

Scalable Design of Heterogeneous Networks

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Abstract—A systematic approach to the analysis and design of a class of large dynamical systems is presented. The approach allows decentralised control laws to be designed independently using only local subsystem models. Design can be conducted using standard techniques, including loopshaping based on Nyquist and Popov plots, H_∞ methods, and μ -synthesis procedures. The approach is applied to a range of network models, including those for consensus, congestion control, electrical power systems, and distributed optimisation algorithms subject to delays.

Index Terms—Decentralised control, large dynamical systems.

I. INTRODUCTION

THE BENEFITS of well designed decentralised control laws to the operation of large dynamical systems are well known. For example, electrical power systems maintain the balance of electrical power using proportional feedback control laws [1], [2], and congestion control in the internet is managed by decentralised network protocols [3]. However the design of such control laws is notoriously difficult as a result of the sheer size of the systems in question, the complexity of subsystem dynamics, and the fact that the system may be changing structurally over time.

In this paper we address the problem of designing decentralised control laws using only local subsystem models. There is a growing literature on the design of decentralised controllers for large systems. Perhaps the first to define a formal notion of distributed design, and the loss of performance it might entail, is [4]. Notable among other approaches are those based on quadratic invariance [5], and see [6] for a survey of recent results. However, these approaches still require that the design process itself be centralised (although the recent paper [7] does show that a partially decentralized design is sometimes possible in this framework). Another burgeoning area is methods based on distributed or scalable optimisation techniques (e.g. [8]), in which the burden of stability verification and design is negotiated locally or more efficiently solved in a centralised manner, e.g. [9], [10], [11].

Our approach is very different to these. We make no claims of optimality, nor do we attempt to rigorously define localised design. We instead conduct design on the basis of decentralised robust stability and performance tests. This is achieved by allowing the subsystems to themselves result from a linear fractional transformation of local dynamics and a local controller, which can then be tuned to satisfy these tests.

The principle argument for conducting network design this way comes from its simplicity and its scalability, two features

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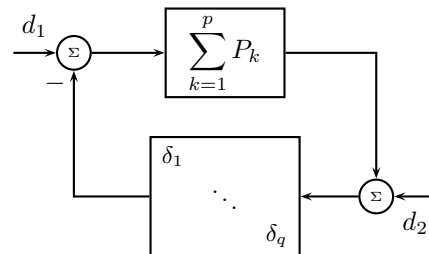


Fig. 1: Studied feedback interconnection.

that we argue are of paramount importance for applications. In particular:

- 1) Controller design using only local network models hugely simplifies the synthesis problems that need to be solved.
- 2) Any controller tunings found will be independent of the rest of the model, making it a valid design in networks of any size.
- 3) No communication network needs to be established to implement the control.
- 4) The design only need change if the local network model changes.

In essence, the approach transfers the burden of analysis and synthesis to the subsystems: provided every subsystem maintains the integrity of its local design requirement, then interconnections of any size are guaranteed to meet these requirements also. These advantages of course come at a price. In particular, working with only local network models certainly introduces conservatism, although this is frequently offset in practice by the extra degree of freedom gained by having a tractable synthesis problem. In light of 1)-4), we feel this is fair price to pay for network applications, where simplicity and robustness are often of primary concern.

This approach is very much in line with some of the early results in the field, c.f. [12], [13], or the passivity based approach of, e.g. [14]. This paper extends some recent results of this type [15], [16] using the strengths of an integral quadratic constraint (IQC) based approach [17], [18], a decomposition structure related to [19], and a generalisation of a relaxation argument from [20] (for a recent development, see [21]). A major strength of the the conditions we derive is that they allow the design to be conducted entirely locally using familiar techniques, including loopshaping, H_∞ methods, and μ -synthesis procedures, and can be applied even when highly detailed subsystem models are used.

More specifically, we present a systematic approach to the analysis and design of simple decentralised control laws for P_k, δ_i in fig. 1. In the context of network models, each P_k, δ_i is an operator that describes the dynamics of a subsystem. This framework can capture the models for several network

applications, including: consensus problems [22], flocking phenomena [23], internet congestion control [3], electrical power systems [2] and distributed optimisation [24], [25]. Directly imposing this structure on the network model allows the subsystems to be described by realistic heterogeneous models, including delays and other higher order dynamics, while maintaining enough structural features to facilitate analysis.

In general the design task is difficult since the feedback interconnection couples all the local design choices. The approach taken in this paper is to relax the standard IQC criteria of Megretski and Rantzer [17] to derive sets of tests that can be checked using only local information. When each P_k is linear and each δ_i captured by an IQC, the decentralised tests take the form:

$$\begin{bmatrix} P_k \\ I \end{bmatrix}^* \begin{bmatrix} -X_k & Y_k^* \\ Y_k & -Z_k \end{bmatrix} \begin{bmatrix} P_k \\ I \end{bmatrix} (j\omega) \geq \epsilon P_k^* P_k (j\omega). \quad (1)$$

The functions X_k, Y_k, Z_k can be determined from the IQCs for their local δ_i 's. Critically satisfaction of the tests for all the P_k 's is sufficient to guarantee robust stability of the network as a whole. Hence each condition in eq. (1) can be used to design the control systems local to each P_k using only local information, so as to be robust to rich classes of uncertainty, while meeting pre-specified performance requirements [17], [26]. We additionally present a dual approach that decomposes the problem by capturing each P_k with an IQC, and a set of frequency domain inequalitys (FDIs) constructed for each δ_i . Satisfaction of these FDIs is also sufficient to meet the standard IQC stability criteria, allowing them to be used to design control systems local to each δ_i as discussed above.

We illustrate the approach by showing how the classical intuition behind several IQCs can be transferred into the network setting using our method. More specifically, we give Nyquist based stability criteria for automatic generation control (AGC) in electrical power systems. In addition we reproduce standard convergence results for gradient based distributed optimisation algorithms, and show how the step sizes can be redesigned to make the algorithms robust to the presence of heterogeneous delays. Furthermore we connect the familiar passivity and small gain type conditions, as well as the Nyquist type conditions in [20] to particular choices of X_k, Y_k, Z_k .

II. NOTATION

Let \mathbf{RL}_∞ be the set of proper rational functions with real coefficients that are bounded on the imaginary axis, and \mathbf{RH}_∞ its subset with no poles in the closed right half plane. Let \mathbf{L}_2 be the set of square integrable functions $x : [0, \infty) \mapsto \mathbb{R}$, and \mathbf{L}_{2e} the set of functions $x : [0, \infty) \mapsto \mathbb{R}$ that need only be square integrable on finite intervals. An operator is a function $F : \mathbf{L}_{2e}^m \mapsto \mathbf{L}_{2e}^n$. An operator is said to be causal if $P_T F = P_T F P_T$ for any $T > 0$, where P_T is the past projection operator¹, and bounded if the operator norm

$$\|F\| = \sup \left\{ \frac{\|Fx\|}{\|x\|} : x \in \mathbf{L}_2^m, x \neq 0 \right\}$$

¹ P_T leaves a function unchanged on the interval $[0, T]$, and equal to zero everywhere else.

is finite. Denote the (negative) feedback interconnection of operators A, B , subject to disturbances d_1, d_2 :

$$\begin{aligned} v &= Aw + d_1 \\ w &= -Bv + d_2 \end{aligned} \quad (2)$$

as $[A, B]$. This interconnection is well posed if the map $(v, w) \mapsto (d_1, d_2)$ defined by eq. (2) has a causal inverse on \mathbf{L}_{2e} , and stable if there exists a $C > 0$ such that

$$\int_0^T (v^T v + w^T w) (t) dt \leq C \int_0^T (d_1^T d_1 + d_2^T d_2) (t) dt$$

for any $T > 0$ and solution to eq. (2).

Definition 1 (IQC): For $\Pi \in \mathbf{RL}_\infty^{(n+m) \times (n+m)}$ where in addition $\Pi(j\omega) = \Pi(j\omega)^*$, define $\text{IQC}(\Pi)$ to be the set of bounded causal operators $D : \mathbf{L}_{2e}^n \mapsto \mathbf{L}_{2e}^n$ such that $D \in \text{IQC}(\Pi)$ if and only if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v} \\ \tau \hat{w} \end{bmatrix}^* \Pi \begin{bmatrix} \hat{v} \\ \tau \hat{w} \end{bmatrix} (j\omega) d\omega \geq 0, \forall w = Dw, v \in \mathbf{L}_2^n,$$

for all $\tau \in [0, 1]$, where the hat symbol denotes the Fourier transform.

Finally, let $\mathbb{Z}_{[a,b]}$ be the set of integers $\{a, a+1, \dots, b\}$, $A \oplus B$ be the direct sum of A and B , and $\bigoplus_{i=1}^n A_i = A_1 \oplus \dots \oplus A_n$.

III. STABILITY CRITERION

Theorem 1 below forms the basis of the approach to scalable design studied in this paper. It gives a condition for testing stability of the interconnection $[P, D]$, when P can be decomposed as

$$P = \sum_{k=1}^p P_k, \quad (3)$$

and it is known that $D \in \text{IQC}(\Pi)$. This result is obtained from a relaxation of the IQC theorem of Megretski and Rantzer ([17], Theorem 1), and when $p = 1$, the two criteria are equivalent. The theorem splits the analysis problem into an FDI for each P_k (eq. (4)), and a coupling constraint involving the multiplier Π and a set of weighting functions (eq. (5)). This is a very general theorem which, in itself, neither gives any hint into, or imposes any restrictions onto, how the problem may be split up. Importantly, we will later show how the interconnection structure itself can always be exploited to provide suitable decompositions satisfying the conditions of the theorem, and this is how we envisage the result being used.

Theorem 1: Let $P = \sum_{k=1}^p P_k$ and $D \in \text{IQC}(\Pi)$, where each P_k is a bounded causal operator and

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2^* \\ \Pi_2 & \Pi_3 \end{bmatrix}.$$

Assume that $[P, \tau D]$ is well posed $\forall \tau \in [0, 1]$. If for each $k \in \mathbb{Z}_{[1,p]}$ there exist $X_k, Z_k \in \mathbf{RL}_\infty$ and an $\epsilon_k > 0$ such that

(i) for each $k \in \mathbb{Z}_{[1,p]}$:

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v} \\ \hat{w} \end{bmatrix}^* \begin{bmatrix} -(X_k + \epsilon_k I) & \Pi_2^* \\ \Pi_2 & -Z_k \end{bmatrix} \begin{bmatrix} \hat{v} \\ \hat{w} \end{bmatrix} (j\omega) d\omega \geq 0, \quad (4)$$

whenever $w \in \mathbf{L}_2^n$ and $v = P_k w$, and
(ii)

$$\sum_{k=1}^p Z_k(j\omega) \geq \Pi_3(j\omega) \quad (5)$$

$$\sum_{k=1}^p \int_{-\infty}^{\infty} \hat{v}_k^* X_k \hat{v}_k(j\omega) d\omega \geq \int_{-\infty}^{\infty} \hat{v}^* \Pi_1 \hat{v}(j\omega) d\omega$$

whenever $w \in \mathbf{L}_2^n$, $v = Pw$ and $v_k = P_k w$, $k \in \mathbb{Z}_{[1,p]}$, then $[P, D]$ is stable.

Proof: We will show that satisfaction of eqs. (4) and (5) guarantee that P satisfies the conditions of the Theorem 1 in [17]. In turn this guarantees that $[P, D]$ is stable.

For stability of $[P, D]$, Theorem 1 of [17] (when combined with Remark 3 from the same paper) requires that there exists an $\epsilon > 0$ such that $\forall w \in \mathbf{L}_2$

$$\int_{-\infty}^{\infty} -\hat{v}^* \Pi_1 \hat{v} + \hat{w}^* \Pi_2 \hat{v} + \hat{v}^* \Pi_2^* \hat{w} - \hat{w}^* \Pi_3 \hat{w} d\omega \geq \int_{-\infty}^{\infty} \epsilon \hat{v}^* \hat{v} d\omega, \quad (6)$$

where $v = Pw$, and the hat symbol indicates the Fourier transform. Satisfaction of eq. (4) guarantees that for any $w \in \mathbf{L}_2$:

$$\int_{-\infty}^{\infty} -\hat{v}_k^* X_k \hat{v}_k + \hat{w}^* \Pi_2 \hat{v}_k \dots$$

$$\dots + \hat{v}_k^* \Pi_2^* \hat{w} - \hat{w}^* Z_k \hat{w} d\omega \geq \int_{-\infty}^{\infty} \epsilon_k \hat{v}_k^* \hat{v}_k d\omega,$$

where $v_k = P_k w$. Therefore

$$\sum_{k=1}^p \int_{-\infty}^{\infty} -\hat{v}_k^* X_k \hat{v}_k + \hat{w}^* \Pi_2 \hat{v}_k \dots$$

$$\dots + \hat{v}_k^* \Pi_2^* \hat{w} - \hat{w}^* Z_k \hat{w} d\omega \geq \sum_{k=1}^p \int_{-\infty}^{\infty} \epsilon_k \hat{v}_k^* \hat{v}_k d\omega. \quad (7)$$

Satisfaction of eq. (5) guarantees that:

$$\int_{-\infty}^{\infty} \hat{v}^* \Pi_1 \hat{v} d\omega \leq \sum_{k=1}^p \int_{-\infty}^{\infty} \hat{v}_k^* X_k \hat{v}_k d\omega \quad (8)$$

$$\int_{-\infty}^{\infty} \hat{w}^* \Pi_3 \hat{w} d\omega \leq \sum_{k=1}^p \int_{-\infty}^{\infty} \hat{w}^* Z_k \hat{w} d\omega.$$

Combining eqs. (7) and (8) shows that

$$\int_{-\infty}^{\infty} -\hat{v}^* \Pi_1 \hat{v} + \hat{w}^* \Pi_2 \hat{v} + \hat{v}^* \Pi_2^* \hat{w} - \hat{w}^* \Pi_3 \hat{w} d\omega \geq \sum_{k=1}^p \int_{-\infty}^{\infty} \epsilon_k \hat{v}_k^* \hat{v}_k d\omega,$$

which guarantees that eq. (6) is satisfied as required. ■

IV. A SCALABLE APPROACH TO CONTROLLER DESIGN

A. Network description

This paper is devoted to analysis and design of network models with dynamics captured by the interconnection

$$\left[\sum_{k=1}^p P_k, \bigoplus_{i=1}^q \delta_i \right]. \quad (9)$$

In this context, eq. (9) represents a general modelling class for describing the dynamics of networks that are composed

of subsystems with dynamics P_k or δ_i . As mentioned in the introduction, and as we shall later demonstrate with our examples, this framework can capture the models for a large class of network applications. The main reason for imposing these structural features up front is that they will define the structure of the analysis and synthesis procedures given in the next section. In particular we will show how to do robustness analysis and controller design on a P_k by P_k (and δ_i by δ_i) basis. Loosely speaking, specifying this structure *a priori* allows the designer to choose the resolution on which they wish to conduct design. This should be contrasted with other approaches, where the notion of structure is typically inherited from more abstract features, such as sparsity patterns.

As well as being rather general, we argue that a large number of models in the literature can be decomposed into eq. (9) in a ‘natural way’. For example, consider the ‘agent-digraph’ interconnection:

$$y = Ax$$

$$\dot{x} = y, x(0) \in \mathbb{R}^q,$$

where $A \in \mathbb{R}^{q \times q}$ is a sparse matrix, and the i th entry of x corresponds to the state of the i th agent. A can always be decomposed into matrices with nonzero 2×2 subblocks, one for each communication between agents, for example

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

This suggests a decomposition of the form eq. (9) in which each P_k would correspond to a communication link, and each δ_i the dynamics of one of the agents. Design on a P_k by P_k basis would then correspond to design on a link by link basis (and δ_i by δ_i to agent by agent design). There are of course other ways to capture this structure. The main argument for eq. (9) is that it can capture structural features for a wide range of other networked systems. It can also provide additional decomposition flexibility, for example one could instead decompose A into matrices that are not necessarily rank 1, or small subnetworks of links.

In the following we will formalise the structure of eq. (9) based on an *incidence matrix* $R \in \mathbb{R}^{q \times p}$, $R_{ik} \in \{0, 1\}$. This matrix not only describes the structure of the interconnection, but also the decomposition structures being used throughout.

Definition 2 (Subspace projection): For $s \in \mathbb{R}^q$ with elements $s_i \in \{0, 1\}$, define the subspace projection operator Q_s as

$$(Q_s x)(t) = \begin{bmatrix} s_1 x_1(t) \\ \vdots \\ s_q x_q(t) \end{bmatrix}.$$

Assumption 1: For each $k \in \mathbb{Z}_{[1,p]}$, $P_k : \mathbf{L}_{2e}^q \mapsto \mathbf{L}_{2e}^q$ is a bounded causal operator that satisfies

$$P_k x = Q_{R_{\bullet k}} P_k Q_{R_{\bullet k}} x, \forall x \in \mathbf{L}_{2e}^q,$$

where $R_{\bullet k}$ denote the k th column of R . That is the k th column of R determines the ‘sparsity’ of P_k , indicating which elements of the inputs and outputs of P_k are dynamically coupled.

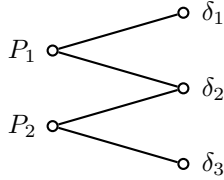


Fig. 2: Bipartite graph for R given by eq. (10). There is one vertex for each P_k , and one for each δ_i , with an edge connecting them if and only if $R_{ik} = 1$.

Assumption 2: For all $i \in \mathbb{Z}_{[1,q]}$, $\delta_i : \mathbf{L}_{2e} \mapsto \mathbf{L}_{2e}$ is a bounded causal operator.

Remark 1: If the component models are linear, these assumptions mean that the dynamics of P_k, δ_i are described by stable transfer functions. For the models for a range of applications, this restriction is too strict. If this is caused by an integrator (as is common in consensus and flocking problems), this issue can often be finessed using loop transformations. We will see two instances of this in the examples. In general this restriction can be accommodated by designing locally stabilising controllers for any unstable components.

These assumptions give the operators P_k and $\bigoplus_{i=1}^q \delta_i$ a local structure with respect to signals in \mathbf{L}_{2e}^q . The example above corresponds to the choice

$$R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad (10)$$

implying

$$P_1 = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \bigoplus_{i=1}^q \delta_i = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix},$$

where a star in the ij th position in the above matrices indicates that the corresponding operator defines a nonzero map between the j th and i th element of its input and output.

Speaking abstractly, the incidence matrix R describes the relationship between objects associated with each P_k , and objects associated with each δ_i . This can be visualised on a bipartite graph, with one vertex for each P_k , and one vertex for each δ_i . An edge is drawn between P_k and δ_i if and only if $R_{ik} = 1$. This is sketched in fig. 2. All notions of locality and decomposition considered in this paper will be derived from this structural picture.

Remark 2: All the presented stability results can be easily extended to the case when the δ_i 's are not necessarily scalar. To do this, all that needs to be done is to perform all the matrix operations described in this paper blockwise, where the size of the blocks is determined by that of the δ_i 's. To describe this precisely leads to a sharp increase in notational complexity, and has hence been omitted. Similarly, it is possible to state the results for the more general interconnection

$$\left[\sum_{k=1}^p P_k, \sum_{i=1}^q \delta_i \right],$$

where the δ_i 's have an overlapping structure similar to the P_k 's (the block diagonal form being a special case of this).

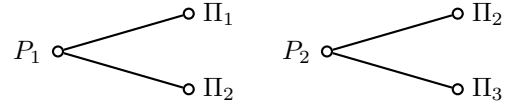


Fig. 3: Illustration of the structure of the tests in Proposition 1 for R given by eq. (10). There is one test for each P_k , dependent only the adjacent Π_i 's. This corresponds to decomposing fig. 2 into a set of p subgraphs, each consisting of the vertices for a given P_k and its neighbouring δ_i 's.

However, this generality is not required for any application we can envisage.

B. The Scalable Stability Criterion

The principal difficulty in using Theorem 1 for performing analysis and design stems from the need to select the weights X_k, Z_k compatibly with the IQC description of the δ_i 's. Motivated by the desire to obtain simple and scalable design conditions, we propose a systematic method for selecting them. Proposition 1 (below) follows from arguably the simplest such choice, and allows stability of eq. (9) to be deduced by checking a test for each P_k . The i th IQC is required to check the k th test if and only if $R_{ik} = 1$. Hence this result decomposes the analysis problem based on the interconnection structure described by R . This corresponds to a type of network decomposition, as sketched in fig. 3. For simplicity we also restrict the operators P_k to be linear.

Definition 3 (Submatrix structure): For $s \in \mathbb{R}^q$ with elements $s_i \in \{0, 1\}$, define $\mathbf{RH}_\infty[s] \subseteq \mathbf{RH}_\infty^{q \times q}$ such that $A \in \mathbf{RH}_\infty[s]$ if and only if

$$A_{ik} \in \begin{cases} \mathbf{RH}_\infty & \text{if both } s_i = 1 \text{ and } s_k = 1, \\ \{0\} & \text{otherwise.} \end{cases}$$

Define $\mathbf{RL}_\infty[s]$ in an analogous manner.

Remark 3: A linear operator meeting assumption 1 will have a transfer function representation in $\mathbf{RH}_\infty[R_{\bullet k}]$.

Proposition 1: Let $R \in \mathbb{R}^{q \times p}$, $P = \sum_{k=1}^p P_k$, and $D = \bigoplus_{i=1}^q \delta_i$, where $R_{ik} \in \{0, 1\}$, $P_k \in \mathbf{RH}_\infty[R_{\bullet k}]$, $\delta_i \in \text{IQC}(\Pi_i)$, and

$$\Pi_i = \begin{bmatrix} A_i & B_i^* \\ B_i & C_i \end{bmatrix} \in \mathbf{RL}_\infty^{2 \times 2}.$$

Define

$$X_k = \bigoplus_{i=1}^q n_i R_{ik} A_i, Y_k = \bigoplus_{i=1}^q R_{ik} B_i, Z_k = \bigoplus_{i=1}^q \frac{R_{ik}}{n_i} C_i, \quad (11)$$

where $n_i = \sum_{k=1}^p R_{ik}$. If $[P, \tau D]$ is well posed $\forall \tau \in [0, 1]$, and for each $k \in \mathbb{Z}_{[1,p]}$ there exists an $\epsilon_k > 0$ such that $\forall \omega \in \mathbb{R}$

$$\begin{bmatrix} P_k \\ I \end{bmatrix}^* \begin{bmatrix} -(X_k + \epsilon_k I) & Y_k^* \\ Y_k & -Z_k \end{bmatrix} \begin{bmatrix} P_k \\ I \end{bmatrix} (j\omega) \geq 0, \quad (12)$$

then $[P, D]$ is stable.

Proof: Since $\delta_i \in \text{IQC}(\Pi_i)$,

$$D \in \text{IQC} \left(\begin{bmatrix} \bigoplus_{i=1}^q A_i & \bigoplus_{i=1}^q B_i^* \\ \bigoplus_{i=1}^q B_i & \bigoplus_{i=1}^q C_i \end{bmatrix} \right).$$

The result will follow from Theorem 1 if the particular choice of X_k, Z_k in eq. (11) satisfies eq. (5) (the inclusion of R_{ik} in the definition of Y_k follows since $Y_k P_k = (\bigoplus_{i=1}^q B_i) P_k$ as a result of the sparsity of P_k). We first establish the inequality in X_k , which follows from a simple generalisation of the well know inequality:

$$\|x + y\|^2 \leq 2 \left(\|x\|^2 + \|y\|^2 \right).$$

Let $V \in \mathbf{L}_2^{q \times p}$, where $V_{\bullet k} = \mathbf{Q}_{R_{\bullet k}} V_{\bullet k}$, and $v = \sum_{k=1}^p V_{\bullet k}$. Substitution of the IQCs from Proposition 1 into eq. (5) shows that we must establish that $\forall i \in \mathbb{Z}_{[1, q]}$:

$$\sum_{k=1}^p \int_{-\infty}^{\infty} \hat{V}_{ik}^* n_i R_{ik} A_i \hat{V}_{ik} d\omega \geq \int_{-\infty}^{\infty} \hat{v}_i^* A_i \hat{v}_i d\omega. \quad (13)$$

It is sufficient to show that this holds frequency by frequency, i.e.

$$\sum_{k=1}^p \hat{V}_{ik}^* n_i R_{ik} A_i \hat{V}_{ik} \geq \hat{v}_i^* A_i \hat{v}_i, \forall \omega \in \mathbb{R}. \quad (14)$$

Now $A_i \geq 0$, otherwise $\delta_i \notin \text{IQC}(\Pi_i)$ (this follows because by Definition 1 it is required that $\tau \delta_i \in \text{IQC}(\Pi)$, $\forall \tau \in [0, 1]$, and the case $\tau = 0$ requires $A_i \geq 0$). Proceeding frequency by frequency, let A_i have Cholesky factorisation $A_i = L_i^* L_i$, and put $\hat{y}_i = L_i \hat{v}_i$ and $\hat{Y}_{ik} = L_i \hat{V}_{ik}$. Substituting this into eq. (13) gives the equivalent statement

$$n_i \sum_{k=1}^p R_{ik} \left\| \hat{Y}_{ik} \right\|^2 \geq \|\hat{y}_i\|^2, \quad (15)$$

where $\|\cdot\|$ gives the 2-norm. By definition, if $R_{ik} \neq 0$ then $\hat{V}_{ik} \neq 0$. Therefore

$$\begin{aligned} \|\hat{y}_i\|^2 &= \left\| \sum_{k=1}^p R_{ik} \hat{Y}_{ik} \right\|^2, \\ &\leq n_i \sum_{k=1}^p R_{ik} \left\| \hat{Y}_{ik} \right\|^2 \end{aligned} \quad (16)$$

by Jensen's inequality, as required.

We will now establish the inequality in Z_k . In fact this is an equality, since $\sum_{k=1}^p \frac{R_{ik}}{n_i} = 1$, and

$$\sum_{k=1}^p \bigoplus_{i=1}^q \frac{R_{ik}}{n_i} C_i = \bigoplus_{i=1}^q C_i \left(\sum_{k=1}^p \frac{R_{ik}}{n_i} \right).$$

For analysis, we envisage Proposition 1 being used in the following way:

- 1) Fix the IQC description of the δ_i 's.
- 2) Select X_k, Y_k, Z_k as defined.
- 3) Test eq. (12).

Critically this process is decoupled in k and can be implemented in a decentralised manner. A serious concern here is that fixing the IQC description could be very conservative. However if well chosen, this is not necessarily the case. Nowhere is this more clearly illustrated than in the passivity based approach to network design, in which passivity is a fixed, predefined, requirement. Proposition 1 essentially generalises this approach to a richer dynamical class of dynamical

properties. We recommend that these be chosen based on intuition about the expected dynamics of the P_k 's, as will be illustrated in the examples.

The real advantage (and one of the main reasons the passivity approach is such a good network control method) of decoupling the analysis in this way is that it enables decentralised design. In this respect eq. (12) defines a set of robust performance problems that can be solved locally and independently. To see this, suppose that

$$P_k = \mathcal{F}_l(H_k, K_k), \quad (17)$$

where H_k is a (possibly unstable) model, and each K_k is a local controller to be chosen. Proposition 1 then defines the following design objectives: for each $k \in \mathbb{Z}_{[1, p]}$, choose K_k such that

- 1) P_k is stable;
- 2) eq. (12) is satisfied.

These design objectives are also decoupled in k , so each K_k can be selected independently. Furthermore, since eq. (12) is an IQC type condition, as a synthesis task it is equivalent to an \mathbf{H}_∞ control problem, and can be solved with standard techniques.

In Section IV-D we will indicate various ways to reduce conservatism by relaxing 1) and 2) through the use of local negotiations. This will not be necessary for any of the examples considered, where IQCs picked using graphical methods will be sufficient. However, for more complex applications, some level of communication or coordination may be appropriate. In this context this process can be viewed iteratively, in which 3) is used to conduct decentralised analysis and design (with scalable guarantees), before returning to 1) and 2) and refining the description of the IQCs (potentially leveraging the ideas in section IV-D).

This entire process is naturally extended to include local measures of robustness and performance. Design with respect to an uncertain $H_k = \mathcal{F}_u(G_k, \Delta_k)$ in eq. (17), where $\Delta_k \in \text{IQC}(\Phi_k)$, can be cast as a robust performance problem of the type considered in [27] (eq. (12) becomes the performance requirement), and tackled with iterative methods. Additional performance requirements, for example \mathbf{L}_2 -gain bounds between signals local to the given P_k , can be appended to this requirement in the usual way. Importantly the IQCs describing the local uncertainty and performance measures for the subsystem P_k are also decoupled in k , and can therefore be optimised locally.

It should also be noted that although the dimension of eq. (12) can be large (it is a $q \times q$ FDI, where q is the number of subsystems δ_i), it's complexity is also locally determined. This follows since

$$\begin{bmatrix} P_k \\ I \end{bmatrix}^* \begin{bmatrix} -(X_k + \epsilon_k I) & Y_k^* \\ Y_k & -Z_k \end{bmatrix} \begin{bmatrix} P_k \\ I \end{bmatrix} \in \mathbf{RL}_\infty [R_{\bullet k}].$$

Therefore eq. (12) can be checked on a lower dimensional space by deleting the rows and columns in the above matrix for each i such that $R_{ik} = 0$ (these rows and columns will be equal to zero, and do not affect positive semidefiniteness). The resulting lower dimensional FDI can be checked by frequency gridding, or by the convex feasibility test obtained

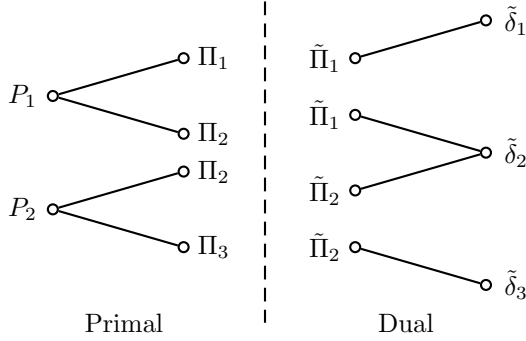


Fig. 4: Illustration of the structure of stability tests for eq. (9) when using Proposition 1 (left decomposition) and Proposition 2 (right decomposition). Analysis with Proposition 1 captures each $\delta_i \in \text{IQC}(\Pi_i)$, leading to a test for each P_k . Analysis with Proposition 2 captures each $P_k \in \text{IQC}(\tilde{\Pi}_k)$, leading to a test for each δ_i . Both correspond to a structured decomposition of the same bipartite graph.

by converting eq. (12) into a linear matrix inequality (LMI) using the Kalman-Yakubovich-Popov lemma as usual (see e.g. [26] for details of the required computations).

C. Dual Scalable Stability Criterion

Proposition 2 below gives an alternative stability criterion for eq. (9). This criterion takes the form of a test for each δ_i , based on an IQC description of the operators P_k . The k th IQC is required to check the i th test if and only if $R_{ik} = 1$. This decomposition of the problem is sketched in fig. 4. Observe the graph decomposition of the dual is analogous to that for the primal, but for a graph structured by R^T in place of R .

This result follows from applying the same steps as in Proposition 1 but to a rearrangement of the principal feedback interconnection eq. (9) in which the δ_i 's appear in the summation, and the P_k 's the direct sum. The underlying structure of this rearrangement is essentially described by R^T , however for notational reasons (discussed at the end of this section) this process introduces a large quantity of redundant variables obscuring this. It is therefore helpful to have a statement of the result in terms of the original problem variables:

Proposition 2: Let $R \in \mathbb{R}^{q \times p}$, $P = \sum_{k=1}^p P_k$, and $D = \bigoplus_{i=1}^q \delta_i$, where $R_{ik} \in \{0, 1\}$, $P_k \in \text{IQC}(\Pi_k)$, $\delta_i \in \mathbf{RH}_\infty$, and

$$\tilde{\Pi}_k = \begin{bmatrix} \bigoplus_{i=1}^q R_{ik} A_{ik} & \bigoplus_{i=1}^q R_{ik} B_{ik}^* \\ \bigoplus_{i=1}^q R_{ik} B_{ik} & \bigoplus_{i=1}^q R_{ik} C_{ik} \end{bmatrix},$$

where $A_{ik}, B_{ik}, C_{ik} \in \mathbf{RL}_\infty$. Define

$$X_i = \bigoplus_{k=1}^p R_{ik} A_{ik}, \quad Y_i = \bigoplus_{k=1}^p R_{ik} B_{ik}, \quad Z_i = \bigoplus_{k=1}^p R_{ik} C_{ik}.$$

If $[P, \tau D]$ is well posed $\forall \tau \in [0, 1]$, and for each $i \in \mathbb{Z}_{[1, q]}$ there exists an $\epsilon_i > 0$ such that $\forall \omega \in \mathbb{R}$

$$\begin{bmatrix} R_{i\bullet}^T \delta_i R_{i\bullet} \\ I \end{bmatrix}^* \begin{bmatrix} -(X_i + \epsilon_i I) & Y_i^* \\ Y_i & -Z_i \end{bmatrix} \begin{bmatrix} R_{i\bullet}^T \delta_i R_{i\bullet} \\ I \end{bmatrix} (j\omega) \geq 0, \quad (18)$$

then $[P, D]$ is stable.

Proof: Define

$$E = \left[\bigoplus_{i=1}^q R_{i1} \quad \dots \quad \bigoplus_{i=1}^q R_{ip} \right].$$

The proof will be given in two parts. First we will establish that stability of the following interconnections are equivalent:

- (i) $[\sum_{k=1}^p P_k, \bigoplus_{i=1}^q \delta_i]$ is stable.
- (ii) $[\sum_{i=1}^q E_i^T \delta_i E_{i\bullet}, \bigoplus_{k=1}^p P_k]$ is stable.

We will then apply Theorem 1 in an analogous manner to Proposition 1 to (ii).

To see the equivalence of (i) and (ii), first observe that

$$\sum_{i=1}^q E_i^T \delta_i E_{i\bullet} = E^T D E.$$

Since P_k and δ_i are bounded causal operators, stability of $[\sum_{k=1}^p P_k, \bigoplus_{i=1}^q \delta_i]$ is equivalent to stability of $[D, E (\bigoplus_{k=1}^p P_k) E^T]$. The equivalence follows since

$$E \left(\bigoplus_{k=1}^p P_k \right) E^T = \sum_{k=1}^p P_k.$$

We now decompose the stability problem on (ii). This may be done exactly as in Proposition 1, but using E in place of R , though extra care must be taken as the blocks of $\tilde{\Pi}_k$ are not scalar. Stacking up the IQCs for the P_k 's shows that $\bigoplus_{k=1}^p P_k \in \text{IQC}(\Phi)$, where

$$\Phi = \begin{bmatrix} \bigoplus_{k=1}^{qp} \text{vec}(R)_k \text{vec}(A)_k & \bigoplus_{k=1}^{qp} \text{vec}(R)_k \text{vec}(B)_k^* \\ \bigoplus_{k=1}^{qp} \text{vec}(R)_k \text{vec}(B)_k & \bigoplus_{k=1}^{qp} \text{vec}(R)_k \text{vec}(C)_k \end{bmatrix}.$$

Here $\text{vec}(\cdot)_k$ denotes the k th element of the vectorised matrix. Decomposing this IQC as in Proposition 1 gives

$$\bar{X}_i = \bigoplus_{k=1}^{pq} n_k E_{ik} \text{vec}(A)_k,$$

with analogous expressions for \bar{Y}_i, \bar{Z}_i . In the above $n_k = \sum_{i=1}^q E_{ik}$, which is in fact always equal to 1. The triple $(\bar{X}_i, \bar{Y}_i, \bar{Z}_i)$ would define a decomposed test on $E_{i\bullet}^T \delta_i E_{i\bullet}$. This test is equivalent to eq. (18). This is because the i th row of E is equal to the i th row of R , but with additional zeros. Removing these does not affect the inequalities, and results in the more compact representation as given. ■

Remark 4: Unlike in Proposition 1, no scaling of the IQCs (through constants analogous to the n_i 's) is required in Proposition 2. This is because this scaling is implicit in performing a test on $R_{i\bullet}^T \delta_i R_{i\bullet}$ as opposed to just δ_i (the $R_{i\bullet}$ 'increases the gain' of δ_i).

Analysis and design with respect to Proposition 2 can be done as described in the previous section, but with the rôles of P_k and δ_i reversed. Just as before, the complexity of eq. (18) is determined locally. It is simple to show that

$$\begin{bmatrix} R_{i\bullet}^T \delta_i R_{i\bullet} \\ I \end{bmatrix}^* \begin{bmatrix} -(X_i + \epsilon_i I) & Y_i^* \\ Y_i & -Z_i \end{bmatrix} \begin{bmatrix} R_{i\bullet}^T \delta_i R_{i\bullet} \\ I \end{bmatrix} \in \mathbf{RL}_\infty [R_{i\bullet}^T].$$

Therefore eq. (18) can be checked on a lower dimensional space by deleting the rows and columns in the above matrix for each k such that $R_{ik} = 0$.

Proposition 2 follows from the same argument as in Proposition 1 after performing a loop rearrangement operation. This

is possible because this operation rewrites the summation on the P_k 's as a direct sum, and vice versa for the δ_i 's, essentially exchanging their rôles for applying Theorem 1 (c.f. interconnection (i) and (ii) in the proof).

This operation is not (quite) an involution, since to make this rewriting possible, copies of the inputs and outputs of P_k 's and δ_i 's need to be introduced for reasons of dimensional compatibility. This means that after applying the operation to the dual interconnection (interconnection (ii)) one does not obtain (i) again, but in fact obtains:

$$\left[\sum_{k=1}^p \bar{E}_{\bullet k} P_k \bar{E}_{\bullet k}^T, \bigoplus_{i=1}^q E_{i\bullet}^T \delta_i E_{i\bullet} \right]. \quad (19)$$

In the above $\bar{E}_{\bullet k}$ is the k th $pq^2 \times q$ blockwise column of \bar{E} , which is given by

$$\bar{E}^T = \left[\bigoplus_{k=1}^{pq} E_{1k} \quad \dots \quad \bigoplus_{k=1}^{pq} E_{qk} \right],$$

and E as in the proof of Proposition 2. The differences between eq. (19) and (i) are only superficial, and questions of stability, and finding IQCs for δ_i v.s. IQCs for $E_{i\bullet}^T \delta_i E_{i\bullet}$, are equivalent. In the context of Proposition 1, it can be shown, for example, that if Proposition 1 is satisfied on (i), then there exist IQCs such that Proposition 1 is satisfied on eq. (19). Much of this (and the notation throughout the paper) can be formalised using the language of incidence structures, though this does not aid the analysis conditions obtained here and will not be pursued further.

D. Reducing Conservatism

Making a systematic choice of the functions X_k, Z_k as proposed in Proposition 1, as opposed to searching for them in some way to satisfy eq. (5) in Theorem 1, certainly introduces conservatism. The advantage gained is that it decouples the robust performance problems that need to be solved. This results in tractable, local, synthesis conditions that can be applied even when realistic component models are considered. We argue that in the context of networks this is a price worth paying, and illustrate by example that the degrees of freedom opened up by having a tractable synthesis problem is often sufficient to overcome any conservatism introduced, even when the components have complex dynamics. Nevertheless, it is of course possible that the problems resulting from this systematic choice are infeasible even when choices of X_k, Z_k satisfying eq. (5) that result in feasible problems exist. In this subsection we will indicate a straightforward way in which conservatism can be reduced in Proposition 1.

In the proof of Proposition 1, it is shown that the given choice of X_k works because of Jensen's inequality, and the given choice of Z_k works because it enforces an equality constraint. These arguments are easily generalised to a class of functions X_k, Z_k :

$$X_k = \bigoplus_{i=1}^q \Omega_{ik} A_i, \quad Z_k = \bigoplus_{i=1}^q \Lambda_{ik} C_i, \quad (20)$$

where

$$\Omega_{ik} \in \begin{cases} \mathbb{R}^+ & \text{if } R_{ik} = 1, \\ \{0\} & \text{otherwise,} \end{cases}$$

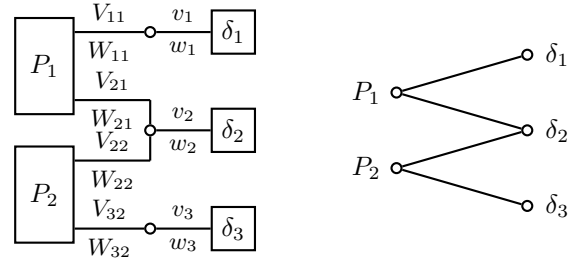


Fig. 5: The left hand part of the figure shows an electrical network interpretation of the interconnection structured by the bipartite graph on the right. Each ‘wire’ leaving a P_k and δ_i represents a port, and is associated with a pair of through and across variables (w_i, v_i) or (W_{ik}, V_{ik}) as labelled. The ports are connected at a set of vertices, one for each δ_i , where KCL and KVL hold (eq. (22)).

and Λ has the same sparsity structure, but with both positive and negative entries allowed. Provided $\forall i \in \mathbb{Z}_{[1,q]}$

$$\sum_{k=1}^p \frac{R_{ik}}{\Omega_{ik}} \leq 1, \quad \sum_{k=1}^p R_{ik} \Lambda_{ik} = 1, \quad (21)$$

the same arguments can still be applied², and these functions may be used in place of their counterparts in Proposition 1. The parameters Ω, Λ can then be treated as optimisation variables much like those in standard IQC analysis. Optimisation of these variables is not decoupled in k because of eq. (21), but it is convex because both eqs. (12) and (21) are convex in Ω, Λ . In addition it is local in the structure of R (eq. (12) is decoupled in the columns of R , and eq. (21) in the rows). This means that their selection can be viewed in terms of local negotiations, in which for example the k th column of each variable is associated with P_k , and neighbouring P_k 's can negotiate over the values in their column locally subject to eq. (21). Alternatively since individually eqs. (12) and (21) are decoupled and convex, optimisation schemes such as the method of alternating projections may be applied to solve them in a distributed manner.

Remark 5: Equations (20) and (21) can be generalised to cover conic multiplier descriptions of A_i, C_i .

V. EXAMPLES

A. Electrical Networks

In this example we will give an electrical network interpretation of the interconnection in eq. (9), and show how to obtain standard stability criteria for the interconnection of passive subsystems using Proposition 1. We will additionally interpret the network decomposition induced by Proposition 1 in terms of a type of tearing.

²Under this choice of X_k it needs to be established that $\|\hat{y}_i\|^2 \leq \sum_{k=1}^p \Omega_{ik} \|\hat{Y}_{ik}\|^2$ in place of eq. (15). This follows from applying Jensen's inequality to $\|\hat{y}_i\|^2 = \left\| \sum \frac{R_{ik}}{\Omega_{ik}} \cdot \Omega_{ik} \hat{Y}_{ik} \right\|^2$, taking the sum over k for which R_{ik} is nonzero. The satisfaction of the constraint involving Z_k is direct.

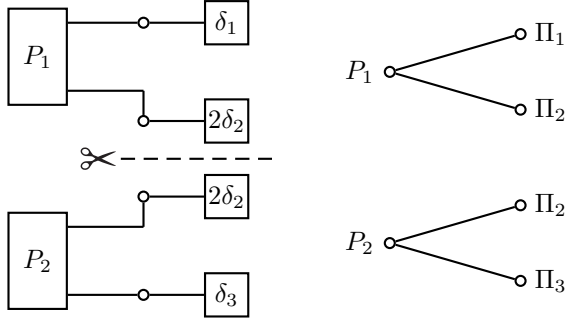


Fig. 6: The left hand figure shows the decomposition of the electrical network induced by the decomposition of the bipartite graph on the right. The network is decomposed into p pieces by tearing each shared δ_i apart equally. Analysis of fig. 5 with Proposition 1 is equivalent to conventional IQC analysis of each of these subnetworks individually.

1) Electrical Network Interpretation of Network Model:

Let $W, V \in \mathbf{L}_{2e}^{q \times p}$, and associate the columns of each W, V with the inputs and outputs of each P_k , i.e.

$$V_{\bullet k} = P_k W_{\bullet k}.$$

In addition, let $v, w \in \mathbf{L}_{2e}^q$, and associate the i th element of these signals with the inputs and outputs of δ_i , i.e.

$$w_i = \delta_i v_i.$$

In the absence of disturbances, eq. (9) defines the following relationships between W, V, w, v :

$$\begin{aligned} w_i + \sum_{k=1}^p R_{ik} W_{ik} &= 0 & \text{(KCL)} \\ V_{ik} &= R_{ik} v_i & \text{(KVL)} \end{aligned} \quad (22)$$

These equations are analogous to Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL). Each δ_i can therefore be thought of as the impedance of a 1-port, and each P_k the admittance of a $(\sum_{i=1}^q R_{ik})$ -port, with the incidence matrix describing how these ports are connected together. This is sketched in fig. 5.

2) *Passivity Conditions:* Suppose that it is known that each δ_i is passive, by which we mean that $\delta_i \in \text{IQC}(\Pi_i)$ where

$$\Pi_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Applying Proposition 1 then states that if for each $k \in \mathbb{Z}_{[1,p]}$ there exists an ϵ_k such that $\forall \omega \in \mathbb{R}$

$$\begin{bmatrix} P_k \\ I \end{bmatrix}^* \begin{bmatrix} -\epsilon_k I & \bigoplus_{i=1}^q R_{ik} \\ \bigoplus_{i=1}^q R_{ik} & 0 \end{bmatrix} \begin{bmatrix} P_k \\ I \end{bmatrix} (j\omega) \geq 0,$$

then the interconnection is stable. This is equivalent to requiring each P_k to be strictly output passive. Hence in this case Proposition 1 reduces to familiar passivity type conditions. This is reassuring since we have shown that eq. (9) essentially has the structure of an electrical network, so we would expect such conditions to hold.

3) *Tearing Interpretation of Network Decomposition:* $\delta_i \in \text{IQC}(\Pi_i)$ if and only if $n_i \delta_i \in \text{IQC}(\bar{\Pi}_i)$, where

$$\bar{\Pi}_i = \begin{bmatrix} \bar{A}_i & \bar{B}_i^* \\ \bar{B}_i & \bar{C}_i \end{bmatrix} = \begin{bmatrix} n_i A_i & B_i^* \\ B_i & \frac{1}{n_i} C_i \end{bmatrix} \in \mathbf{RL}_{\infty}^{2 \times 2}.$$

If interpreted as impedance functions, the impedance of $n_i \delta_i$ is n_i times greater than δ_i . Therefore δ_i is equivalent to the impedance obtained from the parallel interconnection of n_i copies of $n_i \delta_i$. Equivalently $n_i \delta_i$ is the impedance obtained by tearing δ_i into n_i equal pieces. Since the given X_k, Y_k, Z_k could also be calculated through

$$X_k = \bigoplus_{i=1}^q R_{ik} \bar{A}_i, \quad Y_k = \bigoplus_{i=1}^q R_{ik} \bar{B}_i, \quad Z_k = \bigoplus_{i=1}^q R_{ik} \bar{C}_i,$$

testing eq. (12) is equivalent to tearing the full electrical network apart into p pieces, and then testing each individually with the IQC theorem [17]. This is sketched in fig. 6.

B. Small Gain Conditions

In this example we will show how to recover simple small gain like bounds from Proposition 2.

Suppose that it is known that $\|P_k\| \leq \beta_k$. This can be equivalently written as $P_k \in \text{IQC}(\Pi_k)$, where

$$\tilde{\Pi}_k = \begin{bmatrix} \beta_k \bigoplus_{i=1}^q R_{ik} & 0 \\ 0 & -\beta_k^{-1} \bigoplus_{i=1}^q R_{ik} \end{bmatrix}.$$

Applying Proposition 2 then guarantees stability if for each $i \in \mathbb{Z}_{[1,q]}$ there exists an ϵ_i such that $\forall \omega \in \mathbb{R}$

$$R_{i\bullet}^T \delta_i^* (j\omega) R_{i\bullet} \left(\bigoplus_{k=1}^p R_{ik} \beta_k + \epsilon_i \right) R_{i\bullet}^T \delta_i (j\omega) R_{i\bullet} \leq \bigoplus_{k=1}^p R_{ik} \beta_k^{-1}.$$

It is simple to show that this is equivalent to $\forall \omega \in \mathbb{R}$:

$$|\delta_i(j\omega)| < \frac{1}{\sum_{k=1}^p R_{ik} \beta_k}.$$

This condition means that if the product of the gain of δ_i and the sum of the gains for the local P_k 's is less than one for every i , then the interconnection is stable. This is a sufficient condition based on the loop gains of all the short feedback loops within the network.

If information about the gain between the i th input and output of P_k is known, this can be used to give tighter conditions. For example, suppose instead that $P_k \in \mathbf{RH}_{\infty}[R_{\bullet k}]$, and the following bound is available

$$P_k^* P_k (j\omega) \leq \bigoplus_{i=1}^q R_{ik} \beta_{ik}^* \beta_{ik} (j\omega), \quad \forall \omega \in \mathbb{R}.$$

By the same argument, applying Proposition 2 then guarantees stability if for each $i \in \mathbb{Z}_{[1,q]}$

$$|\delta_i(j\omega)| < \frac{1}{\sum_{k=1}^p R_{ik} |\beta_{ik}(j\omega)|}, \quad \forall \omega \in \mathbb{R}.$$

In this case the condition for each δ_i depends on the sum of the gains for the local P_k 's between their i th input and output.

C. Networks with Laplacian Structure

In this example, a network structure relevant for a wide range of applications, including consensus, flocking phenomena, and electrical and mechanical networks will be considered. Proposition 1 will be applied to give simple distributed Nyquist and Popov like conditions. The criteria will be compared with a centralised approach for a small network model.

1) *Network structure*: A weighted Laplacian matrix is a specific matrix representation of a graph, where the i th edge of the graph is associated with a weight δ_i (and an arbitrary direction). In particular for a graph with p vertices and q directed edges, the weighted Laplacian matrix equals $B^T D B$, where $D = \bigoplus_{i=1}^q \delta_i$, $\delta_i > 0$ and

$$B_{ik} = \begin{cases} 1 & \text{if the } i\text{th edge leaves the } k\text{th vertex} \\ -1 & \text{if the } i\text{th edge enters the } k\text{th vertex} \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

In this example we will analyse the interconnection

$$[BLB^T, D], \quad (24)$$

where $L = \bigoplus_{k=1}^p L_k$, $L_k \in \mathbf{RH}_\infty$ and δ_i are bounded causal operators (the restriction that $L_k \in \mathbf{RH}_\infty$ can be relaxed to include integrators, see remark 8). This interconnection can be used to model the dynamics of a range of networks, including simplified models of electrical power systems, consensus algorithms, vehicle platoons, and flocking phenomena, see e.g. [2], [22], [28], [23].

2) *Application of Proposition 1*: To apply Proposition 1 to eq. (24), we must identify some P_k 's that satisfy

$$\sum_{k=1}^p P_k = BLB^T.$$

A choice that reflects the local network structure is $P_k = B_{\bullet k} L_k B_{\bullet k}$, where $R \in \mathbb{R}^{q \times p}$, and

$$R_{ik} = |B_{ik}|.$$

In this case each subsystem P_k captures the dynamics of a single L_k and its local connections to the weights. Proposition 1 can then be applied by characterising $\delta_i \in \text{IQC}(\Pi_i)$, computing the corresponding X_k, Y_k, Z_k , and testing each P_k with eq. (12). This will result in a stability criterion for each L_k . Each test can be checked with only knowledge of the IQCs for the neighbouring δ_i 's on the bipartite graph, or equivalently those associated with the adjacent edges in the Laplacian graph (see fig. 7).

When additional structure is imposed on the IQCs the following stability condition is obtained. This allows stability of eq. (24) to be determined by testing each L_k with a local 'classical multiplier test' (e.g. [29]). In the next section we will compare this condition to a centralised approach based on the Nyquist criterion for a power system model.

Corollary 1: Let $B \in \mathbb{R}^{q \times p}$, $L = \bigoplus_{k=1}^p L_k$, and $D = \bigoplus_{i=1}^q \delta_i$, where $L_k \in \mathbf{RH}_\infty$, $\delta_i \in \text{IQC}(\Pi_i)$,

$$\Pi_i(j\omega) = \begin{bmatrix} 0 & h^* \\ h & -\beta_i^{-1}(h + h^*) \end{bmatrix} (j\omega),$$

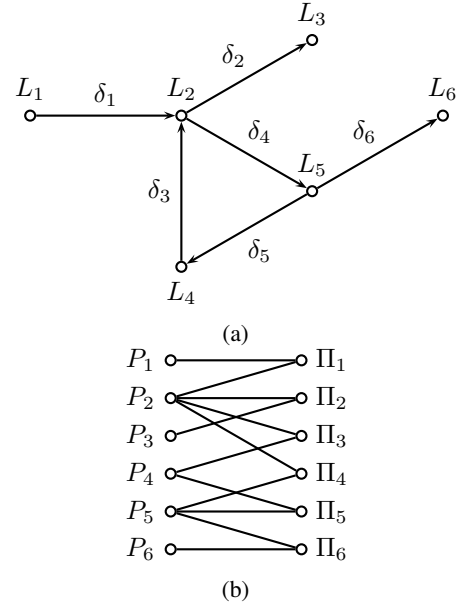


Fig. 7: (a) shows an example of a graph associated with a Laplacian matrix. Each edge is associated with a weight δ_i and arbitrary direction. In eq. (24) each vertex is additionally associated with a dynamical system $L_k \in \mathbf{RH}_\infty$. (b) shows the bipartite graph obtained when describing this model through eq. (9), with $P_k = B_{\bullet k} L_k B_{\bullet k}^T$. Hence in this case the analysis problem is decomposed into 6 independent pieces, each dependent on the dynamics of a single L_k and (the IQCs for) its neighbouring δ_i 's.

$h \in \mathbf{RL}_\infty$, and B satisfies eq. (23). Define

$$\gamma_k = \sum_{i: B_{ik} \neq 0} \beta_i.$$

If for each $k \in \mathbb{Z}_{[1,p]}$

$$\text{Re} \left\{ h(j\omega) \left(L_k(j\omega) + \frac{1}{2} \gamma_k^{-1} \right) \right\} > 0, \quad \forall \omega \in \mathbb{R}, \quad (25)$$

then $[BLB^T, D]$ is stable.

Proof: See Appendix A. ■

Remark 6: Suppose $\delta_i \in \mathbb{R}$, $\delta_i > 0$ (i.e. δ_i are the weights from a Laplacian matrix). Then $\delta_i \in \text{IQC}(\Pi_i)$ with Π_i as in Corollary 1 proved $\beta_i \geq \delta_i$. If $\beta_i \in \mathbb{R}$ is analogous to an admittance, then γ_k is analogous to the parallel admittance of the edges neighbouring the k th vertex in fig. 7(a).

Remark 7 (Popov criterion): Suppose each δ_i is a memoryless static nonlinearity satisfying

$$0 < \delta_i(x) \leq \beta_i x^2, \quad \forall x \in \mathbb{R}.$$

Then $\delta_i \in \text{IQC}(\Pi_i)$ for $h(s) = (1 + \eta s)$, where $\eta \in \mathbb{R}$ (this is the Popov multiplier). Therefore if there exists an η such that $\forall k \in \mathbb{Z}_{[1,p]}$:

$$(1 + \eta j\omega) L_k(j\omega) + \gamma_k^{-1} > 0, \quad \forall \omega \in \mathbb{R},$$

then $[BLB^T, \Delta]$ is stable. For any given η this may be checked on the Popov plot of L_k in the usual way.

Remark 8: In many of the listed applications, it is desirable to model each L_k as a transfer function which includes an

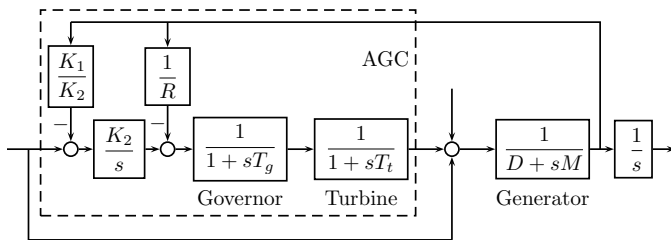


Fig. 8: Block diagram representation of a simple generator model, including the so called AGC, see e.g. [2] (typical parameters in table I).

TABLE I: Typical generating area parameters [2][chapter 2]

M	D	T_g	T_t	R	K_1	K_2
0.16	0.02	0.08	0.40	3.00	0.10	0.30
0.20	0.02	0.06	0.44	2.73	0.08	0.20
0.12	0.02	0.07	0.30	2.82	0.15	0.40

integrator. This cannot be handled directly with IQC analysis as described as an integrator is not in \mathbf{RH}_∞ . A simple ‘fix’ is to encapsulate the integrator in a loop transform [30], and then proceed with analysis as usual on the loop transformed system. For eq. (24) this can be done with an arbitrarily small loopshift, making the analysis of eq. (25) essentially indistinguishable between the transformed and untransformed systems $\forall \omega \neq 0$, see [31][section 4]. Alternatively one may include integrators by performing analysis on the modified L_2 spaces proposed in [32].

3) *Analysis of a power system model:* We will illustrate the conditions in Corollary 1 by applying them to a simplified model of an electrical power system, and comparing the obtained criteria to a centralised approach for a small system. When used to model the dynamics of AC electrical power systems³, each vertex in fig. 7(a) represents a generating area (a collection of geographically close generators and loads) with dynamics L_k , and each edge a transmission line. The transfer function L_k in these models is given in fig. 8, along with some typical model parameters from chapter 2 of [2] in table I. The operators δ_i are positive constants analogous to the admittance of the transmission lines.

Consider the special case of eq. (24) when

$$B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This would correspond to, for example, a power system model with two generating areas connected by a single line. In this case we may quickly assess the stability and robustness of eq. (24) by plotting the Nyquist diagram of the return ratio:

$$\ell(s) = \delta_1 (L_1(s) + L_2(s)).$$

³Such models are suitable for performing local analysis of longer term behaviours in power systems, such as the regulation of the electrical frequency, and the adjustment of generation to pre-determined active power setpoints. In this context Proposition 1 allows the tuning of controllers designed to meet these objectives (for example AGC), see e.g. [1], [2].

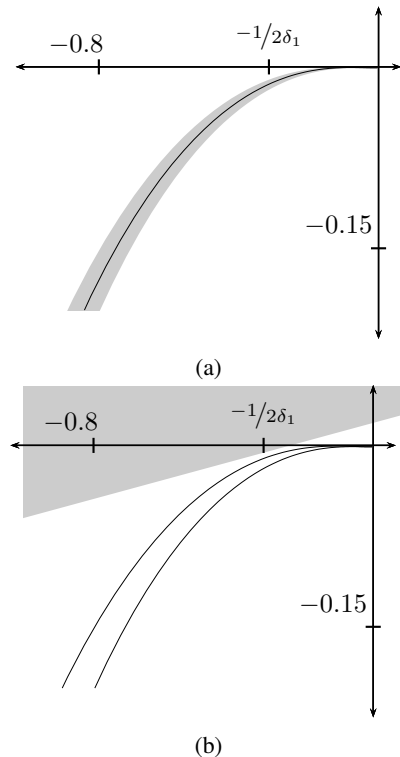


Fig. 9: Sketch of the convexification argument for generating distributed stability conditions. The black curve in (a) is the Nyquist diagram of $\ell(s)/2\delta_1$ using the first two sets of generating area parameters from table I. As this curve makes no encirclements of the $-1/2\delta_1$ this small power system model is stable. The shaded region shows the Nyquist diagram of $\mathcal{L}(s)$. This region includes $-1/2\delta_1$. The two black curves in (b) are the Nyquist diagrams of $L_1(s)$ and $L_2(s)$. Requiring these curves to lie in a halfplane excluding the $-1/2\delta_1$ is a (slightly more conservative) sufficient condition for stability, as this ensures the Nyquist diagram of $\mathcal{L}(s)$ can make no encirclements of $-1/2\delta_1$. Critically though this condition can be checked in a distributed way.

Assuming each L_k is a stable transfer function, stability with a degree of robustness is ensured provided this Nyquist diagram is sufficiently far from the -1 point. This can be easily verified for each of the generating area model parameterisations in table I.

Even in the case of only two L_k ’s, the stability and robustness analysis is slightly complicated by the fact that the summation of dynamics appears in $\ell(s)$. Things are complicated further when considering the synthesis of controllers local to each L_k , as the effect of both controllers appear in the stability condition. For example if we wished to design K_1, K_2 , where

$$L_k = \mathcal{F}_l(H_k, K_k),$$

using a loopshaping method, the design of K_1 would have to take into account that of K_2 (and vice versa).

One could instead replace $\ell(s)$ with the following convex set

$$\mathcal{L}(s) = \{tL_1(s) + (1-t)L_2(s) : t \in [0, 1]\}.$$

Avoiding a halfplane which contains the $-1/2\delta_1$ point with the ‘Nyquist diagram’ of this set is sufficient for stability and robustness. To see this first observe that $\ell(s) \in 2\delta_1\mathcal{L}(s)$. The boundary of the halfplane containing the $-1/2\delta_1$ then provides a separating hyperplane between the $-1/2\delta_1$ point and $\mathcal{L}(s)$, ensuring satisfaction of the Nyquist criterion. This is sketched in fig. 9.

While (slightly) more conservative than the original stability condition, critically we can test this condition by plotting the Nyquist diagram of each L_k individually. Furthermore, we can design the controllers K_k independently. We have hence traded in a little conservatism to gain a simple and decentralised synthesis condition.

This distributed stability criterion can be compactly described as: if for each $k \in \mathbb{Z}_{[1,2]}$

$$\operatorname{Re} \left\{ e^{j\operatorname{sign}(\omega)\theta} \left(L_k(j\omega) + \frac{1}{2}\delta_1^{-1} \right) \right\} > 0, \quad \forall \omega \in \mathbb{R},$$

then the interconnection is stable. Proposition 1 essentially generalises the simple convexification idea behind this condition to large network models. This can be clearly seen in Corollary 1, which shows as a special case ($h = e^{j\operatorname{sign}(\omega)\theta}$) that if $\forall k \in \mathbb{Z}_{[1,p]}$

$$\operatorname{Re} \left\{ e^{j\operatorname{sign}(\omega)\theta} \left(L_k(j\omega) + \frac{1}{2}\gamma^{-1} \right) \right\} > 0, \quad \forall \omega \in \mathbb{R}, \quad (26)$$

then eq. (24) is stable. This result is valid for a B matrix of any size and heterogeneous $L_k \in \mathbf{RH}_\infty$. Observe in particular that the test for each k is identical to those discussed for the $p = 2$ case, but with δ_1 replaced with the parallel admittance of the neighbouring lines. Therefore robust analysis and controller design with respect to diagrams such as fig. 9(b) can be conducted based on local network information, and is valid in a network of any size.

In this example, the network of transmission lines appears in the local stability conditions as a locally determined constant gain. Examining the Nyquist diagrams in fig. 9 shows that the design given by the parameters in table I have a large gain margin, and hence represent a good design for a wide range of network topologies and loadings. The curves do however cross, just, into the upper halfplane indicating that that the decentralised tests will fail if γ_i is large enough. In fact, for the best possible choice of θ the local tests fail for

$$\gamma_1 > 3.88, \quad \gamma_2 > 6.43, \quad \gamma_3 > 3.815 \quad (27)$$

guaranteeing that any network, consisting of an arbitrary number of copies of each type of area, will be stable provided that the parallel admittance seen at any area is no greater than the limit for the corresponding γ . If required, these numbers could be improved by further tuning of the constants K_1, K_2 , though this should be done to respect other features of the AGC problem [2].

Despite the generality of the above claim, it is not unduly conservative. Indeed, it can be easily shown that for

$$\delta_1 \in [5.49, 20.61],$$

the network of just the two generating areas is unstable. For any particular network, including this two area one, the

conservatism can be reduced further by using the methods of Section IV-D. In this case, by associating a larger proportion of the admittance to the criterion for the second area, this network can be guaranteed to be stable for any admittance up to 4.84, just on the basis of the numbers in (27) (i.e. without readjusting θ).

D. More Nyquist type Tests

In the previous example it was shown that when the P_k, δ_i in eq. (9) have special structure, application of Proposition 1 gives conditions that closely resemble applying classical stability criteria to the set of interconnections

$$\left\{ \left[P_k, \bigoplus_{i=1}^q R_{ik}\delta_i \right] : k \in \mathbb{Z}_{[1,p]} \right\}.$$

A strength of Proposition 1 is that these restrictions are not necessary, allowing networks with more complex dynamics (such as models of internet congestion control) to be analysed. In this section we give a graphical stability criterion with similar interpretations to the tests in the previous section that can be applied even when each P_k is unstructured. This example will also serve to demonstrate the inherent robustness of the conditions to classical notions of uncertainty.

1) *The Numerical Range:* The criterion is based on the numerical range.

Definition 4: For $A \in \mathbb{C}^{q \times q}$, define the numerical range $W(A)$ as

$$W(A) = \{x^*Ax \in \mathbb{C} : x \in \mathbb{C}^q, x^*x = 1\}.$$

The numerical range $W(A) \subset \mathbb{C}$ is a compact convex set. $W(A)$ always contains the convex hull of the spectrum of A , with equality when A is a normal matrix ($AA^* = A^*A$). In addition, for scalars a, b , $W(aA + bI) = aW(A) + b$. Outer (and inner) approximations of the boundary of $W(A)$ can be efficiently computed to arbitrary precision. For an extended discussion of the numerical range, see chapter 1 of [33]. When A is rank 1, $W(A)$ is given by an ellipse [34].

Lemma 1: Let $x \in \mathbb{C}^{n \times 1}, y \in \mathbb{C}^{1 \times n}$. Then

$$W(xy) = \{z \in \mathbb{C} : |z - yx| + |z| \leq \|x\| \|y\|\}.$$

Proof: See [20], Lemma 3. ■

2) *Connections to the Previous Example:* The following shows that when $A_i = 0$, the k th condition in Proposition 1 can be checked graphically (equivalently the i th condition in Proposition 2 when $A_{ik} = 0$).

Lemma 2: Let P, Y, Z be compatibly dimensioned matrices with entries in \mathbb{C} , and assume that $P^*P > 0$. Then the following are equivalent:

(i) There exists an $\epsilon > 0$ such that

$$\begin{bmatrix} P \\ I \end{bmatrix}^* \begin{bmatrix} -\epsilon I & Y^* \\ Y & Z + Z^* \end{bmatrix} \begin{bmatrix} P \\ I \end{bmatrix} \geq 0$$

(ii) $\operatorname{Re} \{W(YP + Z)\} > 0$.

Proof: By the definition of the numerical range, $\operatorname{Re} \{W(YP + Z)\} > 0$ is equivalent to

$$(YP + Z) + (YP + Z)^* > 0.$$

Since $P^*P > 0$, this is equivalent to there existing an $\epsilon > 0$ such that

$$YP + P^*Y^* + Z + Z^* \geq \epsilon P^*P.$$

This is equivalent to (i) (expand the expression). ■

If $P^*P \geq 0$, then (ii) \Rightarrow (i). (ii) can be checked graphically by plotting the numerical range across frequency, and then checking whether this plot lies in the right half plane. The major advantage of (ii) is that it can guide loopshaping design: the controllers local to each P_k can be tuned to shape the numerical range plot to push it into the right half plane in the appropriate frequency ranges. Classical notions of robustness can also be incorporated into this thinking, as discussed at the end of the section.

Depending on the form of Y, Z , this test can be made to more closely resemble standard Nyquist type criteria (reinforcing the connection to loopshaping). For example, suppose that $\delta_i \in \text{IQC}(\Pi_i)$, with

$$\Pi_i(j\omega) = \begin{bmatrix} 0 & h(j\omega)^* \\ h(j\omega) & -h(j\omega) - h(j\omega)^* \end{bmatrix},$$

and $\forall i, n_i = 2$. In this case, by Proposition 1 and Lemma 2, if for each $k \in \mathbb{Z}_{[1,p]}$:

$$\text{Re} \left\{ h(j\omega) \left(W(P_k(j\omega)) + \frac{1}{2} \right) \right\} > 0, \quad \forall \omega \in \mathbb{R}, \quad (28)$$

then the interconnection is stable. These tests are of precisely the same form as those in Corollary 1, but with conventional Nyquist plots replaced with ‘Nyquist plots’ of the numerical range of $P_k(j\omega)$. To clarify this, consider again the structured form $P_k = B_{\bullet k} L_k B_{\bullet k}^T$ from the previous example. By Lemma 1

$$W(B_{\bullet k} L_k(j\omega) B_{\bullet k}^T) = \{t L_k(j\omega) \in \mathbb{C} : t \in [0, \gamma_k]\}.$$

Hence eqs. (25) and (28) are equivalent in this case. A possible use of eq. (28) is then when the symmetry in eq. (24) is broken (for example considering stability of $[BL\bar{B}^T, \bigoplus_{i=1}^q \delta_i]$ where $B \neq \bar{B}$).

Remark 9: The ellipse based conditions in [20] arise from a slight generalisation of the above discussion (but using the same IQCs) when applied to the interconnection

$$\left[Q(j\omega) Q(j\omega)^T, I \right],$$

where $Q(j\omega) \in \mathbf{RH}_{\infty}^{q \times p}$, and QQ^T is decomposed as

$$QQ^T = \sum_{k=1}^p Q_{\bullet k} Q_{\bullet k}^T.$$

This generalisation of the model description makes it suitable for models of internet congestion control [15].

3) *Graphical Robustness Guarantees:* The numerical range allows robustness to classical notions of uncertainty to be graphically assessed. For example, satisfying a condition

$$W(A(j\omega)) > \gamma, \quad \forall \omega \in \mathbb{R}$$

guarantees that

$$W(A(j\omega) + \Delta(j\omega)) > 0, \quad \forall \omega \in \mathbb{R}$$

for any $\Delta \in \mathbf{RH}_{\infty}, \|\Delta\| < \gamma$. That is by satisfying the original condition by some margin, stability is guaranteed for a family of additive perturbations. This follows from:

Lemma 3: For square matrices $A, B \in \mathbb{C}$, if $\|B\| < \gamma$ then

$$W(A + B) \subset \{x + y \in \mathbb{C} : x \in W(A), |y| < \gamma\}.$$

Furthermore, for any z such that

$$z \in \{x + y \in \mathbb{C} : x \in W(A), |y| < \gamma\}$$

there exists a C with $\|C\| < \gamma$ such that $z \in W(A + C)$.

Proof: Choose any vector v such that $v^*v = 1$. By Cauchy-Schwartz, $|v^*Bv| \leq \|v\| \|Bv\| < \gamma$. Hence putting $x = v^*Av$ gives the first statement of the lemma. The second follows since any $z = x + y$ can be constructed by observing that if $C = re^{j\theta}I$, where $r < \gamma$ and $\theta \in \mathbb{R}$, then $v^*Cv = re^{j\theta}$ and $\|C\| < \gamma$. ■

E. Distributed optimisation

One of the main applications that we envisage for our results is for analysing the stability and convergence properties of distributed optimisation algorithms when embedded in physical networks. Typically idealised models for the physical systems are used, such as the early reference [3] for rate control in communication networks and, more recently, [35] for frequency control in power networks, for example. The challenge then is to design the dynamics so as to ensure desirable behaviour on the real system, replete with delays and other nontrivial dynamics.

A very general network structure relevant for describing distributed optimisation algorithms is considered here. Proposition 2 will be applied to give simple conditions for establishing convergence of these algorithms. A simple distributed implementation of a gradient descent scheme in the presence of heterogeneous delays will be analysed.

1) *Problem structure:* Let $R \in \mathbb{R}^{q \times p}$, where $R_{ik} \in \{0, 1\}$. Consider the problem of finding the minimiser

$$\bar{x} = \arg \min_{x \in \mathbb{R}^q} \sum_{k=1}^p f_k(\{x_i : R_{ik} \neq 0\}). \quad (29)$$

We will assume throughout that the functions f_k are strongly convex (on a subspace that will be formalised later). The incidence matrix indicates which elements of the vector x are required to compute f_k ($R_{ik} = 1$ if the i th element is required). The use of dynamical systems theory to analyse optimisation algorithms has a long history (for an early reference, see e.g. [36]). Here we have in mind examples such as the single commodity network flow problem where a continuous time setting is natural (for more discussion on continuous and discrete time algorithms, see [37] and the references therein).

We consider the following interconnection:

$$\left[\sum_{k=1}^p \nabla f_k, \bigoplus_{i=1}^q \delta_i \right], \quad (30)$$

Equation (30) can be used to capture the dynamics of a wide class of gradient based distributed optimisation algorithms, with the δ_i 's corresponding to the update rules. By using

integral action in the δ_i 's, the optimal solution of eq. (29) becomes an equilibrium point. Establishing global exponential stability of this point then guarantees convergence of the corresponding algorithm.

2) *Analysis with Proposition 2:* For analysis purposes we consider

$$\left[\sum_{k=1}^p \nabla g_k, \bigoplus_{i=1}^q \delta_i \right], \quad (31)$$

where

$$g_k(\Delta x) = f_k(\Delta x + \bar{x}) - f_k(\bar{x}),$$

and Δx is the deviation in x from equilibrium. The only difference from eq. (30) is that the equilibrium point has been shifted to the origin to allow the nonlinearities to be captured with sector type IQCs.

Let $I_k = \bigoplus_{i=1}^q R_{ik}$. A useful class of IQCs for analysing optimisation algorithms are the Zames-Falb multipliers [38]. Suppose each f_k is strongly convex. That is $\forall x \in \mathbb{R}^q$:

$$\epsilon_k I_k \leq \nabla^2 f_k(x) \leq \beta_k I_k,$$

where $\nabla^2 f_k(x)$ is the hessian of $f_k(x)$, and $\beta_k > \epsilon_k > 0$. Then by Theorem 1 in [39], $\nabla g_k \in \text{IQC}(\tilde{\Pi}_k)$, where

$$\tilde{\Pi}_k = \begin{bmatrix} 0 & (1 - \hat{h}^*) I_k \\ (1 - \hat{h}) I_k & -\beta_k^{-1} (2 - \hat{h} - \hat{h}^*) I_k \end{bmatrix}, \quad (32)$$

$h(t) \geq 0$, $\int_{-\infty}^{\infty} h(t) dt < 1$, and the hat symbol denotes the Fourier transform.

Applying Proposition 2 with the above IQCs results in Corollary 2 below. This corollary allows the convergence and robustness properties of eq. (31) to be verified on the basis of simple multiplier tests. There is one test for each δ_i that depends only on the constants β_k for the neighbouring f_k 's.

Corollary 2: Let $R \in \mathbb{R}^{q \times p}$, $P = \sum_{k=1}^p P_k$, and $D = \bigoplus_{i=1}^q \delta_i$, where $P_k \in \text{IQC}(\tilde{\Pi}_k)$, $\delta_i \in \mathbf{RH}_{\infty}$, and $\tilde{\Pi}_k$ satisfies eq. (32). Define

$$\gamma_i = \sum_{k=1}^p R_{ik} \beta_k.$$

If for each $k \in \mathbb{Z}_{[1,q]}$

$$\text{Re} \left\{ \left(1 - \hat{h}(j\omega) \right) \left(\delta_i(j\omega) + \gamma_i^{-1} \right) \right\} > 0, \quad \forall \omega \in \mathbb{R}, \quad (33)$$

then $[\sum_{k=1}^p P_k, \bigoplus_{i=1}^q \delta_i]$ is stable.

Proof: Let $S = \bigoplus_{k=1}^p \beta_k$ and $Q = RS$. Compute X_i, Y_i, Z_i as in Proposition 2. Hence $X_i = 0$,

$$Y_i = \left(1 - \hat{h} \right) \bigoplus_{k=1}^p R_{ik}, \quad Z_i = - \left(2 - \hat{h} - \hat{h}^* \right) \bigoplus_{k=1}^p Q_{ik},$$

Application of Proposition 2 guarantees stability if $\forall i \in \mathbb{Z}_{[1,q]}$:

$$\begin{bmatrix} R_{i\bullet}^T \delta_i R_{i\bullet} \\ I \end{bmatrix}^* \begin{bmatrix} -\epsilon_i I & Y_i^* \\ Y_i & -Z_i \end{bmatrix} \begin{bmatrix} R_{i\bullet}^T \delta_i R_{i\bullet} \\ I \end{bmatrix} (j\omega) \geq 0$$

This FDI is of the same form as that obtained in the proof of Corollary 1, and eq. (33) follows by the same argument. ■

Remark 10: The given definition of \mathbf{L}_2 -stability only guarantees that $x(t)$ remains bounded. This does not guarantee that $x(t) \rightarrow \bar{x}$ as required to establish convergence. However

under a ‘fading memory’ assumption about the operators ∇g_k (which is satisfied for all the nonlinearities in this section), these stability criteria do guarantee convergence in the required sense (see [40], Theorem 2).

Remark 11: Corollary 2 cannot be used to analyse these algorithms directly, since the corresponding δ_i 's will contain an integrator. However the following loopshifted interconnection can be analysed

$$\left[\sum_{k=1}^p \nabla g_k - \epsilon I, \bigoplus_{i=1}^q \delta_i (1 + \epsilon \delta_i)^{-1} \right],$$

where ϵ may be arbitrarily small. Since ∇f_k are strongly convex, for sufficiently small ϵ , $\nabla g_k - \epsilon I_k \in \text{IQC}(\tilde{\Pi}_k)$.

To illustrate the application of Corollary 2, consider the following (continuous time) gradient descent algorithm:

$$z = \sum_{k=1}^p \nabla f_k(x) \quad (34)$$

$$\dot{x}_i = -\alpha_i z_i, \quad \forall i \in \mathbb{Z}_{[1,q]}.$$

This is a special case of eq. (31), where each δ_i is an integrator scaled by a constant α_i . This may be implemented in a distributed manner by defining the variables $X, Z \in \mathbf{L}_{2e}^{q \times p}$, and associating the inputs and outputs of ∇f_k with a column of these signals. That is

$$Z_{\bullet k} = \nabla f_k X_{\bullet k}.$$

Equation (31) then defines the update rules

$$z_i = - \sum_{k=1}^p R_{ik} Z_{ik}, \quad (35)$$

$$X_{ik} = R_{ik} x_i,$$

which may be implemented locally.

After capturing the integrator as discussed in remark 11, applying Corollary 2 with $h = 0$ gives the following criterion for each $i \in \mathbb{Z}_{[1,q]}$:

$$\frac{\alpha_i}{j\omega + \epsilon \alpha_i} + \gamma_i^{-1} > 0, \quad \forall \omega \in \mathbb{R}.$$

This will be satisfied provided $\alpha_i > 0$. Hence Corollary 2 shows that eq. (34) is guaranteed to converge independently of the size of p and q , provided positive step sizes are used.

Discrete time algorithms can also be analysed by making the usual modifications to IQC analysis (see e.g. [26]). Consider for example the gradient descent algorithm:

$$z = \sum_{k=1}^p \nabla f_k(x)$$

$$x_i^+ = x_i - \alpha_i z_i.$$

The appropriate modification of Corollary 2 (again selecting $h = 0$) requires that $\forall i \in \mathbb{Z}_{[1,q]}$:

$$\frac{\alpha_i}{e^{j\theta} - 1 + \epsilon \alpha_i} + \gamma_i^{-1} > 0, \quad \forall \theta \in [-\pi, \pi].$$

When ϵ is small, this is equivalent to requiring that $0 < \alpha_i < 2\gamma_i^{-1}$. In this case Corollary 2 guarantees convergence provided a local bound on the step sizes is enforced.

Remark 12: More sophisticated update rules (i.e. replace α_i with a transfer function c_i , e.g. a lead-lag compensator), can be locally analysed in an analogous manner. For example, if momentum terms are included, δ_i would have transfer function

$$\frac{1}{M_i s^2 + D_i s},$$

where $M_i > 0, D_i > 0$ (e.g. [41]).

3) *Analysis with delays:* Results for simple cases like the above gradient descent algorithm can be obtained with other (simpler) methods. To give an example of a nontrivial extension that can be obtained, suppose, as would be realistic in network applications, that the update rule linking the subsystems is subject to heterogeneous delays. That is eq. (35) is replaced with

$$\begin{aligned} z_i(t) &= -\sum_{k=1}^p R_{ik} Z_{ik}(t - T_{ik}), \\ X_{ik}(t) &= R_{ik} x_i(t - \bar{T}_{ik}), \end{aligned} \quad (36)$$

where $T, \bar{T} \in \mathbb{R}^{q \times p}, T_{ik} \geq 0, \bar{T}_{ik} \geq 0$ (no additional structural restrictions, such as constant round trip times, are required).

Definition 5: For $x \in \mathbf{L}_2^n$ and $T \in \mathbb{R}^{n \times 1}$ define the delay operator d_T as

$$(d_T x)(t) = \begin{bmatrix} x_1(t - T_1) \\ \vdots \\ x_n(t - T_n) \end{bmatrix}.$$

To analyse convergence in this case it is necessary to verify stability of

$$\left[\sum_{k=1}^p d_{T_{\bullet k}} \nabla g_k d_{\bar{T}_{\bullet k}}, \bigoplus_{i=1}^q \delta_i \right]. \quad (37)$$

By capturing each $d_{T_{\bullet k}} \nabla g_k d_{\bar{T}_{\bullet k}}$ with an IQC, this can also be done using Proposition 2. The following, based on a modification of the IQCs used in eq. (32), is proved in Appendix B.

Corollary 3: Take the variables from Corollary 2, and let $T, \bar{T} \in \mathbb{R}^{q \times p}$, where $T_{ik} \geq 0, \bar{T}_{ik} \geq 0$. Define

$$E_i(s) = \{z \in \mathbb{C} : |z - 1| + |z - f_i - 1| < \gamma_i\},$$

where

$$\begin{aligned} f_i(s) &= \delta_i(s) \sum_{k=1}^p R_{ik} \beta_k e^{-(T_{ik} + \bar{T}_{ik})s}, \\ \gamma_i(s) &= |\delta_i(s)| \sum_{k=1}^p R_{ik} \beta_k. \end{aligned}$$

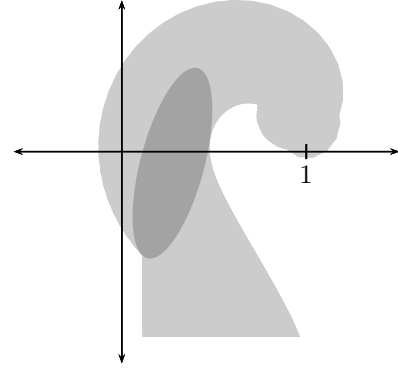
If for each $i \in \mathbb{Z}_{[1,q]}$

$$\operatorname{Re} \left\{ \left(1 - \hat{h}(j\omega)\right) E_i(j\omega) \right\} > 0, \quad \forall \omega \in \mathbb{R}, \quad (38)$$

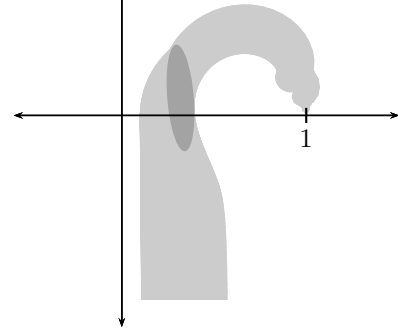
then the interconnection $\left[\sum_{k=1}^p d_{T_{\bullet k}} P_k d_{\bar{T}_{\bullet k}}, \bigoplus_{i=1}^q \delta_i \right]$ is stable.

Proof: See Appendix B. ■

We will show how to use Corollary 3 to tune the step sizes in the continuous time gradient descent algorithm when subject to delays. For illustrative purposes, suppose that the nonzero entries of T_{\bullet} and \bar{T}_{\bullet} are $(0.2, 0.3, 1.1)$ and $(0.6, 1.3, 0.7)$



(a) Equation (39) for $\alpha_i = 0.3$.



(b) Equation (39) for $\alpha_i = 0.15$.

Fig. 10: The light grey regions show the regions given by eq. (39) across frequency. The dark grey ellipses show the regions at a particular frequency $\omega \approx 0.35$ rad/s. Since the grey region in (a) enters the left half plane, eq. (38) is not satisfied. Such figures can be used to tune the parameter α_i to meet eq. (38), as has been done in (b).

respectively, the corresponding values of β_k are all equal to 1, and choose

$$h(t) = \begin{cases} 0.6e^{-.7t} & \text{if } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Figure 10(a) shows the regions

$$\left\{ \left(1 - \hat{h}(j\omega)\right) E_i(j\omega) \right\}, \quad \forall \omega > 0 \quad (39)$$

for these parameter values and α_i equals 0.3. These regions enter the left half plane, and hence the i th condition in eq. (38) is not satisfied for this value of α_i . However this figure can be used to design the parameter α_i such that eq. (38) is satisfied. For example the corresponding regions with $\alpha_i = 0.15$ are shown in fig. 10(b). This plot lies completely in the right half plane, meaning eq. (38) is (robustly) satisfied. Each other α_i could be tuned independently in a similar way to obtain step sizes that will guarantee global convergence of the algorithm.

Remark 13: The region E_i is an ellipse with eccentricity

$$\varepsilon_i = \frac{\sum_{k=1}^p R_{ik} \beta_k e^{-(T_{ik} + \bar{T}_{ik})s}}{\sum_{k=1}^p R_{ik} \beta_k}.$$

Therefore in the undelayed case $T_{\bullet k} = \bar{T}_{\bullet k} = 0$, $\varepsilon_i = 1$ and

at every frequency the ellipses reduce to the lines

$$\{1 + tf_i(j\omega) \in \mathbb{C} : t \in [0, 1]\}.$$

The i th conditions in Corollaries 2 and 3 are then equivalent. As the differences in the delay times grows, at higher frequencies ε_i becomes smaller, and the ellipses become more circular.

VI. FUTURE WORK

A major concern with applying the results in this paper is that they may (in general) be quite conservative. A more detailed investigation of the ideas in section IV-D would be interesting in this respect. There is also considerable flexibility in the network description that has not been investigated. Considering again the digraph example from section IV-A, one could instead decompose A as

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & t & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2-t & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

where $t \in \mathbb{R}$. The conditions obtained from applying proposition 1 to this network description depend on the chosen value of t . It is interesting to consider how to reduce conservatism through such choices. The answer is likely related to the results on chordal matrix decompositions from [8]. In a similar vein, one could also investigate the conservatism introduced by the relaxation in Theorem 1.

VII. CONCLUSIONS

A method for the analysis and design of large dynamical systems was presented. The linear structure of a class of dynamical systems was exploited to derive a set of tests. Critically each test could be checked using only information about local subsystems, and the satisfaction of all such tests was shown to be sufficient for stability of the overall system. Furthermore these tests were interpreted as local design criteria, each defining a local synthesis problem that could be tackled with standard techniques, including loopshaping, \mathbf{H}_∞ and μ -synthesis methods.

APPENDIX A PROOF OF COROLLARY 1

Let $I_k = \bigoplus_{i=1}^q R_{ik}$ and $S = \bigoplus_{i=1}^q \beta_i$. Compute X_k, Y_k, Z_k as in Proposition 1. This gives $X_k = 0, Y_k = hI_k, Z_k = -\frac{1}{2}(h + h^*)I_k S^{-1}$ (note that $n_i = 2$, since each edge in the Laplacian is incident to two vertices). Application of Proposition 1 guarantees stability if there exists an $\epsilon_k > 0$ such that $\forall \omega \in \mathbb{R}$:

$$h \left(B_{\bullet k} L_k B_{\bullet k}^T + \frac{1}{2} I_k S^{-1} \right) + (\cdot)^* \geq \epsilon_k B_{\bullet k} L_k^* L_k B_{\bullet k}^T \|B_{\bullet k}\|^2.$$

Pre and post multiply the above by $S^{\frac{1}{2}*}, S^{\frac{1}{2}}$ (this will not affect positive definiteness), and let $x_k = S^{\frac{1}{2}} B_{\bullet k}$:

$$h \left(x_k L_k x_k^T + \frac{1}{2} I_k \right) + (\cdot)^* \geq \epsilon_k x_k L_k^* L_k x_k^T \|B_{\bullet k}\|^2.$$

By Lemma 2, this is equivalent to

$$\operatorname{Re} \left\{ h(j\omega) \left(W(x_k L_k x_k^T) + \frac{1}{2} \right) \right\} > 0. \quad (40)$$

By Lemma 1,

$$W(x_k L_k x_k^T) = \left\{ t L_k(j\omega) \in \mathbb{C} : t \in [0, \|x_k\|^2] \right\}.$$

The result follows since $\|x_k\|^2 = \gamma_k$.

APPENDIX B PROOF OF COROLLARY 3

The following allows the IQCs for ∇g_k to be modified to cover $d_{T_{\bullet k}} \nabla g_k d_{\bar{T}_{\bullet k}}$.

Lemma 4: Let $s, t \in \mathbb{R}^q$ and Δ be a bounded causal operator, where $s_i \geq 0, t_i \geq 0$. Define $D_1 = \bigoplus_{i=1}^q e^{-j\omega s_i}$, $D_2 = \bigoplus_{i=1}^q e^{-j\omega t_i}$. Define $\bar{\Delta}$ such that $\forall x \in \mathbf{L}_2^q, y = \bar{\Delta}x$, where $\hat{y} = D_1 \hat{w}$, $w = \Delta v$, and $\hat{v} = \frac{1}{1+j\omega} D_2 \hat{x}$. If $\Delta \in \text{IQC}(\Pi)$, where

$$\Pi(j\omega) = \begin{bmatrix} A(j\omega) & B(j\omega)^* \\ B(j\omega) & C(j\omega) \end{bmatrix},$$

then $\bar{\Delta} \in \text{IQC}(\bar{\Pi})$, where

$$\bar{\Pi}(j\omega) = \begin{bmatrix} \frac{1}{1+\omega^2} A(j\omega) & \left(\frac{1}{1+j\omega} D_1 D_2 B(j\omega) \right)^* \\ \left(\frac{1}{1+j\omega} D_1 D_2 B(j\omega) \right) & C(j\omega) \end{bmatrix}.$$

Proof: Since $\Delta \in \text{IQC}(\Pi)$, for all $v \in \mathbf{L}_2^q, w = \Delta v$

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v} \\ \hat{w} \end{bmatrix}^* \Pi \begin{bmatrix} \hat{v} \\ \hat{w} \end{bmatrix} (j\omega) d\omega \geq 0.$$

By definition, for any $x \in \mathbf{L}_2^q, \hat{v} = \frac{1}{1+j\omega} D_2 \hat{x}$, and for any $w \in \mathbf{L}_2^q, \hat{w} = D_1 \hat{w}$. Therefore, for all $x \in \mathbf{L}_2^q, y = \bar{\Delta}x$

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}^* \begin{bmatrix} \frac{D_2}{1+j\omega} & 0 \\ 0 & D_1 \end{bmatrix}^* \Pi \begin{bmatrix} \frac{D_2}{1+j\omega} & 0 \\ 0 & D_1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} d\omega \geq 0.$$

The result follows since $\bar{\Pi}$ is equal to the product of the matrices in the above, and hence $\bar{\Delta} \in \text{IQC}(\bar{\Pi})$ ($\bar{\Pi}$ is a valid multiplier since it is bounded and continuous for all $\omega \in \mathbb{R}$, [17]). ■

The above shows that $\frac{1}{1+j\omega} d_{T_{\bullet k}} \nabla g_k d_{\bar{T}_{\bullet k}} \in \text{IQC}(\tilde{\Pi}_k)$, where

$$\tilde{\Pi} = \begin{bmatrix} 0 & F^* \\ F & -\beta_k^{-1} \bigoplus_{i=1}^q R_{ik} \end{bmatrix},$$

and $F = \frac{1}{1+j\omega} \bigoplus_{i=1}^q R_{ik} e^{-j\omega(T_{ik} + \bar{T}_{ik})}$. Therefore by Proposition 2, the interconnection is stable if for each $i \in \mathbb{Z}_{[1,q]}$, there exists an $\epsilon_i > 0$ such that $\forall \omega \in \mathbb{R}$:

$$\begin{bmatrix} \tilde{\delta}_i \\ I \end{bmatrix}^* \begin{bmatrix} -\epsilon_i I & V_i^* \\ V_i & -W_i \end{bmatrix} \begin{bmatrix} \tilde{\delta}_i \\ I \end{bmatrix} (j\omega) \geq 0, \quad (41)$$

where $\tilde{\delta}_i = (1 + j\omega) R_{i\bullet}^T \delta_i(j\omega) R_{i\bullet}$,

$$V_i = \frac{1 - \hat{h}(j\omega)}{1 + j\omega} \bigoplus_{k=1}^p R_{ik} e^{-j\omega(T_{ik} + \bar{T}_{ik})},$$

and $W_i = \bigoplus_{k=1}^p \left(2 - \hat{h}(j\omega) - \hat{h}(j\omega)^*\right) R_{ik} \beta_k^{-1}$. Multiplying out eq. (41), and applying Lemma 2 as in the proof of Corollary 1 shows that this is equivalent to:

$$\operatorname{Re} \left\{ \left(1 - \hat{h}\right) \left(W(A_i) + 1\right) \right\} > 0,$$

where $A_i = \left(R_i \bullet S^{\frac{1}{2}}\right)^T \delta_i \left(R_i \bullet S^{\frac{1}{2}}\right) D_i$, $S = \bigoplus_{k=1}^p \beta_k$ and $D_i(j\omega) = \bigoplus_{k=1}^p e^{-j\omega(T_{ik} + \bar{T}_{ik})}$. The result then follows by applying Lemma 1 (A_i is rank 1).

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