The augmented base locus of real divisors over arbitrary fields

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Abstract. We show that the augmented base locus coincides with the exceptional locus (i.e. null locus) for any nef $\mathbb{R}$-Cartier divisor on any scheme projective over a field (of any characteristic). Next we prove a semi-ampleness criterion in terms of the exceptional locus generalizing a result of Keel. We also discuss some problems related to augmented base loci of log divisors.

1. Introduction

The base locus of a linear system is a fundamental notion in algebraic and especially birational geometry. The restricted base locus (also called the non-nef locus) and the augmented base locus (also called the non-ample locus) are refinements of the base locus which capture more essential properties of divisors and linear systems. These are closely related to important concepts and problems in birational geometry, eg see [4],[16],[15],[9],[8],[6],[2].

We start with some definitions.

The augmented base locus. Let X be a scheme. An $\mathbb{R}$-*Cartier divisor* is an element of $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ where $\mathrm{Div}(X)$ is the group of Cartier divisors. A $\mathbb{Q}$-*Cartier divisor* is defined similarly by tensoring with $\mathbb{Q}$.

Definition 1.1 Let X be a projective scheme over a field k. The *stable base locus* of a $\mathbb{Q}$-Cartier divisor L is defined as

$$\mathbf{B}(L) = \bigcap_{m\in\mathbb{N},\ mL \text{ Cartier}} \mathrm{Bs}\,|mL|$$

that is, it is the set of those points $x \in X$ such that every section of every mL vanishes where m is a positive integer and mL is Cartier. The base locus, stable base locus, and all the other base loci defined below are considered with the reduced induced structure. The *augmented base locus* of L is defined as

$$\mathbf{B}_+(L) = \bigcap_{m\in\mathbb{N}} \mathbf{B}(mL - A)$$

where A is any ample Cartier divisor.

The augmented base locus of $\mathbb{R}$-Cartier divisors on smooth projective varieties was defined in [8]. For basic properties of the augmented base locus in this

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context see [9][8]. We give a different definition which is more convenient for our purposes (the two definitions agree, by Lemma 3.1 (3) below).

Definition 1.2 Let X be a projective scheme over a field k. Let L be an $\mathbb{R}$-Cartier divisor on X. We can write $L \sim_{\mathbb{R}} \sum t_i A_i$ where $A_1, \dots, A_r$ are very ample Cartier divisors and $t_i \in \mathbb{R}$. The A_i are not necessarily distinct and the expression is obviously not unique. Define $\langle mL \rangle = \sum \lfloor mt_i \rfloor A_i$ which depends on the above expression. Next define the augmented base locus of L as

$$\mathbf{B}_+(L) = \bigcap_{m \in \mathbb{N}} \mathbf{B}(\langle mL \rangle - A)$$

where A is any ample Cartier divisor.

It turns out that $\mathbf{B}_+(L)$ does not depend on the choice of A nor the A_i nor the expression $L \sim_{\mathbb{R}} \sum t_i A_i$ (see Lemma 3.1). In particular, $\mathbf{B}_+(L)$ depends only on the $\mathbb{R}$-linear equivalence class of L.

Relation with the exceptional locus. Before stating our first result we recall the definition of exceptional locus.

Definition 1.3 Let X be a projective scheme over a field k and L an $\mathbb{R}$-Cartier divisor on X. The *exceptional locus* of L (also called the null locus when L is nef) is defined as

$$\mathbb{E}(L) := \bigcup_{L|_V \text{ not big}} V$$

where the union runs over the integral subschemes $V \subseteq X$ with positive dimension.

Theorem 1.4. *Let X be a projective scheme over a field k. Assume that L is a nef $\mathbb{R}$-Cartier divisor with a given expression $L \sim_{\mathbb{R}} \sum t_i A_i$ as in 1.2, and that A is a very ample Cartier divisor on X. Then*

$$\mathbf{B}_+(L) = \mathbf{B}(\langle nL \rangle - A) = \mathrm{Bs}|\langle nL \rangle - A| = \mathbb{E}(L)$$

for any sufficiently divisible $n \in \mathbb{N}$.

The theorem was first proved for X smooth, $\mathbb{Q}$-Cartier L, and k algebraically closed of characteristic zero by Nakamaye [16] using Kodaira type vanishing theorems, and this was generalized to $\mathbb{R}$-Cartier divisors by Ein-Lazarsfeld-Mustaţă-Nakamaye-Popa [8]. Nakamaye's result was extended to log canonical varieties by Cacciola-Lopez [5] again by using Kodaira type vanishing theorems. They also give some applications to the moduli spaces of curves. Related results concerning the restricted volume are proved on normal varieties by Boucksom-Cacciola-Lopez [3].

The theorem was proved by Cascini-M^{c}Kernan-Mustaţă [6] when k is algebraically closed of positive characteristic using techniques related to Keel [12]: the main ingredients are Serre vanishing and the Frobenius. Fujino-Tanaka [10] employ similar arguments on surfaces using Fujita vanishing and the Frobenius.

We will also use Fujita vanishing but not the Frobenius.

A semi-ampleness criterion. The following semi-ampleness result was first proved by Keel [12] when k has positive characteristic. A simplified proof of Keel's result was given by Cascini-McKernan-Mustaţă [6].

Theorem 1.5. *Let X be a projective scheme over a field k. Assume that L is a nef $\mathbb{Q}$-Cartier divisor on X. Then there is a closed subscheme $Z \subseteq X$ such that*

- *the reduced induced scheme associated to Z is equal to $\mathbb{E}(L)$, and*
- *L is semi-ample if and only if $L|_Z$ is semi-ample.*

When k has positive characteristic we can use the Frobenius to show that in fact we can take $Z = \mathbb{E}(L)$. However, when k has characteristic zero in general we cannot take $Z = \mathbb{E}(L)$, by Keel [12, §3]. Although Z is not unique some choice can be calculated for any given X, L. It is interesting to see whether the theorem holds if L is only $\mathbb{R}$-Cartier.

The augmented base locus of log divisors. Let (X, B) be a projective pair over an algebraically closed field k and A a nef and big $\mathbb{R}$-divisor such that $L = K_X + B + A$ is nef. The locus $\mathbf{B}_+(L)$ is closely related to the geometry of X. In Section 6 we recall some results and pose some questions concerning such loci.

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2. Preliminaries

2.1. **Growth of functions.** Let $h\colon \mathbb{Z} \to \mathbb{Z}$ be a function. We say that the *upper growth* of h is like m^d (resp. at most like m^d) if

$$0 < \limsup_{m\to+\infty} \frac{h(m)}{m^d} < +\infty$$

(resp. $\limsup_{m\to+\infty} \frac{h(m)}{m^d} < +\infty$).

2.2. **Divisors.** Let X be a scheme. The group of Cartier divisors on X is denoted by $\mathrm{Div}(X)$. Recall that an $\mathbb{R}$-Cartier divisor (resp. $\mathbb{Q}$-Cartier divisor) is an element of $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$). Such a divisor can be represented as $L = \sum l_i L_i$ where $l_i \in \mathbb{R}$ (resp. $l_i \in \mathbb{Q}$) and L_i are Cartier divisors but this representation is not unique. Two $\mathbb{R}$-Cartier divisors L, L' are $\mathbb{R}$-*linearly equivalent* (resp. $\mathbb{Q}$-*linearly equivalent*) if $L - L' = \sum a_i N_i$ where $a_i \in \mathbb{R}$ (resp. $a_i \in \mathbb{Q}$) and N_i are Cartier divisors linearly equivalent to zero. We denote the equivalence by $L \sim_{\mathbb{R}} L'$ (resp. $L \sim_{\mathbb{Q}} L'$). Note that each $\mathbb{R}$-Cartier divisor $L = \sum_1^r l_i L_i$, determines an $\mathbb{R}$-line bundle $\mathcal{L} := \mathcal{O}_X(L_1)^{l_1} \otimes \cdots \otimes \mathcal{O}_X(L_r)^{l_r}$ in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $f\colon X'\to X$ be a morphism from another scheme X' and L an $\mathbb{R}$-Cartier divisor on X. If f is flat, then one can define the pullback f^*L by taking the pullback of the local defining equations of L (eg, when X is over a field k and f is induced by base change of k to a field k'). However, we cannot define the pullback f^*L in general if f is not flat, although we can define the pullback of the associated $\mathbb{R}$-line bundle $\mathcal{L}$ (i.e. the element in $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R}$ corresponding to L), say $\mathcal{L}' := f^*\mathcal{L}$. If $\mathcal{L}'$ is the $\mathbb{R}$-line bundle associated to some $\mathbb{R}$-Cartier divisor L' (eg, this is the case if X' is an integral scheme, or if X' is projective over a field [14]), then we can define $L' = f^*L$ but this determines L' only up to $\mathbb{R}$-linear equivalence. In this paper, when we talk about pullback of divisors, either the morphism is flat or that pullback is defined up to $\mathbb{R}$-linear equivalence (eg, the restriction $L|_V$ in Definition 1.3).

Now assume X is a projective scheme over a field k. An $\mathbb{R}$-Cartier divisor L on X is:

- *nef* if $L\cdot C\geq 0$ for every curve $C\subseteq X$ (a curve is an integral closed subscheme of dimension one);
- *ample* if $L\sim_{\mathbb{R}}\sum l_iL_i$ with $l_i>0$ and L_i ample Cartier divisors;
- *effective* if $L=\sum l_iL_i$ with $l_i\geq 0$ and L_i effective Cartier divisors;
- *big* if $L\sim_{\mathbb{R}} A+D$ where A is an ample $\mathbb{R}$-Cartier divisor and D is an effective $\mathbb{R}$-Cartier divisor;
- *semi-ample* if $L=\sum l_iL_i$ where $0\leq l_i\in\mathbb{R}$ and L_i are base point free Cartier divisors.

2.3. **The operator $\langle - \rangle$.** Let X be a projective scheme over a field k and L be an $\mathbb{R}$-Cartier divisor with an expression $L\sim_{\mathbb{R}}\sum t_iA_i$ as in 1.2. Let $\pi\colon X'\to X$ be a morphism such that the pullbacks $A_i' := \pi^*A_i$ are defined up to linear equivalence and assume A_i are very ample, eg π is obtained by base change as in 2.7 below or π is a closed embedding. Then we get the expression $L' := \pi^*L\sim_{\mathbb{R}}\sum t_iA_i'$ which we can use to define $\langle mL'\rangle$. Here L' is defined up to $\mathbb{R}$-linear equivalence. It is clear that $\langle mL'\rangle\sim\pi^*\langle mL\rangle$.

For a coherent sheaf $\mathcal{F}$ on X, we often use the notation $\mathcal{F}\langle mL\rangle$ instead of $\mathcal{F}(\langle mL\rangle)$.

2.4. **Pairs.** A pair (X,B) over a field k consists of a normal quasi-projective variety over k and a Weil $\mathbb{R}$-divisor B with coefficients in $[0,1]$ such that K_X+B is $\mathbb{R}$-Cartier where K_X is the canonical divisor. The pair is *klt* if for every projective birational morphism $f\colon Y\to X$ from a normal variety the coefficients of B_Y are all <1 where $K_Y+B_Y=f^*(K_X+B)$.

2.5. **Fujita vanishing theorem.** This is a generalization of Serre vanishing theorem. Let X be a projective scheme over a field k, A an ample Cartier divisor, and $\mathcal{F}$ a coherent sheaf on X. Then there is a number m_0 such that $h^i(\mathcal{F}(mA+L))=0$ for any $i>0$, $m>m_0$, and nef Cartier divisor L [11],[13, Theorem 1.4.35].

2.6. **Restriction to a hyperplane section.** Let X be a projective scheme over a field k, A an effective Cartier divisor, and $\mathcal{F}$ a coherent sheaf on X. Tensoring $\mathcal{F}$ with the exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0$$

gives a sequence

$$0 \to \mathcal{F}(-A) \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_A \to 0$$

which is often not exact on the left. However, if A (considered as a closed subscheme) does not contain any of the finitely many associated points of $\mathcal{F}$, then the latter sequence is also exact on the left. If A is very ample and if k is infinite, then it is a well-known fact that after changing A up to linear equivalence we can make sure that A does not contain any associated point of $\mathcal{F}$.

2.7. **Base loci and base change.** Let X be a projective scheme over a field k and let L be a Cartier divisor on X. Recall that the *base locus* of L is defined as

$$\operatorname{Bs}|L| = \{x \in X \mid \alpha \text{ vanishes at } x \text{ for every } \alpha \in H^0(\mathcal{O}_X(L))\}.$$

As pointed out earlier we consider $\operatorname{Bs}|L|$ (and other loci) with the reduced structure. Recall that $\mathbf{B}(L) = \bigcap_{m\in\mathbb{N}} \operatorname{Bs}|mL|$. If $n, n' \in \mathbb{N}$, then each section $\alpha \in H^0(\mathcal{O}_X(nL))$ gives a section $\alpha^{\otimes n'} \in H^0(\mathcal{O}_X(n'nL))$ hence $\operatorname{Bs}|n'nL| \subseteq \operatorname{Bs}|nL|$. In particular, $\mathbf{B}(L) = \operatorname{Bs}|mL|$ for every sufficiently divisible $m > 0$.

Assume that $k \subseteq k'$ is a field extension and X' is the scheme obtained by base change to k'. Let $\pi\colon X' \to X$ be the corresponding morphism. Since π is flat, we can define the pullback $L' = \pi^*L$. Since

$$H^0(\mathcal{O}_X(L')) = H^0(\mathcal{O}_X(L)) \otimes_k k'$$

we can see that $\pi^{-1}\operatorname{Bs}|L| = \operatorname{Bs}|L'|$. This in turn implies that $\pi^{-1}\mathbf{B}(L) = \mathbf{B}(L')$.

Now assume that L is $\mathbb{R}$-Cartier with a given expression $L \sim_{\mathbb{R}} \sum t_i A_i$ as in 1.2. As pointed out in 2.3, $\langle mL' \rangle \sim \pi^*\langle mL \rangle$ hence $\pi^{-1}\mathbf{B}_+(L) = \mathbf{B}_+(L')$.

With a little more work we can also see that $\pi^{-1}\mathbb{E}(L) \supseteq \mathbb{E}(L')$. Indeed, let V' be a component of $\mathbb{E}(L')$, let W be the closure of $\pi(V')$, and let W' be the scheme obtained from W by base change. If $L|_W$ is big then $L|_W \sim_{\mathbb{R}} A_W + D_W$ where A_W is ample and D_W is effective. But then $L'|_{W'} \sim_{\mathbb{R}} A'_{W'} + D'_{W'}$ where $A'_{W'}$ is ample and $D'_{W'}$ is effective. Now $V' \not\subseteq D'_{W'}$ otherwise $W \subseteq D_W$ which is not possible.Thusby restricting to V' we get $L'|_{V'} \sim_{\mathbb{R}} A'_{V'} + D'_{V'}$ which means that $L'|_{V'}$ is big, a contradiction. Therefore $\pi^{-1}\mathbb{E}(L) \supseteq \mathbb{E}(L')$. Thusif in some situation we want to show that $\mathbf{B}_+(L) \subseteq \mathbb{E}(L)$, then it is enough to show that $\mathbf{B}_+(L') \subseteq \mathbb{E}(L')$ because π is surjective.

3. The augmented base locus is well-defined

In this section, we show that the augmented base locus as defined in Definition 1.2 is well-defined. We also show that the definition agrees with 1.1 and the one in [8].

Lemma 3.1. *Let X be a projective scheme over a field k and L an $\mathbb{R}$-Cartier divisor with a given expression $L \sim_{\mathbb{R}} \sum t_i A_i$ as in 1.2. Then $\mathbf{B}_+(L)$ as defined in 1.2 satisfies the following assertions:*

(1) *$\mathbf{B}_+(L)$ does not depend on the choice of A nor the expression $L \sim_{\mathbb{R}} \sum t_i A_i$;*

(2) *for any positive rational number s we have $\mathbf{B}_+(sL) = \mathbf{B}_+(L)$;*

(3) *$\mathbf{B}_+(L) = \bigcap \mathbf{B}(L-H)$ where H runs over all ample $\mathbb{R}$-Cartier divisors so that $L-H$ is $\mathbb{Q}$-Cartier;*

(4) *if L is $\mathbb{Q}$-Cartier then $\mathbf{B}_+(L)$ coincides with the one defined in 1.1.*

Proof. (1) First we show that $\mathbf{B}_+(L)$ is independent of the choice of A. Indeed let G be any other ample Cartier divisor. Assume $x \notin \bigcap_{m\in\mathbb{N}} \mathbf{B}(\langle mL\rangle - A)$. Then

$$x \notin \mathbf{B}(\langle mL\rangle - A) = \mathbf{B}(\sum \lfloor mt_i \rfloor A_i - A)$$

for some $m > 0$. Thus $x \notin \mathbf{B}(\sum l \lfloor mt_i \rfloor A_i - lA)$ for any sufficiently large $l > 0$. Since

$$\sum \lfloor lmt_i \rfloor A_i - \sum l \lfloor mt_i \rfloor A_i$$

is zero or ample, $x \notin \mathbf{B}(\sum \lfloor lmt_i \rfloor A_i - lA)$, and since lA is sufficiently ample,

$$x \notin \mathbf{B}(\sum \lfloor lmt_i \rfloor A_i - G) = \mathbf{B}(\langle lmL\rangle - G).$$

This shows that

$$\bigcap_{m\in\mathbb{N}} \mathbf{B}(\langle mL\rangle - A) \supseteq \bigcap_{m\in\mathbb{N}} \mathbf{B}(\langle mL\rangle - G).$$

The opposite inclusion $\subseteq$ can be proved similarly hence $\mathbf{B}_+(L)$ is independent of A.

Now we show that $\mathbf{B}_+(L)$ is independent of the expression $L \sim_{\mathbb{R}} \sum t_i A_i$. Indeed assume that $L \sim_{\mathbb{R}} \sum t'_i A'_i$ is another expression. Redefining the indexes we can assume that $A'_i = A_i$. Let $A = \sum A_i + G$ with G ample. Assume that

$$x \notin \bigcap_{m\in\mathbb{N}} \mathbf{B}(\sum \lfloor mt_i \rfloor A_i - A).$$

Then $x \notin \mathbf{B}(\sum \lfloor mt_i \rfloor A_i - A)$ for some m hence $x \notin \mathbf{B}(\sum l \lfloor mt_i \rfloor A_i - lA)$ for any sufficiently large $l > 0$. Arguing as above we can show that $x \notin \mathbf{B}(\sum \lfloor lmt_i \rfloor A_i - A)$. Writing $lmt_i = \lfloor lmt_i \rfloor + u_i$ and $lmt'_i = \lfloor lmt'_i \rfloor + u'_i$, we see that

$$(\sum \lfloor lmt'_i \rfloor A_i - G) - (\sum \lfloor lmt_i \rfloor A_i - A)$$
$$\sim_{\mathbb{R}} \sum u_i A_i - \sum u'_i A_i + A - G$$

is ample hence $x \notin \mathbf{B}(\sum \lfloor lmt'_i \rfloor A_i - G)$ so

$$x \notin \bigcap_{m\in\mathbb{N}} \mathbf{B}(\sum \lfloor mt'_i \rfloor A_i - G).$$

In other words,

$$\bigcap_{m\in\mathbb{N}} \mathbf{B}(\sum \lfloor mt_i \rfloor A_i - A) \supseteq \bigcap_{m\in\mathbb{N}} \mathbf{B}(\sum \lfloor mt'_i \rfloor A_i - G).$$

The opposite inclusion $\subseteq$ can be proved similarly bearing in mind that we are free to change A and G.

(2) It is enough to treat the case when $s \in \mathbb{N}$. It is obvious that $\mathbf{B}_+(sL) \supseteq \mathbf{B}_+(L)$. Assume that $x \notin \mathbf{B}_+(L)$. Let $A = \sum A_i$. Then $x \notin \mathbf{B}(\langle mL \rangle - A)$ for some m hence $x \notin \mathbf{B}(s\langle mL \rangle - sA)$. Since $\langle msL \rangle - A - (s\langle mL \rangle - sA)$ is ample or zero, we see that $x \notin \mathbf{B}(\langle msL \rangle - A)$ which implies that $x \notin \mathbf{B}_+(sL)$. That is, $\mathbf{B}_+(sL) \subseteq \mathbf{B}_+(L)$.

(3) For each $m > 0$,

$$\mathbf{B}(\langle mL \rangle - A) = \mathbf{B}(mL - mH_m) = \mathbf{B}(L - H_m)$$

for some ample $\mathbb{R}$-Cartier divisor H_m. Thus $\mathbf{B}_+(L) \supseteq \bigcap \mathbf{B}(L - H)$. Conversely assume $x \notin \bigcap \mathbf{B}(L - H)$. Then $x \notin \mathbf{B}(L - H)$ for some H. Since $L - H$ is assumed to be $\mathbb{Q}$-Cartier, $mL - mH$ is Cartier for some sufficiently divisible $m > 0$. Since mH is sufficiently ample,

$$\mathbf{B}(mL - mH) \supseteq \mathbf{B}(\langle mL \rangle - A)$$

hence $x \notin \mathbf{B}(\langle mL \rangle - A)$ which implies that $x \notin \mathbf{B}_+(L)$.

(4) We can write $L \sim_{\mathbb{Q}} \sum t_i A_i$ with all the t_i rational numbers. Pick $s \in \mathbb{N}$ so that sL is Cartier, st_i are all integers, and $sL \sim \sum st_i A_i$. Then by (2) and (1) we have

$$\mathbf{B}_+(L) = \mathbf{B}_+(sL) = \bigcap_{m \in \mathbb{N}} \mathbf{B}(\sum \lfloor mst_i \rfloor A_i - sA)$$

$$= \bigcap_{m \in \mathbb{N}} \mathbf{B}(msL - sA) = \bigcap_{m \in \mathbb{N}} \mathbf{B}(mL - A).$$

But this is the same as $\mathbf{B}_+(L)$ in Definition 1.1.

□

4. Growth of cohomology

The next lemma is similar to [6, Lemma 2.2].

Lemma 4.1. *Let X be a scheme projective over a field k. Assume that L is an $\mathbb{R}$-Cartier divisor with a given expression $L \sim_{\mathbb{R}} \sum t_i A_i$ as in 1.2. Let $\mathcal{F}$ be a coherent sheaf on X, let Y be its support, and $d = \dim Y$. Then the upper growth of $h^0(\mathcal{F}\langle mL \rangle)$ is at most like m^d.*

Proof. By 2.3, we can extend k hence assume it is infinite. Let t be a positive integer such that $t_i \leq t$ for every i. By 2.6, we can change the A_i up to linear equivalence so that for each $m > 0$,

$$\mathcal{F}\langle mL \rangle = \mathcal{F}(\sum \lfloor mt_i \rfloor A_i) \subseteq \mathcal{F}(mt \sum A_i).$$

Thusby replacing L with $t \sum A_i$ it is enough to assume that L is an effective very ample Cartier divisor. But then for m sufficiently large $h^0(\mathcal{F}(mL))$ coincides with the Hilbert polynomial of $\mathcal{F}$ with respect to L which is a polynomial of degree m^d.

□

Lemma 4.2. *Let X be a scheme projective over a field k. Assume that A is a very ample Cartier divisor and $\mathcal{F}$ is a coherent sheaf on X. Then we have $h^0(\mathcal{F}) \le h^0(\mathcal{F}(A))$.*

Proof. By extending k we can assume k is infinite. By 2.6, we can change A up to linear equivalence so that $\mathcal{F} \subseteq \mathcal{F}(A)$ which implies the claim. □

Lemma 4.3. *Let X be an integral scheme of dimension d projective over a field k. Assume that L is an $\mathbb{R}$-Cartier divisor with a given expression $L \sim_{\mathbb{R}} \sum t_i A_i$ as in 1.2. Then the following are equivalent:*
(1) *the upper growth of $h^0(\mathcal{O}_X\langle mL\rangle)$ is like m^d;*
(2) *for some coherent sheaf $\mathcal{F}$, the upper growth of $h^0(\mathcal{F}\langle mL\rangle)$ is like m^d;*
(3) *for any coherent sheaf $\mathcal{F}$ whose support is equal to X, the upper growth of $h^0(\mathcal{F}\langle mL\rangle)$ is like m^d;*
(4) *L is big.*

Proof. (1) $\implies$ (4): Let A be an effective ample Cartier divisor. Considering the exact sequence

$$0 \to \mathcal{O}_X(\langle mL\rangle - A) \to \mathcal{O}_X\langle mL\rangle \to \mathcal{O}_A\langle mL\rangle \to 0$$

and applying Lemma 4.1, we deduce that the upper growth of $h^0(\mathcal{O}_X(\langle mL\rangle - A))$ is like m^d. In particular, $\langle mL\rangle - A \sim D$ for some effective Cartier divisor D. Therefore, $mL \sim_{\mathbb{R}} A' + D$ for some ample $\mathbb{R}$-Cartier divisor A' because $mL - \langle mL\rangle$ is zero or ample. Thus$L$ is big.

(4) $\implies$ (3): By definition, $L \sim_{\mathbb{R}} A + D$ where A is an ample $\mathbb{R}$-Cartier divisor and D is an effective $\mathbb{R}$-Cartier divisor. Let $l \in \mathbb{N}$. Then for each $m \in \mathbb{N}$, $\langle mlL\rangle = m\langle lL\rangle + C_m$ for some C_m which is zero or very ample. By Lemma 4.2,

$$h^0(\mathcal{F}(m\langle lL\rangle)) \le h^0(\mathcal{F}\langle mlL\rangle).$$

Moreover, if l is large enough, then $\langle lL\rangle$ is big. Thusby replacing L with lL for some large l and then replacing L with $\langle L\rangle$ allows us to assume that L is Cartier and $\langle mL\rangle = mL$ for each $m > 0$. By replacing A, D we can assume that $L \sim_{\mathbb{Q}} A + D$ and that A, D are $\mathbb{Q}$-Cartier. Replacing L, A, D with multiples we can assume $L \sim A + D$, that A, D are Cartier, and A is very ample.

First assume that $\mathcal{F}$ is generated by global sections. Each global section corresponds to a morphism $\mathcal{O}_X \to \mathcal{F}$. Since X is integral, the morphism is injective if and only if its image is not torsion. Therefore if $\alpha_1, \dots, \alpha_r$ form a basis of $H^0(\mathcal{F})$ and if $\phi_i \colon \mathcal{O}_X \to \mathcal{F}$ corresponds to α_i, then ϕ_i is injective for at least one i otherwise $\mathcal{F}$ would be torsion which is not possible as the support of $\mathcal{F}$ is equal to X. Therefore,

$$h^0(\mathcal{O}_X(mA)) \le h^0(\mathcal{O}_X(mL)) \le h^0(\mathcal{F}(mL))$$

which implies that the upper growth of $h^0(\mathcal{F}(mL))$ is like m^d.

Now we deal with the general case when $\mathcal{F}$ is not necessarily generated by global sections. Tensor $\mathcal{F}$ with $\mathcal{O}_X(-D) \to \mathcal{O}_X$ to get $\mathcal{F}(-D) \to \mathcal{F}$ and let $\mathcal{K}$ and $\mathcal{M}$ be its kernel and image respectively. Then we get exact sequences

$$0 \to \mathcal{K}(mL) \to \mathcal{F}(mL - D) \to \mathcal{M}(mL) \to 0$$

and

$$0 \to \mathcal{M}(mL) \to \mathcal{F}(mL) \to \mathcal{F}(mL) \otimes \mathcal{O}_D \to 0.$$

Note that the support of $\mathcal{K}$ is inside D, so the upper growth of $h^0(\mathcal{K}(mL))$ is at most like m^{d-1} by Lemma 4.1. On the other hand, replacing L, A, D with multiples we can assume $\mathcal{F}(A)$ is generated by global sections. Thus by the last paragraph, the upper growth of

$$h^0(\mathcal{F}(mL - D)) = h^0(\mathcal{F}(A)((m-1)L))$$

is like m^d. This implies the upper growth of $h^0(\mathcal{M}(mL))$ is like m^d which in turn implies the upper growth of $h^0(\mathcal{F}(mL))$ is like m^d.

(3) $\implies$ (2): Obvious.

(2) $\implies$ (1): There is a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$$

of coherent sheaves such that for each $0 < j \leq n$, there exist a closed embedding $f\colon S \to X$ of an integral scheme S and an ideal sheaf $\mathcal{J} \subset \mathcal{O}_S$ such that $\mathcal{F}_j/\mathcal{F}_{j-1} \simeq f_*\mathcal{J}$ (cf. The stacks project [18], section on dévissage of coherent sheaves). Let j be the smallest number such that the upper growth of $h^0(\mathcal{F}_j\langle mL\rangle)$ is like m^d. Let $f\colon S \to X$ and $\mathcal{J}$ be the corresponding embedding and ideal sheaf so that $\mathcal{F}_j/\mathcal{F}_{j-1} \simeq f_*\mathcal{J}$. Then from the exact sequence

$$0 \to H^0(\mathcal{F}_{j-1}\langle mL\rangle) \to H^0(\mathcal{F}_j\langle mL\rangle) \to H^0(f_*\mathcal{J}\langle mL\rangle)$$

we deduce that the upper growth of $h^0(f_*\mathcal{J}\langle mL\rangle)$ is like m^d. By Lemma 4.1, $\dim S = d$, hence $S = X$. But then the upper growth of $h^0(\mathcal{O}_X\langle mL\rangle)$ is like m^d.

□

Proposition 4.4. *Let X be a projective scheme over a field k. Assume that L is a nef $\mathbb{R}$-Cartier divisor with a given expression $L \sim_{\mathbb{R}} \sum t_i A_i$ as in 1.2. Let $\mathcal{F}$ be a coherent sheaf on X, let Y be its support, and $e = \dim Y$. Then*

(1) the upper growth of $h^0(\mathcal{F}\langle mL\rangle)$ is at most like m^e;

(2) the upper growth of $h^i(\mathcal{F}\langle mL\rangle)$ is at most like m^{e-i} for any i;

(3) the upper growth of $h^0(\mathcal{F}\langle mL\rangle)$ is like m^e if and only if $L|_Z$ is big for some component Z of Y with $\dim Z = e$.

Proof. (1) This follows from Lemma 4.1.

(2) We do induction on e. By extending k we can assume that it is infinite. Choose an effective sufficiently ample Cartier divisor H and let $A = 2H$. For each $m > 0$ we can write $\langle mL\rangle \sim_{\mathbb{R}} mL - \sum u_i A_i$ where $u_i \in [0, 1)$ depend on m. Since H is sufficiently ample and L is nef,

$$\langle mL\rangle + H \sim_{\mathbb{R}} mL - \sum u_i A_i + H$$

is ample. Then by Fujita vanishing (2.5) we get $h^i(\mathcal{F}(\langle mL\rangle + A)) = 0$ for every $m > 0$ and $i > 0$. By 2.6, we can choose A so that the sequence

$$0 \to \mathcal{F} \to \mathcal{F}(A) \to \mathcal{F} \otimes \mathcal{O}_A(A) \to 0$$

is exact. The dimension of the support of $\mathcal{F}\otimes\mathcal{O}_A(A)$ is $e-1$. Now using the exact sequence

$$H^{i-1}(\mathcal{F}\otimes\mathcal{O}_A(\langle mL\rangle + A)) \to H^i(\mathcal{F}(\langle mL\rangle)) \to H^i(\mathcal{F}(\langle mL\rangle + A)) = 0$$

for $i>0$, and induction on e we get the result.

(3) As in the proof of Lemma 4.3, there is a filtration

$$0=\mathcal{F}_0\subset\mathcal{F}_1\subset\cdots\subset\mathcal{F}_n=\mathcal{F}$$

of coherent sheaves such that for each $0<j\leq n$, there exist a closed embedding $f\colon S\to X$ of an integral scheme S and an ideal sheaf $\mathcal{J}\subset\mathcal{O}_S$ such that $\mathcal{F}_j/\mathcal{F}_{j-1}\simeq f_*\mathcal{J}$.

Assume that the upper growth of $h^0(\mathcal{F}\langle mL\rangle)$ is like m^e. Let j be minimal with the property that the upper growth of $h^0(\mathcal{F}_j\langle mL\rangle)$ is like m^e. Let $f\colon S\to X$ and $\mathcal{J}$ be the corresponding embedding and ideal sheaf so that $\mathcal{F}_j/\mathcal{F}_{j-1}\simeq f_*\mathcal{J}$. Then the upper growth of $h^0(f_*\mathcal{J}\langle mL\rangle)$ is like m^e.Thusin particular $\mathcal{J}\neq 0$ and since S is integral the support of $f_*\mathcal{J}$ is equal to S. Moreover, since the upper growth of $h^0(f_*\mathcal{J}\langle mL\rangle)$ is like m^e, Lemma 4.1 shows that $\dim S\geq e$. On the other hand, S is a subset of Y because $\mathcal{F}|_{X\setminus Y}=0$ and because of the surjection $\mathcal{F}_j\to f_*\mathcal{J}$. Thus $\dim S\leq e$, hence $\dim S=e$. Now, by Lemma 4.3, $L|_S$ is big and so we can take $Z=S$.

Conversely, assume that there is a component Z of Y of dimension e such that $L|_Z$ is big. In the filtration above, let j be the smallest number such that Z is a component of the support of $\mathcal{F}_j$. Then Z is a subset of the support of the corresponding $f_*\mathcal{J}$ hence $Z\subseteq S$ which in turn implies that $Z=S$ because $e=\dim Z\leq\dim S\leq e$. It is then enough to show that the upper growth of $h^0(f_*\mathcal{J}\langle mL\rangle)$ is like m^e because of the exact sequence

$$0\to H^0(\mathcal{F}_{j-1}\langle mL\rangle)\to H^0(\mathcal{F}_j\langle mL\rangle)\to H^0(f_*\mathcal{J}\langle mL\rangle)\to H^1(\mathcal{F}_{j-1}\langle mL\rangle)$$

and the fact that the upper growth of $h^1(\mathcal{F}_{j-1}\langle mL\rangle)$ is at most like m^{e-1} by (2). Now apply Lemma 4.3.

□

5. Proof of main results

Proof. (of Theorem 1.4) By Noetherian induction we can assume that the theorem already holds for any closed subscheme of X not equal to X.

Step 1. We deal with the first equality in the theorem. By definition, $\mathbf{B}_+(L)\subseteq\mathbf{B}(\langle nL\rangle - A)$ for any $n>0$. Moreover, there are positive integers $m_1,\dots,m_r$ such that

$$\mathbf{B}_+(L)=\mathbf{B}(\langle m_1L\rangle - A)\cap\cdots\cap\mathbf{B}(\langle m_rL\rangle - A)$$

If $n=lm_i$ for some positive integer l, then $\langle nL\rangle - l\langle m_iL\rangle$ is zero or ample hence

$$\mathbf{B}(\langle nL\rangle - A)\subseteq\mathbf{B}(l\langle m_iL\rangle - A)\subseteq\mathbf{B}(l\langle m_iL\rangle - lA)=\mathbf{B}(\langle m_iL\rangle - A)$$

Therefore $\mathbf{B}_+(L)=\mathbf{B}(\langle nL\rangle - A)$ if each $m_i|n$.

For the second equality: for any fixed $n' > 0$ divisible by all the m_i and any sufficiently divisible $l > 0$ we have

$$\mathbf{B}_+(L) = \mathbf{B}(\langle n'L\rangle - A) = \mathrm{Bs}|l\langle n'L\rangle - lA| \supseteq \mathrm{Bs}|\langle ln'L\rangle - A|$$

$$\supseteq \mathbf{B}(\langle ln'L\rangle - A) = \mathbf{B}_+(L).$$

Now take $n = ln'$.

Step 2. The rest of the proof will be devoted to showing $\mathbf{B}_+(L) = \mathbb{E}(L)$. It is obvious that $\mathbf{B}_+(L) \supseteq \mathbb{E}(L)$ so we will focus on the reverse inclusion. If $L|_Z$ is not big for every component Z of X (with the reduced induced structure), then $\mathbf{B}_+(L) \subseteq \mathbb{E}(L) = X$. Thuswe may assume that there is a component Z such that $L|_Z$ is big. Pick such a Z with maximal dimension, say e. Let Y be the union of the other components, again with the induced reduced structure.

There are coherent ideal sheaves $\mathcal{I}, \mathcal{J} \subset \mathcal{O}_X$ such that the support of $\mathcal{I}$ is inside Z but the support of $\mathcal{O}_X/\mathcal{I}$ is inside Y, and the support of $\mathcal{J}$ is inside Y but the support of $\mathcal{O}_X/\mathcal{J}$ is inside Z (cf. [18], section on dévissage of coherent sheaves). Let Y', Z' be the closed subschemes defined by $\mathcal{I}, \mathcal{J}$ respectively. On $Z \setminus Y$ we have $\mathcal{J} = 0$ and $\mathcal{O}_{Z'} = \mathcal{O}_X$. Thus the reduced scheme associated to Z' is nothing but Z. Similarly, one shows that the reduced scheme associated to Y' is Y. By construction, on $Z \setminus Y$ we have $\mathcal{O}_{Z'} = \mathcal{O}_X$ and $\mathcal{I} = \mathcal{O}_X$, and on $Y \setminus Z$ we have $\mathcal{I} = 0$.

Step 3. We would like to find sections of $\mathcal{O}_X(\langle nL\rangle - A)$ which vanish on Y' but not on Z'. Let $\mathcal{I} \to \mathcal{O}_{Z'}$ be the composition $\mathcal{I} \hookrightarrow \mathcal{O}_X \to \mathcal{O}_{Z'}$ and let $\mathcal{K}, \mathcal{L}$ be its kernel and image respectively. Similarly, let $\mathcal{L} \to \mathcal{O}_Z$ be the composition $\mathcal{L} \hookrightarrow \mathcal{O}_{Z'} \to \mathcal{O}_Z$ and let $\mathcal{N}, \mathcal{M}$ be its kernel and image respectively. Then, by Step 2, on $Z \setminus Y$ we have $\mathcal{L} = \mathcal{O}_{Z'}$ and $\mathcal{M} = \mathcal{O}_Z$, and on $Y \setminus Z$ we have $\mathcal{L} = \mathcal{M} = 0$. Therefore the support of $\mathcal{L}, \mathcal{M}, \mathcal{I}$ are all equal to Z, and the support of $\mathcal{K}, \mathcal{N}$ are subsets of Z.

Now we have the exact sequences

$$0 \to \mathcal{K}(\langle nL\rangle - A) \to \mathcal{I}(\langle nL\rangle - A) \to \mathcal{L}(\langle nL\rangle - A) \to 0$$

and

$$0 \to \mathcal{N}(\langle nL\rangle - A) \to \mathcal{L}(\langle nL\rangle - A) \to \mathcal{M}(\langle nL\rangle - A) \to 0.$$

By Proposition 4.4, the upper growth of $h^0(\mathcal{M}(\langle nL\rangle - A))$ is like n^e but the upper growth of $h^1(\mathcal{N}(\langle nL\rangle - A))$ is at most like n^{e-1}. On the other hand, again by Proposition 4.4, the upper growth of $h^0(\mathcal{L}(\langle nL\rangle - A))$ is like n^e but the upper growth of $h^1(\mathcal{K}(\langle nL\rangle - A))$ is at most like n^{e-1}. Therefore for infinitely many $n > 0$ we can lift a nonzero section of $\mathcal{M}(\langle nL\rangle - A)$ to a section of $\mathcal{L}(\langle nL\rangle - A)$ and in turn to a section of $\mathcal{I}(\langle nL\rangle - A)$. In other words, there is a section $\alpha \in H^0(\mathcal{I}(\langle nL\rangle - A))$ whose restriction to Z is nonzero. Since $\mathcal{I}$ is the ideal sheaf of Y', α vanishes on Y' when considered as a section of $\mathcal{O}_X(\langle nL\rangle - A)$ via the injection $\mathcal{I}(\langle nL\rangle - A) \to \mathcal{O}_X(\langle nL\rangle - A)$.

Step 4. From now on we consider α as a section of $\mathcal{O}_X(\langle nL\rangle - A)$. We can think of α as a morphism $\mathcal{O}_X \to \mathcal{O}_X(\langle nL\rangle - A)$ such that if we tensor this with $\mathcal{O}_Z$ then we obtain a nonzero morphism. Let $\alpha_1 := \alpha$ and let $\mathcal{T}_1$ be the kernel of α_1. Let α_2 be the composition

$$\mathcal{O}_X \to \mathcal{O}_X(\langle nL\rangle - A) \to \mathcal{O}_X(2\langle nL\rangle - 2A) \to \mathcal{O}_X(\langle 2nL\rangle - 2A)$$

where the first morphism is α_1, the second one is obtained by tensoring α_1 with $\mathcal{O}_X(\langle nL\rangle - A)$, and the third one comes from the choice of an injective morphism $\mathcal{O}_X \to \mathcal{O}_X(\langle 2nL\rangle - 2\langle nL\rangle)$ (which exists because $\langle 2nL\rangle - 2\langle nL\rangle$ is zero or very ample) and tensoring it with $\mathcal{O}_X(2\langle nL\rangle - 2A)$.

Let $\mathcal{T}_2$ be the kernel of α_2. Obviously, $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Inductively we can define α_i to be the composition

$$\mathcal{O}_X \to \mathcal{O}_X(\langle (i-1)nL\rangle - (i-1)A) \to \mathcal{O}_X(\langle (i-1)nL\rangle + \langle nL\rangle - iA) \to \mathcal{O}_X(\langle inL\rangle - iA)$$

where the first map is α_{i-1}, the second map is obtained by tensoring α_1 with $\mathcal{O}_X(\langle (i-1)nL\rangle - (i-1)A)$, and third one is obtained from the choice of an injective morphism $\mathcal{O}_X \to \mathcal{O}_X(\langle inL\rangle - \langle (i-1)nL\rangle - \langle nL\rangle)$. Again it is obvious that $\mathcal{T}_{i-1} \subseteq \mathcal{T}_i$.

Step 5. By the Noetherian property, there is r such that $\mathcal{T}_r = \mathcal{T}_{r+1} = \cdots$. Since α_1 restricted to Z is nonzero and since Z is integral, we can make sure that the restriction of each α_i to Z is also nonzero: indeed if $U \subset Z$ is a small nonempty open set, then the restriction to U of each map in the definition of α_i is an isomorphism. Therefore each α_i is nonzero hence $\mathcal{T}_r \subsetneq \mathcal{O}_X$.

Now tensor α_r with $\mathcal{O}_X(-\langle rnL\rangle + rA)$ and let $\mathcal{E}$ be its image in $\mathcal{O}_X$. Then we get the exact sequence

$$0 \to \mathcal{T}_r(-\langle rnL\rangle + rA) \to \mathcal{O}_X(-\langle rnL\rangle + rA) \to \mathcal{E} \to 0.$$

Let E be the closed subscheme defined by $\mathcal{E}$, that is, E is the zero subscheme of α_r. Note that since $\alpha_r|_Z$ is nonzero, $Z \not\subseteq E$.

We will argue that $\mathbf{B}(\langle mL\rangle - A) \subseteq E$ if $m > 0$ is sufficiently divisible. By construction, α_r does not vanish outside E hence $\mathbf{B}(\langle rnL\rangle - rA) \subseteq E$. If $m = lrn$, then

$$\mathbf{B}(\langle mL\rangle - A) \subseteq \mathbf{B}(l\langle rnL\rangle - A) \subseteq \mathbf{B}(l\langle rnL\rangle - lA)$$
$$= \mathbf{B}(\langle rnL\rangle - A) \subseteq \mathbf{B}(\langle rnL\rangle - rA) \subseteq E.$$

Step 6. We will assume that $r \gg 0$. Consider the exact sequence

$$0 \to \mathcal{T}_r(\langle mL\rangle - \langle rnL\rangle + rA - aA) \to \mathcal{O}_X(\langle mL\rangle - \langle rnL\rangle + rA - aA) \to \mathcal{E}(\langle mL\rangle - aA) \to 0.$$

Since $\mathcal{T}_r$ does not depend on $r \gg 0$, by Fujita vanishing, we may assume that

$$H^i(\mathcal{T}_r(\langle mL\rangle - \langle rnL\rangle + rA - aA)) = 0$$

and

$$H^i(\mathcal{O}_X(\langle mL\rangle - \langle rnL\rangle + rA - aA)) = 0$$

for any $i > 0$, $m > rn$, and $a \in \{0,1\}$. Therefore $H^i(\mathcal{E}(\langle mL\rangle - aA)) = 0$ if $i > 0$, $m \gg 0$, and $a \in \{0,1\}$ (in this proof we only need to consider $a = 1$ but in the proof of Theorem 1.5 we need to take $a = 0$).

On the other hand, we have the exact sequence

$$0 \to \mathcal{E}(\langle mL\rangle - aA) \to \mathcal{O}_X(\langle mL\rangle - aA) \to \mathcal{O}_E(\langle mL\rangle - aA) \to 0$$

from which we obtain the exact sequence

$$H^0(\mathcal{O}_X(\langle mL\rangle - aA)) \to H^0(\mathcal{O}_E(\langle mL\rangle - aA)) \to H^1(\mathcal{E}(\langle mL\rangle - aA)) = 0$$

if $m \gg 0$ and $0 \le a \le 1$.

Step 7. From the expression $L \sim_{\mathbb{R}} \sum t_i A_i$ we obtain the expression $L|_E \sim_{\mathbb{R}} \sum t_i A_i|_E$. The restriction $L|_E$ (resp. $A_i|_E$) is defined up to $\mathbb{R}$-linear equivalence (resp. linear equivalence). For each $m > 0$ we get $\langle mL\rangle|_E = \langle mL|_E\rangle$. Taking $a = 1$ in Step 6, recalling that $\mathbf{B}(\langle mL\rangle - A) \subseteq E$ if $m > 0$ is sufficiently divisible, and using Step 1, we deduce that

$$\mathbf{B}_+(L) = \mathbf{B}(\langle mL\rangle - A) = \operatorname{Bs}|\langle mL\rangle - A|$$

$$= \operatorname{Bs}|\langle mL|_E\rangle - A|_E| = \mathbf{B}(\langle mL|_E\rangle - A|_E) = \mathbf{B}_+(L|_E)$$

for any sufficiently divisible $m > 0$.

On the other hand, it is easy to see that $\mathbb{E}(L|_E) \subseteq \mathbb{E}(L)$: indeed, if V is a component of $\mathbb{E}(L|_E)$, then $(L|_E)|_V$ is not big, so $L|_V$ is not big, hence $V \subseteq \mathbb{E}(L)$. Finally using the Noetherian induction and the above results we get

$$\mathbb{E}(L) \subseteq \mathbf{B}_+(L) = \mathbf{B}_+(L|_E) = \mathbb{E}(L|_E) \subseteq \mathbb{E}(L)$$

which in particular implies that $\mathbf{B}_+(L) = \mathbb{E}(L)$.

□

Proof. (of Theorem 1.5) We may assume that the theorem holds for every closed subscheme of X other than X itself. Moreover, by replacing L with a multiple we can assume that it is Cartier. If $\mathbb{E}(L) = X$, the theorem is trivial. We thus assume this is not the case. Let E be the subscheme constructed in Step 5 of the proof of Theorem 1.4. We showed that if $m \gg 0$, the map

$$H^0(\mathcal{O}_X(mL)) \to H^0(\mathcal{O}_E(mL))$$

is surjective (by taking $a = 0$). Moreover, we showed that $\mathbb{E}(L|_E) = \mathbb{E}(L)$. Since

$$\mathbf{B}(L) \subset \mathbf{B}_+(L) = \mathbb{E}(L) \subseteq E$$

we have $\mathbf{B}(L) = \mathbf{B}(L|_E)$. Thus L is semi-ample if and only if $L|_E$ is semi-ample. Since the theorem already holds for E by assumption, there is a closed subscheme Z of E whose reduction is $\mathbb{E}(L|_E)$ and such that $L|_E$ is semi-ample if and only if $L|_Z$ is semi-ample. Now L is semi-ample if and only if $L|_Z$ is semi-ample.

□

6. The augmented base locus of log divisors

Assume that X is a normal projective variety of dimension d over an algebraically closed field k, and that $B, A \geq 0$ are $\mathbb{R}$-divisors. Moreover, suppose A is nef and big and $L = K_X + B + A$ is nef.

Theorem 6.1. *Assume $L^d = 0$. Then $\mathbf{B}_+(L) = X$ is covered by rational curves C with $L \cdot C = 0$.*

The theorem was proved by Cascini-Tanaka-Xu [7] and independently by M^{c}Kernan, when k has positive characteristic. A short proof of this in any characteristic was given in [1]. Now if $L^d > 0$, what can we say about $\mathbf{B}_+(L)$? For example, is it again covered by rational curves intersecting L trivially? We give a couple of examples to shed some light on this question.

Example 6.2 Let E be an elliptic curve over an algebraically closed field k and let $X = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(1))$. The surjection $\mathcal{O}_E \oplus \mathcal{O}_E(1) \to \mathcal{O}_E$ defines a section of the projection $X \to E$ whose image will be denoted by E again. Moreover, there is a birational contraction $X \to Z$ which contracts only E. Let $B = E$ and A be the pullback of a sufficiently ample divisor on Z. Let $L = K_X + B + A$. By construction, $\mathbf{B}_+(L) = \mathbb{E}(L) = E$ which is not covered by rational curves but at least it is covered by curves intersecting L trivially.

Example 6.3 There is a well-known example of a smooth projective surface S over an algebraically closed field k, which is ruled over an elliptic curve Z, containing a curve M (a section of $S \to Z$) such that the Kodaira dimension of M as a divisor on S is zero, $K_S + 2M \sim 0$, and if $M \cdot C = 0$ for some curve C then $C = M$ (see Shokurov [17, Example 1.1] for such an example). Let $X = \mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(1))$. The surjection $\mathcal{O}_S \oplus \mathcal{O}_S(1) \to \mathcal{O}_S$ defines a section of the projection $\pi\colon X \to S$ whose image will be denoted by S again. Moreover, there is a birational contraction $X \to Z$ which contracts only S to a point. Let $B = S + 3\pi^*M$ and let A be the pullback of a sufficiently ample divisor on Z. Let $L = K_X + B + A$. Then $\mathbf{B}_+(L) = \mathbb{E}(L) \subseteq S$. Moreover, since

$$L|_S = (K_X + S + 3\pi^*M + A)|_S \sim K_S + 3M \sim M$$

is not big, $\mathbf{B}_+(L) = \mathbb{E}(L) = S$. But there is no family of curves C covering S with the property $L \cdot C = 0$.

These examples show that we need to put some reasonably strong condition on X, B, A to be able to say something interesting about $\mathbf{B}_+(L)$.

Question 6.4. *Assume that (X, B) is a projective klt pair over an algebraically closed field k and A a nef and big $\mathbb{R}$-divisor. Assume that $L = K_X + B + A$ is nef and that $L^d > 0$. Is it true that $\mathbf{B}_+(L)$ is covered by rational curves C with $L \cdot C = 0$?*

Assume that $k = \mathbb{C}$. Then L in the question is semi-ample by the base point free theorem hence it defines a contraction $X \to Y$. Moreover, it is well-known that the fibres of $X \to Y$ are covered by rational curves. Note that $\mathbf{B}_+(L)$ is nothing but the union of the fibres of $X \to Y$.

Now assume that k has characteristic $p > 5$ and $\dim X \leq 3$. One can show that L is again semi-ample (if $\dim X = 2$, this holds for any p [19]). We sketch the proof. Since A is nef and big, we can change the situation so that it is ample [1, Lemma 8.2]. Using boundedness of the length of extremal rays [12],[1, 3.3] one can show that $L = \sum r_i(K_X + B_i + A_i)$ where $r_i > 0$, $\sum r_i = 1$, B_i, A_i are effective $\mathbb{Q}$-divisors, A_i is ample, (X, B_i) is klt, and $K_X + B_i + A_i$ is nef and big. Now applying [1, Theorem 1.4],[20] each $K_X + B_i + A_i$ is semi-ample hence L is also semi-ample. Thus L defines a contraction $X \to Y$. In particular, $\mathbf{B}_+(L)$ is covered by a family of curves intersecting L trivially. Using the results of [1] it does not seem hard to prove that the fibres of $X \to Y$ are actually covered by rational curves.

Assume that k has positive characteristic and $\dim X \geq 4$. It seems hard to answer the question in this case because of the lack of resolution of singularities. However, if we replace the klt condition with strongly F-regular, then it is likely that one can actually answer the question.

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