ON THE QUADRATIC INVARIANT OF BINARY SEXTICS

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Abstract. We provide a geometric characterisation of binary sextics with vanishing quadratic invariant.

1. Introduction

Classical invariant theory was formulated and developed by Cayley, Salmon, Sylvester, Young and others in the second half of the 19^{th} century. After a quiet period which lasted for the most part of the 20^{th} century, the theory has reappeared in algebraic geometry as the *geometric invariant theory* [13], in representation theory of $GL(2)$, and in other areas of modern mathematics [2]. Each of these areas replaces the classical terminology by its own language. In this paper we shall nevertheless follow the old–fashioned terminology of [8]. This is in line with other modern expositions of the subject [14, 12].

Despite the recent developments, some of the problems left over from the 19^{th} century remain open. One class of such problems has to do with finding the interpretation of the vanishing of invariants and covariants. This should be expressed in terms of the underlying projective geometry of roots on the Riemann sphere. The aim of this paper is to solve a problem from this category, and interpret the vanishing of the *apolar* invariant for a binary sextic, and more generally for binary quantics of even degree.

Consider a polynomial of degree six in x with complex coefficients $\boldsymbol{\psi} = (\psi_0, \dots, \psi_6)$

$$\psi(x) = \psi_0 x^6 + 6\psi_1 x^5 + 15\psi_2 x^4 + 20\psi_3 x^3 + 15\psi_4 x^2 + 6\psi_5 x + \psi_6. \tag{1.1}$$

Substituting

$$x = \frac{a\tilde{x} + b}{c\tilde{x} + d}, \quad \text{where} \quad ad - bc = 1,$$

and multiplying the resulting expression by $(c\tilde{x} + d)^6$ to clear the denominators gives the polynomial $\tilde{\psi}(\tilde{x})$ with the coefficients $\tilde{\boldsymbol{\psi}}$ given by a linear transformation of $\boldsymbol{\psi}$. The function

$$\mathcal{I}(\boldsymbol{\psi}) = 2\psi_0\psi_6 - 12\psi_1\psi_5 + 30\psi_2\psi_4 - 20\psi_3^2 \tag{1.2}$$

is an *invariant* of the sextic, as $\mathcal{I}(\boldsymbol{\psi}) = \mathcal{I}(\tilde{\boldsymbol{\psi}})$.

Vanishing of any invariant of a binary quantic – the precise definitions are given in the next Section – describes some geometric property of the configurations of the roots of the quantic regarded as points on the two–dimensional sphere $\mathbb{CP}^1 = \mathbb{C} + \{\infty\}$. The analog of the quadratic invariant (1.2) can be constructed for any polynomial of even degree - see formula (2.4). In this paper we find a geometric interpretation of the condition $\mathcal{I} = 0$. This natural question does not seem to have been answered by the classical invariant theorists in the 19^{th} and early 20^{th} centuries, except when the the quantic has degree two or four: A generic quartic has four distinct roots, and the condition $\mathcal{I} = 0$ implies that their cross ratio is a cube root of unity. This is the equianharmonic condition. The roots of the quartic, when viewed as points on the Riemann sphere, can in this case be transformed into vertices of a regular tetrahedron by a Möbius transformation. The answer for binary sextics appears not to be known, and trying to understand this case in the context of twistor theory of G_2

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structures [4, 3] is one motivation for this paper. It turns out that the sextic case can be reduced to the quartic in a sense made precise by the following Theorem

Theorem 1.1. *Let X_1, X_2, X_3, X_4 be four points on a two–dimensional sphere such that the stereographic projection of one of the roots of the sextic* (1.1) *from any of these four points lies in the centroid of the projections of the remaining five roots. Then $\mathcal{I}(\boldsymbol{\psi}) = 0$ if and only if the points $X_1, \dots, X_4$ can be transformed into vertices of a regular tetrahedron by a Möbius transformation (if they are distinct), or if at least three of these points coincide.*

First we will need to establish that there are indeed only four points, up to multiplicity, with the property stated in this Theorem. This, together with the rest of the proof with be presented in Section 3.

The invariant (1.2) is quadratic in the coefficients of the sextic, and thus given any five points on the sphere there will generically exist two points which complement these five points to roots of a sextic with $\mathcal{I} = 0$. In Proposition 4.2 we shall characterise the non–generic configurations of five points such that any choice of a distinct sixth point yields a sextic with $\mathcal{I} \neq 0$. It will be shown that any such non-generic configuration is projectively equivalent to vertices of a square pyramid.

The sextic case is special in some ways, but the general method in the paper together with an inductive argument applies to binary quantics of any even degree.

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2. Quantics and invariants

A binary quantic is a homogeneous polynomial in two variables which we shall call (x, y). We shall consider binary quantics of even degree

$$\psi(x, y) = \sum_{k=0}^{2n} \binom{2n}{k} \psi_k x^{2n-k} y^k. \tag{2.3}$$

The coefficients of the quantic $\boldsymbol{\psi} = (\psi_0, \dots, \psi_{2n})$ are assumed to be complex numbers. There exists a unique, up to an overall scale, quadratic invariant

$$\mathcal{I}(\psi) = 2 \sum_{k=0}^{2n} (-1)^{2n-k} \binom{2n}{k} \psi_k \psi_{2n-k} \tag{2.4}$$

one can associate to the quantic[1]. The invariance of $\mathcal{I}$ is to be understood in the following way: Consider the linear action of $GL(2, \mathbb{C})$ on $\mathbb{C}^2$ given by the change of variables

$$x = a\tilde{x} + b\tilde{y}, \qquad y = c\tilde{x} + d\tilde{y}, \qquad ad - bc \neq 0.$$

Given a binary quantic $\psi(x, y)$, let $\widetilde{\psi}(\tilde{x}, \tilde{y})$ be a binary quantic given by

$$\begin{aligned} \widetilde{\psi}(\tilde{x}, \tilde{y}) &= \sum_{k=0}^{2n} \binom{2n}{k} \psi_k \, (a\tilde{x} + b\tilde{y})^{2n-k} \, (c\tilde{x} + d\tilde{y})^k \\ &= \tilde{\psi}_0 \tilde{x}^{2n} + 2n\tilde{\psi}_1 \tilde{x}^{2n-1}\tilde{y} + n(2n-1)\tilde{\psi}_2 \tilde{x}^{2n-2}\tilde{y}^2 + \dots + \tilde{\psi}_{2n}\tilde{y}^{2n}. \end{aligned}$$

This induces an irreducible embedding $GL(2, \mathbb{C}) \subset GL(2n+1, \mathbb{C})$, as the $(2n+1)$ coefficients of $\widetilde{\psi}$ are linear homogeneous functions of the coefficients of ψ.

[1]Sylvester [18] calls it the *quadrinvariant.* In his latter works, see e.g. [19], he proposed an analogy between classical invariant theory and molecular chemistry. A binary sextic would correspond to an atom with six free valent electrons, and the quadratic invariant is then the bi-atomic molecule.

Definition 2.1. *A covariant of a binary quantic is a polynomial* $I = I(\psi_0, \dots, \psi_{2n}, x, y)$ *such that*

$$I(\tilde{\psi_0}, \dots, \tilde{\psi_{2n}}, \tilde{x}, \tilde{y}) = (ad - bc)^w I(\psi_0, \dots, \psi_{2n}, x, y).$$

The number w *is called the weight of the covariant. A covariant which only depends on the coefficients of the quantic, and not on* (x, y) *is called an invariant.*

Thus the degree–two invariant (2.4) has weight $2n$. There are other invariants of degree higher than two [8]. In the case of the binary sextic there are four more invariants, of degree 4, 6, 10 and 15 respectively connected with $\mathcal{I}$ by a syzygy of degree 30.

2.1. **Transvectants.** Let $V_m = \mathrm{Sym}^m(\mathbb{C}^{2\vee})$ be the $(m+1)$–dimensional complex vector space of binary quantics of degree m, where $\mathbb{C}^{2\vee}$ is the dual of $\mathbb{C}^2$. Given two binary quantics $\phi \in V_n$ and $\psi \in V_m$, the kth transvectant is a map $< \, , \, >_k \colon V_m \times V_n \to V_{m+n-2k}$ given by a quantic of degree $n + m - 2k$

$$< \phi, \psi >_k := \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{\partial^k \phi}{\partial x^{k-j} \partial y^j} \frac{\partial^k \psi}{\partial x^j \partial y^{k-j}}. \tag{2.5}$$

Thus, for any $k \leq \min(m, n)$, transvectants are covariants of weight k and degree two. One of the results in the classical invariant theory is that all covariants and invariants arise from the transvectants operations [8, 14].

Definition 2.2. *The quantic* $\phi \in V_n$ *is apolar to* $\psi \in V_m$ *where* $m \geq n$ *if* $< \psi, \phi >_n = 0$.

In the special case $n = m$ the apolarity condition is given by

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \psi_k \phi_{n-k} = 0.$$

Any quantic of an odd degree is apolar to itself. A quantic of an even degree is apolar to itself iff the quadratic invariant $\mathcal{I}$ given by (2.4) vanishes.

3. Characterisation of $\mathcal{I}$

Let ψ be a sextic (1.1) which is generic in the sense that its six complex roots are distinct, and let $\mathcal{I}(\psi)$ be given by (1.2).

Proposition 3.1. *Let* $P_1, \dots, P_6$ *be six points on the sphere* $\mathbb{CP}^1$ *corresponding to the roots of a sextic* ψ. *Then* $\mathcal{I}(\psi) = 0$ *if and only if the four roots of the quartic* $< \kappa, \rho >_1$ *can be transformed into vertices of a regular tetrahedron (if they are distinct) or contain a root of multiplicity at least three. Here* κ *and* ρ *are any quintic and a linear form respectively such that* $\psi(x) = \kappa(x)\rho(x)$.

Proof. Any quintic corresponding to distinct points $P_1, \dots, P_5$ associates four points on the sphere to a given point P_6, such that the quartic defining the four points is a transvectant of the quintic with a linear form corresponding to P_6. Given $\psi(x, y) = \kappa(x, y)\rho(x, y)$, and using $< \rho, \rho >_1 = 0$ we compute

$$\mathcal{I}(\psi) = < \psi, \psi >_6 = - < \delta, \delta >_4 = -\mathcal{I}(\delta), \quad \text{where} \quad \delta = < \kappa, \rho >_1 .$$

Thus $\mathcal{I}(\psi) = 0$ iff $\mathcal{I}(\delta) = 0$, where δ is the quartic defined above. But $\mathcal{I}(\delta) = 0$ if at least three of the roots of δ coincide, or if the cross ratio of the four distinct roots is a cube root of unity (the equianharmonic case). For completeness we give the proof of this fact, well known in the 19^{th} century literature. Set $y = 1$, consider a general quartic

$$\delta = \delta_0 x^4 + 4\delta_1 x^3 + 6\delta_2 x^2 + 4\delta_3 x + \delta_4$$

with four distinct roots. This has $\mathcal{I}(\delta) = 2\delta_0\delta_4 - 8\delta_1\delta_3 + 6(\delta_2)^2$. Now consider the quartic corresponding to a regular tetrahedron with one vertex at ∞. This quartic is represented by $(x-1)(x-\omega)(x-\omega^2)$ where $\omega^3 = 1$, and we find $\mathcal{I} = 0$. The cross ratio of the roots is given by ω, and conversely any quartic with the cross ratio of four roots given by a cube root of unity is projectively equivalent to the tetrahedral quartic.

□

We can now give the proof of the result stated in the Introduction. In the proof, and in the remaining part of the paper we shall write

$$\psi = P_1 P_2 \dots P_m \tag{3.6}$$

to denote a binary quantic defined (up to an overall non–zero multiple) by its roots corresponding to points $P_1, P_2, \dots, P_m$ on the sphere.

Proof of Theorem 1.1. Consider the quartic $\delta =< \kappa, \rho >_1$ introduced in Proposition 3.1. Let its four roots correspond to the (not necessarily distinct) points $X_1, \dots, X_4$. Thus, using the notation (3.6), the quartic δ is apolar (in the sense of Definition 2.2) to a quartic $XXXX$, where X is any of the roots X_i.

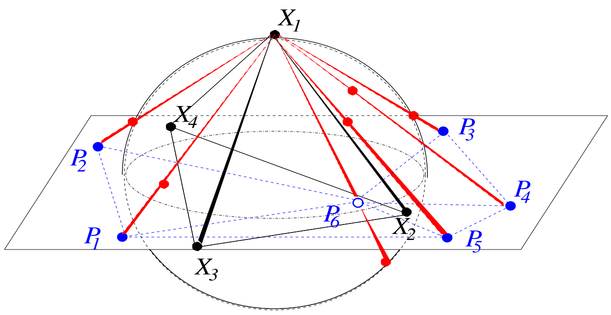

Figure 1. *Stereographic projection of P_6 to the centroid of $P_1, \dots, P_5$.*

Equivalently the quintic $\kappa = P_1P_2P_3P_4P_5$ is apolar to the quintic $\chi = P_6X_iX_iX_iX_i$ for any $i = 1, \dots, 4$. This happens if and only if the stereographic projection of P_6 from any of the four points X_i lies in the centroid of the projections of the five points $P_1, \dots, P_5$. This fact was observed in [16, 20]. Indeed, if the homogeneous coordinates of the projected points P_i are $(1, -x_i)$ where $i = 1, \dots, 6$ and the coordinates of the north pole X are $(0, 1)$ then $< P_i, X >= 1, < P_i, P_j >= x_i - x_j$, so

$$< \kappa, \chi >_5 = \sum_{i=1}^{5} (x_i - x_6) = 0, \qquad \text{and} \quad x_6 = \frac{1}{5}\sum_{i=1}^{5} x_i.$$

We note that the centroid of the set of points is not invariant under the projective transformations, but is invariant under the subgroup of affine transformations $x \to \alpha x + \beta$ preserving the north pole of the stereographic projection. The statement in the Theorem 1.1 is nevertheless projectively invariant.

□

This proof extends to binary quantics of arbitrary even degree. Assume we know a geometrical interpretation for vanishing of $\mathcal{I}$ for a binary quantic ψ_{2n-2} of degree $2n-2$. We can now characterise the self–apolarity of a binary quantic ψ_{2n} of degree $2n$ by the following inductive argument: Given $2n-1$ distinct points $(P_1, \dots, P_{2n-1})$ on the sphere, let κ be the unique (up to a non–zero multiple) binary quantic of degree $(2n-1)$ with these points corresponding to its roots. The points $(P_1, \dots, P_{2n-1}, P_{2n})$ are roots of a binary quantic ψ_{2n} with $\mathcal{I}(\psi_{2n}) = 0$ if a linear form ρ with a root corresponding to the point P_{2n} is such that the binary quantic $\psi_{2n-2} =< \kappa, \rho >$ has $\mathcal{I}(\psi_{2n-2}) = 0$.

Given $2n$ unordered points $\psi_{2n} = \{P_1, P_2, \ldots, P_{2n}\}$ on the sphere, split them into a set of $2n-1$ points $\kappa = \{P_1, P_2, \ldots, P_{2n-1}\}$ together with one point $\{P_{2n}\}$. Let $\delta = \{X_1, X_2, \ldots, X_{2n-2}\}$ be the points on the sphere such that the stereographic projection of P_{2n} from any X_i is the centroid of the stereographic projections of $P_1, \ldots, P_{2n-1}$. Then the set of points ψ is self-apolar iff the set of points δ is self–apolar. The set δ consist of the roots of the polynomial equation

$$< \kappa, \chi >_{2n-1} = 0, \quad \text{where,} \quad \chi = P_{2n} X X \ldots X \tag{3.7}$$

counted with multiplicity.

Example. The multiplicities of the elements of δ can depend on the choice of the point P_{2n} from the set ψ. Consider the sextic

$$\psi = (x-1)(x-\omega)(x-\omega^2)(x-\omega^3)(x-\omega^4), \quad \text{where} \quad \omega^5 = 1 \tag{3.8}$$

corresponding to a pentagonal pyramid with one root placed at ∞. This has $\mathcal{I} = 0$. Thus $\psi = \{1, \omega, \omega^2, \omega^3, \omega^4, \infty\}$. Taking $P_6 = \infty$ gives the quartic (3.7) to be x^4 which has one quadrupole root $x = 0$ and thus is self–apolar. Choosing instead $P_6 = 1$ gives the polynomial (3.7)

$$x^4 + 6x^3 + 6x^2 + 6x + 6 = 0$$

which has four distinct roots with equianharmonic cross ratio.

3.1. **Canonical form.** Almost all sextics can be put in the Sylvester Canonical Form [18]

$$\psi = Cu^6 + Av^6 + Bw^6 + uvw(u-v)(v-w)(w-u), \tag{3.9}$$

where $u + w + v = 0$ - thus to obtain (1.1) set $u = x, v = 1, w = -x - 1$ in this formula[2]. In this form the quadratic invariant (1.2) is

$$\mathcal{I}(\boldsymbol{\psi}) = 2CA + 2CB + 2BA - 2.$$

If the sextic does not have a root at ∞ we can assume that $C + B \neq 0$, and solve for $A = (1 - CB)/(C + B)$. This gives a canonical form of a generic self-apolar sextic. It parametrises a non–singular open orbit in the space of all self-apolar sextics. Setting $B = (b+c)/(3b-3c), C = (6-b-c)/(3b-3c)$ gives

$$\psi = x^6 + 2bx^5 + 5bx^4 + \frac{20}{6}(b+c)x^3 + 5cx^2 + 2cx + \frac{1}{36}(b+c)^2 + \frac{1}{4}(b-c)^2, \tag{3.10}$$

so the self-apolar sextic depends on two parameters. In general the binary quantic of degree $2n$ with $\mathcal{I} = 0$ depends on $2n - 4$ arbitrary parameters, up to the Möbius transformation.

3.2. **Catalectant.** We shall finish off this section giving an alternative interpretation of the condition $\mathcal{I} = 0$, which brings up another quartic invariant of binary sextics.

The seven–dimensional complex vector space of binary sextics V_6 belongs to the space of endomorphisms of the four–dimensional space of binary cubics V_3, where the endomorphism corresponding to a sextic ψ is given by the transvectant $\phi \to < \psi, \phi >_3$, where $\phi \in V_3$. Consider a complex eigenvalue problem

$$< \psi, \phi >_3 = \lambda \phi. \tag{3.11}$$

The eigenvalue λ has weight three under the $GL(2)$ action on V_6. It is also an eigenvalue, in the ordinary sense, of a 4 by 4 matrix corresponding to ψ by choosing a basis of V_3. If

[2]The exceptional sextics which can not be put in the canonical form are classified in [6].

$\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the four eigenvalues of ψ then the characteristic polynomial is a quartic[3]

$$\begin{aligned}\chi_\psi(\lambda) &= 8(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)\\ &= 8\lambda^4 + 4\mathcal{I}\,\lambda^2 - \mathcal{J}. \end{aligned} \tag{3.12}$$

Such quartic is canonically associated to every sextic, and has weight 12. The degree four invariant $\mathcal{J} = << \psi, \psi >_3, < \psi, \psi >_3 >_6$ appearing in (3.12) is the *catalectant* of the sextic. Its zero set is the closure of the locus of sextics expressible as the sum of three 6th powers [7]. Equivalently, the catalectant vanishes if the sextic admits an apolar cubic [8]. The latter result follows directly from setting $\lambda = 0$ in (3.12). If $\mathcal{I} = 0$ and $\mathcal{J} \neq 0$, then the roots of (3.12) form a *harmonic* set. This corresponds to a square on an equator in $\mathbb{CP}^1$.

4. Maximally separated quintics

In this section we shall consider a problem of recovering a sextic $P_1P_2\dots P_6$ with $\mathcal{I} = 0$ from a quintic. Let us project stereographically the sphere to the complex plane from one of the roots - say $P_6 = \infty$. Given four points $P_1, \dots, P_4$ on the plane, we can now look for a fifth point P_5 such that $\mathcal{I} = 0$. Rewriting the sextic (1.1) with the root corresponding to P_6 at ∞ as

$$\psi = (x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)$$

and comparing the coefficients of various powers of x we find that

$$\mathcal{I} = a\, {x_5}^2 + 2b\, x_5 + c, \tag{4.13}$$

where (a, b, c) are polynomials in $(x_1, \dots, x_4)$. This expression has at most two roots $x_5 \neq \infty$, thus given five distinct points on $\mathbb{CP}^1$ there exist at most two points such that the invariant $\mathcal{I}$ of the sextic defining the resulting six distinct points vanishes. Generically, if $a \neq 0$, there will be two such points.

Example. Consider a quintic corresponding to four points on the base of an equatorial regular pentagon, and a point at infinity. Equation (4.13) gives two possibilities for the sixth s.t. $\mathcal{I} = 0$, one of which is gives rise to a pentagonal pyramid (3.8)(Figure 2).

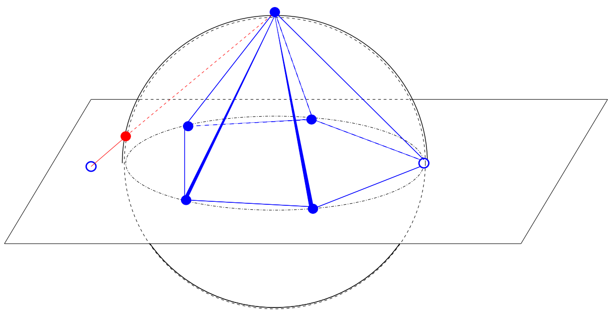

Figure 2. *Regular pentagon with $x_5 = 1$ and its companion with $x_5 = -3$.*

The next example is of a non-generic type

Example–Regular Octahedron. Assume that the six points P_i form the vertices of the regular octahedron, and project from one of its vertices P_6 which does not belong to the square. The resulting sextic is $\psi = x^5 - x$, and $\mathcal{I} = 1/3$. We can transform the four corners of the square $P_1, \dots, P_4$ to $\pm 1, \pm i$, and look for a point P_5 such that $\mathcal{I} = 0$. The resulting sextic is

$$(x^4 - 1)(x - x_5)$$

[3]This is the equation $\mathbf{R}$ on page 24 in [18], used to construct the canonical form (3.9).

and we find that $\mathcal{I} = 1/3$ for any value of x_5. Thus there is no P_5 which complements the five vertices of the square pyramid to an octahedron with $\mathcal{I} = 0$. This example corresponds to $a = b = 0, c \neq 0$ in (4.13).

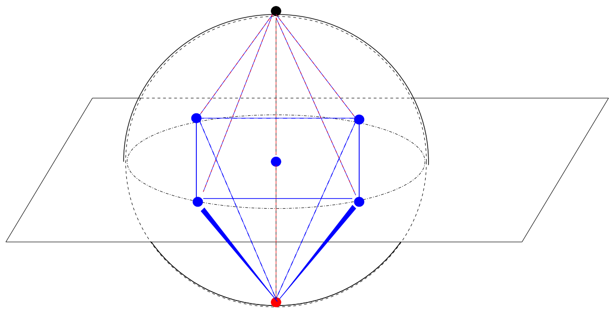

Figure 3. *Five maximally separated points.*

Definition 4.1. *We shall call five distinct points on the sphere maximally separated if* $\mathcal{I} \neq 0$ *for the sextic with six distinct roots resulting from any choice of the sixth point.*

Proposition 4.2. *Any maximally separated five points on the sphere can be transformed into vertices of a square pyramid by a Möbius transformation.*

Proof. Stereographicaly project from any of the five maximally separated points. The remaining four points are the distinct roots of a quartic

$$\gamma(x) = x^4 + 4b_1x^3 + 6b_2x^2 + 4b_3x + b_4.$$

Now consider the sextic ψ with one root at infinity given by $\psi = \gamma(x)(x - x_5)$. Computing (4.13) and equating a and b to zero gives $b_2 = b_1^2, b_3 = b_1^3$, and now $\mathcal{I} \neq 0$ unless all four roots of γ coincide which we have excluded. The roots of the resulting quartic are of the form $\alpha \pm \beta, \alpha \pm i\beta$, where α and β are complex numbers depending on b_1 and b_4. These roots are *harmonically separated*, with cross ratio equal to $-1, 2$ or $1/2$. Thus the corresponding points can be transformed to vertices of a square. We can use the remaining freedom in the Möbius transformations to set two roots to $\pm i$. Then either $b_1 = 0$ in which case the remaining two roots are at ± 1, or $b_1 = \pm 1$ in which case the roots are at $2 \pm i$ or $-2 \pm i$. Both squares are equivalent under the Mobiüs transformations. Thus the resulting five points form a square pyramid (Figure 3).

□

5. Motivation: Twistor theory of G_2 structures

We shall close the paper explaining the motivation of characterising binary sextics with $\mathcal{I} = 0$ coming from twistor theory of G_2 structures. For the sake of the following discussion twistor theory is a correspondence [15] between global algebraic geometry of curves in complex two–folds or three-folds, and local differential geometry on the moduli spaces of these curves.

Let $\mathcal{Z}$ be a complex two–fold with a family of rational curves $L_m \cong \mathbb{CP}^1$ parametrised by points $m \in M$, where M is some complex manifold. For any m, the embedding of L_m in $\mathcal{Z}$ is, to the first order, described by the normal bundle $N(L_m) = T\mathcal{Z}/TL_m$. This is a holomorphic line bundle $\mathcal{O}(k)$, for some integer k which we shall assume to be positive and even. The obstruction group $H^1(L_m, N(L_m)) = 0$ vanishes, and the Kodaira deformation theorem [11] states that there exists a canonical isomorphism

$$T_mM \cong H^0(L_m, N(L_m)) = \mathrm{Sym}^k(\mathbb{C}^{2\vee})$$

between vectors tangent to M and binary quantics of degree k. This is where the quadratic invariant (2.4) becomes relevant: if k is even, we can define a holomorphic conformal structure $[g]$ on M by declaring a vector field $U \in \Gamma(TM)$ to be null iff the corresponding quantic has $\mathcal{I}_2(U) = 0$. This is a quadratic condition, and the holomorphic light-cone of any point in M is a surface of co–dimension one in M, so $[g]$ is indeed well defined.

Let us now restrict to the case $k = 6$, where vector fields correspond to binary sextics, and $\dim_{\mathbb{C}} M = 7$, [4] . In this case there exists a skew–symmetric three–form $\Psi \in \Lambda^3(M)$ given by

$$\Psi(U, V, W) =<< U, V >_3, W >_3$$

where $<, >_3$ is the third transvectant (2.5) of two binary sextics, and we use the same symbols to denote vector fields and corresponding sextics. This three–form is compatible with $[g]$ is a sense that

$$(U \lrcorner \Psi) \wedge (U \lrcorner \Psi) \wedge \Psi = 0, \quad \text{iff} \quad \mathcal{I}_2(U) = 0.$$

The invariants $\mathcal{I}_2$ and Ψ have weights six and nine respectively, so changing a metric $g \in [g]$ yields $g \rightarrow \Omega^6 g, \Psi \rightarrow \Omega^9 \Psi$ where Ω is a non–vanishing function on M. Therefore the structure group of TM reduces to $GL(2, \mathbb{C}) \subset {G_2}^{\mathbb{C}} \times \mathbb{C}$, where ${G_2}^{\mathbb{C}}$ is the complexification of the exceptional Lie group G_2.

There are only few known examples of this construction which lead to positive–definite G_2 structures on Riemannian manifolds $M_{\mathbb{R}}$. Bryant's weak G_2 holonomy metric [1] on $M_{\mathbb{R}} = SO(5)/SO(3)$ arises from a family of $Sp(4)$ invariant rational sextics [5]. Another example corresponds to $\mathcal{Z} = \mathbb{CP}^2$, and M being the homogeneous space $PSL(3, \mathbb{C})/\mathbb{C}^*$ of ternary cuspidal cubics in $\mathcal{Z}$. The cuspidal cubics are rational, but singular. The Kodaira theory nevertheless applies as the contact lifts of the cuspidal cubics to $T\mathbb{CP}^2$ are smooth. There exist three real slices of M, one of which is $M_{\mathbb{R}} = SU(2, 1)/U(1)$. In [3] it is shown that $M_{\mathbb{R}}$ admits a G_2 structure which is co–calibrated [3].

6. Conclusions

We have found a geometric interpretation of vanishing of the quadratic invariant associated with a binary sextic, and more generally with any binary quantic of even degree. The result should have an interpretation in the theory of hyperelliptic curves, as projective equivalence classes of binary sextics with distinct roots correspond to points on the moduli space of genus two algebraic curves. Igusa [10] considered a zero locus of various invariants for sextics, but the case of vanishing quadratic invariant has not been analysed in this work.

The problem addressed in this paper can also be reformulated in the context of the representation theory of $SO(3, \mathbb{C})$ which is related to $SL(2, \mathbb{C})$ by the homomorphism $SL(2, \mathbb{C})/\mathbb{Z}_2 \cong SO(3, \mathbb{C})$. The isomorphism $\mathbb{C}^3 = \text{Sym}^2(\mathbb{C}^{2\vee})$ identifies complex vectors in $\mathbb{C}^3$ with symmetric 2 by 2 matrices with complex coefficients. The *null* vectors correspond to rank one matrices, and this gives rise to a holomorphic conformal structure on $\mathbb{C}^3$. The seven–dimensional space $\text{Sym}^6(\mathbb{C}^{2\vee})$ of binary sextics is identified with a subspace of $\text{Sym}^3(\mathbb{C}^{3\vee})$ which consist of harmonic ternary cubics, i. e. those forms $\Psi_{ijk} Z^i Z^j Z^k, i, j, k = 1, \ldots, 3$ which satisfy $\delta^{ij} \Psi_{ijk} = 0$. Hitchin [9] showed that a generic harmonic ternary cubic in $\mathbb{CP}^2$ passes through two sets of six points corresponding to six axes of a regular icosahedron in $\mathbb{C}^3$. In this formulation the quadratic invariant is given by the norm of the cubic Ψ_{ijk} taken w. r. t. the conformal structure defined above. It is however not clear what is the geometric meaning of its vanishing in terms of Hitchin's icosahedron.

Appendix: Two component spinors

A convenient way to represent binary quantics and the associated invariants uses the two–component spinor notation [17]. Let the capital letters $A, B, \dots$ denote indices taking values 0 and 1. The general quantic (2.3) is represented by a *symmetric spinor* of valence $2n$. The Fundamental Theorem of Algebra states that any such spinor factorises into valence one spinors

$$\psi_{AB\cdots C} = \alpha_{(A}\beta_B \dots \gamma_{C)},$$

where the round brackets on the RHS denote the symmetrisation. The binary quantic (2.3) is then given by

$$\begin{aligned}\psi &= \psi_{AB\dots C}\pi^A\pi^B \dots \pi^C \\ &= (\alpha_0 x + \alpha_1 y)(\beta_0 x + \beta_1 y)\dots(\gamma_0 x + \gamma_1 y) \sim (x - x_1)(x - x_2)\dots(x - x_{2n}),\end{aligned}$$

where $\pi^0 = x, \pi^1 = y$ and $\psi_0 = \psi_{00\dots0}, \quad \psi_1 = \psi_{10\dots0}, \quad \dots, \quad \psi_{2n} = \psi_{11\dots1}$. Thus the complex numbers $x_1 = -\alpha_1/\alpha_0, x_2 = -\beta_1/\beta_0, \dots, x_{2n} = -\gamma_1/\gamma_0$ are the roots of the inhomogeneous polynomial of degree $2n$ obtained by setting $y = 1$. The invariant (2.4) is in this notation given by

$$\mathcal{I} = \psi_{AB\dots C}\psi^{AB\dots C},$$

where the indices are lowered by the anti–symmetric matrix ε_{AB} with $\varepsilon_{01} = 1$, so that

$$\psi_{AB\dots C} = \psi^{PQ\dots R}\varepsilon_{PA}\varepsilon_{QB}\dots\varepsilon_{RC}.$$

The kth transvectant (2.5) is

$$< \psi, \phi >_k = \varepsilon^{A_1B_1}\dots\varepsilon^{A_kB_k}\psi_{A_1\dots A_kA_{k+1}\dots A_m}\phi_{B_1\dots B_kB_{k+1}\dots B_n}\pi^{A_{k+1}}\dots\pi^{A_m}\pi^{B_{k+1}}\dots\pi^{B_n}.$$

Using the spinor notation gives a simple proof of the following algebraic interpretation of the condition $\mathcal{I} = 0$

Lemma .1. *An even degree quantic $\psi \in V_{2n}$ with distinct roots is a sum of $(2n)^{th}$ powers of its factors iff $\mathcal{I}(\psi) = 0$.*

Proof. Let $\alpha_A, \beta_A, \dots, \gamma_A$ be homogeneous coordinates of the points in $\mathbb{CP}^1$ corresponding to the roots of ψ so that the condition stated in the Lemma becomes[4]

$$\alpha_{(A}\beta_B \dots \gamma_{C)} = c_1\ \alpha_A\alpha_B\dots\alpha_C + c_2\ \beta_A\beta_B\dots\beta_C + \dots + c_{2n}\ \gamma_A\gamma_B\dots\gamma_C, \tag{A1}$$

where $c_1, c_2, \dots, c_{2n}$ are some constants determined by the roots. The condition $\mathcal{I} = 0$ is then equivalent to

$$\frac{\partial^{2n}\psi}{\partial\alpha\partial\beta\dots\partial\gamma} = 0, \quad \text{where} \quad \frac{\partial}{\partial\alpha} := \alpha^A\frac{\partial}{\partial\pi^A} \text{ etc.}$$

Thus

$$\frac{\partial^{2n-1}\psi}{\partial\beta\dots\partial\gamma} = \tilde{c}_1(\alpha_A\pi^A)$$

for some constant $\tilde{c}_1$, as both sides are homogeneous of degree one in π^A. The successive $(2n-2)$ integrations yield (A1), with $c_1 = (2n)!^{-1} {<\alpha, \beta>_1}^{-1} \dots {<\alpha, \gamma>_1}^{-1}(\tilde{c}_1)$ etc.

□

[4]Formula (A1) holds for any binary quantic of odd degree, as then $\mathcal{I}$ vanishes identically.

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