7. We use the notation from the proof of Lemma 2.4.6. Arguing as in that proof ι-invariantly, we can again assume Ω is standard on $U'(p)$. The formulas for L_1, L_2 are invariant under the change $x_i \mapsto -x_i$, $i \le l$, so L_1, L_2 are ι-invariant. This proves property (2) from Proposition 2.4.2. Next, we already know L_1 intersects L_2 transversely at a single point. This point has co-ordinates $x_1 = 0, \ldots, x_{n-1} = 0$, $x_n = z_2$. (Recall z_2 is a root of $h(x_n) - t$.) This intersection point is ι-invariant, and it obviously belongs to the connected component of the point p in E^ι, so property (3) from Proposition 2.4.2 holds. Property (4) is true because E^ι locally around π is given by $x_1 = \ldots = x_l = 0$, and so $L_i \cap \Sigma$ are transverse Lagrangians for the same reason that the L_i are. By their local construction, the L_i do not intersect the connected components of E_t^ι other than Σ.

It remains to explain property (5). Let $S^{n-1} \subset \mathbb{R}^n$ be the standard unit sphere. Let ι_0 be the involution on S^n which changes the sign of the first k co-ordinates on $\mathbb{R}^n$. It naturally extends to an involution ι_0 on T^*S^n. It is not hard to check there is an (ι, ι_0)-equivariant diffeomorphism $V(L_1) \to V(S^n)$ where $V(L_1)$ is an ι-invariant tubular neighbourhood of $L_1 \subset X$ and $V(S^n)$ is an ι_0-invariant tubular neighbourhood of the zero-section in T^*S^n. Then there is also an (ι, ι_0)-equivariant symplectomorphism $V(L_1) \to V(S^n)$, by an equivariant analogue of the Weinstein tubular neighbourhood theorem. The Dehn twist in T^*S^n is ι_0-equivariant by definition. Its pullback via the equivariant symplectomorphism $V(L_1) \to V(S^n)$ is the desired ι-equivariant Dehn twist inside E_t. $\square$

2.4.2 A_2 fibrations of divisors from projective embeddings

One way of constructing an A_2 fibration is to embed all its fibres E_t as divisors $E_t = X_t \subset Y$ in a single Kähler manifold Y. This idea can be used to prove Lemma 2.4.1, and now we will run such an argument ι-invariantly to prove Proposition 2.4.2.

Proof of Proposition 2.4.2. Let us recall the setting. We are given a very ample line bundle $\mathscr{L} \to Y$ over a Kähler manifold Y, and a holomorphic involution $\iota\colon Y \to Y$ which lifts to an involution on $\mathscr{L}$. This means ι induces a linear involution on $H^0(Y, \mathscr{L})^*$ splitting it into the direct sum of ± 1 eigenspaces denoted by $H^0(Y, \mathscr{L})^*_\pm$. The projectivisations of these eigenspaces are denoted by $\Pi_\pm \subset \mathbb{P}H^0(Y, \mathscr{L})^*$. We also denote $\mathbb{P}^N := \mathbb{P}H^0(Y, \mathscr{L})^*$, and the ι-induced involution on $\mathbb{P}^N$ by $\iota_{\mathbb{P}^N}$. The fixed locus of $\iota_{\mathbb{P}^N}$ is $\Pi_+ \sqcup \Pi_- \subset \mathbb{P}^N$.

Because $\mathscr{L}$ is very ample, we have an embedding $Y \subset \mathbb{P}^N$, $\mathscr{L} = \mathscr{O}_Y(1) := \mathscr{O}_{\mathbb{P}^N}(1)|_Y$, Y is invariant under $\iota_{\mathbb{P}^N}$ and $\iota_{\mathbb{P}^N}|_Y = \iota$, and also

$$Y^\iota = (Y \cap \Pi_+) \sqcup (Y \cap \Pi_-).$$

Let $\tilde{\Sigma}$ be the given connected component of Y^ι (smooth by assumption), and $\dim \tilde{\Sigma} = l$. Then $\tilde{\Sigma} \subset \Pi_\varepsilon$ where ε is one of the two symbols: $+$ or $-$. We will also denote by ε the correspondingly signed number ± 1.

Choose homogeneous co-ordinates $(x_0 : \ldots : x_l : x_{l+1} : \ldots : x_N)$ on $\mathbb{P}^N$ with the following properties:

1. $\iota_{\mathbb{P}^N}(x_0 : \ldots : x_l : x_{l+1} : \ldots : x_N) = (\varepsilon x_0 : \ldots : \varepsilon x_l : \pm x_{l+1} : \ldots : \pm x_{N+1})$;
2. $(1:0:\ldots:0) \in \tilde{\Sigma}$
3. the plane spanned by $(x_0, \ldots, x_l)$ (other co-ordinates are set to 0) is the tangent plane to $\tilde{\Sigma}$ at $(1:0:\ldots:0)$;
4. for some $n \geq l$, the plane spanned by $(x_0, \ldots, x_n)$ (other co-ordinates are set to 0) is the tangent plane to Y at $(1:0:\ldots:0)$.

The third property implies that $x_0, \ldots, x_l$, seen as sections in $H^0(\mathscr{O}_{\mathbb{P}^N}(1))$, belong to the ε-eigenspace of ι. This is in agreement with the first property. So co-ordinates with the above properties exist.

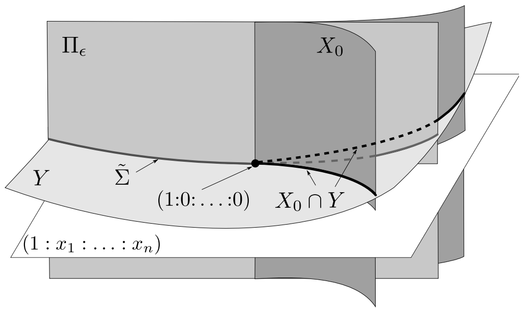


Fig. 2.4 A divisor X_0 from the family X_t constructed in the proof of Proposition 2.4.2.

In the affine chart $x_0 = 1$, the co-ordinates $(x_1, \ldots, x_n)$ serve as local co-ordinates for Y near the origin. In the chart $x_0 = 1$, write (see Figure 2.4):

$$X_t := x_1^3 + x_2^2 + \ldots + x_n^2 - t.$$

We want X_t to be a section of $\mathscr{O}_{\mathbb{P}^N}(d)$, so in projective co-ordinates we set

$$X_t := x_0^{d-3} x_1^3 + x_0^{d-2}(x_2^2 + \ldots + x_n^2) - t x_0^d.$$

From property (1) of the co-ordinates x_i, we see that $X_t \circ \iota = \varepsilon^d X_t$ as polynomials. In other words:

(a) if d is even, $X_t \in H^0(\mathscr{O}_{\mathbb{P}^N}(d))_+$;

(b) if d is odd, $X_t \in H^0(\mathscr{O}_{\mathbb{P}^N}(d))_\varepsilon$.

For all t, the divisors $\{X_t = 0\}$ and $\{X_t = 0\} \cap Y$ are reducible and hence singular. We want to smooth the family $\{X_t = 0\} \cap Y$ so that a generic divisor in this t-family becomes non-singular.

Suppose d is even. Then the linear system $H^0(\mathscr{O}_{\mathbb{P}^N}(d))_+$ has no base locus as it contains all monomials x_i^d. Then $H^0(\mathscr{O}_Y(d))_+ = H^0(Y, \mathscr{L}^{\otimes d})_+$ has no base locus too. By Bertini's theorem in characteristic 0, there exists $F \in H^0(\mathscr{O}_{\mathbb{P}^n}(d))_+$ such that the divisor $\{F = 0\} \cap Y$ is smooth.

Suppose d is odd. Then the linear systems $H^0(\mathscr{O}_{\mathbb{P}^N}(d))_\pm$ have non-empty base loci, namely $\Pi_\mp$ (see the proof of Lemma 2.1.8 below). Therefore it is not a priori clear that these linear systems contain a smooth divisor. This condition is included in the assumptions of Proposition 2.4.2, Case (b). Let $F \in H^0(\mathscr{O}_{\mathbb{P}^n}(d))_\varepsilon$ be a polynomial such that $\{F = 0\} \cap Y$ is smooth.

The rest of the proof is the same for even and odd d. For all generic $\delta \in \mathbb{C}$, the divisors $\{X_t + \delta F = 0\} \cap Y$ are smooth except for a finite number of t's. Recall that $(x_1, \ldots, x_n)$ is a holomorphic chart for Y around $(1:0:\ldots:0)$. There is another chart $\tilde{x}_1, \ldots, \tilde{x}_n$ in which the divisors $\{X_t + \delta F = 0\} \cap Y$ are given by:

$$h(\tilde{x}_1) + \tilde{x}_2^2 + \ldots + \tilde{x}_n^2 - t + c = 0$$

where $h(\tilde{x}_1)$ is close to $\tilde{x}_1^3$ (when δ is small) and c is a small constant. Moreover, the change of co-ordinates from x_i to $\tilde{x}_i$ is ι-equivariant. This follows from an equivariant version of the holomorphic Morse splitting lemma [9].

Consider the family $\{X_t + \delta F = 0\} \cap Y$ of divisors in Y, $t \in D$. They are ι-invariant and belong to the linear system $|\mathscr{L}^{\otimes d}|$. Let $E \to D$ be the total space of this family, see Definition 2.3.6. It may be singular; if it is, remove its singular locus to get E_0. The involution ι turns $E_0 \to D$ into an involutive fibration in the sence of Definition 2.4.5. So by Lemma 2.4.7, a smooth divisor in the family $\{X_t + \delta F = 0\} \cap Y$ has a pair of Lagrangian spheres (L_1, L_2) that satisfy properties (2)—(5) of Proposition 2.4.2. It is easy to see that Lemma 2.4.7 constructs L_1, L_2 which are $|\mathscr{L}^{\otimes d}|$-vanishing.

It remains to check ι satisfies property (1). We have to show that the smooth divisors $\{X_t + \delta F = 0\} \cap Y$ intersect $\Sigma = \tilde{\Sigma} \cap Y$ transversely. Suppose $X := \{X_t + \delta F = 0\} \cap Y$ intersects Σ non-transversely at one point p, so $T_p\Sigma \subset T_pX$ (the tangent spaces are taken inside Y). This means T_pX contains $\dim \Sigma$ positive $(+1)$ eigenvalues of $d\iota$. Then the same must hold for all intersection points $X \cap \Sigma$, and hence $T_p\Sigma \subset T_pX$ for any $p \in X \cap \Sigma$. But in a neighbourhood of $(1:0:\ldots:0)$ the intersection $X\cap\Sigma$ is transverse, which is easily verified in the local chart $(x_1,\ldots,x_n)$ from above. So X intersects Σ transversely everywhere. Similarly, every other connected component of Y^ι either intersects X transversely or is contained in X. □

2.5 Proofs of the theorems about Lagrangian spheres in divisors

Proof of Theorem 2.1.7. Apply Proposition 2.4.2 to $Y,\mathscr{L},\tilde{\Sigma}$ given by the hypothesis of Theorem 2.1.7. Proposition 2.4.2 returns an $|\mathscr{L}^{\otimes d}|$-divisor $X\subset Y$ and $|\mathscr{L}^{\otimes d}|$-vanishing Lagrangian spheres $L_1,L_2\subset X$ satisfying the conditions enumerated there. Because $|\mathscr{L}^{\otimes d}|$-vanishing spheres are unique up to symplectomorphism (Lemma 2.3.8), it suffices to show that τ_{L_1} has infinite order in $Symp(X)/Ham(X)$. To show this, we compute the Lefschetz number of $\tau_{L_1}^{2k}\tau_{L_2}^{2k}|_{X^\iota}=\tau_{L_1^\iota}^{2k}\tau_{L_2^\iota}^{2k}$ on $H^*(X^\iota)$, where X^ι is the fixed locus of the involution ι on X. Recall that $\Sigma=\tilde{\Sigma}\cap X$ is a connected component of X^ι. We are given that $\dim\tilde{\Sigma}$ is even, so $\dim\Sigma=\dim\tilde{\Sigma}-1$ is odd. Let $X^\iota=\Sigma\sqcup\Sigma_0$ where Σ_0 is all other connected components. We identify $H^*(X^\iota)$ with $H_*(X^\iota)$ via Poincaré duality.

Consider the homology classes $[L_1^\iota],[L_2^\iota]\in H_*(\Sigma)$, $[L_1^\iota]\cdot[L_2^\iota]=1$. Using the Picard-Lefschetz formula (see Subsection 2.3.4) and property (5) from Proposition 2.4.2, we write down the actions of Dehn twists on the 2-dimensional vector space $\operatorname{span}\{[L_1^\iota],[L_2^\iota]\}\subset H_*(X^\iota)$. Let $s=\dim_{\mathbb{C}}\Sigma$ and $\varepsilon=(-1)^{\frac{1}{2}s(s-1)}$.

$$(\tau_{L_1^\iota})_*^{2k}:\begin{pmatrix}1 & k(1+(-1)^{s-1})\varepsilon\\ 0 & 1\end{pmatrix},\qquad (\tau_{L_2^\iota})_*^{2k}:\begin{pmatrix}1 & 0\\ k\left(1+(-1)^{s-1}\right)\varepsilon & 1\end{pmatrix}.$$

Now since $s=\dim_{\mathbb{C}}\Sigma$ is odd, we see that

$$STr\left((\tau_{L_1^\iota})_*^{2k}(\tau_{L_2^\iota})_*^{2k}|_{\operatorname{span}\{[L_1^\iota],[L_2^\iota]\}}\right)=-4k^2-2.$$

(The negative signs appear because we are computing the supertrace). If s were even, we would get the constant 2 instead.

We can extend $[L_1^\iota],[L_2^\iota]$ to a basis of $H_*(X^\iota)$ in which all other elements have zero intersection with $[L_1^\iota],[L_2^\iota]$. By the Picard-Lefschetz formula, $(\tau_{L_i^\iota})_*$ acts by Id on the rest of such basis. Consequently, the Lefschetz number is

$$L\left((\tau_{L_1^\iota})^{2k}(\tau_{L_2^\iota})^{2k}\right)=-4k^2+c,$$

where c is a constant independent of k. By Proposition 2.1.4,

$$\dim_\Lambda HF^*(\tau_{L_1}^{2k}\tau_{L_2}^{2k}) \geq |-4k^2+c|. \tag{2.35}$$

Suppose $\tau_{L_1}^{2k}$ is Hamiltonian isotopic to Id for some $k>0$. Then $\tau_{L_2}^{2k}$ is also Hamiltonian isotopic to Id, because by Lemma 2.3.8 there is a symplectomorphism of X taking L_1 to L_2. Then the product $\tau_{L_1}^{2k}\tau_{L_2}^{2k}$ is also Hamiltonian isotopic to Id. Since k can be taken arbitrarily large, this contradicts to the growth of dimensions in Equation (2.35). Consequently τ_{L_1} has infinite order in the group $Symp(X)/Ham(X)$. $\square$

Next we prove Lemma 2.1.8. It follows from a strong Bertini theorem which we now quote.

Theorem 2.5.1 ([34, Corollary 2.4])**.** *Let Y be a compact smooth complex manifold and S an effective linear system of divisors on Y. Let B be the base locus of S. If B is reduced and non-singular, and* $\dim B < \frac{1}{2}\dim Y$*, then a generic divisor in S is smooth.* $\square$

If B is disconnected, the dimensional inequality must hold for every connected component of B.

Proof of Lemma 2.1.8. We repeat the beginning of the proof of Proposition 2.4.2. We have $Y \subset \mathbb{P}^N$ and $\mathscr{L}^{\otimes d} = \mathscr{O}_{\mathbb{P}^N}(d)|_Y$. The involution ι acts on sections of $\mathscr{L}$ and so acts on $\mathbb{P}^N$ by a linear involution $\iota_{\mathbb{P}^N}$, and $Y \subset \mathbb{P}^N$ is invariant under it. Pick homogeneous co-ordinates $(x_0 : \ldots : x_N)$ such that

$$\iota_{\mathbb{P}^N}(x_0 : \ldots x_l : x_{l+1} : \ldots : x_N) = (x_0 : \ldots : x_l : -x_{l+1} : -x_N).$$

Recall that d is odd by assumption. Then $H^0(\mathscr{O}_{\mathbb{P}^N}(d))_+$ consists of degree-d polynomials which are sums of monomials of the following form:

$$x_0^{odd} \ldots x_l^{odd} x_{l+1}^{even} \ldots x_N^{even}.$$

Here *even* or *odd* denote the parity of a power. The base locus of the linear system $\mathbb{P}H^0(\mathscr{O}_{\mathbb{P}^N}(d))_+$ is given by

$$x_0 = 0,\ \ldots,\ x_l = 0$$

and so coincides with Π_-. The base locus B of $\mathbb{P}H^0(Y, \mathscr{L}^{\otimes d})_+$ is therefore $\Pi_- \cap Y$. It is smooth because Y^ι is smooth. We are also given that $\dim B < \frac{1}{2}\dim Y$ by hypothesis. Finally, we know that $\iota_{\mathbb{P}^N}|_{\Pi_-} = \mathrm{Id}$, so Y intersects Π_- cleanly (i.e. transversely in the normal direction to $\Pi_- \cap Y$), and hence $B = \Pi_- \cap Y$ is reduced. Consequently, Lemma 2.1.8

follows from Theorem 2.5.1. (The case when the signs symbols $+$ and $-$ are interchanged is analogous.) □

We now return to divisors in Grassmannians and prove Theorem 2.1.2. Let $Gr(k,n) \subset \mathbb{P}^N$ be the Plücker embedding; the anti-canonical class of $Gr(k,n)$ equals $\mathscr{O}_{\mathbb{P}^N}(n)|_{Gr(k,n)}$ [75, Proposition 1.9]. Consequently, a smooth divisor $X \subset Gr(k,n)$ in the linear system $\mathscr{O}_{\mathbb{P}^N}(d)|_{Gr(k,n)}$ satisfies the W^+ condition, see Definition 2.2.1, if and only if $d \leq n$ or $d \geq k(n-k)+n-2$, and X is monotone (Fano) if and only if $d < n$.

Proof of Theorem 2.1.2. We have already mentioned this theorem is easy and essentially known when $k(n-k)$ is even. (The sphere $L \subset X$ is non-trivial in $H_n(X)$ by Lemmas 2.4.1 and 2.3.8. Then apply Corollary 2.3.13(2).) We will now prove the hard case when $k(n-k)$ is odd using the general Theorem 2.1.7. Denote $k = 2p+1$, $n = 2q$.

Consider a linear involution on $\mathbb{C}^{2q}$ with $q+l$ positive eigenvalues and $q-l$ negative eigenvalues for some l. It induces a non-degenerate involution ι on $Gr(2p+1,2q)$ whose fixed locus is

$$Gr(2p+1,2q)^\iota = \bigsqcup_{t=0}^{2p+1} Gr(t,q+l) \times Gr(2p+1-t,q-l).$$

This fixed locus consists of $(2p+1)$-planes that admit a frame in which t vectors lie in the positive eigenspace of the involution on $\mathbb{C}^{2q}$, and the remaining $2p+1-t$ vectors lie in the negative eigenspace. We compute:

$$\begin{aligned} &\dim Gr(t,q+l) + \dim Gr(2p+1-t,q-l) \\ &\qquad -\frac{1}{2}\dim Gr(2p+1,2q) = -\frac{1}{2}(1+2p-2t)(1+2p+2l-2t). \quad (2.36) \end{aligned}$$

For this paragraph, set $l = 0$. Then the expression (2.36) is less than 0 for any $t \in \mathbb{Z}$. This means $\dim Gr(2p+1,2q)^\iota < \frac{1}{2}\dim Gr(2p+1,2q)$. (The left-hand side is disconnected, and we mean that the inequality holds for each of its connected components.) Therefore we can apply Lemma 2.1.8 to either of the two linear systems $\mathbb{P}H^0(Y,\mathscr{L}^{\otimes d})_\pm$. In order to apply Theorem 2.1.7, it remains to check that $Gr(2p+1,2q)^\iota$ contains a connected component of even dimension. A computation shows that a connected component of $Gr(2p+1,2q)^\iota$ has dimension of parity

$$\dim Gr(t,q) + \dim Gr(2p+1-t,q) \equiv q-1 \mod 2$$

independently of t. We will now consider the case when q is odd, and will discuss the case when q is even in the next paragraph. If d is odd, apply Theorem 2.1.7(b) taking either of the two sign symbols $+$ or $-$. If d is even, apply Theorem 2.1.7(a) (this case is easier and does not require the computation of dimensions we have made). This proves Theorem 2.1.2 for $Gr(2p+1,2q)$ in the case when q is odd.

Now suppose q is even. Set $l=1$ until the end of the proof. Recall that $Gr(2p+1,2q)^\iota = (\Pi_+ \sqcup \Pi_-) \cap Gr(2p+1,2q)$. The only case when (2.36) fails to be less than zero is when

$$1+2p-2t=-1.$$

This happens for a unique $t\in\mathbb{Z}$. So either $\dim Gr(2p+1,2q)\cap\Pi_+ < \frac{1}{2}\dim Gr(2p+1,2q)$, or the same holds with Π_- taken instead. (As above, we mean that the inequality holds for each connected component of the left hand side.) A computation shows that a connected component of $Gr(2p+1,2q)^\iota$ has dimension of parity

$$\dim Gr(t,q+1)+\dim Gr(2p+1-t,q-1)\equiv q \mod 2 \equiv 0 \mod 2$$

Therefore we can apply Lemma 2.1.8 and Theorem 2.1.7 taking that symbol $+$ or $-$ for which the inequality $\dim Gr(2p+1,2q)\cap\Pi_\mp < \frac{1}{2}\dim Gr(2p+1,2q)$ holds. Theorem 2.1.2 is proved in all cases. □

Proof of Corollaries 2.1.3, 2.1.9. These corollaries follow from Theorems 2.1.2, 2.1.7 and Lemma 2.3.14. □

2.6 Growth of Lagrangian Floer cohomology and ring structures

2.6.1 Dehn twists around spheres with deformed cohomology

The main theorems of this chapter have been proved; this last section is devoted to an additional observation on the relation between the Floer cohomology of a Lagrangian sphere and its associated Dehn twist. Keating [60] has recently obtained an exact sequence involving iterated Dehn twists in the Fukaya category of a symplectic manifold, extending Seidel's original exact sequence [95]. In this subsection we use it to prove Proposition 2.6.1, which is stated below. Then we apply it to compute Floer cohomology rings of vanishing spheres in some divisors.

Let X be a compact monotone symplectic manifold. Denote by $\mathscr{F}(X)$ its monotone Fukaya category over $\mathbb{C}$, which is a collection of A_∞ categories $\mathscr{F}(X)_\lambda$, $\lambda \in \mathbb{C}$, corresponding to the eigenvalues of multiplication with $c_1(X)$ in $QH^*(X)$. Our aim is to prove the following.

Proposition 2.6.1. *Let X be a monotone symplectic manifold,* $\dim_{\mathbb{R}} X = 4k$ *for some $k \geq 1$, $L_1 \subset X$ be a Lagrangian sphere and $L_2 \subset X$ another monotone Lagrangian which intersects L_1 transversely, once. Assume L_1, L_2 are included into the same summand $\mathscr{F}(X)_\lambda$. Suppose that* $\dim HF^*(\tau_{L_1}^k L_2, L_2) > 2$ *for some $k \in \mathbb{N}$. Then there is an isomorphism of rings* $HF^*(L_1, L_1) \cong \mathbb{C}[x]/x^2$.

We will use the language of A_∞ categories and refer to [96] for the relevant definitions. All A_∞ algebras and modules in this section are assumed to be minimal.

Definition 2.6.2. Let A be a strictly unital $\mathbb{Z}/2$-graded A_∞ algebra with unit $1 \in A$, M a right A_∞ module over A and N a left A_∞ module over A. Fix an augmentation, i.e. a vector space splitting $A = (1) \oplus \bar{A}$. The k-truncated bar complex is the vector space

$$(M \otimes_A N)_k := \bigoplus_{j=0}^{k-1} M \otimes \bar{A}^{\otimes j} \otimes N$$

with the differential that on the jth summand equals

$$\sum_{\substack{j+2=p+q+r,\\ p,r\geq 0,\ q\geq 2}} (-1)^{\maltese}(-1)^r(\mathrm{Id}^{\otimes p} \otimes \mu^q \otimes \mathrm{Id}^{\otimes r}). \tag{2.37}$$

Here $\maltese \in \{0,1\}$ depends on the gradings of the arguments: if the input is $m \otimes x_1 \otimes \ldots \otimes x_{k-1} \otimes n$, where $m \in M$, $x_i \in A$, $n \in N$, then $\maltese$ is the sum of gradings of the last r elements of the input. If we put $p = 0$ in (2.37), we get the summand $\mu^q \otimes \mathrm{Id}^{\otimes r}$ which involves the module structure map $\mu^q \colon M \otimes A^{\otimes(q-1)} \to M$. Similarly, when we put $r = 0$ in (2.37), μ^q is understood to be the module structure map $\mu^q \colon A^{\otimes(q-1)} \otimes N \to N$. When $p, r > 0$, μ^q denotes the algebra structure map $A^{\otimes q} \to A$ composed with the augmentation $A \to \bar{A}$.

Theorem 2.6.3 (Keating, [60, Lemma 7.2 and Remark 6.6])**.** *Suppose $L_1, L, L_2 \subset X$ are three Lagrangian submanifolds which are objects of $\mathscr{F}(X)_\lambda$, and L is a sphere. Then there is an exact sequence of vector spaces below.* $\square$

$$\begin{tikzcd} HF^*(L_1, L_2) \arrow[rr] & & HF^*(\tau_L^k L_1, L_2) \arrow[dl] \\ & H\big(Hom(L, L_1) \otimes_{Hom(L,L)} Hom(L_2, L)\big)_k \arrow[ul] & \end{tikzcd}$$

Here the Hom-spaces denote Floer complexes seen as the morphism spaces of the Fukaya category; for example, $Hom(L,L) = CF^*(L,L)$ has an A_∞ algebra structure whose definition was sketched in Chapter 1.

Note that [60] states this theorem for exact X and over $\mathbb{Z}/2$; in particular, it does not mention the signs in (2.37). The proof uses a theorem of Seidel [96, Corollary 17.17] which says that $\tau_L L_1$ is quasi-isomorphic to the cone of a certain evaluation map, as an object of the (category of twisted complexes over the) Fukaya category. This allows to write $\tau_L^k L_1$ as an iterated cone, which automatically provides some exact sequence of the type above. Keating proves Theorem 2.6.3 by simplifying the iterated cone in a purely algebraic way: by identifying and killing some acyclic sub-complexes in it. We know that the initial Seidel's theorem holds for the monotone Fukaya category and over $\mathbb{C}$ (see e.g. Oh [80] for the homological version), and the proof of Theorem 2.6.3 works in the monotone case and over $\mathbb{C}$ by virtue of being purely algebraic. The signs in (2.37) will be enforced for algebraic reasons, and it is a matter of book-keeping to check that they are the ones that we expect to see in a bar complex. In addition to Theorem 2.6.3, we will need some auxiliary lemmas.

Lemma 2.6.4 (Formality). *Every A_∞ algebra whose cohomology ring is $\mathbb{C}[x]/(x^2-1)$ is quasi-isomorphic to the A_∞ algebra $\mathbb{C}[x]/(x^2-1)$ with vanishing higher multiplications: $\mu^j = 0$, $j > 2$.*

Proof. The Hochschild cohomology of the associative algebra $\mathbb{C}[x]/(x^2-1)$ is concentrated in degree 0; this is proved in [56, Proposition 2.2] when x has even degree and in [59] when x has odd degree. The lemma then follows from [58, Corollary 4]; see also [100, Section 3]. □

Lemma 2.6.5. *Take the A_∞ algebra $\mathbb{C}[x]/(x^2-1)$ with vanishing $\mu^j, j > 2$. Every strictly unital A_∞ module M over this algebra with vanishing μ^1 necessarily has vanishing $\mu^j, j > 2$.*

Proof. Take the minimal j such that $\mu^j(m, x^{\otimes(j-1)}) \neq 0$ for some $m \in M$. If $j > 1$, the A_∞ relation for the tuple $(m, x^{\otimes(j-1)}, 1)$ gives $\mu^j(m, x^{\otimes(j-1)}) = 0$, a contradiction. □

Lemma 2.6.6 ([60, Lemma 3.1]). *Let (M,A,N) be a c-unital A_∞ category consisting of an A_∞ algebra A, a left A_∞ module M and a right A_∞ module N. Let A' be a strictly unital A_∞ algebra quasi-isomorphic to A. Then there are strictly unital A_∞ modules M', N' over A' such that the category (M,A,N) is quasi-isomorphic to (M',A',N'). The underlying Hom spaces of (M,A,N) and (M',A',N') are the same.* □

Lemma 2.6.7 (Cf. [60, Lemma 7.3]). *Let (M,A,N) and (M',A',N') be two strictly unital A_∞ categories consisting of an algebra, a left module and a right module. If they are* *quasi-isomorphic, the associated bar complexes* $(M \otimes_A N)_k$ *and* $(M' \otimes_{A'} N')_k$ *are quasi-isomorphic.* □

Remark 2.6.1. Let $\dim_{\mathbb{R}} X = 2n$. Suppose $L \subset X$ is a Lagrangian sphere. The $\mathbb{Z}/2$-graded Floer chain complex $CF^*(L,L)$ can be realised as a 2-dimensional vector space $\mathbb{C} \oplus \mathbb{C}$ with two generators: the unit 1, $\deg 1 = 0$ and the second generator x, $\deg x \equiv n \mod 2$. The differential has degree 1. If n is even, Floer's differential must vanish and $HF^*(L,L)$ is a unital 2-dimensional commutative algebra. Up to isomorphism, this leaves only two possibilities: $\mathbb{C}[x]/x^2$ or $\mathbb{C}[x]/(x^2-1)$. If n is odd, $HF^*(L,L)$ can also vanish.

The minimal Chern number of X is the maximal integer N such that $c_1(X)$ is divisible by N in integral cohomology $H^2(X;\mathbb{Z})$. The Floer cohomology of a Lagrangian sphere can be made $\mathbb{Z}/2N$ graded, and our generators have gradings $\deg 1 = 0$, $\deg x \equiv n \mod 2N$. If $n \neq 0 \mod N$, for grading reasons we obtain $x^2 = 0$ and $HF^*(L,L) \cong \mathbb{C}[x]/x^2$.

Proof of Proposition 2.6.1. We want to prove that $HF^*(L_1,L_1) \cong \mathbb{C}[x]/x^2$. Suppose this is not the case, then by Remark 2.6.1, $HF^*(L_1,L_1;\mathbb{C}) \cong \mathbb{C}[x]/(x^2-1)$. Recall that n is even.

Inside $\mathscr{F}(X)_\lambda$, take the subcategory consisting of the A_∞ algebra $Hom(L_1,L_1)$, its left module $Hom(L_1,L_2)$ and its right module $Hom(L_2,L_1)$. Because $|L_1 \cap L_2| = 1$, $Hom(L_1,L_2)$ and $Hom(L_2,L_1)$ are 1-dimensional as vector spaces. Denote their generators by

$$Hom(L_1,L_2) = \langle m \rangle, \quad Hom(L_2,L_1) = \langle n \rangle.$$

By Lemma 2.6.4, the A_∞ algebra $Hom(L_1,L_1)$ is quasi-isomorphic to the associative algebra $\mathbb{C}[x]/(x^2-1)$ with trivial higher multiplications. By Lemma 2.6.6 and Lemma 2.6.5, modules $Hom(L_1,L_2)$ and $Hom(L_2,L_1)$ are quasi-isomorphic to those with trivial higher multiplications over $\mathbb{C}[x]/(x^2-1)$. The module μ^2-operations, however, must be non-trivial because $x^2 = 1$:

$$\mu^2(m,x) = \varepsilon_m m, \quad \mu^2(x,n) = \varepsilon_n n \qquad \text{where} \quad \varepsilon_m, \varepsilon_n = \pm 1.$$

Lemma 2.6.7 allows to compute the homology of the bar complex

$$B_k := \big(Hom(L_1,L_2) \otimes_{Hom(L_1,L_1)} Hom(L_2,L_1)\big)_k$$

using the simple associative model we obtained. In this model, the bar complex B_k is based on the k-dimensional vector space

$$\bigoplus_{j=1}^{k-1} m \otimes x^{\otimes j} \otimes n.$$

The differential comes only from $\mu^2(m,x)$ and $\mu^2(x,n)$:

$$\partial(m \otimes x^{\otimes j} \otimes n) = \big((-1)^j \varepsilon_n + \varepsilon_m\big)\, m \otimes x^{\otimes (j-1)} \otimes n.$$

Note that $(-1)^{\maltese} = 1$ because we are given $\deg x = 0$ and may assume $\deg n = 0$. We see that $\dim H(B_k) = 0$ or 1, depending on the parity of k. By the exact sequence of Theorem 2.6.3, we get $\dim HF^*(\tau_{L_1}^k L_2, L_2; \mathbb{C}) \leq 2$, which contradicts to the hypothesis. □

Remark 2.6.2. If $HF^*(L_1,L_1;\mathbb{C}) \cong \mathbb{C}[x]/x^2$, it might still happen that $Hom(L_1,L_1)$ is formal, for example when X is exact. Running the above proof, from $x^2 = 0$ we conclude that $\mu^2(m,x) = \mu^2(x,n) = 0$. So the differential on the k-dimensional model for B_k written above vanishes, and $\dim H(B_k) = k$. This agrees with the growth of $\dim HF^*(\tau_{L_1}^k L_2, L_2)$.

2.6.2 Floer cohomology rings of Lagrangian spheres in divisors

We now combine Proposition 2.6.1 with previous results (Propositions 2.1.6 and 2.4.2) to compute the ring $HF^*(L,L;\mathbb{C})$ for vanishing Lagrangian spheres L in certain divisors; we use the notation from Subsection 2.1.7. We also provide a corollary which specialises to divisors in Grassmannians.

Proposition 2.6.8. *In addition to the conditions of Theorem 2.1.7 (a) or (b), suppose X is Fano and $\dim_{\mathbb{C}} X$ is even. Then there is a ring isomorphism $HF^*(L,L;\mathbb{C}) \cong \mathbb{C}[x]/x^2$.*

Corollary 2.6.9. *Let $X \subset Gr(k,n)$ be a smooth divisor of degree $3 \leq d < n$, $\dim_{\mathbb{C}} X$ even. Let $L \subset X$ be an $|\mathscr{O}(d)|$-vanishing Lagrangian sphere. Then there is a ring isomorphism $HF^*(L,L;\mathbb{C}) \cong \mathbb{C}[x]/x^2$.*

The possibility ruled out by these two statements is the deformed ring $HF^*(L,L) \cong \mathbb{C}[x]/(x^2-1)$. An example of a sphere with $HF^*(L,L) \cong \mathbb{C}[x]/(x^2-1)$ is the antidiagonal $L \subset \mathbb{P}^1 \times \mathbb{P}^1$. Note that for this sphere, τ_L has order 2 in $\pi_0 Symp(\mathbb{P}^1 \times \mathbb{P}^1)$ [97]. It seems natural to ask whether there is a general relation between the isomorphism $HF^*(L,L) \cong \mathbb{C}[x]/(x^2-1)$ and τ_L being of finite order (both cases are rare). Observe that for many, but not all, pairs (k,n) Corollary 2.6.9 follows the grading consideration in Remark 2.6.1.

Proof of Proposition 2.6.8. As in the beginning of the proof of Theorem 2.1.7, take X, L_1, L_2 as constructed in Proposition 2.4.2. By Lemma 2.3.8, it suffices to prove that $HF^*(L_1,L_1) \cong \mathbb{C}[x]/x^2$.

From the Picard-Lefschetz formula (Lemma 2.3.12), given that $|L_1 \cap L_2| = 1$ and $\dim L_i^\iota$ is odd, we get the equality $[\tau_{L_1^\iota}^k L_2^\iota] = [L_2^\iota] - \varepsilon k [L_1^\iota]$ in the homology of the fixed locus $H_*(X^\iota)$.

Consequently, $[\tau^k_{L^\iota_1} L^\iota_2] \cdot [L^\iota_2] = -\varepsilon k$. By Proposition 2.1.6, $\dim HF^*(\tau^k_{L_1} L_2, L_2) \geq k$. By Proposition 2.6.1, $HF^*(L_1, L_1) \cong \mathbb{C}[x]/x^2$. $\square$

Proof of Corollary 2.6.9. Repeat the proof of Theorem 2.1.2 but refer to Proposition 2.6.8 instead of Theorem 2.1.7. Recall the condition $d < n$ means that X is Fano. $\square$

Chapter 3

The closed-open string map for circle-invariant Lagrangians

This chapter is based on the author's preprint [108].

3.1 Introduction

3.1.1 Overview of main results

Let X be a compact monotone symplectic manifold, $L \subset X$ be a monotone Lagrangian submanifold, and $\mathbb{K}$ be a field. We assume L satisfies the usual conditions making its Floer theory well-defined over $\mathbb{K}$: namely, L has minimal Maslov number at least 2, and is oriented and spin if $\operatorname{char}\mathbb{K} \neq 2$. As explained briefly in Chapter 1, one can define a unital algebra over $\mathbb{K}$ called the Floer cohomology $HF^*(L,L)$, which is invariant under Hamiltonian isotopies of L. A larger amount of information about L is captured by the Fukaya A_∞ algebra of L, and given this A_∞ algebra, one can build another associative unital algebra called the Hochschild cohomology $HH^*(L,L)$. There is the so-called (full) closed-open string map

$$\mathscr{C}\mathscr{O}^*\colon\ QH^*(X) \to HH^*(L,L),$$

which is a map of unital algebras, where $QH^*(X)$ is the (small) quantum cohomology of X. This map is of major importance in symplectic topology, particularly in light of Abouzaid's split-generation criterion, one of whose versions in the case $\operatorname{char}\mathbb{K} = 2$ says the following: if the closed-open map is *injective*, then L split-generates the Fukaya category $\mathscr{F}uk(X)_w$, where $w = w(L) \in \mathbb{K}$ is the so-called obstruction number of L. (When $\operatorname{char}\mathbb{K} \neq 2$, the hypothesis should say that $\mathscr{C}\mathscr{O}^*$ is injective on a relevant eigensummand of $QH^*(X)$; we will recall this later.)

Split-generation of the Fukaya category $\mathscr{F}uk(X)_w$ by a Lagrangian submanifold L is an algebraic phenomenon which has important geometric implications. For example, in this case L must have non-empty intersection with any other monotone Lagrangian submanifold L' which is a non-trivial object in $\mathscr{F}uk(X)_w$, namely such that $HF^*(L',L') \neq 0$ and $w(L') = w$. Another application, though not discussed here, is that split-generation results are used in proofs of Homological Mirror Symmetry.

The present chapter contributes with new calculations of the closed-open map, motivated by the split-generation criterion and the general lack of explicit calculations known so far. (The closed-open map is defined by counting certain pseudo-holomorphic disks with boundary on L, which makes it extremely hard to compute in general.)

There is a simplification of the full closed-open map, called the "zeroth-order" closed-open map, which is a unital algebra map

$$\mathscr{C}\mathscr{O}^0\colon\ QH^*(X) \to HF^*(L,L).$$

It is the composition of $\mathscr{C}\mathscr{O}^*$ with the canonical projection $HH^*(L,L) \to HF^*(L,L)$, and if $\mathscr{C}\mathscr{O}^0$ is injective, so is $\mathscr{C}\mathscr{O}^*$ (but not vice versa). Although $\mathscr{C}\mathscr{O}^0$ generally carries less information than $\mathscr{C}\mathscr{O}^*$, it is sometimes easier to compute. For example, we compute $\mathscr{C}\mathscr{O}^0$ when L is the real locus of a complex toric Fano variety X, see Theorem 3.1.12. This map turns out to be non-injective in many cases, e.g. for $\mathbb{R}P^{2n+1} \subset \mathbb{C}P^{2n+1}$ over a characteristic 2 field. The aim of the present chapter is to study the higher order terms of the full closed-open map $\mathscr{C}\mathscr{O}^*$, and to find examples where $\mathscr{C}\mathscr{O}^*$ is injective but $\mathscr{C}\mathscr{O}^0$ is not.

Specifically, let us consider the following setting: a loop γ of Hamiltonian symplectomorphisms preserves a Lagrangian L setwise. Let $S(\gamma) \in QH^*(X)$ be the Seidel element of γ, then from Charette and Cornea [27] one can see that $\mathscr{C}\mathscr{O}^0(S(\gamma)) = 1_L$, the unit in $HF^*(L,L)$. Our main result, Theorem 3.1.7, is a tool for distinguishing $\mathscr{C}\mathscr{O}^*(S(\gamma))$ from the Hochschild cohomology unit in $HH^*(L,L)$; this way it captures a non-trivial piece of the full closed-open map $\mathscr{C}\mathscr{O}^*$ not seen by $\mathscr{C}\mathscr{O}^0$. We apply Theorem 3.1.7 to show that $\mathscr{C}\mathscr{O}^*$ is injective for some real Lagrangians in toric manifolds, and also for monotone toric fibres which correspond to (non-Morse) A_2-type critical points of the Landau-Ginzburg superpotential.

After this work had appeared in the form of the preprint [108], a paper of Evans and Lekili [40] proved split-generation for all orientable real toric Lagrangians and all monotone toric fibres, therefore superseding many applications that we provide. However, not entirely all applications: the Lagrangian non-formality theorems, soon to be stated, do not directly follow from [40]; as well as split-generation for non-orientable real Lagrangians (in the examples we provide). Also, the scope and flavour of our general theorem on S^1-actions is quite different from that of [40] which works with homogeneous Lagrangians.

We will now mention our examples regarding real Lagrangians, and postpone all discussion of monotone toric fibres, along with an introductory part, to Section 3.4.

Proposition 3.1.1. *Let $\mathbb{K}$ be a field of characteristic 2 and $\mathbb{R}P^n$ be the standard real Lagrangian in $\mathbb{C}P^n$. Then $\mathscr{CO}^*\colon QH^*(\mathbb{C}P^n) \to HH^*(\mathbb{R}P^n, \mathbb{R}P^n)$ is injective for all n. In contrast, $\mathscr{CO}^0\colon QH^*(\mathbb{C}P^n) \to HF^*(\mathbb{R}P^n, \mathbb{R}P^n)$ is injective if and only if n is even.*

Corollary 3.1.2. *Over a field of characteristic 2, $\mathbb{R}P^n$ split-generates $\mathscr{F}uk(\mathbb{C}P^n)_0$.*

As hinted above, this corollary leads to a result on non-displaceability of $\mathbb{R}P^n$ from other monotone Lagrangians. This has already been known due to Biran and Cornea [18, Corollary 8.1.2], and Entov and Polterovich [38]. On the other hand, we can extract another interesting consequence about projective spaces from our main computation of the closed-open map.

Proposition 3.1.3. *The Fukaya A_∞ algebra of the Lagrangian $\mathbb{R}P^{4n+1} \subset \mathbb{C}P^{4n+1}$ is not formal over a characteristic 2 field, for any $n \geq 0$.*

Here, formality means the existence of a quasi-isomorphism with the associative algebra $HF^*(\mathbb{R}P^{4n+1}, \mathbb{R}P^{4n+1}) \cong \mathbb{K}[u]/(u^{4n+2}-1)$, considered as an A_∞ algebra with trivial higher-order structure maps. In particular, the Fukaya A_∞ algebra of the equator $S^1 \subset S^2$ is not formal in characteristic 2, although S^1 is topologically formal in any characteristic; we devote a separate discussion to it in Section 3.3.

Remark 3.1.1. We recall that [40] proves a "parallel" theorem saying that the Fukaya A_∞ algebra of the symplectic 2-sphere (which is, by definition, the Fukaya A_∞ algebra of the sphere considered as the Lagrangian antidiagonal in its square product), is not formal in characteristic 2. The relation between the A_∞ algebra of a toric manifold and the one of its real Lagrangian seems not to have been fully explored, although one might expect them to be quasi-isomorphic in characteristic 2. This isomorphism on cohomology level was established by Haug, and we recall it later.

Remark 3.1.2. The non-formality of Fukaya categories of some other surfaces and in characteristics other than 2 has already been discussed in the literature; see for example [67].

Below is another example of split-generation which we can prove using the same methods.

Proposition 3.1.4. *Let $\mathbb{K}$ be a field of characteristic 2, $X = Bl_{\mathbb{C}P^1}\mathbb{C}P^9$ the blow-up of $\mathbb{C}P^9$ along a complex line which intersects $\mathbb{R}P^9$ in a circle, and let $L \subset X$ be the blow-up of $\mathbb{R}P^9$ along that circle. Then $\mathscr{CO}^*\colon QH^*(X) \to HH^*(L,L)$ is injective although $\mathscr{CO}^0\colon QH^*(X) \to HF^*(L,L)$ is not. Consequently, L split-generates $\mathscr{F}uk(X)_0$.*

(The manifold $Bl_{\mathbb{C}P^1}\mathbb{C}P^9$ is the first instance among $Bl_{\mathbb{C}P^k}\mathbb{C}P^n$ for which L is monotone of minimal Maslov number at least 2, and such that $\mathscr{C}\mathscr{O}^0$ is not injective — the last requirement makes the use of our general results essential in this example.) In general, it is known that the real Lagrangian in a toric Fano variety is not displaceable from the monotone toric fibre: this was proved by Alston and Amorim [7]. Proposition 3.1.4 implies a much stronger non-displaceability result, like the one which has been known for $\mathbb{R}P^n \subset \mathbb{C}P^n$.

Corollary 3.1.5. *Let $\mathbb{K}$ and $L \subset X$ be as in Proposition 3.1.4, and $L' \subset X$ any other monotone Lagrangian, perhaps equipped with a local system $\pi_1(L) \to \mathbb{K}^\times$, with minimal Maslov number at least 2 and such that $HF^*(L', L') \neq 0$. Then $L \cap L' \neq \emptyset$.*

Here $HF^*(L', L')$ denotes the Floer cohomology of L' with respect to the local system ρ, so a better notation would be $HF^*((L', \rho), (L', \rho))$. For brevity, we decided to omit ρ from our notation of Floer and Hochschild cohomologies throughout the chapter, when it is clear that a Lagrangian is equipped with such a local system. The point of allowing local systems in Corollary 3.1.5 is to introduce more freedom in achieving the non-vanishing of $HF^*(L', L')$.

Note that Corollary 3.1.5 does not require that the obstruction number of L' matches the one of L, namely zero. If $w(L') \neq 0$, we can pass to $X \times X$ noticing that $w(L' \times L') = 2w(L') = 0$ and similarly $w(L \times L) = 0$, so we have well-defined Floer theory between the two product Lagrangians. This trick was observed by Abreu and Macarini [4] and has also been used in [7]. So it suffices to show that $L \times L$ split-generates $\mathscr{F}uk(X \times X)_0$; this follows from Proposition 3.1.4 by the general fact that the hypothesis of the split-generation criterion is "preserved" under Künneth isomorphisms. The most suitable reference seems to be Ganatra [52], as explained later; see also [3, 8]. As in the case with $\mathbb{R}P^n$, we also prove a non-formality statement.

Proposition 3.1.6. *The Fukaya A_∞ algebra of the Lagrangian $Bl_{\mathbb{R}P^1}\mathbb{R}P^9 \subset BL_{\mathbb{C}P^1}\mathbb{C}P^9$ from Proposition 3.1.4 is not formal over a characteristic 2 field.*

Although we cannot prove that $\mathscr{C}\mathscr{O}^*$ is injective for the real locus of an arbitrary toric Fano variety, we are able to do this in a slightly wider range of examples which we postpone to Section 3.3. We will prove Proposition 3.1.1 and Corollary 3.1.2 at the end of the introduction, and the remaining statements from above will be proved in Section 3.3. Now we are ready state the main theorem, with some comments coming after.

Theorem 3.1.7. *Let X be a compact monotone symplectic manifold, $L \subset X$ a monotone Lagrangian submanifold of minimal Maslov number at least 2, possibly equipped with a local system $\rho\colon H_1(L) \to \mathbb{K}^\times$. If $\operatorname{char}\mathbb{K} \neq 2$, assume L is oriented and spin.*

Let $\gamma = \{\gamma_t\}_{t \in S^1}$ be a loop of Hamiltonian symplectomorphisms of X, and denote by $S(\gamma) \in QH^*(X)$ the corresponding Seidel element. Suppose the loop γ preserves L setwise, that is, $\gamma_t(L) = L$. Denote by $l \in H_1(L)$ the homology class of an orbit $\{\gamma_t(q)\}_{t \in S^1}$, $q \in L$. Finally, assume $HF^*(L,L) \neq 0$.

(a) Then $\mathscr{CO}^0(S(\gamma)) = \rho(l) \cdot 1_L$ where $1_L \in HF^*(L,L)$ is the unit.

(b) Suppose there exists no $a \in HF^*(L,L)$ such that

$$\mu^2(a, \Phi(y)) + \mu^2(\Phi(y), a) = \rho(l) \cdot \langle y, l \rangle \cdot 1_L \quad \textit{for each} \quad y \in H^1(L). \quad (*)$$

Then $\mathscr{CO}^*(S(\gamma)) \in HH^*(L,L)$ is linearly independent from the Hochschild cohomology unit.

(c) More generally, suppose $Q \in QH^*(X)$ and there exists no $a \in HF^*(L,L)$ such that

$$\mu^2(a, \Phi(y)) + \mu^2(\Phi(y), a) = \rho(l) \cdot \langle y, l \rangle \cdot \mathscr{CO}^0(Q) \quad \textit{for each} \quad y \in H^1(L). \quad (**)$$

Then $\mathscr{CO}^*(S(\gamma) * Q)$ and $\mathscr{CO}^*(Q)$ are linearly independent in Hochschild cohomology $HH^*(L,L)$.

Here μ^2 is the product on $HF^*(L,L)$, $\langle -, - \rangle$ is the pairing $H^1(L) \otimes H_1(L) \to \mathbb{K}$, and $S(\gamma) * Q$ is the quantum product of the two elements. Next,

$$\Phi\colon\ H^1(L) \to HF^*(L,L)$$

is the PSS map of Albers [5], which is canonical and well-defined if $HF^*(L,L) \neq 0$. Its well-definedness in a setting closer to ours was studied by e.g. Biran and Cornea [19], and later we discuss it in more detail. Note that Φ is not necessarily injective, although in our applications, when $HF^*(L,L) \cong H^*(L)$, it will be. Finally, in the theorem we have allowed L to carry an arbitrary local system, which modifies the Fukaya A_∞ structure of L by counting the same punctured holomorphic disks as in the case without a local system with coefficients which are the values of ρ on the boundary loops of such disks. The algebras $HF^*(L,L)$, $HH^*(L,L)$ get modified accordingly, although their dependence on ρ is not reflected by our notation, as mentioned earlier. We allow non-trivial local systems in view of our application to toric fibres, and will only need the trivial local system $\rho \equiv 1$ for applications to real Lagrangians.

Outline of proof. It has been mentioned earlier that part (a) of Theorem 3.1.7 is an easy consequence of the paper by Charette and Cornea [27]. The proof of parts (b) and (c) also starts by using a result from that paper, and then the main step is an explicit computation of $\mathscr{CO}^1(S(\gamma))|_{CF^1(L,L)}\colon CF^1(L,L)\to CF^0(L,L)$ on cochain level, which turns out to be dual to taking the γ-orbit of a point: this is Proposition 3.2.6. The final step is to check whether the computed nontrivial piece of the Hochschild cocycle $\mathscr{CO}^*(S(\gamma))$ survives to cohomology; this is controlled by equations $(*)$, $(**)$.

3.1.2 The split-generation criterion

We will now discuss the split-generation criterion in more detail, particularly because we wish to pay attention to both $\operatorname{char}\mathbb{K}=2$ and $\operatorname{char}\mathbb{K}\neq 2$ cases. We continue to denote by $L\subset X$ a monotone Lagrangian submanifold with minimal Maslov number at least 2, which is oriented and spin if $\operatorname{char}\mathbb{K}\neq 2$. If $\operatorname{char}\mathbb{K}=2$, we allow L to be non-orientable. Consider the quantum multiplication by the first Chern class as an endomorphism of quantum cohomology, $-*c_1(X)\colon QH^*(X)\to QH^*(X)$. If $\mathbb{K}$ is algebraically closed, we have an algebra decomposition $QH^*(X)=\oplus_w QH^*(X)_w$ where $QH^*(X)_w$ is the generalised w-eigenspace of $-*c_1(X)$, $w\in\mathbb{K}$.

Recall that $w(L)\in\mathbb{K}$ denotes the obstruction number of L, i.e. the count of Maslov index 2 disks with boundary on L. By an observation of Auroux, Kontsevich and Seidel, $\mathscr{CO}^0(2c_1)=2w(L)\cdot 1_L$, which in $\operatorname{char}\mathbb{K}\neq 2$ implies that $\mathscr{CO}^0(c_1)=w(L)\cdot 1_L$, see e.g. [103]. Now suppose that $\operatorname{char}\mathbb{K}=2$ and $c_1(X)$ lies in the image of $H^2(X,L;\mathbb{K})\to H^2(X;\mathbb{K})$, which is true if L is orientable (because the Maslov class goes to twice the Chern class under $H^2(X,L;\mathbb{Z})\to H^2(X;\mathbb{Z})$, and the Maslov class of an orientable manifold is integrally divisible by two). In this case, the same argument shows again that $\mathscr{CO}^0(c_1)=w(L)\cdot 1_L$. This way one deduces the following lemma, which is well-known but usually stated only for $\operatorname{char}\mathbb{K}\neq 2$.

Lemma 3.1.8. *For $\mathbb{K}$ of any characteristic, if L is orientable, then $\mathscr{CO}^0\colon QH^*(X)\to HF^*(L,L)$ vanishes on all summands except maybe $QH^*(X)_{w(L)}$.* □

(If $w(L)$ is not an eigenvalue of $-*c_1(X)$, then $\mathscr{CO}^0$ vanishes altogether, and it follows that $HF^*(L,L)=0$. Recall that L is required to be monotone.) The same vanishing statement is expected to hold for the full map $\mathscr{CO}^*$. Keeping this vanishing in mind, we see that the "naıve" version of the split-generation criterion stated in the introduction, that $\mathscr{CO}^*\colon QH^*(X)\to HH^*(L,L)$ is injective, can only be useful when $\operatorname{char}\mathbb{K}=2$ and L is non-orientable. In other cases it must be replaced by a more practical criterion which does not ignore the eigenvalue decomposition; we will now state both versions of the criterion. Let $\mathscr{F}uk(X)_w$ denote the Fukaya category whose objects are monotone Lagrangians in X with minimal Maslov number at least 2, oriented and spin if $\mathrm{char}\,\mathbb{K} \neq 2$, and whose obstruction number equals $w \in \mathbb{K}$.

Theorem 3.1.9. *Let $L_1, \ldots, L_n \subset X$ be Lagrangians which are objects of $\mathscr{F}uk(X)_w$, and $\mathscr{G} \subset \mathscr{F}uk(X)_w$ be the full subcategory generated by $L_1, \ldots, L_n$. Then $\mathscr{G}$ split-generates $\mathscr{F}uk(X)_w$ if either of the two following statements hold.*

(a) $\mathrm{char}\,\mathbb{K} \neq 2$, *and* $\mathscr{C}\mathscr{O}^*|_{QH^*(X)_w} \colon QH^*(X)_w \to HH^*(\mathscr{G})$ *is injective.*

(b) $\mathbb{K}$ *is arbitrary, and* $\mathscr{C}\mathscr{O}^* \colon QH^*(X) \to HH^*(\mathscr{G})$ *is injective.* □

In the monotone case, this theorem is due to Ritter and Smith [87] and Sheridan [103]. It is more common to only state part (a), but it is easy to check the same proof works for part (b) as well. (In part (a), we could also allow $\mathrm{char}\,\mathbb{K} = 2$, if L is orientable.) Theorem 3.1.9 is most easily applied when $QH^*(X)_w$ is 1-dimensional: because $\mathscr{C}\mathscr{O}^*$ is unital, it automatically becomes injective; we are going to apply this theorem in more complicated cases. Before we proceed, let us mention one easy corollary of split-generation. We say that $L_1, \ldots, L_n$ split-generate the Fukaya category when $\mathscr{G}$ does.

Lemma 3.1.10. *If Lagrangians $L_1, \ldots, L_n \subset X$ split-generate $\mathscr{F}uk(X)_w$, and $L \subset X$ is another Lagrangian which is an object of $\mathscr{F}uk(X)_w$ with $HF^*(L,L) \neq 0$, then L has non-empty intersection, and non-zero Floer cohomology, with at least one of the Lagrangians L_i.* □

3.1.3 $\mathscr{C}\mathscr{O}^0$ for real toric Lagrangians

Let X be a (smooth, compact) toric Fano variety with minimal Chern number at least 2, i.e. $\langle c_1(X), H_2(X;\mathbb{Z})\rangle = N\mathbb{Z}$, $N \geq 2$. As a toric manifold X has a canonical anti-holomorphic involution $\tau \colon X \to X$. Its fixed locus is the so-called real Lagrangian $L \subset X$ which is smooth [35, p. 419], monotone and whose minimal Maslov number equals the minimal Chern number of X [54]. When speaking of such real Lagrangians, we will always be working over a field $\mathbb{K}$ of characteristic 2. In particular, there is the Frobenius map:

$$\mathscr{F} \colon QH^*(X) \to QH^{2*}(X), \quad \mathscr{F}(x) = x^2.$$

Because $\mathrm{char}\,\mathbb{K} = 2$, $\mathscr{F}$ is a map of unital algebras. We have reflected in our notation that $\mathscr{F}$ multiplies the $\mathbb{Z}/2N$-grading by two. A classical theorem of Duistermaat [35] constructs, again in $\mathrm{char}\,\mathbb{K} = 2$, the isomorphisms $H^i(L) \cong H^{2i}(X)$. We can package these isomorphisms into a single isomorphism of unital algebras,

$$\mathscr{D} \colon H^{2*}(X) \xrightarrow{\cong} H^*(L).$$

Let us now recall a recent theorem of Haug [54].

Theorem 3.1.11. *If* $\operatorname{char}\mathbb{K} = 2$*, then* $HF^*(L,L) \cong H^*(L)$ *as vector spaces. Using the identification coming from a specific perfect Morse function from [54], and also indentifying* $QH^*(X) \cong H^*(X)$*, the same map*

$$\mathscr{D}\colon\ QH^{2*}(X) \xrightarrow{\cong} HF^*(L,L)$$

is again an isomorphism of unital algebras. $\square$

It turns out that it is possible to completely compute $\mathscr{C}\mathscr{O}^0$ for real toric Lagrangians. This is a rather quick corollary of the works of Charette and Cornea [27], Hyvrier [57], and McDuff and Tolman [74]; we explain it in Section 3.3.

Theorem 3.1.12. *The diagram below commutes.*

$$\begin{tikzcd} QH^*(X) \arrow[r, "\mathscr{F}"] \arrow[dr, "\mathscr{C}\mathscr{O}^0"'] & QH^{2*}(X) \arrow[d, "\mathscr{D}"', "\cong"] \\ & HF^*(L,L) \end{tikzcd}$$

In particular, $\mathscr{C}\mathscr{O}^0$ is injective if and only if $\mathscr{F}$ is injective.

3.1.4 Split-generation for the real projective space

We conclude the introduction by proving Proposition 3.1.1 and Corollary 3.1.2. The crucial idea is that when n is odd, the kernel of $\mathscr{C}\mathscr{O}^0\colon\ QH^*(\mathbb{C}P^n) \to HF^*(\mathbb{R}P^n,\mathbb{R}P^n)$ is the ideal generated by the Seidel element of a non-trivial Hamiltonian loop preserving $\mathbb{R}P^n$; this allows to apply Theorem 3.1.7 and get new information about $\mathscr{C}\mathscr{O}^*$. Recall that $QH^*(X) \cong \mathbb{K}[x]/(x^{n+1}-1)$ and $w(\mathbb{R}P^n) = 0$, because the minimal Maslov number of $\mathbb{R}P^n$ equals $n+1$ (when $n = 1$, we still have $w(S^1) = 0$ for $S^1 \subset S^2$).

Proof of Proposition 3.1.1. If n is even, the Frobenius map on $QH^*(\mathbb{C}P^n)$ is injective, so by Theorem 3.1.12, $\mathscr{C}\mathscr{O}^0\colon\ QH^*(\mathbb{C}P^n) \to HF^*(\mathbb{R}P^n,\mathbb{R}P^n)$ is injective, and hence $\mathscr{C}\mathscr{O}^*$ too.

Now suppose n is odd and denote $n = 2p-1$. Given $\operatorname{char}\mathbb{K} = 2$, we have $QH^*(\mathbb{C}P^n) \cong \mathbb{K}[x]/(x^p+1)^2$, so $\ker\mathscr{F} = \ker\mathscr{C}\mathscr{O}^0$ is the ideal generated by x^p+1. Consider the Hamiltonian loop γ on $\mathbb{C}P^n$ which in homogeneous co-ordinates $(z_1 : \ldots : z_{2p})$ is the rotation $\left(\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix}\right)$, $t \in [0,\pi]$, applied simultaneously to the pairs $(z_1,z_2),\ldots,(z_{2p-1},z_{2p})$. Note that t runs to π, not 2π. This loop is Hamiltonian isotopic to the loop

$$(z_0 : \ldots : z_{2p-1}) \mapsto (e^{2it}z_0 : z_1 : \ldots : e^{2it}z_{2p-1} : z_{2p}), \quad t \in [0,\pi],$$

so $S(\gamma) = x^p$, see [74]. The loop γ obviously preserves the real Lagrangian $\mathbb{R}P^n \subset \mathbb{C}P^n$, and its orbit l is a generator of $H_1(\mathbb{R}P^n) \cong \mathbb{K}$. Taking $y \in H^1(\mathbb{R}P^n)$ to be the generator, we get $\langle y, l\rangle = 1$, and the right-hand side of equation $(**)$ from Theorem 3.1.7 equals $\mathscr{C}\mathscr{O}^0(Q)$. On the other hand, the product on $HF^*(\mathbb{R}P^n, \mathbb{R}P^n)$ is commutative by Theorem 3.1.11, so the left-hand side of $(**)$ necessarily vanishes. We conclude that the hypothesis of Theorem 3.1.7(c) is satisfied for any $Q \notin \ker \mathscr{C}\mathscr{O}^0$.

Let us prove that $\mathscr{C}\mathscr{O}^*(P) \neq 0$ for each nonzero $P \in QH^*(\mathbb{C}P^n)$. If $\mathscr{C}\mathscr{O}^0(P) \neq 0$, we are done, so it suffices to suppose that $\mathscr{C}\mathscr{O}^0(P) = 0$. It means that $P = (x^p + 1) * Q = (S(\gamma) + 1) * Q$ for some $Q \in QH^*(\mathbb{C}P^n)$. Note that if $Q \in \ker \mathscr{C}\mathscr{O}^0 = \ker \mathscr{F}$ then $P \in (\ker \mathscr{F})^2 = \{0\}$. So if $P \neq 0$, then $\mathscr{C}\mathscr{O}^0(Q) \neq 0$, and thus $\mathscr{C}\mathscr{O}^*(P) \neq 0$ by Theorem 3.1.7(c) and the observation earlier in this proof. $\square$

Remark 3.1.3. When n is even, $c_1(\mathbb{C}P^n)$ is invertible in $QH^*(\mathbb{C}P^n)$, so the 0-eigenspace $QH^*(\mathbb{C}P^n)_0$ is trivial; but L is non-orientable, so this does not contradict Lemma 3.1.8. On the other hand, when n is odd, L is orientable but $c_1(\mathbb{C}P^n)$ vanishes in $\operatorname{char} \mathbb{K} = 2$, so the whole $QH^*(\mathbb{C}P^n)$ is its 0-eigenspace; this is again consistent with Lemma 3.1.8.

Proof of Corollary 3.1.2. This follows from Proposition 3.1.4 and Theorem 3.1.9(b). $\square$

The same trick of finding a real Hamiltonian loop whose Seidel element generates $\ker \mathscr{C}\mathscr{O}^0$ works for some other toric manifolds which have "extra symmetry" in addition to the toric action, like a Hamiltonian action of $SU(2)^{\dim_{\mathbb{C}} X/2}$ which was essentially used above. As already mentioned, we will provide more explicit examples in Section 3.3.

3.2 Proof of Theorem 3.1.7

Let X be a monotone symplectic manifold and $w \in \mathbb{K}$. We recall that the objects in the monotone Fukaya category $\mathscr{F}uk(X)_w$ are monotone Lagrangian submanifolds $L \subset X$ with minimal Maslov number at least 2, oriented and spin if $\operatorname{char} \mathbb{K} \neq 2$, equipped with local systems $\rho\colon\ \pi_1(L) \to \mathbb{K}^\times$, whose count of Maslov index 2 disks (weighted using ρ) equals w; see the references in Chapter 1. There is a notion of bounding cochains from [47], generalising the notion of a local system, and our results are expected carry over to them as well.

3.2.1 A theorem of Charette and Cornea

Suppose $\gamma = \{\gamma_t\}_{t \in S^1}$ is a loop of Hamiltonian symplectomorphisms on X. As explained by Seidel in [96, Section (10c)], the loop γ gives rise to a natural transformation $\gamma^\sharp$ from the identity functor on $\mathscr{F}uk(X)_w$ to itself. Any such natural transformation is a cocycle of the

Hochschild cochain complex $CC^*(\mathscr{F}uk(X)_w)$ [96, Section (1d)]. Denote the corresponding Hochschild cohomology class by $[\gamma^\sharp] \in HH^*(\mathscr{F}uk(X)_w)$. We denote, as earlier, the closed-open map by $\mathscr{C}\mathscr{O}^*\colon\ QH^*(X) \to HH^*(\mathscr{F}uk(X)_w)$ and the Seidel element by $S(\gamma) \in QH^*(X)$. The following theorem was proved by Charette and Cornea [27].

Theorem 3.2.1. *If we take for $\mathscr{F}uk(X)_w$ the Fukaya category of Lagrangians with trivial local systems only, then $\mathscr{C}\mathscr{O}^*(S(\gamma)) = [\gamma^\sharp]$.* □

Now take a Lagrangian $L \subset X$ which is an object of $\mathscr{F}uk(X)_w$, with a possibly non-trivial local system $\rho\colon\ \pi_1(L) \to \mathbb{K}^\times$. Assume L is preserved by the Hamiltonian loop γ, and denote by $l \in H_1(L)$ the homology class of an orbit of γ on L. Let $CC^*(L,L)$ denote the Hochschild cochain complex of the A_∞ algebra $CF^*(L,L)$, and let $HH^*(L,L)$ be its Hochschild cohomology. (The definition of Hochschild cohomology is reminded later in this section.) We shall now recall the proof of Theorem 3.2.1, specialised to the A_∞ algebra of L rather than the full Fukaya category. We have two reasons to recall this proof: first, we wish to see how Theorem 3.2.1 gets modified in the presence of a local system on L; second, we shall remind the definition $\gamma^\sharp$ in the process. Eventually, for later use we need a form of Theorem 3.2.1 expressed by Formula (3.1) below, which takes the local system into account.

Pick some Floer datum $\{H_s, J_s\}_{s\in[0,1]}$ and perturbation data defining an A_∞ structure on Floer's complex $CF^*(L,L)$ [96]. Recall that the maps

$$\mathscr{C}\mathscr{O}^k(S(\gamma))\colon\ CF^*(L,L)^{\otimes k} \to CF^*(L,L)$$

count 0-dimensional moduli space of disks satisfying a perturbed pseudo-holomorphic equation (with appropriately chosen perturbation data) with $k+1$ boundary punctures (k inputs and one output) and one interior marked point. These disks satisfy the Lagrangian boundary condition L, and their interior marked point is constrained to a cycle dual to $S(\gamma)$, see Figure 3.1(a) (in this figure, we abbreviate the datum $\{H_s, J_s\}$ simply to H). A disk u is counted with coefficient $\pm\rho(\partial u)$ where the sign $\pm$ comes from an orientation on the moduli space and $\rho(\partial u) \in \mathbb{K}^\times$ is the monodromy of the local system. The collection of maps $\mathscr{C}\mathscr{O}^*(S(\gamma)) := \{\mathscr{C}\mathscr{O}^k\}_{k\geq 0}$ is a cochain in $CC^*(L,L)$, if all perturbation data are chosen consistently with gluing.

The argument of Charette and Cornea starts by passing to a more convenient definition of the closed-open map in which $\mathscr{C}\mathscr{O}^k$ count holomorphic disks with $k+1$ boundary punctures and one interior puncture (instead of a marked point). We can view the neighbourhood of the interior puncture as a semi-infinite cylinder, then the pseudo-holomorphic equation restricts on this semi-infinite cylinder to a Hamiltonian Floer equation with some Floer datum $\{F_t, J_t\}_{t\in S^1}$. We input the PSS image of $S(\gamma)$ to the interior puncture, see Figure 3.1(b), given

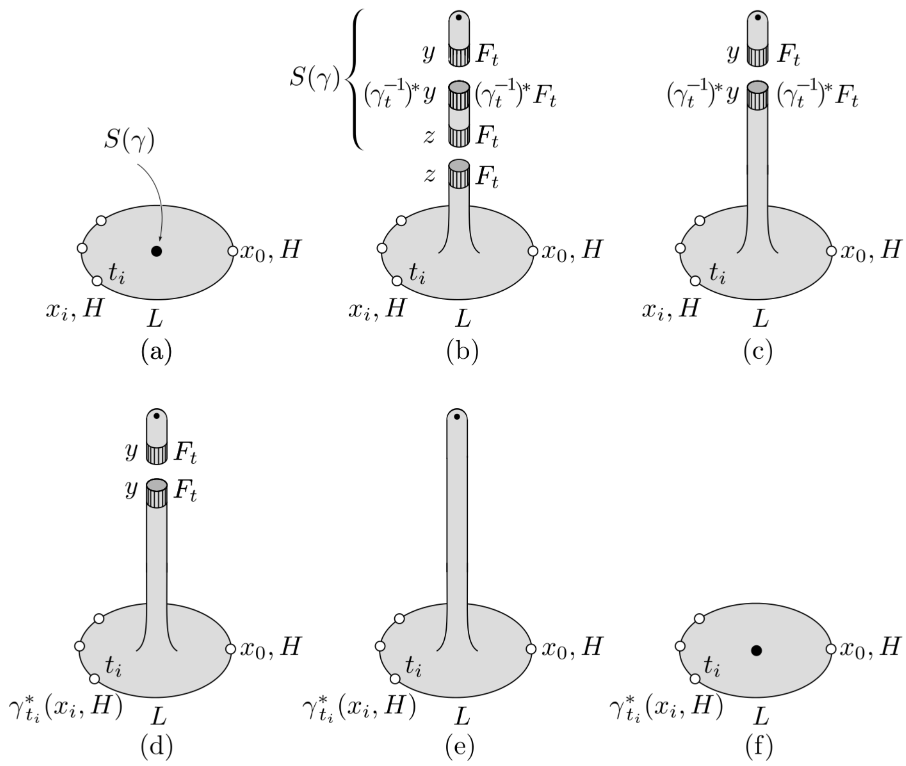


Fig. 3.1 A computation of $\mathscr{C}\mathscr{O}^*(S(\gamma))$ by Charette and Cornea.

as a linear combination of some Hamiltonian orbits z (in the figure, we abbreviate the datum $\{F_t, J_t\}$ simply to F_t).

The PSS image of $S(\gamma)$ counts configurations shown in the upper part of Figure 3.1(b), consisting of disks with one output puncture (say, asymptotic to an orbit y), and a cylinder counting continuation maps from $(\gamma_t^{-1})^* y$ as an orbit of Floer's complex with datum pulled back by the loop γ_t^{-1}, see [93, Lemmas 2.3 and 4.1], to another orbit z of the original Floer's complex with datum $\{F_t, J_t\}$. Let us glue the z-orbits together, passing to Figure 3.1(c), and then substitute each lower punctured pseudo-holomorphic disk u in Figure 3.1(c) by another disk u' defined as follows: $u'(re^{2\pi it}) = \gamma_t^{-1} \circ u(re^{2\pi it})$. Here $re^{2\pi it}$ is a point on the domain of the disk; we are assuming that the interior puncture is located at $0 \in \mathbb{C}$ and the output puncture at $1 \in \mathbb{C}$. Let us look at the effect of this substitution.

First, $[\partial u] = [\partial u'] + l \in H_1(L)$ so the count of configurations in Figure 3.1(c) (before substitution) is equal to the count of configurations in Figure 3.1(d) (after substitution) multiplied by $\rho(l)$.

Second, u' satisfies the same boundary condition L because $\gamma_t L = L$, but the perturbation data defining the pseudo-holomorphic equation get pulled back accordingly. In particular, the Lagrangian Floer datum $\{H_s,J_s\}_{s\in[0,1]}$ and the asymptotic chord at a strip-like end corresponding to the boundary puncture at $t_i\in S^1$ get pulled back by γ_{t_i}.

Third, near the interior puncture u' satisfies the Hamiltonian Floer equation with original datum $\{F_t,J_t\}$ and asymptotic orbit y. So we can glue the y-orbits, passing to Figure 3.1(e), and Figure 3.1(f) is another drawing of the same domain we got after gluing: namely, the disk with $k+1$ boundary punctures and one interior unconstrained marked point, fixed at $0\in\mathbb{C}$. This interior marked point comes from one on the upper disk in Figure 3.1(b) (that disk defines the unit in Hamiltonian Floer cohomology), where that point serves to stabilise the domain. Summing up, for $x_i\in CF^*(L,L)$ we obtain:

$$\mathscr{C}\mathscr{O}^k(S(\gamma))(x_1\otimes\ldots\otimes x_k)=\rho(l)\cdot\sum\sharp\mathscr{M}^\gamma(x_1,\ldots,x_k;x_0)\cdot x_0 \tag{3.1}$$

where $\mathscr{M}^\gamma(x_1,\ldots,x_k;x_0)$ is the 0-dimensional moduli space of disks shown in Figure 3.1(f) which satisfy the inhomogeneous pseudo-holomorphic equation defined by domain- and modulus-dependent perturbation data in the sense of [96] such that:

- the disks carry the unconstrained interior marked point fixed at $t=0$, the output boundary puncture fixed at $t_0=1$, and k free input boundary punctures at $t_i\in S^1$, $i=1,\ldots,k$;
- on a strip-like end corresponding to a boundary puncture $t_i\in S^1$, perturbation data restrict to the Floer datum which is the γ_{t_i}-pullback of the original Floer datum $\{H_s,J_s\}_{s\in[0,1]}$, and the asymptotic chord for this strip must be the γ_{t_i}-pullback of the asymptotic chord x_i of the original Floer datum.
- the data must be consistent with gluing strip-like ends at $t_i\in S^1$ to strip-like ends of punctured pseudo-holomorphic disks carrying γ_{t_i}-pullbacks of the original perturbation data defining the A_∞ structure on $CF^*(L,L)$.

The counts $\sharp\mathscr{M}^\gamma$ are signed and weighted by ρ as usual; the last condition guarantees that $\mathscr{C}\mathscr{O}^*(S(\gamma))$ is a Hochschild cocycle. When $\rho\equiv 1$, Formula (3.1) coincides with the formula from [96, Section (10c)] defining the natural transformation $[\gamma^\sharp]$.

Remark 3.2.1. The fixed interior marked point at $t=0$ and the fixed boundary marked point at $t_0=1$ make sure our disks have no automorphisms, so the positions $t_i\in S^1$ of the other boundary punctures are uniquely defined.

Before proceeding, note that we are already able to compute $\mathscr{C}\mathscr{O}^0(S(\gamma))\in HF^*(L,L)$.

Corollary 3.2.2. *If $\{\gamma_t\}_{t\in S^1}$ is a Hamiltonian loop such that $\gamma_t(L)=L$, then $\mathscr{C}\mathscr{O}^0(S(\gamma))=\rho(l)\cdot 1_L$.*

Proof. When $k = 0$, the moduli space in Formula (3.1) is exactly the moduli space defining the cohomological unit in $CF^*(L,L)$, see e.g. [103, Section 2.4]. □

Proof of Theorem 3.1.7(a). This is the homology-level version of Corollary 3.2.2. □

3.2.2 The PSS maps in degree one

Our goal will be to compute a "topological piece" of $\mathscr{C}\mathscr{O}^1(S(\gamma))$. This subsection introduces some background required for the computation: in particular, we recall that there is a canonical map $\Phi\colon H^1(L) \to HF^*(L,L)$ which was used in the statement of Theorem 3.1.7. This is the Lagrangian PSS map of Albers [5], and the fact it is canonical was discussed, for instance, by Biran and Cornea [19, Proposition 4.5.1(ii)] in the context of Lagrangian quantum cohomology.

First, recall that once the Floer datum is fixed, the complex $CF^*(L,L)$ acquires the Morse $\mathbb{Z}$-grading. This grading is not preserved by the Floer differential or the A_∞ structure maps, but is still very useful. Assume that the Hamiltonian perturbation, as part of the Floer datum, is chosen to have a unique minimum x_0 on L, which means that $CF^0(L,L)$ is one-dimensional and generated by x_0. We denote by $1_L \in CF^0(L,L)$ the chain-level cohomological unit, which is proportional to x_0. Now pick a metric and a Morse-Smale function f on L with a single minimum; together they define the Morse complex which we denote by $C^*(L)$. Consider the "Maslov index 0" versions of the PSS maps

$$\Psi\colon\ CF^*(L,L) \to C^*(L), \quad \Phi\colon\ C^*(L) \to CF^*(L,L) \tag{3.2}$$

defined as in the paper of Albers [5], with the difference that Φ, Ψ count configurations with Maslov index 0 disks only. For example, the map Ψ counts configurations consisting of a Maslov index 0 pseudo-holomorphic disk with boundary on L and one input boundary puncture, followed by a semi-infinite gradient trajectory of f which outputs an element of $C^*(L)$. Similarly, Φ counts configurations in which a semi-infinite gradient trajectory is followed by a Maslov index 0 disk with an output boundary puncture. The maps Ψ, Φ preserve $\mathbb{Z}$-gradings on the two complexes.

Let $d_0\colon\ CF^*(L,L) \to CF^{*+1}(L,L)$ be the "Morse" part of the Floer differential counting the contribution of Maslov index 0 strips, see Oh [79]. Denote by $d_{Morse}\colon\ C^*(L) \to C^{*+1}(L)$ the usual Morse differential. The lemma below is a version of [5, Theorem 4.11].

Lemma 3.2.3. Φ, Ψ *are chain maps with respect to* d_0 *and* d_{Morse}*, and are cohomology inverses of each other.* □

Lemma 3.2.4. *Suppose $HF^*(L,L) \neq 0$. If $y \in C^1(L)$ is a Morse cocycle (resp. coboundary) then $\Phi(y)$ is a Floer cocycle (resp. coboundary).*

Proof. This follows from Oh's decomposition of the Floer differential [79]. □

Consequently, if $HF^*(L,L) \neq 0$, we get a map

$$\Phi\colon\ H^1(L) \to HF^*(L,L).$$

For Ψ, we have a weaker lemma using [79] (this lemma is not true for coboundaries instead of cocycles).

Lemma 3.2.5. *If $y \in CF^*(L,L)$ is a Floer cocycle, then $\Psi(y) \in C^*(L)$ is a Morse cocycle.* □

3.2.3 Computing the topological part of $\mathscr{C}\mathscr{O}^1(S(\gamma))$

We continue to use the above conventions and definitions, namely we use the $\mathbb{Z}$-grading on $CF^*(L,L)$, the maps Φ, Ψ, and the choice of the Hamiltonian perturbation on L with a unique minimum x_0. From now on, we assume $HF^*(L,L) \neq 0$. Recall that $\mathscr{C}\mathscr{O}^*(S(\gamma))$ is determined via Formula (3.1) by the moduli spaces $\mathscr{M}^\gamma(x_1,\dots,x_k;x_0)$. The connected components of $\mathscr{M}^\gamma(x_1,\dots,x_k;x_0)$ corresponding to disks of Maslov index μ have dimension $|x_0| + k + \mu - \sum_{i=1}^k |x_i|$ where $|x_i|$ are the $\mathbb{Z}$-gradings of the $x_i \in CF^*(L,L)$. Consequently, $\mathscr{C}\mathscr{O}^k(S(\gamma))\colon\ CF^*(L,L)^{\otimes k} \to CF^*(L,L)$ is a sum of maps of degrees

$$-k - mN_L, \quad m \geq 0,$$

where N_L is the minimal Maslov number of L. In particular, the restriction $\mathscr{C}\mathscr{O}^1(S(\gamma))$ to $CF^1(L,L)$ is of pure degree -1, that is, its image lands in $CF^0(L,L)$:

$$\mathscr{C}\mathscr{O}^1(S(\gamma))|_{CF^1(L,L)}\colon\ CF^1(L,L) \to CF^0(L,L).$$

Moreover, this map is determined by the moduli space consisting of Maslov index 0 disks only, and can be computed in purely topological terms. This is the main technical computation which we now perform.

Proposition 3.2.6. *Suppose $HF^*(L,L) \neq 0$. If $x \in CF^1(L,L)$ is a Floer cocycle, then*

$$\mathscr{C}\mathscr{O}^1(S(\gamma))(x) = \langle \Psi(x), l \rangle \cdot \rho(l) \cdot 1_L.$$

Here $\Psi(x) \in H^1(L)$ and $\langle -,- \rangle$ denotes the pairing $H^1(L) \otimes H_1(L) \to \mathbb{K}$.

Proof. All disks with boundary on L we consider in this proof are assumed to have Maslov index 0. We identify the domains of all disks that appear with the unit disk in $\mathbb{C}$, and their boundaries are identified with the unit circle $S^1 \subset \mathbb{C}$. In the subsequent figures, punctured marked points will be drawn by circles filled white, and unpunctured marked points by circles filled black. According to Formula (3.1), for a generator $x \in CF^1(L,L)$ we have

$$\mathscr{CO}^1(S(\gamma))(x) = \rho(l) \cdot \sharp \mathscr{M}^\gamma(x;x_0) \cdot x_0,$$

where $\mathscr{M}^\gamma(x;x_0)$ consists of (perturbed pseudo-holomorphic) Maslov index 0 disks whose domains are shown in Figure 3.2(a).

Step 1. Perturbation data producing bubbles with unpunctured points

Recall that the domains appearing in the moduli space $\mathscr{M}^\gamma(x;x_0)$ are disks with the interior marked point 0 and boundary punctures $1,t$, where $t \in S^1 \setminus \{1\}$. For further use, we will choose perturbation data defining $\mathscr{M}^\gamma(x;x_0)$ whose bubbling behaviour as $t \to 1$ differs from the standard one. Usually, the perturbation data would be chosen so as to be compatible, as $t \to 1$, with the gluing shown in Figure 3.2(a)→(b), where the bubble meets the principal disk along a puncture, meaning that near this puncture it satisfies a Floer equation and shares the asymptotic with the corresponding puncture of the principal disk. On the other hand, we will use perturbation data consistent with gluing shown in Figure 3.2(a)→(c), where the bubble is attached to the principal disk by an unpunctured marked point.

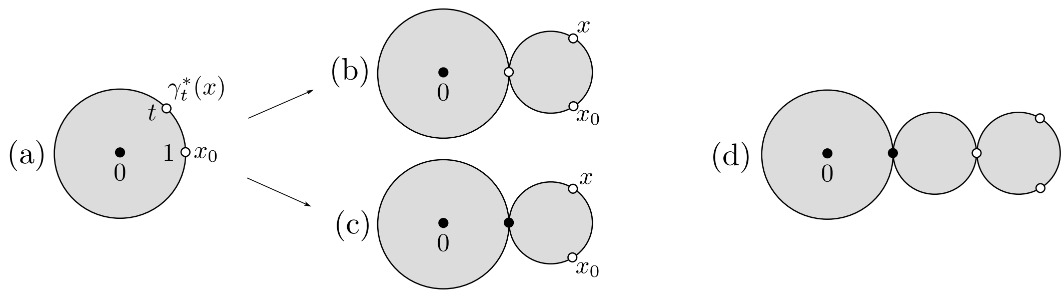


Fig. 3.2 Two types of gluings for $\mathscr{M}^\gamma(x;x_0)$, and a way to interpolate between the glued perturbation data.

Let us explain how to define both types of data more explicitly. The domain in Figure 3.2(a), with free parameter t close to 1, is bi-holomorphic to the domain shown in Figure 3.3, whose boundary marked points are fixed at 1 and some $t_0 \in S^1$, upon which a stretching procedure along the strip labelled (b) is performed. This stretching procedure changes the complex structure on the disk by identifying the strip with $[0,1] \times [0,1]$, removing it, and gluing back the longer strip $[0,1] \times [0,r]$. The parameter $r \in [1,+\infty)$ is free and replaces the free parameter t, so that tending $r \to +\infty$ translates into the collision of two marked points $t \to 1$.

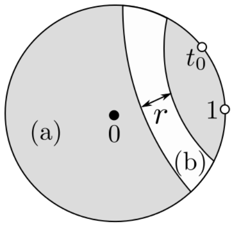


Fig. 3.3 Collision of two boundary marked points seen as stretching the strip (b) with parameter $r \to +\infty$.

In order to get perturbation data which are consistent with the usual bubbling shown in Figure 3.2(a)→(b), one requires the perturbed pseudo-holomorphic equation to coincide, on the strip $[0,1] \times [0,r]$, with the usual Floer equation defining the Floer differential, which uses a Hamiltonian perturbation translation-invariant in the direction of $[0,r]$. In order to get perturbation data producing the bubbling pattern Figure 3.2(a)→(c), we simply put an unperturbed pseudo-holomorphic equation on the strip $[0,1] \times [0,r]$, without using a Hamiltonian perturbation at all.

Both ways of defining perturbation data are subject to appropriate gluing and compactness theorems, which precisely say that as we tend $r \to +\infty$, the solutions bubble in one of the two corresponding ways shown in Figure 3.2. While the standard choice is used, for example, to prove that $[\gamma^\sharp]$ (obtained from the counts of various $\mathscr{M}^\gamma$) is a Hochschild cocycle, the other choice will be more convenient for our computations. Note that the two different types of perturbation data give the same count $\sharp\mathscr{M}^\gamma(x;x_0)$: this is proved by interpolating between them using the two-parametric space of perturbation data obtained from gluing together the disks in Figure 3.2(d) with different length parameters. Recall that all disks in $\mathscr{M}^\gamma(x;x_0)$ have Maslov index 0, so no unnecessary bubbling occurs. (Since we do not want to compute the moduli spaces $\mathscr{M}^\gamma$ other than $\mathscr{M}^\gamma(x;x_0)$, we do not have to worry about extending our unusual type of perturbation data to the other moduli spaces.)

In addition, we will assume that the Hamiltonian perturbation vanishes over the principal disk in Figure 3.2(c), making this disk J-holomorphic and hence constant, because the disk has Maslov index 0. Such configurations can be made consistent with gluing: for this, one just needs to make the Hamiltonian perturbation vanish over subdomain (a) in Figure 3.3, for all t close to 1. Note that regularity can be achieved by perturbing the pseudo-holomorphic equation over the subdomain to the right of the strip (b) in Figure 3.3.

Step 2. A one-dimensional cobordism from $\mathscr{M}^{\gamma}(x;x_0)$

In what follows, we will use the theory of holomorphic pearly trees developed by Sheridan in his Morse-Bott definition of the Fukaya category [103], which performs the analysis based on explicit perturbations of the pseudo-holomorphic equations (a setup which extends Seidel's setup of Fukaya categories from [96]). Although [103] considers exact Lagrangians instead of monotone ones, all the analysis works equally well in the non-exact case if we only consider disks having Maslov index 0, because here unpunctured disk bubbles cannot occur just like in the exact case. The definition of Floer's differential, as part of the Fukaya category, using holomorphic pearly trees is more classical and has been carried out in detail in the works of Biran and Cornea [17, 19, 18]. Techniques for dealing with holomorphic pearly trees (or "clusters") with disks of arbitrary Maslov index have appeared in [32, 25], but we will not actually need to appeal to them.

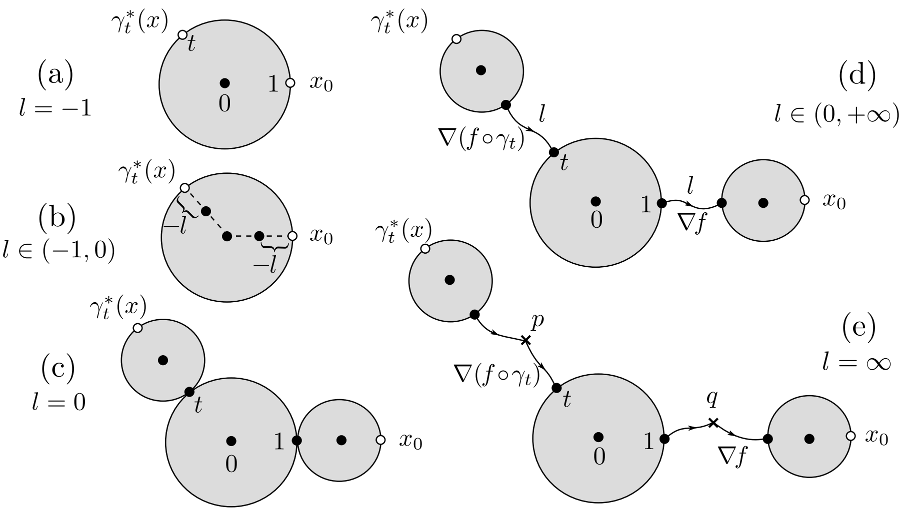


Fig. 3.4 The domains for $s \in (0,2\pi)$, $l \in [-1,\infty]$, where $t = e^{is}$.

We will now define a family of domains depending on two parameters $s \in [0,2\pi]$, $l \in [-1,+\infty]$. When $s \notin \{0,2\pi\}$, the domains are shown in Figure 3.4(a)–(e), where we denote $t = e^{is}$; we discuss the case $s \in \{0,2\pi\}$ later. When $l = -1$ the domain is the disk from the definition of $\mathscr{M}^{\gamma}(x;x_0)$ we have just recalled. When $l \in (-1,0)$, the domain is the same disk (called principal) with two additional interior marked points whose position is determined by the parameter l: the first point lies on the line segment $[0,t]$, the second one lies on the line segment $[0,1]$, and both points have distance $1+l$ from 0. When $l = 0$, the domain consists of the principal disk with marked points $0,1,t$, and two bubble disks attached to the principal disk at points 1 and t. The first bubble disk has marked points $0,1$ and a boundary puncture at -1, the second one has marked points at $0,-1$ and a boundary puncture at 1. When $0 < l < \infty$, the domain contains the same three disks, now disjoint from each other, plus two line segments of length l connecting the bubble disks to the principal one along the boundary marked points at which the disks used to be attached to each other. When $l = \infty$, we replace each line segment by two rays $[0, +\infty) \sqcup (-\infty, 0]$.

When $s = 0$ or $s = 2\pi$, the domains obtain extra bubbles as those discussed above in the definition of $\mathscr{M}^\gamma(x; x_0)$, which correspond to the parameter $t = e^{is} \in S^1$ approaching $1 \in S^1$ from the two sides. These domains are shown in Figure 3.5: as l goes from -1 to 0, the two interior marked points move along the punctured paths. Observe that these points are crossing the node between the two disks at some intermediate value of l; this does not cause any difficulty with the definitions because these marked points are only used to represent varying perturbation data consistent with the types of bubbling we prescribe in the figures. We will soon mention what these varying data are in terms of stretching certain strips inside a fixed disk. When $l > 0$, the length of the paths equals l. When $l = \infty$, one introduces broken lines $[0, +\infty) \sqcup (-\infty, 0]$ like in Figure 3.4(e).

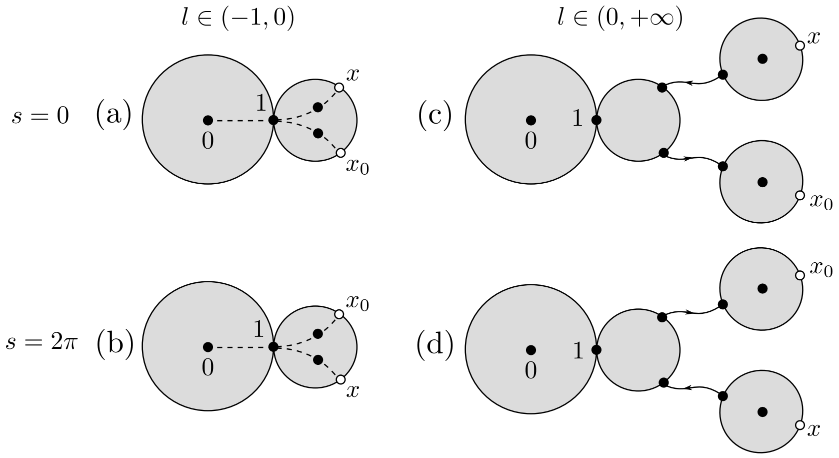


Fig. 3.5 The domains for $s \in \{0, 2\pi\}$, $l \in [-1, \infty]$.

Having specified the domains, we briefly explain how to equip the disks with suitable perturbed pseudo-holomorphic equations, and line segments with suitable gradient equations to get a moduli space of solutions. When $l = -1$, we choose the equations defining $\mathscr{M}^\gamma(x; x_0)$ as discussed above, which is consistent with bubbling at the unpunctured point as $s \to 0$ or $s \to 2\pi$. When $-1 < l < 0$, we choose the equations with the same properties as for $\mathscr{M}^\gamma(x; x_0)$, which are additionally consistent with bubbling at unpunctured points as $l \to 0$. When $l \geq 0$, we choose the equation on the disk with the input puncture (marked x in Figure 3.4) to be the γ_t-pullback of the one appearing in the definition of the PSS map Ψ, and the equation on the disk with an output puncture (marked x_0) to be exactly the equation from the PSS map Φ. On the line segments and rays, the equation is the gradient equation of

f or its pullback by γ_t, as shown in Figure 3.4. On all disks without boundary punctures, we require the perturbation to vanish, which makes these disks constant, since they have Maslov index 0. (This can be made compatible with gluing, like in the case $s \to 0, 2\pi$ discussed in Step 1, and it is not hard to see that the freedom to vary f is enough to achieve regularity of solutions. We will later explain in a bit more detail why the configurations are regular for $l = \infty$.) When $s = 0$ and $l < 0$, we require the perturbation data on the twice-punctured disk in Figure 3.5(a) to be obtained by π-rotation from the perturbation data on the similar disk for $s = 2\pi$ (and the same parameter l), if we identify the two punctures with points $1, -1$ of the unit disk. When $s = 0$ and $l \geq 0$, we make a similar symmetric choice.

Finally, if x is a generator of $CF^1(L,L)$, we specify that the input puncture must be asymptotic to the γ_t-pullback of x (as usual, if x is a linear combination of generators, we take the disjoint union of the relevant moduli spaces). The output puncture must be asymptotic to the unique generator $x_0 \in CF^0(L,L)$. When $l = \infty$, the first pair of rays in Figure 3.4(e) must be asymptotic to a point p such that $\gamma_t^{-1}(p) \in C^1(L)$ (that is, p is an index 1 critical point of $f \circ \gamma_t$) and the second pair of rays must be asymptotic to $q \in C^0(L)$; we assume q is the unique minimum of f. The interior marked points on the disks are unconstrained.

The moduli space over the two-dimensional family of domains we have just specified is 1-dimensional, by our choice of indices, and its boundary consists of:

- solutions whose domains have parameter $l = -1$ or $l = \infty$,
- solutions whose domains have parameter $s = 0$ or $s = 2\pi$.

We claim that solutions of the second type cancel pairwise. Indeed, recall that the disks without boundary punctures in Figures 3.5(a)–(d) are constant, and the perturbation data on the punctured disks for $s = 0, 2\pi$ are chosen in a way to provide the same solutions, after a π-rotation on each disk. Let us describe more explicitly what happens when $l < 0$, as the case when $l \geq 0$ is clear enough from Figures 3.5(c),(d). We can represent the domains shown in Figure 3.4(a), with free $l < 0$ and free small $s > 0$, where $t = e^{is}$, by a disk with fixed boundary punctures, stretched with length parameters $-1/l$ and $1/s$ along the three strips shown in Figure 3.6(a). The stretching procedure was described earlier, and our choice of perturbation data says that the stretched strips, and the sub-domain to the left of the $1/s$-strip, carry an unperturbed pseudo-holomorphic equation. So for $s = 0$ we get the disks shown in Figure 3.6(b), with the unpunctured boundary marked point attached to a constant disk, which means this boundary marked point is unconstrained. (As usual, the domain is considered up to complex automorphisms, so the unconstrained point does not prevent us from having rigid solutions.) This way, Figures 3.5(a) and 3.6(b) are drawings of the same solution, for any $l < 0$. If we rotate the disk in Figure 3.6(b) by π, we get precisely the disk with perturbation data we would have got for $s = 2\pi$, except that the boundary marked point is on the different side of the boundary. But since that point is unconstrained, its position does not actually matter, and for $s = 0, 2\pi$ we get a pair of the same solutions. The pairs of solutions with

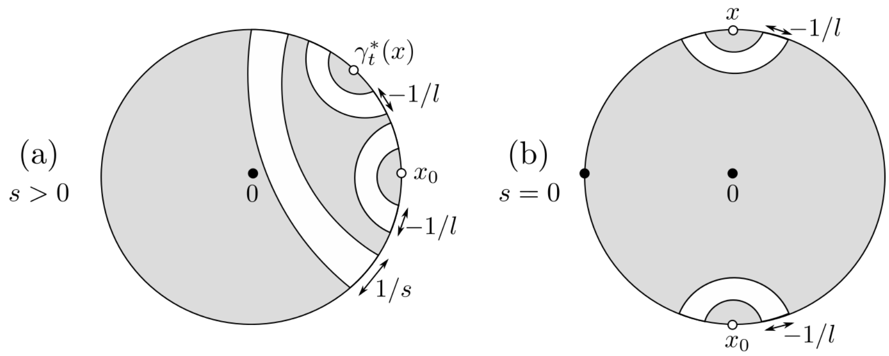


Fig. 3.6 Left: the domains for $l < 0$, $s > 0$ seen as a fixed disk with three stretched strips. Right: the same domains for $s = 0$ when the principal disk is constant.

$s = 0, 2\pi$ contribute with different signs because they correspond to the opposite sides of the boundary of the moduli space of domains. The outcome is that the count of configurations in Figure 3.4(a), i.e. $\sharp\mathscr{M}^\gamma(x;x_0)$, equals the count of configurations in Figure 3.4(e), and it remains to compute the latter.

Step 3. A Morse-theoretic computation

Let us look at Figure 3.4(e). Recall that $q \in L$ is the minimum of f, so the semi-infinite flowline of ∇f flowing into q must be constant. Second, we have arranged the principal disk to be constant, as well. So the configurations in Figure 3.4(e) reduce to those shown in Figure 3.7.

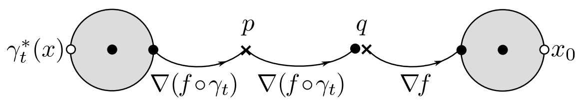


Fig. 3.7 The domains when $l = +\infty$ and the principal disk together with a flowline are constant. Here p is an index 1 critical point of $f \circ \gamma_t$, and q is the minimum of f.

The free parameter $t = e^{is} \in S^1 \setminus \{1\}$ is "unseen" by the domain after the principal disk became ghost (i.e. constant), but the equations still depend on it. First, consider the left disk and the left flowline in Figure 3.7, forgetting the rest of the configuration. Those disk and flowline satisfy the γ_t-pullback of the equation defining the PSS map Ψ, so for each t the linear combination of points p appearing as limits of such configurations equals $\gamma_t(\Psi(x))$, where $\Psi(x) \in C^1(L)$ is the PSS image which is a linear combination of index 1 critical points of f, so that $\gamma_t(\Psi(x))$ is a combination of critical points of $f\circ\gamma_t$. Let us now add back the middle flowline, still forgetting the right flowline and the right disk, and count the resulting configurations. The middle flowline is a semi-infinite flowline of $\nabla(f\circ\gamma_t)$ ending at the point q; note that q is not a critical point of $f\circ\gamma_t$ when $t\neq 1\in S^1$. Suppose for the moment that we allow the right end of the middle flowline to be free (not constrained to q) and denote the moduli space of such configurations by P. Then there is the evaluation map at the right end of the flowline, $\mathrm{ev}\colon P\to L$. Its image is the unstable manifold of the input critical points, which we determined to be the linear combination $\gamma_t(\Psi(x))$, with respect to the function $f\circ\gamma_t$. Consequently, if we denote by $C_{\Psi(x)}\subset L$ the disjoint union of (oriented, codimension 1) unstable manifolds of the Morse cochain $\Psi(x)\in C^1(L)$ with respect to f, then

$$P=(S^1\setminus\{1\})\times C_{\Psi(x)},\qquad \mathrm{ev}(t,z)=\gamma_t(z).$$

Those configurations which evaluate at $q\in L$ are the intersection points $C_{\Psi(x)}\cap l$, where $l=\{\gamma_t(q)\}_{t\in S^1}$ is the orbit of q. By perturbing γ_t and f, the intersections can be easily made transverse, and we get:

$$\sharp(P\times_{\mathrm{ev}}\{q\})=[C_{\Psi(x)}]\cdot[l]=\langle\Psi(x),l\rangle.$$

Recall this is the count of the part of confugurations in Figure 3.7 which end up at q. Finally, the count of the rightmost flowlines (emerging from q) plus the right disks in Figure 3.7 equals $1_L\in CF^0(L,L)$. Indeed, the unstable manifold of the minimum q is the whole manifold L (minus a codimension 2 subset), so the count is the same as the count of the rightmost disks only, and the latter by definition produces 1_L.

Putting everything together, we get the statement of Proposition 3.2.6. One last thing is to argue that the moduli space we computed in Figure 3.7 is regular. According to [103], the regularity of moduli spaces consisting of pseudo-holomorphic disks and flowlines is equivalent to the regularity of the separate disks and flowlines not constrained to satisfy the incidence conditions, plus the transversality of the evaluation maps which account for the incidence conditions. As constant disks are known to be regular and we have observed that the evaluation map controlling incidence with the constant disk is transverse, the moduli space is indeed regular. The proof of Proposition 3.2.6 is complete. □

3.2.4 Checking non-triviality in Hochschild cohomology

In this subsection we prove Theorem 3.1.7(b), (c) (recall that part (a) was proved earlier, see Corollary 3.2.2). We have computed in Proposition 3.2.6 the map $\mathscr{C}\mathscr{O}^1(S(\gamma))|_{CF^1(L,L)}$, and it remains to see when the result survives to something non-trivial on the level of Hochschild cohomology, thus distinguishing $\mathscr{C}\mathscr{O}^*(S(\gamma)) \in HH^*(L,L)$ from the unit in $HH^*(L,L)$.

First, let us quickly recall the definition of Hochschild cohomology. Let A be an A_∞ algebra, and assume it is $\mathbb{Z}/2$ graded if $\operatorname{char}\mathbb{K} \neq 2$. The space of Hochschild cochains is $CC^*(A,A) = \prod_{k\geq 0} Hom(A^{\otimes k},A)$. If A is $\mathbb{Z}/2$-graded then $CC^*(A,A)$ is $\mathbb{Z}/2$-graded: $CC^r(A,A) = \prod_{k\geq 0} Hom(A^{\otimes k},A[r-k])$. If $h = \{h^k\}_{k\geq 0} \in CC^*(A)$, $h^k\colon A^{\otimes k} \to A$, then the Hochschild differential of h is the sequence of maps

$$
\begin{aligned}
&(\partial h)^k(a_k,\ldots,a_1) = \\
&\textstyle\sum_{i+j\leq k}(-1)^{(r+1)(|a_1|+\ldots+|a_i|+i)}\cdot \mu^{k+1-i}(a_k,\ldots a_{i+j+1},h^j(a_{i+j},\ldots,a_{i+1}),a_i,\ldots,a_1)+ \\
&\textstyle\sum_{i+j\leq k}(-1)^{r+1+|a_1|+\ldots+|a_i|+i}\cdot h^{k+1-i}(a_k,\ldots a_{i+j+1},\mu^j(a_{i+j},\ldots,a_{i+1}),a_i,\ldots,a_1).
\end{aligned}
$$

Here r is the $\mathbb{Z}/2$-degree of h. (When $k=0$, the agreement is that $Hom(A^{\otimes 0},A) = A$, so h^0 is an element of A.) If $\operatorname{char}\mathbb{K} = 2$, we do not need the gradings as the signs do not matter.

Let us return to the A_∞ algebra $CF^*(L,L)$. We continue to use the $\mathbb{Z}$-grading on the vector space $CF^*(L,L)$ keeping in mind this grading is not respected by the A_∞ structure. If L is oriented, the reduced $\mathbb{Z}/2$-grading is preserved by the A_∞ structure so $CF^*(L,L)$ is a $\mathbb{Z}/2$-graded A_∞ algebra. If L is not oriented, we must suppose $\operatorname{char}\mathbb{K} = 2$.

Proof of Theorem 3.1.7(b). Because the map $\mathscr{C}\mathscr{O}^*$ is unital [103, Lemma 2.3], the Hochschild cohomology unit is realised by the cochain $1_{HH} := \mathscr{C}\mathscr{O}^*(1) \in CC^*(L,L)$, where 1 is the unit in $QH^*(X)$. The A_∞ category $CF^*(L,L)$ need not be strictly unital, so the maps $(1_{HH})^k = \mathscr{C}\mathscr{O}^k(1)$ need not vanish for $k>0$. However, because the identity Hamiltonian loop preserves L and has homologically trivial orbits on it, Proposition 3.2.6 applies to $1 = S(\mathrm{Id}) \in QH^*(X)$ and says that $(1_{HH})^1(x) = 0$ for any Floer cochain $x \in CF^1(L,L)$.

Suppose $\mathscr{C}\mathscr{O}^*(S(\gamma)) + \alpha\cdot 1_{HH}$ is the coboundary of an element $h \in CC^*(L,L)$, for some $\alpha \in \mathbb{K}$. By equating $(\partial h)^0$ with $\mathscr{C}\mathscr{O}^0(S(\gamma)) + \alpha\cdot(1_{HH})^0$, see Corollary 3.2.2, we get:

$$\mu^1(h^0) = 1_L + \alpha\cdot 1_L.$$

Here μ^1 is the Floer differential, and the assumption that $HF^*(L,L) \neq 0$ implies that the Floer cohomology unit 1_L cannot be killed by the Floer differential. Therefore, we cannot solve the above equation unless $\alpha = -1$ and $\mu^1(h^0) = 0$. Next, by equating $(\partial h)^1$ with $\mathscr{C}\mathscr{O}^1(S(\gamma)) + \alpha\cdot(1_{HH})^1$, see Proposition 3.2.6, and using $\alpha = -1$, for any Floer cochain

$x \in CF^1(L,L)$ we get

$$(-1)^{|h^0|+1}\mu^2(x,h^0) + (-1)^{(|h^0|+1)(|x|+1)}\mu^2(h^0,x) \\ + \mu^1(h^1(x)) + (-1)^{|h^1|+1}h^1(\mu^1(x)) = \rho(l)\cdot\langle\Psi(x),l\rangle\cdot 1_L.$$

Now suppose x is a Floer cocycle, so the last summand of the left-hand side vanishes. If $\operatorname{char}\mathbb{K} = 2$, redenote $a := h^0 \in CF^*(L,L)$. If $\operatorname{char}\mathbb{K} \neq 2$, let $a \in CF^{odd}(L,L)$ be the odd degree part of h^0. By computing the signs in the above equality we get, for any Floer cocycle $x \in CF^1(L,L)$:

$$\mu^2(x,a) + \mu^2(a,x) + \mu^1(h^1(x)) = \rho(l)\cdot\langle\Psi(x),l\rangle\cdot 1_L.$$

Recall that $\mu^1(h^0) = 0$ so $\mu^1(a) = 0$ as well, and we get the following equality for Floer cohomology classes $[x],[a] \in HF^*(L,L)$ and $[\Psi(x)] \in H^1(L)$:

$$\mu^2([x],[a]) + \mu^2([a],[x]) = \rho(l)\cdot\langle[\Psi(x)],l\rangle\cdot 1_L \in HF^*(L,L).$$

Now put $x = \Phi(y)$, where $y \in C^1(L)$ is a Morse cochain and Φ is the map from (3.2). The above equality means that for all $[y] \in H^1(L)$,

$$\mu^2(\Phi([y]),[a]) + \mu^2([a],\Phi([y])) = \rho(l)\cdot\langle[y],l\rangle\cdot 1_L \in HF^*(L,L).$$

This is exactly the equality prohibited by the hypothesis of Theorem 3.1.7(b), so Theorem 3.1.7(b) is proved. □

Proof of Theorem 3.1.7(c). Note that $\mathscr{CO}^0(S(\gamma)*Q) = \mathscr{CO}^0(S(\gamma))\cdot\mathscr{CO}^0(Q) = 1_L\cdot\mathscr{CO}^0(Q) = \mathscr{CO}^0(Q)$ (here the dot denotes the μ^2 product), so the only possible linear relation between $\mathscr{CO}^*(S(\gamma)*Q)$ and $\mathscr{CO}^*(Q)$ is that

$$\mathscr{CO}^*((S(\gamma)-1)*Q) = 0,$$

where 1 is the unit in $QH^*(X)$. We have $\mathscr{CO}^*((S(\gamma)-1)*Q) = \mathscr{CO}^*(S(\gamma)-1)\star\mathscr{CO}^*(Q)$, where the symbol $\star$ denotes the Yoneda product in Hochschild cohomology. Recall that if $\phi = \{\phi^k\}_{k\geq 0}, \psi = \{\psi^k\}_{k\geq 0} \in CC^*(L,L)$ are Hochschild cochains, the $k = 1$ part of their Yoneda product by definition equals

$$(\phi\star\psi)^1(x) = \pm\mu^2(\phi^1(x),\psi^0) \pm \mu^2(\phi^0,\psi^1(x)).$$

There is an explicit formula for the signs which we do not need. Let us apply this formula to $\mathscr{C}\mathscr{O}^*(S(\gamma)-1)$ and $\mathscr{C}\mathscr{O}^*(Q)$. We know that $(\mathscr{C}\mathscr{O}^*(S(\gamma)-1))^0=0$, and

$$(\mathscr{C}\mathscr{O}^*(S(\gamma)-1))^1(x)=\mathscr{C}\mathscr{O}^1(S(\gamma))(x)$$

is given by Proposition 3.2.6 for any Floer cocycle $x\in CF^1(L,L)$. Using this, we get:

$$\mathscr{C}\mathscr{O}^1((S(\gamma)-1)*Q)(x)=\rho(l)\cdot\langle\Psi(x),l\rangle\cdot\mathscr{C}\mathscr{O}^0(Q).$$

From this point, the rest of the proof follows the one of Theorem 3.1.7(b). □

3.3 The closed-open map for real toric Lagrangians

In this section, after a short proof of Theorem 3.1.12, we look for further examples of real toric Lagrangians where Theorem 3.1.7 can be effectively applied. We also discover that Proposition 3.2.6, after additional work, allows to show that the Fukaya A_∞ algebra of some of the considered Lagrangians is not formal. In particular, we prove the results about real toric Lagrangians stated in Section 3.1 (except for Proposition 3.1.1 and Corollary 3.1.2, which have been proved therein).

3.3.1 A proof of Theorem 3.1.12

Let X be a compact, smooth toric Fano variety, and $D\subset X$ be a toric divisor corresponding to one of the facets of the polytope defining X. There is a Hamiltonian circle action γ on X associated with D, which comes from the toric action by choosing a Hamiltonian which achieves maximum on D. A theorem of McDuff and Tolman [74] says the following.

Theorem 3.3.1. *We have $S(\gamma)=D^*$, where $D^*\in QH^*(X)$ is the Poincaré dual of D.* □

The loop γ never preserves the real Lagrangian $L\subset X$, but if we parametrise $\gamma=\{\gamma_t\}_{t\in[0,1]}$ then $\gamma_{1/2}(L)=L$, see [54]. Consequently, $\alpha=\{\gamma_t(L)\}_{t\in[0,1/2]}$ is a loop of Lagrangian submanifolds, and moreover we have $\alpha^2=\{\gamma_t(L)\}_{t\in[0,1]}$ in the space of Lagrangian loops. There is an associated Lagrangian Seidel element $S_L(\alpha)\in HF^*(L,L)$, which counts pseudo-holomorphic disks with rotating Lagrangian boundary condition α, and a single boundary puncture which evaluates to an element of $HF^*(L,L)$. A theorem of Hyvrier [57, Theorem 1.13], based on the disk doubling trick, computes $S_L(\alpha)$.

Theorem 3.3.2. *We have $S_L(\alpha)=[L\cap D]^*$, where $L\cap D$ is the clean intersection that has codimension 1 in L, and $[L\cap D]^*\in H^1(L)\subset HF^*(L,L)$ is its dual class.* □

The inclusion $H^1(L) \subset HF^*(L,L)$ is the PSS map Φ from Section 3.2, which is injective because $HF^*(L,L) \cong H^*(L)$ by Theorem 3.1.11.

Proof of Theorem 3.1.12. It suffices to prove that $\mathscr{C}\mathscr{O}^0(D^*) = \mathscr{D}(\mathscr{F}(D^*))$, where $D \subset X$ is a toric divisor as above and $D^* \in QH^*(X)$ is its dual class, because such D^* generate $QH^*(X)$ as an algebra [74]. Let γ be the Hamiltonian loop corresponding to D as above, and α be the Lagrangian loop as above, such that $\alpha^2 = \{\gamma_t(L)\}_{t\in[0,1]}$. It follows from Theorem 3.2.1 that $\mathscr{C}\mathscr{O}^0(S(\gamma)) = S_L(\alpha^2)$, and the latter can be rewritten as $\mathscr{F}(S_L(\alpha))$, where $\mathscr{F}$ is the Frobenius map on $HF^*(L,L)$. By Theorem 3.3.1, $S(\gamma) = D^*$, and by Theorem 3.3.2, $S_L(\alpha) = [L\cap D]^*$. Finally, if we look at Haug's construction [54] of the Duistermaat isomorphism $\mathscr{D}$, we will see that $[L\cap D]^* = \mathscr{D}(D^*)$. Putting everything together, we get $\mathscr{C}\mathscr{O}^0(D^*) = \mathscr{F}(\mathscr{D}(D^*))$. Because $\mathscr{D}$ is a ring map, it commutes with the Frobenius maps on $HF^*(L,L)$ and $QH^*(X)$, and the theorem follows. □

3.3.2 Split-generation for toric varieties with Picard rank 2

It is known that the unique toric variety with Picard number 1 is the projective space. By a theorem of Kleinschmidt [63], see also [31], every n-dimensional toric Fano variety whose Picard group has rank 2 (i.e. whose fan has $n+2$ generators) is isomorphic to the projectivisation of a sum of line bundles over $\mathbb{C}P^{n-k}$:

$$X(a_1,\dots,a_k) := \mathbb{P}_{\mathbb{C}P^{n-k}}(\mathscr{O}\oplus\mathscr{O}(a_1)\oplus\ldots\oplus\mathscr{O}(a_k)),\ a_i \geq 0,\ \sum_{i=1}^{k} a_i \leq n-k-1. \tag{3.3}$$

(The imposed conditions on a_i are equivalent to X being toric Fano). The $n+2$ vectors in $\mathbb{Z}^n$ generating the fan of $X(a_1,\dots,a_k)$ are the columns of the following matrix:

$$\left(\begin{array}{c|cc} & -1 & a_1 \\ & \vdots & \vdots \\ I_{n\times n} & -1 & a_k \\ & 0 & -1 \\ & \vdots & \vdots \\ & 0 & -1 \end{array}\right) \tag{3.4}$$

The minimal Chern number of $X(a_1,\dots,a_k)$ equals $\gcd(k+1, n-k+1-\sum a_i)$, see [85]. Some of these varieties provide further examples where, using Theorems 3.1.12 and 3.1.7, we can prove the injectivity of $\mathscr{C}\mathscr{O}^*$ and deduce split-generation.

Theorem 3.3.3. *Let $X := X(a_1,\dots,a_k)$ be as above, $L \subset X$ the real Lagrangian, $\mathbb{K}$ a field of characteristic 2. Suppose all a_i are odd and* $\gcd(k+1, n-k+1-\sum a_i) \geq 2$.

(a) If $n-k+1$ is odd, then $\mathscr{C}\mathscr{O}^0\colon\ QH^*(X) \to HF^*(L,L)$ is injective.

(b) *If $n-k+1$ is even, k is even and the numbers a_i come in equal pairs, then $\mathscr{CO}^*\colon\ QH^*(X)\to HH^*(L,L)$ is injective while $\mathscr{CO}^0$ is not.*

In both cases L split-generates $\mathscr{F}uk(X)_0$.

Proof. Let $x,y\in H^2(X)$ be the generators corresponding to the last two columns of the matrix (3.4). They generate $QH^*(X)$ as an algebra and satisfy the following relations when $\operatorname{char}\mathbb{K}=2$:

$$x(x+y)^k=1,\quad y^{n-k+1}(x+y)^{-\sum a_i}=1.$$

(For brevity, we no longer use the symbol $*$ to denote the quantum product.) If $n-k+1$ is odd, one can show that the Frobenius endomorphism $\mathscr{F}$ on $QH^*(X)$ is an isomorphism, so $\mathscr{CO}^0$ is injective by Theorem 3.1.12. It follows that $\mathscr{CO}^*$ is also injective, and split-generation follows from Theorem 3.1.9. Part (a) is proved.

In the rest of the proof we work with the case (b), so let us redenote: $n-k+1=2r$, $k=2q$, $\sum a_i=2p$. The rewritten relations in $QH^*(X)$ are:

$$x(x+y)^{2q}=1,\quad y^{2r}(x+y)^{-2p}=1. \tag{3.5}$$

Lemma 3.3.4. *For the ring given by relations (3.5), the kernel of the Frobenius endomorphism $\mathscr{F}$ is the ideal generated by $y^r(x+y)^{-p}+1$.*

Proof. Equations (3.5) are equivalent to

$$x^{-p}=y^{2rq},\quad y^{4rq+2r}+y^{4rp+2p}+1=0,$$

where the second equation is rewritten from the second equation in (3.5) using the substitution $x^{-p}=y^{2rq}$. This means if we denote

$$R(y)=y^{2rq+r}+y^{2rq+p}+1,$$

then $R(y)=y^r(x+y)^{-p}+1$. Denote $g=\gcd(2rq,p)$ and let $\alpha,\beta\in\mathbb{Z}$ be such that

$$-2rq\cdot\alpha+p\cdot\beta=g.$$

Consider the map $\phi\colon\ \mathbb{K}[u]\to QH^*(X)$ given by $u\mapsto x^\alpha y^\beta$; this map is onto because we get

$$\phi(u^{p/g})=y,\quad \phi(u^{2qr/g})=x^{-1} \tag{3.6}$$

using the given relations (note that the powers p/g, $2qr/g$ are integral). Further, $\ker\phi$ is obviously the ideal generated by $V(u)^2$ where $V(u):=R(u^{p/g})$, and we conclude that ϕ

provides an isomorphism

$$\phi\colon\ \mathbb{K}[u]/V(u)^2 \xrightarrow{\cong} QH^*(X), \quad V(u) = u^{\frac{p}{g}(2rq+r)} + u^{\frac{p}{g}(2rq+p)} + 1. \qquad (3.7)$$

It is clear that $V(u)$ generates the kernel of the Frobenius map on $\mathbb{K}[u]/V(u)^2$. Because $V(u)$ corresponds to $y^r(x+y)^{-p}+1$ under ϕ, Lemma 3.3.4 follows. □

We continue the proof of Theorem 3.3.3(b). It turns out that, similarly to the case of $\mathbb{R}P^n \subset \mathbb{C}P^n$ studied in the introduction, the generator of $\ker\mathscr{F}$ from Lemma 3.3.4 equals $S(\gamma)+1$ for a real Hamiltonian loop γ on X which preserves L setwise and has homologically non-trivial orbits on it. To construct γ, we will need the additional assumption that the a_i come in equal pairs, so we assume the sequence $(a_i)_{i=1}^{2q}$ is $(a_1,a_1,\ldots,a_q,a_q)$.

Recall that X, being a toric manifold, is a quotient of $\mathbb{C}^{2r+2q+1}$ minus some linear subspaces determined by the fan, by an action of $(\mathbb{C}^*)^2$. Using the common notation, this action is given by $z \mapsto t_1^{v_1}t_2^{v_2}z$, where $z \in \mathbb{C}^{2r+2q+1}$ and v_1, v_2 are the vectors in $\mathbb{Z}^{2r+2q+1}$ given by the following two rows:

$$\begin{array}{ccccc|ccc|cc}
\multicolumn{5}{c|}{2q\ \textit{entries}} & \multicolumn{3}{c|}{2r-1\ \textit{entries}} & \multicolumn{2}{c}{2\ \textit{entries}} \\
a_1 & a_1 & \ldots & a_q & a_q & -1 & \ldots & -1 & 0 & -1 \\
-1 & -1 & \ldots & -1 & -1 & 0 & \ldots & 0 & -1 & 0
\end{array} \qquad (3.8)$$

Let $(z_1,\ldots,z_{2r+2q+1})$ be co-ordinates on $\mathbb{C}^{2r+2q+1}$. The action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^{2r+2q+1}$ commutes with the action of $G = SU(2)^q \times SU(2r)$, where the $SU(2)$ factors act respectively on $(z_1,z_2),\ldots,(z_{2q-1},z_{2q})$, and $SU(2r)$ acts on $(z_{2q+1},\ldots,\ z_{2q+2r-1},z_{2q+2r+1})$, note we have omitted z_{2q+2r}. (If we view X as a projective bundle over $\mathbb{C}P^{2r-1}$ as in (3.3), the co-ordinates on which $SU(2r)$ acts are the homogeneous co-ordinates on the base.) Denote by $G^{\mathbb{R}} = SO(2)^q \times SO(2r)$ the real form of G. Because all a_i are odd, the action of $(-1,+1) \in (\mathbb{C}^*)^2$ coincides with the action of $-I \in G$. Consequently, the action of G descends to a Hamiltonian action of $G/\pm I$ on X. Its real form $G^{\mathbb{R}}/\pm I$ preserves the real Lagrangian $L \subset X$, and we let γ be the S^1-subgroup of $G^{\mathbb{R}}/\pm I$ defined as follows. This subgroup lifts to the path from I to $-I$ in $G^{\mathbb{R}}$ which is the image of the rotation $\left(\begin{smallmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{smallmatrix}\right) \in SO(2)$, $t \in [0,\pi]$, under the diagonal inclusions $SO(2) \subset SO(2)^q \times SO(2)^r \subset SO(2)^q \times SO(2r) = G^{\mathbb{R}}$. Recall that we are assuming $\operatorname{char}\mathbb{K} = 2$.

Lemma 3.3.5. *The homology class of γ-orbits on L is non-zero in $H_1(L;\mathbb{K})$.*

Proof. Indeed, L is a real projective bundle over $\mathbb{R}P^{2r-1}$, and the orbits project to the non-trivial cycle on the base, provided $\operatorname{char}\mathbb{K} = 2$. □

Lemma 3.3.6. *We have $S(\gamma)+1=y^r(x+y)^{-p}+1$ (which is the generator of* $\ker\mathscr{F}$ *from Lemma 3.3.4).*

Proof. Inside the complex group $G/\pm I$, the loop γ is homotopic to the loop γ' lifting to the path from I to $-I$ in G which is the image of the path $\left(\begin{smallmatrix} e^{it} & 0 \\ 0 & e^{-it}\end{smallmatrix}\right)\in SU(2)$, $t\in[0,\pi]$, under the diagonal inclusions $SU(2)\subset SU(2)^q\times SU(2)^r\subset SU(2)^q\times SU(2r)=G$. By using the action of $\mathbb{C}^*\subset(\mathbb{C}^*)^2$ corresponding to the first vector in (3.8), we see that γ' descends to the same Hamiltonian loop in X as the loop γ'' in G which acts on $\mathbb{C}^{2r+2q+1}$ as follows:

$$\begin{aligned}(z_1,\ldots,z_{2r+2q+1})\mapsto(&e^{it\frac{a_1+1}{2}}z_1,e^{it\frac{a_1-1}{2}}z_2,\ldots,e^{it\frac{a_q+1}{2}}z_{2q-1},e^{it\frac{a_q-1}{2}}z_{2q},\\ &z_{2q+1},e^{-it}z_{2q+2}\ldots,e^{-it}z_{2r+2q-2},z_{2r+2q-1},z_{2r+2q},e^{-it}z_{2r+2q+1}),\quad t\in[0,2\pi].\end{aligned}$$

Note that here t runs through $[0,2\pi]$, hence the $\frac{1}{2}$-factors. Because all a_i are odd, γ'' is now a closed loop in G, not only in $G/\pm I$. So by [74] its Seidel element $S(\gamma'')\in QH^*(X)$ can be computed as the quantum product of powers of the divisors corresponding to the co-ordinates on $\mathbb{C}^{2r+2q+1}$, where the powers are the multiplicities of rotations. Given $\operatorname{char}\mathbb{K}=2$, and recalling that $S(\gamma'')=S(\gamma')=S(\gamma)$, we get:

$$S(\gamma)=(x+y)^{\frac{a_1+1}{2}}(x+y)^{\frac{a_1-1}{2}}\ldots(x+y)^{\frac{a_q+1}{2}}(x+y)^{\frac{a_q-1}{2}}y^{-1}\ldots y^{-1}=(x+y)^py^{-r}.$$

This element squares to 1 by (3.5) (in agreement with the fact γ has order 2 in $\pi_1(G/\pm I)\cong\mathbb{Z}/2$), so it also equals $y^r(x+y)^{-p}$, which proves Lemma 3.3.6. □

We conclude the proof of Theorem 3.3.3(b). By Lemmas 3.3.4 and 3.3.6, $\ker\mathscr{F}$ is the ideal generated by $S(\gamma)+1$. Suppose $P\in QH^*(X)$ such that $\mathscr{CO}^*(P)=0\in HH^*(L,L)$. Then $\mathscr{CO}^0(P)=0$, so $P\in\ker\mathscr{F}$ by Theorem 3.1.12. Consequently $P=(S(\gamma)+1)*Q$, and if $P\neq0$ then $Q\notin\ker\mathscr{F}$ (because otherwise we would get $P\in(\ker\mathscr{F})^2=\{0\}$). Apply Theorem 3.1.7(b) to the product $(S(\gamma)+1)*Q$; the left hand side of $(**)$ vanishes because μ^2 is commutative on $HF^*(L,L)$ [54], and the right hand side is non-trivial for some y by Lemma 3.3.5 and because $\mathscr{CO}^0(Q)\neq0$. It follows that $\mathscr{CO}^*(P)\neq0$. We have shown that $\mathscr{CO}^*$ is injective, and split-generation follows from Theorem 3.1.9. Note that $w(L)=0$ holds for all real Lagrangians, as Maslov index 2 disks come in pairs because of the action of the anti-holomorphic involution, see [54]. □

The following corollary in particular implies Proposition 3.1.4 from the introduction.

Corollary 3.3.7. *Let $X=Bl_{\mathbb{C}P^{2q-1}}\mathbb{C}P^{2r+2q-1}$, and $L\subset X$ be the real Lagrangian (diffeomorphic to $Bl_{\mathbb{R}P^{2q-1}}\mathbb{R}P^{2r+2q-1}$). Assume* $\gcd(2q+1,2r-2q)\geq2$ *and that either r or q are* *odd. Then $\mathscr{CO}^*\colon QH^*(X)\to HF^*(L,L)$ is injective, although $\mathscr{CO}^0$ is not. Consequently, L split-generates $\mathscr{F}uk(X)_0$.*

Proof of Proposition 3.1.4. Take X as in (3.3) with $a_1=\ldots=a_k=1$, then $X=Bl_{\mathbb{C}P^{k-1}}\mathbb{C}P^n$, see e.g. [36, Proposition 11.14]. The additional hypotheses of the current corollary make sure X satisfies all conditions of Theorem 3.3.3(b), which together with the split-generation criterion (Theorem 3.1.9(b)) implies the corollary. □

In order to deduce non-displaceability results between the real Lagrangian L and other Lagrangians with arbitrary obstruction numbers, we need the following lemma.

Lemma 3.3.8. *Suppose* $\operatorname{char}\mathbb{K}=2$, *$L$ is an object of $\mathscr{F}uk(X)_w$ and $\mathscr{CO}^*\colon QH^*(X)\to HH^*(L,L)$ is injective. Then $L\times L$ split-generates $\mathscr{F}uk(X\times X)_0$.*

Note that by Lemma 3.1.8, the hypothesis of Lemma 3.3.8 can only hold if $QH^*(X)=QH^*(X)_w$ or L is non-orientable.

Proof. First, observe that $w(L\times L)=2w(L)=0$. By [103], the injectivity of $\mathscr{CO}^*$ is equivalent to the fact that the open-closed map $\mathscr{OC}^*\colon HH_*(L,L)\to QH^*(X)$ hits the unit $1\in QH^*(X)$. By Ganatra [52, Remark 11.1], there is a commutative diagram

$$\begin{tikzcd} CC_*(L,L)\otimes CC_*(L,L) \arrow[r] \arrow[d, "\mathscr{OC}^*\otimes\mathscr{OC}^*"] & CC_*^{split}(L\times L,L\times L) \arrow[d, "\mathscr{OC}^*_{\mathrm{prod}}"] \\ CF^*(X)\otimes CF^*(X) \arrow[r, "="'] & CF^*(X\times X) \end{tikzcd}$$

where $\mathscr{OC}^*_{\mathrm{prod}}$ is the open-closed map on the product, and CF^* are Hamiltonian Floer complexes, whose cohomology is quantum cohomology. Although [52] works with exact manifolds, wrapped Fukaya category and symplectic cohomology, the arguments required for this diagram carry over to the monotone setup. Here $CC_*^{split}(L\times L,L\times L)$ indicates that the A_∞ structure on $L\times L$ is computed using a split Hamiltonian perturbation and a product almost complex structure; such a choice can be made regular. If $\mathscr{OC}^*$ hits the unit, then $\mathscr{OC}^*\otimes\mathscr{OC}^*$ and $\mathscr{OC}^*_{\mathrm{prod}}$ also do. The latter fact implies that $\mathscr{CO}^*$ is injective on the product, and split-generation follows from Theorem 3.1.9(b). □

Corollary 3.1.5 from the introduction is a particular case of the following.

Corollary 3.3.9. *Let $\mathbb{K}$ be a field of characteristic 2 and $L\subset X$ be as in Theorem 3.3.3(a) or (b), or as in Corollary 3.3.7. Suppose $L'\subset X$ another monotone Lagrangian, perhaps equipped with a local system $\pi_1(L)\to\mathbb{K}^\times$, with minimal Maslov number at least 2 and such that $HF^*(L',L')\neq 0$. Then $L\cap L'\neq\emptyset$.*

Proof. If $w(L') = 0$, this follows from the fact L split-generates $\mathscr{F}uk(X)_0$ and Lemma 3.1.10. If $w(L') \neq 0$, we have that $w(L' \times L') = 2w(L') = 0$, so $L' \times L'$ is an object of $\mathscr{F}uk(X \times X)_0$ which is split-generated by $L \times L$ by Lemma 3.3.8. Then $(L \times L) \cap (L' \times L') \neq 0$ by Lemma 3.1.10, and so $L \cap L' \neq 0$. $\square$

3.3.3 An application to non-formality

Recall that if $\mathscr{A} \to \mathscr{A}'$ is a quasi-isomorphism of A_∞ categories, it induces an isomorphism $HH^*(\mathscr{A}) \to HH^*(\mathscr{A}')$, see e.g. Seidel [99, (1.14)]. We will need an explicit chain-level formula for this isomorphism, which can be obtained by combining Seidel's argument with Ganatra's functoriality formulas [52, Section 2.9], and this requires a short account. We are assuming the reader is familiar with the basic language of A_∞ categories from e.g. [96, 103, 52], so that we can skip some basic definitions and present the other ones rather informally. For simplicity, we are working with $\operatorname{char}\mathbb{K} = 2$ so we won't have to worry about signs, and restrict to A_∞ algebras rather than categories.

Recall that if $\mathscr{A}$ is an A_∞ algebra, its Hochschild cohomology $HH^*(\mathscr{A})$ can be seen as Hochschild cohomology $HH^*(\mathscr{A}, \mathscr{A})$ of $\mathscr{A}$ as an $\mathscr{A} - \mathscr{A}$ bimodule. If $F\colon \mathscr{A} \to \mathscr{A}'$ is a quasi-isomorphism between A_∞ algebras, it induces quasi-isomorphisms

$$CC^*(\mathscr{A}, \mathscr{A}) \xrightarrow{F_*} CC^*(\mathscr{A}, F^*\mathscr{A}') \xleftarrow{F^*} CC^*(\mathscr{A}', \mathscr{A}'), \tag{3.9}$$

which proves that $HH^*(\mathscr{A}, \mathscr{A}) \cong HH^*(\mathscr{A}', \mathscr{A}')$. Chain-level formulas for the two intermediate quasi-isomorphisms, which we will now recall, were written down e.g. by Ganatra [52, Section 2.9] (in the context of Hochschild homology, but these are easily adjusted to cohomology).

If $\mathscr{B}, \mathscr{B}'$ are two $\mathscr{A} - \mathscr{A}$ bimodules, a morphism $G\colon \mathscr{B} \to \mathscr{B}'$ is a sequence of maps $G^k\colon \mathscr{A}^{\otimes i} \otimes \mathscr{B} \otimes \mathscr{A}^{\otimes j} \to \mathscr{B}'$, $i + j + 1 = k$, satisfying a sequence of relations which we informally write down as $\sum_\star G^\star(\mathrm{Id}^{\otimes\star} \otimes \mu^\star_{\mathscr{A}\ or\ \mathscr{B}} \otimes \mathrm{Id}^{\otimes\star}) = \sum_\star \mu^\star_{\mathscr{B}'}(\mathrm{Id}^{\otimes\star} \otimes G^\star \otimes \mathrm{Id}^{\otimes\star})$. Here $\star$ are positive integers which are mutually independent but are such that the total number of inputs on both sides of the equation is the same; the sum is over all such possibilities; and the structure map on the left is $\mu^\star_{\mathscr{A}}$ or $\mu^\star_{\mathscr{B}}$ depending on whether one of its arguments is in $\mathscr{B}$. We will keep this informal style of notation, in which all valency integers are replaced by $\star$, further. The induced map $G_*\colon CC^*(\mathscr{A}, \mathscr{B}) \to CC^*(\mathscr{A}, \mathscr{B}')$ is defined by

$$(G_*(h))^\star = \sum\nolimits_\star G^\star(\mathrm{Id}^{\otimes\star} \otimes h^\star \otimes \mathrm{Id}^{\otimes\star}) \tag{3.10}$$

where $h^\star\colon \mathscr{A}^{\otimes\star} \to \mathscr{B}$ and $(G_*(h))^\star\colon \mathscr{A}^{\otimes\star} \to \mathscr{B}'$. If G is a quasi-isomorphism, so is G_*.

If $\mathscr{A}, \mathscr{A}'$ are two A_∞ algebras, a morphism $F\colon \mathscr{A} \to \mathscr{A}'$ is a sequence of maps $F^\star\colon \mathscr{A}^{\otimes\star} \to \mathscr{A}'$ such that $\sum_\star \mu^\star_{\mathscr{A}'}(F^\star \otimes \ldots \otimes F^\star) = \sum_\star F^\star(\mathrm{Id}^{\otimes\star} \otimes \mu^\star_{\mathscr{A}} \otimes \mathrm{Id}^{\otimes\star})$. Next, if $\mathscr{B}$ is an $\mathscr{A}' - \mathscr{A}'$ bimodule, its two-sided pull-back $F^*\mathscr{B}$ is an $\mathscr{A} - \mathscr{A}$ bimodule based on the same vector space $\mathscr{B}$, whose structure maps are [52, Section 2.8]

$$\mu^\star_{F^*\mathscr{B}} = \sum\nolimits_\star \mu^\star_{\mathscr{B}}(F^\star \otimes \ldots \otimes F^\star \otimes \mathrm{Id}_{\mathscr{B}} \otimes F^\star \otimes \ldots \otimes F^\star) \tag{3.11}$$

There is also a morphism $F^*\colon CC^*(\mathscr{A}', \mathscr{B}) \to CC^*(\mathscr{A}, F^*\mathscr{B})$ defined by

$$(F^*(h))^\star = \sum\nolimits_\star h^\star(F^\star \otimes \ldots \otimes F^\star) \tag{3.12}$$

where $h^\star\colon (\mathscr{A}')^{\otimes\star} \to \mathscr{B}$ and $(F^*(h))^\star\colon \mathscr{A}^{\otimes\star} \to \mathscr{B}$. The total number of inputs here can be zero, and $F^*(h)^0 = h^0$. If F is a quasi-isomorphism, so is F^*.

If, again, $F\colon \mathscr{A} \to \mathscr{A}'$ is a morphism of A_∞ algebras, let $F^*\mathscr{A}'$ be the $\mathscr{A} - \mathscr{A}$ bimodule which is the pull-back of $\mathscr{A}'$ seen as an $\mathscr{A}' - \mathscr{A}'$ bimodule.

Lemma 3.3.10. *The same sequence of maps $F^\star\colon \mathscr{A}^{\otimes\star} \to \mathscr{A}'$ provides a morphism of $\mathscr{A} - \mathscr{A}$ bimodules $\mathscr{A} \to F^*\mathscr{A}'$, also denoted by F.*

Proof. We must check that $\sum_\star F^\star(\mathrm{Id}^{\otimes\star} \otimes \mu^\star_{\mathscr{A}} \otimes \mathrm{Id}^{\otimes\star}) = \sum_\star \mu^\star_{F^*\mathscr{A}}(\mathrm{Id}^{\otimes\star} \otimes F^\star \otimes \mathrm{Id}^{\otimes\star})$. If we apply formula (3.11) to rewrite the right-hand sum, the unique Id-factor in (3.11), which in our case is $\mathrm{Id}_{\mathscr{A}'}$, gets applied to the $F^\star$-factor. So our right-hand sum equals $\sum_\star \mu^\star_{\mathscr{A}'}(F^\star \otimes \ldots \otimes F^\star \otimes \ldots \otimes F^\star)$ which is exactly the condition that F is a morphism of A_∞ algebras $\mathscr{A} \to \mathscr{A}'$. □

This lemma explains the precise meaning of (3.9): if $F\colon \mathscr{A} \to \mathscr{A}'$ is a morphism of A_∞ algebras, then the first map F_* from (3.9) is the push-forward of F considered as a morphism of modules $\mathscr{A} \to F^*\mathscr{A}'$, given by formula (3.10). The second map in (3.9) is the pull-back as in (3.12). Next, if $\mathscr{F}\colon \mathscr{A} \to \mathscr{A}'$ is a quasi-isomorphism, then the $\mathscr{A} - \mathscr{A}$ bimodule morphism F from Lemma 3.3.10 is also a quasi-isomorphism, hence there is an $\mathscr{A} - \mathscr{A}$ bimodule quasi-isomorphism $G\colon F^*\mathscr{A}' \to \mathscr{A}$ which is the cohomological inverse of F, so we have quasi-isomorphisms:

$$CC^*(\mathscr{A}, \mathscr{A}) \xleftarrow{G_*} CC^*(\mathscr{A}, F^*\mathscr{A}') \xleftarrow{F^*} CC^*(\mathscr{A}', \mathscr{A}'). \tag{3.13}$$

Their composition acts on Hochschild cochains by:

$$(G_*F^*(h))^\star = \sum\nolimits_\star G^\star(\mathrm{Id}^{\otimes\star} \otimes h^\star(F^\star \otimes \ldots \otimes F^\star) \otimes \mathrm{Id}^{\otimes\star})$$

where $h^\star\colon (\mathscr{A}')^{\otimes\star} \to \mathscr{A}'$ and $(G_*F^*(h))^\star\colon \mathscr{A}^{\otimes\star} \to \mathscr{A}$. In particular, $(G_*F^*(h))^0 = G^1(h^0)$, and if $h^0 = 0 \in \mathscr{A}'$ then

$$(G_*F^*(h))^1(u) = G^1(h^1(F^1(u))), \quad u \in \mathscr{A}'. \tag{3.14}$$

Note that $G^1\colon \mathscr{A}' \to \mathscr{A}$, $F^1\colon \mathscr{A} \to \mathscr{A}'$ are chain maps with respect to $\mu^1_{\mathscr{A}}$, $\mu^1_{\mathscr{A}'}$ and are cohomology inverses of each other.

Assume $L \subset X$ is a Lagrangian preserved by a Hamiltonian loop γ which together satisfy the conditions of Theorem 3.1.7 (those conditions which are common to all parts of the theorem). Assume the A_∞ algebra $CF^*(L,L)$ is formal, i.e. there is an A_∞ quasi-isomorphism $F\colon HF^*(L,L) \to CF^*(L,L)$. Denote

$$h := (G_*F^*)(S(\gamma)-1) \in CC^*(HF^*(L,L), HF^*(L,L)), \tag{3.15}$$

where G is the cohomological inverse of F, and G_*, F^* are as in (3.13). So h is a Hochschild cochain for the accociative algebra $HF^*(L,L)$. Then by Corollary 3.2.2, $h^0 = 0$, Proposition 3.2.6 and equation (3.14), $h^1(x) = \rho(l) \cdot \langle \Psi(F^1(x)), l\rangle \cdot G^1(1_L)$. Let us additionally assume that L is wide, i.e. there is a vector space isomorphism between $H^*(L)$ and $HF^*(L,L)$, and that L admits a perfect Morse function. These conditions enable us to identify $CF^*(L,L) \cong HF^*(L,L)$ as vector spaces. Because G^1 is cohomologically unital, $G^1(1_L) = 1_L \in HF^*(L,L)$, so

$$h^1(x) = \rho(l) \cdot \langle \Psi(F^1(x)), l\rangle \cdot 1_L \in HF^*(L,L). \tag{3.16}$$

Under our identifications, Ψ becomes an isomorphism between the vector spaces below, and F^1 can be considered as algebra isomorphism from $HF^*(L,L)$ to itself:

$$HF^*(L,L) \xrightarrow[\text{algebra iso.}]{F^1} HF^*(L,L) \xrightarrow[\text{v. space iso.}]{\Psi} H^*(L). \tag{3.17}$$

We now turn the discussion to Hochschild cohomology of monic algebras. Let $f(u) \in \mathbb{K}[u]$ be a polynomial and $A := \mathbb{K}[u]/(f)$ be the quotient algebra, called a monic algebra. This is an algebra in the ordinary associative sense, but we can also consider it as an A_∞ algebra by equipping it with trivial higher structure maps. The Hochschild cohomology algebra $HH^*(A)$ was computed by Holm [56]. Recall that Hochschild cohomology of ungraded associative algebras is $\mathbb{Z}$-graded (unlike Hochschild cohomology of non-$\mathbb{Z}$-graded A_∞ algebras): cochains $A^{\otimes k} \to A$ are said to have degree k, and the differential has degree 1.

By [56, Proposition 2.2],

$$HH^k(A) = \begin{cases} A, & \text{if } k = 0 \\ Ann_A(f') & \text{if } k > 0 \text{ is odd} \\ A/(f') & \text{if } k > 0 \text{ is even.} \end{cases}$$

For example, an explicit computation shows that every Hochschild cocycle $h \in CC^1(A)$, i.e. $h\colon A \to A$, must be of form

$$h(u^m) = amu^{m-1}, \quad \text{for a fixed } a \in A. \tag{3.18}$$

So any cocycle $h \in CC^1(A)$ is completely determined by a single element $a = h(u) \in A$, and we must meet an additional condition that $h(f(u)) = h(0) = 0$, which is equivalent to $a \in Ann_A(f')$. As the differential $CC^0(A) \to CC^1(A)$ vanishes, we get an isomorphism $HH^1(A) \to Ann_A(f')$, $h \mapsto h(u)$.

We will further assume that $\operatorname{char}\mathbb{K} = 2$ and $f' = 0$. The latter condition means that f is a sum of even powers of u. Denote by

$$\psi\colon\ HH^1(A) \to A$$

the isomorphism $\phi(h) = h(u)$ from above. Note that if $s(u) \in A$ is an arbitrary element given by a polynomial with derivative $s'(u)$, then by (3.18) we get

$$\psi(h) = s'(u) \cdot h(s(u)). \tag{3.19}$$

For $k > 1$, we also have isomorphisms $\psi\colon\ HH^k(A) \to A$, all of which we denote by the same letter by abusing notation; we will not need an explicit formula for these isomorphisms when $k > 1$.

Moreover, [56, Lemma 5.1] computes the Yoneda product on $HH^*(A)$. In particular, given $h_1, h_2 \in HH^1(A)$, their Yoneda product $h_1 \star h_2$ is determined by

$$\psi(h_1 \star h_2) = \psi(h_1) \cdot \psi(h_2) \textstyle\sum_{j\, odd} f_{2j} u^{2j-2} \in A, \tag{3.20}$$

where $f = \sum_j f_j u^j$, $f_j \in \mathbb{K}$.

The two strands of discussion can be combined in the following theorem.

Theorem 3.3.11. *Let $\mathbb{K}$ be a field of characteristic 2, $L \subset X$ a Lagrangian preserved by a Hamiltonian loop γ which together satisfy the conditions of Theorem 3.1.7 (those conditions* *which are common to all parts of the theorem). Assume there is an algebra isomorphism $HF^*(L,L) \cong \mathbb{K}[u]/(f)$ where $f(u) = \sum_{j\geq 0} f_j u^j$ is a polynomial, and also that L is wide and admits a perfect Morse function, so that we can identify the vector spaces $HF^*(L,L) \cong CF^*(L,L)$, and $\Psi\colon HF^*(L,L) \to H^*(L)$ becomes an isomorphism of vector spaces. Further, assume:*

- *$f' = 0$, and $\sum_{j\, odd} f_{2j} u^{2j-2}$ is invertible in $\mathbb{K}[u]/(f)$;*
- *$\langle \Psi(r(u)), l\rangle = 1$ for an element $r(u) \in \mathbb{K}[u]/(f) \cong HF^*(L,L)$ which generates $HF^*(L,L)$ as an algebra;*
- *$S(\gamma)^2 = 1 \in QH^*(X)$.*

Then the Fukaya A_∞ algebra of L is not formal over $\mathbb{K}$.

Proof. Supposing $CF^*(L,L)$ is formal, let h be as in (3.15) and F^1 be as in (3.17). Then there exists $s(u) \in HF^*(L,L)$ (we view this element as a polynomial in $\mathbb{K}[u]/(f)$) such that $F^1(s(u)) = r(u)$. Then by (3.16), $h^1(s(u)) = \rho(l)\cdot 1 \in HF^*(L,L)$, so by (3.19), $\psi(h^1) = \rho(l)\cdot s'(u) \in HF^*(L,L)$. Further, note that $h\star h = 0$ because $(S(\gamma)+1)^2 = 0$, so (3.20) yields $\rho(l)^2\cdot(s'(u))^2 \sum_{j\, odd} f_{2j}u^{2j-2} = 0 \in HF^*(L,L)$. By hypothesis, this implies $(s'(u))^2 = 0$, so $s'(u) \in \ker\mathscr{F}$ where $\mathscr{F}\colon \mathbb{K}[u]/(f) \to \mathbb{K}[u]/(f)$ is the Frobenius endomorphism. In general, over $\operatorname{char}\mathbb{K} = 2$ it is always true that $s'(u)$ is a sum of even powers of u, so $s'(u)$ is a square of another polynomial: $s'(u) = (t(u))^2$. Then $t(u)^2 \in \ker\mathscr{F}$, which implies $t(u) \in \ker\mathscr{F}$ because $\ker\mathscr{F}$, being an ideal in $\mathbb{K}[u]/(f)$, is necessarily prime. Consequently, $s'(u) = 0$. So $s(u)$ is a sum of even powers of u, hence the subalgebra generated by $s(u)$ lies in the subalgebra of $\mathbb{K}[u]/(f)$ generated by u^2, which is smaller than the whole $\mathbb{K}[u]/(f)$: for example, it does not contain the element u. (Recall that f is also a sum of even powers of u.) On the other hand, we know that F^1 is an algebra isomorphism, $F^1(s(u)) = r(u)$ and $r(u)$ generates the whole $HF^*(L,L)$ by hypothesis, so $s(u)$ should also generate $HF^*(L,L)$, which is a contradiction. □

Proof of Proposition 3.1.3. Take the real loop γ preserving $\mathbb{R}P^{4n+1}$ defined in the proof of Proposition 3.1.1 (see Section 3.1), and denote $L = \mathbb{R}P^{4n+1}$, $X = \mathbb{C}P^{4n+1}$. Recall that, if $x \in H^2(X)$ is the generator, then $QH^*(X) \cong \mathbb{K}[x]/(x^{4n+2}+1)$ and $S(\gamma) = x^{2n+1}$, so $S(\gamma)^2 = 1$. Also recall that $l \in H_1(L) \cong \mathbb{K}$ is non-zero. By Theorem 3.1.11, we have $HF^*(L,L) \cong \mathbb{K}[u]/(u^{4n+2}+1)$ where $u \in CF^1(L,L) \cong \mathbb{K}$, and one sees that $\langle\Psi(u), l\rangle = 1$. Now apply Theorem 3.3.11 taking $r(u) = u$ to conclude the proof. □

Proposition 3.3.12. *Let $X = Bl_{\mathbb{C}P^{2q-1}}\mathbb{C}P^{2r+2q-1}$, $L \subset X$ be the real Lagrangian (diffeomorphic to $Bl_{\mathbb{R}P^{2q-1}}\mathbb{R}P^{2r+2q-1}$). Assume that* $\gcd(2q+1, 2r-2q) \geq 2$ *and that either r or q are odd. Then the A_∞ algebra of L is not formal over a characteristic 2 field.*

Proof. We recall that all real Lagrangians are wide by Theorem 3.1.11 and admit a perfect Morse function by [54]. The fact that $\gcd(2q+1,2r-2q)\geq 2$ means we are in the situation of Theorem 3.3.3(b), with $k=2q=2p$, $a_1=\ldots=a_k=1$. We have already seen (3.7) that $HF^*(L,L)\cong\mathbb{K}[u]/(f)$ with $f'=0$, and it is easy to check that $\sum_{j\ odd} f_{2j}u^{2j-2}$ is invertible provided that either r or q is odd (otherwise this element would vanish). Moreover, via (3.6) and Haug's isomorphism (Theorem 3.1.11), $u^{p/g}$ corresponds to the generator of $CF^1(L,L)\cong\mathbb{K}^2$ such that $\langle\Psi(u),l\rangle=1$. Now apply Theorem 3.3.11 taking $r(u)=u^{p/g}$. $\square$

3.3.4 Non-formality of the equator on the sphere

Proposition 3.1.3 says in particular that the A_∞ algebra of an equatorial circle on S^2 is not formal over $\operatorname{char}\mathbb{K}=2$. This is an especially simple case which can be verified by hand, and it is worth discussing it in more detail. Let $L_1\subset S^2$ be a fixed equator, and $L_2,L_3,\ldots$ be a sequence of its small Hamiltonian perturbations; assume $|L_i\cap L_j|=2$ for each i,j. Then $CF^0(L_i,L_j)\cong\mathbb{K}$ is generated by an element which we denote by 1 (this is the cohomological unit), and $CF^1(L_i,L_j)\cong\mathbb{K}$ is generated by an element which we denote by u (we use the same letter for all i,j). Of the two intersection points $L_i\cap L_j$, the point u is the one at which T_uL_j is obtained from T_uL_i by a small positive rotation with respect to the ω-induced orientation on S^2. Consider the A_∞ structure maps between the consequtive Lagrangians:

$$\mu^k\colon\ CF^*(L_k,L_{k+1})\otimes\ldots\otimes CF^*(L_1,L_2)\to CF^*(L_1,L_{k+1}) \tag{3.21}$$

given by counting immersed polygons as in [96, 98]. These define an A_∞ algebra structure of L, because all the L_i differ small perturbations and we can canonically identify the spaces $CF^*(L_i,L_{i+1})$ with each other. The structure maps will depend on the particular arrangement of the L_i, although up to quasi-isomorphism they give the same A_∞ algebra.

Remark 3.3.1. The fact the A_∞ algebra of L defined using the count of polygons is quasi-isomorphic to the one defined using Hamiltonian perturbations seems not to have been written down in detail but is widely accepted. An approach is sketched in [98, Remark 7.2], and also performed in [102] in a slightly different setup.

Let us compute some of the A_∞ structure maps using a specific choice of the L_i. Fix a Hamiltonian H whose flow is the rotation of $S^2\subset\mathbb{R}^3$ around an axis which is not orthogonal to the plane intersecting S^2 along the equator L_1. Let $L_2,L_3,\ldots$ be obtained from L_1 by applying that rotation by small but consequtively increasing angles, i.e. L_i are time-t_i push-offs of L_1 under the flow of H, $0=t_1<t_2<t_3\ldots$. The first four resulting circles L_i are represented in Figure 3.8(a). The pairwise intersections of the L_i are contained in two opposite patches of the sphere; those patches are shown in the top and bottom of Figure 3.8(a)

together with the L_i on them, which are depicted by straight lines. Both patches are drawn as if we look at them from the same point "above" the sphere, so that the positive rotation (with respect to the orientation on S^2) is counter-clockwise on the upper patch and clockwise on the lower patch. For this particular choice of perturbations, and for each $i < j$, all degree-one points $u \in CF^*(L_i, L_j)$ are located on the upper patch, and all points $1 \in CF^*(L_i, L_j)$ are on the lower patch.

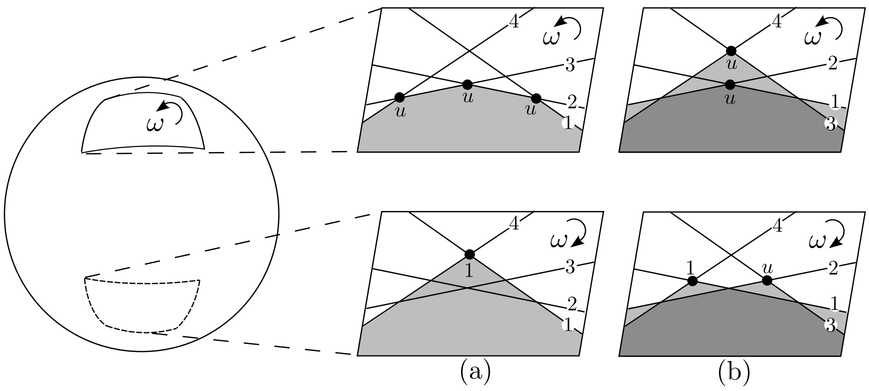


Fig. 3.8 Two different configurations (a) and (b) consisting of four small Hamiltonian push-offs $L_1, \ldots, L_4$ (marked by numbers) of an equatorial circle on S^2. The image of the disk contributing to $\mu^3(u,u,u) = 1$ is shaded.

We claim that in this model we get:

$$\mu^3(u,u,u) = 1,\ \mu^3(u,u,1) = 0,\ \mu^3(u,1,u) = 0,\ \mu^3(1,u,u) = 0.$$

For grading reasons, $\mu^k(u,\ldots,u)$ is a multiple of 1, and is determined by counting Maslov index 2 disks. There is a unique such disk; for $k = 3$ it is shown in gray shade in Figure 3.8(a) on the two patches; away from the patches this disc is just a strip between L_1 and L_4. Also for grading reasons, the only other products which can possible be non-trivial are $\mu^k(u,\ldots,u,1,u\ldots,u) \in \{0,u\}$, where exactly one input is 1. It possible to check that these vanish for our configuration of the circles L_i, at least when $k = 3$. Now note that

$$\mu^2(1+u,1+u) = 0, \quad \mu^3(1+u,1+u,1+u) = 1.$$

The latter equality exhibits a non-trivial Massey product, seen as an element of $\mathbb{K}[u]/(1+u) \cong \mathbb{K}$. The presence of a non-trivial Massey product is invariant under quasi-isomorphisms. To see this, recall that the analogous fact for dg algebras is easy, and any A_∞ algebra is quasi-isomorphic to a dg algebra. Moreover, the Massey products for the A_∞ and dg models satisfy a simple relation [71, Theorem 3.1 and Corollary A.5], in particular, if triple Massey products of an A_∞ algebra are non-trivial, they remain non-trivial for its dg-model. This gives us an alternative proof of the fact that the A_∞ algebra of the equator on S^2 is not formal.

For any other arrangement of the L_i, we will necessarily have $\mu^3(1+u,1+u,1+u)=1$ modulo $1+u$ because of invariance of Massey products, meaning that $\mu^3(1+u,1+u,1+u)\in\{1,u\}$. For example, another possible configuration of $L_1,\dots,L_4$ is shown in Figure 3.8(b); it is simply obtained from the earlier configuration by changing the ordering of the L_i. In this new model, the maps μ^k from (3.21) are now:

$$\mu^3(u,u,u)=1,\ \mu^3(u,u,1)=0,\ \mu^3(u,1,u)=u,\ \mu^3(1,u,u)=u.$$

The unique disk contributing to $\mu^3(u,u,u)$ is shown in Figure 3.8(b) by gray shade. It is an immersed disk, and the domain over which it self-overlaps has darker shade. Note that the degree-one generators $u\in CF^1(L_1,L_2)$, $CF^1(L_3,L_4)$, $CF^1(L_1,L_4)$ correspond to the intersection points on the upper patch, and the degree-one generator $u\in CF^1(L_2,L_3)$ corresponds to the intersection point on the lower patch. We see that we again get $\mu^3(1+u,1+u,1+u)=u$.

The existence of the Massey product above crucially required $\operatorname{char}\mathbb{K}=2$, because otherwise we would not get $\mu^2(1+u,1+u)=0$, which is necessary to speak of the triple Massey product of $1+u$ with itself. If $\operatorname{char}\mathbb{K}\neq 2$, then $HF^*(L_1,L_1)\cong\mathbb{K}[u]/(u^2-1)\cong\mathbb{K}[u]/(u-1)\oplus\mathbb{K}[u]/(u+1)$ is a direct sum of fields, whose Hochschild cohomology as an ordinary algebra vanishes except in degree zero [59], in contrast to the case $\operatorname{char}\mathbb{K}=2$. So any A_∞ algebra over $\mathbb{K}[u]/(u-1)\oplus\mathbb{K}[u]/(u+1)$ is formal by [58] or [100, Section 3], in particular the A_∞ algebra of the equator on S^2 is formal. For example, the product $\mu^3(1+u,1+u,1+u)$ can be made to vanish after a formal diffeomorphism. Because of the non-trivial Massey product in characteristic 2, such a formal diffeomorphism, say over $\mathbb{Q}$, will necessarily involve division by 2, and cannot be realised by any geometric choice of the push-offs L_i.

In comparison, the topological A_∞ algebra of the circle is formal over a field of any characteristic. Indeed, the topological A_∞ algebra is $\mathbb{Z}$-graded, so if we make this algebra to be based on the cohomology ring $H^*(S^1)\cong\mathbb{K}[x]/x^2$ where $|x|=1$, the only possibly non-trivial products will be $\mu^k(x,\dots,x,1,x,\dots,x)$ for grading reasons. On the other hand, every A_∞ algebra is quasi-isomorphic to a strictly unital one over a field of any characteristic [96, Lemma 2.1], in which those products vanish by definition when $k\geq 3$.

3.4 The closed-open map for monotone toric fibres

3.4.1 The mechanism of Theorem 3.1.7 for toric fibres

Let X be an n-dimensional compact toric Fano variety, and $T \subset X$ the unique monotone toric fibre. Evans and Lekili [40] proved (after this work had appeared in the form of the preprint [108]) that the Fukaya category $\mathscr{F}uk(X)_w$ is split-generated by several copies of T, equipped with the local systems corresponding to the critical points of the Landau-Ginzburg superpotential with critical value $w \in \mathbb{K}$. (We will recall the definition of the superpotential for toric varieties in the next subsection; the common references are [30, 45].)

Prior to that, split-generation by toric fibres had been proved only in the case when the superpotential is Morse, see Ritter [86]. (For Ritter, proving split-generation requires considerable effort even in the Morse case, if W has several critical points with the same critical value. However, the difficulty is mainly related to the fact that he allows some non-compact toric varieties, where the injectivity of $\mathscr{C}\mathscr{O}^*$ is no longer a criterion for split-generation and one must look at $\mathscr{O}\mathscr{C}^*$ instead. If we work with compact manifolds, checking that $\mathscr{C}\mathscr{O}^*$ is injective for an arbitrary Morse potential is easy: see Corollary 3.4.3). An example of a toric Fano variety with non-Morse superpotential over $\mathbb{C}$ has been obtained by Ostrover and Tyomkin [82], and one can check that the superpotential in their case has an A_3 singularity.

To complete the literature overview, we should mention the work in progress by Abouzaid, Fukaya, Oh, Ohta and Ono [2] that will prove the split-generation result for toric manifolds that are not necessarily Fano.

Because the toric fibre T is invariant under all the Hamiltonian loops coming from the torus action, it is an obvious example where Theorem 3.1.7 can be put to the test. It turns out that it does allow to prove split-generation away from the Morse case, though not too far from it: the superpotential is required to have at worst A_2 singularities, and an extra condition $\operatorname{char}\mathbb{K} \neq 2, 3$ is required, see Corollary 3.4.5.

The idea behind our approach is the observation that the ability to solve equation $(*)$ from Theorem 3.1.7 depends on whether W is Morse or not. Equip T with a local system ρ which corresponds to a critical point of W; then (T, ρ) is wide, which means we can identify the vector spaces $HF^*(T, \rho) \cong H^*(T)$ via the PSS map Φ. For convenience, let us rewrite equation $(*)$:

$$\mu^2(a,y) + \mu^2(y,a) = \rho(l) \cdot \langle y, l \rangle \cdot 1_T \quad \text{for each} \quad y \in H^1(T). \quad (*)$$

Recall that Theorem 3.1.7(b) can be applied if there exists no $a \in HF^*(T,\rho)$ making $(*)$ hold. The Floer cohomology algebra of (T,ρ) is a Clifford algebra determined by the Hessian of W at the point ρ, so the left-hand side of $(*)$ is equal to $\mathrm{Hess}_\rho W(a,y)\cdot 1_T$, at least when $a \in H^1(T)$; we are using informal notation for the moment. Therefore, finding an element a solving $(*)$ reduces to finding an a such that

$$\mathrm{Hess}_\rho W(a,-) = \mathrm{const}\cdot\langle -,l\rangle. \tag{3.22}$$

The ability to find such an a depends on how degenerate $\mathrm{Hess}_\rho W$ is. If ρ is a Morse point of W, such an a can always be found, so Theorem 3.1.7(b) does not apply. However, the Morse case can actually be covered by Theorem 3.1.7(a), as we explain below. On the other hand, when $\mathrm{Hess}_\rho W$ has kernel, we will have some elements $l \in H_1(T)$ for which equation (3.22) has no solution a. If we consider the S^1-action whose orbit is such an element l, Theorem 3.1.7(b) can be applied to the Seidel element of this S^1-action to reveal some new information on $\mathscr{CO}^*$ which is not seen by $\mathscr{CO}^0$. This information turns out to be sufficient only when the superpotential has A_2 singularities, however, there is a possible way of improvement which we speculate upon in the end of this section.

3.4.2 The results

Recall that the Landau-Ginzburg superpotential of X is a Laurent polynomial $W\colon\ (\mathbb{K}^\times)^n \to \mathbb{K}$ is given by

$$W(x_1,\dots,x_n) = \sum_e \sum_{j=1}^n x_j^{e^j}$$

where the first sum is over the outer normals $e \in \mathbb{Z}^n$ to the facets of the polyhedron defining X, and $e^j \in \mathbb{Z}$ are their co-ordinates. (More commonly, the superpotential is written down with a Novikov parameter, but we can ignore it because we will only be working with the monotone torus T.) We identify $(\mathbb{K}^\times)^n$ with the space of all local systems $H_1(T;\mathbb{Z}) \to \mathbb{K}^\times$. For $\rho \in (\mathbb{K}^\times)^n$, we write (T,ρ) for the torus equipped with this local system. Also, we will abbreviate $HF^*(T,\rho) = HF^*((T,\rho),(T,\rho))$, and the same for Hochschild cohomology. It is known, see for example [82, Proposition 3.3], that

$$QH^*(X) \cong \mathbb{K}[x_1^{\pm 1},\dots,x_n^{\pm 1}]/Jac(W) = \mathscr{O}(Z), \tag{3.23}$$

where the Jacobian ideal $Jac(W)$ is generated by $(\partial W/\partial x_1,\dots,\partial W/\partial x_n)$, and Z is the subscheme of $\mathrm{Spec}\,\mathbb{K}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ defined by the ideal sheaf $Jac(W)$. (The available references typically prove this isomorphism over $\mathbb{C}$, but one can track that the proofs work over $\mathbb{Z}$ and hence over finite fields.) Then Z is a 0-dimensional scheme supported at the critical points of W,

$$\{\rho_1, \dots \rho_q\} = \operatorname{Crit} W, \quad \rho_i \in (\mathbb{K}^\times)^n.$$

The obstruction number of the torus is given by

$$w(T,\rho) = W(\rho).$$

Under the isomorphism (3.23), the quantum product is the usual product on $\mathscr{O}(Z)$, and the first Chern class of X is given by the function W itself. The generalised eigenspace decomposition with respect to $- * c_1(X)$ is simply the decomposition into the local rings at the points ρ_i:

$$\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/Jac(W) \cong \bigoplus_{\rho_i \in \operatorname{Crit} W} \mathscr{O}_{\rho_i}(Z),$$

the eigenvalue of the ρ_i-summand being the critical value $W(\rho_i)$. From Lemma 3.1.8, we see that $HF^*(T,\rho) = 0$ if $\rho \notin \operatorname{Crit} W$. On the other hand, it is known that (T,ρ_i) is wide for $\rho_i \in \operatorname{Crit} W$, i.e. $HF^*(T,\rho_i)$ is isomorphic as a vector space to $H^*(T)$.

Lemma 3.4.1. *Under the isomorphism (3.23), the map $\mathscr{CO}^0\colon QH^*(X) \to HF^*(T,\rho_i)$ is given by*

$$\mathscr{CO}^0(f) = f(\rho_i) \cdot 1_T.$$

Here $f(x_1, \dots x_n) \in QH^*(X)$, $f(\rho_i) \in \mathbb{K}$ is the value of the function at $\rho_i \in \operatorname{Crit} W$, and $1_T \in HF^*(T,\rho_i)$ is the unit.

Proof. Because $\mathscr{CO}^0$ is a map of algebras, it suffices to prove the lemma when $f = x_k$ is a linear function, $1 \le k \le n$. By [74], $f = S(\gamma)$ for a Hamiltonian loop γ coming from the Hamiltonian torus action, such that the value of the local system ρ_i on an orbit of γ equals the kth co-ordinate ρ_i^k, which is the same as the value $f(\rho_i)$. So $\mathscr{CO}^0(f) = f(\rho_i) \cdot 1_T$ by Theorem 3.1.7(a). □

Corollary 3.4.2. *For $\rho_i \neq \rho_j \in \operatorname{Crit} W$, the map $\mathscr{CO}^*|_{\mathscr{O}_{\rho_i}(Z)} \to HH^*(T,\rho_j)$ vanishes.*

Proof. Let $f \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be such that $f(\rho_i) \neq 0$ and $f(\rho_j) = 0$. Then, as an element of $\mathscr{O}_{\rho_i}(Z)$, f is invertible. If the corollary does not hold, $\mathscr{CO}^*(f)$ is also invertible. On the other hand, $\mathscr{CO}^0(f) = 0 \in HF^*(T,\rho_j)$ by Lemma 3.4.1. The map $HH^*(T,\rho_j) \to HF^*(T,\rho_j)$, which takes a Hochschild cochain to its zeroth-order term, is a map of unital algebras, by the formula for the Yoneda product and because the Hochschild cohomology unit is represented by a cochain whose zeroth-order term is the Floer cohomology unit (this follows, for example, from the unitality of $\mathscr{CO}^*$). We have determined that f lies in the kernel of

$HH^*(T,\rho_j) \to HF^*(T,\rho_j)$, but that contradicts the fact that f is invertible. This implies the corollary. $\square$

For $w \in \mathbb{K}$, denote
$$\mathrm{Crit}_w W = \{\rho \in \mathrm{Crit}\, W : \ W(\rho) = w\}$$
the set of all critical points of W with the same critical value w. We will sometimes denote the restrictions of $\mathscr{C}\mathscr{O}^0$ and $\mathscr{C}\mathscr{O}^*$ to subalgebras of $QH^*(X)$ by the same symbol, when it is otherwise clear that we are considering a restriction.

Corollary 3.4.3. *If* $\mathrm{char}\, \mathbb{K} \neq 2$*, the map*
$$\mathscr{C}\mathscr{O}^0 : \ QH^*(X)_w \longrightarrow \bigoplus_{\rho_i \in \mathrm{Crit}_w W} HF^*(T,\rho_i)$$
is injective if and only if all points of $\mathrm{Crit}_w W$ *are Morse.*

Proof. By Corollary 3.4.2, $\mathscr{C}\mathscr{O}^0$ is injective if and only if its restrictions $\mathscr{C}\mathscr{O}^0 : \ \mathscr{O}_{\rho_i}(Z) \to HF^*(T,\rho_i)$ are injective for each ρ_i. The map $\mathscr{O}_{\rho_i}(Z) \to \mathbb{K}$ which takes $f \in \mathscr{O}_{\rho_i}(Z)$ to its value $f(\rho_i)$ is injective if and only if $\mathscr{O}_{\rho_i}(Z)$ is a field, which is equivalent to the fact that ρ_i is a Morse point of W when $\mathrm{char}\, \mathbb{K} \neq 2$. Now apply Lemma 3.4.1. $\square$

Proposition 3.4.4. *Suppose* $\mathrm{char}\, \mathbb{K} \neq 2,3$ *and* W *has an* A_2 *singularity at a point* ρ*, then* $\mathscr{C}\mathscr{O}^* : \ \mathscr{O}_\rho(Z) \to HH^*(T,\rho)$ *is injective.*

Proof. After an integral linear change of co-ordinates, we may assume that the Hessian of W at ρ is the diagonal matrix: $\mathrm{Hess}_\rho W = \mathrm{diag}(1,\dots,1,0)$. We claim that $\mathscr{O}_\rho(Z)$ is generated, as a vector space, by the two elements 1 and x_n, where the linear function x_n corresponds to the kernel of $\mathrm{Hess}_\rho W$. Indeed, after a further non-linear change of coordinates with the identity linear part, we can bring W to the canonical form
$$W(\tilde{x}_1,\dots,\tilde{x}_n) = W(\rho) + \sum_{i=1}^{n-1} (\tilde{x}_s - \rho^i)^2 \ + (\tilde{x}_n - \rho^n)^3.$$
Here $\rho^i \in \mathbb{K}$ are the co-ordinates of ρ. Then $Jac(W) = ((\tilde{x}_1-\rho^1),\dots,(\tilde{x}_{n-1}-\rho^{n-1}),(\tilde{x}_n-\rho^n)^2)$, so $\mathscr{O}_\rho(Z)$ is generated, as a vector space, by 1 and $\tilde{x}_n$. Because x_n, as a function of $\tilde{x}_1,\dots,\tilde{x}_n$, equals $\tilde{x}_n$ plus terms of order at least 2, it is easy to see that the elements $1, x_n$ also generate the vector space $\mathscr{O}_\rho(Z)$.

Let us identify $HF^*(T,\rho)$ with $H^*(T)$ via the PSS map Φ. Recall that, in general, $HF^*(T,\rho)$ is the algebra generated by $y_1,\dots,y_n \in H^1(T)$ with relations
$$y_p y_q + y_q y_p = \partial^2_{x_p x_q} W(\rho).$$

In particular, in our case we get $y_p y_n + y_n y_p = 0$ for any $1 \leq p \leq n$, so $y_n \in HF^1(T,\rho)$ anti-commutes with any element of $HF^*(T,\rho)$ of odd degree. Consequently, the left-hand side of equation $(*)$ from Theorem 3.1.7 vanishes if we put $y = y_n$, and allow a to be of arbitrary odd degree.

Returning to our generator $x_n \in \mathscr{O}_\rho(Z)$, we have $x_n = S(\gamma)$ for a Hamiltonian S^1-action (coming from the toric action) such that the element $y_n \in HF^1(T,\rho)$ is dual to the orbit $l \in H_1(T)$ of γ, so that $\langle y_n, l\rangle = 1$. So if we put $y = y_n$, the right-hand side of equation $(*)$ from Theorem 3.1.7 becomes $\rho^n \cdot 1_T \neq 0$. Hence $(*)$ has no solution, and Theorem 3.1.7(b) says that $\mathscr{C}\mathscr{O}^*(x_n)$ and $1_{HH} = \mathscr{C}\mathscr{O}^*(1)$ are linearly independent. □

Combining the above discussion with the split-generation criterion, we get the following corollary.

Corollary 3.4.5. *Suppose* $\operatorname{char}\mathbb{K} \neq 2,3$ *and each critical point* $\rho_i \in \mathrm{Crit}_w W$ *is either Morse or an* A_2 *singularity. Then the copies of the monotone toric fibre with the local systems* $\{(T,\rho_i)\}_{\rho_i \in \mathrm{Crit}_w W}$ *split-generate* $\mathscr{F}uk(X)_w$. □

3.4.3 A way of extending Theorem 3.1.7

It is in fact not surprising that Theorem 3.1.7 turned out to be efficient only for A_2 singularities. The main result on which Theorem 3.1.7 is based upon is Proposition 3.2.6, which computes the *linear* part $\mathscr{C}\mathscr{O}^1$ of the closed-open map, while the only non-Morse singularity whose local Jacobian is generated as a vector space by constant and *linear* functions is the A_2 singularity (for which the Jacobian is generated by 1 and x_n as above). One could extend the computation in Proposition 3.2.6 to all orders of $\mathscr{C}\mathscr{O}^*$ when applied to products of 1-cochains on L; we conjecture that the following holds.

Conjecture 3.4.6. *The restriction*

$$\mathscr{C}\mathscr{O}^k(S(\gamma))|_{CF^1(L,L)^{\otimes k}}\colon\ CF^1(L,L)^{\otimes k} \to CF^0(L,L)$$

equals

$$\rho(l)\cdot (l^*)^{\otimes k} \cdot 1_L \tag{3.24}$$

on tensor products of Floer 1-cocycles. Here $l^*\colon\ CF^1(L,L) \to \mathbb{K}$ *is given by* $l^*(x) = \langle \Psi(x), l\rangle$, *and* $l \in H_1(L)$ *is the orbit of* γ.

Remark 3.4.1. As in Proposition 3.2.6, part of the formula is the fact that the image of this restriction necessarily lands in $CF^0(L,L)$: this follows for degree reasons. We state the formula as a conjecture rather than a theorem because in the case of several inputs, some of them may collide in a 1-parametric moduli space. This issue is new compared to the collision of the input and output points encountered in the proof of Proposition 3.2.6 as the former produces bubbles that do not immediately cancel. So an extra argument is required, which we have not checked in detail.

This is a chain level computation, and whether it survives to something non-trivial in Hochschild cohomology will be governed by equations generalising equation $(*)$ from Theorem 3.1.7; those equations will be determined by the A_∞ structure maps on L up to order $k+1$. When L is the monotone toric fibre, the structure maps have been related to higher-order partial derivatives of W by Cho [29], and intuitively, the more degenerate the superpotential is, the more non-trivial information from (3.24) survives to Hochschild cohomology. Consequently, these observations are a possible starting point for proving split-generation results for toric Fano varieties with other degenerate superpotentials. However, further development of this discussion seems both complicated and not particularly demanded, given the general results of [40, 2].

Chapter 4

Low-area Floer theory and non-displaceability

This chapter is based on author's joint work with Renato Vianna [110].

4.1 Introduction

4.1.1 The Chekanov family of tori

A classical problem in symplectic topology, originating from Arnold's conjectures and still inspiring numerous advances in the field, is to understand whether two given Lagrangian submanifolds L_1, L_2 are (Hamiltonian) non-displaceable, meaning that there exists no Hamiltonian diffeomorphism that would map L_1 to a Lagrangian disjoint from L_2. One of the tools to approach this problem is Floer theory; historically, most applications of this theory have been constrained to monotone (or exact) Lagrangians, as their Floer theory is foundationally easier to set up, and usually easier to compute. More recent developments have led to non-displaceability results about some non-monotone Lagrangians. For example, Fukaya, Oh, Ohta and Ono [49] found a continuous family of non-displaceable Lagrangian tori $\hat{T}_a \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ by means of Floer cohomology with bulk deformations, whose general theory was developed in [47, 48]. For other recent methods of proving non-displaceability, see e.g. [4, 22, 114]. (When we say a single Lagrangian is non-displaceable, we mean it is non-displaceable from itself.)

We will later recall the definition of the tori $\hat{T}_a \subset \mathbb{C}P^1 \times \mathbb{C}P^1$; now we mention that they are invariant under the $\mathbb{Z}/2$-action on $\mathbb{C}P^1 \times \mathbb{C}P^1$ permuting the factors, and descend to a family of Lagrangian tori $T_a \subset \mathbb{C}P^2$: these tori will be our main example. The parameter $a \in (0,1)$ can be understood as the least area of a holomorphic Maslov index 2 disk with boundary on T_a; such disks were determined by Auroux [10] and Wu [115]. When $a = 1/3$, the torus T_a is the monotone Chekanov torus [76], known to be non-displaceable [28]; for other values of a, the tori are not monotone. One can show that T_a is displaceable when $a > 1/3$; in contrast, when $a < 1/3$ the tori T_a are expected to exhibit "rigid" behaviour. Unlike the case of $\hat{T}_a$, the statement below is still a conjecture.

Conjecture 4.1.1. *For each $a \in (0, 1/3)$, the Lagrangian torus $T_a \subset \mathbb{C}P^2$ is Hamiltonian non-displaceable.*

Recall that [49] proved that the tori $\hat{T}_a \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ are non-displaceable, for $a \in (0, 1/2]$, by showing that they have non-vanishing Floer cohomology after a suitable choice of a bulk deformation and a bounding cochain. On the other hand, we check in Proposition 4.3.8 that the Floer cohomology of the tori $T_a \subset \mathbb{C}P^2$ vanishes for each $a \neq 1/3$ with respect to any bulk deformation by a class in $H^2(\mathbb{C}P^2)$, and any bounding cochain (this includes local systems). It rules out all ways for conventional Floer theory to prove any rigidity result about these tori, except for the possibility of bulk deformation by $H^4(\mathbb{C}P^2)$; however this remaining possibility would require the knowledge of holomorphic disks of all Maslov indices with boundary on T_a (not only index 2), and such a computation seems out of reach. We introduce a new approach, called low-area Floer theory, and prove the following.

Theorem 4.1.2. *For each $a \in (0, 1/9]$, the torus $T_a \subset \mathbb{C}P^2$ is Hamiltonian non-displaceable from the monotone Clifford torus T_{Cl}.*

Although we were unable to solve Conjecture 4.1.1 (see Remark 4.3.2), this theorem is the first non-displaceability result about a continuous family of Lagrangian submanifolds in $\mathbb{C}P^2$. We would also like to point out that our proof only uses classical transversality theory for holomorphic curves, as opposed to virtual perturbations required to set up conventional Floer theory for non-monotone Lagrangians.

Remark 4.1.1. An interesting detail of the proof is that we have to work over the coefficient group $\mathbb{Z}/8$, and it is impossible to use a field, or the group $\mathbb{Z}$, instead. To place this into context, recall that conventional Floer cohomology over finite fields can detect non-displaceable monotone Lagrangians unseen by characteristic zero fields: the simplest example is $\mathbb{R}P^n \subset \mathbb{C}P^n$, see e.g. [50]; a more sophisticated example, where the characteristic of the field to take is not so obvious, is the Chiang Lagrangian studied by Evans and Lekili [39], see also J. Smith [106]. However, there are no examples in conventional Floer theory that would require working over a torsion group which is not a field.

Floer theory for monotone Lagrangians has abundant algebraic structure, a particular example of which are the open-closed and closed-open string maps. Below we recall a non-displaceability criterion for a pair of monotone Lagrangians formulated in terms of these string maps, due to Biran and Cornea. On a more technical level, our main finding can be summarised as follows: it is possible define a low-area version of the string maps for a non-monotone Lagrangian, and prove a version of Biran-Cornea's theorem under an additional assumption on the areas of holomorphic disks involved. This method is novel and can prove non-displaceability in examples having no clear alternative proof, like that of showing $HF^*(K,L) \neq 0$.

4.1.2 The context from usual Floer theory

We will now discuss Biran-Cornea's non-displaceability criterion for monotone Lagrangians, to motivate the definitions we are going to introduce in the non-monotone context. We will use the language of pearly trajectories (see Chapter 1 and references therein), and mention some facts without proof as we will not need them later. The definitions that will be introduced in the non-monotone context will actually be more elementary.

Recall that one way of defining the Floer cohomology $HF^*(L)$ of a monotone Lagrangian $L \subset X$ uses the pearl complex of Biran and Cornea; its differential counts pearly trajectories consisting of certain configurations of Morse flowlines on L interrupted by holomorphic disks with boundary on L. Also recall that the basic fact—if $HF^*(L) \neq 0$, then L is non-displaceable,—has no intrinsic proof within the language of pearly trajectories. Instead, the proof uses the isomorphism relating $HF^*(L)$ to the (historically, more classical) version of Floer cohomology that uses Hamiltonian perturbations. Nevertheless, there is a different non-displaceability statement whose proof is carried out completely in the language of holomorphic disks. That statement employs an additional structure, namely the maps

$$\mathscr{OC}\colon\ HF^*(L) \to QH^*(X), \quad \mathscr{CO}\colon\ QH^*(X) \to HF^*(L)$$

defined by counting suitable pearly trajectories in the ambient symplectic manifold X. These maps are frequently called the open-closed and the closed-open (string) map, respectively; note that Biran and Cornea denote them by i_L, j_L. The statement we referred to above is the following one.

Theorem 4.1.3 ([19, Theorem 2.4.1])**.** *For two monotone Lagrangian submanifolds $L, K \subset X$, suppose the composition*

$$HF^*(L) \xrightarrow{\mathscr{OC}} QH^*(X) \xrightarrow{\mathscr{CO}} HF^*(K) \tag{4.1}$$

does not vanish. Then L and K are Hamiltonian non-displaceable. $\square$

In this work we restrict ourselves to dimension four, so let us first discuss the monotone setting of Theorem 4.1.3 in this dimension. Assume that $H_1(X)=0$, then there are three possible ways for (4.1) not to vanish. The first way is via the topological part of (4.1):

$$HF^0(L) \xrightarrow[\mu=0]{\mathscr{OC}} QH^2(X) \xrightarrow[\mu=0]{\mathscr{CO}} HF^2(K).$$

In this case, as indicated by the $\mu=0$ labels, the relevant string maps necessarily factor through $QH^2(X)$ and are topological, i.e. involve pearly trajectories containing only constant Maslov index 0 disks. The composition above computes the homological intersection $[L]\cdot[K]$ inside X, where $[L],[K]\in H_2(X)$; it vanishes in the cases we are interested in. Here we use the Morse $\mathbb{Z}$-grading which only exists on cochain level, so formally we should be using Morse cochains instead of the HF^* but we skip this point for brevity. We use the cohomological convention: pearly trajectories of total Maslov index μ contribute to the degree $-\mu$ part of $\mathscr{CO}$, and to the degree $\dim L-\mu$ part of $\mathscr{OC}$ on cochain level.

The second possibility for $\mathscr{CO}\circ\mathscr{OC}$ not to vanish is via the contribution of pearly trajectories whose total Maslov index sums to two; the relevant parts of the string maps factorise as shown below. Again, the $\mu=0$ parts vanish when $[K]=[L]=0\in H_2(X)$ so we are not interested in this possibility either.

$$\begin{aligned} HF^0(L) &\xrightarrow[\mu=0]{\mathscr{OC}} QH^2(X) \xrightarrow[\mu=2]{\mathscr{CO}} HF^0(K),\\ HF^2(L) &\xrightarrow[\mu=2]{\mathscr{OC}} QH^2(X) \xrightarrow[\mu=0]{\mathscr{CO}} HF^2(K). \end{aligned}$$

The remanining part of $\mathscr{CO}\circ\mathscr{OC}$ breaks as a sum of three compositions factoring as follows:

$$\begin{tikzcd} & QH^0(X) \arrow[dr, "\mu=0", "\cong"'] & \\ HF^2(L) \arrow[ur, "\mu=4"] \arrow[r, "\mu=2"] \arrow[dr, "\cong", "\mu=0"'] & QH^2(X) \arrow[r, "\mu=2"] & HF^0(K) \\ & QH^4(X) \arrow[ur, "\mu=4"'] & \end{tikzcd} \tag{4.2}$$

The labels here indicate the total Maslov index of holomorphic disks present in the corresponding pearly trajectories; the $\mu=0$ parts are isomorphisms. Therefore, to compute $\mathscr{CO}\circ\mathscr{OC}|_{HF^2(L)}$ we would need to know the Maslov index 4 disks. We wish to avoid this, keeping in mind that the Maslov index 2 disks bounded by the tori T_a are known, but the Maslov index 4 disks are not. It turns out that the Maslov index 2 disks can be "singled out" if we only consider those ones whose boundary is non-zero in $H_1(L;\mathbb{Z})$ or $H_1(K;\mathbb{Z})$. This means we consider the composition

$$HF^2(L) \xrightarrow[\mu=2]{\mathscr{OC}^{(2)}} QH^2(X) \xrightarrow[\mu=2]{\mathscr{CO}^{(2)}} HF^0(K) \tag{4.3}$$

where the modified maps $\mathscr{OC}^{(2)}$, $\mathscr{CO}^{(2)}$ by definition count pearly trajectories contributing to the middle row of (4.2), i.e. containing a single disk, of Maslov index 2, with the additional condition that the boundary of that disk is homologically non-trivial. (In known examples, all holomorphic Maslov index 2 disks satisfy this condition.) The superscript "(2)" reflects that we are only considering Maslov index 2 trajectories, ignoring the Maslov index 0 and 4 ones; the condition about non-zero boundaries is not reflected by our notation. If the composition (4.3) does not vanish, then K, L are non-displaceable. A proof of this modified non-displaceability criterion can be recovered from the more general theorem about non-monotone Lagrangians which we shall soon state.

The above modification of the string maps will be implanted from the beginning into the definitions we give in the non-monotone setting, although it is possible to give the definitions without such a modification.

4.1.3 Non-displaceability using low-area Floer theory

In this subsection we formulate our main non-displaceability result. Fix an Abelian group of coefficients; it will be used in all (co)homology groups when the coefficients are omitted. Let $L, K \subset X$ be two orientable Lagrangian surfaces in a compact symplectic four-manifold X, where K is monotone, but L is not necessarily. Fix a tame almost complex structure J and points $p_L \in L$, $p_K \in K$. Denote

$$a = \min\{\omega(u) \mid u\colon\ (D, \partial D) \to (X, L) \text{ is } J\text{-holomorphic}, \mu(u) = 2\}. \tag{4.4}$$

Let $\{D_i^L\}_i \subset (X, L)$ be the images of all J-holomorphic Maslov index 2 disks of area a such that $p_L \in \partial D_i^L$ and *whose boundary is non-zero in* $H_1(L;\mathbb{Z})$ (their number is finite). Assume that

$$\sum_i \partial[D_i^L] = 0 \in H_1(L) \tag{4.5}$$

and the disks are regular. Then let

$$\mathscr{OC}^{(2)}_{low}([p_L]) \in H_2(X)$$

be any element whose image under the map $H_2(X) \to H_2(X, L)$ equals $\sum_i [D_i^L]$.

Similarly, for the monotone Lagrangian K let $\{D_j^K\}_j$ be the set of holomorphic Maslov index 2 disks with boundary on K such that $p_K \in \partial D_j^K$ and *whose boundary is non-zero in* $H_1(L;\mathbb{Z})$. Assume that $\sum_j \partial[D_j^K] = 0 \in H_1(K)$ and the disks are regular. Then let

$$\mathscr{OC}^{(2)}([p_K]) \in H_2(X)$$

be any element whose image under the map $H_2(X) \to H_2(X,K)$ equals $\sum_i [D_j^K]$. Recall that the areas $\omega(D_j^K)$ are equal and determined by the monotonicity constant of X; denote these areas

$$b = \omega(D_j^K) \in \mathbb{R}.$$

Finally, returning to L, denote

$$A = \min\{\omega(u) > a \mid u\colon\ (D,\partial D) \to (X,L) \text{ is } J\text{-holomorphic}, \mu(u) = 2\}. \tag{4.6}$$

Theorem 4.1.4. *In the above setup, suppose $a+b<A$ and the homological intersection number $\mathscr{OC}^{(2)}_{low}([p_L]) \cdot \mathscr{OC}^{(2)}([p_K])$ is non-zero. Then L and K are Hamiltonian non-displaceable.*

As hinted above, our proof of Theorem 4.1.4 follows the proof of [19, Theorem 2.4.1] with several modifications involved. The condition $a+b<A$, which does not arise when both Lagrangians are monotone, is used in the proof when the disks D_i^L and D_j^K are glued to an annulus of area $a+b$; the condition makes sure higher-area Maslov index 2 disks on L cannot bubble off this annulus. This condition will translate to $a<1/9$ in Theorem 4.1.2.

Remark 4.1.2. Recall that, for a two-dimensional monotone Lagrangian K equipped with the trivial local system, we have $\sum_j \partial[D_j^K] = 0$ if and only if $HF^*(K) \neq 0$, and in the latter case $HF^*(K) \cong H^*(K)$. Indeed, $\sum_j \partial[D_j^K]$ computes the Poincaré dual of the Floer differential $d([p_K])$ where $[p_K]$ is the generator of $H^2(K)$. If we pick a perfect Morse function on K, then p_K is geometrically realised by its maximum. Recall that if $d([p_K]) = 0$, then by duality the unit is not hit by the differential, hence $HF^*(K) \neq 0$. The condition $\sum_i \partial[D_i^L] = 0$ is a natural low-area version of the non-vanishing of Floer cohomology.

Remark 4.1.3. Observe that the homology class $\mathscr{OC}^{(2)}([p_K]) \in H_2(X)$ is defined up to the kernel of $H_2(X) \to H_2(X,K)$, i.e. up to the image of $H_2(K) \to H_2(X)$, and the same applies to $\mathscr{OC}^{(2)}_{low}([p_L])$. The usual definitions of string maps using pearly trajectories, as referred to in Theorem 4.1.3, do not have this ambiguity, but this is not a contradiction: recall that there is no canonical identification between $HF^*(K)$ and $H^*(K)$, even when they are abstractly isomorphic [19, Section 4.5]. In particular, $HF^*(K)$ is only $\mathbb{Z}/2$-graded and the element $[p_K] \in HF^*(K)$ corresponding to the degree 2 generator of $H^2(K)$ is defined up to adding a multiple of the unit $1_K \in HF^*(K)$. Recall that $\mathscr{OC}(1_K)$ is dual to $[K] \in H_2(X)$, and this matches with the fact that $\mathscr{OC}([p_K])$, as well as $\mathscr{OC}^{(2)}([p_K])$, is defined up to the image $H_2(K) \to H_2(X)$. Theorem 4.1.4 is true for any choice of $\mathscr{OC}^{(2)}([p_K])$ and $\mathscr{OC}^{(2)}_{low}([p_L])$.

One can show that if both L, K are monotone and $[L] \cdot [K] = 0$, then $\mathscr{OC}^{(2)}([p_L]) \cdot \mathscr{OC}^{(2)}([p_K]) \neq 0$ if and only if the composition (4.3) is non-zero; compare Lemma 4.2.1.

Remark 4.1.4. Charette [26] defined quantum Reidemeister torsion for monotone Lagrangians whose Floer cohomology vanishes. While it is possible this definition generalises to the non-monotone setting, making our tori $T_a \subset \mathbb{C}P^2$ valid candidates as far as classical Floer theory is concerned, it is shown in [26, Corollary 4.1.2] that quantum Reidemeister torsion is always trivial for tori.

The structure of the rest of the chapter is as follows. In Section 4.2 we prove Theorem 4.1.4, and discuss a version of this theorem for a pair of non-monotone Lagrangians. Notice that when stating Theorem 4.1.4, we defined the homology class $\mathscr{OC}^{(2)}_{low}([p_L])$ "from scratch", without providing any definition of Floer cohomology that, in the monotone case, underlies the definition of the open-closed string map. This was done to keep the introduction concise, and in Section 4.2 we sketch a definition of low-area Floer cohomology for non-monotone Lagrangians in all dimensions. We also explain a general setting when this definition could be useful.

In Section 4.3 we recall the definition of the tori $T_a \subset \mathbb{C}P^2$; prove Theorem 4.1.2 and a related result for $\mathbb{C}P^1 \times \mathbb{C}P^1$; and explain why Floer theory with bulk deformation does not readily apply to the T_a.

4.2 Proof of Theorem 4.1.4, and a discussion of low-area Floer theory

4.2.1 Proof of Theorem 4.1.4

Our proof essentially follows [19, Theorem 2.4.1] with some differences: we check that certain unwanted bubbling, impossible in the monotone case, does not occur in our setting given that $a+b<A$; we include an argument which "singles out" the contribution of Maslov index 2 disks with non-trivial boundary from that of Maslov index 4 disks; and relate the homology 2-cycles $\mathscr{OC}([p_K])$, $\mathscr{OC}^{(2)}_{low}([p_L])$ defined "from scratch" in Subsection 4.1.2 to the ones appearing in the Biran-Cornea's pearly trajectory definition of the open-closed maps. To keep the proof shorter we refer to [19] for the precise definitions of the moduli spaces we use.

Suppose there exists a Hamiltonian diffeomorphism ϕ such that $\phi(K) \cap L = \emptyset$, and redenote $\phi(K)$ by K, so that $K \cap L = \emptyset$. We may use the original almost complex structure J (or its small perturbation, to make other curves appearing in the argument regular), for which the area-a holomorphic Maslov index 2 disks with boundary on the non-monotone Lagrangian L are as in the setup (in particular, regular). This way, we only need to refer to the well known fact that $\mathscr{OC}([p_K]) \in H_2(X)$ is invariant under Hamiltonian isotopies of the monotone Lagrangian K (because the Maslov index 2 disks with boundary on K do not bubble). We do not need to prove that $\mathscr{OC}^{(2)}_{low}([p_L])$ is invariant; see Section 4.2.2 for further discussion of the invariance.

Pick generic metrics and Morse functions f_1, f_2 on L, K. We assume that the functions f_1, f_2 are perfect (it simplifies the proof, but is not essential); such exist because L, K are two-dimensional and orientable. Consider the moduli space $\mathscr{M}$ of configurations ("pearly trajectories") of the three types shown in Figure 4.1, with the additional condition that the total boundary homology classes of these configurations are non-zero both in $H_1(L;\mathbb{Z})$ and $H_1(K;\mathbb{Z})$. (By writing 'total' we mean that if the configuration's boundary on a single Lagrangian has two components, their sum must be non-zero.) The figure prescribes the

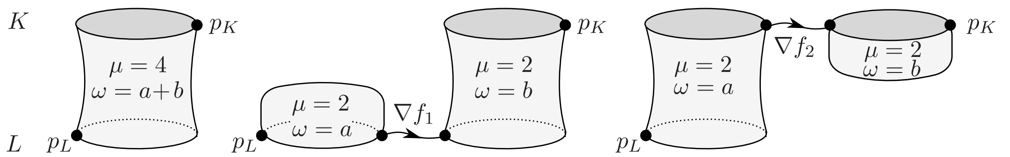


Fig. 4.1 The moduli space $\mathscr{M}$ consists of pearly trajectories of these types.

Maslov index and the area of each holomorphic curve. The conformal parameter of each annulus is allowed to take any value $R \in (0, +\infty)$; recall that the domain of an annulus with conformal parameter R can be realised as $\{z \in \mathbb{C} : 1 \leq |z| \leq e^R\}$. There is also a time-length parameter l associated to each flowline. Configurations with a contracted flowline (i.e. one with $l = 0$) correspond to interior points of $\mathscr{M}$, because gluing the disk to the annulus is identified with l becoming negative. The curves pass through fixed points $p_K \in K$, $p_L \in L$ as shown. Finally, the two marked points on each annulus must be the images of fixed points on the domain; for example, we can fix the marked points to be 1 and e^R for a domain as above.

Recall that the Fredholm index of unparametrised holomorphic annuli without marked points and with free conformal parameter equals the Maslov index. Computing the rest of the indices and using the regularity of the disks, one shows $\mathscr{M}$ is a smooth 1-dimensional oriented manifold [18, Section 8.2]. (The non-constant annuli will be regular for a generic J, and the appearance of constant annuli is a priori excluded because K and L are disjoint.)

The space $\mathscr{M}$ can be compactified by adding configurations with broken flowlines as well as configurations corresponding to the conformal parameter R of the annulus becoming 0 or $+\infty$. We describe each of the three types of configurations separately and determine their signed count.

(i) The configurations with broken flowlines are shown in Figure 4.2. As before, they are subject to the condition that the total boundary homology classes of the configuration are non-zero both in $H_1(L;\mathbb{Z})$ and $H_1(K;\mathbb{Z})$. The annuli have a certain conformal parameter R_0 and the breaking is an index 1 critical point of f_i [18, Section 8.2.1, Item (a)].

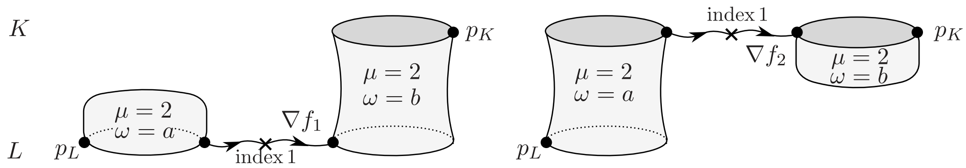


Fig. 4.2 Configurations with broken flowlines, called type (i).

The count of the sub-configurations consisting of the disk and the attached flowline vanishes: this is a Morse-theoretic restatement of the hypothesis that $\sum_i \partial[D_i^L] = \sum_j \partial[D_j^K] = 0$. Hence (by perfectness of the f_i) the count of the whole configurations in Figure 4.2 also vanishes, at least if we ignore the condition of non-zero total boundary. Separately, the count of configurations in Figure 4.2 whose total boundary homology class is zero either in L or K, also vanishes. Indeed, suppose for example that the $\omega = a$ disk in Figure 4.2 (left) has boundary homology class $l \in H_1(L;\mathbb{Z})$ and the lower boundary of the annulus has class $-l$; then the count of the configurations in the figure with that disk and that annulus equals the homological intersection $(-l)\cdot l = 0$. We conclude that the count of configurations in the above figure whose total boundary homology classes are *non-zero*, also vanishes.

(ii) The configurations with $R = 0$ contain a curve whose domain is an annulus with a contracted path connecting the two boundary components. The singular point of this domain must be mapped to an intersection point $K \cap L$, so these configurations do not exist if $K \cap L = \emptyset$ [18, Section 8.2.1, Item (c)].

(iii) The configurations with $R = +\infty$ correspond to an annulus breaking into two disks, one with boundary on K and the other with boundary on L [18, Section 8.2.1, Item (d)]. One of the disks can be constant, and the possible configurations are shown in Figure 4.3.

In fact, there is another potential annulus breaking at $R = +\infty$ that we have ignored: the one into a Maslov index 4 disk on one Lagrangian and a (necessarily constant) Maslov index 0 disk on the other Lagrangian, see Figure 4.4. This broken configurations cannot arise from the configurations in $\mathscr{M}$ by the non-zero boundary condition imposed on the elements of this

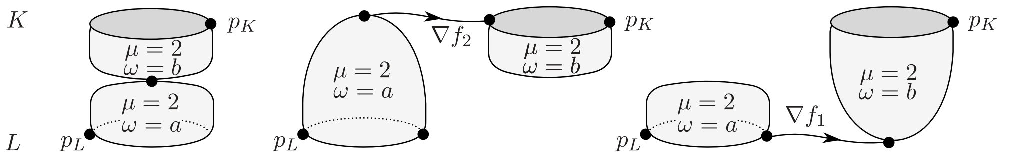


Fig. 4.3 The limiting configurations when $R = +\infty$, called type (iii).

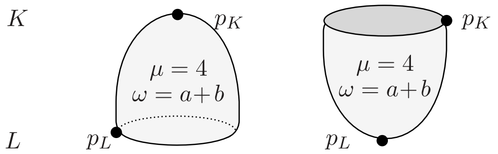
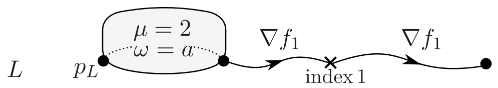


Fig. 4.4 The limiting configurations for $R = +\infty$ which are impossible by the non-zero boundary condition.

moduli space. The fact that a Maslov index 0 disk has to be constant is due to the generic choice of J; see below.

Lemma 4.2.1. *The count of configurations in Figure 4.3 equals $\mathscr{OC}^{(2)}_{low}([p_L]) \cdot \mathscr{OC}([p_K])$ as defined in Section 3.1.*

Proof. In the right-most configuration in Figure 4.3, forget the $\omega = b$ disk so that one endpoint of the ∇f_1-flowline becomes free; let C^L be the singular 2-chain on L swept by these endpoints. We claim that $\partial C^L = \sum_i \partial D_i^L$ on chain level. Indeed, the boundary ∂C^L corresponds to zero-length flowlines that sweep $\sum_i \partial D_i^L$, and to flowlines broken at an index 1 critical point of f_1, shown below:



The endpoints of these configurations sweep the zero 1-chain. Indeed, we are given that $\sum_i \partial [D_i^L] = 0$ so the algebraic count of the appearing index 1 critical points represents a null-cohomologous Morse cocycle, therefore this count equals zero by perfectness of f_1. It follows that $\partial C^L = \sum_i \partial D_i^L$.

Similarly, define the 2-chain C^K on K, $\partial C^K = \sum_j \partial D_j^K$, by forgetting the $\omega = a$ disk in the second configuration of type (iii) above, and repeating the construction. It follows that the homology class $\mathscr{OC}^{(2)}_{low}([p_L])$ from Subsection 4.1.2 can be represented by the cycle $(\cup_i D_i^L) \cup C^L$, and similarly $\mathscr{OC}([p_K])$ can be represented by $(\cup_j D_j^K) \cup C^K$. Note that $\mathscr{OC}^{(2)}_{low}([p_L])$, $\mathscr{OC}([p_K])$ were defined up to adding a multiple of $[L], [K] \in H_2(X)$ respectively, see Remark 4.1.3, and here we have picked specific representatives. However, the intersection number $\mathscr{OC}^{(2)}_{low}([p_L]) \cdot \mathscr{OC}([p_K])$ does not depend on the choice if $L \cap K = \emptyset$. This intersection number can be expanded into four chain-level intersections:

$$\mathscr{OC}^{(2)}_{low}([p_L])\cdot\mathscr{OC}([p_K]) = (\cup_i D_i^L)\cdot(\cup_j D_j^K) + (\cup_i D_i^L)\cdot C^K + C^L\cdot(\cup_j D_j^K) + C^L\cdot C^K.$$

The last summand vanishes because $L\cap K=\emptyset$, and the other summands correspond to the three configurations of type (iii) pictured earlier. □

Remark 4.2.1. Note that the equality between the intersection number $\mathscr{OC}^{(2)}_{low}([p_L])\cdot\mathscr{OC}([p_K])$ and the count of the $R=+\infty$ boundary points of $\mathscr{M}$ holds integrally, i.e. with signs. This follows from the general set-up of orientations of moduli spaces in Floer theory, which are consistent with taking fibre products and subsequent gluings. For example, in our case the signed intersection points between a pair of holomorphic disks can be seen as the result of taking fibre product along evaluations at interior marked points; therefore these intersection signs agree with the orientations on the moduli space of the glued annuli.

If the moduli space $\mathscr{M}$ is completed by the above configurations (i)—(iii), it becomes compact. Indeed, by the condition $a+b<A$, Maslov index 2 disks on L with area higher than a cannot bubble. Disks of Maslov index $\mu\ge 4$ cannot bubble (for finite R) on either Lagrangian because the rest of the configuration would contain an annulus of Maslov index $\mu\le 0$ passing through a fixed point on the Lagrangian, and such configurations have too low index to exist generically (the annuli can be equipped with a generic domain-dependent perturbation of J, hence are regular). Similarly, holomorphic disks of Maslov index $\mu\le 0$ cannot bubble as they do not exist for generic perturbations of the initial almost complex structure J. (This is true for simple disks by the index formula, and follows for non-simple ones from the decomposition theorems [66, 65], as such disks must have an underlying simple disk with $\mu\le 0$.) Finally, side bubbles of Maslov index 2 disks (not carrying a marked point with a p_K or a p_L constraint) cannot occur because the remaining Maslov index 2 annulus, with both the p_K and p_L constraints, would not exist generically; and, as usual, sphere bubbles cannot happen in a 1-dimensional moduli space because they are a codimension 2 phenomenon.

By the compactness of $\mathscr{M}$, the signed count of its boundary points (i)—(iii) equals zero. We therefore conclude from Lemma 4.2.1 and the preceding discussion that $\mathscr{OC}^{(2)}_{low}([p_L])\cdot\mathscr{OC}([p_K])=0$, which contradicts the hypothesis of Theorem 4.1.4. □

4.2.2 Further discussion of low-area Floer theory

This informal subsection will not be used further, and may be skipped by the reader. Here we discuss a version of Theorem 4.1.4 for a pair of non-monotone Lagrangians; and a version of low-area Floer cohomology in all dimensions with a setup in which it could be useful.

Suppose $L, K \subset X$ are non-monotone Lagrangians, where $\dim X = 4$. It makes sense to speak of $\mathscr{OC}^{(2)}_{low}([p_L])$ and $\mathscr{OC}^{(2)}_{low}([p_K])$ if the boundary homology classes of the least area holomorphic Maslov index 2 disks cancel as in Equation (4.5) both for K and L. To get a valid version of Theorem 4.1.4 in this setting, in its statement one replaces $\mathscr{OC}([p_K])$ by $\mathscr{OC}^{(2)}_{low}([p_K])$, puts b to be the least area of holomorphic Maslov index 2 disks on K, and

$$\begin{aligned} A = \min\{ \quad & \min\{\omega(u) > a \mid u\colon\ (D,\partial D) \to (X,L) \text{ is } J\text{-holomorphic}, \ \mu(u) = 2\}, \\ & \min\{\omega(u) > b \mid u\colon\ (D,\partial D) \to (X,K) \text{ is } J\text{-holomorphic}, \ \mu(u) = 2\} \quad \}. \end{aligned}$$

The proof follows the same lines, but there are two steps which require additional attention. First, one must prove that $\mathscr{OC}^{(2)}_{low}([p_L])$ is invariant under the choice of an almost complex structure (and therefore, under Hamiltonian isotopies of L). This is obvious when L is monotone since Maslov index 2 disks cannot bubble in this case. For the same reason, the invariance is obvious if a turns out to be the least positive area among all topological (not necessarily J-holomorphic) Maslov index 2 disks, i.e. if

$$a = \inf\{\omega(u) > 0 \mid u \in H_2(X,L),\ \mu(u) = 2\}. \tag{4.7}$$

The tori $T_a \subset \mathbb{C}P^2$ fall into this case, for $a \le 1/5$. In Section 4.3, we will describe the areas of some Maslov index 2 disks whose boundaries span $H_1(T_a;\mathbb{Z})$; using that information, the lemma below can be easily checked (we omit the proof).

Lemma 4.2.2. *Equation (4.7) holds for the torus $T_a \subset \mathbb{C}P^2$ if and only if $a \le 1/5$.* □

If Equation (4.7) does not hold, one has to be more careful in showing that $\mathscr{OC}^{(2)}_{low}([p_L])$ is invariant under the choice of J. Suppose we change $J = J_0$ in a family $\{J_t\}_{t\in[0,1]}$ of almost complex structures. The J_t-holomorphic area-a disks cannot bubble as long as they are of least area among J_t-holomorphic Maslov index 2 disks for the current t; however, regardless of bubbling there is a possibility of birth-death phenomena which can create a pair of oppositely-oriented moduli spaces of holomorphic Maslov index 2 disks of area $\varepsilon < a$. The algebraic count of such area ε disks through a point in L will necessarily be zero, in each relative homology class, for each fixed J_t. As we change t further, at some time $t = t_0$, an area a Maslov index 2 disk can bubble into an area ε Maslov 2 index disk, plus a Maslov index 0 stable disk.

Lemma 4.2.3. *The count of area-a Maslov index 2 disks through a fixed point in L in any fixed relative homology class stays invariant under the bubbling above.*

Sketch of proof. The general wall-crossing formula says that under such bubbling, the change in the count of area-a Maslov index 2 disks is computed by fibre products of moduli spaces of Maslov 0 stable disks with moduli spaces of area ε Maslov index 2 disks; see e.g. [10, Section 3.2]. These fibre products are taken along the evaluation at a boundary marked point; by the algebraic cancellation of the area ε Maslov index 2 disks, the fibre products vanish. □

The invariance of $\mathscr{OC}^{(2)}_{low}([p_L])$ essentially follows from Lemma 4.2.3. To make the above sketch proof rigorous, one needs to refer to (some version of) virtual perturbation theory, whose purpose is to make the moduli spaces of Maslov 0 stable disks regular: this was assumed in the sketch above, but does not hold a priori because of the existence of multiply covered disks. The mentioned fibre-product formula is also expected to be part of any virtual perturbation theory package. We choose not to discuss further details here, in particular we will not upgrade the above sketch into a complete proof. Note that our argument only requires the formal existence of the fibre-product formula; in particular, we do not need to know what the Maslov index 0 disks actually evaluate to, after a virtual perturbation.

A similar issue appears at another place in the proof. Namely, algebraically cancelling holomorphic Maslov index 2 disks of area less than $a+b$ may bubble from the annuli, as we change their conformal parameter. (By the condition $a+b<A$, only algebraically cancelling disks do arise.) To rule out these disks altogether, it is enough to assume the topological condition

$$\begin{aligned} A = \min\{ \quad &\min\{\omega(u) > a \mid u \in H_2(X,L),\ \mu(u) = 2\}, \\ &\min\{\omega(u) > b \mid u \in H_2(X,K),\ \mu(u) = 2\} \quad \}. \end{aligned} \tag{4.8}$$

Without this assumption, another argument like Lemma 4.2.3 is required. The upshot is that if Equations (4.7), (4.8) are true, a version of Theorem 4.1.4 from the beginning of this subsection can be proved without additional effort, i.e. without referring to virtual perturbation theory.

It is worthwhile to note that when $K=L$, Equation (4.7) and the condition $a+b<A$ (which specialises to $2a<A$) already imply Equation (4.8). Therefore, let us record that the proof of the theorem below works without the use of virtual perturbation theory.

Theorem 4.2.4. *Let L be as in the setting of Theorem 4.1.4; assume* $2a<A$ *and Equation (4.7) holds. If* $\mathscr{OC}^{(2)}_{low}([p_L])\cdot\mathscr{OC}^{(2)}_{low}([p_L])\neq 0$, *then L is non-displaceable.* □

Unfortunately, the intersection number in this theorem turns out to vanish when $L=T_a$, see Remark 4.3.2 below.

One can state generalisations of Theorem 4.1.4 to higher dimensions (they would require the use of higher index disks at least on one of the Lagrangians). We will not discuss them in this paper; instead, we would like to sketch a possible definition of low-area Floer cohomology of a non-monotone Lagrangian submanifold in any dimension, which uses least area holomorphic Maslov index 2 disks and naturally underlies the definition of the low-area open-closed map $\mathscr{OC}^{(2)}_{low}$, although the latter was defined without reference to low-area Floer cohomology.

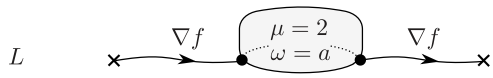
Suppose $L \subset X$ is an orientable, spin Lagrangian submanifold in a symplectic manifold (X, ω). Pick a tame almost complex structure J on X. Denote

$$a = \min\{\omega(u) \mid u\colon\ (D, \partial D) \to (X, L) \text{ is } J\text{-holomorphic}\}. \tag{4.9}$$

(This minimum exists by Gromov compactness.) For simplicity, assume L admits a perfect Morse function f, choose a metric g on L, and let $CF^*(L)$ be the corresponding Morse complex (with the trivial differential, by perfectness).

Definition 4.2.5. Assume all J-holomorphic disks $(D, \partial D) \to (X, L)$ having area a are of Maslov index 2 and regular. Define the differential $d_{low}\colon\ CF^*(L) \to CF^{*-1}(L)$ to count rigid pearly trajectories on L that contain a single J-holomorphic disk, which moreover has area a.

The figure below shows what a pearly trajectory for d_{low} looks like; it connects a critical point of f with another one, of index lower by 1.



Lemma 4.2.6. *We have* $(d_{low})^2 = 0$. □

The above lemma is a variation on [18, Section 3.3] and we omit its proof.

Definition 4.2.7. We define $HF^*_{low}(L)$ to be the homology of $(CF^*(L), d_{low})$.

Remark 4.2.2. It is possible to drop the condition that f is a perfect Morse function by restricting d_{low} to the space of Morse cocycles.

The lemma below is easy; it can also be proved without assuming Equation (4.7) by referring to virtual perturbation theory as in Lemma 4.2.3.

Lemma 4.2.8. *Suppose Equation (4.7) holds, then the cohomology $HF^*_{low}(L)$ does not depend on a generic choice of J, f, g as above.* □

We see no general reason to expect that the non-vanishing of $HF^*_{low}(L)$ should imply that L is non-displaceable, because there is no corresponding low-area version of Floer cohomology with Hamiltonian perturbations (or that of a pair of Lagrangian submanifolds). On the other hand, $HF^*_{low}(L)$ naturally underlies the definition of $\mathscr{OC}^{(2)}_{low}$ in the same way as the usual Floer cohomology for monotone Lagrangians underlies the usual string maps. For example, the equality $HF^*_{low}(L) = H^*(L)$ is equivalent to Equation (4.5), and if that holds the homology class $\mathscr{OC}^{(2)}_{low}([p_L])$ can be defined and may be used to prove non-displaceability results.

In higher dimensions, our definition of $HF^*_{low}(L)$ may not be the only possible one: it also seems meaningful to define d_{low} by allowing trajectories of total Maslov index possibly higher than 2, but of total area bounded by some number a such that holomorphic (or, better, topological) disks of positive area bounded by a behave in a monotone manner, that is, have the property that their area is proportional to their Maslov index. A potential example is a monotone Lagrangian $L \subset T^*S$, where S is itself a Lagrangian in a symplectic manifold X. If we rescale L to lie close to the zero-section S, we obtain a non-monotone Lagrangian embedding $L \subset X$ in a neighbourhood of S whose topological low-area disks come ones in T^*S and behave in a monotone manner. It will be seen in Remark 4.3.6 that the tori $T_a \subset \mathbb{C}P^2$ come from this setup, with $S = \mathbb{R}P^2$. Similarly, the tori $\hat{T}_a \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ realise the above setup with $S = S^2$.

4.3 The tori T_a are non-displaceable from the Clifford torus

In this section we recall the definition of the tori $\hat{T}_a \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ and $T_a \subset \mathbb{C}P^2$ appearing in the introduction, and prove Theorem 4.1.2 along with a similar result for $\hat{T}_a \subset \mathbb{C}P^1 \times \mathbb{C}P^1$. We also check that Floer cohomology with bulk deformations vanishes for the T_a.

4.3.1 Definition of the tori

We choose to define the T_a following [115]; we shall use the *coupled spin* system [84, Example 6.2.4] on $\mathbb{C}P^1 \times \mathbb{C}P^1$. Consider $\mathbb{C}P^1 \times \mathbb{C}P^1$ as the configuration space of the double pendulum composed of two unit length rods: the endpoint of the first rod is attached to the origin $0 \in \mathbb{R}^3$ around which the rod can freely rotate; the second rod is attached to the other endpoint of the first rod and can also freely rotate around it, see Figure 4.5.

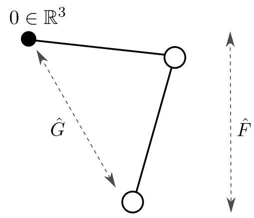


Fig. 4.5 The double pendulum defines two functions $\hat{F}, \hat{G}$ on $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Define two functions $\hat{F}, \hat{G}\colon\ \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{R}$ to be, respectively, the z-coordinate of the free endpoint of the second rod, and its distance from the origin, normalised by $1/2$. In formulas,

$$\begin{aligned}
&\mathbb{C}P^1 \times \mathbb{C}P^1 = \{x_1^2 + y_1^2 + z_1^2 = 1\} \times \{x_2^2 + y_2^2 + z_2^2 = 1\} \subset \mathbb{R}^6,\\
&\hat{F}(x_1, y_1, z_1, x_2, y_2, z_2) = \tfrac{1}{2}(z_1 + z_2),\\
&\hat{G}(x_1, y_1, z_1, x_2, y_2, z_2) = \tfrac{1}{2}\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2 + (z_1 + z_2)^2}.
\end{aligned}$$

The function $\hat{G}$ is not smooth along the anti-diagonal Lagrangian sphere $S^2_{\mathrm{ad}} \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ (corresponding to the folded pendulum), and away from it the functions $\hat{F}$ and $\hat{G}$ Poisson commute. The image of the "moment map" $(\hat{F}, \hat{G})\colon\ \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{R}^2$ is the triangle shown in Figure 4.6.

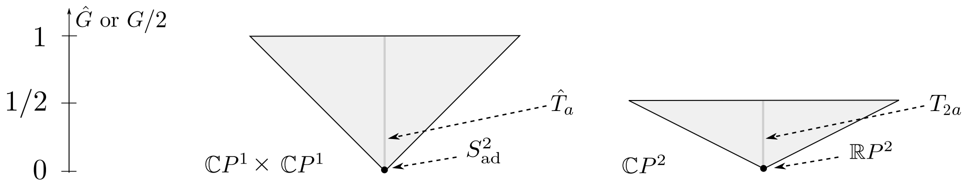


Fig. 4.6 The images of the "moment maps" on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2$, and the lines above which the tori $\hat{T}_a, T_a$ are located.

Definition 4.3.1. For $a \in (0,1)$, the Lagrangian torus $\hat{T}_a \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ is the pre-image of $(0,a)$ under the map $(\hat{F}, \hat{G})$.

The functions $(\hat{F}, \hat{G})$ are invariant under the $\mathbb{Z}/2$-action on $\mathbb{C}P^1 \times \mathbb{C}P^1$ that swaps the two $\mathbb{C}P^1$ factors. This involution defines a 2:1 cover $\mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^2$ branched along the diagonal of $\mathbb{C}P^1 \times \mathbb{C}P^1$, so the functions $(\hat{F}, \hat{G})$ descend to functions on $\mathbb{C}P^2$ which we denote by (F, G); the image of $(F, G/2)\colon\ \mathbb{C}P^2 \to \mathbb{R}^2$ is shown in Figure 4.6. Note that the quotient of the Lagrangian sphere S^2_{ad} is $\mathbb{R}P^2 \subset \mathbb{C}P^2$. Being branched, the 2:1 cover cannot be made symplectic, so it requires some care to explain with respect to which symplectic form the tori $T_a \subset \mathbb{C}P^2$ are Lagrangian. One solution is to consider $\mathbb{C}P^2$ as the symplectic cut [68] of $T^*\mathbb{R}P^2$, as explained by Wu [115]. It is natural to take $(F, G/2)$, not (F, G), as the "moment map" on $\mathbb{C}P^2$.

We normalise the symplectic forms ω on $\mathbb{C}P^2$ and $\hat{\omega}$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$ so that $\omega(H) = 1$ and $\hat{\omega}(H_1) = \hat{\omega}(H_2) = 1$, where $H = [\mathbb{C}P^1]$ is the generator of $H_2(\mathbb{C}P^2)$, and $H_1 = [\{\mathrm{pt}\} \times \mathbb{C}P^1]$, $H_2 = [\mathbb{C}P^1 \times \{\mathrm{pt}\}]$ in $H_2(\mathbb{C}P^1 \times \mathbb{C}P^1)$.

Definition 4.3.2. For $a \in (0,1)$, the Lagrangian torus $T_a \subset \mathbb{C}P^2$ is the pre-image of $(0, a/2)$ under $(F, G/2)$, i.e. the image of $\hat{T}_a$ under the 2:1 branched cover $\mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^2$.

Remark 4.3.1. There is an alternative way to define the tori $\hat{T}_a$ and T_a. It follows from the work of Oakley and Usher [76] that the above defined tori are Hamiltonian isotopic to the so-called Chekanov-type tori introduced by Auroux [10]:

$$\hat{T}_a \cong \{([x:w],[y:z]) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \setminus \{z=0\} \cup \{w=0\} : \tfrac{xy}{wz} \in \hat{\gamma}_a,\ \left|\tfrac{x}{w}\right| = \left|\tfrac{y}{z}\right|\},$$
$$T_a \cong \{[x:y:z] \in \mathbb{C}P^2 \setminus \{z=0\} : \tfrac{xy}{z^2} \in \gamma_a,\ \left|\tfrac{x}{z}\right| = \left|\tfrac{y}{z}\right|\},$$

where $\hat{\gamma}, \gamma \subset \mathbb{C}$ are closed curves that enclose a domain not containing $0 \in \mathbb{C}$. The area of this domain is determined by a and must be such that the areas of holomorphic disks computed in [10] match Table 4.1; see below. (Curves that enclose domains of the same area not containing $0 \in \mathbb{C}$ give rise to Hamiltonian isotopic tori.) The advantage of this presentation is that the tori T_a are immediately seen to be Lagrangian. Yet another way of defining the tori is by Biran's circle bundle construction [16] over a monotone circle in the symplectic sphere which is the preimage of the top side of the triangles in Figure 4.6; see again [76].

4.3.2 Holomorphic disks

We start by recalling the theorem of Fukaya, Oh, Ohta and Ono mentioned in the introduction.

Theorem 4.3.3 ([49, Theorem 3.3]). *For $a \in (0, 1/2]$, the torus $\hat{T}_a \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ is non-displaceable.* □

Theorem 4.3.4. *Inside $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2$, all fibres corresponding to interior points of the "moment polytopes" shown in Figure 4.6, except for the tori $\hat{T}_a$ when $a \in (0, 1/2]$, and T_a when $a \in (0, 1/3]$, are displaceable.*

Proof. Recall the method of probes due to McDuff [72] which is a mechanism for displacing certain toric fibres. Horizontal probes displace all the fibres except the $\hat{T}_a$ or T_a, $a \in (0,1)$. Vertical probes over the segment $\{0\} \times (0, 1/2]$ displace the T_a for $a > 1/2$, and probes over the segment $\{0\} \times (0, 1]$ to displace the $\hat{T}_a$ for $a > 1/2$. When $1/3 \le a < 1/2$, the method of probes cannot not displace T_a; this will be proved by Georgios Dimitroglou Rizell in an appendix to the forthcoming revision of [110]. □

The Maslov index 2 holomorphic disks for the tori $\hat{T}_a$ and T_a, with respect to some choice of an almost complex structure for which the disks are regular, were computed, respectively, by Fukaya, Oh, Ohta and Ono [49] and Wu [115]. Their results can also be recovered using the alternative presentation of the tori from Remark 4.3.1. Namely, Chekanov and Schlenk [28] determined Maslov index 2 holomorphic disks for the monotone Chekanov tori $T_{1/3} \subset \mathbb{C}P^2$ and $T_{1/2} \subset \mathbb{C}P^1 \times \mathbb{C}P^1$, and the combinatorics of these disks stays the same for the Chekanov-type tori from Remark 4.3.1 if one uses the standard complex structures on $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ [10, Proposition 5.8, Corollary 5.13]. We summarise these results in the statement below.

$T_a \subset \mathbb{C}P^2$			
Disk class	#	*Area*	$\mathfrak{PO}$ *term*
$H-2\beta-\alpha$	1	a	$t^a z^{-2} w^{-1}$
$H-2\beta$	2	a	$t^a z^{-2}$
$H-2\beta+\alpha$	1	a	$t^a z^{-2} w$
β	1	$(1-a)/2$	$t^{(1-a)/2} z$

$\hat{T}_a \subset \mathbb{C}P^1 \times \mathbb{C}P^1$			
Disk class	#	*Area*	$\mathfrak{PO}$ *term*
$H_1-\beta-\alpha$	1	a	$t^a z^{-1} w^{-1}$
$H_1-\beta$	1	a	$t^a z^{-1}$
$H_2-\beta$	1	a	$t^a z^{-1}$
$H_2-\beta+\alpha$	1	a	$t^a z^{-1} w$
β	1	$1-a$	$t^{1-a} z$

Table 4.1 The homology classes of all Maslov index two J-holomorphic disks on the tori; the number of such disks through a generic point on the torus; their areas; the corresponding monomials in the superpotential function: all for some regular almost complex structure J. Here α, β denote some fixed homology classes in $H_2(\mathbb{C}P^2, T_a)$ or $H_2(\mathbb{C}P^1 \times \mathbb{C}P^1, \hat{T}_a)$.

Proposition 4.3.5 ([10, 28, 49, 115]). *There exist almost complex structures on $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ for which the enumerative geometry of Maslov index 2 holomorphic disks with boundary on T_a, resp. $\hat{T}_a$, is as shown in Table 4.1, and these disks are regular.* □

4.3.3 Proof of Theorem 4.1.2

We now have all the ingredients to prove Theorem 4.1.2 using Theorem 4.1.4. Take the almost complex structure J from Proposition 4.3.5, then the parameter a of the torus $T_a \subset \mathbb{C}P^2$ satisfies Equation (4.4) whenever $a < 1/3$. Let $\{D_i\}_i \subset (\mathbb{C}P^2, T_a)$ be the images of all J-holomorphic Maslov index 2 disks of area a such that $p \in \partial D_i$, for a fixed point $p \in T_a$. We work over the coefficient group $\mathbb{Z}/8$. According to Table 4.1,

$$\sum_i \partial[D_i] = -8 \cdot \partial\beta = 0 \in H_1(T_a; \mathbb{Z}/8).$$

Moreover, according to Table 4.1 we have

$$\mathscr{OC}^{(2)}_{low}([p_{T_a}]) = 4H \in H_2(\mathbb{C}P^2;\mathbb{Z}/8)$$

as defined in Subsection 4.1.2. Note that the next to the smallest area A from Equation (4.6) equals $A = (1-a)/2$. It is well known that the monotone Clifford torus T_{Cl} bounds three Maslov index 2 J-holomorphic disks passing through a generic point, belonging to classes of the form β_1, β_2, $H-\beta_1-\beta_2 \in H_2(\mathbb{C}P^2, T_{Cl};\mathbb{Z})$ [30], see also [10, Proposition 5.5], and having area $b = 1/3$. So we obtain

$$\mathscr{OC}^{(2)}([p_{T_{Cl}}]) = H \in H_2(\mathbb{C}P^2;\mathbb{Z}/8).$$

Proof of Theorem 4.1.2. Since

$$\mathscr{OC}^{(2)}_{low}([p_{T_a}]) \cdot \mathscr{OC}^{(2)}([p_{T_{Cl}}]) = 4 \neq 0 \mod 8,$$

we are in shape to apply Theorem 4.1.4, provided that:

$$a+b = a+1/3 < A = \tfrac{1-a}{2}$$

i.e. $a < 1/9$. The case $a = 1/9$ follows by continuity. □

Remark 4.3.2. We are unable to prove that the tori T_a are non-displaceable from themselves using Theorem **??** because $\mathscr{OC}^{(2)}_{low}([p_{T_a}]) \cdot \mathscr{OC}^{(2)}_{low}([p_{T_a}]) = 16 \equiv 0 \mod 8$.

Remark 4.3.3. It is instructive to see why the argument cannot be made to work over $\mathbb{C}$ or $\mathbb{Z}$. Then $\sum_i \partial[D_i] = -8 \cdot \partial\beta$ is non-zero, but this can be fixed by introducing a local system $\rho\colon \pi_1(T_a) \to \mathbb{C}^\times$ taking $\alpha \mapsto -1$, $\beta \mapsto +1$. By definition, ρ is multiplicative, so for example, $\rho(\alpha+\beta) = \rho(\alpha)\rho(\beta)$. Then $\sum_i \rho(\partial[D_i]) \cdot \partial[D_i]$ equals

$$-(-2\partial\beta - \partial\alpha) + 2(-2\partial\beta) - (-2\partial\beta + \partial\alpha) = 0 \in H_1(T_a;\mathbb{C}).$$

However, in this case $\mathscr{OC}^{(2)}_{low}([p_{T_a}];\rho) = \sum_i \rho(\partial[D_i])[D_i]$ vanishes in $H_2(\mathbb{C}P^2;\mathbb{C})$, because the H-classes from Table 4.1 cancel in this sum.

4.3.4 A similar theorem for $\mathbb{C}P^1 \times \mathbb{C}P^1$

Using our technique, we can prove a similar non-displaceability result inside $\mathbb{C}P^1 \times \mathbb{C}P^1$, but it is probably less novel.

Theorem 4.3.6. *For each $a \in (0,1/4]$, the torus $\hat{T}_a \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ is Hamiltonian non-displaceable from the monotone Clifford torus $T_{Cl} \subset \mathbb{C}P^1 \times \mathbb{C}P^1$.*

Remark 4.3.4. We believe this theorem can be obtained by a short elaboration on [49]: for the bulk-deformation $\mathfrak{b}$ used in [49], there should exist local systems on $\hat{T}_a$ and T_{Cl} such that $HF^{\mathfrak{b}}(\hat{T}_a, T_{Cl}) \neq 0$, for $a \in (0,1/2]$. Alternatively, in addition to $HF^{\mathfrak{b}}(\hat{T}_a, \hat{T}_a) \neq 0$ as proved in [49], one can show that $HF^{\mathfrak{b}}(T_{Cl}, T_{Cl}) \neq 0$ for some local system, and apply a version of Theorem 4.1.3 using the unitality of the string maps and the semi-simplicity of the deformed quantum cohomology $QH^{\mathfrak{b}}(\mathbb{C}P^2)$. Our proof only works for $a \leq 1/4$, but is based on much simpler transversality foundations.

We work over the coefficient group $\mathbb{Z}/4$. By looking at Table 4.1, we see that for $a < 1/2$ we have

$$\mathscr{OC}^{(2)}_{low}([p_{\hat{T}_a}]) = 2(H_1 + H_2) \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z}/4),$$

and $A = 1 - a$. One easily shows that

$$\mathscr{OC}^{(2)}([p_{\hat{T}_{Cl}}]) = H_1 + H_2 \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z}/4),$$

since the Clifford torus bounds Maslov index 2 disks of area $b = 1/2$, passing once through each point of T_{Cl}, in classes of the form β_1, β_2, $H_1 - \beta_1$, $H_2 - \beta_2$ [30], see also [10, Section 5.4]. We cannot directly apply Theorem 4.1.4 because

$$\mathscr{OC}^{(2)}_{low}([p_{\hat{T}_a}]) \cdot \mathscr{OC}^{(2)}([p_{T_{Cl}}]) = 4 \equiv 0 \mod 4.$$

Nonetheless, we can form the cycle $H_2 \in H_2(\mathbb{C}P^1 \times \mathbb{C}P^1, \mathbb{Z}/4)$ using the disks in classes β_2 and $H_2 - \beta_2$ bounded by T_{Cl}.

Claim 4.3.7. *The fact that $\mathscr{OC}^{(2)}_{low}([p_{\hat{T}_a}]) \cdot H_2 = 2 \neq 0 \mod 4$ implies that T_a is non-displaceable from T_{Cl} provided that:*

$$a + b = a + 1/2 < A = 1 - a,$$

i.e. $a < 1/4$. (The case $a = 1/4$ follows by continuity.)

Sketch of proof. The idea is to consider the moduli space $\mathscr{M}$ as in Section 4.2, with the additional condition that the sum of the T_{Cl}-components of the boundaries of the curves in Figure 4.1 equals $[\partial\beta_2] \in H_1(T_{Cl}; \mathbb{Z}/2)$ (note the $\mathbb{Z}/2$ coefficients here). The reason the configurations of type (i) from Section 4.2 cancel in this modified setting is: for each class in $H_1(T_{Cl}; \mathbb{Z}/2)$, the boundaries of disks through a point on T_{Cl} in that $\mathbb{Z}/2$-class (e.g. β_2 and $-\beta_2$) cancel in $H_1(T_{Cl}; \mathbb{Z}/4)$, in fact even integrally. Then, as in the original proof, $0 = \sharp\partial\mathscr{M}$ equals the count of configurations of type (iii), and the latter equals

$\mathscr{OC}^{(2)}_{low}([p_{\hat{T}_a}]) \cdot H_2$ analogously to Lemma 4.2.1. Therefore the displaceability would imply $\mathscr{OC}^{(2)}_{low}([p_{\hat{T}_a}]) \cdot H_2 = 0$. □

4.3.5 The superpotentials

We conclude by an informal discussion of the superpotentials of the tori we study. The Landau-Ginzburg superpotential (further called "potential") associated to a Lagrangian 2-torus and an almost complex structure J is a Laurent series in two variables which combinatorially encodes the information about all J-holomorphic index 2 disks through a point on L. We refer to [10, 46, 49, 115] for the definitions; in the setting of Proposition 4.3.5, the potentials are given by

$$\mathfrak{PO}_{\mathbb{C}P^2} = t^{(1-a)/2}z + \frac{t^a}{z^2 w} + 2\frac{t^a}{z^2} + \frac{t^a w}{z^2} = t^{(1-a)/2}z + t^a \frac{(1+w)^2}{z^2 w}; \tag{4.10}$$

$$\mathfrak{PO}_{\mathbb{C}P^1\times\mathbb{C}P^1} = t^{1-a}z + \frac{t^a}{zw} + 2\frac{t^a}{z} + \frac{t^a w}{z} = t^{1-a}z + t^a \frac{(1+w)}{zw} + t^a \frac{(1+w)}{z}. \tag{4.11}$$

(These functions are sums of monomials corresponding to the disks as shown in Table 4.1.) Here t is the formal parameter of the Novikov ring Λ_0 associated with a ground field $\mathbb{K}$, usually assumed to be of characteristic zero:

$$\Lambda_0 = \left\{ \sum a_i t^{\lambda_i} \mid a_i \in \mathbb{K},\ \lambda_i \in \mathbb{R}_{\geq 0},\ \lambda_i \leq \lambda_{i+1},\ \lim_{i\to\infty} \lambda_i = \infty \right\}.$$

Let $\Lambda_\times$ be the field of elements of Λ_0 with nonzero constant term $a_0 t^0$. We can view $(\Lambda_\times)^2$ as the space of local systems $\pi_1(L) \to \Lambda_\times$ on a Lagrangian torus L; or equivalently [46, Remark 5.1] as the space $(\mathbb{C}^\times)^2 \cdot \exp(H_1(L;\Lambda_0))$ of exponentials of elements in $H_1(L;\Lambda_0)$, the so-called bounding cochains from the works of Fukaya, Oh, Ohta and Ono [45–47], extended by $\mathbb{C}^\times$-valued local systems. In turn, the potential can be seen as a function $(\Lambda_\times)^2 \to \Lambda_0$, and its critical points correspond to local systems $\sigma \in (\Lambda_\times)^2$ such that $HF^*(L,\sigma) \neq 0$ [46, Theorem 5.9]

If the potential has no critical points, it can sometimes be "improved" by introducing a bulk deformation $\mathfrak{b} \in H^{2k}(X;\Lambda_0)$ which deforms the function; critical points of the deformed potential correspond to local systems $\sigma \in (\Lambda_\times)^2$ such that $HF^{\mathfrak{b}}(L,\sigma) \neq 0$ [46, Theorem 8.4]. This was the strategy of [49] for proving that the tori $\hat{T}_a \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ are non-displaceable. When $\mathfrak{b} \in H^2(X;\Lambda_0)$, the deformed potential is still determined by Maslov index 2 disks (if $\dim X = 2n > 4$, this will be the case for $\mathfrak{b} \in H^{2n-2}(X;\Lambda_0)$), see e.g. [46, Theorem 8.2].

For bulk-deformation classes in other degrees, the deformed potential will use disks of all Maslov indices, and its computation becomes out of reach.

In contrast to the $\hat{T}_a$, the potential for the tori T_a does not acquire a critical point after we introduce a degree 2 bulk-deformation class $\mathfrak{b} \in H^2(\mathbb{C}P^2, \Lambda_0)$.

Proposition 4.3.8. *Unless $a = 1/3$, for any bulk deformation class $\mathfrak{b} \in H^2(\mathbb{C}P^2, \Lambda_0)$, the deformed potential $\mathfrak{PO}^{\mathfrak{b}}$ for the torus $T_a \subset \mathbb{C}P^2$ has no critical point in $(\Lambda_\times)^2$.*

Proof. Let $Q \subset \mathbb{C}P^2$ be the quadric which is the preimage of the top side of the traingle in Figure 4.6, so $[Q] = 2H$. Then $\mathfrak{b}$ must be Poincaré dual to $c \cdot [Q]$ for some $c \in \Lambda_0$. Among the holomorphic disks in Table 4.1, the only disk intersecting Q is the β-disk intersecting it once [115]. Therefore the deformed potential

$$\mathfrak{PO}^{\mathfrak{b}}_{\mathbb{C}P^2} = t^{(1-a)/2} e^c z + t^a \frac{(1+w)^2}{z^2 w}$$

differs from the usual one by the e^c factor by the monomial corresponding to the β-disk, compare [49]. Its critical points are given by

$$w = 1,\ z^3 = 8t^{(3a-1)/2} e^{-c}.$$

Unless $3a - 1 = 0$, the t^0-term of z has to vanish, so $z \notin \Lambda_\times$. □

Remark 4.3.5. If one ignores possible issues with multivalued perturbations, it is possible, at least formally, to speak of critical points of the potential and its bulk deformations using a ground field $\mathbb{K}$ of any characteristic (or even a ground ring). Local systems are then no longer exponentials of bounding cochains, but exist in their own right; similarly, the e^c-factor which is the result of bulk deformation above can be considered as an arbitrary element of $\Lambda_\times$. We see that $\mathfrak{PO}^{\mathfrak{b}}_{\mathbb{C}P^2}$ still has no critical points over any ground field when $a \neq 1/3$.

Keeping an informal attitude, let us drop the monomial $t^{(1-a)/2} z$ from Equation (4.10) of $\mathfrak{PO}_{\mathbb{C}P^2}$; denote the resulting function by $\mathfrak{PO}_{\mathbb{C}P^2,low}$. For $a < 1/3$, it reflects the information about least area holomorphic disks with boundary on $T_a \subset \mathbb{C}P^2$,

$$\mathfrak{PO}_{\mathbb{C}P^2,low} = t^a \frac{(1+w)^2}{z^2 w}. \tag{4.12}$$

Now, this function has plenty of critical points. Over $\mathbb{C}$, it has the critical line $w = -1$, and if one works over $\mathbb{Z}/8$ then the point $(1,1)$ is also a critical point, reflecting the fact the boundaries of the least area holomorphic Maslov index 2 disks on T_a cancel modulo 8, with the trivial local system.

Remark 4.3.6. If we remove the quadric $Q \subset \mathbb{C}P^2$ which is the preimage of the top side of the triangle in Figure 4.6, we get the unit cotangent bundle $D^*\mathbb{R}P^2 \subset T^*\mathbb{R}P^2$. The tori T_a, considered as Lagrangian submanifolds of $T^*\mathbb{R}P^2$, become monotone for each a, and (4.12) becomes their potential in the usual sense. The fact it has a critical point implies, this time by the standard machinery, that the tori $T_a \subset T^*\mathbb{R}P^2$ are non-displaceable. They are quotients of the non-displaceable tori in T^*S^2 [6] under a free $\mathbb{Z}/2$-action.

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