ALPHA INVARIANTS AND K-STABILITY FOR GENERAL POLARISATIONS OF FANO VARIETIES

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Abstract. We provide a sufficient condition for polarisations of Fano varieties to be K-stable in terms of Tian's alpha invariant, which uses the log canonical threshold to measure singularities of divisors in the linear system associated to the polarisation. This generalises a result of Odaka-Sano in the anti-canonically polarised case, which is the algebraic counterpart of Tian's analytic criterion implying the existence of a Kähler-Einstein metric. As an application, we give new K-stable polarisations of a general degree one del Pezzo surface. We also prove a corresponding result for log K-stability.

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1. Introduction

A central problem in complex geometry is to find necessary and sufficient conditions for the existence of a constant scalar curvature Kähler (cscK) metric in a given Kähler class. One of the first sufficient conditions is due to Tian, who introduced the alpha invariant. The alpha invariant $\alpha(X, L)$ of a polarised variety (X, L) is defined as the infimum of the log canonical thresholds of $\mathbb{Q}$-divisors in the linear system associated to L, measuring singularities of these divisors. Tian [31] proved that if X is a Fano variety of dimension n with canonical divisor K_X, the lower bound $\alpha(X, -K_X) > \frac{n}{n+1}$ implies that X admits a Kähler-Einstein metric in $c_1(X) = c_1(-K_X)$.

The Yau-Tian-Donaldson conjecture states that the existence of a cscK metric in $c_1(L)$ for a polarised manifold (X, L) is equivalent to the algebro-geometric notion of K-stability, related to geometric invariant theory. This conjecture has recently been proven in the case that $L = -K_X$ [8, 6, 7, 32]. By work of Donaldson [9] and Stoppa [28], it is known that the existence of a cscK metric in $c_1(L)$ implies that (X, L) is K-stable, provided the automorphism group of X is discrete. Odaka-Sano [22] have given a direct algebraic proof that $\alpha(X, -K_X) > \frac{n}{n+1}$ implies that $(X, -K_X)$ is K-stable. This provides the first algebraic proof of K-stability of varieties of dimension greater than one.

On the other hand, few sufficient criteria are known for K-stability in the general case. We give a sufficient condition for general polarisations of Fano varieties to be K-stable. A fundamental quantity will be the slope of a polarised variety (X, L), defined as

$$\mu(X, L) = \frac{-K_X.L^{n-1}}{L^n} = \frac{\int_X c_1(X).c_1(L)^{n-1}}{\int_X c_1(L)^n}. \tag{1}$$

The slope is therefore a topological quantity which, after rescaling L, can be assumed equal to 1. Our main result is then as follows.

Theorem 1.1. *Let (X, L) be a polarised $\mathbb{Q}$-Gorenstein log canonical variety with canonical divisor K_X. Suppose that*

(i) $\alpha(X, L) > \frac{n}{n+1}\mu(X, L)$ *and*
(ii) $-K_X \geq \frac{n}{n+1}\mu(X, L)L$.

Then (X, L) is K-stable.

Here, for divisors H, H', we write $H \geq H'$ to mean $H - H'$ is nef. Note that when $L = -K_X$, the slope of (X, L) is equal to 1, the second condition is vacuous and this theorem is then due to Odaka-Sano. The condition that X is log canonical ensures that $\alpha(X, L) \geq 0$, while the condition that X is $\mathbb{Q}$-Gorenstein ensures that $-K_X$ exists as a $\mathbb{Q}$-Cartier divisor. The second condition implies that X is either Fano or numerically Calabi-Yau, see Remark 3.5. By proving a continuity result for the alpha invariant, we also show in Corollary 4.3 that provided the inequality in the second condition is strict, the conditions to apply Theorem 1.1 are *open* when varying the polarisation.

Theorem 1.1 gives the first non-toric criterion for K-stability of general polarisations of Fano varieties. On the analytic side, a result of LeBrun-Simanca [14] states that the condition that a polarised variety (X, L) admits a cscK metric is an *open* condition when varying L, provided the automorphism group of X is discrete. As the existence of a cscK metric in $c_1(L)$ implies K-stability, this gives an analytic proof that in the situation of 1.1, K-stability is an open condition again in the case that the automorphism group of X is discrete. On the other hand, our result can also be used to give *explicit* K-stable polarisations, see for example Theorem 1.2.

Many computations [4, 3] of alpha invariants have been done for anti-canonically polarised Fano varieties. Cheltsov [3], building on work of Park [25], has calculated alpha invariants of del Pezzo surfaces. As a corollary, Cheltsov's results imply that general anti-canonically polarised del Pezzo surfaces of degrees one, two and three are K-stable. Following the method of proof of Cheltsov, we give new examples of K-stable polarisations of a general del Pezzo surface X of degree one. Noting that X is isomorphic to a blow-up of $\mathbb{P}^2$ at 8 points in general position, we denote by H the hyperplane divisor, E_i the 8 exceptional divisors and $L_\lambda = 3H - \sum_{i=1}^{7} E_i - \lambda E_8$ arising from this isomorphism.

Theorem 1.2. *(X, L_λ) is K-stable for*

$$\frac{19}{25} \approx \frac{1}{9}(10 - \sqrt{10}) < \lambda < \sqrt{10} - 2 \approx \frac{29}{25}. \tag{2}$$

We note that Theorem 1.1 is merely *sufficient* to prove K-stability. It would be interesting to know exactly which polarisations of a general degree one del Pezzo surface are K-stable. Analytically, a result of Arezzo-Pacard [1] implies that (X, L_λ) admits a cscK metric for λ sufficiently small. In particular, by work of Donaldson [9] and Stoppa [28], this implies (X, L_λ) is K-stable for λ sufficiently small. However, using the technique of slope stability, Ross-Thomas [27, Example 5.30] have shown that there are polarisations of such an X which are K-*un*stable.

The recent proof of the Yau-Tian-Donaldson conjecture [8, 6, 7, 32] in the case $L = -K_X$ has emphasised the importance of log K-stability. This concept extends K-stability to pairs (X, D) and conjecturally corresponds to cscK metrics with cone singularities along D. With this in mind, we extend Theorem 1.1 to the log setting as follows.

Theorem 1.3. *Let $((X, D); L)$ consist of a $\mathbb{Q}$-Gorenstein log canonical pair (X, D) with canonical divisor K_X, such that D is an effective integral reduced Cartier divisor on a polarised variety (X, L). Denote $\mu_\beta((X, D); L) = \frac{-(K_X+(1-\beta)D).L^{n-1}}{L^n}$. Suppose that*

(i) $\alpha((X, D); L) > \frac{n}{n+1}\mu_\beta((X, D); L)$ *and*
(ii) $-(K_X + (1-\beta)D) \geq \frac{n}{n+1}\mu_\beta((X, D); L)L$.

Then $((X, D); L)$ is log K-stable with cone angle β along D.

Notation and conventions: By a polarised variety (X, L) we mean a normal complex projective variety X together with an ample line bundle L. We often use the same letter to denote a divisor and the associated line bundle, and mix multiplicative and additive notation for line bundles.

2. Prerequisites

2.1. **K-stability.** K-stability of a polarised variety (X, L) is an algebraic notion conjecturally equivalent to the existence of a constant scalar curvature Kähler metric in $c_1(L)$, which requires the so-called Donaldson-Futaki invariant to be positive for all non-trivial test configurations.

Definition 2.1. A *test configuration* for a normal polarised variety (X, L) is a normal polarised variety $(\mathcal{X}, \mathcal{L})$ together with

- a proper flat morphism $\pi : \mathcal{X} \to \mathbb{C}$,
- a $\mathbb{C}^*$-action on $\mathcal{X}$ covering the natural action on $\mathbb{C}$,
- and an equivariant very ample line bundle $\mathcal{L}$ on $\mathcal{X}$

such that the fibre $(\mathcal{X}_t, \mathcal{L}_t)$ over t is isomorphic to (X, L) for one, and hence all, $t \in \mathbb{C}^*$.

Definition 2.2. We say that a test configuration is *almost trivial* if X is $\mathbb{C}^*$-isomorphic to the product configuration away from a closed subscheme of codimension at least 2.

Definition 2.3. We will later be interested in a slightly modified version of test configurations. In particular, we will be interested in the case where we have a proper flat morphism $\pi : \mathcal{X} \to \mathbb{P}^1$ with target $\mathbb{P}^1$ rather than $\mathbb{C}$ such that $\mathcal{L}$ is just *relatively* semi-ample over $\mathbb{P}^1$, that is, a multiple of the restriction to each fibre over $\mathbb{P}^1$ is basepoint free. We call this a *semi-test configuration.*

As the $\mathbb{C}^*$-action on $(\mathcal{X}, \mathcal{L})$ fixes the central fibre $(\mathcal{X}_0, \mathcal{L}_0)$, there is an induced action on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$ for all k. Denote by $w(k)$ the total weight of this action, which is a polynomial in k of degree $n+1$ for $k \gg 0$, where n is the dimension of X. Denote the Hilbert polynomial of (X, L) as

$$\mathcal{P}(k) = \chi(X, L^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}) \tag{3}$$

and denote also the total weight of the $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$ as

$$w(k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}). \tag{4}$$

Definition 2.4. We define the *Donaldson-Futaki invariant* of a test configuration $(\mathcal{X}, \mathcal{L})$ to be

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{b_0 a_1 - b_1 a_0}{a_0^2}. \tag{5}$$

We say (X, L) is *K-stable* if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) > 0$ for all test configurations which are not almost trivial.

Remark 2.5. For more information on the following remarks, or for a more detailed discussion of K-stability, see [27].

- The definition of K-stability is independent of scaling $L \to L^r$. In particular, it makes sense for pairs (X, L) where X is a variety and L is a $\mathbb{Q}$-line bundle.
- If one expands $\frac{w(k)}{k\mathcal{P}(k)} = f_0 + f_1 k^{-1} + O(k^{-2})$, the Donaldson-Futaki invariant is given by f_1.
- One should think of test configuration as geometrisations of the one-parameter subgroups that are considered when applying the Hilbert-Mumford criterion to GIT stability. In fact, asymptotic Hilbert stability implies K-semistability, since the Donaldson-Futaki invariant appears as the leading coefficient in a polynomial associated with asymptotic Hilbert stability.
- The notion of almost trivial test configurations was introduced by Stoppa [29] to resolve a pathology noted by Li-Xu [15, Section 2.2].

Conjecture 2.1. *(Yau-Tian-Donaldson) A smooth polarised variety (X, L) admits a constant scalar curvature metric in $c_1(L)$ if and only if (X, L) is K-stable.*

Remark 2.6. This conjecture as stated has recently been proven by Chen-Donaldson-Sun [8, 6, 7] and separately Tian [32] in the case $L = -K_X$ (so X is *Fano*). It is expected to hold in the general case, with possibly some slight modifications to the definition of K-stability, see [30].

2.2. **Odaka's Blowing-up Formalism.** In [20], Odaka shows that to check K-stability, it suffices to check the positivity of the Donaldson-Futaki invariant on *semi*-test configurations arising from flag ideals.

Definition 2.7. A flag ideal on X is a coherent ideal sheaf $\mathcal{I}$ on $X \times \mathbb{A}^1$ of the form $\mathcal{I} = I_0 + (t)I_1 + \ldots + (t^N)$ with $I_0 \subseteq I_1 \subseteq \ldots \subseteq I_{N-1} \subseteq \mathcal{O}_X$ a sequence of coherent ideal sheaves. The ideal sheaves I_j thus correspond to subschemes $Z_0 \supseteq Z_1 \supseteq \ldots \supseteq Z_{N-1}$ of X. Flag ideals can be equivalently characterised by being $\mathbb{C}^*$-invariant with support on $X \times \{0\}$.

Remark 2.8. The flag ideal $\mathcal{I}$ naturally induces a coherent ideal sheaf on $X \times \mathbb{P}^1$, which we also denote by $\mathcal{I}$. Blowing-up $\mathcal{I}$ on $X \times \mathbb{P}^1$, we get a map

$$\pi : \mathcal{B} = Bl_{\mathcal{I}}(X \times \mathbb{P}^1) \to X \times \mathbb{P}^1. \tag{6}$$

Denote by E the exceptional divisor of the blow-up $\pi : \mathcal{B} \to X \times \mathbb{P}^1$, that is, $\mathcal{O}(-E) = \pi^{-1}\mathcal{I}$. Abusing notation, write $\mathcal{L} - E$ to denote $(p_1 \circ \pi)^* L \otimes \mathcal{O}(-E)$, where $p_1 : X \times \mathbb{P}^1 \to X$ is the natural projection. Note that the induced map from $\mathcal{B} \to \mathbb{P}^1$ is flat by [27, Remark 5.2]. There is a natural $\mathbb{C}^*$ action on $X \times \mathbb{P}^1$, acting trivially on X, which lifts to an action on $\mathcal{B}$. With this action, provided $\mathcal{L} - E$ is relatively semi-ample over $\mathbb{P}^1$ and $\mathcal{B}$ is normal, we have that $(\mathcal{B}, \mathcal{L} - E)$ is a *semi*-test configuration.

Theorem 2.9. [20, Corollary 3.11] *Assume that (X, L) is a normal polarised variety. Then (X, L) is K-stable if and only if* $\mathrm{DF}(\mathcal{B}, \mathcal{L}^r - E) > 0$ *for all $r > 0$ and for all flag ideals $\mathcal{I} \neq (t^N)$ with $\mathcal{B}$ normal and Gorenstein in codimension one and with $\mathcal{L}^r - E$ relatively semi-ample over $\mathbb{P}^1$.*

Remark 2.10. That $\mathcal{B}$ can be assumed normal was noted by Odaka-Sano [22, Proposition 2.1]. The condition that $\mathcal{I} \neq (t^N)$ is to ensure $\mathcal{B}$ is not almost trivial, see Definition 2.2.

Remark 2.11. As a general test configuration $(\mathcal{X}, \mathcal{L})$ is $\mathbb{C}^*$-isomorphic to $(X \times \mathbb{A}^1, L^r)$ away from the central fibre, it is $\mathbb{C}^*$-birational to $(X \times \mathbb{A}^1, L)$. In particular, it is dominated by a blow-up of $X \times \mathbb{A}^1$ along a flag ideal. Odaka shows that one can choose a flag ideal such that the Donaldson-Futaki invariant of the two test configurations are equal. In order to use the machinery of intersection theory, one must also compactify $X \times \mathbb{A}^1$ to $X \times \mathbb{P}^1$.

Remark 2.12. In the case the flag is of the form $\mathcal{I} = I_0 + (t)$, blowing-up $\mathcal{I}$ on $X \times \mathbb{A}^1$ leads to *deformation to the normal cone.* In [26], Ross-Thomas study test configurations arising from this process. Stability with respect to test configurations arising from blow-ups of the form $\mathcal{I} = I_0 + (t)$ is called *slope stability.* Note that Panov-Ross [24, Example 7.8] have shown that the blow-up of $\mathbb{P}^2$ at 2 points is slope stable but is not K-stable. One must therefore consider more general flag ideals to check K-stability.

One benefit of this formalism is that, for test configurations arising from flag ideals, there is an explicit intersection-theoretic formula for the Donaldson-Futaki invariant.

Theorem 2.13. [20, Theorem 3.2] *For a semi-test configuration of the form $(\mathcal{B} = Bl_{\mathcal{I}} X \times \mathbb{P}^1, \mathcal{L} - E)$ arising from a flag ideal $\mathcal{I}$ with $\mathcal{B}$ normal and Gorenstein in codimension one, the Donaldson-Futaki invariant is given by (up to multiplication by a positive constant)*

$$\mathrm{DF} = -n(L^{n-1}.K_X)(\mathcal{L} - E)^{n+1} + (n+1)(L^n)(\mathcal{L} - E)^n.(\mathcal{K}_X + K_{\mathcal{B}/X \times \mathbb{P}^1}). \quad (7)$$

Here we have denoted by $\mathcal{K}_X$ the pull back of K_X to $\mathcal{B}$. The intersection numbers $L^{n-1}.K_X$ and L^n are computed on X, while the remaining intersection numbers are computed on $\mathcal{B}$. Replacing L and $\mathcal{L}$ by L^r and $\mathcal{L}^r$ respectively in formula 7 gives the formula for the Donaldson-Futaki invariant of a test configuration of the form $(\mathcal{B}, \mathcal{L}^r - E)$.

Note that $\mathcal{K}_X + K_{\mathcal{B}/X \times \mathbb{P}^1} = K_{\mathcal{B}/\mathbb{P}^1}$. The benefit of splitting this into two terms is that positivity of the contribution from the second term, the relative canonical divisor over $X \times \mathbb{P}^1$, can be controlled under assumptions on the singularities of X.

2.3. **Log Canonical Thresholds.** The log canonical threshold of a pair (X, D) is a measure of singularity, related to the complex singularity exponent. It takes into consideration both the singularities of X and D. See [12] for more information on log canonical thresholds.

Definition 2.14. Let X be a normal variety and let $D = \sum d_i D_i$ be a divisor on X such that $K_X + D$ is $\mathbb{Q}$-Cartier, where D_i are prime divisors. Let $\pi : Y \to X$ be a log resolution of singularities, so that Y is smooth and $\pi^{-1}D \cup E$ has simple normal crossing support where E is the exceptional divisor. We can then write

$$K_Y - \pi^*(K_X + D) \equiv \sum a(E_i, (X, D))E_i \tag{8}$$

where E_i is either an exceptional divisor or $E_i = \pi_*^{-1} D_i$ for some i. That is, either E_i is exceptional or the proper transform of a component of D. We usually abbreviate $a(E_i, (X, D))$ to a_i. We say the pair (X, D) is

- *log canonical* if $a(E_i, (X, D)) \geq -1$ for all E_i,
- *Kawamata log terminal* if $a(E_i, (X, D)) > -1$ for all E_i.

By [12, Lemma 3.10] these notions are independent of log resolution.

We will later need a form of inversion of adjunction for log canonicity.

Theorem 2.15. [11] *Let $D = D' + D''$ be a $\mathbb{Q}$-divisor with D' an effective reduced normal Cartier divisor and D'' an effective $\mathbb{Q}$-divisor which has no common components with D'. Then (X, D) is log canonical on some open neighbourhood of D' if and only if $(D', D''|_{D'})$ is log canonical.*

Definition 2.16. We say a variety X is *log canonical* if $(X, 0)$ is log canonical. Note in particular that log canonical varieties are normal by assumption.

For a not necessarily log canonical pair (X, D) we can still use the idea of log canonicity to measure singularities.

Definition 2.17. Let X be a normal variety, and let D be a $\mathbb{Q}$-divisor. The *log canonical threshold* of a pair (X, D) is

$$\operatorname{lct}(X, D) = \sup\{\lambda \in \mathbb{Q}_{>0} \mid (X, \lambda D) \text{ is log canonical}\}. \tag{9}$$

One can generalise the log canonical threshold of a divisor to general coherent ideal sheaves as follows.

Definition 2.18. Let $I \subset \mathcal{O}_X$ be a coherent ideal sheaf, and let D be an effective $\mathbb{Q}$-divisor on X. We say that $\pi : Y \to X$ is a *log resolution* of I and D if Y is smooth and there is an effective divisor F on Y with $\pi^{-1}I = \mathcal{O}_Y(-F)$ such that $F \cup E \cup \tilde{D}$ has simple normal crossing support, where $\tilde{D}$ is the proper transform of D. Let $\pi : Y \to X$ be such a log resolution and assume the pair (X, D) is log canonical. For a real number $c \in \mathbb{R}$, we define the *discrepancy* of $((X, D); cI)$ to be

$$a(E_i, ((X, D); cI)) = a(E_i, (X, D)) - c\,\operatorname{val}_{E_i}(I). \tag{10}$$

Here by $\operatorname{val}_{E_i}(I)$ we mean the valuation of the ideal I on E_i, while the $a(E_i, (X, D))$ are as in Definition 2.14. We say $((X, D); cI)$ is *log canonical* if $a(E_i, ((X, D); cI)) \geq -1$ for all E_i appearing in a log resolution of I and D. The *log canonical threshold* of $((X, D); I)$ is then defined as

$$\operatorname{lct}((X, D); I) = \sup\{\lambda \in \mathbb{Q}_{>0} \mid ((X, D); \lambda I) \text{ is log canonical}\}. \tag{11}$$

Remark 2.19. [12, Proposition 8.5] For a proper birational map $f : X' \to X$, we have that

$$\operatorname{lct}((X,D);cI) \leq \min_{E_i \subset X'} \left\{ \frac{1 + \operatorname{val}_{E_i} K_{X'/X} - \operatorname{val}_{E_i} D}{c \operatorname{val}_{E_i}(I)} \right\}, \tag{12}$$

where our convention for the appearance of the E_i is as in Definition 2.14. Equality is achieved on a log resolution, where this is essentially a rephrasing of the definition of the log canonical threshold.

Definition 2.20. Let (X, L) be a log canonical polarised variety. We define the *alpha invariant* of (X, L) to be

$$\alpha(X,L) = \inf_{m \in \mathbb{Z}_{>0}} \inf_{D \in |mL|} \operatorname{lct}(X, \frac{1}{m} D). \tag{13}$$

In particular, for $c > 0$ the alpha invariant satisfies the scaling property

$$\alpha(X, cL) = \frac{1}{c} \alpha(X, L). \tag{14}$$

This definition of the alpha invariant is the algebraic counterpart of Tian's original definition. For further details on the following *analytic* definition, see [4, Appendix A].

Definition 2.21. Let h be a singular hermitian metric on L, written locally as $h = e^{-2\phi}$. We define the *complex singularity exponent* $c(h)$ to be

$$c(h) = \sup\{\lambda \in \mathbb{R}_{>0} | \text{ for all } z \in X, h^{\lambda} = e^{-2\lambda\phi} \text{ is } L^1 \text{ in a neighbourhood of } z\}. \tag{15}$$

We then define Tian's *alpha invariant* $\alpha^{an}(X, L)$ of (X, L) to be

$$\alpha^{an}(X, L) = \inf_{h \text{ with } \Theta_{L,h} \geq 0} c(h) \tag{16}$$

where the infimum is over all singular hermitian metrics h with curvature $\Theta_{L,h} \geq 0$. For a compact subgroup G of $\operatorname{Aut}(X, L)$, one defines α^{an} similarly, however considering only G-invariant metrics.

Theorem 2.22. [4, Appendix A] *The alpha invariant $\alpha(X, L)$ defined algebraically equals Tian's alpha invariant $\alpha^{an}(X, L)$. That is,*

$$\alpha(X, L) = \alpha^{an}(X, L). \tag{17}$$

Remark 2.23. As every divisor $D \in L$ gives rise to a singular hermitian metric, one sees that $\alpha(X, L) \geq \alpha^{an}(X, L)$. Equality follows from approximation techniques for plurisubharmonic functions.

The main consequence of the definition of the alpha invariant is the following theorem of Tian, which states that certain lower bounds on the alpha invariant imply the existence of a Kähler-Einstein metric.

Theorem 2.24. [31, Theorem 2.1] *Let G be a compact subgroup of* $\operatorname{Aut}(X)$ *and suppose X is a smooth Fano variety with $\alpha_G(X, -K_X) > \frac{n}{n+1}$. Then X admits a Kähler-Einstein metric.*

3. Alpha Invariants and K-stability

In this section we provide a *sufficient* condition for polarised varieties (X, L) of dimension n to be K-stable. A fundamental quantity will be the *slope* of a polarised variety.

Definition 3.1. We define the *slope* of (X, L) to be

$$\mu(X, L) = \frac{-K_X.L^{n-1}}{L^n} = \frac{\int_X c_1(X).c_1(L)^{n-1}}{\int_X c_1(L)^n}. \tag{18}$$

The slope of a polarised variety is thus a topological quantity which, after rescaling L, may be assumed equal to 1. Note in particular that $\mu(X, -K_X) = 1$.

Remark 3.2. In [27], Ross-Thomas defined a similar quantity, which they also call the slope, defined to be $\frac{n}{2}$ times our definition.

Theorem 3.3. *Let (X, L) be a polarised $\mathbb{Q}$-Gorenstein log canonical variety with canonical divisor K_X. Suppose that*

(i) $\alpha(X, L) > \frac{n}{n+1}\mu(X, L)$ *and*
(ii) $-K_X \geq \frac{n}{n+1}\mu(X, L)L$.

Then (X, L) is K-stable.

Remark 3.4. Here, for divisors H, H', we write $H \geq H'$ to mean $H - H'$ is nef. Note that both conditions are independent of positively scaling L. For $L = -K_X$, the second condition is vacuous, so in this case, this is the algebraic counterpart of a theorem of Tian (Theorem 2.24) and in this case is due to Odaka-Sano [22, Theorem 1.4]. The condition that X is log canonical ensures that $\alpha(X, L) \geq 0$, while the condition that X is $\mathbb{Q}$-Gorenstein ensures that $-K_X$ exists as a $\mathbb{Q}$-Cartier divisor.

Remark 3.5. If $\mu(X, L) = 0$, i.e. $L^{n-1}.K_X = 0$, the second condition requires $-K_X$ to be nef. Suppose $-K_X$ is nef but not numerically equivalent to zero, and suppose $L^{n-1}.K_X = 0$. Then, by the Hodge Index Theorem [16, Theorem 1] we would have $L^{n-2}.K_X^2 < 0$, contradicting the fact that $-K_X$ is nef. In particular, for the second condition of the theorem to hold, X must either be numerically Calabi-Yau or Fano. In the Calabi-Yau case, this theorem also follows from a theorem due to Odaka [21, Theorem 1.1].

A Corollary of Theorem 3.3 is that the automorphism group of (X, L) is discrete. Indeed, if $\mathrm{Aut}(X, L)$ were to admit a one parameter subgroup, this would give two test configurations with Donaldson-Futaki invariants of opposite sign. But K-stability requires strict positivity of the Donaldson-Futaki invariant, a contradiction.

Corollary 3.6. *If the criteria of Theorem 3.3 are satisfied, then* $\mathrm{Aut}(X, L)$ *is discrete.*

To prove Theorem 3.3, we first establish an *upper* bound on the alpha invariant.

Proposition 3.7. *(c.f.* [22, Proposition 3.1]*) Let $\mathcal{B}$ be the blow-up of $X \times \mathbb{P}^1$ along a flag ideal, with $\mathcal{B}$ normal and Gorenstein in codimension one, $\mathcal{L} - E$ relatively semi-ample over $\mathbb{P}^1$ and notation as in Remark 2.8. Denote the natural map arising from the composition of the blow-up map and the projection map by $\Pi : \mathcal{B} \to \mathbb{P}^1$. Denote also*

- *the discrepancies as:* $K_{\mathcal{B}/X\times\mathbb{P}^1} = \sum a_i E_i$*,*
- *the multiplicities of* $X \times \{0\}$ *as:* $\Pi^*(X\times\{0\}) = \Pi_*^{-1}(X\times\{0\}) + \sum b_i E_i$*,*
- *the exceptional divisor as:* $\Pi^{-1}\mathcal{I} = \mathcal{O}_{\mathcal{B}}(-\sum c_i E_i) = \mathcal{O}_{\mathcal{B}}(-E)$*.*

Then

$$\alpha(X,L) \leq \min_i \left\{ \frac{a_i - b_i + 1}{c_i} \right\}. \tag{19}$$

Proof. We are seeking an upper bound on the alpha invariant, where this upper bound is related to the flag ideal $\mathcal{I} = I_0 + (t)I_1 + \ldots + (t^N)$ on $X\times\mathbb{P}^1$. As the divisors considered in the definition of the alpha invariant are divisors on X, we pass from $\mathcal{I}$ to its first component I_0. The choice of I_0 is because I_0 is a subsheaf of the full flag ideal $\mathcal{I}$.

Let $\pi_0 : Bl_{I_0}X \to X$ be the blow-up of I_0 with exceptional divisor E_0. We claim $\pi_0^* L - E_0$ is semi-ample. This is equivalent to $I_0^m(mL)$ being base-point free for some m. However, as $\mathcal{L} - E$ is semi-ample restricted each fibre, we know that $\mathcal{I}^m\pi^*(mL)$ is base-point free on each fibre of $X\times\mathbb{P}^1$. As I_0 is a subsheaf of $\mathcal{I}$, the result follows.

Choose m sufficiently large and divisible such that $H^0(Bl_{I_0}X, m(\pi_0^*L - E_0)) = H^0(X, I_0^m(mL))$ has a section, which exists since multiples of semi-ample line bundles have sections. Let D be in the linear series $H^0(X, I_0^m(mL))$. We show that

$$\alpha(X,L) \leq m\,\mathrm{lct}(X,D) \leq \min_i \left\{ \frac{a_i - b_i + 1}{c_i} \right\}. \tag{20}$$

For general ideal sheaves I, J, [19, Property 1.12] states that $I \subset J$ implies $\mathrm{lct}(X,I) \leq \mathrm{lct}(X,J)$. We therefore see that

$$\mathrm{lct}(X,D) = \mathrm{lct}(X,I_D) \leq \frac{1}{m}\,\mathrm{lct}(X,I_0). \tag{21}$$

Note that $X\times\{0\}$ is a divisor on $X\times\mathbb{A}^1$. By a basic form of inversion of adjunction of log canonicity, we have

$$\mathrm{lct}(X,I_0) = \mathrm{lct}((X\times\mathbb{P}^1, X\times\{0\}); I_0). \tag{22}$$

One can see this by taking a log resolution of $((X\times\mathbb{P}^1, X\times\{0\}); I_0)$ of the form $\tilde{X}\times\mathbb{P}^1 \to X\times\mathbb{P}^1$, where $\tilde{X}\to X$ is a log resolution of (X, I_0). Note that for all divisors E_i over X, we have

$$\mathrm{val}_{E_i}(I_0) \geq \mathrm{val}_{E_i}(\mathcal{I}). \tag{23}$$

In particular, we see that

$$\mathrm{lct}((X\times\mathbb{P}^1, X\times\{0\}); I_0) \leq \mathrm{lct}((X\times\mathbb{P}^1, X\times\{0\}); \mathcal{I}) \tag{24}$$

$$\leq \min_i \left\{ \frac{a_i - b_i + 1}{c_i} \right\}. \tag{25}$$

The last inequality is by Remark 2.19.

□

The final ingredients of the proof of Theorem 3.3 are the following lemmas on computing the positivity of terms in Odaka's formula for the Donaldson-Futaki invariant, which are due to Odaka-Sano. We repeat their proof for the reader's convenience.

Lemma 3.8. [22, Lemma 4.2] *Let L and R be ample and nef divisors respectively on X, with $p^*L = \mathcal{L}$ and $p^*R = \mathcal{R}$ where $p : \mathcal{B} \to X$ is the natural map arising from the composition of the blow-up map $\mathcal{B} \to X \times \mathbb{P}^1$ and the projection $X \times \mathbb{P}^1 \to \mathbb{P}^1$. Suppose that $\mathcal{L} - E$ is semi-ample on the blow-up $Bl_{\mathcal{I}} X \times \mathbb{P}^1$ for some flag ideal $\mathcal{I}$. Then*

$$(\mathcal{L} - E)^n.\mathcal{R} \leq 0. \tag{26}$$

Proof. Firstly note that, because $\mathcal{R}$ and $\mathcal{L}$ are the pull back of ample and nef divisors respectively from X, which has dimension n, we have $\mathcal{L}^n.\mathcal{R} = 0$. Now note that we have the equality

$$-(\mathcal{L} - E)^n.\mathcal{R} = \mathcal{L}^n.\mathcal{R} - (\mathcal{L} - E)^n.(\mathcal{R} - E) - (\mathcal{L} - E)^n.E \tag{27}$$

$$= E.\mathcal{R}.(\mathcal{L}^{n-1} + \mathcal{L}^{n-2}.(\mathcal{L} - E) + \ldots + (\mathcal{L} - E)^{n-1}). \tag{28}$$

As $\mathcal{L}$ is nef and the restriction of $\mathcal{L} - E$ to the central fibre of the map $\mathcal{B} \to \mathbb{P}^1$ is semi-ample, hence nef, and as $E.\mathcal{R}$ is a non-zero effective cycle with support in the central fibre, the result follows. □

Lemma 3.9. [22, Lemma 4.7] *Let $E = \sum c_i E_i$ be the exceptional divisor of the blow-up $\mathcal{B} \to X \times \mathbb{P}^1$. Then*

$$(\mathcal{L} - E)^n.E_i \geq 0. \tag{29}$$

Moreover, strict positivity holds for some E_i, that is,

$$(\mathcal{L} - E)^n.E > 0. \tag{30}$$

Proof. Since each E_i has support in the central fibre, and $\mathcal{L} - E$ restricted to the central fibre is semi-ample, hence nef, we have that $(\mathcal{L} - E)^n.E_i \geq 0$.

To show $(\mathcal{L} - E)^n.E > 0$, we first show $(\mathcal{L} - E)^n.(\mathcal{L} + nE) > 0$. Note that we have the equality of polynomials

$$(x - y)^n(x + ny) = x^{n+1} - \sum_{i=1}^{n}(n + 1 - i)(x - y)^{n-i}x^{i-1}y^2. \tag{31}$$

In fact, the polynomials $(x - y)^{n-i}x^{i-1}y^2$ for $1 \leq i \leq n$ are linearly independent over $\mathbb{Q}$, and for all $0 < s < n$, the monomial $x^s y^{n+1-s}$ can be written as a linear combination of these polynomials with coefficients in $\mathbb{Z}$.

Note that $\mathcal{L}^{n+1} = 0$, as $\mathcal{L}$ is the pull back of an ample line bundle from X, which has dimension n. In particular, we can write

$$(\mathcal{L} - E)^n.(\mathcal{L} + nE) = -E^2. \left(\sum_{i=1}^{n}(n + 1 - i)(\mathcal{L} - E)^{n-i}.\mathcal{L}^{i-1}\right). \tag{32}$$

Let $s = \dim(\mathrm{Supp}(\mathcal{O}/\mathcal{I}))$, where $\mathcal{I} = I_0 + (t)I_1 + \ldots + (t^N)$. By dividing $\mathcal{I}$ by a power of t if necessary, which does not change the resulting blow-up $Bl_{\mathcal{I}} X \times \mathbb{P}^1$ and hence does not change the Donaldson-Futaki invariant of the associated semi-test configuration, we can assume $s < n$. Perturbing the coefficients in equation (32), we get

$$(\mathcal{L}-E)^n.(\mathcal{L}+nE) = -E^2. \left(\sum_{i=1}^{n}(n + 1 - i + \epsilon_i)(\mathcal{L} - E)^{n-i}\mathcal{L}^{i-1}\right) - \epsilon'(\mathcal{L}^s.(-E)^{n+1-s}) \tag{33}$$

where $0 < |\epsilon_i| \ll 1$ and $0 < \epsilon' \ll 1$.

The following lemma then shows that $(\mathcal{L}-E)^n.(\mathcal{L}+nE)>0$.

Lemma 3.10. [22, Lemma 4.7]
(i) $-E^2.(\mathcal{L}-E)^{n-i}.\mathcal{L}^{i-1}\geq 0$ *for all* $0<i<n$.
(ii) $\mathcal{L}^s.(-E)^{n+1-s}<0$.

Proof. (i) Cutting $\mathcal{B}$ by general elements of $|r\mathcal{L}|$ and $|r(\mathcal{L}-E)|$ for $r\gg 0$, we can assume $\dim X=2$. In this case, the required equation becomes $-E^2.(\mathcal{L}-E)\geq 0$. Note that $\mathcal{L}-E$ is semi-ample restricted to fibres of $\mathcal{B}\to\mathbb{P}^1$ and E has support in the central fibre. In particular, $E.(\mathcal{L}-E)$ is an effective cycle with support in fibres of the blow-up map $\mathcal{B}\to X\times\mathbb{P}^1$. Since $-E$ is relatively ample over fibres of $\mathcal{B}\to X\times\mathbb{P}^1$, we have $-E.(E.(\mathcal{L}-E))\geq 0$ and the result follows.

(ii) Again cutting $\mathcal{B}$ by general elements of $\mathcal{L}^r$ for $r\gg 0$, we can assume $s=0$. The required result then follows by relative ampleness of $-E$ over fibres of $\mathcal{B}\to X\times\mathbb{P}^1$.

□

Finally, since $(\mathcal{L}-E)^n.(\mathcal{L}+nE)>0$ and $(\mathcal{L}-E)^n.\mathcal{L}\leq 0$ by Lemma 3.8, we have $(\mathcal{L}-E)^n.E>0$ as required.

□

Proof. (of Theorem 3.3) We show that the Donaldson-Futaki invariant is positive for all semi-test configurations of the form $(\mathcal{B},\mathcal{L}^r-E)$ arising from flag ideals. We assume $r=1$ for notational simplicity, the proof in the general case is essentially the same. The idea is to first split formula 2.13 for the Donaldson-Futaki invariant into two terms, which we consider separately. We split the Donaldson-Futaki invariant as

$$\mathrm{DF}(\mathcal{B},\mathcal{L}-E)=\mathrm{DF}_{num}+\mathrm{DF}_{disc}, \tag{34}$$

$$\mathrm{DF}_{num}=(\mathcal{L}-E)^n.(-n(L^{n-1}.K_X)\mathcal{L}+(n+1)(L^n)\mathcal{K}_X), \tag{35}$$

$$\mathrm{DF}_{disc}=(\mathcal{L}-E)^n.((n+1)(L^n)K_{\mathcal{B}/X\times\mathbb{P}^1}+n(L^{n-1}.K_X)E). \tag{36}$$

Our second hypothesis in Theorem 3.3 is

$$-K_X\geq\frac{n}{n+1}\mu(X,L)L. \tag{37}$$

In particular, $n(L^{n-1}.K_X)\mathcal{L}-(n+1)(L^n)\mathcal{K}_X$ is nef. So, by Lemma 3.8, $\mathrm{DF}_{num}\geq 0$.

As $(\mathcal{L}-E)^n.E>0$ by Lemma 3.9, it suffices to show that there exists an $\epsilon>0$ such that

$$(n+1)(L^n)K_{\mathcal{B}/X\times\mathbb{P}^1}+n(L^{n-1}.K_X)E\geq\epsilon E. \tag{38}$$

Here we mean that each coefficient of E_i is non-negative in the difference of the divisors. As L^n is positive, this is equivalent to showing

$$K_{\mathcal{B}/X\times\mathbb{P}^1}-\frac{n}{n+1}\mu(X,L)E\geq\epsilon E. \tag{39}$$

By the first assumption in 3.3, namely that $\alpha(X,L)>\frac{n}{n+1}\mu(X,L)$, we see that

$$K_{\mathcal{B}/X\times\mathbb{P}^1}-\frac{n}{n+1}\mu(X,L)E>K_{\mathcal{B}/X\times\mathbb{P}^1}-\alpha(X,L)E. \tag{40}$$

But by the upper bound on the alpha invariant, Proposition 3.7, we see that

$$K_{\mathcal{B}/X\times\mathbb{P}^1}-\alpha(X,L)E\geq K_{\mathcal{B}/X\times\mathbb{P}^1}-\min_i\{\frac{a_i-b_i+1}{c_i}\}E. \tag{41}$$

Here we have used notation as in Proposition 3.7. Finally, we see that

$$\begin{aligned} K_{\mathcal{B}/X\times\mathbb{P}^1} - \tfrac{n}{n+1}\mu(X,L)E &> K_{\mathcal{B}/X\times\mathbb{P}^1} - \min_i\{\tfrac{a_i-b_i+1}{c_i}\}\sum c_iE_i \\ &= \sum a_iE_i - \min_i\{\tfrac{a_i-b_i+1}{c_i}\}\sum c_iE_i \\ &= \sum(\tfrac{a_i-b_i+1}{c_i} - \min_i\{\tfrac{a_i-b_i+1}{c_i}\} + \tfrac{b_i-1}{c_i})c_iE_i \\ &\geq 0. \end{aligned}$$

The result follows as $(\mathcal{L}-E)^n.E > 0$, by Lemma 3.9.

□

Remark 3.11. One can marginally strengthen Theorem 3.3 as follows. The positivity of the alpha invariant is used in the proof of Theorem 3.3 as it appears as a coefficient of the exceptional divisor E. In particular, one has a term with positive contribution of the form $(\alpha(X,L) - \frac{n}{n+1}\mu(X,L))E$. By the proof of Lemma 3.9, we have that $(\mathcal{L}-E)^n.(\mathcal{L}+nE) > 0$. Using this, one can use the positivity of the contribution of the term $(\alpha(X,L) - \frac{n}{n+1}\mu(X,L))E$ to slightly weaken the requirement that $-K_X \geq \frac{n}{n+1}\mu(X,L)L$. However, the resulting hypothesis still implies that $-K_X$ is either ample or numerically trivial. We therefore omit the details.

Remark 3.12. For any compact subgroup $G \subset \mathrm{Aut}(X,L)$, Odaka-Sano [22, Section 2.2] have defined a form of stability, which they call *G-equivariant K-stability* and conjecture to be equivalent to K-stability.

Definition 3.13. Let $G \subset \mathrm{Aut}(X,L)$ be compact, and define a G-test configuration $(\mathcal{X},\mathcal{L})$ be a test configuration equipped with an extension of the natural G-action on $(\mathcal{X},\mathcal{L})|_{\pi^{-1}(\mathbb{A}^1-\{0\})}$ to $(\mathcal{X},\mathcal{L})$ which commutes with the $\mathbb{C}^*$-action on $(\mathcal{X},\mathcal{L})$. We say (X,L) is *G-equivariantly K-stable* if the Donaldson-Futaki invariant of all G-test configuration is strictly positive for all non-trivial test configurations.

As G-test configurations give rise to G-invariant flag ideals, Theorem 3.3 can be adapted to G-equivariant K-stability as follows.

Corollary 3.14. *Let (X,L) be a polarised $\mathbb{Q}$-Gorenstein log canonical variety with canonical divisor K_X, and let $G \subset \mathrm{Aut}(X,L)$ be a compact subgroup. Suppose that*

(i) $\alpha_G(X,L) > \frac{n}{n+1}\mu(X,L)$ *and*
(ii) $-K_X \geq \frac{n}{n+1}\mu(X,L)L$.

Then (X,L) is G-equivariantly K-stable.

4. Examples

By showing that the alpha invariant of a line bundle is a continuous function of the line bundle, we first show that the conditions of Theorem 3.3 are *open* when varying the polarisation. To prove this, we need a lemma regarding adding ample divisors and alpha invariants.

Lemma 4.1. *Let (X,L) be a log canonical polarised variety, and let D be an ample $\mathbb{Q}$-divisor on X. Then $\alpha(X,L+D) \leq \alpha(X,L)$.*

Proof. Take any divisor $D' \in |m'L|$. We find a divisor $F \in |p(L+D)|$ such that $\mathrm{lct}(X,\frac{1}{p}F) \leq \mathrm{lct}(X,\frac{1}{m'}D')$. Suppose that mD is a $\mathbb{Z}$-divisor. Let $F = mD' + mm'D \in |mm'(L+D)|$. As $F - mD' = mm'D$ is ample, hence effective, the discrepancies satisfy

$$a(E_i,(X,F)) \leq a(E_i,(X,mD')) \tag{42}$$

for all divisors E_i over X [13, Lemma 2.27], so we have

$$\operatorname{lct}(X, \frac{1}{mm'}F) \leq \operatorname{lct}(X, \frac{1}{m'}D'). \tag{43}$$

Therefore $\alpha(X, L + D) \leq \alpha(X, L)$. □

Using this we can show that the alpha invariant of a polarised variety (X, L) is a continuous function of L.

Proposition 4.2. *Let (X, L) be a polarised klt variety and D be a $\mathbb{Q}$-divisor on X. Then for all $\epsilon > 0$ there exists a $\delta > 0$ depending on D such that $|\alpha(X, L) - \alpha(X, L + \delta D)| < \epsilon$.*

Proof. Firstly, suppose both $\gamma L + D$ and $\gamma L - D$ are ample for some $0 < \gamma < 1$. By the inverse linearity property of the alpha invariant noted in Definition 2.20, we then have

$$\alpha(X, L) = (1 + \gamma)\alpha(X, (1 + \gamma)L). \tag{44}$$

Lemma 4.1 implies that subtracting ample divisors raises the alpha invariant. Applying Lemma 4.1 by subtracting $\gamma L - D$ from $(1 + \gamma)L$, we see that

$$(1 + \gamma)\alpha(X, (1 + \gamma)L) \leq (1 + \gamma)\alpha(X, L + D). \tag{45}$$

This in particular implies

$$\alpha(X, L) - \alpha(X, L + D) \leq \gamma\alpha(X, L + D). \tag{46}$$

On the other hand, since $\gamma L + D$ is ample, applying Lemma 4.1 by adding $\gamma L + D$ to $(1 - \gamma)L$ gives

$$\alpha(X, L) = (1 - \gamma)\alpha(X, (1 - \gamma)L) \tag{47}$$

$$\geq (1 - \gamma)\alpha(X, L + D). \tag{48}$$

Therefore

$$|\alpha(X, L) - \alpha(X, L + D)| \leq \gamma\alpha(X, L + D). \tag{49}$$

Note that equation (48) implies that $\alpha(X, L + D) \leq \frac{1}{1-\gamma}\alpha(X, L)$, so we have

$$|\alpha(X, L) - \alpha(X, L + D)| \leq \frac{\gamma}{1 - \gamma}\alpha(X, L). \tag{50}$$

Since ampleness is an open condition, there exists a $c > 0$ such that both $L + cD$ and $L - cD$ are ample.

We now show continuity of the alpha invariant at L. Given $\epsilon > 0$ let $\delta = \frac{c\epsilon}{2\alpha(X,L)+\epsilon}$. Then both $(\delta c^{-1})L + \delta D$ and $(\delta c^{-1})L - \delta D$ are ample. In our situation $\gamma = \delta c^{-1} = \frac{\epsilon}{2\alpha(X,L)+\epsilon} < 1$, so we can apply equation (50). Noting that $\frac{\delta c^{-1}}{1-\delta c^{-1}} = \frac{\epsilon}{2\alpha(X,L)}$, we therefore have

$$|\alpha(X, L) - \alpha(X, L + \delta D)| \leq \frac{\epsilon}{2} < \epsilon. \tag{51}$$

□

Corollary 4.3. *Suppose (X, L) is a klt $\mathbb{Q}$-Gorenstein polarised variety such that*

(i) $\alpha(X, L) > \frac{n}{n+1}\mu(X, L)$ *and*
(ii) $-K_X > \frac{n}{n+1}\mu(X, L)L$.

Note that both inequalities are strict. Then for all $\mathbb{Q}$-divisors D, there exists an $\epsilon > 0$ such that $L + \epsilon D$ is K-stable.

Proof. This follows by Proposition 4.2 and continuity of intersections of divisors. □

We now apply Theorem 3.3 to a general degree one del Pezzo surface X. Here the genericity condition means that $|-K_X|$ contains no cuspidal curves, so $\alpha(X, -K_X) = 1$ ([3], Theorem 1.7). Note that X is isomorphic to the blow-up of $\mathbb{P}^2$ at a configuration of 8 points in general position. Denote by H the hyperplane divisor pulled back from $\mathbb{P}^2$, let E_i be the 8 exceptional divisors arising from an isomorphism $X \cong Bl_{p_1,\dots,p_8}\mathbb{P}^2$ and let $L_\lambda = 3H - \sum_{i=1}^{7} E_i - \lambda E_8$.

Theorem 4.4. *(X, L_λ) is K-stable for*

$$\frac{19}{25} \approx \frac{1}{9}(10-\sqrt{10}) < \lambda < \sqrt{10} - 2 \approx \frac{29}{25}. \tag{52}$$

To prove this result, we first obtain a lower bound for $\alpha(X, L_\lambda)$ using the following two lemmas.

Lemma 4.5. [33, Corollary 6] *Let S be a smooth variety, let $p \in S$, and let D, B be effective $\mathbb{Q}$-divisors on S with $p \in S, p \in B$ such that (S, D) is not log canonical at p but (S, B) is log canonical at p. Then, for all $c \in [0, 1) \cap \mathbb{Q}$,*

$$(S, \frac{1}{1-c}(D - cB)) \tag{53}$$

is not log canonical at p.

Lemma 4.6. [18, Lemma 2.4 (i)] *Let (S, D) be pair consitisting of a smooth surface S and an effective $\mathbb{Q}$-divisor D such that (S, D) is not log canonical at p. Then* $\operatorname{mult}_p D > 1$.

Proposition 4.7. *For X be a general del Pezzo surface of degree one as above, and $L_\lambda = 3H - \sum_{i=1}^{7} E_i - \lambda E_8$ with $\lambda \geq 0$, we have*

$$\alpha(X, L_\lambda) \geq \min\left\{\frac{1}{2-\lambda}, 1\right\}. \tag{54}$$

Proof. Suppose for contradiction $\omega < \min\{\frac{1}{2-\lambda}, 1\}$, and there exists an effective $\mathbb{Q}$-divisor D with $mD \in |mL_\lambda|$ for some m such that $(S, \omega D)$ is not log canonical at some point $p \in X$. Write $D = aC + \Omega$, where $C \in |-K_X|$ is a $\mathbb{Z}$-divisor with $p \in C$, and $C \not\subseteq \operatorname{Supp}(\Omega)$. Note that since X is a *general* degree one del Pezzo surface, we have that ωC is log canonical by ([25], Proposition 3.2). Since $\Omega = D - aC$, we see that

$$\begin{aligned} \Omega.H &= (D - aC).H && (55)\\ &= (1-a)(-K_X).H + (1-\lambda)E_8.H && (56)\\ &= 3(1-a). && (57) \end{aligned}$$

But since H is ample and Ω is effective, $\Omega.H \geq 0$. Thus $a \leq 1$, and in particular, $\omega a < 1$.

By Lemma 4.5, we see that $(S, \frac{1}{1-\omega a}(\omega D - \omega a C))$ is not log canonical at p. Note that $\omega D - \omega a C = \omega\Omega$. Therefore $\operatorname{mult}_p(\frac{\omega}{1-\omega a}\Omega) > 1$ by Lemma 4.6. But since $C \not\subseteq \operatorname{Supp}(\Omega)$, we have that

$$\omega C.\Omega \geq \omega \operatorname{mult}_p \Omega > 1 - \omega a. \tag{58}$$

Thus

$$\begin{aligned}\omega(2-\lambda) &= \omega D.C && (59)\\ &= \omega(aC.C+\Omega.C) && (60)\\ &> \omega a + 1 - \omega a && (61)\\ &= 1. && (62)\end{aligned}$$

But this implies $\omega > \frac{1}{2-\lambda}$, a contradiction.

□

Using this lower bound we can prove Theorem 4.4.

Proof. (of Theorem 4.4) For Theorem 3.3 to apply, the two equations that must be satisfied are

(i) $\alpha(X, L_\lambda) > \frac{2}{3}\mu(X, L_\lambda)$ and
(ii) $-K_X - \frac{2}{3}\mu(X, L_\lambda)L_\lambda$ is nef.

In our case $\mu(X, L_\lambda) = \frac{2-\lambda}{2-\lambda^2}$. By Proposition 4.7, for $\lambda \leq 1$, we have $\alpha(X, L_\lambda) \geq \frac{1}{2-\lambda}$ and the first condition is always satisfied. When $\lambda \geq 1$, we have $\alpha(X, L_\lambda) \geq 1$ and the first condition requires $2 - 3\lambda^2 + 2\lambda > 0$, which is true for $\lambda < \frac{1}{3}(1+\sqrt{7})$.

For the second condition to apply, we require

$$3H - \sum_{i=1}^{7} E_i - \frac{6-4\lambda-\lambda^2}{2+2\lambda-3\lambda^2}E_8 \tag{63}$$

to be nef. Note for $\lambda = 1$ this holds.

By the cone theorem [13, Theorem 3.7] applied to a del Pezzo surface, to check when a line bundle on a del Pezzo surface is nef, it suffices to check it has non-negative intersection with all curves of negative self-intersection. However, by the adjunction formula, all curves C on a del Pezzo surface of negative self-intersection are exceptional, that is, $C.C = -1$. Therefore, to check when a line bundle is nef on a del Pezzo surface, it suffices to check it has non-negative intersection with all exceptional curves. For the blow-up of $\mathbb{P}^2$ at 8 points, from [17, Theorem 26.2] we know that the exceptional curves are the proper transforms of:

- points which are blown up, with class E_i,
- lines through pairs of points, with class $H - E_i - E_j$,
- conics through 5 points, with class $2H - \sum_5 E_i$,
- cubics through 7 points, vanishing doubly at E_j for some j, with class $3H - E_j - \sum_7 E_i$,
- quartics through 8 points, vanishing doubly at E_j, E_k, E_l, with class $4H - E_i - E_j - E_k - \sum_8 E_l$,
- quintics through 8 points, vanishing doubly at 6 points, with class $5H - E_j - E_k - 2\sum_6 E_i$,
- sextics through 8 points, vanishing doubly at 7 points and triply at another, with class $6H - 3E_j - 2\sum_7 E_i$.

For a line bundle of the form $W = 3H - \sum_{i=1}^{7} E_i - \delta E_8$, the first condition requires $\delta \geq 0$, the second and third conditions require $\delta \leq 2$, the fourth, fifth and sixth require $\delta \leq \frac{3}{2}$, while the seventh condition requires $\delta \leq \frac{4}{3}$. In particular, $\delta = \frac{4}{3}$ is the maximal value of δ with W nef.

The equation that therefore must be satisfied for $3H - \sum_{i=1}^{7} E_i - \frac{6-4\lambda-\lambda^2}{2+2\lambda-3\lambda^2} E_8$ to be nef is

$$0 \leq \frac{6-4\lambda-\lambda^2}{2+2\lambda-3\lambda^2} \leq \frac{4}{3}. \tag{64}$$

As $\lambda > 0$, the condition that $6-4\lambda-\lambda^2 \geq 0$ requires $\lambda < \sqrt{10}-2 \approx \frac{29}{25}$. The upper bound is equivalent to

$$9\lambda^2 - 20\lambda + 10 \leq 0, \tag{65}$$

which is true for $\frac{1}{9}(10-\sqrt{10}) \leq \lambda \leq \frac{1}{9}(10+\sqrt{10}) \approx \frac{29}{20}$. Therefore, the range for which both conditions required to apply Theorem 3.3 are satisfied is $\frac{1}{9}(10-\sqrt{10}) < \lambda < \sqrt{10}-2$.

□

Remark 4.8. Note that the lower bound for $\alpha(X, L_\lambda)$ may not be sharp. A more delicate analysis of the $\mathbb{Q}$-divisors linearly equivalent to L_λ may provide a sharper lower bound. However, both the upper and lower bounds obtained in Theorem 4.4 were given by the requirement that $-K_X \geq \frac{2}{3}\mu(X, L_\lambda)L_\lambda$. Since we calculated exactly for which λ that this condition holds, we have calculated precisely the range of λ for which Theorem 3.3 applies.

Remark 4.9. Analytically, Arezzo-Pacard [1, Theorem 1.1] have shown that if a general (X, L) admits a constant scalar curvature Kähler metric in $c_1(L)$, and $\pi : Y \to X$ is the blow-up of X at a point p, then $(Y, \pi^*L - \epsilon E)$ admits a constant scalar curvature Kähler metric in $c_1(\pi^*L - \epsilon E)$ for sufficiently small ϵ, provided $\mathrm{Aut}(X, L)$ is discrete. As the existence of a cscK metric in $c_1(L)$ implies K-stability by work of Stoppa [28, Theorem 1.2] and Donaldson [9], this in particular implies that (X, L_λ) as in Theorem 4.4 is K-stable for λ sufficiently small. Theorem 4.4 shows that (X, L_λ) is K-stable for $\frac{1}{9}(10-\sqrt{10}) < \lambda < \sqrt{10}-2$. On the other hand, using the techniques of slope stability, Ross-Thomas [26, Example 5.30] have shown that there are polarisations of a general degree one del Pezzo surface X which are K-*un*stable. It would be interesting to know exactly which polarisations of a general degree one del Pezzo surface are K-stable.

5. Log Alpha Invariants and Log K-stability

In this section we extend Theorem 3.3 to K-stability with cone singularities along an anti-canonical divisor, which conjecturally corresponds to the existence of cscK metrics with cone singularities along a divisor. For a general introduction to log K-stability, see [23].

Definition 5.1. Let (X, L^r) be a normal polarised variety, and let D be an effective integral reduced divisor on X. We define a *log test configuration* for $((X, D); L^r)$ to be a pair of test configurations $(\mathcal{X}, \mathcal{L})$ for (X, L^r) and $(\mathcal{Y}, \mathcal{L}|_{\mathcal{Y}})$ for $(D, L^r|_D)$ with a compatible $\mathbb{C}^*$ action. We denote by $((\mathcal{X}, \mathcal{Y}); \mathcal{L})$ the data of a log test configuration. Denote the Hilbert polynomials of (X, L) and $(D, L|_D)$ respectively as

$$\mathcal{P}(k) = \chi(X, L^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}), \tag{66}$$

$$\tilde{\mathcal{P}}(k) = \chi(D, L|_D^k) = \tilde{a}_0 k^{n-1} + \tilde{a}_1 k^{n-2} + O(k^{n-3}). \tag{67}$$

Denote also the total weights of the $\mathbb{C}^*$-actions on $H^0(\mathcal{X}_0, \mathcal{L}_0^k)$ and $H^0(\mathcal{Y}_0, \mathcal{L}^{k_0}|_{\mathcal{Y}_0})$ respectively as

$$w(k) = b_0 k^n + b_1 k^{n-1} + O(k^{n-2}), \tag{68}$$

$$\tilde{w}(k) = \tilde{b}_0 k^{n+1} + \tilde{b}_1 k^n + O(k^{n-1}). \tag{69}$$

We define the *log Donaldson-Futaki invariant* of $((\mathcal{X}, \mathcal{Y}); \mathcal{L})$ with cone angle $2\pi\beta$ for $0 \leq \beta \leq 1$ to be

$$\mathrm{DF}_\beta((\mathcal{X}, \mathcal{Y}); \mathcal{L}) = 2(b_0 a_1 - b_1 a_0) + (1-\beta)(a_0 \tilde{b}_0 - b_0 \tilde{a}_0). \tag{70}$$

We say that $((X, D); L)$ is *log K-stable with cone angle* $2\pi\beta$ if $\mathrm{DF}_\beta((\mathcal{X}, \mathcal{Y}); \mathcal{L}) > 0$ for all log test configurations $((\mathcal{X}, \mathcal{Y}); \mathcal{L})$ with $\mathcal{X}, \mathcal{Y}$ normal, Gorenstein in codimension one and such that $((\mathcal{X}, \mathcal{Y}); \mathcal{L})$ is not almost trivial.

Note that the usual Donaldson-Futaki invariant for the test configuration $(\mathcal{X}, \mathcal{L})$ is $\frac{b_0 a_1 - b_1 a_0}{a_0^2}$, we have multiplied by $2a_0^2$ for ease of notation. Since K-stability is independent of positively scaling L, this makes no difference to the definition of K-stability.

Odaka-Sun [23] have extended the blowing-up formalism of Odaka to the log case. Recall that to certain flag ideals $\mathcal{I}$ on $X \times \mathbb{A}^1$ one can associate a semi-test configuration by the following method. Blowing-up $\mathcal{I}$ on $X \times \mathbb{P}^1$, we get a map

$$\pi : \mathcal{B} = Bl_{\mathcal{I}}(X \times \mathbb{P}^1) \to X \times \mathbb{P}^1. \tag{71}$$

Denote $\mathcal{B}_{(D\times\mathbb{P}^1)} = Bl_{\mathcal{I}|_{(D\times\mathbb{P}^1)}}(D \times \mathbb{P}^1)$ and let E be the exceptional divisor of the blow-up $\pi : \mathcal{B} \to X \times \mathbb{P}^1$, that is, $\mathcal{O}(-E) = \pi^{-1}\mathcal{I}$. Abusing notation, write $\mathcal{L} - E$ to denote $(p_1 \circ \pi)^* L \otimes \mathcal{O}(-E)$, where $p_1 : X \times \mathbb{P}^1 \to X$ is the natural projection.

Theorem 5.2. [23, Corollary 3.6] *A normal polarised variety (X, L) is log K-stable with cone angle $2\pi\beta$ if and only if $\mathrm{DF}_\beta((\mathcal{B}, \mathcal{B}_{(D\times\mathbb{P}^1)}); \mathcal{L}^r - E) > 0$ for all $r > 0$ and for all flag ideals $\mathcal{I}$ such that $\mathcal{B}, \mathcal{B}_{(D\times\mathbb{P}^1)}$ are normal and Gorenstein in codimension one, $\mathcal{L}^r - E$ is relatively semi-ample over $\mathbb{P}^1$ and $\mathcal{I} \neq (t^N)$.*

Moreover, there is an explicit formula for the log Donaldson-Futaki invariant for log test configurations arising from flag ideals.

Theorem 5.3. [23, Theorem 3.7] *With all notation as above, we have*

$$\mathrm{DF}_\beta((\mathcal{B}, \mathcal{B}_{(D\times\mathbb{P}^1)}); \mathcal{L} - E) = -n(L^{n-1}.(K_X + (1-\beta)D))(\mathcal{L} - E)^{n+1} + \tag{72}$$

$$+ (n+1)(L^n)(\mathcal{L} - E)^n.(\mathcal{K}_X + (1-\beta)\mathcal{D} + (K_{\mathcal{B}/((X,(1-\beta)D)\times\mathbb{P}^1)})_{exc}). \tag{73}$$

Here we have denoted by $\mathcal{K}_{\mathcal{B}/((X,(1-\beta)D)\times\mathbb{P}^1)})_{exc}$ the exceptional terms of $K_{\mathcal{B}} - \pi^*(K_{X\times\mathbb{P}^1} + (1-\beta)D)$, and $\mathcal{K}_X$ the pull back of K_X to $\mathcal{B}$. The intersection numbers $L^{n-1}.K_X$ and L^n are computed on X, while the remaining intersection numbers are computed on $\mathcal{B}$. Replacing L and $\mathcal{L}$ by L^r and $\mathcal{L}^r$ respectively in formula 72 gives the formula for the Donaldson-Futaki invariant of a test configuration of the form $(\mathcal{B}, \mathcal{L}^r - E)$.

We can extend the definition of the alpha invariant to the log setting as follows.

Definition 5.4. Let $((X, D); L)$ consist of a log canonical pair (X, D) with L an ample $\mathbb{Q}$-line bundle. We define the log alpha invariant of $((X, D); L)$ to be

$$\alpha((X, D); L) = \inf_{m\in\mathbb{Z}_{>0}} \inf_{F\in|mL|} \mathrm{lct}((X, D); \frac{1}{m}F). \tag{74}$$

Berman [2, Section 6] has provided an analytic counterpart to the log alpha invariant as follows.

Definition 5.5. Let $((X,D);L)$ consist of a Kawamata log terminal pair (X,D) with X smooth, L an ample $\mathbb{Q}$-line bundle and $D=\sum d_i D_i$ a simple normal crossing divisor with $D_i=\{f_i=0\}$. Let h be a singular hermitian metric on L, written locally as $h=e^{-2\phi}$. We define the *complex singularity exponent* $c_D(h)$ to be

$$c(h)=\sup\left\{\lambda\in\mathbb{R}_{>0}|\text{ for all } z\in X, h^{\lambda}=e^{-2\lambda\phi}\prod|f_i|^{-\lambda d_i}\text{ is } L^1\text{ near } z\right\}. \quad (75)$$

We then define Tian's *log alpha invariant* $\alpha^{an}((X,D);L)$ of $((X,D);L)$ to be

$$\alpha^{an}((X,D);L)=\inf_{h\text{ with }\Theta_{L,h}\geq 0}c(h) \quad (76)$$

where the infimum is over all singular hermitian metrics h with curvature $\Theta_{L,h}\geq 0$. For a compact subgroup G of $\mathrm{Aut}(X,L)$, one defines α^{an} similarly, however considering only G-invariant metrics.

Theorem 5.6. [2, Section 6] *The log alpha invariant $\alpha((X,D);L)$ defined algebraically equals Tian's log alpha invariant $\alpha^{an}((X,D);L)$. That is,*

$$\alpha((X,D);L)=\alpha^{an}((X,D);L) \quad (77)$$

We can now extend Theorem 3.3 to the log setting.

Theorem 5.7. *Let $((X,D);L)$ consist of a $\mathbb{Q}$-Gorenstein log canonical pair (X,D) with canonical divisor K_X, such that D is an effective integral reduced Cartier divisor on a polarised variety (X,L). Denote $\mu_\beta((X,D);L)=\frac{-(K_X+(1-\beta)D).L^{n-1}}{L^n}$ Suppose that*

(i) $\alpha((X,D);L)>\frac{n}{n+1}\mu_\beta((X,D);L)$ *and*
(ii) $-(K_X+(1-\beta)D)\geq\frac{n}{n+1}\mu_\beta((X,D);L)L.$

Then $((X,D);L)$ is log K-stable with cone angle β along D.

Remark 5.8. In the case $L=-K_X$, and $D\in|-K_X|$, this result is due to Odaka-Sun [23, Theorem 5.6]. Again in the case $L=-K_X$, this is the analytic counterpart of a result of Berman ([2], Theorem 3.11) and Jeffres-Mazzeo-Rubinstein ([10], Lemma 6.9). Explicit examples are given in [5].

To prove this theorem, we extend Proposition 3.7 to the log setting.

Proposition 5.9. *Let $\mathcal{B}$ be the blow-up of $X\times\mathbb{P}^1$ along a flag ideal, with $\mathcal{B}$ normal and Gorenstein in codimension one, $\mathcal{L}-E$ relatively semi-ample over $\mathbb{P}^1$ and notation as in Remark 2.8. Denote the natural map arising from the composition of the blow-up map and the projection map by $\Pi:\mathcal{B}\to\mathbb{P}^1$. Denote also*

- *the discrepancies as:* $K_{\mathcal{B}/X\times\mathbb{P}^1}=\sum a_iE_i$,
- *the multiplicities of $X\times\{0\}$ as:* $\Pi^*(X\times\{0\})=\Pi_*^{-1}(X\times\{0\})+\sum b_iE_i$,
- *the multiplicities of D as:* $\Pi^*D=\Pi_*^{-1}D+\sum d_iE_i$,
- *the exceptional divisor as:* $\Pi^{-1}\mathcal{I}=\mathcal{O}_{\mathcal{B}}(-\sum c_iE_i)=\mathcal{O}_{\mathcal{B}}(-E)$.

Then

$$\alpha((X,(1-\beta)D);L)\leq\min_i\left\{\frac{a_i-b_i+1-(1-\beta)d_i}{c_i}\right\}. \quad (78)$$

Proof. Let $\pi_0 : Bl_{I_0}X \to X$ be the blow-up of I_0 with exceptional divisor E_0. As in Proposition 3.7, we have that $\pi_0^*L - E_0$ is semi-ample. Choose m sufficiently large and divisible such that $H^0(Bl_{I_0}X, m(\pi_0^*L - E_0)) = H^0(X, I_0^m(mL))$ has a section, and let F be such a section. We show that

$$\alpha((X,(1-\beta)D);L) \leq m\,\mathrm{lct}((X,D);F) \leq \min_i \left\{ \frac{a_i - b_i + 1 - (1-\beta)d_i}{c_i} \right\}. \quad (79)$$

Since for general ideal sheaves I, J, we have $I \subset J$ implies $\mathrm{lct}(X,I) \leq \mathrm{lct}(X,J)$, we see that

$$\mathrm{lct}((X,D);F) = \mathrm{lct}((X,D);I_F) \leq \frac{1}{m}\,\mathrm{lct}((X,D);I_0). \quad (80)$$

Since (X,D) is log canonical, using inversion of adjunction of log canonicity (Theorem 2.15), we have

$$\begin{aligned} \mathrm{lct}((X,(1-\beta)D);I_0) &= \mathrm{lct}((X\times\mathbb{P}^1, X\times\{0\} + (1-\beta)D\times\mathbb{P}^1);I_0) && (81)\\ &\leq \mathrm{lct}((X\times\mathbb{P}^1, X\times\{0\} + (1-\beta)D\times\mathbb{P}^1);\mathcal{I}) && (82)\\ &\leq \min_i \left\{ \frac{a_i - b_i + 1 - (1-\beta)d_i}{c_i} \right\}. && (83) \end{aligned}$$

The last inequality is by Remark 2.19.

□

Using this we can prove Theorem 5.7.

Proof. (of Theorem 5.7) We treat the case $r = 1$ for notational simplicity, the general case is similar. The log Donaldson-Futaki invariant is given by

$$\begin{aligned} \mathrm{DF}_\beta((\mathcal{B},\mathcal{B}_{(D\times\mathbb{P}^1)});\mathcal{L}-E) &= -n(L^{n-1}.(K_X + (1-\beta)D))(\mathcal{L}-E)^{n+1} + && (84)\\ &+ (n+1)(L^n)(\mathcal{L}-E)^n.(\mathcal{K}_X + (1-\beta)\mathcal{D} + (K_{\mathcal{B}/((X,(1-\beta)D)\times\mathbb{P}^1)})_{exc}). && (85) \end{aligned}$$

For ease of notation, we let $K'_X = K_X + (1-\beta)D$ and $\mathcal{K}' = \mathcal{K}_X + (1-\beta)\mathcal{D}$. We split the log Donaldson-Futaki invariant into two terms as

$$\begin{aligned} &\mathrm{DF}_\beta((\mathcal{B},\mathcal{B}_{(D\times\mathbb{P}^1)});\mathcal{L}-E) = \mathrm{DF}_{\beta,num} + \mathrm{DF}_{\beta,disc}, && (86)\\ &\mathrm{DF}_{\beta,num} = (\mathcal{L}-E)^n.(-n(L^{n-1}.K'_X)\mathcal{L} + (n+1)(L^n)\mathcal{K}'_X), && (87)\\ &\mathrm{DF}_{\beta,disc} = (\mathcal{L}-E)^n.((n+1)(L^n)(K_{\mathcal{B}/((X,(1-\beta)D)\times\mathbb{P}^1)})_{exc}) + n(L^{n-1}.K'_X)E). && (88) \end{aligned}$$

Since $-(K_X + (1-\beta)D) \geq \frac{n}{n+1}\mu_\beta((X,D);L)L$, Lemma 3.8 implies that $\mathrm{DF}_{\beta,num} \geq 0$.

To prove $\mathrm{DF}_{\beta,disc} > 0$, since $(\mathcal{L}-E)^n.E > 0$ by Lemma 3.9 it suffices to prove the existence of an $\epsilon > 0$ such that

$$\mathcal{K}_{\mathcal{B}/((X,(1-\beta)D)\times\mathbb{P}^1)})_{exc} - \frac{n}{n+1}\mu_\beta((X,D);L)E \geq \epsilon E. \quad (89)$$

By Proposition 5.9 and the first hypothesis of the theorem, we have that

$$(K_{\mathcal{B}/((X,(1-\beta)D)\times\mathbb{P}^1})_{exc} - \frac{n}{n+1}\mu_\beta((X,D);L)E > \tag{90}$$

$$(K_{\mathcal{B}/((X,(1-\beta)D)\times\mathbb{P}^1})_{exc} - \alpha((X,D);L)E \tag{91}$$

$$\geq \sum(a_i - (1-\beta)d_i)E_i - \min_i\left\{\frac{a_i - b_i + 1 - (1-\beta)d_i}{c_i}\right\}\sum c_i E_i \tag{92}$$

$$= \sum\left(\frac{a_i - b_i - (1-\beta)d_i + 1}{c_i} - \min_i\left\{\frac{a_i - b_i - (1-\beta)d_i}{c_i} + 1\right\} + \frac{b_i - 1}{c_i}\right)c_i E_i \tag{93}$$

$$\geq 0. \tag{94}$$

The result follows. □

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