# Realizability Interpretations for Intuitionistic Set Theories

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The candidate confirms that the work submitted is his/her own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

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# Abstract

The present thesis investigates the validity of some interesting principles such as *the Axiom of Choice*, **AC**, in the general extensional realizability structure $\mathtt{V}(\mathcal{A})$ for an arbitrary *applicative structure*, $\mathcal{A}$, generalising the result by *Rathjen* in [28] established for the specific realizability model $\mathtt{V}(\mathcal{K}_1)$, *the Fan Theorem*, **FT**, and *the principle of Bar Induction*, **BI**, in the particular realizability structures over *the Graph Model*, $\mathtt{V}(\mathbf{P}\omega)$, and over *the Scott $D_\infty$ Model*, $\mathtt{V}(D_\infty)$, since, in the literature, little is known about these realizability models and most investigations are carried out in the realizability models built over Kleene's first and second models.

After an introduction and some background material, given in the first two chapters, I introduce the notion of extensional realizability over an arbitrary applicative structure, $\mathcal{A}$, and I show that variants of the axiom of choice hold in $\mathtt{V}(\mathcal{A})$. Next, the focus switches from considering the general realizability structure $\mathtt{V}(\mathcal{A})$ generated on an arbitrary applicative structure, $\mathcal{A}$, to the specific realizability universes, $\mathtt{V}(D_\infty)$ and $\mathtt{V}(\mathbf{P}\omega)$ to investigate some interesting properties including the validity of **FT** and **BI** in these universes.

For the remainder of the thesis, a proof of the soundness of realizability with truth, as it leads to different applications than that without truth, for the theories **CZF** and **CZF + REA**, is given and an investigation of many choice principles is carried out in the truth realizability universe $\mathtt{V}^*(\mathcal{A})$ for an arbitrary applicative structure, $\mathcal{A}$.

# Contents

# Chapter 1

# Introduction

Realizability interpretations are of fundamental importance to the study of intuitionistic set theories. They are used to extract some useful computational information from constructive proofs. This thesis mainly employs these realizability semantics to work out whether some interesting principles hold in the realizability structures, $\mathtt{V}(\mathcal{A})$ for the general extensional realizability and $\mathtt{V}^*(\mathcal{A})$ where realizability is combined with truth in the background model $\mathtt{V}$, for arbitrary or specific *applicative structure* $\mathcal{A}$.

Firstly, the general realizability structure, $\mathtt{V}(\mathcal{A})$ is introduced to investigate a generalization of the result by *Rathjen*, given in [28] establishing the validity of variants of the Choice axiom in $\mathtt{V}(\mathcal{A})$, to an arbitrary applicative structure, $\mathcal{A}$.
Next the *Brouwerian Principles* of the Fan Theorem, **FT**, and Bar Induction, **BI**, are addressed in the specific realizability universes $\mathtt{V}(\mathbf{P}\omega)$ and $\mathtt{V}(D_\infty)$, where $\mathbf{P}\omega$ is the *Plotkin-Scott Graph model* and $D_\infty$ is the well known *Scott model*. Finally, the realizability with truth structure $\mathtt{V}^*(\mathcal{A})$ that was introduced in [30], is used to show that the axioms of the theories **CZF**, and **CZF** + **REA** hold in $\mathtt{V}^*(\mathcal{A})$. Moreover, this realizability structure also validates varity of choice principles if they hold in $\mathtt{V}$.

## 1.1 A brief history of intuitionism and realizability semantics

In this section, we give a brief history of intuitionistic logic and realizability interpretations. More details can be found in [41], [42], [44], [21] and [30].
At the very beginning of the 20th century and in the early days of logic, *Brouwer* developed his idea of intuitionism. As we know from papers and text books, Brouwer, the father of intuitionism, participated little in intuitionistic logic but he guided his successors [44].

Brouwer believed that mathematical objects are mentally constructed and emphasized that they are only meaningful if one can understand them mentally. Intuitionistis believe that some laws of traditional logic are untrustworthy. More precisely, they reject the *Law of Excluded Middle*, **LEM**, $(p \vee \neg p)$. Let us explain this by a way of an example. Let $G$ be *Goldbach's conjecture* that every even number greater than or equal 6 is a sum of two odd primes. By a quick check for small numbers his conjecture is confirmed. However, it is still difficult to prove the conjecture. The current knowledge of mathematics has not provided a proof of $G$ nor of $\neg G$. The question is now, *can we confirm* $G \vee \neg G$*?* If yes, then a construction that decides which of the two alternatives holds and gives a proof of it should be provided. Of course, in this case, we cannot picture such a construction and therefore have no base for admitting that $G \vee \neg G$ is correct.

*Glivenko* and *Kolmogorov* were the first to considered the logic of intuitionism more formally. In 1928 Heyting independently formalized intuitionistic predicate logic and theories of arithmetic and set theory. For details see [39].

A core method in the study of intuitionistic theories is *Realizability interpretations* which have been developed over the last 70 years with many different facts across different areas of mathematics, logic and computer science [44]. The study of these interpretations was started by *Kleene* [16] in 1945. It appears in the literature that *Tharp* [38] was the first to give a realizability definition for set theory. His realizers were codes for $\Sigma_1$ definable partial class functions. This form of realizability is in fact a direct extension of *Kleene's* 1945 number realizability in that $e \Vdash \forall x\phi(x)$ reads as $e$ realizes $\forall x\phi(x)$, is the index $e$ of a $\Sigma_1$ partial function with $\{e\}(x) \Vdash \phi(x)$ , for all $x$. Likewise, $e \Vdash \exists x\phi(x)$ if $e$ is of the form $\langle a, e' \rangle$ where $e'$ is a realizer for $\phi(a)$.

In defining realizability, *Kleene* aimed at expressing the concept that $e$ provides a computational witness for the constructive content of $\phi$. Following his version several other notions of realizability arose. A considerably different realizability notion was proposed by *Kreisel and Troelstra* [20]. They defined a notion of realizability for second order *Heyting Arithmetic*. The clauses of their definition that are related to second order quantifiers are:

$$e \Vdash \forall X\phi(X) \Leftrightarrow \forall X\, e \Vdash \phi(X), e \Vdash \exists X\phi(X) \Leftrightarrow \exists X\, e \Vdash \phi(X).$$

Note that the realizing numbers just pass through quantifiers and thus this notion of realizability does not seem to give any constructive meaning to second order quantifiers, from which one infers that the collection of sets of natural numbers is generically visualized. As stated above, intuitionistically the only way to establish the truth of the formula $\forall X\phi(X)$ is to provide a proof of it. A collection of objects is said to be generic when no member of the collection has the power to make differences to a proof. *Friedman* applied the latter notion of realizability to systems of higher order arithmetic [13]. Moreover, *Kleene's slash* [17, 18] notion of realizability was extended by *Myhill* [23, 24] to several intuitionistic set theories.

## 1.2 Intuitionistic Zermelo-Fraenkel Set Theory, IZF

**IZF** is a constructive set theory. One approach to **IZF** is to begin with a classical theory and throw out the *Law of Excluded Middle*, **LEM**, and any axioms that imply it and see what is left over. So, we start with classical Zermelo-Fraenkel set theory, **ZF** and replace classical predicate logic with intuitionistic predicate logic. But we also need to modify the *Foundation Axiom* since it implies **LEM**.

The Axiom of Foundation asserts that $\in$ is well-founded *i.e.*

$$x \neq \emptyset \Rightarrow \exists y(y \in x \wedge y \cap x = \emptyset)$$

and this is replaced by transfinite induction on $\in$ "the set induction schema"

$$\forall x((\forall y \in x)\phi(y) \rightarrow \phi(x)) \longrightarrow \forall z\phi(z).$$

This is classically equivalent to the axiom of foundation, capturing the idea that all sets are built up from the empty set.

The Axiom of Foundation is the only axiom of **ZF** that intuitionistically implies the **LEM** since $\mathbf{ZF}^{-} \subseteq \mathbf{IZF}$, where $\mathbf{ZF}^{-}$ is **ZF** without foundation and **IZF** does not prove **LEM** as **IZF** + **CT**, where **CT** is *Church's Thesis*, are consistent but $\mathbf{IZF} + \mathbf{LEM} \vdash \neg CT$. However there is one more axiom which needs to be discussed, the *Axiom of Replacement*.
Two classically equivalent forms of *Replacement* are:

(a) $\forall x \in a \exists! y\phi(x, y) \longrightarrow \exists b \forall x \in a \exists y \in b\phi(x, y)$ *Replacement* .

(b) $\forall x \in a \exists y\phi(x, y) \longrightarrow \exists b \forall x \in a \exists y \in b\phi(x, y)$ *Collection* .

Friedman has shown that replacement $\not\Rightarrow$ collection in intuitionistic set theories, and since we aim to discard as little as possible of **ZF**, we shall formulate **IZF** with collection instead of replacement.
So, the precise axiomatic formulation of **IZF** is as follows:

(i) *Extensionality* $\forall x \forall y[\forall z(z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$

(ii) *Pairing* $\forall x \forall y \exists z(x \in z \wedge y \in z)$

(iii) *Union* $\forall x \exists y \forall z \forall w \quad [(w \in z \wedge z \in x) \Rightarrow w \in y]$

(iv) *Infinity* $\exists x \forall u[u \in x \leftrightarrow (\emptyset = u \vee (\exists v \in x) u = v + 1)]$
where $v + 1 = v \cup \{v\}$

(v) *Separation* $\forall x \exists y \forall z[z \in y \leftrightarrow z \in x \wedge \phi(z)]$ for each formula $\phi$ with y not free in $\phi(z)$.

(vi) *Powerset* $\forall x \exists y \forall z[z \in y \leftrightarrow \forall s(s \in z \to s \in x)]$

(vii) Instead of Replacement **IZF** has **collection**.

(viii) Instead of Foundation **IZF** has the **Set Induction Schema**. [8]

## 1.3 Constructive Zermelo-Fraenkel Set Theory, CZF

The general concept of *Constructive Set Theory* arose in a paper by *Myhill* 1975 (see [25]) where a particular axiom system *CST* is introduced. He aimed at developing constructive set theory in order to align the principles with Bishop's notion of what functions and sets are. Furthermore, he aimed to make "these principles to be such as to make the process of formalization completely trivial, as it is in the classical case" ([25], p. 347) [32].

**CZF** is a constructive theory based on intutionistic first order logic with equality and has the same first order language as that for classical **ZF** with $\in$ being the only non-logical symbol. The non-logical axioms of **CZF** are *Extensionality*, *Pairing*, *Union*, *Set Induction Scheme* and *Infinity* in their usual forms in addition to the following axiom schemas:

(i) **Bounded Separation Scheme**

$$\forall x \exists y \forall a[a \in y \leftrightarrow a \in x \wedge \phi(a)]$$

for any bounded formula $\phi$ with y not free in $\phi$.

(ii) **Subset Collection Scheme**
$\forall x \forall y \exists z \forall u [(\forall a \in x)(\exists b \in y)\psi(a,b,u) \rightarrow (\exists c \in z)((\forall a \in x)(\exists b \in c)\psi(a,b,u)$
$\wedge (\forall b \in c)(\exists a \in x)\psi(a,b,u))]$ for any formula $\psi$.

(iii) **Strong Collection Scheme**
$\forall x[(\forall a \in x)\exists b \phi(a,b) \rightarrow \exists y[(\forall a \in x)(\exists b \in y)\phi(a,b) \wedge (\forall b \in y)(\exists a \in x)\phi(a,b)]]$
for all formulas $\phi$. [1], [2], [3] and [28].

## 1.4 Applicative Structures

To define a realizability semantics, a notion of realizing functions must be available. An especially elegant approach to realizability is built on domains of computation allowing for self application and recursion that have been variably known as *applicative structures*, *partial combinatory algebras* or *Schönfinkel algebras*. These domains are best constructed as models of a theory **PCA** described in the next chapter [31].

It is usually useful to endow *pcas* with additional structure such as the natural numbers, pairing and definition by integer cases. These tools can in fact be constructed in any *pca*, for details see ([8], Theorem 4.2.9) and for convenience we also include the proof in the next chapter.

## 1.5 Axiom of Choice, AC

The axiom of choice is distinguished from the other axioms of set theory, by the fact of being the only one that is ever mentioned in everyday mathematics and thus it is worth devoting part of this thesis to investigating the axiom in the realizability models $\mathtt{V}(\mathcal{A})$ and $\mathtt{V}^*(\mathcal{A})$. Discussions about the axiom of choice date back to the early part of the 20th century. By using the axiom of choice *Zermelo* proved (in 1904), that every set can be well-ordered. He also argued that the axiom of choice is a constructive principle, however; prominent analysts of the day criticized his view.

In constructive mathematics, unsurprisingly the axiom of choice has an ambiguous status. It is considered to be a direct conclusion of the constructive semantics of the quantifiers, *i.e.* every proof of $(\forall a \in x)\,(\exists b \in y)\ \phi(a, b)$ must produce a function $f : x \longrightarrow y$ such that $(\forall a \in x)\ \phi(a, f(a))$. In contrast to this, it has been shown that adding the full axiom of choice to extensional constructive set theories results in constructively rejected instances of *the Law of Excluded Middle* (see [10] and [26] Proposition 3.2).

Generally, a proof of a statement of the form $(\forall a \in x)\ (\exists b \in y)\ \phi(a, b)$ in an extensional intuitionistic set theory only supplies a function $F$ which given input a proof $p$ that witnesses $a \in x$, produces $F(p) \in y$ and a proof of $\phi(a, F(p))$. So essentially such a function cannot be recognised as a function of $a$ alone. Thus, choice holds over sets that possess *a canonical proof function* where a constructive function $g$ is a canonical proof function for $x$ if for all $a \in x$, $g(a)$ is a constructive proof for $a \in x$. Sets with canonical proof functions naturally "built-in" are called *bases* (see [40], p. 841).

In this thesis, we investigate many forms of the axiom of choice that have been considered to be constructive. These forms include countable and dependent choices as well as relativized dependent choice in addition to some weaker form of the axiom. A stronger form of the axiom of choice, which will also be discussed in this thesis, is *the presentation axiom*, **PAx**, which asserts that "every set is the surjective image of a set over which the axiom of choice holds" [26].

Countable choice and dependent choice follow from **PAx** and it is known to be validated by several realizability semantics. In the present thesis we also establish their validity in the general realizability structure, $\mathtt{V}(\mathcal{A})$, for extensional intuitionistic set theories bulit on an arbitrary applicative structure, $\mathcal{A}$, assuming that they hold in the background theory $\mathtt{V}$. We also establish a simillar result in the truth realizability universe, $\mathtt{V}^*(\mathcal{A})$.

## 1.6 Brouwerian Principles

In *Brouwerian Mathematics*, the principle of *Bar Induction*, **BI**, occupies a central place. The other principle at the core of Brouwerian mathematics is the *Fan Theorem*, **FT**. *Brouwer* appeals to his principle of **BI** to justify **FT** [32].

Little is known of whether these principles hold true in the realizability interpretations of intuitionistic theories. In this thesis we give a detailed introduction to these principles and investigate what the general extensional realizability structures over $\mathbf{P}\omega$ and $D_\infty$ models can say about their validity.

## 1.7 Overview of the thesis

Having set out the general motivation, we now proceed to describe what is contained in the thesis.

We start in chapter 2 by giving some background on applicative structures. As well as fixing notation, this material is needed for the subsequent chapters. A summary of useful definitions, facts and properties of *applicative structures* in general is given and we treat in particular *the Graph Model*, $\mathbf{P}\omega$, and *Scott* $D_\infty$ *Models* as applicative structures.

The third chapter is concerned with an examination of several forms of the axiom of choice in the general realizability model built over an arbitrary applicative structure, $\mathcal{A}$, for extensional intuitionistic set theories, $\mathtt{V}(\mathcal{A})$. We will show that certain forms of the axiom of choice hold in these realizability structures. In particular, this is a generalisation of the result by *Rathjen* [28] that validates those choice principles over $\mathtt{V}(\mathcal{K}_1)$.

In Chapter 4 we will also show that there is an infinite base $A \subseteq \mathtt{V}(D_\infty)$ such that $A$ is isomorphic to its own function space.

Chapter 5, provides a clear picture of how the world $\mathtt{V}(\mathcal{A})$ of the realizability universe over an arbitrary applicative structure $\mathcal{A}$, thinks of *Baire Space*. This is needed to determine which functions from $\mathbb{N}$ to $\mathbb{N}$ can be represented in $\mathbf{P}\omega$ and $D_\infty$ which is used later on to establish the validity of *Bar Induction*, **BI** and the *Fan Theorem*, **FT** in the special realizability structures $\mathtt{V}(\mathbf{P}\omega)$ and $\mathtt{V}(D_\infty)$.

The final two chapters are devoted to looking at the realizability (class) structure, $\mathtt{V}^*(\mathcal{A})$, where we combine extensional realizability for intuitionistic set theory with truth in the background universe, $\mathtt{V}$. In the literature, [30] and [33] this form of realizability has been used to show that almost all "reasonable" intuitionistic set theories have the disjunction and numerical existence properties and are closed under Church's rule (and more). In these papers only the first Kleene algebra is being used. An interesting question therefore is whether other applicative structures can be put to use to establish further derived rules for intuitionistic set theories. Berg and Moerdijk [43] have employed sheaf models to show that specific intuitionistic set theories are closed under the bar rule and the fan rule. In these chapters we will give a generalisation of the result by *Rathjen* (see [30], Theorem 6.1) of the soundness theorem for **CZF** and **CZF + REA** in the particular realizability universe $\mathtt{V}^*(\mathcal{K}_1)$ built over the *first Kleene* algebra, $\mathcal{K}_1$ where we show that the theories **CZF** and **CZF + REA** are also sound when moving from $\mathtt{V}^*(\mathcal{K}_1)$ to $\mathtt{V}^*(\mathcal{A})$ for any applicative structure, $\mathcal{A}$. We also show that various choice principles hold in the realizability model $\mathtt{V}^*(\mathcal{A})$ which also extends the result established by *Rathjen* in [33] for the particular realizability structure $\mathtt{V}^*(\mathcal{K}_1)$.

# Chapter 2

# Partial Combinatory Algebras

In this chapter, we set out definitions and review some basic facts and properties about *pcas* and *applicative structures* in general, the graph model, $\mathbf{P}\omega$, and Scott's $D_\infty$ models in particular since the main concerns of this thesis are to give realizability models built on these *applicative structures*. Most of the material of this chapter was covered in [4], [5], [8], [14], [29], [31] and [36].

## 2.1 Partial Combinatory Algebras (*pcas*)

For any notion of realizability, the point of departure is a domain of computation known as *a partial combinatorial algebra pca*, *pca* also known by the name *"applicative structure"*. The idea of using these as domains of computation came from a remark of *Feferman* in [12]. A *pca* can be generally viewed as an "abstract machine" in which we can perform certain computations.

**Definition 2.1.1.** *A pca is a structure* $(A, \bullet)$ *where* $\bullet$ *is a partial binary operation on* $A$ *with* $A$ *having at least two elements and there are elements* $\mathbf{k}$ *and* $\mathbf{s} \in A$ *such that :*

*(i)* $\mathbf{k} \bullet x \bullet y = x$

*(ii)* $\mathbf{s} \bullet x \bullet y \downarrow$

*(iii)* $\mathbf{s} \bullet x \bullet y \bullet z \simeq x \bullet z \bullet (y \bullet z)$ *for all* $x, y, z \in A$*, where* $\simeq$ *means if both sides are defined then they are equal.*

$\mathbf{k}xy$ is shorthand for $(\mathbf{k} \bullet x) \bullet y$ and we assume association to the left. [31]

**Examples.**

(a) Kleene's first model, $\mathcal{K}_1$, is the best known *pca*. $\mathcal{K}_\mathbf{1} = (\mathbb{N}, \bullet)$ in which the universe of computation $|\mathcal{K}_1| = \mathbb{N}$ and application $\bullet$ is Turing machine application, $\{a\}(b) \simeq z$, where $\{a\}$ is the partial computable function encoded by $a$ and if the computations terminate on input $b$, it gives an output $z$.

(b) Kleene's second model, $\mathcal{K}_2$, is the *pca* with universe *Baire Space* $\mathbb{N}^\mathbb{N}$. To describe this *pca*, we first review some terminology.

**Definition 2.1.2.** *Let $\alpha, \beta, ...$ range over functions in $\mathbb{N}^\mathbb{N}$ and let $\sigma, \tau, ...$ range over finite sequences of natural numbers. Assume that each $n \in \mathbb{N}$ codes a finite sequence $\sigma$.*

*(i) For finite sequences $\sigma$ and $\tau$, write $\sigma \subset \tau$ for $\sigma$ is an initial part of $\tau$, $\sigma * \tau$ for the list concatenation of $\sigma$ and $\tau$ and $\langle\ \rangle$ for the empty sequence.*

*(ii) If $\tau$ is the sequence $\langle n_0, ..., n_k\rangle$, then the length of $\tau$ is $k+1$ and is denoted by $lh(\tau)$.*

*(iii) For $\alpha \in \mathbb{N}^\mathbb{N}, \alpha[0] = \langle\ \rangle$ and $\alpha[n] = \langle \alpha(0), ..., \alpha(n-1)\rangle$ for $n > 0$.*

*(iv) For $\alpha \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$, $\langle n\rangle * \alpha$ produces a new function $\beta \in \mathbb{N}^\mathbb{N}$ with $\beta(0) = n$ and $\beta(k+1) = \alpha(k)$.*

Moreover, the following two operations on $\mathbb{N}^\mathbb{N}$ are required for the definition of application on $\mathcal{K}_2$.

**Definition 2.1.3.** *Let $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ and $n, m, l \in \mathbb{N}$. Define,*

*(i) $\alpha \diamond \beta = m \iff \exists n[\alpha(\beta[n]) = m + 1 \wedge \forall l < n\, \alpha(\beta[l]) = 0]$.*

*(ii) $(\alpha|\beta)(n) = \alpha \diamond (\langle n\rangle * \beta)$.*

*Note that the function $\alpha|\beta$ is not total so we cannot define application on $\mathbb{N}^\mathbb{N}$ by $\alpha|\beta$. Thus, application is defined as:*

$$\alpha \bullet \beta = \gamma \iff \forall n\ (\alpha|\beta)(n) = \gamma(n)$$

*[29]*

**Definition 2.1.4.** *A pca is said to be* extensional *if it satisfies*

$$\forall a, b[\forall x(ax = bx) \rightarrow a = b]$$

.

### 2.1.1 The Theory PCA

axioms

(i) $\mathbf{k}xy = x$

(ii) $\mathbf{s}xy\downarrow$

(iii) $\mathbf{s}xyz \simeq xz(yz)$ for all $x, y, z \in A$

## 2.2 The Lambda Calculus

The *lambda calculus* is a theory of *functions* as *formulas.* It is a system for manipulating functions as *expressions.* [36]
For example, the function $x \mapsto x^3$ evaluated at $x = 5$ is written in the lambda calculus as the formula/expression $(\lambda x.x^3)(5)$. The lambda calculus is a *formal language* whose expressions are called *lambda terms.*

**Definition 2.2.1.** *Let $\mathcal{V}$ be an infinite set of variables, denoted by $x_i$ for $i \in \omega$. The set of lambda terms, $\Lambda$, is defined inductively as follows:*

*(i)* $x_i \in \Lambda$ *where* $x_i \in \mathcal{V}$.

*(ii) If* $M \in \Lambda$ *then,* $(\lambda x.M) \in \Lambda$.

*(iii) If* $M, N \in \Lambda$ *then,* $(MN) \in \Lambda$.

**Notation.**

(1) For variables $x, y$, and a term $M$, write $M\{y/x\}$ for the result of renaming $x$ as $y$ in $M$, where $y$ replaces all occurrences of $x$ and only applied if $y$ does not occur in $M$.

(2) Write $M[N/x]$ for the result of substituting the variable $x$ by the lambda term $N$ in $M$, where $x$ has to be free and if , for example, $M \equiv \lambda x.yx$ and $N \equiv \lambda z.xz$ then, we have to rename the bound variable $x$ in $M$ before the substitution to avoid the unintended capture of free variables.

**Definition 2.2.2** (**$\alpha$-equivalence**). *$\alpha$-equivalence is denoted by $=_\alpha$ and is defined to be the smallest congruence relation on lambda terms such that for all terms $M$ and for all variables $y$ with $y$ not occurring in $M$*

$$\lambda x.M =_\alpha \lambda y.(M[y/x])$$

The process of evaluating lambda terms by plugging arguments into functions is called $\beta$-reduction, and is formally defined as follows:

$$\beta := \{(\lambda x.\, M)N, M[N/x]) : M, N \text{ are } \lambda - \text{ terms } \}$$

**Definition 2.2.3.** *Let $M$ and $N$ be $\lambda$-terms then, we write $M =_\beta N$ if $M$ is reducible to $N$ using $\beta$-reduction.* *[36]*

**Theorem 2.2.4** (**Introduction of $\lambda$-terms**). *Let $t$ be a term and $x$ be a variable. Then, for any $t$ and $x$, one can construct a term, $\lambda x.t$, such that* $\mathbf{PCA} \vdash \lambda x.t \downarrow$ *and* $\mathbf{PCA} \vdash (\lambda x.t)x \simeq t$*, where the free variables of $\lambda x.t$ are those of $t$ excluding $x$.* *[8]*

**Proof.** (i) $\lambda x.x = \mathbf{I} = \mathbf{skk} \downarrow$ by the axioms of **PCA**.

(ii) $\lambda x.t$ is $\mathbf{k}t$ for $t$ constant or variable different from $x$.

(iii) $\lambda x.uv$ is $\mathbf{s}(\lambda x.u)(\lambda x.v) \downarrow$, since $\lambda x.u$ and $\lambda x.v$ are defined using the induction hypothesis, $\mathbf{s}xy \downarrow$ is an axiom of **PCA**, and $\mathbf{s}(\lambda x.u)(\lambda x.v) \simeq (\lambda x.u)x((\lambda x.v)x)x \simeq uv$. [8]

□

**Theorem 2.2.5 (The Recursion Theorem).** *There is a term* $\mathbf{R}$ *such that* **PCA** *proves:* $\mathbf{R}f\downarrow$ *and* $[g = \mathbf{R}f \to \forall x(gx \simeq fgx)]$

**Proof.** Take $\mathbf{R}$ to be $\lambda f.tt$ where $t = \lambda yx.f(yy)x$. Then,

$$\begin{aligned} \mathbf{R}f &\simeq (\lambda f.tt)f \\ &\simeq tt \end{aligned}$$

But,

$$\begin{aligned} tt &\simeq (\lambda y.(\lambda x.f(yy)x))t \\ &\simeq (\lambda x.f(tt))x \\ &\simeq f(tt) \end{aligned}$$

So, if $\mathbf{R}f\downarrow$ and $g = \mathbf{R}f$ then,

$$g \simeq tt \simeq f(tt) \simeq fg$$

Thus, $gx = fgx$. □

**Lemma 2.2.6** ([8], exercise 4, p. 107)**.** *The axiom* $\mathbf{k} \neq \mathbf{s}$ *in* **PCA** *can be replaced by* $\exists a, b(a \neq b)$*.*

**Proof.** Assume towards a contradiction that $\mathbf{k} = \mathbf{s}$. Then we have:
$\mathbf{k}xyz = xz$ as $\mathbf{k}xy = x$ and since $\mathbf{s}xyz \simeq xz(yz)$, we conclude that $xz = xz(yz)$ Now, let $x = z = \lambda v.v$ and let $y = \mathbf{k}u$ for any $u$. Then, $\mathbf{k}xyz = xz = \lambda v.v(\lambda v.v) = \lambda v.v$ and $\mathbf{s}xyz = xz(yz) = \lambda v.v(\mathbf{k}uz) = \mathbf{k}uz = u$ for any $u$.
So, $u = \lambda v.v$ for all $u \in pca$.
Thus, $\forall a, b(a = b)$ contrary to the assumption. □

**Remark** The models of the theory **PCA** are precisely the *pcas*.
**PCA** is an elegant theory but we need more structure in our theory of rules. In particular we need pairing, natural numbers, and some form of definition by cases. To attain this we extend **PCA** to the theory **PCA**$^+$. [8]

### 2.2.1 The Theory $\mathbf{PCA}^+$

language

that of **PCA** augmented by additional constants $\mathbf{p}$, $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{d}$, $\mathbf{S_N}$, $\mathbf{P_N}$, 0 and the predicate symbol $\mathbf{N}$.

axioms

those of **PCA** together with the following

(i) $\mathbf{p}xy\downarrow$ and $\mathbf{p_0}(\mathbf{p}xy) = x$ and $\mathbf{p_1}(\mathbf{p}xy) = y$.

(ii) Axioms for natural numbers:

(a) $\mathbf{N}(0)$ and $\forall x(\mathbf{N}(x) \to \mathbf{N}(\mathbf{S_N}(x))$ and $\mathbf{P_N}(\mathbf{S_N}(x)) = x$ and $\mathbf{S_N}(x) \neq 0)$.

(b) $\forall x(\mathbf{N}(x))$ and $x \neq 0 \to \mathbf{N}(\mathbf{P_N}(x))$ and $\mathbf{S_N}(\mathbf{P_N}(x)) = x$.

(iii) Definition by cases:

(a) $\mathbf{N}(a)$ and $\mathbf{N}(b)$ and $a = b \to \mathbf{d}(a, b, x, y) = x$.

(b) $\mathbf{N}(a)$ and $\mathbf{N}(b)$ and $a \neq b \to \mathbf{d}(a, b, x, y) = y$.

[31]

**Theorem 2.2.7** ([8], section 4.2.9)**.** $\mathbf{PCA}^+$ *is conservative over* **PCA**.

**Proof.** To show this, it is sufficient to show how natural numbers are defined in the theory **PCA** with appropriately defined successor and predecessor functions in addition to suitable definitions for pairing, projection functions and definition by integer cases.

(i) Define $\mathbf{I} = \mathbf{skk}$. Then

$$\mathbf{PCA} \vdash \mathbf{I}x = x \text{ since ,}$$

$$\begin{aligned} \mathbf{I}x &= \mathbf{skk}x \\ &= \mathbf{k}x(\mathbf{k}x) \quad \text{by definition of } \mathbf{s} \\ &= x \quad \text{by definition of } \mathbf{k}. \end{aligned}$$

(ii) Define the truth values as $\mathbf{T} = \mathbf{k}$ and $\mathbf{F} = \mathbf{kI}$.

Observe that $\mathbf{F}xy = y$ since,

$$\begin{aligned} \mathbf{F}xy &= \mathbf{kI}xy \\ &= (\mathbf{kI}x)y \quad \text{since association is to the left} \\ &= \mathbf{I}y \\ &= y \end{aligned}$$

(iii) Define pairing and projection functions as follows

$$\begin{aligned} \mathbf{p} &:= \lambda xyz.zxy \\ \mathbf{p}_0 &:= \lambda x.x\mathbf{k} \\ \mathbf{p}_1 &:= \lambda x.x\mathbf{F} \end{aligned}$$

To verify that they satisfy the pairing axioms, we need to check the following

(a) $\mathbf{p}xy = \lambda z.zxy \downarrow$ since by (2.2.4) $\lambda x.t \downarrow$ in **PCA**.

(b)

$$\begin{aligned} \mathbf{p}_0(\mathbf{p}xy) &= \mathbf{p}xy\mathbf{k} \\ &= \mathbf{k}xy \\ &= x \end{aligned}$$

(c)

$$\begin{aligned} \mathbf{p}_1(\mathbf{p}xy) &= \mathbf{p}xy\mathbf{F} \\ &= \mathbf{p}xy(\mathbf{kI}) \\ &= (\mathbf{kI}x)y \\ &= \mathbf{I}y \\ &= y \end{aligned}$$

The pairing function $\mathbf{p}$ is injective because suppose that

$$\begin{aligned} \mathbf{p}xy &= \mathbf{p}x'y' \\ \text{Then} \quad \lambda z.zxy &= \lambda z.zx'y' \\ \text{hence} \quad \mathbf{p}_0(\lambda z.zxy) &= \mathbf{p}_0(\lambda z.zx'y') \quad \text{by applying } \mathbf{p}_0 \text{ to both sides} \\ \text{So} \quad (\lambda x.x\mathbf{k})(\lambda z.zxy) &= (\lambda x.x\mathbf{k})(\lambda z.zx'y') \quad \text{by } \mathbf{p}_0 \text{ definition} \\ \text{Therefore} \quad (\lambda z.zxy)\mathbf{k} &= (\lambda z.zx'y')\mathbf{k} \\ \text{which gives} \quad \mathbf{k}xy &= \mathbf{k}x'y'. \\ \text{So} \quad x &= x'. \end{aligned}$$

And now, if we apply $\mathbf{p}_1$ to both sides instead we obtain the following

$$\begin{aligned} \mathbf{p}_1(\lambda z.zxy) &= \mathbf{p}_1(\lambda z.zx'y') \\ \text{Hence} \quad (\lambda x.x\mathbf{F})(\lambda z.zxy) &= (\lambda x.x\mathbf{F})(\lambda z.zx'y') \quad \text{by the definition of } \mathbf{p}_1 \\ \text{So} \quad (\lambda z.zxy)\mathbf{F} &= (\lambda z.zx'y')\mathbf{F} \\ \text{which gives} \quad \mathbf{F}xy &= \mathbf{F}x'y' \\ \text{Therefore} \quad \mathbf{kI}xy &= \mathbf{kI}x'y' \\ \text{i. e.} \quad (\mathbf{kI}x)y &= (\mathbf{kI}x')y'. \\ \text{So that} \quad \mathbf{I}y &= \mathbf{I}y' \\ \text{giving} \quad y &= y'. \end{aligned}$$

Thus, $\mathbf{p}$ is indeed injective.

(iv) Defining natural numbers

There are different ways of defining natural numbers one of which is the following:

Set 0 to be $\underline{0} = \mathbf{I}$ and let $n+1$ to be $\underline{n+1} = \mathbf{pF}\underline{n}$.

We may define successor and predecessor by using the following terms

$$\mathbf{S_N} = \lambda x.\mathbf{pF}x \text{ and } \mathbf{P_N} = \mathbf{p}_1.$$

which satisfy the corresponding axioms of $\mathbf{PCA}^+$.

(v) Definition by integer cases

We provide a definition by integer cases in steps.

Step 1 Construct a term that decides for each numeral whether it is zero or not. This term is denoted by $\mathbf{Z}$ and is defined by $\mathbf{Z} := \mathbf{p}_0 = \lambda x.x\mathbf{k}$.

Then, we have:

$$\begin{aligned} \mathbf{Z}\underline{0} &= \mathbf{p}_0\mathbf{I} \\ &= \mathbf{Ik} \\ &= \mathbf{k} \\ &= \mathbf{T} \end{aligned}$$

and

$$\begin{aligned} \mathbf{Z}\underline{n} &= \mathbf{p}_0\underline{n} \quad \text{for } n \neq 0 \\ &= \underline{n}\mathbf{k} \\ &= \mathbf{p}(\mathbf{F}, \underline{n-1})\mathbf{k} \\ &= \mathbf{k}\ \mathbf{F}\underline{n-1} \\ &= \mathbf{F}. \end{aligned}$$

Now, observe that for any $A, B, C$,

$$\mathbf{PCA} \vdash ABC = B \quad \text{if } A = \mathbf{T}$$

$$\mathbf{PCA} \vdash ABC = C \quad \text{if } A = \mathbf{F}$$

where $\mathbf{PCA} \vdash B \downarrow$ and $\mathbf{PCA} \vdash C \downarrow$.

This holds because, for $A = \mathbf{T}$, we have:

$$\mathbf{PCA} \vdash \mathbf{k}BC = B$$

is an axiom.

And for $A = \mathbf{F} = \mathbf{kI}$, we have:

$$\mathbf{PCA} \vdash \mathbf{kI}BC = C$$

since in the theory **PCA**, the following holds:

$$\begin{aligned} \mathbf{kI}BC &= (\mathbf{kI}B)C \quad \text{as association is to the left} \\ &= \mathbf{I}C \\ &= C \end{aligned}$$

Step 2  Let (**if** $A$ **then** $B$ **else** $C$) be the term $A(\lambda x.B)(\lambda x.C)x$. Then if $A = \mathbf{T}$, then

$$\begin{aligned} \textbf{if } A \textbf{ then } B \textbf{ else } C &= \mathbf{T}(\lambda x.B)(\lambda x.C)x \\ &= [\mathbf{k}(\lambda x.B)(\lambda x.C)]x \quad \text{since association is to the left} \\ &= (\lambda x.B)x \\ &= B \end{aligned}$$

regardless whether $C$ is defined or not.
If $A = \mathbf{F}$, then

$$\begin{aligned} \textbf{if } A \textbf{ then } B \textbf{ else } C &= \mathbf{F}(\lambda x.B)(\lambda x.C)x \\ &= [\mathbf{kI}(\lambda x.B)](\lambda x.C)x \text{ since association is to the left} \\ &= \mathbf{I}(\lambda x.C)x \\ &= (\lambda x.C)x \\ &= C \end{aligned}$$

Step 3  Define the term $\mathbf{D}$ by:
$\mathbf{D}(x,y) = \textbf{if } \mathbf{Z}x \textbf{ then } \mathbf{Z}y \textbf{ else } \big(\textbf{if } \mathbf{Z}y \textbf{ then } \mathbf{Z}x \textbf{ else } \mathbf{D}(\mathbf{p}_1 x, \mathbf{p}_1 y)\big)$.
Next, observe that $\mathbf{D}(\underline{n}, \underline{m})$ is either $\mathbf{T}$ or $\mathbf{F}$ depending on whether $n = m$ or not. To see this consider the following cases and use induction on $n + m$ to verify them:

(a) If $n = m = 0$. Then $\mathbf{Z}\underline{0} = \mathbf{T}$ and by the above observation $\mathbf{D}(\underline{n}, \underline{m}) = \mathbf{Z}\underline{m} = \mathbf{T}$.

(b) If $n > 0$ and $m > 0$. Then $\mathbf{Z}\underline{n} = \mathbf{Z}\underline{m} = \mathbf{F}$ and hence $\mathbf{D}(\underline{n}, \underline{m}) = \mathbf{D}(\mathbf{p}_1\underline{n}, \mathbf{p}_1\underline{m}) = \mathbf{D}(\underline{n-1}, \underline{m-1})$, by definition of $\mathbf{D}$ and $\mathbf{p}_1$. Now by the induction hypothesis, $\mathbf{D}(\underline{n-1}, \underline{m-1})$ is either $\mathbf{T}$ or $\mathbf{F}$. In fact, we arrive at

$$\mathbf{D}(\underline{n}, \underline{m}) = \mathbf{D}(\underline{n-1}, \underline{m-1}) = \begin{cases} \mathbf{T} & \text{if } n-1 = m-1 \\ \mathbf{F} & \text{otherwise} \end{cases}$$

(c) If $n = 0$ and $m > 0$. Then $\mathbf{D}(\underline{n}, \underline{m}) = \mathbf{Z}\underline{m} = \mathbf{F}$.

(d) If $n > 0$ and $m = 0$. Then $\mathbf{D}(\underline{n}, \underline{m}) = \mathbf{Z}\underline{n} = \mathbf{F}$.

Therefore, the constant $\mathbf{d}$ in $\mathbf{PCA}^+$ used for definition by integer cases is interpreted by the constant $\mathbf{d}'$ in $\mathbf{PCA}$ defined as follows:

$$\mathbf{d}' := \lambda abxy.\mathbf{D}(a,b)xy = \begin{cases} \mathbf{T}xy \; a, b \in \mathbf{N} \text{ with } a = b \\ \mathbf{F}xy \; a, b \in \mathbf{N} \text{ with } a \neq b \end{cases}$$

(vi) The last part of the proof is to interpret the predicate $\mathbf{N}$. Since the set of numerals is not defined by a formula of $\mathbf{PCA}$, we cannot interpret $\mathbf{N}$ as the set of numerals and thus we cannot give a direct interpretation of $\mathbf{PCA}^+$ in $\mathbf{PCA}$. Instead however we can use a model-theoretic argument as follows:
Let $A$ be a formula of $\mathbf{PCA}$ such that

$$\mathbf{PCA} \nvdash A$$

Then, there is a Kripke model $M$ of $\mathbf{PCA}$ such that

$$M \not\models A$$

Next, expand $M$ to a model $M^*$ in which the new constants of $\mathbf{PCA}^+$ are interpreted to be the above defined combinatorial terms. The predicate $\mathbf{N}$ is interpreted to hold on the interpretation of the numerals in $M$ and hence we obtain a Kripke model $M^*$ of $\mathbf{PCA}^+$ with $\mathbf{PCA}^+ \nvdash A$.

□

**Notation.** In this thesis we shall use the term *applicative structure* for a $\mathbf{PCA}^+$ structures.

## 2.3 The Plotkin-Scott Graph Model P$\omega$

Independently Plotkin (1972) and Scott (1974) constructed a $\mathbf{PCA}^+$ model $\mathbf{P}\omega$ with universe the power set of the set of natural numbers partially ordered by inclusion, $\mathbf{P}\omega = \{x : x \subseteq \omega\}$. [5] and [8]

The construction relies upon the fact that natural numbers can encode finite subsets of $\omega$.

**Notation.** Write $z = (x, y)$ for the pairing function on $\mathbb{N}$ and $(z)_0, (z)_1$ for the first and second projections respectively.

### 2.3.1 Binary Representation of $\mathbb{N}$

Number representations are relative to a base $a$, and to represent a number in base $a$, we need $a$ distinct digits. So, to represent a natural number $n \in \mathbb{N}$ in binary, we use base 2 with 0 and 1 as digits. To convert a natural number $n$ to its binary form we start with $n$ and divide by 2 and calculate the reminder. For example, let $n = 100$, the sequence 1100100 represents $n$ in binary. Conversely, one can find the number represented by a sequence of 0$s$ and 1$s$ by multiplying every digit by the corresponding power of 2 and summing the results.

$$a_{n-1}...a_1a_0 = \sum_{i=0}^{n-1} a_i 2^i.$$

For example, the sequence 1100100 represents the number 100 since,

$$0 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 + 0 \times 2^3 + 0 \times 2^4 + 1 \times 2^5 + 1 \times 2^6 = 100.$$

**Definition 2.3.1 (Coding of ordered pairs and finite subsets).** *(i) Define a pairing function* $\mathbf{p} : \mathbb{N}^2 \to \mathbb{N}$ *by*

$$(a, b) = \frac{(a+b)(a+b+1)}{2} + a + 1.$$

*(ii) To any* $n \in \mathbb{N}$ *assign the finite subset* $e_n := \{i$*: the ith digit in the binary representation of* $n$ *is* $1\}$*. Observe that this is a surjective function onto the finite subsets of* $\mathbb{N}$*.*

*[8] and [5]*

**Example.** To calculate $e_{100}$, we know from the previous section that $100 = 1100100$ in binary representation and hence $e_{100} = \{2, 5, 6\}$.
Therefore, $\mathbb{N}$ satisfies the above conditions and hence, for $x \in \mathbb{N}$ and $Y \subseteq \mathbb{N}$ we write $x \subset Y$ to mean that $e_x \subseteq Y$. [8]
**Remark.** $\mathbf{P}\omega$ can be constructed for any codings $(n, m)$ and $e_n$. [5]

**Definition 2.3.2** (**Application in the Graph Model**)**.** *Application in* $\mathbf{P}\omega$ *is defined by*

$$X \bullet Y := \{z \in \mathbb{N} : \exists y \subset Y \;\; (y, z) \in X\}.$$

*[8]*

**Theorem 2.3.3** ([8], Theorem 7.2.4 and [15])**.** *With the application defined above* $\mathbf{P}\omega$ *is a non-extensional pca.*

**Proof.** (a) Define $\mathbf{k} := \{(a, (b, c)) : c \in a\}$. Then we have

$$\begin{aligned}
\mathbf{k}XY &= (\{(a, (b, c)) : c \in a\}X)Y. \\
&= \{(b, c) : c \in X\}Y. \\
&= \{c : c \in X\}. \\
&= X \;\; \text{verifying the axiom for } \mathbf{k}.
\end{aligned}$$

(b) Define $\mathbf{s} := \{(a, (b, (c, d))) : \exists q[\exists c_1 \subset c\,((c_1, (q, d)) \in a) \text{ and } \forall p \in q\, \exists c_1 \subset c\,((c_1, c) \in b)]\}$, and let $M = \exists c_1 \subset c\,((c_1, (q, d)) \in a) \text{ and } \forall p \in q\, \exists c_1 \subset c\,((c_1, p) \in b)$. Then we have

$$\begin{aligned}
\mathbf{s}XYZ &= ((\mathbf{s}X)Y)Z. \\
&= (\{(b, (c, d)) : \exists x \subset X\ \exists qM\}Y)Z. \\
&= \{(c, d) : \exists x \subset X\ \exists y \subset Y\ \exists qM\}Z. \\
&= \{d : \exists x \subset X\ \exists y \subset Y\ \exists z \subset Z\ \exists qM\}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned} XZ(YZ) &= \{d : \exists e \subset YZ\ (e,d) \in XZ\}. \\ &= \{d : \exists z \subset Z \exists e \subset YZ[(z,(e,d)) \in X\ \wedge\ \forall f \in e \exists z_f \subset Z(z_f, f) \in Y]\}. \\ &= \mathbf{s}XYZ \text{ satisfying the axiom for } \mathbf{s}. \end{aligned}$$

Thus, with the above defined constants **k** and **s** is a *pca*. To show non-extensionality, let $A := \{(\emptyset, a), (\{a\}, a)\}$ and let $B := \{(\emptyset, a)\}$. Then, for all $X \in \mathbf{P}\omega$, we have

$$A \bullet X = B \bullet X,$$

while clearly $A \neq B$. □

## 2.4 Scott's $D_\infty$ Models

**Definition 2.4.1.** *Let $D$ be a set partially ordered by $\leq$, i.e. $\leq$ is a reflexive, antisymmetric, and transitive binary relation.*

*(i) An inhabited subset $X \subseteq D$ is said to be directed if for every $x_1, x_2 \in X$, there is $x_3 \in X$ such that $x_1 \leq x_3$ and $x_2 \leq x_3$.*

*(ii) $D$ is a complete partial order ($CPO$) if,*

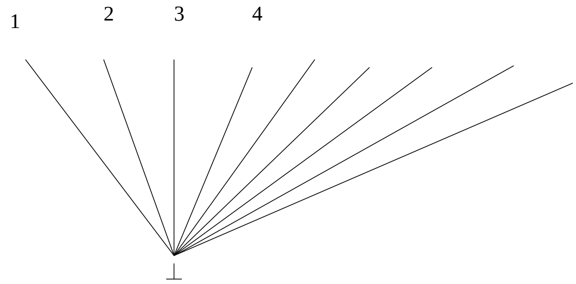
- *(a) There is a* bottom *element $\bot \in D$ such that $\forall x \in D, \bot \leq x$.*
- *(b) For each directed subset $X \subseteq D$, there is a least upper bound sup$(X) \in D$. [5]*

*(iii) $D$ is an $\omega - CPO$ if every $\omega$-chain, $x_1 \leq x_2 \leq x_3, \ldots$ has a supremum in $D$.*

Thus, every $CPO$ is an $\omega - CPO$.

**Definition 2.4.2.** *Let $\mathbb{N}^+$ be $\mathbb{N} \cup \{\bot\}$, where $\bot \neq a$ for all $a \in \mathbb{N}$ and consider the following partial order $\preceq$ defined on $\mathbb{N}^+$ for all $x, y \in \mathbb{N}$*

$$x \preceq y \iff (x = \bot \text{ and } y \in \mathbb{N})\ OR\ (x = y)$$




**Notation.** We will use $\mathbb{N}^+$ to mean $(\mathbb{N}^+, \preceq)$.

**Examples.**

(1) $\mathbb{N}^+$ is a $CPO$, see ([14], Lemma 12.9).

(2) Let $X$ be a set. Then, $(\mathcal{P}(X), \subseteq)$ is a $CPO$ because $\perp = \emptyset$ and clearly every directed subset has a supremum in $P(X)$ by the definition of a direct subset and since the ordering is inclusion.

**Definition 2.4.3.** *let $(D_1, \leq)$ and $(D_2, \leq)$ be CPOs. A map $\theta : (D_1, \leq) \to (D_2, \leq)$ is order continuous if $\theta$ is an order-preserving map from $D_1$ to $D_2$ such that for any directed subset $X \subseteq D_1$ we have:*
*$\theta(supX) = sup\{\theta(x)|x \in X\}$.*

**Notation.** write $1_{D_i}$ for the identity function on $Di$.

**Definition 2.4.4 (Scott Topology).** *The Scott topology is a natural topology on $(CPOs)$ where a subset $U \in D$ ($D$ is a CPO) is open iff*

*(i) It is upward closed. If $u \in U$ and $u \leq u'$ then $u' \in U$.*

*(ii) For a directed subset $X \subseteq D$ whose supremum is in $U$, $U \cap X \neq \emptyset$.*

It is clear that $\emptyset$ and $D$ are open. For open subsets $U_1, U_2 \in D$, condition $(i)$ is clearly satisfied for $U_1 \cap U_2$ since it is true for any $u_1 \in U_1$ and any $u_2 \in U_2$. For condition $(ii)$, since $U_1$, and $U_2$ are open, $U_1 \cap X \neq \emptyset$ and $U_2 \cap X \neq \emptyset$. So, there are $x_1 \in U_1 \cap X$ and $x_2 \in U_2 \cap X$, and $\exists z \in X[x_1 \leq z$ and $x_2 \leq z]$ since $X$ is directed but $U_1$ and $U_2$ are upward closed as they are open, and hence $z \in U_1 \cap U_2$ thus, $z \in [X \cap (U_1 \cap U_2)]$. Let $\mathcal{O} = O_1, O_2, ..., O_n, ...$ be an infinite collection of open sets

and let $o \in \bigcup \mathcal{O}$ then, $o \in O_i$ for some $i$. Now, if $o \leq o'$ then $o' \in O_i$ as $O_i$ is open and hence $o' \in \bigcup \mathcal{O}$ and this showed that $\bigcup \mathcal{O}$ is upward closed.
Let $X \subseteq \bigcup \mathcal{O}$ be directed whose suprema is in $\bigcup \mathcal{O}$ then $sup(X) \in O_i$ for some $i$ and hence $X \cap O_i \neq \emptyset$ since $O_i$ is open and thus $X \cap \bigcup \mathcal{O} \neq \emptyset$.
**Notation.** Henceforth, $D_1 = (D_1, \leq), D_2 = (D_2, \leq), ...$ will range over $CPOs$ equipped with the Scott topology.

**Proposition 2.4.5.** *An order-continuous map of* $(CPOs)$ *is precisely a continuous map for this topology.*

**Proof.** Let $\theta : D_1 \to D_2$ be an order-continuous map between $CPOs$ and let $V \subseteq D_2$ be open. Need to show that $\theta^{-1}(V) = \{u \in D_1 | \theta(u) \in V\}$ is open.
$\theta^{-1}(V)$ is upward closed
Let $u \in \theta^{-1}(V)$ and let $u^{'} \geq u$. Then, as $u^{'} \geq u$ and since $\theta$ is order-preserving map, we have $\theta(u^{'}) \geq \theta(u)$.
But $\theta(u) \in V$ and $V$ is upward closed since it is open. So, $\theta(u^{'}) \in V$, hence $u^{'} \in \theta^{-1}(V)$.
Thus, $\theta^{-1}(V)$ is upward closed.
We need to show $\theta^{-1}(V) \cap X \neq \emptyset$, for a directed $X \subseteq D_1$ with $supX \in \theta^{-1}(V)$
Since $supX \geq x$ for any $x \in X$ and as $\theta$ is an order-preserving map we have:
$\theta(supX) \geq \theta(x)$ , for any $x \in X$.
We have $\theta(supX) = sup(\theta(X)) = sup\{\theta(x) | x \in X\} \in V$.
But $\theta(X)$ is directed since $\theta$ is order-preserving and $X$ is assumed to be directed. So, $sup(\theta(X))$ is in $V$ and hence $\theta(X) \cap V \neq \emptyset$ since $V$ is open. Thus, for some $x_0 \in X$ we have, $\theta(x_0) \in V$ since $V$ is open.
Thus, $x_0 \in \theta^{-1}(V)$, and hence $\theta^{-1}(V) \cap X \neq \emptyset$.
Therefore, $\theta$ is continuous.
Conversely, If $\theta : D_1 \to D_2$ is a continuous map for this topology, then we show that $\theta$ is order-continuous.
**Claim** For any $D$, the set $I_x = \{y \in D : y \not\leq x\}$ is open.
Let $y_1 \in I_x$. Then $y_1 \not\leq x$ so for $y_2 \geq y_1$, $y_2 \in I_x$ since otherwise $y_2 \leq x$ but $y_1 \leq y_2$, so $y_1 \leq x$ and hence, $y_1 \notin I_x$ contrary to the assumption. Therefore, $I_x$ is upward

closed. Now, suppose that $sup(U) \in I_x$ for a directed subset $U$. Then $sup(U) \nleq x$. Assume for a contradiction that $\forall y \in U[y \leq x]$. Then also $sup(U) \leq x$ because $x$ is an upper bound for $U$ contrary to the assumption $sup(U) \in I_x$. Thus, there is $y_0 \in U$ with $y_0 \nleq x$.

Now we use the claim above to show that $\theta$ is order-continuous.

<u>$\theta$ is order-preserving</u>

Suppose that $u_1 \leq u_2$ and assume towards a contradiction that $\theta(u_1) \nleq \theta(u_2)$. Then, $\theta(u_1) \in I_{\theta(u_2)} = \{v \in D_2 : v \nleq \theta(u_2)\}$. So, $u_1 \in \theta^{-1}(I_{\theta(u_2)})$, and since the subset $I_{\theta(u_2)}$ is open in $D_2$ (by the above claim), its pre-image is open in $D_1$ since $\theta$ is continuous, and hence $u_2 \in \theta^{-1}(I_{\theta(u_2)})$ by condition $(i)$ for open sets.

Thus, $\theta(u_2) \in I_{\theta(u_2)}$ contradiction.

Therefore, $\theta$ is order-preserving and hence, $\forall x \in X[\theta(x) \leq \theta(sup(X))]$ and in particular, $sup(\theta(X)) \leq \theta(sup(X))$.

Now, assume that $\theta(sup(X)) \nleq sup(\theta(X))$. Then, $\theta(sup(X)) \in I_{sup(\theta(X))}$, and hence $sup(X) \in \theta^{-1}(I_{sup(\theta(X))})$ and as the subset $I_{sup(\theta(X))}$ is open (by the claim above), and as $\theta$ is continuous, $\theta^{-1}(I_{sup(\theta(X))})$ is open. So, $\theta^{-1}(I_{sup(\theta(X))}) \cap X \neq \varnothing$, by condition $(ii)$ for open sets. Thus, for some $x_0 \in X[\theta(x_0) \nleq sup(\theta(X))]$ contrary to the assumption.

Therefore, $\theta$ is order-continuous. □

**Notation.** The set of all continuous functions with respect to the *Scott topology* is denoted by $[D \to D]$.

**Definition 2.4.6.** *Let $D_1, D_2$ be CPOs. For $f, g \in [D_1 \longrightarrow D_2]$, define*

$$f \leq g \Leftrightarrow (\forall x \in D_1)\, f(x) \leq_{D_2} g(x)$$

**Lemma 2.4.7.** *Let $D_1, D_2$ be CPOs. Then $[D_1 \longrightarrow D_2]$ is a CPO and for all directed subset $H \subseteq [D_1 \longrightarrow D_2]$, the following holds $(\forall x \in D_1)(supH)(x) = sup\{h(x) : h \in H\}$.*

**Proof.** Suppose that $H \subseteq [D_1 \longrightarrow D_2]$ is directed and for every $x \in D_1$ define

$$H(x) = \{h(x) : h \in H\}.$$

Then $H(x)$ is a directed subset of $D_2$ and hence it has a supremum. Now, define a function $f : D_1 \longrightarrow D_2$ such that

$$f(x) = sup(H(x)).$$

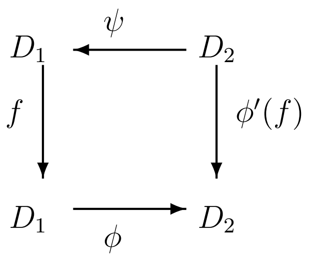
Then $f \in [D_1 \longrightarrow D_2]$ and $f = supH$. For detial see ([14], Lemma 12.18). $\square$

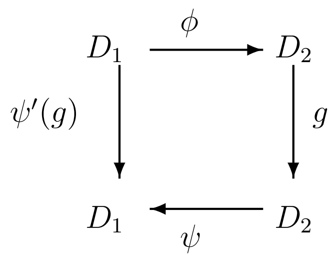
**Definition 2.4.8.** *Let $D_1, D_2$ be $CPOs$. A pair of functions $(\phi, \psi)$ is an injection-projection (ip) pair between $D_1, D_2$ ($\phi : [D_1 \to D_2], \psi : [D_2 \to D_1]$) if $\psi \circ \phi = 1_{D_1}$ and $\phi \circ \psi \leq 1_{D_2}$.*

**Notation.** Write $(\phi, \psi) : D_1 \to D_2$ where $\phi$ is the injection and $\psi$ is the projection.

**Lemma 2.4.9.** *Let $(\phi, \psi)$ be an ip-pair between $D_1$ and $D_2$. Then, there exists an ip pair $(\phi', \psi')$ such that $(\phi', \psi') : [D_1 \to D_1] \longrightarrow [D_2 \to D_2]$ with $\phi'(f) = \phi \circ f \circ \psi$ and $\psi'(g) = \psi \circ g \circ \phi$ for $f \in [D_1 \to D_1]$ and $g \in [D_2 \to D_2]$.*

**Proof.** Consider the diagrams:




$\phi'$ is continuous since $\phi, f$ and $\psi$ are continuous and similarly, $\psi'$ is continuous. Furthermore,

$$\begin{aligned} \psi' \circ \phi'(f) &= \psi'(\phi'(f)) \\ &= \psi \circ \phi'(f) \circ \phi \\ &= \psi \circ \phi \circ f \circ \psi \circ \phi \\ &= f \quad (\text{ since } \psi \circ \phi = 1_{D_1}) \end{aligned}$$

So, $\psi' \circ \phi' = 1_{[D_1 \to D_1]}$.

For $g \in [D_2 \longrightarrow D_2]$ we have

$$\begin{aligned} \phi' \circ \psi'(g) &= \phi'(\psi'(g)) \\ &= \phi \circ \psi'(g) \circ \psi \\ &= \phi \circ \psi \circ g \circ \phi \circ \psi \\ &\leq 1_{D_2} \circ g \circ 1_{D_2} \\ &\leq g \ \ (\text{ since } \phi \circ \psi \leq 1_{D_2}) \end{aligned}$$

So, $\phi' \circ \psi' \leq 1_{[D_2 \to D_2]}$. Therefore, $(\phi', \psi')$ is indeed an *ip* pair. □

**Definition 2.4.10 (Construction of $D_\infty$).** *We define $D_n$ by recursion and take $D_\infty$ to be the limit of the $D_n$s. So, start with a CPO $D$,*
*Base case Define $(\phi_0, \psi_0) : D \to [D \to D]$ by*

$$\begin{aligned} \phi_0(x) &= \lambda y \in D.x \qquad (\textit{the constant map with value } x) \\ \psi_0(f) &= f(\bot_D) \end{aligned}$$

*Since, $\psi_0 \circ \phi_0(x) = \psi_0(\lambda y \in D.x) = \lambda y \in D.x(\bot_D) = x$, $\psi_0 \circ \phi_0 = I_D$ and $\phi_0 \circ \psi_0(f) = \phi_0(f(\bot_D)) = \lambda y \in D.f(\bot_D) \leq f$, $\phi_0 \circ \psi_0 \leq I_D$ and thus this is indeed an ip pair known as the standard projection of $[D \to D]$ on $D$. Now, set $D_0 = D$*
*Successor case*

$$\begin{aligned} D_{n+1} &= [D_n \to D_n] \\ (\phi_{n+1}, \psi_{n+1}) &= (\phi'_n, \psi'_n) \end{aligned}$$

*So, a sequence ,$D_n$, is formed with $D_0 = D$, $D_1 = [D_0 \to D_0]$, ..., $D_{n+1} = [D_n \to D_n]$.*
*$D_\infty$ is the set of all infinite sequences*

$$x = \langle x_0, x_1, x_2, ... \rangle$$

*where $x_i \in D_i$ and $x_i = \psi_i(x_{i+1}) \forall i \in \omega$.*

**Notation.** Write $\mathbf{x}$ for $x \in D_\infty$ and write $x_n$ for the $n^{th}$ component of $\mathbf{x}$.

$$D_0 \underset{\psi_0}{\overset{\phi_0}{\rightleftarrows}} D_1 \underset{\psi_1}{\overset{\phi_1}{\rightleftarrows}} D_2 \underset{\psi_2}{\overset{\phi_2}{\rightleftarrows}} D_3 \quad ...$$

**Definition 2.4.11.** *For* $\mathbf{x}, \mathbf{y} \in D_\infty$, *define (point-wise) the ordering*

$$\mathbf{x} \leq \mathbf{y} \Longleftrightarrow \forall n \in \omega[x_n \leq_{D_n} y_n].$$

**Proposition 2.4.12.** *With the ordering defined above* (2.4.11), $D_\infty$ *is a* $CPO$.

**Proof.** Suppose that $X \subseteq D_\infty$ is directed. Define

$$Y := \langle sup(X_0), sup(X_1), sup(X_2), ...\rangle.$$

We claim that $Y \in D_\infty$ and $supX = Y$.

Finally, the bottom element of $D_\infty$ is $\perp_{D_\infty} = \langle \perp_{D_n} \rangle$. For details see ([14], Lemma 12.36). ☐

**Definition 2.4.13 (injection-projection pairs from $D_n$ to $D_m$).** *For any* $n, m \in \mathbb{N}$, *define* $K_{nm} : D_n \longrightarrow D_m$ *inductively (by following the arrows above) as follows*

$$K_{nm} = \begin{cases} K_{(n-1)m} \circ \psi_{(n-1)} & n > m \\ I_{D_n} & n = m \\ \phi_{(m-1)} \circ K_{n(m-1)} & n < m \end{cases}$$

*So, for* $n < m$, *we have an ip pair* $(K_{nm}, K_{mn})$: $D_n \to D_m$.

**Definition 2.4.14 (injection-projection pairs between $D_n$ and $D_\infty$).** *For* $n < \omega$, *we define*
$K_{n\infty} : D_n \longrightarrow D_\infty$, *and* $K_{\infty n} : D_\infty \longrightarrow D_n$ *by* $K_{n\infty}(x) = \langle K_{ni}(x)\rangle_{i\in\mathbb{N}}$, *and* $K_{\infty n}(\mathbf{x}) = x_n$.

**Lemma 2.4.15.** $(K_{n\infty}, K_{\infty n})$ *is an ip-pair. Furthermore, for all* $x \in D_n$ *we have*

$$K_{m\infty}(K_{nm}(x)) = K_{n\infty}(x) \quad \textit{for } n \leq m \qquad [14]$$

**Proof.** Clearly, $K_{\infty n}$ is continuous since for any ascending sequence $\mathbf{x_p}$ in $D_\infty$ we have, $K_{\infty n}(sup_{\mathbf{p}}\mathbf{x_p}) = sup_n(K_{\infty n}(\mathbf{x_p}))$.

As for the continuity of $K_{n\infty}$, suppose that $\langle x_p \rangle$ is an ascending sequence in $D_n$. Then we have the following:

$$\begin{aligned}(K_{n\infty}(sup_p^{D_n} x_p))_n &= \langle K_{ni}(sup_p^{D_n} x_p)\rangle_{i\in\mathbb{N}} \\ &= sup_p\langle K_{ni}x_p\rangle_{i\in\mathbb{N}} \text{ since, for all } iK_{ni} \text{ is continuous} \\ &= supK_{n\infty}x_p.\end{aligned}$$

$K_{\infty n} \circ K_{n\infty} = 1_{D_n}$ as, $K_{\infty n} \circ K_{n\infty}(x) = K_{\infty n}(\langle K_{ni}(x)\rangle_{i\in\mathbb{N}}) = x$.

For, $K_{n\infty} \circ K_{\infty n} \leq 1_{D_\infty}$, we distinguish two cases

(i) For $m \leq n$. We have

$$\begin{aligned}((K_{n\infty} \circ K_{\infty n})(\mathbf{x}))_m &= (K_{n\infty}(x_n))_m \\ &= (\langle K_{ni}(x_n)\rangle_{i\in\mathbb{N}})_m \\ &= x_m.\end{aligned}$$

(ii) For $m > n$. This follows by induction on $m$ as follows:

<u>Base case</u>

$$\begin{aligned}((K_{n\infty} \circ K_{\infty n})(\mathbf{x}))_{n+1} &= (K_{n\infty}(x_n))_{n+1} \\ &= (\langle K_{ni}(x_n)\rangle_{i\in\mathbb{N}})_{n+1} \\ &= (\langle K_{n0}(x_n), K_{n1}(x_n), ..., x_n, K_{n(n+1)}(x_n), ...\rangle)_{n+1} \\ \text{So }, x_{n+1} &= K_{n(n+1)}(x_n) \\ &= \phi_n(x_n), \text{ as } K_{n(n+1)} \text{ is } \phi_n \circ K_{nn} = \phi_n \\ &= \phi_n(\psi_n(x_{n+1})) \\ &\leq x_{n+1} \text{ since } \phi_n, \psi_n \text{ are injection-projection pairs .}\end{aligned}$$

Induction Step

$$
\begin{aligned}
((K_{n\infty} \circ K_{\infty n})(\mathbf{x}))_{(n+k)+1} &= (K_{n\infty}(x_n))_{(n+k)+1} \\
&= (\langle K_{ni}(x_n)\rangle_{i\in\mathbb{N}})_{(n+k)+1} \\
&= (\langle K_{n0}(x_n), K_{n1}(x_n), ..., K_{n(n+k)}(x_n), K_{n((n+k)+1)} \\
&= (x_n), ...\rangle)_{(n+k)+1} \\
&= K_{n(n+k+1)}(x_n) \\
&= \phi_{n+k} \circ K_{n(n+k)}(x_n) \\
&= \phi_{n+k}(K_{n(n+k)}(x_n)) \\
&= \phi_{n+k} \circ \phi_{n+k-1} \circ K_{n(n+k-1)}(x_n) \\
&= \phi_{n+k} \circ \phi_{n+k-1} \circ \phi_{n+k-2} \circ ... \circ \phi_n \circ K_{nn}(x_n) \\
&= \phi_{n+k} \circ \phi_{n+k-1} \circ ... \circ \phi_n(x_n) \\
&= \phi_{n+k} \circ \phi_{n+k-1} \circ ... \circ \phi_{n+1} \circ \phi_n(\psi_n(x_{n+1})).
\end{aligned}
$$

But $\phi_n(\psi_n(x_{n+1})) \leq x_{n+1}$ since $\phi_n, \psi_n$ are injection-projection pairs . Now, let $y_{n+1}$ be such that $\phi_n(\psi_n(x_{n+1})) = y_{n+1} \leq x_{n+1}$ and hence we have, $\phi_{n+k} \circ \phi_{n+k-1} \circ ... \circ \phi_{n+1} \circ \phi_n(\psi_n(x_{n+1})) = \phi_{n+k} \circ \phi_{n+k-1} \circ ... \circ \phi_{n+1}(\psi_{n+1}(y_{n+2}))$. Similarly, $\phi_{n+1}(\psi_{n+1}(y_{n+2})) \leq y_{n+2}$.

Iterating this we are finally left with $((K_{n\infty} \circ K_{\infty n})(\mathbf{x}))_{(n+k)+1} \leq x_{n+k+1}$.

For $K_{m\infty}(K_{nm}(x)) = K_{n\infty}(x)$ for $n \leq m$, we consider two cases

(i) If $n < m$, $m = n + k$ say. Then,

$$
\begin{aligned}
K_{m\infty} \circ K_{nm}(x) &= \langle K_{m0} \circ K_{nm}(x), K_{m1} \circ K_{nm}(x), ..., K_{mn} \circ K_{nm}(x), \\
&\quad K_{m(n+1)} \circ K_{nm}(x), ..., K_{m(n+k)} \circ K_{nm}(x), ...\rangle \\
&= \langle K_{n0}(x), K_{n1}(x), ..., K_{nn}(x), K_{n(n+1)}(x), ..., K_{nm}(x), ...\rangle \\
&= K_{n\infty}(x).
\end{aligned}
$$

(ii) If $n = m$, then,

$$\begin{aligned} K_{m\infty}(K_{nm}(x)) &= K_{m\infty}(K_{mm}(x)) \\ &= K_{m\infty}(I_{D_m})(x) \\ &= K_{m\infty}(x) \\ &= K_{n\infty}(x) \quad \text{as } n = m. \end{aligned}$$

□

**Remark.** By lemma (2.4.15), $K_{n\infty}$ isomorphically embeds $D_n$ into $D_\infty$. And by the second part of (2.4.15), we obtain the following inclusions

$$D_0 \subseteq D_1 \subseteq D_2 \subseteq ... \subseteq D_\infty.$$

Thus, any $x \in D_n$ may be identified with $K_{n\infty}(x) \in D_\infty$ and hence $x$ is being regarded as if it was an element of $D_\infty$.

**Definition 2.4.16** (**Application on** $D_\infty$)**.** *Application on* $D_\infty$ *is defined by* $\mathbf{a} \bullet \mathbf{b} = \sup_{n<\omega} K_{n\infty}(a_{n+1}(b_n))$. *[14]*

### 2.4.1 Properties of Scott's $D_\infty$ models

**Lemma 2.4.17.**

$$K_{n\infty}(x_{n+1}(y_n)) \leq K_{(n+1)\infty}(x_{n+2}(y_{n+1}))$$

*for any* $\mathbf{x}, \mathbf{y} \in D_\infty$.

**Proof.** See [14], lemma 12.42. □

**Corollary 2.4.18.** *Let* $r \geq 0$ *then for any* $\mathbf{x}, \mathbf{y} \in D_\infty$*, the following hold:*

*(i)* $\mathbf{xy} = \sup_{n \geq r} K_{n\infty}(x_{n+1}(y_n))$.

*(ii)* $(\mathbf{xy})_r = \sup_{n \geq r} K_{nr}(x_{n+1}(y_n))$.

*(iii)* $(\mathbf{xy})_r \geq x_{r+1}(y_r)$.

**Proof.** See [14], corollary 12.42.1. □

**Lemma 2.4.19.** *Let $x \in D_{n+1}$ and $y \in D_n$, then the following hold:*

*(i)* $\psi_{n-1}(x(y)) \geq \psi_n(x)(\psi_{n-1}(y))$ *for* $n \geq 1$.

*(ii)* $\phi_n(x(y)) = \phi_{n+1}(x)(\phi_n(y))$ *for* $n \geq 0$.

**Proof.** See ([14], lemma 12.29). □

**Definition 2.4.20** ([14] Definition 10.5)**.** *Let $\langle D, \bullet \rangle$ be a pca then,*

*(i) For $a, b \in D$, $a$ is said to be* extensionally equivalent *to $b$ written as $a \sim b$ iff*

$$(\forall c \in D)\, a \bullet c = b \bullet c$$

*(ii) The* extensional-equivalent-class *containing $a \in D$ is*

$$\tilde{a} = \{b \in D :\ b \sim a\}$$

**Definition 2.4.21** ([14], Definition 11.19)**.** *Let $\langle D, \bullet \rangle$ be a pca and let $\Lambda$ maps $D$ to $D$, a (syntax-free) $\lambda$-model is a triple $\langle D, \bullet, \Lambda \rangle$ such that the following holds:*

*(i)* $\Lambda(a) \sim a\, (\forall a \in D)$.

*(ii)* $a \sim b \Rightarrow \Lambda(a) = \Lambda(b)\, \forall a, b \in D$.

*(iii)* $(\exists e \in D)(\forall a \in D)$, *we have,* $e \bullet a = \Lambda(a)$.

**Theorem 2.4.22** ([14], Theorem 11.30)**.** *Let $\mathcal{A} = \langle D, \bullet \rangle$ be an extensional pca. Then $\mathcal{A}$ can be made into a $\lambda$-model $\langle D, \bullet, \Lambda \rangle$ in a unique way, namely by defining $\Lambda(a) = a$.*

**Proof.** Since we are assuming that $\langle D, \bullet \rangle$ is an extensional *pca*, every $\tilde{a}$ has one member and hence $\Lambda(a) = a$ is the only way to define $\Lambda$ such that $\Lambda(a) \in \tilde{a}$ and this $\Lambda$ clearly satisfy *Definition* (2.4.21) (*i*) to (*iv*) □

**Proposition 2.4.23** ([14], Exercise 12.30)**.** *For $n \geq 2$, let*

$$k_n = \lambda x \in D_{n-1}.\lambda y \in D_{n-2}.\psi_{n-2}(x).$$

*Then*

*(i)* $k_n \in D_n$ *for* $n \geq 2$.

*(ii)* $\psi_1(k_2) = I_{D_0}$ *and* $\psi_0(\psi_1(k_2)) = \perp_0$.

*(iii)* $\psi_n(k_{n+1}) = k_n$ *for* $n \geq 2$.

**Proof.** (i)

$k_n \in D_n \iff$ $(a)$ $\forall d \in D_{n-1}$ $k_n(d)$ is continuous ( which implies $k_n(d) \in D_{n-1}$ and hence $k_n \in (D_{n-1} \rightarrow D_{n-1})$).

$(b)$ $k_n$ is continuous .

(a) Let $d \in D_{n-1}$. Then,

$$k_n(d) = \lambda y \in D_{n-2}.\psi_{n-2}(d).$$

This is a constant function on $D_{n-2}$ with value $\psi_{n-2}(d)$ and hence $k_n(d)$ is continuous and thus $k_n \in (D_{n-1} \rightarrow D_{n-1})$.

(b) $k_n$ is continuous iff $k_n$ is an order-preserving map and $k_n(supX) = sup\{k_n(x) : x \in X\}$ for any directed $X \subseteq D_{n-1}$. Let $d_1 \leq d_2 \in D_{n-1}$. Then, $k_n(d_1) = \lambda y \in D_{n-2}.\psi_{n-1}(d_1)$.
$k_n(d_2) = \lambda y \in D_{n-2}.\psi_{n-1}(d_2)$.
Notice that $\forall y \in D_{n-2}[\psi_{n-1}(d_1) \leq \psi_{n-1}(d_2)]$ by continuity of $\psi_{n-1}$ and hence $k_n$ is an order-preserving map.
Next, let $X \subseteq D_{n-1}$ be directed. Then,

$$\begin{aligned} k_n(supX) &= \lambda y \in D_{n-2}.\psi_{n-2}(supX) \\ &= sup\{\lambda y \in D_{n-2}.\psi_{n-2}(x) : x \in X\} \quad \text{by continuity of } \psi_{n-2} \\ &= sup\{k_n(x) : x \in X\} \end{aligned}$$

Thus, $k_n$ is continuous and hence $k_n \in D_n$.

(ii) $\psi_1(k_2) = \psi_0 \circ k_2 \circ \phi_0$. Let $a \in D_0$. Then,

$$\begin{aligned} \psi_0 \circ k_2 \circ \phi_0(a) &= \psi_0(k_2(\phi_0(a))) \\ &= \psi_0(\lambda y \in D_0.\psi_0(\phi_0(a))) \\ &= \psi_0(\lambda y \in D_0.a) \quad \text{since } \psi_0 \circ \phi_0 = I_{D_0} \\ &= \lambda y \in D_0.a(\perp_0) \quad \text{by definition of } \psi_0 \\ &= a. \end{aligned}$$

Thus, $\psi_1(k_2) = I_{D_0}$. And,

$$\begin{aligned} \psi_0(\psi_1(k_2)) &= (\psi_1(k_2))(\perp_0) \quad \text{by } \psi_0 \text{ definition} \\ &= I_{D_0}(\perp_0) \quad \text{by first part of } (ii) \\ &= \perp_0. \end{aligned}$$

(iii) $\psi_n(k_{n+1}) = \psi_{n-1} \circ k_{n+1} \circ \phi_{n-1}$. Now, let $a \in D_{n-1}$. Then,

$$\begin{aligned} \psi_{n-1} \circ k_{n+1} \circ \phi_{n-1}(a) &= \psi_{n-1}(\lambda y \in D_{n-1}.\psi_{n-1}(\phi_{n-1}(a))) \\ &= \psi_{n-1}(\lambda y \in D_{n-1}.I_{D_1}(a)) \\ &= \psi_{n-1}(\lambda y \in D_{n-1}.a) \\ &= \psi_{n-2} \circ (\lambda y \in D_{n-1}.a) \circ \phi_{n-2}. \end{aligned}$$

Let $b \in D_{n-2}$. Then,

$$\begin{aligned} \psi_{n-2} \circ (\lambda y \in D_{n-1}.a) \circ \phi_{n-2}(b) &= \psi_{n-2} \circ (\lambda y \in D_{n-1}.a(\phi_{n-2}(b))) \\ &= \psi_{n-2}(a). \end{aligned}$$

Therefore,

$$\begin{aligned} \psi_{n-1} \circ k_{n+1} \circ \phi_{n-1}(a) &= \lambda a \in D_{n-1}.\lambda b \in D_{n-2}.\psi_{n-2}(a) \\ &= k_n. \end{aligned}$$

□

**Definition 2.4.24** ([14], Definition 12.43)**.** *Let $Var$ be the set of variables. For any* *valuation* $\rho : Var \longrightarrow D_\infty$, *the interpretation of combinations of variables in* $D_\infty$ *is inductively defined as follows:*

*(i)* $[\![x]\!]_\rho = \rho(x)$, *for any variable* $x$.

*(ii)* $[\![PQ]\!]_\rho = [\![P]\!]_\rho \bullet [\![Q]\!]_\rho$, *where* $P$, $Q$ *are combination of variables.*

**Definition 2.4.25** ([14], Definition 12.44)**.** *If* $M$ *is an arbitrary combination of variables, then besides* $[\![M]\!]_\rho \in D_\infty$, *we additionaly have the map* $\rho$ *generates a "rough" interpretation, denoted by* $[\![M]\!]^n_\rho$, *in every* $D_n$, *defined as follows:*

*(i)* $[\![x]\!]^n_\rho = (\rho(x))_n$.

*(ii)* $[\![PQ]\!]^n_\rho = [\![P]\!]^{n+1}_\rho([\![Q]\!]^n_\rho)$.

**Lemma 2.4.26.** *Let* $\rho$ : *variables* $\longrightarrow D_\infty$ *and let* $M$ *be any combination of variables. For* $n, k \geq 0$, *the following holds*

*(i)* $[\![M]\!]_\rho = \sup_{n\geq k} K_{n\infty}([\![M]\!]^n_\rho)$.

*(ii)* $([\![M]\!]_\rho)_k = \sup_{n\geq k} K_{nk}([\![M]\!]^n_\rho)$.

**Proof.** See [14], lemma 12.48. □

**Lemma 2.4.27** ([14], Example 12.49)**.** *Let* $M$ *be the combination of variables* $ac(bc)$ *with the interpretation* $\rho(a) = \mathbf{x}, \rho(b) = \mathbf{y}$, *and* $\rho(c) = \mathbf{z}$. *Then,* (2.4.26) *part* $(i)$ *yields*

$$[\![ac(bc)]\!]_\rho = \mathbf{xz}(\mathbf{yz}) = \sup_{n\geq 0} K_{n\infty}(x_{n+2}(z_{n+1})(y_{n+1}(z_n)))$$

.

**Proof.** Lemma (2.4.24) part $(i)$ says that $[\![M]\!]_\rho = \sup_{n\geq r} K_{n\infty}([\![M]\!]^n_\rho)$, and we know from definition (2.4.25) that

$$\begin{aligned} [\![M]\!]_\rho &= [\![a]\!]_\rho[\![c]\!]_\rho([\![b]\!]_\rho[\![c]\!]_\rho) \\ &= \mathbf{xz}(\mathbf{yz}) \end{aligned}$$

Also,

$$\begin{aligned}
\sup_{n\geq r} K_{n\infty}([\![M]\!]^n_\rho) &= \sup_{n\geq r} K_{n\infty}([\![ac]\!]^{n+1}_\rho([\![(bc)]\!]^n_\rho)) \text{ by definition (2.4.25)} \\
&= \sup_{n\geq r} K_{n\infty}([\![a]\!]^{n+2}_\rho([\![c]\!]^{n+1}_\rho)([\![b]\!]^{n+1}_\rho([\![c]\!]^n_\rho))) \\
&= \sup_{n\geq r} K_{n\infty} x_{n+2}(z_{n+1})(y_{n+1}(z_n)).
\end{aligned}$$

□

**Definition 2.4.28.** *By using the $k_n$ defined in proposition* (2.4.23)*, we can define the combinatorial constant* $\mathbf{k} \in D_\infty$ *by*

$$\mathbf{k} := \langle \bot_0, I_{D_0}, k_2, k_3, ...\rangle$$

**Lemma 2.4.29.** *[14] The combinatorial constant* $\mathbf{k}$ *defined in definition* (2.4.28) *is in $D_\infty$ and for any $\mathbf{x}, \mathbf{y} \in D_\infty$, we have*

$$\mathbf{kxy} = \mathbf{x}$$

**Proof.** By proposition (2.4.23), it can be seen that $\mathbf{k} \in D_\infty$. For $\mathbf{kxy} = \mathbf{x}$, and applying definition (2.4.25) part $(ii)$ to the following combination of variables

$M \equiv abc$ and $\rho(a) = \mathbf{k}, \rho(b) = \mathbf{x}, \rho(c) = \mathbf{y}$. Hence,

$$
\begin{aligned}
(\mathbf{kxy})_r &= \sup_{n \geq r} K_{nr}(k_{n+2}(x_{n+1})(y_n)) \text{ by (2.4.26) part } (ii) \\
&= \sup_{n \geq r} K_{nr}(\psi_n(x_{n+1})) \quad \text{by the definition of } k_{n+2} \text{ given in (2.4.23)} \\
&= sup\, K_{(r+k)r}(x_{r+k}) \quad \text{where } n = r + k \\
&= sup\, K_{((r+k)-1)r} \circ \psi_{(r+k)-1}(x_{r+k}) \quad \text{by the definition of } K_{(n+k)r} \\
&= sup\, K_{((r+k)-1)r}(x_{(r+k)-1}) \\
&= sup\, K_{((r+k)-2)r} \circ \psi_{(r+k)-2}(x_{(r+k)-1}) \\
&= K_{((r+k)-2)r}(x_{(r+k)-2}) \\
&= .. \\
&= .. \\
&= .. \\
&= sup\, K_{rr} \circ \psi_r(x_{r+1}) \\
&= sup\, \{x_r\} \\
&= x_r
\end{aligned}
$$

□

**Proposition 2.4.30** ([14], Exercise 12.31)**.** *For $n \geq 3$, let*

$$s_n = \lambda x \in D_{n-1}.\lambda y \in D_{n-2}.\lambda z \in D_{n-3}.x(\phi_{n-3}(z))(y(z)).$$

*Then*

*(i) $s_n \in D_n$ for any $n \geq 3$.*

*(ii) $\psi_2(s_3) = \lambda x \in D_1.\lambda y \in D_0.x(\perp_0)$ and $\psi_1(\psi_2(s_3)) = I_{D_0}$.*

*(iii) $\psi_n(s_{n+1}) = s_n$ for any $n \geq 3$.*

**Proof.** (i)

$$
\begin{aligned}
s_n \in D_n \iff & (a)\ \forall d \in D_{n-1} s_n(d) \text{ is continuous ( this implies } s_n(d) \in D_{n-1} \\
& \text{and hence } s_n \in (D_{n-1} \to D_{n-1})) \\
& (b)\ s_n \text{ is continuous}
\end{aligned}
$$

(a) Let $d \in D_{n-1}$. Then,

$s_n(d) = \lambda y \in D_{n-2}.\lambda z \in D_{n-3}.d(\phi_{n-3}(z))(y(z))$

Since $d \in D_{n-1}, y \in D_{n-2}$ and $\phi_{n-3}$ are all continuous functions and as the application defined on $D_\infty$ is continuous, $s_n(d)$ is continuous and hence $s_n \in (D_{n-1} \to D_{n-1})$.

(b) $s_n$ is continuous iff $s_n$ is an order-preserving map and $s_n(supX) = sup\{s_n(x) : x \in X\}$ for any directed $X \subseteq D_{n-1}$. Now, let $d_1 \leq d_2 \in D_{n-1}$. Then,

$s_n(d_1) = \lambda y \in D_{n-2}.\lambda z \in D_{n-3}.d_1(\phi_{n-3}(z))(y(z))$ and

$s_n(d_2) = \lambda y \in D_{n-2}.\lambda z \in D_{n-3}.d_2(\phi_{n-3}(z))(y(z))$

Notice that by the definition of ordering on $D_{n-1}$ and $D_{n-2}$, we have:

$$d_1(\phi_{n-3}(z))(y(z)) \leq d_2(\phi_{n-3}(z))(y(z))$$

So, $s_n(d_1) \leq s_n(d_2)$ showing that $s_n$ is order-preserving.

Next, let $X \subseteq D_{n-1}$ be a directed subset. Then,

$$\begin{aligned} s_n(supX) &= \lambda y \in D_{n-2}.\lambda z \in D_{n-3}.supX(\phi_{n-3}(z))(y(z)) \\ &= sup\{\lambda y \in D_{n-2}.\lambda z \in D_{n-3}.x(\phi_{n-3}(z))(y(z)) : x \in X\} \\ &\quad \text{by the continuity of application on } D_\infty \\ &= sup\{s_n(x) : x \in X\} \end{aligned}$$

(ii) $\psi_2(s_3) = \psi_1 \circ s_3 \circ \phi_1$. Let $b \in D_1$ then,.

$$\begin{aligned} \psi_2(s_3) &= \psi_1 \circ \lambda x \in D_2.\lambda y \in D_1.\lambda z \in D_0.x(\phi_0(z))(y(z)) \circ \phi_1(b) \\ &= \psi_1\big(\lambda y \in D_1.\lambda z \in D_0.\phi_1(b)(\phi_0(z))(y(z))\big) \\ &= \psi_0 \circ \lambda y \in D_1.\lambda z \in D_0.\phi_1(b)(\phi_0(z))(y(z)) \circ \phi_0. \end{aligned}$$

Let $a \in D_0$. Then,

$$\begin{aligned}
\psi_2(s_3) &= \psi_0 \circ \lambda y \in D_1.\lambda z \in D_0.\phi_1(b)(\phi_0(z))(y(z)) \circ \phi_0(a) \\
&= \psi_0 \circ \lambda z \in D_0.\phi_1(b)(\phi_0(z))(\phi_0(a)(z)) \\
&= \psi_0 \circ \lambda z \in D_0.\phi_1(b)(\phi_0(z))(a) \\
&= \psi_0 \circ \lambda z \in D_0.\phi_0 \circ b \circ \psi_0(\phi_0(z))(a) \\
&= \psi_0 \circ \lambda z \in D_0.\phi_0(b)(z)(a) \\
&= \psi_0\big(\lambda z \in D_0.(b)(z)\big) \\
&= \lambda z \in D_0.(b)(z)(\bot_0) \\
&= b(\bot_0)
\end{aligned}$$

$$\begin{aligned}
\psi_1(\psi_2(s_3)) &= \psi_0 \circ \psi_2(s_3) \circ \phi_0 \\
&= \psi_0 \circ \lambda x \in D_1.\lambda y \in D_0.x(\bot_0) \circ \phi_0
\end{aligned}$$

Let $a \in D_0$. Then,

$$\begin{aligned}
\psi_1(\psi_2(s_3)) &= \psi_0 \circ \lambda x \in D_1.\lambda y \in D_0.x(\bot_0) \circ \phi_0(a) \\
&= \psi_0 \circ (\lambda x \in D_1.\lambda y \in D_0.x(\bot_0))(\lambda p \in D_0.a) \\
&= \psi_0(\lambda y \in D_0.(\lambda p \in D_0.a)(\bot_0)) \\
&= \psi_0(\lambda p \in D_0.a) \\
&= \lambda p \in D_0.a(\bot_0) \\
&= a
\end{aligned}$$

Therefore, $\psi_1(\psi_2(s_3)) = I_{D_0}$

(iii) $\psi_n(s_{n+1}) = \psi_{n-1} \circ s_{n+1} \circ \phi_{n-1}$. If $a \in D_{n-1}$ then $\phi_{n-1}(a) \in D_n$ and so we have the following:

$$\begin{aligned}
\psi_n(s_{n+1}) &= \psi_{n-1} \circ s_{n+1}(\phi_{n-1}(a)) \\
&= \psi_{n-1}(\lambda y \in D_{n-1}.\lambda z \in D_{n-2}.\phi_{n-1}(a)(\phi_{n-2}(z))(y(z))) \\
&= \psi_{n-1}(\lambda y \in D_{n-1}.\lambda z \in D_{n-2}.\phi_{n-2}(a(z))(y(z))) \quad \text{by } (2.4.19) \\
&= \psi_{n-2} \circ (\lambda y \in D_{n-1}.\lambda z \in D_{n-2}.\phi_{n-2}(a(z))(y(z))) \circ \phi_{n-2}
\end{aligned}$$

If $b \in D_{n-2}$ then $\phi_{n-2}(b) \in D_{n-1}$ and hence we have the following:

$$\begin{aligned}
\psi_n(s_{n+1}) &= \psi_{n-2} \circ (\lambda y \in D_{n-1}.\lambda z \in D_{n-2}.\phi_{n-2}(a(z))(y(z))) \circ \phi_{n-2}(b) \\
&= \psi_{n-2}(\lambda z \in D_{n-2}.\phi_{n-2}(a(z))(\phi_{n-2}(b)(z))) \\
&= \psi_{n-3} \circ (\lambda z \in D_{n-2}.\phi_{n-2}(a(z))(\phi_{n-2}(b)(z))) \circ \phi_3
\end{aligned}$$

If $c \in D_{n-3}$ then $\phi_{n-3}(c) \in D_{n-2}$ and hence we get the following:

$$\begin{aligned}
\psi_n(s_{n+1}) &= \psi_{n-3}(\phi_{n-2}(a(\phi_{n-3}(c)))(\phi_{n-2}(b)(\phi_{n-3}(c)))) \\
&= \psi_{n-3}(\phi_{n-2}(a(\phi_{n-3}(c)))(\phi_{n-3}(b(c)))) \quad \text{by } (2.4.19) \\
&= \psi_{n-3}(\phi_{n-3}(a(\phi_{n-3}(c)))(b(c))) \quad \text{by } (2.4.19) \\
&= \psi_{n-3} \circ \phi_{n-3}(a(\phi_{n-3}(c)))(b(c))) \\
&= I_{D_{n-3}}(a(\phi_{n-3}(c))(b(c))) \\
&= \lambda a \in D_{n-1}.\lambda b \in D_{n-2}.\lambda c \in D_{n-3}.a(\phi_{n-3}(c))(b(c)) \\
&= s_n
\end{aligned}$$

□

**Definition 2.4.31.** *Using $s_n$ introduced in proposition* (2.4.30)*, we can define the combinatorial constant* $\mathbf{s} \in D_\infty$ *by*

$$\mathbf{s} := \langle \bot_0, I_{D_0}, \psi_2(s_3), s_3, s_4, ... \rangle$$

**Lemma 2.4.32.** *[14] The combinatorial constant* $\mathbf{s}$ *defined in definition* (2.4.31) *is in $D_\infty$ and for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in D_\infty$, we have*

$$\mathbf{sxyz} = \mathbf{xz(yz)}$$

**Proof.** By proposition (2.4.30), it can be seen that $\mathbf{s} \in D_\infty$. As for $\mathbf{sxyz} = \mathbf{xz}(\mathbf{yz})$, we apply definition (2.4.25) part $(ii)$ to the following combination of variables $M \equiv uabc$ with $\rho(u) = \mathbf{s}, \rho(a) = \mathbf{x}, \rho(b) = \mathbf{y}, \rho(c) = \mathbf{z}$. Hence, we obtain the following

$$\begin{aligned}(\mathbf{sxyz})_r &= \sup_{n\geq r} K_{nr}(s_{n+3}(x_{n+2})(y_{n+1})(z_n)) \\ &= \sup_{n\geq r} K_{nr}(x_{n+2}(\phi_n(z_n))(y_{n+1}(z_n))) \quad \text{by the definition of } s_{n+3} \\ &\quad \text{given in } proposition (2.4.30)\end{aligned}$$

Now, $\phi_n(z_n) = \phi_n(\psi_n(z_{n+1})) \leq z_{n+1}$ using the fact that $\phi_n \circ \psi_n \leq I_{D_{n+1}}$. So,

$$\begin{aligned}(\mathbf{sxyz})_r &\leq \sup_{n\geq r} K_{nr}(x_{n+2}(z_{n+1})(y_{n+1}(z_n))) \\ &= (\mathbf{xz}(\mathbf{yz}))_r \quad \text{by}(2.4.25)\end{aligned}$$

Thus, $(\mathbf{sxyz})_r \leq (\mathbf{xz}(\mathbf{yz}))_r$.

Next, we are aiming to show that

$$(\mathbf{sxyz})_r \geq (\mathbf{xz}(\mathbf{yz}))_r.$$

Starting with the left hand side,

$$\begin{aligned}(\mathbf{sxyz})_r &= \sup_{n\geq r+1} K_{nr}(x_{n+2}(\phi_n(z_n))(y_{n+1}(z_n))) \text{ from step two above} \\ &= \sup_{n\geq r+1} K_{(n-1)r}[\psi_{n-1}(x_{n+2}(\phi_n(z_n))(y_{n+1}(z_n)))] \\ &\quad \text{by the definition of } K_{nr} \text{ given in definition (2.4.31)} \\ &\geq \sup_{n\geq r+1} K_{(n-1)r}[\psi_n(x_{n+2}(\phi_n(z_n))(\psi_{n-1}(y_{n+1}(z_n)))] \text{ by (2.4.19)} \\ &\geq sup K_{(n-1)r}[\psi_{n+1}(x_{n+2})(\psi_n(\phi_n(z_n)))(\psi_n(y_{n+1})(\psi_{n-1}(z_n)))] \quad \text{by lemma (2.4.19)} \\ &= \sup_{n\geq r+1} K_{(n-1)r}[x_{n+1}(z_n)(y_n(z_{n-1}))] \\ &\quad \text{by the definition of } D_\infty \text{ and the fact that } \psi_n \circ \phi_n = I_{D_n} \\ &= (\mathbf{xz}(\mathbf{yz}))_r \quad \text{by Lemma (2.4.27)}\end{aligned}$$

□

**Theorem 2.4.33.** *Scott's $D_\infty$ model with the application defined in definition (2.4.16) is a pca.*

**Proof.** Take the combinatorial constants $\mathbf{k}$ and $\mathbf{s}$ to be as given in definitions

(2.4.28) and (2.4.31). □

**Proposition 2.4.34.** *[14] $D_\infty$ is an extensional pca.*

**Proof.** To show that $D_\infty$ is extensional, we must show that if $\mathbf{xz} = \mathbf{yz}$ for all $\mathbf{z}$, then $\mathbf{x} = \mathbf{y}$.

Now, $\mathbf{x} = \mathbf{y} \iff \forall r \geq 0(x_{r+1} = y_{r+1})$. For $x_0 = \psi_0(x_1) = \psi_0(y_1) = y_0$.

And to show inductively that the functions $x_{r+1}$ and $y_{r+1}$ are equal, it is enough to show that

$$x_{r+1}(d) = y_{r+1}(d) \text{ for all } d \in D_r$$

For $d \in D_r$, define $\mathbf{z} \in D_\infty$ by $\mathbf{z} = K_{r\infty}(d)$. Then,

$$z_n = K_{rn}(d) \text{ for } n \geq 0$$

Thus,

$$\begin{aligned}
(\mathbf{xz})_r &= \sup_{n \geq r} K_{nr}(x_{n+1}(z_n)) \quad \text{by (2.4.18)} \\
&= \sup_{n \geq r} K_{nr}(x_{n+1}(K_{rn}(d))) \\
&= \sup_{n \geq r} (K_{(n-1)r} \circ \psi_{n-1}(x_{n+1}(\phi_{n-1} \circ K_{r(n-1)}(d)))) \quad \text{by definition of } K_{nr} \text{ and } K_{rn} \\
&= \sup_{n \geq r} (\psi_r \circ ... \circ \psi_{n-2} \circ \psi_{n-1} \circ x_{n+1} \circ \phi_{n-1} \circ \phi_{n-2} \circ ... \circ \phi_r)(d) \\
&\quad \text{by repeating the previous step} \\
&= \sup_{n \geq r} (\psi_r \circ ... \circ \psi_{n-2} \circ (\psi_n(x_{n+1})) \circ \phi_{n-2} \circ ... \circ \phi_r)(d), \quad \text{using definition} \\
&\quad \psi_n(x_{n+1}) = \psi_{n-1} \circ x_{n+1} \circ \phi_{n-1} \\
&= \sup_{n \geq r} (\psi_r \circ ... \circ \psi_{n-2} \circ x_n \circ \phi_{n-2} \circ ... \circ \phi_r)(d) \quad \text{as for } \mathbf{x} \in D_\infty, \psi_n(x_{n+1}) = x_n \\
&= \sup_{n \geq r} x_{r+1}(d) \quad \text{by repeating the last two steps} \\
&= x_{r+1}(d).
\end{aligned}$$

Likewise, $(\mathbf{yz})_r = y_{r+1}(d)$. Therefore, if $\mathbf{xz} = \mathbf{yz}$, then $x_{r+1}(d) = y_{r+1}(d)$ as desired.

□

**Theorem 2.4.35** ([14], Theorem 12.55)**.** *$D_\infty$ is an extensional $\lambda$-model.*

**Proof.** By (2.4.33), (2.4.34) and (2.4.22) □

# Chapter 3

# Preservation of Choice Principles under Realizability

## 3.1 Realizability

This chapter presents realizability models for intuitionistic set theories. These models are used to prove some properties, such as the disjunction and the existence properties, of intuitionistic set theories that cannot be shown to hold otherwise. It is also used to extract computational information from constructive proofs. In $\mathtt{V}(\mathcal{K}_2)$, for example, from the realizer for $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, a program verifying the continuity of $f$ can be extracted from the realizer obtained from a constructive proof of $f$ is continuous.

In 1945, Kleene developed realizability semantics for intuitionistic arithmetic and later for other theories. Kreisel and Troelstra [20] gave a definition of realizability for higher order Heyting arithmetic which got extended to systems of set theory by Myhill [23] and subsequently by Friedman [13]. Later further realizability models were developed by Beeson [7, 8] for non-extensional set theories. The extensional version of this realizability, already indicated by Beeson, only required small additions for the atomic formulas and was implemented by McCarty [22]. [22] is mainly concerned with realizability for intuitionistic Zermelo-Fraenkel set theory, **IZF**. As this approach employs transfinite iterations of the powerset operation through all the ordinals in defining the realizability (class) structure $\mathtt{V}(\mathcal{A})$ for any $\mathbf{PCA}^+$ structure $\mathcal{A}$, it was not clear whether this semantics could be developed internally in **CZF**. Moreover, in addition to the powerset axiom the approach in [22] also uses the unrestricted separation axioms. As **CZF** lacks the powerset axiom and has only bounded separation it was not clear whether **CZF** was sufficient as background theory. The development of this kind of realizability on the basis of **CZF** was carried out by Rathjen in [28]. A question that remained open after [28] was the preservation of various choice principles, i.e., which choice principles holding in the background universe $V$ are preserved when moving to $\mathtt{V}(\mathcal{A})$ for an arbitrary $\mathbf{PCA}^+$ structure $\mathcal{A}$. Preservation of several choice principles was shown to hold for the special case of the first Kleene algebra, $\mathcal{K}_1$, assuming a classical background theory by McCarty [22], and on the basis of **CZF** in [28]. Especially it was shown that **CZF** augmented by the presentation axiom suffices to validate that the presentation axiom holds in $V(\mathcal{K}_1)$ whereas [22] uses the full axiom of choice. The purpose of this chapter is to scan the proofs in [28] dealing with $\mathtt{V}(\mathcal{K}_1)$ and show that they can be amended to also work for $\mathtt{V}(\mathcal{A})$.

### 3.1.1 Axioms of Choice

Axioms of countable choice and dependent choices, in many texts on constructive mathematics, are considered as constructive principles.

The weakest choice axiom, denoted by $\mathbf{AC}^{\omega,\omega}$, asserts that there is a function $f : \omega \to \omega$ with $\forall i \in \omega\, \theta(i, f(i))$ whenever $\forall i \in \omega \exists j \in \omega\, \theta(i, j)$ for some formula $\theta$.

The *Axiom of Countable Choice,* $\mathbf{AC}_\omega$, asserts that for some formula $\theta$, if $\forall i \in \omega \exists x\, \theta(i, x)$, then there is a function $f$ with domain $\omega$ such that $\forall i \in \omega\, \theta(i, f(i))$. Obviously, $\mathbf{AC}_\omega$ implies $\mathbf{AC}^{\omega,\omega}$.

The *Axiom Scheme of Dependent Choices* **DC** is a useful axiom to have in set theory which may be given by the following scheme:
For any formula $\phi$, whenever $(\forall a \in x)(\exists b \in x)\phi(a, b)$ and $x_0 \in x$, then there exists a function $f : \omega \to x$ such that $f(0) = x_0$ and $(\forall n \in \omega)\, \phi(f(n), f(n + 1))$.

A very useful Axiom Scheme is the *Relativized Dependent Choices Axiom*, **RDC**, which states that for arbitrary formulas $\phi$ and $\psi$, if

$$\forall x[\phi(x) \to \exists y(\phi(y) \wedge \psi(x, y))]$$

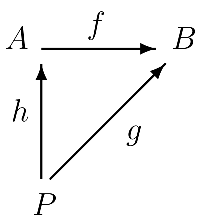
and $\phi(a_0)$, then there exists a function $f$ whose domain is $\omega$ with $f(0) = a_0$ and

$$(\forall n \in \omega)[\phi(f(n)) \wedge \psi(f(n), f(n+1))].$$

How DC and RDC are related: *RDC* implies *DC*, see [27] Lemma 3.4, and on the basis of **CZF** + *DC*, this was explained in p. 61

### 3.1.2 The Presentation Axiom PAx

Let $\mathcal{C}$ be a category and let $P$ be an object in $\mathcal{C}$. Then, $P$ is called *projective* in $\mathcal{C}$ if for any objects $A, B$ in $\mathcal{C}$ and morphisms
$A \xrightarrow{f} B$, $P \xrightarrow{g} B$, with f an epimorphism, there is a morphism $P \xrightarrow{h} A$ such that the diagram below commutes .




Now, taking $\mathcal{C}$ to be the category of sets, then it follows easily that a set $P$ is projective if for all $P$-indexed family $(X_i)_{i \in P}$ of inhabited sets $X_i$ there is a function $f$ with domain $P$ such that for all $i \in P$, $f(i) \in X_i$.
The *presentation axiom* **PAx** asserts that each set is the surjective image of a projective set. Projective sets are often called *bases.* [33]

### 3.1.3 Realizability Structures

Realizability interpretations are important semantics for the study of intuitionistic set theories and to be able to define a realizability interpretation, a notion of realizing functions (partial functions serving as realizers for the formulas of the theory) must be to hand. An elegant and general approach to realizability builds on *applicative structures*.

Assume that we can formalize (in **CZF**) the notion of an applicative structure and let $\mathcal{A}$ be an arbitrary fixed applicative structure which in fact is a set. Furthermore, let $|\mathcal{A}|$ denotes the carrier set of $\mathcal{A}$ but sometimes we will overload notation and write just $\mathcal{A}$ for $|\mathcal{A}|$.

Write $\langle x, y\rangle$ for the ordered pair of $x$ and $y$.

**Definition 3.1.1** ([28], Definition 3.1)**.** *Ordinals are transitive sets with transitive elements. Lower case Greek letters will be used to range over ordinals. For* $\mathcal{A} \models$ $\mathbf{PCA}^+$*, let*

$$\begin{aligned} V(\mathcal{A})_\alpha &= \bigcup_{\beta\in\alpha} \mathcal{P}(|\mathcal{A}| \times V(\mathcal{A})_\beta) \\ V(\mathcal{A}) &= \bigcup_{\alpha} V(\mathcal{A})_\alpha. \end{aligned}$$

In **CZF**, it is not clear whether $\mathtt{V}(\mathcal{A})$ can be formalized since the power set axiom is not among its axioms. However, this is possible and follows from [28, Lemma 3.4]. Provably in **CZF**, we also have the following result.

**Lemma 3.1.2** ([28], Lemma 3.5)**.** *(i) For any* $\beta \in \alpha \quad \mathtt{V}(\mathcal{A})_\beta \subseteq \mathtt{V}(\mathcal{A})_\alpha$.

*(ii) For a subset* $U \subseteq |\mathcal{A}| \times \mathtt{V}(\mathcal{A})$*, we have* $U \in \mathtt{V}(\mathcal{A})$.

**Proof.** Since $V(\mathcal{A})_\alpha = \bigcup_{\beta\in\alpha} \mathcal{P}(|\mathcal{A}| \times \mathtt{V}(\mathcal{A})_\beta)$, $(i)$ is immediate.

$(ii)$ Let $U \subseteq |\mathcal{A}| \times \mathtt{V}(\mathcal{A})$. Then,

$$\forall u \in U \exists \alpha \exists r \in |\mathcal{A}| \exists x \in \mathtt{V}(\mathcal{A})_\alpha [u = \langle r, x\rangle]$$

Now, the scheme of strong collection implies:

$$\exists O[\forall u \in U \exists \alpha \in O \exists r \in |\mathcal{A}| \exists x \in \mathtt{V}(\mathcal{A})_\alpha u = \langle r, x\rangle]$$

where $O$ is a set of ordinals.

Let $O' = \{\alpha + 1 : \alpha \in O\}$ where $\alpha + 1 = \alpha \cup \{\alpha\}$, and let $\gamma = \bigcup O'$. Notice that $\gamma$ is also an ordinal and that $\forall \alpha \in O[\alpha \in \gamma]$.

Therefore,

$$\forall u \in U \exists \alpha \in \gamma \exists r \in |\mathcal{A}| \exists x \in \mathtt{V}(\mathcal{A})_\alpha u = \langle r, x \rangle$$

Hence, $U \subseteq \bigcup_{\alpha\in\gamma} \mathcal{P}(|\mathcal{A}| \times \mathtt{V}(\mathcal{A})_\alpha)$. Thus, $U \in \mathcal{P}(|\mathcal{A}| \times \mathtt{V}(\mathcal{A})_\alpha) = \mathtt{V}(\mathcal{A})_\gamma \subseteq \mathtt{V}(\mathcal{A})$. □

Now, we define a notion of realizability over $\mathtt{V}(\mathcal{A})$ for extensional set theories. For $r \in |\mathcal{A}|$ we define what it means for $r$ to realize a sentence $\phi$ whose parameters are in $\mathtt{V}(\mathcal{A})$, which is written as $r \Vdash \phi$.
For $r \in |\mathcal{A}|$, write $(r)_0$ for $\mathbf{p}_0 r$ and $(r)_1$ for $\mathbf{p}_1 r$.

**Definition 3.1.3** ([29], definition 4.1)**.** *Bounded and unbounded quantifiers are syntactically considered as different types of quantifiers. If $r \in |\mathcal{A}|$ and $a, b \in \mathtt{V}(\mathcal{A})$ then, $r \Vdash \phi$ for a sentence $\phi$ with parameters in $\mathtt{V}(\mathcal{A})$ is defined inductively on the complexity of $\phi$ as follows:*

*(i)* $r \Vdash a \in b \iff \exists c[\langle (r)_0, c\rangle \in b \wedge (r)_1 \Vdash a = c]$.

*(ii)* $r \Vdash a = b \iff \forall g, f[(\langle g, f\rangle \in a \rightarrow (r)_0 g \Vdash f \in b) \wedge (\langle g, f\rangle \in b \rightarrow (r)_1 g \Vdash f \in a)]$.

*(iii)* $r \Vdash \phi \wedge \psi \iff (r)_0 \Vdash \phi$ *and* $(r)_1 \Vdash \psi$.

*(iv)* $r \Vdash \phi \vee \psi \iff [(r)_0 = \mathbf{0} \wedge (r)_1 \Vdash \phi] \vee [(r)_0 = \mathbf{1} \wedge (r)_1 \Vdash \psi]$.

*(v)* $r \Vdash \neg\phi \iff \forall k \in |\mathcal{A}| \quad \neg k \Vdash \phi$.

*(vi)* $r \Vdash \phi \rightarrow \psi \iff \forall k \in |\mathcal{A}|[k \Vdash \phi \rightarrow rk \Vdash \psi]$.

*(vii)* $r \Vdash \forall x \in a\phi \iff \forall \langle k, h\rangle \in a \quad rk \Vdash \phi[x/h]$.

*(viii)* $r \Vdash \exists x \in a\phi \iff \exists h(\langle (r)_0, h\rangle \in a \wedge (r)_1 \Vdash \phi[x/h])$.

*(ix)* $r \Vdash \forall x\phi \iff \forall h \in \mathtt{V}(\mathcal{A}) \quad r \Vdash \phi[x/h]$.

*(x)* $r \Vdash \exists x\phi \iff \exists h \in \mathtt{V}(\mathcal{A}) \quad r \Vdash \phi[x/h]$.

Notice that $(i)$ and $(ii)$ are definitions by transfinite recursion. In particular, the (Class) functions

$$F_{\in}(x,y) = \{r \in |\mathcal{A}| : r \Vdash x \in y\}$$
$$G_{=}(x,y) = \{r \in |\mathcal{A}| : r \Vdash x = y\}$$

can be (simultaneously) defined on $\mathtt{V} \times \mathtt{V}$ by recursion on the relation

$$\langle c,d\rangle \lhd \langle a,b\rangle \;\Leftrightarrow\; \big(c = a \wedge d \in \mathbf{TC}(b)\big) \vee \big(d = b \wedge c \in \mathbf{TC}(a)\big)$$

where $\mathbf{TC}(x)$ is the transitive closure of a set $x$. Moreover, It was shown in [34] Lemma 7.1 that **CZF** proves transfinite recursion on $\lhd$. More precisely, this principle is a consequence of Strong Collection (or Replacement) together with Set Induction.

**Lemma 3.1.4.** *For any $x, y, z \in \mathtt{V}(\mathcal{A})$, there are elements $\mathbf{i}_r, \mathbf{i}_s, \mathbf{i}_t, \mathbf{i}_0, \mathbf{i}_1 \in |\mathcal{A}|$ such that:*

*(i)* $\mathbf{i}_r \Vdash x = x$.

*(ii)* $\mathbf{i}_s \Vdash x = y \to y = x$.

*(iii)* $\mathbf{i}_t \Vdash (x = y \wedge y = z) \to x = z$.

*(iv)* $\mathbf{i}_0 \Vdash (x = y \wedge y \in z) \to x \in z$.

*(v)* $\mathbf{i}_1 \Vdash (x = y \wedge z \in x) \to z \in y$.

*Furthermore, for every* **CZF***-formula $\phi(u, v_1, ..., v_n)$ with $FV(\phi) \subseteq u, v_1, ..., v_n$, there is an element $\mathbf{i}_\phi \in |\mathcal{A}|$ such that:*

$$\forall x, y, z_1, ..., z_n[\mathbf{i}_\phi \Vdash \phi(x, \vec{z}) \wedge x = y \to \phi(y, \vec{z})] \text{ where } \vec{z} = z_1, ..., z_n\,.$$

**Proof.** [28, Lemma 4.2]. □

**Theorem 3.1.5** ([28], Theorem 4.3)**.** *Let $\mathcal{P}$ be a proof of a* **CZF***-formula $\phi(u_1, ..., u_n)$ (with $FV(\phi)$ among $u_1, ..., u_n$) in intuitionistic predicate logic with* *equality. Then, there is a realizer* $r_{\mathcal{P}} \in |\mathcal{A}|$ *such that:*

$$r_{\mathcal{P}} \Vdash \forall u_1 ... \forall u_n \phi(u_1, ..., u_n)$$

*is provable in* **CZF**.

**Proof.** First, we find realizers for the following logical principles that relate bounded and unbounded quantification:

$$\forall u \in a \phi(u) \leftrightarrow \forall u[u \in a \rightarrow \phi(u)]$$

$$\exists u \in a \phi(u) \leftrightarrow \exists u[u \in a \wedge \phi(u)].$$

We have: $r \Vdash \forall u[u \in a \rightarrow \phi(u)]$

$$\Leftrightarrow \quad \forall x \in \mathrm{V}(\mathcal{A})\, r \Vdash x \in a \rightarrow \phi(x)$$

$$\Leftrightarrow \quad \forall x \in \mathrm{V}(\mathcal{A})\, \forall e \in |\mathcal{A}|[e \Vdash x \in a \rightarrow re \Vdash \phi(x)]$$

$$\Leftrightarrow \quad \forall x \in \mathrm{V}(\mathcal{A})\, \forall e \in |\mathcal{A}|[\exists c(\langle (e)_0, c\rangle \in a \wedge (e)_1 \Vdash x = c) \rightarrow re \Vdash \phi(x)]$$

$$\Rightarrow \quad \forall c \forall e \in |\mathcal{A}|[(\langle (e)_0, c\rangle \in a \wedge (e)_1 \Vdash c = c) \rightarrow re \Vdash \phi(c)]$$

$$\Rightarrow \quad \forall \langle f, c\rangle \in a \quad r(\mathbf{p} f \mathbf{i}_r) \Vdash \phi(c)$$

$$\Rightarrow \quad \lambda f.r(\mathbf{p} f \mathbf{i}_r) \Vdash \forall u \in a \phi(u).$$

Conversely, if $r \Vdash \forall u \in a\phi(u)$, then, equivalently $\forall \langle k, h\rangle \in a \;\; rk \Vdash \phi(h)$, and this implies that $\forall x \in \mathrm{V}(\mathcal{A}) \forall f \in |\mathcal{A}|[\exists c(\langle (f)_0, c\rangle \in a \wedge (f)_1 \Vdash x = c) \rightarrow \mathbf{i}_\phi(\mathbf{p}(r(f)_0)(f)_1) \Vdash \phi(x)]$.

Now, let $\mathbf{R} := \mathbf{p}(\lambda r.\lambda f.r(\mathbf{p} f \mathbf{i}_r))(\lambda r.\lambda f.\mathbf{i}_\phi(\mathbf{p}(r(f)_0)(f)_1)))$.

Then $\mathbf{R} \Vdash \forall \vec{q} \forall u(\forall v \in u\phi(v) \leftrightarrow \forall v[v \in u \rightarrow \phi(v)])$, where $\forall \vec{q}$ quantifies over the remaining $FV(\phi)$. Similarly, one can find $\mathbf{R}'$ such that:

$\mathbf{R}' \Vdash \forall \vec{q} \exists u(\exists v \in u\phi(v) \leftrightarrow \exists v[v \in u \wedge \phi(v)])$.

We skip the remaining laws of intuitionistic predicate logic. □

**Theorem 3.1.6 (The Soundness Theorem for CZF).** *For each axiom A of* **CZF**, *there is a closed application term t such that:*

$$\mathbf{CZF} \vdash (t \Vdash A)$$

**Proof.** This is shown in [28, Theorem 5.1]. To give the flavour, we only show how to find a realizer for the *Extensionality* and *Bounded Separation* axioms.

**Extensionality**: $\forall x \forall y[\forall a(a \in x \leftrightarrow a \in y) \longrightarrow x = y]$.

We want to find $r \in |\mathcal{A}|$ such that $r \Vdash \forall x \forall y[\forall a(a \in x \leftrightarrow a \in y) \longrightarrow x = y]$.

By the definition of realizability, we get these equivalences:

$$
\begin{aligned}
& r \Vdash \forall x \forall y[\forall a(a \in x \leftrightarrow a \in y) \longrightarrow x = y] \\
\iff \; & \forall z \in \mathtt{V}(\mathcal{A})\, r \Vdash \forall y[\forall a(a \in z \leftrightarrow a \in y) \longrightarrow z = y] \\
\iff \; & \forall z, u \in \mathtt{V}(\mathcal{A})\, r \Vdash \forall a(a \in z \leftrightarrow a \in u) \longrightarrow z = u \\
\iff \; & \forall z, u \in \mathtt{V}(\mathcal{A}) \forall f \in |\mathcal{A}|[f \Vdash \forall a(a \in z \leftrightarrow a \in u) \text{ implies } rf \Vdash z = u] \\
\iff \; & \forall z, u \in \mathtt{V}(\mathcal{A}) \forall f \in |\mathcal{A}|[\forall g \in \mathtt{V}(\mathcal{A})(f \Vdash g \in z \leftrightarrow g \in u) \text{ implies } rf \Vdash z = u].
\end{aligned}
$$

Now, we deal separately with $\forall g \in \mathtt{V}(\mathcal{A})(f \Vdash g \in z \leftrightarrow g \in u)$, which is equivalent to:

(i) $\forall g \in \mathtt{V}(\mathcal{A})[(f)_0 \Vdash g \in z \to g \in u]$.

(ii) $\forall g \in \mathtt{V}(\mathcal{A})[(f)_1 \Vdash g \in u \to g \in z]$.

So, we have

$$
\begin{aligned}
(i) \iff \; & \forall g, h \in |\mathcal{A}|[h \Vdash g \in z \text{ then } (f)_0 h \Vdash g \in u] \\
\iff \; & \forall g, h \in |\mathcal{A}|[\exists k(\langle (h)_0, k\rangle \in z \wedge (h)_1 \Vdash g = k) \text{ implies } (f)_0 h \Vdash g \in u] \\
\iff \; & \forall k, h \in |\mathcal{A}|[(\langle (h)_0, k\rangle \in z \wedge (h)_1 \Vdash k = k] \text{ implies } (f)_0 h \Vdash k \in u].
\end{aligned}
$$

Consequently, if $\langle l, k\rangle \in z$, then (i) yields that $(f)_0 \mathbf{p} l \mathbf{i}_r \Vdash k \in u$. Similarly, (ii) implies that if $\langle l, k\rangle \in u$, then $(f)_1 \mathbf{p} l \mathbf{i}_r \Vdash k \in z$. As a result, $r = \lambda y.\mathbf{p}(\lambda x.\mathbf{p}_0 y(\mathbf{p} x \mathbf{i}_r))(\lambda x.\mathbf{p}_1 y(\mathbf{p} x \mathbf{i}_r))$ is a realizer for *Extensionality.*

**Bounded Separation**: $\forall x \exists y \forall a[a \in y \leftrightarrow a \in x \wedge \phi(a)]$, where $\phi$ is a bounded formula with parameters in $\mathtt{V}(\mathcal{A})$.

We need to find $r_1, r_2 \in |\mathcal{A}|$ such that $\forall a \in \mathtt{V}(\mathcal{A}) \quad \exists b \in \mathtt{V}(\mathcal{A})$ such that:

$$r_1 \Vdash \forall u \in b[u \in a \wedge \phi(u)]$$

$$r_2 \Vdash \forall u \in a[\phi(u) \to u \in b].$$

Let $a \in \mathtt{V}(\mathcal{A})$ and define

$$Sep^{\mathcal{A}}(a, \phi) := \{\langle \mathbf{p} fg, x \rangle : f, g \in \mathcal{A} \wedge \langle g, x \rangle \in a \wedge f \Vdash \phi(x)\}.$$

Observe that $Sep^{\mathcal{A}}(a, \phi)$ is a set and moreover it belongs to $\mathtt{V}(\mathcal{A})$, for details see [28] Corollary 4.7. Let $r_1 := \lambda u.\mathbf{p}(\mathbf{p}(u)_1\mathbf{i}_r))u)_0$ and $r_2 := \lambda u.\lambda v.\mathbf{p}(\mathbf{p}vu)\mathbf{i}_r$. With $b := Sep^{\mathcal{A}}(a, \phi)$, these terms provide realizers for the formulas above. For details see [28, Theorem 5.1]. □

**Definition 3.1.7** (**Representing** $\omega$ **in** $\mathtt{V}(\mathcal{A})$)**.** *Let $\mathcal{A}$ be an applicative structure, and recall that the zero of $\mathcal{A}$ is denoted by $\mathbf{0}$ and $\mathbf{1} = \mathbf{S_N 0}$. Then, $\omega$ is represented in $\mathtt{V}(\mathcal{A})$ by $\overline{\omega}$ given by an injection of $\omega$ into $\mathtt{V}(\mathcal{A})$ defined as follows:*
*Set $\underline{0} = \mathbf{0}$, where $0$ is the least element of $\omega$ (the empty set) and for every $n \in \omega$, let $\underline{n+1} = \mathbf{S_N}\underline{n}$. Then, define $\overline{n} := \{\langle \underline{m}, \overline{m} \rangle : m \in n\}$ and take $\overline{\omega} = \{\langle \underline{n}, \overline{n} \rangle : n \in \omega\}$.*

To verify that $\overline{\omega} \in \mathtt{V}(\mathcal{A})$ note that $\overline{\omega} \subseteq |\mathcal{A}| \times \mathtt{V}(\mathcal{A})$, which implies that $\overline{\omega} \in \mathtt{V}(\mathcal{A})$ by (3.1.2) and clearly $\mathbf{N}(\underline{n})$ holds for each $n \in \omega$.
Furthermore, we will now show that if $n \neq m$ then $\underline{n} \neq \underline{m}$ for all $n, m \in \omega$, using the applicative axiom on section (2.2.1).
To show this let $n \neq m$. Then either one of them is 0 and the other is a successor or both are successors.

(i) Suppose that $n = 0$ and $m = k + 1$ for $k \in \omega$. Then, $\underline{n} = \mathbf{0}$ and $\underline{m} = S_N \underline{k}$ and (by Axiom (ii)(a) for a $\mathbf{PCA}^+$) $S_N\underline{k} \neq \mathbf{0}$. Hence $S_N\underline{k} \neq \mathbf{0}$ which implies that $\underline{n} \neq \underline{m}$.

(ii) Suppose that $n = k + 1$ and $m = l + 1$ for $k, l \in \omega$. By Ax (ii)(b) for a $\mathbf{PCA}^+$ we have $P_N\underline{n} \downarrow$, $P_N\underline{m} \downarrow$, $P_N(S_N\underline{k}) = \underline{k}$ and $P_N(S_N\underline{l}) = \underline{l}$. Since $n \neq m$ entails $k \neq l$ we can inductively assume that $\underline{k} \neq \underline{l}$. Therefore, $\underline{n} = S_N\underline{k} \neq S_N\underline{l} = \underline{m}$.

The contrapositive of the above implies that:

$$\mathbf{PCA}^+ \vdash \underline{n} = \underline{m} \implies n = m$$

and the converse of this is trivial and thus we obtain the following proposition:

**Proposition 3.1.8.** *For any $n, m \in \omega$ we have,*

$$n = m \Longleftrightarrow \mathbf{PCA}^+ \vdash \underline{n} = \underline{m}.$$

We often write $\mathtt{V}(\mathcal{A}) \models \phi$ to convey that there exists $a \in |\mathcal{A}|$ such that $a \Vdash \phi$.

**Proposition 3.1.9.** *Membership and equality on $\overline{\omega}$ are realizably absolute. In other words, for all $n, m \in \omega$ we have:*

*(i)* $n = m \iff \mathtt{V}(\mathcal{A}) \models \overline{n} = \overline{m}$.

*(ii)* $n \in m \iff \mathtt{V}(\mathcal{A}) \models \overline{n} \in \overline{m}$.

**Proof.** We prove (i) and (ii) simultaneously by induction on $n + m$.

(i) If $n = m$ then $\mathbf{i}_r \Vdash \overline{n} = \overline{m}$. Now, suppose that $r \Vdash \overline{n} = \overline{m}$. Then we have:

$$\forall f, d[(\langle f, d\rangle \in \overline{n} \to (r)_0 f \Vdash d \in \overline{m}) \wedge (\langle f, d\rangle \in \overline{m} \to (r)_1 f \Vdash d \in \overline{n})]. \tag{3.1}$$

Since $\langle \underline{k}, \overline{k}\rangle \in \overline{n}$ holds for all $k \in n$, applying the induction hypothesis to (3.1) we get $\forall k \in n\ k \in m$. By symmetry we also deduce $\forall i \in m\ i \in n$. Hence $n = m$.

(ii) If $n \in m$, then $\langle \underline{n}, \overline{n}\rangle \in \overline{m}$, and hence $\mathbf{p}\underline{n}\mathbf{i}_r \Vdash \overline{n} \in \overline{m}$.

Now, suppose that $e \Vdash \overline{n} \in \overline{m}$. Then there exists $c$ such that $\langle e_0, c\rangle \in \overline{m} \wedge e_1 \Vdash \overline{n} = c$. This implies that $c = \overline{k}$ for some $k \in m$. So the induction hypothesis from part (i) yields $n = k$, and therefore $n \in m$.

□

### 3.1.4 Absoluteness Properties

**Definition 3.1.10.** *For $a, b \in \mathtt{V}(\mathcal{A})$, define $\{a, b\}_{\mathcal{A}} := \{\langle \mathbf{0}, a\rangle, \langle \mathbf{1}, b\rangle\}$ and let*

$$\langle a, b\rangle_{\mathcal{A}} := \{\langle \mathbf{0}, \{a, a\}_{\mathcal{A}}\rangle, \langle \mathbf{1}, \{a, b\}_{\mathcal{A}}\rangle\}.$$

**Lemma 3.1.11 (Internal Pairing in $\mathtt{V}(\mathcal{A})$).** *If $a, b, x \in \mathtt{V}(\mathcal{A})$ then:*

*(i)* $\mathtt{V}(\mathcal{A}) \models x \in \{a, b\}_{\mathcal{A}} \leftrightarrow x = a \vee x = b$.

*(ii)* $\mathtt{V}(\mathcal{A}) \models x \in \langle a, b\rangle_{\mathcal{A}} \leftrightarrow x = \{a, a\}_{\mathcal{A}} \vee x = \{a, b\}_{\mathcal{A}}$.

**Proof.** (i): $e \Vdash x \in \{a, b\}_{\mathcal{A}}$. Then there exists $c$ such that $\langle (e)_0, c\rangle \in \{a, b\}_{\mathcal{A}}$ and $(e)_1 \Vdash x = c$. But, $\langle (e)_0, c\rangle \in \{a, b\}_{\mathcal{A}}$ implies that $\langle (e)_0, c\rangle = \langle \mathbf{0}, a\rangle$ or $\langle (e)_0, c\rangle = \langle \mathbf{1}, b\rangle$, and hence we obtain:

$$[(e)_0 = \mathbf{0} \wedge (e)_1 \Vdash x = a] \vee [(e)_0 = \mathbf{1} \wedge (e)_1 \Vdash x = b].$$

Thus, $e \Vdash x = a \vee x = b$.

Conversely, suppose that $e \Vdash x = a \vee x = b$. Then retracing the steps of the foregoing proof backwards shows that $e \Vdash x \in \{a, b\}_{\mathcal{A}}$. And therefore $\mathbf{p}(\lambda x.x)(\lambda x.x)$ provides a realizer for (i).

(ii): First assume that $e \Vdash x \in \langle a, b\rangle_{\mathcal{A}}$. Then there exists $c$ such that $\langle (e)_0, c\rangle \in \langle a, b\rangle_{\mathcal{A}}$ and $(e)_1 \Vdash x = c$. But $\langle (e)_0, c\rangle \in \langle a, b\rangle_{\mathcal{A}}$ implies that either $\langle (e)_0, c\rangle = \langle \mathbf{0}, \{a, a\}_{\mathcal{A}}\rangle$ or $\langle (e)_0, c\rangle = \langle \mathbf{1}, \{a, b\}_{\mathcal{A}}\rangle$, and hence:

$$[(e)_0 = \mathbf{0} \wedge (e)_1 \Vdash x = \{a, a\}_{\mathcal{A}}] \vee [(e)_0 = \mathbf{1} \wedge (e)_1 \Vdash x = \{a, b\}_{\mathcal{A}}].$$

Thus, $e \Vdash x = \{a, a\}_{\mathcal{A}} \vee x = \{a, b\}_{\mathcal{A}}$.

Conversely, if $e \Vdash x = \{a, a\}_{\mathcal{A}} \vee x = \{a, b\}_{\mathcal{A}}$, then we have

$$[(e)_0 = \mathbf{0} \wedge (e)_1 \Vdash x = \{a, a\}_{\mathcal{A}}] \vee [(e)_0 = \mathbf{1} \wedge (e)_1 \Vdash x = \{a, b\}_{\mathcal{A}}].$$

Thus, $\langle (e)_0, c\rangle \in \langle a, b\rangle_{\mathcal{A}} \wedge (e)_1 \Vdash x = c$ for some $c \in \mathtt{V}(\mathcal{A})$. So, $e \Vdash x \in \langle a, b\rangle_{\mathcal{A}}$. Therefore $\mathbf{p}(\lambda x.x)(\lambda x.x)$ is also a realizer for (ii). □

### 3.1.5 Axioms of choice and $\mathtt{V}(\mathcal{A})$

It follows from [22] and [28] that arguing in **CZF**, the principles **DC**, **RDC**, and **PAx** hold in $\mathtt{V}(\mathcal{K}_1)$ assuming their validity in the background universe $\mathtt{V}$. Moreover, $\mathbf{AC}^{\omega,\omega}$holds in $\mathtt{V}(\mathcal{K}_1)$ regardless of whether it holds in $\mathtt{V}$. Here we show that $\mathcal{K}_1$ can actually be replaced by any applicative structure $\mathcal{A}$.

**Theorem 3.1.12.** *Let $\mathcal{A}$ be any applicative structure. Then:*

*(i)* (**CZF**) $\quad \mathtt{V}(\mathcal{A}) \models \mathbf{AC}^{\omega,\omega}$.

*(ii)* ($\mathbf{CZF} + \mathbf{AC}_\omega$) $\quad \mathtt{V}(\mathcal{A}) \models \mathbf{AC}_\omega$.

*(iii)* ($\mathbf{CZF} + \mathbf{DC}$) $\quad \mathtt{V}(\mathcal{A}) \models \mathbf{DC}$.

*(iv)* ($\mathbf{CZF} + \mathbf{RDC}$) $\quad \mathtt{V}(\mathcal{A}) \models \mathbf{RDC}$.

*(v)* ($\mathbf{CZF} + \mathbf{PAx}$) $\quad \mathtt{V}(\mathcal{A}) \models \mathbf{PAx}$.

**Proof.** (i) Suppose that

$$e \Vdash \forall i \in \bar{\omega} \exists j \in \bar{\omega}\, \theta(i,j).$$

Then, $\forall \langle a, x\rangle \in \bar{\omega}\; ea \Vdash \exists j \in \bar{\omega}\, \theta(x,j)$, and hence

$$\forall \langle a, x\rangle \in \bar{\omega}\, \exists y\, [\langle (ea)_0, y\rangle \in \bar{\omega} \,\wedge\, (ea)_1 \Vdash \theta(x,y)].$$

Now, because for $\langle r, s\rangle \in \bar{\omega}$, $s$ is uniquely determined by $r$, the above entails that there exists a function $f : \omega \to \omega$ such that for all $n \in \omega$,

$$\langle (e\underline{n})_0, \overline{f(n)}\,\rangle \in \bar{\omega} \quad \text{and} \quad (e\underline{n})_1 \Vdash \theta(\bar{n}, \overline{f(n)}\,). \tag{3.2}$$

now define

$$g \,:=\, \{\langle \underline{n}, \langle \bar{n}, \overline{f(n)}\,\rangle_{\mathcal{A}}\rangle \mid n \in \omega\}.$$

Clearly, $g \in \mathtt{V}(\mathcal{A})$. We first prove that $g$ picks the right things and care about its functionality later. As

$$\mathbf{p}\underline{n}\mathbf{i}_r \Vdash \langle \bar{n}, \overline{f(n)}\,\rangle_{\mathcal{A}} \in g \tag{3.3}$$

it follows from (3.2) and (3.3) that with

$$h \,:=\, \lambda u.\mathbf{p}(\mathbf{p}(\mathbf{p}((eu)_0\mathbf{i}_r), \mathbf{p}u\mathbf{i}_r), (eu)_1)$$

we have for all $n \in \omega$ that

$$h\underline{n} \Vdash \exists y\, [y \in \bar{\omega} \,\wedge\, \langle \bar{n}, y\rangle_{\mathcal{A}} \in g \,\wedge\, \theta(\bar{n}, y)]. \tag{3.4}$$

As for functionality of $g$, assume that $x, y, z \in \mathtt{V}(\mathcal{A})$ and

$$d \Vdash \langle x, y\rangle_{\mathcal{A}} \in g \ \wedge \ \langle x, z\rangle_{\mathcal{A}} \in g.$$

Then there exist $y', z' \in \mathcal{A}$ such that $\langle d_{0,0}, y'\rangle \in g$, $\langle d_{1,0}, z'\rangle \in g$, and

$$d_{0,1} \Vdash y' = \langle x, y\rangle_{\mathcal{A}} \quad \wedge \quad d_{1,1} \Vdash z' = \langle x, z\rangle_{\mathcal{A}}. \tag{3.5}$$

Moreover, there exist $n, m \in \omega$ such that $y' = \langle \bar{n}, \overline{f(n)}\,\rangle_{\mathcal{A}}$ and $z' = \langle \bar{m}, \overline{f(m)}\,\rangle_{\mathcal{A}}$. Thus it follows from Lemma 3.1.11 that $\mathtt{V}(\mathcal{A}) \models \bar{n} = \bar{m}$, and therefore $n = m$ by Lemma 3.1.9. As a result, one can effectively construct a realizer $d' \in \mathcal{A}$ from $d$ such that $d' \Vdash y = z$, showing functionality of $g$.

(ii) Validating $\mathbf{AC}_\omega$ in $\mathtt{V}(\mathcal{A})$ is very similar to the proof of (i). Suppose that

$$e \Vdash \forall i \in \bar{\omega} \exists y\, \theta(i, y).$$

Then, $\forall \langle a, x\rangle \in \bar{\omega}\ ea \Vdash \exists y\, \theta(x, y)$, and hence

$$\forall \langle a, x\rangle \in \bar{\omega}\, \exists z \in \mathtt{V}(\mathcal{A})\, ea \Vdash \theta(x, z)].$$

Now, invoking $\mathbf{AC}_\omega$ in $\mathtt{V}$ that there exists a function $F : \omega \to \mathtt{V}(\mathcal{A})$ such that for all $n \in \omega$,

$$e\underline{n} \Vdash \theta(\bar{n}, F(n)). \tag{3.6}$$

Now define

$$G := \{\langle \underline{n}, \langle \bar{n}, F(n)\rangle_{\mathcal{A}}\rangle \mid n \in \omega\}.$$

Clearly, $G \in \mathtt{V}(\mathcal{A})$. The rest of the proof proceeds similarly as in (i).

(iii) Let $t, u \in \mathtt{V}(\mathcal{A})$, and suppose the following:

$$e \Vdash \forall x \in t\, \exists y \in t \quad \phi(x, y) \tag{3.7}$$

and

$$e^* \Vdash u \in t \tag{3.8}$$

Then, (by the definition of realizability) (3.7) is equivalent to:

$$\forall \langle a, x\rangle \in t\, \exists y[\langle (ea)_0, y\rangle \in t \wedge (ea)_1 \Vdash \phi(x, y)].$$

From (3.8) we conclude that there exists $u_0$ such that

$$\langle (e^*)_0, u_0\rangle \in t \wedge (e^*)_1 \Vdash u = u_0.$$

Thus, for all $a \in |\mathcal{A}|$ and for all $z$ in $\mathtt{V}(\mathcal{A})$, if $\langle a, z\rangle \in t$, then $ea\downarrow$ and there is a $q'$ in $\mathtt{V}(\mathcal{A})$ such that $\langle (ea)_0, q'\rangle \in t \wedge (ea)_1 \Vdash \phi(z, q')$.
Externally, define $\phi^{\Vdash}$ by:

$$\phi^{\Vdash}(\langle a, z\rangle, \langle b, q\rangle) \quad \Leftrightarrow \quad b = (ea)_0 \wedge (ea)_1 \Vdash \phi(z, q).$$

By the validity of **DC** in $\mathtt{V}$, there exists a function $F : \omega \longrightarrow t$ with:

$$F(0) = \langle (e^*)_0, u_0\rangle \text{ and for each } n \in \omega, \quad \phi^{\Vdash}(F(n), F(n+1)).$$

Notice that, we think of $\omega \in \mathcal{A}$ as in (3.1.7).
Next, we need to internalize $F$ and show that it provides the function required for the validity of **DC** in $\mathtt{V}(\mathcal{A})$. If $x$ is an ordered pair $\langle u, v\rangle$, we use $(x)^s_0$ and $(x)^s_1$ to denote its standard set-theoretic projections, i.e., $(x)^s_0 = u$ and $(x)^s_1 = v$.
Let $\overline{F}$ (the internalization of $F$) be defined as follows:

$$\overline{F} := \{\langle \mathbf{p}(\underline{n}, (F(n))^s_0), \langle \overline{n}, (F(n))^s_1\rangle_{\mathcal{A}}\rangle : n \in \omega\}.$$

Clearly, $\overline{F} \in \mathtt{V}(\mathcal{A})$ as $\mathbf{p}(\underline{n}, (F(n))^s_0) \in |\mathcal{A}|$ and $\langle \overline{n}, (F(n))^s_1\rangle_{\mathcal{A}} \in \mathtt{V}(\mathcal{A})$ (by internal pairing properties).
Now, we need to check that $\overline{F}$ is internally a function from $\overline{\omega}$ to $t$.
Firstly, we show that $\mathtt{V}(\mathcal{A})$ thinks that $\overline{F}$ is a binary relation with domain $\overline{\omega}$ and range a subset of $t$ using properties of internal pairing in $\mathtt{V}(\mathcal{A})$.
To prove that $\overline{F}$ is realizably functional, suppose that:

$$h \Vdash \langle \overline{n}, x\rangle_{\mathcal{A}} \in \overline{F} \tag{3.9}$$

and

$$k \Vdash \langle \overline{n}, y \rangle_{\mathcal{A}} \in \overline{F}. \tag{3.10}$$

Then, (3.9) is equivalent to the existence of an element $c \in \mathtt{V}(\mathcal{A})$ such that:

$$\langle (h)_0, c \rangle \in \overline{F} \wedge (h)_1 \Vdash \langle \overline{n}, x \rangle_{\mathcal{A}} = c. \tag{3.11}$$

$\langle (h)_0, c \rangle \in \overline{F}$ yields that $(h)_0$ must have the form $\mathbf{p}(\underline{m}, (F(m))_0^s)$ and $c$ be of the form $\langle \overline{m}, (F(m))_1^s \rangle_{\mathcal{A}}$ for some $m \in \omega$. But (3.11) entails that $\mathtt{V}(\mathcal{A}) \Vdash \bar{n} = \bar{m}$ using (3.1.11) and hence $n = m$ by Lemma 3.1.9. Hence, $h_{00} = \underline{n}$ and $c = \langle \overline{n}, (F(n))_1^s \rangle_{\mathcal{A}}$, where $h_{00}$ is an abbreviation for $((h)_0)_0$. Thus,

$$(h)_1 \Vdash \langle \overline{n}, x \rangle_{\mathcal{A}} = \langle \overline{n}, (F(n))_1^s \rangle_{\mathcal{A}}. \tag{3.12}$$

Likewise (3.10) yields

$$(k)_1 \Vdash \langle \overline{n}, y \rangle_{\mathcal{A}} = \langle \overline{n}, (F(n))_1^s \rangle_{\mathcal{A}}. \tag{3.13}$$

Using Lemma 3.1.11 it follows from (3.12) and (3.13) that we get a realizer $e'$ such that $e' \Vdash x = y$ and $e'$ can be computably obtained from $e, h, k$, showing that $\overline{F}$ is realizably functional.

Next, to verify that $\mathtt{V}(\mathcal{A}) \models \overline{F} \subseteq \overline{\omega} \times t$, suppose that $h \Vdash \langle x, y \rangle_{\mathcal{A}} \in \overline{F}$. Then, using arguments as before:

$$(h)_1 \Vdash \langle x, y \rangle_{\mathcal{A}} = \langle \overline{n}, (F(n))_1^s \rangle_{\mathcal{A}}$$

with $h_{00} = \underline{n}$. We also have $\mathbf{p}(h_{00}, \mathbf{i}_r) \Vdash \bar{n} \in \bar{\omega}$ and, thanks to the definition of $\overline{F}$, $\mathbf{p}(h_{01}, \mathbf{i}_r) \Vdash (F(n))_1^s \in t$. Thus we can computably obtain $h^*$ from $h$ such that $h^* \Vdash x \in \bar{\omega} \ \wedge \ y \in t$.

Finally, we need to show the realizability of $\overline{F}(0) = u$ (where 0 stands for the empty set in the sense of $\mathtt{V}(\mathcal{A})$ which really can be taken to be the empty set) and of $\forall u \in \overline{\omega} \phi(\overline{F}(u), \overline{F}(u+1))$.

As for the realizability of $\overline{F}(0) = u$, suppose that $r \Vdash \langle 0, u_0 \rangle_{\mathcal{A}} \in \overline{F}$. Then, there exists $c$ such that $\langle (r)_0, c \rangle \in \overline{F} \wedge (r)_1 \Vdash \langle 0, u_0 \rangle_{\mathcal{A}} = c$. $\langle (r)_0, c \rangle \in \overline{F}$ entails that $(r)_0$ has the form $\mathbf{p}(\underline{n}, (F(n))_0^s)$ and $c$ has the form $\langle \overline{n}, (F(n))^s_1 \rangle_{\mathcal{A}}$. Hence, $(r)_1 \Vdash \langle 0, u_0 \rangle_{\mathcal{A}} = \langle \overline{n}, (F(n))^s_1 \rangle_{\mathcal{A}}$. As the latter implies $\mathtt{V}(\mathcal{A}) \models 0 = \bar{n}$ by the internal pairing properties, this forces $n = 0$ by (3.1.9), and hence $\mathtt{V}(\mathcal{A}) \models u_0 = (F(0))^s_1$, so that $(r)_1 = \mathbf{i}_r$ and $\mathbf{p}(\mathbf{p}(\mathbf{0}, (e^*)_0), \mathbf{i}_r) \Vdash \langle 0, u_0 \rangle_{\mathcal{A}} \in \overline{F}$ *i.e.* $\Vdash \overline{F}(0) = u_0$ and since $(e^*)_1 \Vdash u = u_0$, there is a realizer $e'$ that can be computably obtained from $r$ and $e^*$ such that $e' \Vdash \overline{F}(0) = u$.

Next, we deal with the realizability of $\forall u \in \overline{\omega}\, \phi(\overline{F}(u), \overline{F}(u+1))$. Since for all $n \in \omega$ we have $[\phi^{\Vdash}(F(n), F(n+1))]$,

$$((F(n+1))^s_0 = (e(F(n))^s_0)_0 \text{ and} \tag{3.14}$$

$$(e(F(n))^s_0)_1 \Vdash \phi((F(n))^s_1, (F(n+1))^s_1). \tag{3.15}$$

Using the recursion theorem for applicative structures, we computably obtain $\rho \in |\mathcal{A}|$ from $e, e^*$ such that

$$\rho\underline{0} = (e^*)_0 \text{ and } \rho(\underline{n+1}) = (e\rho(\underline{n}))_0.$$

Using induction on $n$, it follows that $\rho\underline{n} = (F(n))^s_0$ for all $n \in \omega$. Further, by induction on $n$, it can be shown that:

(a) $\mathbf{p}(\mathbf{p}(\underline{n}, \rho\underline{n}), \mathbf{i}_r) \Vdash \langle \overline{n}, (F(n))^s_1 \rangle_{\mathcal{A}} \in \overline{F}$.

(b) $(e(\rho\underline{n}))_1 \Vdash \phi((F(n))^s_1, (F(n+1))^s_1)$.

To verify this, let $n = 0$ and we show that:

(a) $\mathbf{p}(\mathbf{p}(\mathbf{0})(\rho(\mathbf{0})))(\mathbf{i}_r) \Vdash \langle \overline{0}, (F(0))^s_1 \rangle_{\mathcal{A}} \in \overline{F}$, *i.e* $\mathbf{p}(\mathbf{p}(\mathbf{0})((e^*))_0)(\mathbf{i}_r) \Vdash \langle \overline{0}, u_0 \rangle_{\mathcal{A}} \in \overline{F}$, holds by the above argument.

(b) $(e\rho\mathbf{0})_1 \Vdash \phi((F(0))^s_1, (F(1))^s_1)$, *i.e* $(e(e^*)_0))_1 \Vdash \phi(u_0, (F(1))^s_1)$.
To this end, since $\forall n \in \omega[\phi^{\Vdash}(F(n), F(n+1))]$, we have:

$$\begin{aligned}
(e(F(n))^s_0)_1 &\Vdash \phi((F(n))^s_1, (F(n+1))^s_1) \\
(e(F(0))^s_0)_1 &\Vdash \phi((F(0))^s_1, (F(1))^s_1) \\
(e(e^*)_0)_1 &\Vdash \phi(u_0, (F(1))^s_1)
\end{aligned}$$

Next, we do the induction step, so assume the result for $n$ and we show that:

(a) $\mathbf{p}(\mathbf{p}(\underline{n+1})(\rho(\underline{n+1})))(\mathbf{i}_r) \Vdash \langle\overline{n+1}, (F(n+1))^s_1\rangle_{\mathcal{A}} \in \overline{F}$.

To show this, assume that $r \Vdash \langle\overline{n+1}, (F(n+1))^s_1\rangle_{\mathcal{A}} \in \overline{F}$. Then, $r \Vdash \langle\overline{n+1}, (F(n+1))^s_1\rangle_{\mathcal{A}} \in \overline{F} \Leftrightarrow \exists c[\langle(r)_0, c\rangle \in \overline{F} \wedge (r)_1 \Vdash \langle\overline{n+1}, (F(n+1))^s_1\rangle_{\mathcal{A}} = c]$ and hence, $(r)_0$ must have the form $(\underline{m}, (F(m))^s_0)$ and $c$ must have the form $\langle\overline{m}, (F(m))^s_1\rangle_{\mathcal{A}}$ for some $m \in \omega$. So, $(r)_1 \Vdash \langle\overline{n+1}, (F(n+1))^s_1\rangle_{\mathcal{A}} = \langle\overline{m}, (F(m))^s_1\rangle_{\mathcal{A}}$ which implies that a realizer $\hat{r}$ can be calculated such that $\hat{r} \Vdash \overline{n+1} = \overline{m}$ which, by (3.1.9), yields that $n+1 = m$ and by (3.1.8) we obtain that $\underline{n+1} = \underline{m}$. So, by the induction hypothesis we have $\mathbf{p}(\mathbf{p}(\underline{n+1})(\rho(\underline{n+1})))(\mathbf{i}_r) \Vdash \langle\overline{n+1}, (F(n+1))^s_1\rangle_{\mathcal{A}} \in \overline{F}$.

(b) $(e\rho(\underline{n+1}))_1 \Vdash \phi((F(n+1))^s_1, (F(n+2))^s_1)$. Towards this goal, we know that $(e(F(n+1))^s_0)_1 \Vdash \phi((F(n+1))^s_1, (F(n+2))^s_1)$ by (3.15). Note that:

$$\begin{aligned}
\rho(\underline{0}) = (e^*)_0 &= (F(0))^s_0 \\
\rho(\underline{1}) &= (e(F(0))^s_0)_0 \\
\rho(\underline{2}) &= (e(e(F(0))^s_0)_0)_0 \\
\rho(\underline{n+1}) &= (\underbrace{e(e...(e}_{\text{n-times}}(F(0))^s_0)_0)_0)_0
\end{aligned}$$

and

$$\begin{aligned}
(F(1))^s_0 &= (e(F(0))^s_0)_0 \\
(F(2))^s_0 &= (e(e(F(0))^s_0)_0)_0 \\
(F(3))^s_0 &= (e(e(e(F(0))^s_0)_0)_0)_0 \\
(F(n+1))^s_0 &= (\underbrace{e(e...(e}_{\text{n-times}}(F(0))^s_0)_0)_0)_0
\end{aligned}$$

Thus, clearly $\rho(\underline{n+1}) = (F(n+1))^s_0$. Therefore, $(e(\rho(\underline{n+1})))_1 \Vdash \phi((F(n+1))^s_1, (F(n+2))^s_1)$.

(iv) Given part $(iii)$ of this theorem, working in **CZF**+**RDC**, it is enough to show that $\mathtt{V}(\mathcal{A})$ validates the following schema,

$$\forall x(\phi(x) \rightarrow \exists y[\phi(y) \wedge \psi(x,y)]) \wedge \phi(a_0) \longrightarrow \exists s(a_0 \in s \wedge \forall x \in s \exists y \in s[\phi(y) \wedge \psi(x,y)]).$$

So, let $a_0 \in \mathtt{V}(\mathcal{A})$ and suppose the following hold:

$$e \Vdash \forall x(\phi(x) \to \exists y[\phi(y) \wedge \psi(x,y)]) \text{ and} \tag{3.16}$$

$$r \Vdash \phi(a_0) \tag{3.17}$$

Now, we have:

$$\begin{aligned} (3.16) \quad &\Leftrightarrow \quad \forall x \in \mathtt{V}(\mathcal{A})\ e \Vdash \phi(x) \to \exists y[\phi(y) \wedge \psi(x,y)]. \\ &\Leftrightarrow \quad \forall f \in |\mathcal{A}| \forall x \in \mathtt{V}(\mathcal{A})[f \Vdash \phi(x) \to ef \Vdash \exists y(\phi(y) \wedge \psi(x,y))]. \\ &\Leftrightarrow \quad \forall f \in |\mathcal{A}| \forall x \in \mathtt{V}(\mathcal{A})[f \Vdash \phi(x) \to \exists y \in \mathtt{V}(\mathcal{A}) ef \Vdash \phi(y) \wedge \psi(x,y)]. \end{aligned}$$

Thus, for all $f$ in $|\mathcal{A}|$ and for all $x \in \mathtt{V}(\mathcal{A})$ we have

$$f \Vdash \phi(x) \to \exists y \in \mathtt{V}(\mathcal{A})\,[(ef)_0 \Vdash \phi(y) \wedge (ef)_1 \Vdash \psi(x,y)].$$

Let $\mathbf{N} = \{\underline{n} \mid n \in \omega\}$. By applying **RDC** to the above, we conclude that there are functions $i : \mathbf{N} \longrightarrow \mathcal{A}$, $j : \mathbf{N} \longrightarrow \mathcal{A}$ and $l : \omega \longrightarrow \mathtt{V}(\mathcal{A})$ with $i(\underline{0}) = r$, $l(0) = a_0$ and for all $n$ in $\omega$, we have:

$$i(\underline{n}) \Vdash \phi(l(n)) \text{ and } j(\underline{n}) \Vdash \psi(l(n), l(n+1)),$$

$$i(\underline{n+1}) = (ei(\underline{n}))_0 \text{ and } j(\underline{n}) = (ei(\underline{n}))_1.$$

Using the recursion theorem for $\mathcal{A}$, one can explicitly calculate $t_i, t_j \in |\mathcal{A}|$ from $e$ and $r$ such for all $n \in \omega$, $i(\underline{n}) = t_i\underline{n}$ and $j(\underline{n}) = t_j\underline{n}$. And thus the function $h : \mathbf{N} \longrightarrow \mathcal{A}$ defined by $h(\underline{n}) = \mathbf{p}(\underline{n}, \mathbf{p}(i(\underline{n}), j(\underline{n})))$ for some $n \in \omega$ is representable in $|\mathcal{A}|$ via an element $t_h$ computable from $e$ and $r$ as well, i.e., $h(\underline{n}) = t_h\underline{n}$ for all $n \in \omega$.

Now, set

$$B := \{\langle h(\underline{n}), l(n)\rangle : n \in \omega\}.$$

$B \in \mathtt{V}(\mathcal{A})$, since $h(\underline{n}) \in |\mathcal{A}|$ and $l(n) \in \mathtt{V}(\mathcal{A})$.

Now, we need to find a realizer $e^*$ such that:

$$e^* \Vdash a_0 \in B. \tag{3.18}$$

As

$$e^* \Vdash a_0 \in B \Leftrightarrow \exists c[\langle (e^*)_0, c\rangle \in B \wedge (e^*)_1 \Vdash a_0 = c]$$

and $\langle (e^*)_0, c\rangle \in B$ iff $\langle (e^*)_0, c\rangle = \langle h(\underline{n}), l(n)\rangle$, so $(e^*)_0 = h(\underline{n})$ and $c = l(n) = a_0$. Since $l$ is a function, $n$ must be 0 and thus $e^* = \mathbf{p}(h(\underline{0}), \mathbf{i}_r)$. So, $(h(\underline{0}), \mathbf{i}_r) \Vdash a_0 \in B$.

Furthermore, for $\langle k, u\rangle \in B$ we have $k = h(\underline{n})$ for some $n \in \omega$. Consequently, $(h(\underline{n}))_0 = \underline{n} = (k)_0$ and $u = l((k)_0)$, hence $\langle h(\mathbf{S_N}((k)_0)), l((k)_0 + 1)\rangle \in B$. Moreover, since $(k)_1 = \mathbf{p}(i(\underline{n}), j(\underline{n}))$ we have $k_{1,0} = i(\underline{n})$, thus $k_{1,0} \Vdash \phi(l(n))$, so $k_{1,0} \Vdash \phi(u)$ and $k_{1,1} = j(\underline{n})$ which implies $k_{1,1} \Vdash \psi(u, l((k)_0 + 1))$.

Therefore,

$$\forall \langle k, u\rangle \in B\, \exists v\, [\langle h(\mathbf{S_N}((k)_0)), v\rangle \in B \;\wedge\; k_{1,0} \Vdash \phi(u) \wedge k_{1,1} \Vdash \psi(u, v)]. \quad (3.19)$$

From (3.18) and (3.19), it is clear that there exists a realizer $\grave{e}$ computed from $e$ and $r$ such that:

$$\grave{e} \Vdash a_0 \in B \wedge \forall x \in B \exists y \in B[\phi(x) \wedge \psi(x, y)]$$

and hence,

$$\grave{e} \Vdash \exists s(a_0 \in s \wedge \forall x \in s \exists y \in s[\phi(x) \wedge \psi(x, y)])$$

which completes the proof of $(iv)$.

(v) Let $s \in \mathtt{V}(\mathcal{A})$. We are aiming to find a set $B^* \in \mathtt{V}(\mathcal{A})$ such that $\mathtt{V}(\mathcal{A})$ believes that $B^*$ is a base that maps onto $s$.

Since **PAx** holds in the background model $\mathtt{V}$, we can choose a base $B$ and a surjective map $j : B \longrightarrow s$. As $s$ is a set of ordered pairs, we may define:

$$j_0 : B \longrightarrow \mathcal{A} \text{ and } j_1 : B \longrightarrow \mathtt{V}(\mathcal{A})$$

by

$$j_0(u) = 1^{st}(j(u)) \text{ and } j_1(u) = 2^{nd}(j(u))$$

where these functions denote the standard pojections of ordered pairs in set theory. Using transfinite recursion, for any set $x$, we define:

$$x^{st} = \{\langle \mathbf{0}, y^{st}\rangle : y \in x\}.$$

$x^{st} \in \mathtt{V}(\mathcal{A})$ is straightforwardly proved by $\in$-induction as follows:
Inductively assume that $y^{st} \in \mathtt{V}(\mathcal{A})$ for all $y \in x$. As $\mathbf{0} \in \mathcal{A}$ this implies that $\{\langle \mathbf{0}, y^{st}\rangle \mid y \in x\} \subseteq |\mathcal{A}| \times \mathtt{V}(\mathcal{A})$, and thus $x^{st} \in \mathtt{V}(\mathcal{A})$ by (3.1.2) part *(ii)*.
To complete the proof we need the following facts.

**Proposition 3.1.13.**

*(i)*

$$x = y \quad \textit{iff} \quad \mathtt{V}(\mathcal{A}) \models x^{st} = y^{st}.$$

*(ii)*

$$x \in y \quad \textit{iff} \quad \mathtt{V}(\mathcal{A}) \models x^{st} \in y^{st}.$$

**Proof.** We show (i) and (ii) by simultaneous $\in$-induction as follows:

(i): The implication from left to right is immediate. As for the other direction, suppose that $e \Vdash x^{st} = y^{st}$. Then,

$$\forall \langle f, u\rangle \in x^{st}\ ((e)_0 f \Vdash u \in y^{st}) \wedge \forall \langle f, u\rangle \in y^{st}\ ((e)_1 f \Vdash u \in x^{st}).$$

If $z \in x$ then $\langle \mathbf{0}, z^{st}\rangle \in x^{st}$, thus $\mathtt{V}(\mathcal{A}) \models z^{st} \in y^{st}$ and thus inductively $z \in y$. By a symmetric argument, $z \in y$ yields $z \in x$. Hence $x = y$.

(ii): Again the left to right direction is obvious. Suppose $\mathtt{V}(\mathcal{A}) \models x^{st} \in y^{st}$. Then there exists $z \in y$ such that $\mathtt{V}(\mathcal{A}) \models x^{st} = z^{st}$, and therefore inductively $x = z$, and hence $x \in y$.

□

As a result, the map taking $x$ to $x^{st}$ is an injection from $\mathtt{V}$ to $\mathtt{V}(\mathcal{A})$.
Next, let

$$B^* := \{\langle j_0(u), \langle (j_0(u))^{st}, u^{st}\rangle_{\mathcal{A}}\rangle : u \in B\}.$$

Note that the map $u \longmapsto \langle j_0(u), \langle (j_0(u))^{st}, u^{st}\rangle_{\mathcal{A}}\rangle$ injects $B$ onto $B^*$ and hence $B^*$ is a base (in the sense of the ground universe).

Define

$$l : B \longrightarrow \mathtt{V}(\mathcal{A})$$

such that $l(u) = \langle (j_0(u))^{st}, u^{st}\rangle_{\mathcal{A}}$ and let

$$j^* := \{\langle j_0(u), \langle l(u), j_1(u)\rangle_{\mathcal{A}}\rangle : u \in B\}.$$

As $j_0(u) \in \mathcal{A}$ and $l(u), j_1(u) \in \mathtt{V}(\mathcal{A})$, it follows that $j^* \in \mathtt{V}(\mathcal{A})$. Now, we claim that:

$$\mathtt{V}(\mathcal{A}) \models j^* \text{ maps } B^* \text{ onto } s. \tag{3.20}$$

Firstly, we verify that $\mathtt{V}(\mathcal{A}) \models j^* \subseteq B^* \times s$. Towards this goal, assume that $e \Vdash \langle b, c\rangle_{\mathcal{A}} \in j^*$. Then there exists $f$ such that $\langle (e)_0, f\rangle \in j^*$ and $(e)_1 \Vdash \langle b, c\rangle_{\mathcal{A}} = f$. However, $\langle (e)_0, f\rangle \in j^*$ means that there exists $u \in B$ with $(e)_0 = j_0(u)$ and $f = \langle l(u), j_1(u)\rangle_{\mathcal{A}}$, and hence $(e)_1 \Vdash \langle b, c\rangle_{\mathcal{A}} = \langle l(u), j_1(u)\rangle_{\mathcal{A}}$.
So, we need to find realizers $r \Vdash l(u) \in B^*$ and $r^* \Vdash j_1(u) \in s$.
Well, $r \Vdash l(u) \in B^* \Leftrightarrow \exists c[\langle (r)_0, c\rangle \in B^* \wedge (r)_1 \Vdash l(u) = c]$. However, $\langle (r)_0, c\rangle \in B^*$ implies $(r)_0 = j_0(u)$ and $c = \langle (j_0(u))^{st}, u^{st}\rangle_{\mathcal{A}} = l(u)$ for some $u \in B$, and hence letting $r := \mathbf{p}(j_0(u), \mathbf{i}_r)$ we get get $r \Vdash l(u) \in B^*$.
$r^* \Vdash j_1(u) \in s$ is equivalent to the existence of a $c$ such that $\langle (r^*)_0, c\rangle \in s \wedge (r^*)_1 \Vdash j_1(u) = c$. It follows from $\langle (r^*)_0, c\rangle \in s$ that $(r^*)_0 = j_0(u)$ and $c = j_1(u)$ and hence, with $r^* := \mathbf{p}(j_0(u), \mathbf{i}_r)$ we have $r^* \Vdash j_1(u) \in s$.
Therefore, a realizer $e^*$ can be computed from $e$ such that

$$e^* \Vdash b \in B^* \wedge c \in s$$

which shows that $\mathtt{V}(\mathcal{A}) \models j^* \subseteq B^* \times s$.
To verify that $j^*$ is realizably total on $B^*$, assume $e \Vdash \langle c, d\rangle_{\mathcal{A}} \in B^*$. Then there exists $f$ such that $\langle (e)_0, f\rangle \in B^* \wedge (e)_1 \Vdash \langle c, d\rangle_{\mathcal{A}} = f$. But $\langle (e)_0, f\rangle \in B^*$ has the form $\langle j_0(u), \langle (j_0(u))^{st}, u^{st}\rangle_{\mathcal{A}}\rangle$ for some $u \in B$. So, $(e)_0 = j_0(u)$ and $f = \langle (j_0(u))^{st}, u^{st}\rangle_{\mathcal{A}} = l(u)$ for some $u \in B$. Thus, for some $u \in B$ we have, $(e)_0 = j_0(u)$ and $(e)_1 \Vdash \langle c, d\rangle_{\mathcal{A}} = l(u)$.

Since $\mathbf{p}(j_0(u), \mathbf{i}_r) \Vdash j_1(u) \in s$ and

$$\mathbf{p}(j_0(u), \mathbf{i}_r) \Vdash \langle l(u), j_1(u)\rangle_{\mathcal{A}} \in j^* \tag{3.21}$$

a realizer $\hat{e}$ can be computed from $e$ such that:

$$\hat{e} \Vdash \langle c, d\rangle_{\mathcal{A}} \textit{ is in the domain of } j^*.$$

To verify (3.21) let $r \Vdash \langle l(u), j_1(u)\rangle_{\mathcal{A}} \in j^*$. Then,
$r \Vdash \langle l(u), j_1(u)\rangle_{\mathcal{A}} \in j^* \Leftrightarrow \exists c[\langle (r)_0, c\rangle \in j^* \wedge (r)_1 \Vdash \langle l(u), j_1(u)\rangle_{\mathcal{A}} = c]$.
From $\langle (r)_0, c\rangle \in j^*$, it follows that $(r)_0 = j_0(u)$ and $c = \langle l(u), j_1(u)\rangle_{\mathcal{A}}$ for some $u \in B$, and hence $\mathbf{p}(j_0(u), \mathbf{i}_r) \Vdash \langle l(u), j_1(u)\rangle_{\mathcal{A}} \in j^*$.
Hence, since we have already verified that $\mathtt{V}(\mathcal{A}) \models j^* \subseteq B^* \times s$, we can infer that $\mathtt{V}(\mathcal{A}) \models B^*$ is the domain of $j^*$.
Next, we need to show that $j^*$ is realizably functional. To this end, assume that:

$$f \Vdash \langle b, c\rangle_{\mathcal{A}} \in j^* \tag{3.22}$$

$$h \Vdash \langle b, d\rangle_{\mathcal{A}} \in j^*. \tag{3.23}$$

So by (3.22) there is a $q$ such that $\langle (f)_0, q\rangle \in j^* \wedge (f)_1 \Vdash \langle b, c\rangle_{\mathcal{A}} = q$ which entails that $(f)_0$ has the form $j_0(u)$ and $q$ has the form $\langle l(u), j_1(u)\rangle_{\mathcal{A}}$ for some $u$ in $B$, so that $(f)_1 \Vdash \langle b, c\rangle_{\mathcal{A}} = \langle l(u), j_1(u)\rangle_{\mathcal{A}}$.
And from (3.23) we get that there is a $q'$ such that $\langle (h)_0, q'\rangle \in j^* \wedge (h)_1 \Vdash \langle b, d\rangle_{\mathcal{A}} = q'$, and similarly we obtain that $(h)_0$ has the form $j_0(v)$ and $q'$ has the form $\langle l(v), j_1(v)\rangle_{\mathcal{A}}$ for some $v \in B$ and hence, $(h)_1 \Vdash \langle b, d\rangle_{\mathcal{A}} = \langle l(v), j_1(v)\rangle_{\mathcal{A}}$.
Therefore a realizer $r$ can be extracted such that $r \Vdash l(u) = l(v)$, so $\mathtt{V}(\mathcal{A}) \models l(u) = l(v)$. By the definition of $l$, we have:

$$\begin{aligned}
\mathtt{V}(\mathcal{A}) &\models l(u) = l(v) \\
\mathtt{V}(\mathcal{A}) &\models \langle (j_0(u))^{st}, u^{st}\rangle_{\mathcal{A}} = \langle (j_0(v))^{st}, v^{st}\rangle_{\mathcal{A}} \\
&\Leftrightarrow \mathtt{V}(\mathcal{A}) \models u^{st} = v^{st} \\
&\Leftrightarrow u = v \quad \text{by (3.1.13).}
\end{aligned}$$

Therefore, there is a realizer $\grave{e}$ computable from $f$, $h$ such that $\grave{e} \Vdash c = d$.

Next, we need to show that $j^*$ is realizably surjective. To this end, suppose that $e \Vdash x \in s$. Then there exists a $c$ such that $\langle (e)_0, c\rangle \in s \wedge (e)_1 \Vdash x = c$. $\langle (e)_0, c\rangle \in s$ implies that $\langle (e)_0, c\rangle$ has the form $\langle j_0(u), j_1(u)\rangle$ for some $u \in B$ because $j : B \longrightarrow s$ maps $B$ onto $s$. Furthermore, since $\mathbf{p}(j_0(u), \mathbf{i}_r) \Vdash l(u) \in B^*$ and $\mathbf{p}(j_0(u), \mathbf{i}_r) \Vdash \langle l(u), j_1(u)\rangle_{\mathcal{A}} \in j^*$, it follows that a realizer $\tilde{e}$ can be calculated such that

$$\tilde{e} \Vdash x \textit{ in the range of } j^*.$$

This completes the proof of (3.20).

Finally, we need to verify that $\mathtt{V}(\mathcal{A})$ believes that $B^*$ is a base. To verify this, suppose that:

$$e \Vdash \forall x \in B^* \exists y \phi(x, y) \text{ for some formula } \phi. \tag{3.24}$$

Now, we are aiming to compute a realizer $e^{**}$ calculable from $e$ satisfying:

$$e^{**} \Vdash \exists H[\mathbf{Fun}(H) \wedge \mathbf{dom}(H) = B^* \wedge \forall x \in B^* \phi(x, H(x))] \tag{3.25}$$

$$\begin{aligned} \text{Note that } e \Vdash \forall x \in B^* \exists y \phi(x, y) \quad &\Leftrightarrow \quad \forall \langle q, c\rangle \in B^* \quad eq \Vdash \exists y \phi(c, y) \\ &\Leftrightarrow \quad \forall \langle q, c\rangle \in B^* \exists d \in \mathtt{V}(\mathcal{A}) \quad eq \Vdash \phi(c, d). \end{aligned}$$

Hence, from (3.24) it follows that:

$$\forall \langle q, c\rangle \in B^* \exists y \in \mathtt{V}(\mathcal{A}) \quad eq \Vdash \phi(c, y).$$

Now, because $B^*$ is a base in the ground universe, there is a function

$$F : B^* \longrightarrow \mathtt{V}(\mathcal{A})$$

such that

$$\forall \langle q, c\rangle \in B^* \quad eq \Vdash \phi(c, F(\langle q, c\rangle)).$$

Next, we need an internalization of $F$ namely $\tilde{F}$, defined by:

$$\tilde{F} := \{\langle \mathbf{p}(eq, q), \langle c, F(\langle q, c\rangle)\rangle_{\mathcal{A}}\rangle : \langle q, c\rangle \in B^*\}.$$

Since, $eq \in \mathcal{A}$, $\mathbf{p}(eq, q) \in \mathcal{A}$, $c \in \mathtt{V}(\mathcal{A})$, $\langle q, c\rangle \in B^* \in \mathtt{V}(\mathcal{A})$ and also $F(\langle q, c\rangle) \in \mathtt{V}(\mathcal{A})$, we can deduce that $\tilde{F} \in \mathtt{V}(\mathcal{A})$.

First, we need to show that $\mathtt{V}(\mathcal{A}) \models \mathbf{dom}(\tilde{F}) = B^*$. Towards this goal, suppose that $h \Vdash x \in B^*$. Then there exist $c$ such that $\langle (h)_0, c\rangle \in B^*$ and $(h)_1 \Vdash x = c$. $\langle (h)_0, c\rangle \in B^*$ yields that

$$\mathbf{p}(\mathbf{p}(e(h)_0, (h)_0), \mathbf{i}_r) \Vdash \langle c, F(\langle (h)_0, c\rangle)\rangle_{\mathcal{A}} \in \tilde{F},$$

from which we can effectively construct a realizer $\hat{h}$ such that $\hat{h} \Vdash x \in \mathbf{dom}(\tilde{F})$.

Conversely, assume that $d \Vdash \langle x, y\rangle_{\mathcal{A}} \in \tilde{F}$. Then there exists $\langle q, c\rangle \in B^*$ such that $\langle (d)_0, \langle c, F(\langle q, c\rangle)\rangle_{\mathcal{A}}\rangle \in \tilde{F}$ where $q = ((d)_0)_1$ and $(d)_1 \Vdash \langle x, y\rangle_{\mathcal{A}} = \langle c, F(\langle q, c\rangle)\rangle_{\mathcal{A}}$. Consequently, $\mathbf{p}(((d)_0)_1, \mathbf{i}_r) \Vdash c \in B^*$. Therefore we can calculate an index $d^*$ from $d$ such that $d^* \Vdash x \in B^*$.

Finally, we have all the pieces to construct $\hat{e}$ from $e$ such that

$$\hat{e} \Vdash \mathbf{dom}(\tilde{F}) = B^*.$$

Next, it remains to show that $\tilde{F}$ is realizably functional. To this end, suppose:

$$f \Vdash \langle b, c\rangle_{\mathcal{A}} \in \tilde{F} \tag{3.26}$$

$$h \Vdash \langle b, d\rangle_{\mathcal{A}} \in \tilde{F}. \tag{3.27}$$

(3.26) and (3.27) provide $\langle q, x\rangle, \langle q', y\rangle \in B^*$ such that $((f)_0)_1 = q$, $((h)_0)_1 = q'$, and

$$\begin{aligned} (f)_1 &\Vdash \langle b, c\rangle_{\mathcal{A}} = \langle x, F(\langle q, x\rangle)\rangle_{\mathcal{A}} \wedge \\ (h)_1 &\Vdash \langle b, d\rangle_{\mathcal{A}} = \langle y, F(\langle q', y\rangle)\rangle_{\mathcal{A}}. \end{aligned}$$

The latter yields $\mathtt{V}(\mathcal{A}) \models x = y$. Since $\langle q, x\rangle, \langle q', y\rangle \in B^*$ there exist $u, v \in B$ satisfying

$$x = \langle (j_0(u))^{st}, u^{st}\rangle_{\mathcal{A}} \text{ and } q = j_0(u)$$

as well as

$$y = \langle (j_0(v))^{st}, v^{st}\rangle_{\mathcal{A}} \text{ and } q' = j_0(v).$$

As $\mathtt{V}(\mathcal{A})\models x = y$, the above implies

$$\mathtt{V}(\mathcal{A}) \Vdash q^{st} = (q')^{st} \wedge u^{st} = v^{st},$$

and so by Proposition 3.1.13, we arrive at $q = q'$ and $u = v$, which also yields $x = y$ and $F(\langle q, x\rangle) = F(\langle q', y\rangle)$. Thus, also taking (3.28) into account, we can construct a realizer $\nu$ such that $\nu f h \Vdash c = d$. This verifies the functionality of $\tilde{F}$, so $\mathtt{V}(\mathcal{A})\models\tilde{F}$ *is a function*.

In sum, taking all the foregoing together, we can calculate in $\mathcal{A}$ a realizer $e^{**}$ from $e$ such that (3.25) holds.

This completes the proof.

□

Therefore, by the recursion theorem for applicative structures and utilizing the fact that applicative structures have a copy of natural numbers, Theorem 10.1 in [28] was generalised to work for an arbitrary applicative structure, $\mathcal{A}$, (the previous theorem). It worths mentioning here that since the *full axiom of choice* implies **LEM**, see Theorem 1.1 in [8], it cannot be realized in $\mathtt{V}(\mathcal{A})$.

# Chapter 4

# Realizability Structure Over Scott $D_\infty$ Models

## 4.1 Realizability in $\mathtt{V}(D_\infty)$

For sets $A, B$ we use the notation $A \equiv B$ to convey that there exists a bijection between $A$ and $B$. Moreover, for ease of notation and since we are not mixing $\mathbf{x} \in D_\infty$ with its components $x_n$, in what follows we shall write $x$ for $\mathbf{x}$.

**Theorem 4.1.1.** *Let $A \subseteq \mathtt{V}(D_\infty)$ be the set:*

$$A := \{\langle x, x^{st}\rangle : x \in D_\infty\}$$

*Then, we have*

*(i)* $\mathtt{V}(D_\infty) \models A \equiv A \longrightarrow A \;\wedge\; \exists f\,[f : \mathbb{N} \longrightarrow A \;\wedge\; f$ *is injective*$]$.

*(ii) If* $\mathtt{V} \models A$ *is a base then* $\mathtt{V}(D_\infty) \models A$ *is a base.*

**Proof.** (i): First we'd like to find a realizer for the statement $\exists g\,[g : \mathbb{N} \longrightarrow A \wedge g$ is injective$]$.
Define $G : \mathbb{N} \longrightarrow A$ by:

$$G := \{\langle \underline{n}, \langle \bar{n}, \underline{n}^{st}\rangle_{D_\infty}\rangle : n \in \mathbb{N}\}$$

(where perhaps $\underline{n}^{st}$ less ambiguously should be written as $(\underline{n})^{st}$).

Clearly, $G \in \mathtt{V}(D_\infty)$ as $\underline{n} \in D_\infty$ and $\langle \bar{n}, \underline{n}^{st} \rangle_{D_\infty} \in \mathtt{V}(D_\infty)$ by the internal pairing properties.

Now, we need to check that $G$ is internally a function from $\mathbb{N}$ into $A$.
Firstly, using properties of internal pairing in $\mathtt{V}(D_\infty)$, we show that $\mathtt{V}(D_\infty)$ believes that $G$ is a binary relation whose domain is $\bar{\omega}$ and range a subset of $A$. To show that $G$ is realizably functional, assume that:

$$h \Vdash \langle a, x \rangle_{D_\infty} \in G \tag{4.1}$$

$$k \Vdash \langle a, y \rangle_{D_\infty} \in G. \tag{4.2}$$

By (4.1) there exists a $b$ such that $\langle (h)_0, b \rangle \in G$ and $(h)_1 \Vdash \langle a, x \rangle_{D_\infty} = b$. As $\langle (h)_0, b \rangle \in G$, $(h)_0$ must be of the form $\underline{n}$ and $b = \langle \bar{n}, \underline{n}^{st} \rangle_{D_\infty}$ for some $n \in \mathbb{N}$. Thus,

$$(h)_1 \Vdash \langle a, x \rangle_{D_\infty} = \langle \bar{n}, \underline{n}^{st} \rangle_{D_\infty}. \tag{4.3}$$

Similarly, from (4.2) we infer the existence of an $l \in \mathbb{N}$ such that

$$(k)_1 \Vdash \langle a, y \rangle_{D_\infty} = \langle \bar{l}, \underline{l}^{st} \rangle_{D_\infty}. \tag{4.4}$$

Using the properties of internal pairing (Lemma 3.1.11), we derive from (4.3) and (4.4) that $\mathtt{V}(D_\infty) \models \bar{n} = a \wedge \bar{l} = a$ and thus $n = l$ by Lemma 3.1.4 and Proposition 3.1.9. Moreover, by Proposition 3.1.8 $\underline{n} = \underline{l}$ and thus $\underline{n}^{st} = \underline{l}^{st}$ using Proposition 3.1.13. Therefore, there is an application term $t$ such that such that $t(h, k) \Vdash x = y$, showing that $G$ is realizably functional.
Note also that from (4.1) alone we deduced (4.3) for some $n \in \mathbb{N}$, and hence there exists an application term $s$ showing that $s \Vdash G \subseteq \bar{\omega} \times A$.

Finally, we need to verify that $G$ is realizably injective. To this end, assume that:

$$h \Vdash \langle a, x \rangle_{D_\infty} \in G \tag{4.5}$$

$$k \Vdash \langle b, x \rangle_{D_\infty} \in G. \tag{4.6}$$

By (4.5) there exists a $c$ such that $\langle (h)_0, c \rangle \in G$ and $(h)_1 \Vdash \langle a, x \rangle_{D_\infty} = c$. As

$\langle (h)_0, c\rangle \in G$, $(h)_0$ must be of the form $\underline{n}$ and $c = \langle \bar{n}, \underline{n}^{st}\rangle_{D_\infty}$ for some $n \in \mathbb{N}$. Thus,

$$(h)_1 \Vdash \langle a, x\rangle_{D_\infty} = \langle \bar{n}, \underline{n}^{st}\rangle_{D_\infty}. \tag{4.7}$$

Similarly, from (4.6) we infer the existence of an $l \in \mathbb{N}$ such that

$$(k)_1 \Vdash \langle b, x\rangle_{D_\infty} = \langle \bar{l}, \underline{l}^{st}\rangle_{D_\infty}. \tag{4.8}$$

Using the properties of internal pairing (Lemma 3.1.11), we derive from (4.7) and (4.8) that $\mathtt{V}(D_\infty) \models \underline{n}^{st} = x \wedge \underline{l}^{st} = x$ and thus $n = l$ by Lemma 3.1.4 and Proposition 3.1.13. Therefore, in view of (4.7) and (4.8), and using Proposition 3.1.9 we can infer that $a = b$ is realized, i.e., we can define an application term $t'$ such that such that $t'(h, k) \Vdash a = b$, showing that $G$ is realizably injective.

Next, we need to show that there is a bijection between $A$ and $A \longrightarrow A$ in $\mathtt{V}(D_\infty)$. First, for each $e \in D_\infty$ we define an object

$$f_e \quad := \quad \{\langle e, \langle x^{st}, (ex)^{st}\rangle_{D_\infty}\rangle \mid x \in D_\infty\}. \tag{4.9}$$

In the definition of $f_e$ above we make use of the fact that $D_\infty$ is a total applicative structure namely that $ex$ is defined for all $x \in D_\infty$. One easily checks that $f_e \in \mathtt{V}(D_\infty)$ for all $e \in D_\infty$.

We first show that there is an application term $t$ such that

$$t \Vdash f_e \textit{ is a function from } A \textit{ to } A \tag{4.10}$$

holds for all $e \in D_\infty$. To show that $f_e$ is realizably functional, assume that:

$$h \Vdash \langle a, u\rangle_{D_\infty} \in f_e \tag{4.11}$$

$$k \Vdash \langle a, v\rangle_{D_\infty} \in f_e. \tag{4.12}$$

By (4.11) there exists a $b$ such that $\langle (h)_0, b\rangle \in f_e$ and $(h)_1 \Vdash \langle a, u\rangle_{D_\infty} = b$. As $\langle (h)_0, b\rangle \in f_e$, we conclude that $(h)_0 = e$ and $b = \langle x^{st}, (ex)^{st}\rangle_{D_\infty}$ for some $x \in D_\infty$. Thus,

$$(h)_1 \Vdash \langle a, u\rangle_{D_\infty} = \langle x^{st}, (ex)^{st}\rangle_{D_\infty}. \tag{4.13}$$

Similarly, from (4.12) we infer the existence of a $y \in D_\infty$ such that

$$(k)_1 \Vdash \langle a, v\rangle_{D_\infty} = \langle y^{st}, (ey)^{st}\rangle_{D_\infty}. \tag{4.14}$$

Using the properties of internal pairing (Lemma 3.1.11), we derive from (4.13) and (4.14) that $\mathtt{V}(D_\infty) \models x^{st} = y^{st}$ and thus $x = y$ by Proposition 3.1.13, and thus also $(ex)^{st} = (ey)^{st}$ which in view of (4.13) and (4.14) entails that $\mathtt{V}(D_\infty) \models u = v$. More precisely, we can distill an application term $s$ such that $s(h, k) \Vdash u = v$.

Concentrating solely on (4.11), the arguments above also show that $f_e$ realizably has domain and range $A$. Moreover, it is also clear that realizably every element of $A$ is in the domain of $f_e$. As a result there exists an application term $t$ such that $t \Vdash f_e$ *is a function from* $A$ *to* $A$ holds for all $e \in D_\infty$, verifying (4.10).

Now let

$$F \;:=\; \{\langle e, \langle e^{st}, f_e\rangle_{D_\infty}\rangle \mid e \in D_\infty\}. \tag{4.15}$$

Since $f_e \in \mathtt{V}(D_\infty)$ for all $e \in D_\infty$ we have $F \in \mathtt{V}(D_\infty)$. We want to show that $F$ is a bijection between $A$ and $A \longrightarrow A$ in $\mathtt{V}(D_\infty)$. Functionality of $F$ in $\mathtt{V}(D_\infty)$ can be shown in the same vein as for $f_e$. It is also clear that $F$ realizably has domain $A$ and from (4.10) it follows that $F$ realizably has a range consisting of functions from $A$ to $A$. In sum, there is an application term $r$ such that

$$r \Vdash F \text{ is a function from } A \text{ to } A \to A.$$

It remains to verify that $F$ realizably furnishes a bijection.

We first address injectivity. To this end, assume that:

$$h \Vdash \langle a, u\rangle_{D_\infty} \in F \tag{4.16}$$

$$k \Vdash \langle b, u\rangle_{D_\infty} \in F. \tag{4.17}$$

Then there exists an $e \in D_\infty$ such that $e = (h)_0$ and $(h)_1 \Vdash \langle a, u\rangle_{D_\infty} = \langle e^{st}, f_e\rangle_{D_\infty}$.

Similarly, from (4.17) we infer the existence of a $d \in D_\infty$ such that $d = (k)_0$ and $(k)_1 \Vdash \langle b, u\rangle_{D_\infty} = \langle d^{st}, f_d\rangle_{D_\infty}$. As a result of the above we obtain $\mathtt{V}(D_\infty) \models f_e = f_d$. Unraveling the definitions of $f_e$ and $f_d$, this then implies that $\mathtt{V}(D_\infty) \models (ex)^{st} =$

$(dx)^{st}$ and hence $ex = dx$ for all $x \in D_\infty$ by Proposition 3.1.13. Owing to the extensionality of $D_\infty$ this can only hold if $e = d$ or equivalently, if $e^{st} = d^{st}$ by Proposition 3.1.13 which by the above entails $s'(h,k) \Vdash a = b$ for some application term $s'$. Consequently we can construct an application term $s$ such that

$$s \Vdash F \textit{ is injective.}$$

To show surjectivity, assume that we have

$$\tilde{d} \Vdash g \text{ is a function from } A \text{ to } A. \tag{4.18}$$

Then there is an application term $t$ such that

$$t\tilde{d} \Vdash \forall u \in A \, \exists v \in A \, g(u) = v. \tag{4.19}$$

Let $d := t\tilde{d}$. Unraveling (4.19) we get that for all $x \in D_\infty$,

$$(dx)_1 \Vdash g(x^{st}) = ((dx)_0)^{st}. \tag{4.20}$$

Let $e := \lambda z.(dz)_0$ in $D_\infty$. One then constructs an application term $s$ such that for all $x \in D_\infty$,

$$s(\tilde{d}, x) \Vdash g(x^{st}) = f_e(x^{st}).$$

Since extensionality holds in $\mathtt{V}(D_\infty)$ one finds application terms $s', s''$ such that $s'\tilde{d} \Vdash g = f_e$ and furthermore $s''\tilde{d} \Vdash g = F(e^{st})$. As a result we can concoct an application term $r$ such that $r \Vdash F$ *maps onto* $A \to A$.

(ii) We need to find a realizer $\tilde{e}$ such that:

$\tilde{e} \Vdash A$ is a base . Towards this goal, suppose that:

$$e \Vdash \forall u \in A \exists v \, \phi(u, v) \text{ for some formula } \phi \tag{4.21}$$

The aim is to construct a realizer $\hat{e} = t\tilde{e}$ for some application term $t$, satisfying:

$$\hat{e} \Vdash \exists H[\mathbf{Fun}(H) \wedge \mathbf{dom}(H) = A \wedge \forall u \in A \, \phi(u, H(u))] \tag{4.22}$$

$$\text{Since } e \Vdash \forall u \in A \exists v \phi(u,v) \Leftrightarrow \forall \langle x, x^{st}\rangle \in A \quad ex \Vdash \exists v \phi(x^{st}, v)$$
$$\Leftrightarrow \forall \langle x, x^{st}\rangle \in A \exists a \in \mathtt{V}(D_\infty) \quad ex \Vdash \phi(x^{st}, a)$$

Hence, from (4.21) it follows that:

$$\forall \langle x, x^{st}\rangle \in A \exists v \in \mathtt{V}(D_\infty) \quad ex \Vdash \phi(x^{st}, v)$$

Now, since $A$ is a base in the background universe, $\mathtt{V}$, there is a function $F : A \longrightarrow \mathtt{V}(D_\infty)$ such that:

$$\forall \langle x, x^{st}\rangle \in A \quad ex \Vdash \phi(x^{st}, F(\langle x, x^{st}\rangle))$$

Next, we need to internalize $F$ and show that this internalization, $\tilde{F}$, defined below; provides the function required to show that $A$ is a base. Set

$$\tilde{F} := \{\langle \mathbf{p}(ex)(x), \langle x^{st}, F(\langle x, x^{st}\rangle)\rangle_{D_\infty}\rangle : \langle x, x^{st}\rangle \in A\}$$

Since, $x, ex \in D_\infty$ we have $\mathbf{p}(ex)(x) \in D_\infty$. Moreover, $x^{st}, F(\langle x, x^{st}\rangle) \in \mathtt{V}(D_\infty)$ entails that $\langle x^{st}, F(\langle x, x^{st}\rangle)\rangle_{D_\infty}$ and hence $\tilde{F} \in \mathtt{V}(D_\infty)$.
Next, we need to show that the following hold:

(a) $\mathbf{dom}(\tilde{F}) = A$.

To this end, suppose that $h \Vdash u \in A$. Then,

$$h \Vdash u \in A \Leftrightarrow \exists c[\langle (h)_0, c\rangle \in A \wedge (h)_1 \Vdash u = c.]$$

Now, let $r \Vdash \langle (h)_0^{st}, F(\langle (h)_0, (h)_0^{st}\rangle)\rangle_{D_\infty} \in \tilde{F}$. Then, we can find a $b \in \mathtt{V}(D_\infty)$ such that $\langle (r)_0, b\rangle \in \tilde{F} \ \wedge \ (r)_1 \Vdash \langle (h)_0^{st}, F(\langle (h)_0, (h)_0^{st}\rangle)\rangle_{D_\infty} = b$.
But $\langle (r)_0, b\rangle \in \tilde{F}$ implies that $(r)_0$ is of the form $\mathbf{p}(ex)(x)$ and $b$ is of the form $\langle x^{st}, F(\langle x, x^{st}\rangle)\rangle_{D_\infty}$ for some $\langle x, x^{st}\rangle \in A$ and thus taking $\langle (h)_0, ((h)_0)^{st}\rangle \in A$, it follows that $\mathbf{p}(\mathbf{p}(e(h)_0)((h)_0))(\mathbf{i}_r) = r$. Therefore, we conclude that there is an application term, $\hat{h}$, constructible from $h$ such that $\hat{h} \Vdash u \in \mathbf{dom}(\tilde{F})$.

Conversely, assume that $d \Vdash \langle u, v\rangle_{D_\infty} \in \tilde{F}$. Then, this is equivalent to the existence of $f \in D_\infty$ such that $\langle (d)_0, f\rangle \in \tilde{F} \ \wedge \ (d)_1 \Vdash \langle u, v\rangle_{D_\infty} = f$.
$\langle (d)_0, f\rangle \in \tilde{F}$ entails that $(d)_0$ has the form $\mathbf{p}(ex)(x)$ and $f$ is of the form $\langle x^{st}, F(\langle x, x^{st}\rangle)\rangle_{D_\infty}$ for some $\langle x, x^{st}\rangle \in A$. Hence, we have $d \Vdash \langle u, v\rangle_{D_\infty} \in \tilde{F}$ iff

there exists $\langle x, x^{st}\rangle \in A$ such that $\langle (d)_0, \langle x^{st}, F(\langle x, x^{st}\rangle)\rangle_{D_\infty}\rangle \in \tilde{F}$ where $x = d_{01}$ and $(d)_1 \Vdash \langle u, v\rangle_{D_\infty} = \langle x^{st}, F(\langle x, x^{st}\rangle)\rangle_{D_\infty}$.

Let $s \Vdash x^{st} \in A$. Then, this entails the existing of a $c \in \mathtt{V}(D_\infty)$ such that $\langle (s)_0, c\rangle \in A$ and $(s)_1 \Vdash x^{st} = c$. From $\langle (s)_0, c\rangle \in A$, it follows that $(s)_0 = x$ and $c = x^{st}$ so that $\mathbf{p}(x)(\mathbf{i}_r) \Vdash x^{st} \in A$.

Thus, we infer the existence of an application term $\hat{d}$ with $\hat{d}d \Vdash u \in A$ and hence we can cook up an application term, $\hat{e}$, with $\hat{e} \Vdash \mathbf{dom}(\tilde{F}) = A$.

(b) Functional realizability of $\tilde{F}$. To establish this, let:

$$f \Vdash \langle u, v\rangle_{D_\infty} \in \tilde{F} \tag{4.23}$$

$$k \Vdash \langle u, w\rangle_{D_\infty} \in \tilde{F} \tag{4.24}$$

Then, we have (4.23) iff $\exists c$ such that $\langle (f)_0, c\rangle \in \tilde{F} \wedge (f)_1 \Vdash \langle u, v\rangle_{D_\infty} = c$. Now, from $\langle (f)_0, c\rangle \in \tilde{F}$, we obtain $(f)_0 = \mathbf{p}(ex)(x)$ and $c = \langle x^{st}, F(\langle x, x^{st}\rangle)\rangle_{D_\infty}$ for some $\langle x, x^{st}\rangle \in A$ and hence (4.23) is equivalent to the existence of $\langle x, x^{st}\rangle \in A$ such that $f_{01} = x$ and $f_1 \Vdash \langle u, v\rangle_{D_\infty} = \langle x^{st}, F(\langle x, x^{st}\rangle)\rangle_{D_\infty}$.

Similarly, one can infer from (4.24) that there exists $\langle y, y^{st}\rangle \in A$ such that $k_{01} = y$ and $k_1 \Vdash \langle u, w\rangle_{D_\infty} = \langle y^{st}, F(\langle y, y^{st}\rangle)\rangle_{D_\infty}$. Thus, $\mathtt{V}(D_\infty) \models x^{st} = y^{st}$ which implies by proposition 3.1.13 that $\mathtt{V}(D_\infty) \models x = y$. In consequent, by functionality of $F$ in the background universe $\mathtt{V}$, we deduce that

$$F(\langle x, x^{st}\rangle) = F(\langle y, y^{st}\rangle)$$

Therefore, there exists an application term $s$ such that $s(f, k) \Vdash v = w$ which establishes functionality of $\tilde{F}$.

Using this together with

(i) $r \Vdash \mathbf{dom}(\tilde{F}) = A$

(ii) $\forall \langle x, x^{st}\rangle \in A \quad ex \Vdash \phi(x^{st}, F(\langle x, x^{st}\rangle))$

we can construct the realizer, $\hat{e}$, in (4.22) and this completes the proof of $(ii)$.

□

**Remarks.** $(i)$ The previous construction would not work for $\mathtt{V}(\mathbf{P}\omega)$. The reason is that $F$ can not be shown to be injective in $\mathtt{V}(\mathbf{P}\omega)$ since $\mathbf{P}\omega$ is not extensional.
$(ii)$ By inspection of the proof of Theorem 4.1.1, one can see that the theorem can be generalised to hold for any total extensional applicative structure not only for the particular applicative structure $D_\infty$.

# Chapter 5

# Brouwerian Principles

The purpose of this chapter is to study the key tenets of the *Brouwerian Mathematical World* known as *Brouwerian principles* namely, *The Fan Theorem*, (**FT**), and *the principle of Bar Induction*, (**BI**), in the realizability structures $\mathtt{V}(\mathbf{P}\omega)$ and $\mathtt{V}(D_\infty)$. These principles were introduced by Brouwer to preserve the gist of traditional analysis. In particular, he hoped to save the theorem that a continuous function from a closed interval of a finite length to the real line is uniformly continuous and he established this theorem as a consequence of his bar theorem [40]. The objective here is to determine whether these principles get realized in the realizability structures $\mathtt{V}(D_\infty)$ and $\mathtt{V}(\mathbf{P}\omega)$.
We first start by introducing these principles.

## 5.1 The Fan Theorem, FT

*The Fan Theorem* is denoted by **FT** and considered to be central to *Brouwer's Intuitionistic Analysis*. **FT** is a crucial corollary to *The Bar Theorem* mentioned in the next section, and all of the known applications of the Bar Theorem already follow from **FT**.
**FT** also holds classically and it is equivalent to *König's Lemma*. [40]

**Definition 5.1.1** ([29], Definition 4.2)**.** *Let* $2^{\mathbb{N}}$ *denotes the set of all* 01*-sequences* $\alpha : \mathbb{N} \longrightarrow \{0,1\}$ *and set* $2^*$ *to be the set of all finite such sequences. Write* $s \subseteq t$ *for*

*$s, t \in 2^*$ to mean that $s$ is an initial segment of $t$.*

*A bar $\mathbf{B}$ of $2^*$ is a subset of $2^*$ such that:*

$$\forall \alpha \in 2^{\mathbb{N}}\ \exists n \quad \alpha[n] \in \mathbf{B}$$

*where $\alpha[n+1] := \langle \alpha(0), ..., \alpha(n) \rangle$ and $\alpha[0] := \langle\ \rangle$.*

*The bar $\mathbf{B}$ is* decidable *if it additionally satisfies:*

$$\forall t \in 2^*(t \in \boldsymbol{B} \vee t \notin \boldsymbol{B}).$$

*The decidable Fan Theorem $\mathbf{FT_D}$ states that each decidable bar $\mathbf{B}$ of $2^*$ is uniform, in other words:*

$$\exists m \in \mathbb{N}[\forall \alpha \in 2^{\mathbb{N}}\ \exists k \leq m \quad \alpha[k] \in \mathbf{B}]$$

*The statement of* The Full Fan Theorem*, $\mathbf{FT}$, is the following:*

*Each bar $\mathbf{B}$ of $2^*$ is uniform.*

## 5.2 Bar Induction Principle BI

As mentioned in the previous section, Brouwer obtained $\mathbf{FT_D}$ by appealing to a principle called *decidable* Bar Induction, $\mathbf{BI_D}$. Moreover, the reasons for accepting **BI** provide the best understanding for the intuitionistic motivations that led to accepting **FT**. It is well known that **BI** occupies a notable place in the literature since it has a leading role in developing *Brouwer's Intuitionistic Mathematics*.

Let $\mathbb{N}^*$ be the set of all finite sequences of $\mathbb{N}$. For $m \in \mathbb{N}$ and $s \in \mathbb{N}^*$ such that $s = \langle s_0, ..., s_k \rangle$, $s * \langle m \rangle$ denotes the sequence $\langle s_0, ..., s_k, m \rangle$.

A bar $\mathbf{B}$ of $\mathbb{N}^*$ is defined in the same way as that of $2^*$, *i.e.* $\mathbf{B} \subseteq \mathbb{N}^*$ such that:

$$\forall \alpha \in \mathbb{N}^{\mathbb{N}}\ \exists n \quad \alpha[n] \in \mathbf{B}. \tag{5.1}$$

The bar $\mathbf{B}$ is *decidable* if

$$\forall t \in \mathbb{N}^*(t \in \mathbf{B} \vee t \notin \mathbf{B}). \tag{5.2}$$

### 5.2.1 Decidable Bar Induction $\mathbf{BI_D}$

$\mathbf{BI_D}$ states that for any decidable bar $\mathbf{B}$ of $\mathbb{N}^*$ and any arbitrary class $G$

$$\text{if} \quad \forall t \in \mathbb{N}^*(t \in \mathbf{B} \longrightarrow t \in G) \tag{5.3}$$

$$\text{and} \quad \forall t \in \mathbb{N}^*[(\forall k \in \mathbb{N},\ \ t * \langle k \rangle \in G) \longrightarrow t \in G] \tag{5.4}$$

$$\text{then} \quad \langle\ \rangle \in G. \tag{5.5}$$

### 5.2.2 Monotone Bar Induction $\mathbf{BI_M}$

$\mathbf{BI_M}$ states that for any bar $\mathbf{B}$ of $\mathbb{N}^*$ and any arbitrary class $G$

$$\text{if} \quad \forall s, t \in \mathbb{N}^*(s \in \mathbf{B} \longrightarrow s * t \in \mathbf{B}), \tag{5.6}$$

$$\forall t \in \mathbb{N}^*(t \in \mathbf{B} \longrightarrow t \in G) \tag{5.7}$$

$$\text{and} \quad \forall t \in \mathbb{N}^*[(\forall k \in \mathbb{N},\ \ t * \langle k \rangle \in G) \longrightarrow t \in G], \tag{5.8}$$

$$\text{then} \quad \langle\ \rangle \in G \tag{5.9}$$

$\mathbf{BI_M}$ entails $\mathbf{BI_D}$, for details see([11], Theorem 3.7). Furthermore , $\mathbf{BI_D}$ entials $\mathbf{FT_D}$ and $\mathbf{BI_M}$ entials $\mathbf{FT}$, this was shown in [19], Ch1 Section 6.10 . [29]
To understand the idea behind this principle, consider the tree of all finite sequences of natural numbers $\mathbf{T}$. Then conditions (5.1- 5.3) assert that any path through $\mathbf{T}$ meets the bar $\mathbf{B}$ at a node $n$ and hence $G$ is hit at $n$ as well. Condition (5.4) states that if $G$ holds at all extensions of a node then it also holds at that node and this allows the property $G$ to "sneak" towards the root of $\mathbf{T}$. Furthermore, we may think of this principle as an induction statement over a well-founded relation and to justify this, we argue as follows:
Given condition (5.4), the failure of $G$ to hold at the empty sequence would yield a construction of an infinite sequence of natural numbers:

$$n_0, n_1, n_2, ...$$

such that at every initial segment $\langle n_0, n_1, n_2, ... \rangle$, $G$ does not hold and therefore defining such a function $\alpha$ by

$$\alpha(k) := n_k$$

would contradict conditions (5.1) and (5.3). [9]

The **BI** principle follows classically from an adequate axiom of dependent choices. [9]

The unrestricted form of the principle of bar induction, *i.e.* omitting requirement (5.2), is denoted by **BI**. [32]

## 5.3 Fan Theorem and Realizability Structures over P$\omega$ and $D_\infty$

We are aiming to investigate whether the realizability models over the applicative structures $\mathbf{P}\omega$ and $D_\infty$ validate **FT** or **BI** assuming their validity in the background universe $\mathtt{V}$. The first step in the investigation will certainly be what *Baire space* looks like in $\mathtt{V}(\mathbf{P}\omega)$ and $\mathtt{V}(D_\infty)$. However, in the following section, we shall describe in the general case of what $\mathtt{V}(\mathcal{A})$ thinks of Baire space for an arbitrary applicative structure, $\mathcal{A}$.

### 5.3.1 Baire Space in $\mathtt{V}(\mathcal{A})$

Let $\underline{n}$ be the $n$-th numeral in the applicative structure $\mathcal{A}$, i.e., the interpretation of the term $\mathbf{S_N} \ldots \mathbf{S_N} 0$ (with $n$-many symbols $\mathbf{S_N}$) in $\mathcal{A}$. Below we identify $\mathbf{N}$ with its interpretation in $\mathcal{A}$.

Recalling definition (3.1.7) we have,

$$\overline{n} := \{\langle \underline{m}, \overline{m}\rangle : m \in n\}$$

$$\overline{\omega} = \{\langle \underline{n}, \overline{n}\rangle : n \in \mathbb{N}\}.$$

And define,

$$\begin{aligned} \langle a, b\rangle_{\mathcal{A}} &:= \{\langle \underline{0}, \{a, a\}_{\mathcal{A}}\rangle, \langle \underline{1}, \{a, b\}_{\mathcal{A}}\rangle\} \\ &= \{\langle \underline{0}, \{\langle \underline{0}, a\rangle, \langle \underline{1}, a\rangle\}\rangle, \langle \underline{1}, \{\langle \underline{0}, a\rangle, \langle \underline{1}, b\rangle\}\rangle\} \end{aligned}$$

where $a, b \in \mathtt{V}(\mathcal{A})$ and $\{a, b\}_{\mathcal{A}} := \{\langle \underline{0}, a\rangle, \langle \underline{1}, b\rangle\}$.

A realizer $e \in \mathcal{A}$ can be constructed such that

$$e \Vdash \langle a, b\rangle_{\mathcal{A}} \text{ is the ordered pair of } a, b$$

and moreover $e$ does not depend on $a, b$.

This can be seen from the proof of Lemma (3.1.11).

**Definition 5.3.1.** *$f \in \mathcal{A}$ is said to be of Type 1 if for every $n \in \omega$, $f\underline{n}\downarrow$ and there is some $m \in \omega$ such that:*

$$f\underline{n} = \underline{m}.$$

**Definition 5.3.2.** *Given a Type 1 $f \in \mathcal{A}$ we can construct $f_{\mathbb{B}} \in \mathtt{V}(\mathcal{A})$ such that:*

$$f_{\mathbb{B}} := \{\langle \underline{n}, \langle \overline{n}, \overline{m}\rangle_{\mathcal{A}}\rangle : n \in \omega \text{ and } f\underline{n} = \underline{m}\},$$

*where $\langle \overline{n}, \overline{m}\rangle_{\mathcal{A}}$ is the internal pairing operation introduced in Definition 3.1.10.*

**Proposition 5.3.3.** *For every $f \in \mathcal{A}$ of Type 1, there is a realizer (in $\mathcal{A}$) for the statement that $f_{\mathbb{B}}$ is a function from $\omega$ to $\omega$.*

**Proof.** Suppose that $e$ is a realizer for the statement that $\langle a, b\rangle_{\mathcal{A}}$ is the ordered pair of $a$ and $b$.

To show that $f_{\mathbb{B}}$ is (internally) a function from $\omega$ to $\omega$, we need to show that $f_{\mathbb{B}} \subseteq \overline{\omega} \times \overline{\omega}$.

Towards this goal, let

$$\begin{aligned}
d \quad &\Vdash \quad \forall n \in \overline{\omega}\ \exists m \in \overline{\omega}\ \exists y \in f_{\mathbb{B}}\ \ y = \langle n, m\rangle \\
&\iff \quad \forall \langle f_1, c\rangle \in \overline{\omega}\ \ df_1 \Vdash \exists m \in \overline{\omega}\ \exists y \in f_{\mathbb{B}}\ \ y = \langle c, m\rangle \\
&\iff \quad \forall \langle f_1, c\rangle \in \overline{\omega}\ \exists c'\, (\langle (df_1)_0, c'\rangle \in \overline{\omega} \wedge (df_1)_1 \Vdash \exists y \in f_{\mathbb{B}}\ \ y = \langle c, c'\rangle) \\
&\iff \quad \forall \langle f_1, c\rangle \in \overline{\omega}\ \exists c'\, \langle (df_1)_0, c'\rangle \in \overline{\omega}\ \exists c^*\, (\langle (df_1)_{10}, c^*\rangle \in f_{\mathbb{B}} \wedge (df_1)_{11} \Vdash c^* = \langle c, c'\rangle)
\end{aligned}$$

Now, $\langle f_1, c\rangle \in \overline{\omega} \iff f_1 = \underline{n}$ and $c = \overline{n}$ for some $n \in \omega$ and hence

$$d\underline{n} \Vdash \exists m \in \overline{\omega}\ \exists y \in f_{\mathbb{B}}\ \ y = \langle \overline{n}, m\rangle$$

and, $\langle (d\underline{n})_0, c' \rangle \in \overline{\omega} \iff (d\underline{n})_0 = \underline{m}$ and $c' = \overline{m}$ for some $m \in \omega$ and

$$(d\underline{n})_{11} \Vdash c^* = \langle \overline{n}, \overline{m} \rangle$$

$\langle (d\underline{n})_{10}, c^* \rangle \in f_{\mathbb{B}} \iff (d\underline{n})_{10} = \underline{n}$ and $c^* = \langle \overline{n}, \overline{m} \rangle_{\mathcal{A}}$ for some $n \in \omega$ and

$$(d\underline{n})_{11} \Vdash c^* = \langle c, c' \rangle$$

Next,

$$\begin{aligned} (d\underline{n})_1 &= \mathbf{p}(d\underline{n})_{10}(d\underline{n})_{11} \\ &= \mathbf{p}(\underline{n})(e) \\ \text{and } d\underline{n} &= \mathbf{p}(d\underline{n})_0(d\underline{n})_1 \\ &= \mathbf{p}(f\underline{n})(\mathbf{p}(\underline{n})(e)) \text{ as } \underline{m} = f\underline{n} \end{aligned}$$

Thus, $d = \lambda x.\mathbf{p}(fx)(\mathbf{p}(x)(e))$. Also, if $\langle \underline{n}, \langle \overline{n}, \overline{m} \rangle_{\mathcal{A}} \rangle, \langle \underline{n'}, \langle \overline{n'}, \overline{m'} \rangle_{\mathcal{A}} \rangle \in f_{\mathbb{B}}$ and $\mathtt{V}(\mathcal{A}) \models \overline{n} = \overline{n'}$, then, $n = n'$ by (3.1.9) part $(i)$.
This yields that $m = m'$ and therefore, $f_{\mathbb{B}}$ is realizably functional and one easily construct a realizer for the statement

$$\forall x, y \in f_{\mathbb{B}}[(x)_0 = (y)_0 \longrightarrow (x)_1 = (y)_1]$$

We can combine these realizers to obtain a realizer for the statement that $f_{\mathbb{B}}$ is a function. □

**Proposition 5.3.4.** *Suppose* $\mathtt{V}(\mathcal{A}) \models f$ *is a function from* $\overline{\omega}$ *to* $\overline{\omega}$. *Then, there is a Type 1* $g \in \mathcal{A}$ *such that*

$$\mathtt{V}(\mathcal{A}) \models f = g_{\mathbb{B}}$$

**Proof.** As there is a realizer stating that $f$ is a function, we can find realizers $a, b,$ and $c \in \mathcal{A}$ such that:

$$a \Vdash \forall n \in \overline{\omega}\ \exists m \in \overline{\omega}\ \langle n, m \rangle \in f \tag{5.10}$$

$$b \Vdash \forall x \in f\ \exists n, m \in \overline{\omega}\ x = \langle n, m \rangle \tag{5.11}$$

$$c \Vdash \forall n, m, m' [\langle n, m \rangle \in f \wedge \langle n, m' \rangle \in f \longrightarrow m = m'] \tag{5.12}$$

So we have,

$$(5.10) \iff \forall\langle f_1, c\rangle \in \overline{\omega} \;\; af_1 \Vdash \exists m \in \overline{\omega}\; \langle c, m\rangle \in f$$
$$\iff \forall\langle f_1, c\rangle \in \overline{\omega}\; \exists c'\, (\langle (af_1)_0, c'\rangle \in \overline{\omega} \wedge ((af_1)_1)_1 \Vdash \langle c, c'\rangle \in f)$$

Now, $\langle f_1, c\rangle \in \overline{\omega} \iff f_1 = \underline{n}$ for some $n \in \omega$ and $c = \bar{f}_1 = \overline{n}$.
$\langle (af_1)_0, c'\rangle \in \overline{\omega} \iff (af_1)_0$ and $c' = \overline{(af_1)_0}$.
Let $g := \lambda x.(ax)_0$. Then, by the above $g$ must be of Type 1.
Next, each element of $g_{\mathbb{B}}$ is of the form:
$\langle \underline{n}, \langle \overline{n}, \overline{g\underline{n}}\rangle_{\mathcal{A}}\rangle = \langle \underline{n}, \langle \overline{n}, \overline{(a\underline{n})_0}\rangle_{\mathcal{A}}\rangle$ and $(a\underline{n})_1 \Vdash \langle \overline{n}, \overline{(a\underline{n})_0}\rangle_{\mathcal{A}} \in f$.
Thus, we obtain a realizer for $g \subseteq f$.
Next, since we have:

$$(5.11) \iff \forall\langle l, k\rangle \in f \;\; bl \Vdash \exists n, m \in \overline{\omega}\; k = \langle n, m\rangle$$
$$\iff \forall\langle l, k\rangle \in f\; \exists q\, (\langle (bl)_0, q\rangle \in \overline{\omega} \wedge (bl)_1 \Vdash \exists m \in \overline{\omega}\; k = \langle q, m\rangle)$$
$$\iff \forall\langle l, k\rangle \in f\; \exists q\, \langle (bl)_0, q\rangle \in \overline{\omega} \wedge \exists t\, (\langle (bl)_{10}, t\rangle \in \overline{\omega} \wedge (bl)_{11} \Vdash k = \langle q, t\rangle)$$

Then, for all $\langle l, x\rangle$ in $f$, we have:

$$bl \Vdash \exists u, v \in \overline{\omega}\; x = \langle u, v\rangle$$

Let $\underline{n} = (bl)_0$, $\underline{m} = (bl)_{10}$. Then,
$(a\underline{n})_1 \Vdash \langle \overline{n}, \overline{(a\underline{n})_0}\rangle_{\mathcal{A}} \in f$ and also, $(bl)_{11} \Vdash \langle \overline{(bl)_0}, \overline{(bl)_{10}}\rangle_{\mathcal{A}} = \langle \overline{n}, \overline{m}\rangle_{\mathcal{A}} \in f$.
Next, we need to find a realizer for $\overline{m} = \overline{(a\underline{n})_0}$ which gives a realizer for $x = \langle \overline{n}, \overline{(a\underline{n})_0}\rangle_{\mathcal{A}}$, but by (5.12) a realizer for $\overline{m} = \overline{(a\underline{n})_0}$ can be constructed and thus one can compute a realizer for $f \subseteq g$.
Therefore, $\mathtt{V}(\mathcal{A}) \models f = g_{\mathbb{B}}$ as required. □

**Proposition 5.3.5.** $\mathtt{V}(\mathcal{A}) \models \{\langle f, f_{\mathbb{B}}\rangle : f \in \mathcal{A}$ *and* $\forall x \in \mathbf{N}\; fx \in \mathbf{N}\}$ *is the set of all functions from* $\omega$ *to* $\omega$.

**Proof.** The proof is immediate using the previous two propositions. □

Therefore, based on what the internal *Baire Space* of $\mathtt{V}(\mathcal{A})$ for any applicative structure, $\mathcal{A}$, looks like from the previous section, the investigation of **FT** in $\mathtt{V}(\mathbf{P}\omega)$

and $\mathtt{V}(D_\infty)$ reduces to investigating the set:

$$\{\langle f, f_{\mathbb{B}}\rangle : f \in \mathcal{A} \text{ and } \forall x \in \mathbf{N}\ f \bullet x \in \mathbf{N}, \}$$

where $\mathbf{N}$ are the natural numbers of $\mathcal{A}$.
In other words, we would like to investigate where the following hold for $\mathbf{P}\omega$ and $D_\infty$:

$$\forall g : \mathbf{N} \longrightarrow \mathbf{N}\ \exists f \in \mathcal{A}\ \ \forall x[x \in \mathbf{N} \quad f \bullet x = g(x)] \tag{5.13}$$

The upshot of the above is that if the given applicative structure, $\mathcal{A}$ satisfies (5.13), then we have:

$$\mathtt{V}(\mathcal{A}) \models \mathbf{FT}.$$

### 5.3.2 The Fan Theorem in $\mathtt{V}(\mathbf{P}\omega)$

We have earlier shown the development of a **PCA** as a $\mathbf{PCA}^+$ in Theorem (2.2.7) and one interesting part was deciding how the given applicative structure thinks of $\omega$. For the applicative structure, $\mathbf{P}\omega$, $\mathbf{P}\omega$ thinks of $n \in \mathbb{N}$ as the singleton subset $\{n\} \in \mathbf{P}\omega$, see ([8] Theorem 7.2.4). Based on the discussion from section (5.3.1), the problem of investigating whether $\mathtt{V}(\mathbf{P}\omega)$ proves/disproves **FT** reduces to the problem of determining which functions $h : \mathbb{N} \longrightarrow \mathbb{N}$ can be represented in $\mathbf{P}\omega$.
Let $h : \mathbb{N} \longrightarrow \mathbb{N}$ be any given function. Then, a copy of this function in $\mathbf{P}\omega$ is denoted by $\hat{h} : \mathbf{P}\omega \longrightarrow \mathbf{P}\omega$ where $\hat{h}(X) = \{h(n) : n \in X\}$ and consequently, $\hat{h}(\{n\}) = \{h(n)\}$.
**Claim.** We claim that $\hat{h}$ is represented in $\mathbf{P}\omega$ by the element $X_h \in \mathbf{P}\omega$ defined by:

$$X_h := \{(2^n, h(n)) : n \in \omega\}.$$

To see this, note that $X_h \bullet \{n\} = \{z \in \mathbb{N} : \exists y \subset \{n\}\, (y, z) \in X_h\}$ by definition, we have $y \subset \{n\}$ iff $y = e_0$ or $y = e_{2^n}$. But, there is no $k_i$ such that $0 = \sum_{i \leq r} 2^{k_i}$ and thus we omit the possibility that $y = e_0$. Therefore, $X_h \bullet \{n\} = \{h(n)\} = \hat{h}(\{n\})$, which proves the claim.
Therefore, assuming that people in $\mathtt{V}(\mathbf{P}\omega)$ thinks that every infinite path through a tree hits a bar $\mathbf{B}$. Then since all paths in $\mathtt{V}$ are also represented in $\mathtt{V}(\mathbf{P}\omega)$, $\mathbf{B}$ is also a bar in $\mathtt{V}$. Thus, using **FT** in $\mathtt{V}$, there exists a uniform bound.

**Theorem 5.3.6.** *(i)* $\mathtt{V}(\mathbf{P}\omega) \models \mathbf{FT_D}$ *if* $\mathbf{FT_D}$ *holds in the background universe* $\mathtt{V}$.

*(ii)* $\mathtt{V}(\mathbf{P}\omega) \models \mathbf{BI_D}$ *if* $\mathbf{BI_D}$ *holds in the background universe* $\mathtt{V}$.

**Proof.** Since $\mathbf{FT_D}$ follows from $\mathbf{BI_D}$ it suffices to construct a realizer for $\mathbf{BI_D}$. First, since finite sequences of natural numbers (elements of $\mathbb{N}^*$) can be coded by the elements of $\mathbb{N}$ and for all natural number $n$ in $\omega$, $n$ codes a finite sequence $s$ in $\mathbb{N}^*$, we may therefore identify $\mathbb{N}^*$ with $\mathbb{N}$.

Next, suppose there exists realizers $e_1, e_2, e_3$ and $e_4 \in \mathbf{P}\omega$ such that

$$e_1 \Vdash \forall \alpha \in \mathbb{N}^{\mathbb{N}}\ \exists n \in \mathbb{N}\ \ \mathbf{B}(\alpha[n]) \tag{5.14}$$

$$e_2 \Vdash \forall s \in \mathbb{N}\ \ (\mathbf{B}(s) \vee \neg \mathbf{B}(s)) \tag{5.15}$$

$$e_3 \Vdash \forall s \in \mathbb{N}\ \ (\mathbf{B}(s) \longrightarrow G(s)) \tag{5.16}$$

$$e_4 \Vdash \forall s \in \mathbb{N}\ \ [(\forall k \in \mathbb{N}\ G(s * \langle k \rangle)) \longrightarrow G(s)] \tag{5.17}$$

By the realizability definition, (5.14) is equivalent to:

$$\forall \langle f, c \rangle \in (\mathbb{N}^{\mathbb{N}})^{\mathtt{V}(\mathbf{P}\omega)}\ \ e_1 f \Vdash \exists n \in \mathbb{N}\ \ \mathbf{B}(c[n])$$

$\iff$ $\forall \langle f, f_{\mathbb{B}} \rangle \in (\mathbb{N}^{\mathbb{N}})^{\mathtt{V}(\mathbf{P}\omega)}\ \ e_1 f \Vdash \exists n \in \mathbb{N}\ \ \mathbf{B}(f_{\mathbb{B}}[n])$, since $\langle f, c \rangle$ in the internal Baire Space of $\mathtt{V}(\mathbf{P}\omega)$, and thus has the form $f_{\mathbb{B}} = \{\langle \underline{n}, \langle \overline{n}, \overline{m} \rangle_{\mathbf{P}\omega} \rangle$, for some Type 1 $f \in \mathbf{P}\omega$ and some $n, m \in \omega\}$.

$\iff$ $\forall \langle f, f_{\mathbb{B}} \rangle \in (\mathbb{N}^{\mathbb{N}})^{\mathtt{V}(\mathbf{P}\omega)} \exists d \in \mathtt{V}(\mathbf{P}\omega)\ \ \big(\langle (e_1 f)_0, d \rangle \in (\mathbb{N})^{\mathtt{V}(\mathbf{P}\omega)} \wedge (e_1 f)_1 \Vdash \mathbf{B}(f_{\mathbb{B}}[d])\big)$.

However, $\langle (e_1 f)_0, d \rangle \in (\mathbb{N})^{\mathtt{V}(\mathbf{P}\omega)}$ entails that $d = \overline{(e_1 f)_0}$.Thus, we arrive at

$$(e_1 f)_1 \Vdash \mathbf{B}(f_{\mathbb{B}}[\overline{(e_1 f)_0}]). \tag{5.18}$$

And, (5.15) is equivalent to:

$\forall \langle g, d \rangle \in (\mathbb{N})^{\mathtt{V}(\mathbf{P}\omega)}\ \ e_2 g \Vdash (\mathbf{B}(d) \vee \neg \mathbf{B}(d))$.

$\langle g, d \rangle \in (\mathbb{N})^{\mathtt{V}(\mathbf{P}\omega)} \iff \langle g, d \rangle$ is of the form $\langle \underline{n}, \overline{n} \rangle$ for some $n \in \omega$.

Thus, for all $\langle \underline{n}, \overline{n} \rangle \in (\mathbb{N})^{\mathsf{V}(\mathbf{P}\omega)}$ we have

$$\begin{aligned} e_2\underline{n} \quad &\Vdash \quad \mathbf{B}(\overline{n}) \ \vee \ \neg\mathbf{B}(\overline{n}) \\ &\iff \quad [(e_2\underline{n})_0 = \mathbf{0} \wedge (e_2\underline{n})_1 \Vdash \mathbf{B}(\overline{n})] \ \vee \\ &\qquad\quad [(e_2\underline{n})_0 = \mathbf{1} \wedge (e_2\underline{n})_1 \Vdash \neg\mathbf{B}(\overline{n})] \end{aligned} \tag{5.19}$$

Define $\tilde{\mathbf{B}}(s) :\iff (e_2\underline{s})_0 = \mathbf{0}$, and define a function $\psi : \mathbf{N} \longrightarrow \mathbf{P}\omega$ as follows:

$$\psi(\underline{s}) = \begin{cases} (e_3\underline{s})(e_2\underline{s})_1 & \text{if } (e_2\underline{s})_0 = \mathbf{0} \\ (e_4\underline{s})(\lambda\underline{u}.\psi(\underline{s * \langle u \rangle}) & \text{if } (e_2\underline{s})_0 \neq \mathbf{0} \end{cases}$$

Let $\tilde{G}(s) :\iff \psi(\underline{s}) \downarrow \ \wedge \ \psi(\underline{s}) \Vdash G(\overline{s})$. But, since $\mathbf{P}\omega$ is total $\psi(\underline{s}) \downarrow$ is trivial.

**Claim.** We claim that the above defined $\tilde{\mathbf{B}}$ and $\tilde{G}$ are the bar and the class which supply the validity of $\mathbf{BI_D}$ in $\mathbf{P}\omega$. In other words, the following hold

(a) $\forall \alpha \in \mathbb{N}^{\mathbb{N}} \ \exists n \in \mathbb{N} \ \ \tilde{\mathbf{B}}(\alpha[n])$.

(b) $\forall s \in \mathbb{N} \ \ (\tilde{\mathbf{B}}(s) \vee \neg\tilde{\mathbf{B}}(s))$.

(c) $\forall s \in \mathbb{N} \ \ (\tilde{\mathbf{B}}(s) \longrightarrow \tilde{G}(s))$.

(d) $\forall s \in \mathbb{N} \ \ [(\forall k \in \mathbb{N} \ \tilde{G}(s * \langle k \rangle)) \longrightarrow \tilde{G}(s)]$.

**proof of the claim.**

(a) Since $\forall \alpha \in \mathbb{N}^{\mathbb{N}}$, there exists $f \in \mathbf{P}\omega$ such that $\underline{\alpha[n]} = f \bullet \underline{n}$ for all $n$ in $\mathbb{N}$. $\mathbf{B}(\overline{\alpha[n]})$ holds in $\mathsf{V}(\mathbf{P}\omega)$ by (5.18). Thus, by (5.19) $(e_2\underline{\alpha[n]})_0 = \mathbf{0}$ so that $\tilde{\mathbf{B}}(\alpha[n])$ holds.

(b) Since $\tilde{\mathbf{B}}(s) :\iff (e_2\underline{s})_0 = \mathbf{0}$, it is immediate that $\forall s \in (\mathbb{N})^{\mathsf{V}(\mathbf{P}\omega)}(\tilde{\mathbf{B}}(s) \ \vee \ \neg\tilde{\mathbf{B}}(s))$ using (**??**).

(c) Suppose that $\tilde{\mathbf{B}}(s)$ then, $(e_2\underline{s})_0 = \mathbf{0}$ holds and hence $(e_2\underline{s})_1 \Vdash \mathbf{B}(\overline{s})$ so it follows from (5.16) that $(e_3\underline{s})(e_2\underline{s})_1 \Vdash G(\overline{s})$. Consequently, by definition of $\psi$ it follows $\psi(\underline{s}) \Vdash G(\overline{s})$ and therefore by definition of $\tilde{G}$, $\tilde{G}(s)$.

(d) Suppose that $\forall k \in \mathbb{N} \ \tilde{G}(s * \langle k \rangle)$ then, the definition of $\tilde{G}$ entails

$$\psi(\underline{s * \langle u \rangle}) \downarrow \ \wedge \ \psi(\underline{s * \langle u \rangle}) \Vdash G(\overline{s * \langle u \rangle})$$

Moreover, unravelling (5.17) yields

$$\begin{aligned}(5.17) \iff & \forall\langle f,c\rangle \in (\mathbb{N})^{\mathtt{V}(\mathbf{P}\omega)}\, e_4 f \Vdash [(\forall k \in \mathbb{N}\, G(c * \langle k\rangle)) \longrightarrow G(c)] \\ \iff & e_4\underline{s} \Vdash [(\forall k \in \mathbb{N}\, G(\overline{s} * \langle k\rangle)) \longrightarrow G(\overline{s})],\ \text{since } \langle f,c\rangle \in (\mathbb{N})^{\mathtt{V}(\mathbf{P}\omega)} \\ & \text{ has to be of the form } \langle \underline{s}, \overline{s}\rangle \text{ for some } s \text{ in } \omega \\ \iff & \forall r \in \mathbf{P}\omega[(r \Vdash \forall k \in \mathbb{N}\, G(\overline{s} * \langle k\rangle)) \longrightarrow (e_4\underline{s})r \Vdash G(\overline{s})].\end{aligned}$$

However, $r = \lambda\underline{u}.\psi(\underline{s * \langle u\rangle})$. Therefore,

$$(e_4\underline{s})(\lambda\underline{u}.\psi(\underline{s * \langle u\rangle})) \Vdash G(\overline{s}),$$

and thus $\psi(\underline{s}) \Vdash G(\overline{s})$ for all $s$ in $\omega$ which by definition of $\tilde{G}$ entails $\tilde{G}(s)$.

Invoking $\mathbf{BI_D}$ in $\mathtt{V}$, and using the claim above we can infer $\tilde{G}(\langle\ \rangle)$ which by the definition of $\tilde{G}$ entails that $\psi(\underline{\langle\ \rangle}) \Vdash G(\overline{\langle\ \rangle})$. Thus, $\mathbf{BI_D}$ holds in the special realizability model $\mathtt{V}(\mathbf{P}\omega)$. □

**Remark.** Given the fact that we can represent Baire space in $\mathbf{P}\omega$, it is very likely that the principle of $\mathbf{BI_M}$ hold true in $\mathtt{V}(\mathbf{P}\omega)$. It would be very interesting to investigate this and see how the proof will proceed but, due to time constrains we are not carrying out this investigation in the present thesis.

### 5.3.3 The Fan Theorem in $\mathtt{V}(D_\infty)$

Due to the complicated construction of Scott $D_\infty$ models we will first be addressing simple $D_\infty$ models where we start with a simple complete partial order $D_0$ and look at its $D_\infty$ version in investigating (5.13). The simplest $D_\infty$ model we shall be considering in the present work is $\mathbb{N}^+_\infty$ with $D_0 = \mathbb{N}^+$ from example (2.4.2) but, we first give some definitions and fix notation that will be useful later.

#### 5.3.3.1 Some Basic Definitions and Useful Notations

**Definition 5.3.7.** *(i) A set $R$ is said to be regular if $R$ is inhabited (i.e.$\exists r\ r \in R$), transitive (i.e. $x \in y \in R \Rightarrow x \in R$) set and whenever $b \in R$ and $S \subseteq b \times R$*

*satisfies* $\forall x \in b \exists y \in R \;\; \langle x, y \rangle \in S$ *then, there exists* $C \in R$ *such that*

$$\forall x \in b \exists y \in C \;\; \langle x, y \rangle \in S$$
$$\text{and } \forall y \in C \exists x \in b \;\; \langle x, y \rangle \in S$$

*Write* $\mathbf{Reg}(R)$ *for* $R$ *is regular.*

*(ii)* $R$ *is said to be a union closed regular set, denoted by* $\bigcup$*-closed* $\mathbf{Reg}(R)$*, if* $R$ *is regular and moreover* $\forall r \in R \quad \bigcup r \in R$.

[35]

**Definition 5.3.8.** *Let* $R$ *be a* $\bigcup$*-closed regular set with* $\omega \in R$*, and define the local power set of* $\omega$ *relative to* $R$ *denoted by* $\mathbf{P}_R\omega$ *as follows*

$$\mathbf{P}_R\omega = \{C \in R : C \subseteq \omega\}$$
$$\text{or equivalently}$$
$$\mathbf{P}(\omega) \cap R$$

*which is indeed a set.*

We also have the following useful lemma.

**Lemma 5.3.9.** $(\mathbf{P}_R\omega, \subseteq)$ *is an* $\omega - CPO$.

**Proof.** Let $f : \omega \longrightarrow \mathbf{P}_R\omega$ be an $\omega$-chain then,

$$f(i) \in \mathbf{P}_R\omega \Rightarrow f(i) \subseteq \omega \text{ and } f(i) \in R$$

Now, $range(f) \in R$ since $\omega \in R$ and $R$ is regular.
Next, let

$$\begin{aligned} b &:= \bigcup range(f) \\ &= \bigcup \{f(i) : i \in \omega\} \\ &= \sup_{i \in \omega} f(i), \text{ as the ordering here is inclusion } \quad [37] \end{aligned}$$

□

Note that $\mathbf{P}_R\omega$ is not a $CPO$ since, if we consider an arbitrary $X \subseteq \mathbf{P}_R\omega$ then $X \subseteq \mathbf{P}_R\omega \not\Rightarrow supX \in \mathbf{P}_R\omega$.
As stated earlier, the first step in investigating the validity of $\mathbf{FT_D}$ in $\mathtt{V}(D_\infty)$ is to decide how $D_\infty$ thinks of natural numbers. However, in the literature there are various approaches to modelling $\omega$ in $D_\infty$. In the present thesis, depending on what $D_\infty$ is, we consider the two approaches contained in the following definition.

**Definition 5.3.10.** *(i) If $D_\infty$ is guaranteed to contain a copy of $\omega$, for instance, by taking $D_0 = \mathbb{N}^+$ defined in Chapter 1 or $D_0 = \mathbf{P}_R\omega$ for $\mathbf{Reg}(R)$, starting with this approach, one can model $\omega$ by:*

$$\mathbf{N}(\mathbf{x}) \Longleftrightarrow \exists n \in \omega \quad K_{0\infty}(n) = \mathbf{x}$$

*with successor $\mathbf{S_N}$ defined in the usual way. This, gives the required behaviour on the natural numbers $\mathbf{N}$ of $D_\infty$. The same would work if $D_0$ consists of the subsets of $\omega$ in some regular set $A$ with $\omega \in A$ (e.g. $D_0 = \mathbf{P}_A\omega$, the local power set of $\omega$ relative to the regular set $A$) where the natural numbers are modelled by:*

$$\mathbf{N}(\mathbf{x}) \Longleftrightarrow \exists n \in \omega \quad K_{0\infty}(\{n\}) = \mathbf{x}$$

*with $\mathbf{S_N} := K_{1\infty}(s_1)$ such that $s_1 = \lambda X \in D_0.\{y+1 : y \in X\} \in D_1 = [D_0 \longrightarrow D_0]$ gives the required behaviour on the natural numbers $\mathbf{N}$ of $D_\infty$. [37]*

*(ii) Modelling $\omega$ by the* Church numerals, *in this approach $\omega$ is modelled by:*

$$\mathbf{N}(\mathbf{x}) \Longleftrightarrow \exists n \in \omega \quad \mathbf{x} = \lambda \mathbf{f}.\mathbf{f}^n$$

*This gives the required behaviour on the natural numbers $\mathbf{N}$ of $D_\infty$ with successor, predecessor and test for zero that satisfy the corresponding axioms for natural numbers in the theory $\mathbf{PCA}^+$, see* Section (2.2.1) *are defined as* *follows:*

$$\begin{aligned}\mathbf{S}_{CN} &= \lambda nfx.\, f(nfx)\\ \mathbf{P}_{CN} &= \lambda nfx.\, n(\lambda gh.\, h(gf))(\lambda u.x)(\lambda u.u)\\ \mathbf{Tzero}_{CN} &= \lambda nxy.\, n(\lambda z.y)x\end{aligned}$$

*[37]*

Firstly, consider part $(i)$ of definition (5.3.10). To investigate (5.13) we need to start with a $CPO, D_0$, such that its $D_\infty$ version contains $\omega$. Note that the simple structure is $D_\infty = \mathbb{N}^+_\infty$ where we take the $CPO$ $D_0 = \mathbb{N}^+$ would work in this case. However, it is informative here to give the next definitions together with the following lemma with the use of *classical logic*.

**Definition 5.3.11** ([14], p.133)**.** *Every partial function $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ can be extended to a total function $\phi^+ : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ defined as follows:*

$$\phi^+(n) = \begin{cases} \phi(n) & \textit{for } \phi(n)\downarrow \\ \bot & \textit{otherwise} \end{cases}$$

*and $\phi^+(\bot) = \bot$.*

**Definition 5.3.12** ([14], p.133)**.** *There are two kinds of constant functions $\psi$ : $\mathbb{N}^+ \longrightarrow \mathbb{N}^+$ defined as follows for all $p \in \mathbb{N}$*

*(a) $\psi' : \psi'(n) = p$ $(\forall n \in \mathbb{N})$ and $\psi'(\bot) = \bot$*

*(b) $\psi'' : \psi''(n) = p$ $(\forall n \in \mathbb{N})$ and $\psi''(\bot) = p$*

**Remark.** If $\phi$ from definition (5.3.11) is a constant function with value $p \in \mathbb{N}$ then, $\psi'$ is of the form $\phi^+$ where as, for all such functions $\phi$, $\psi''$ is never of the form $\phi^+$.

**Lemma 5.3.13** ([14], exercise 12.12)**.** *If $\theta : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ is a function, then $\theta$ is continuous $\Longleftrightarrow$ $\theta$ has either forms $\phi^+$ from definition* (5.3.11) *or $\psi''$ from definition* (5.3.12)

**Proof.**

By definition , $\theta$ is continuous $\iff$ (1) $\theta$ is order-preserving

(2) for all directed $X \subseteq \mathbb{N}^+$

$\theta(supX) = sup\{\theta(x) : x \in X\}$

(a) For $x, y \in \mathbb{N}^+$, $x \preceq y \iff x = \perp \wedge y \in \mathbb{N}$ or $x = y$.

Case 1 If $x = \perp \wedge y \in \mathbb{N}$, then

$\theta(x) \preceq \theta(y) \iff \theta(x) = \perp \wedge \theta(y) \in \mathbb{N}$ so that $\theta$ is of the form $\phi^+$ or $\theta(x) = \theta(y)$ in which case $\theta$ is of the form $\psi''$.

Case 2 If $x = y$, then

$\theta(x) \preceq \theta(y) \iff \theta(x) = \perp \wedge \theta(y) \in \mathbb{N}$ which cannot happen since $\theta$ is a function , or $\theta(x) = \theta(y)$, in which case $\theta$ satisfies any function including $\phi^+$ *and* $\psi''$.

(b) $\theta(supX) = sup\{\theta(x) : x \in X\}$ for $X$ directed.

$X \subseteq \mathbb{N}^+$ is directed $\iff$ (1) $X$ is a singleton

(2) $X$ is $\{\perp, n\}$ with $n \in \mathbb{N}$

Case 1 If $X$ is a singleton, *i.e* $X = \{x\}$ for $x \in \mathbb{N}^+$, then

$\theta(supX) = \theta(x) = sup\{\theta(x)\}$.

Case 2 If $X$ is $\{\perp, n\}$, for some $n \in \mathbb{N}$, then

$\theta(supX) = \theta(sup\{\perp, n\}) = \theta(n) = sup\{\theta(\perp), \theta(n)\} \Leftrightarrow \theta(\perp) = \perp$ in which case $\theta$ has the form $\phi^+$, or $\theta(\perp) = \theta(n)$, in which case $\theta$ has the form $\psi''$.

□

Next, let $h : \mathbb{N} \longrightarrow \mathbb{N}$ be a given function. Then $h$ may be extended to a function $h^+ : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ defined as follows:

$$h^+(x) = \begin{cases} h(x) & \text{if } x \in \mathbb{N} \\ \perp & \text{otherwise} \end{cases}$$

Note that $h^+$ is continuous and hence $h^+ \in D_1$ by (5.3.13) and by the embedding of $D_1$ into $D_\infty$, $h^+$ can be viewed as an element of $D_\infty$. Thus, when $n \in \mathbb{N}$ is modelled by its embedding into $D_\infty$ ($K_{0\infty}(n)$), every function $h : \mathbb{N} \longrightarrow \mathbb{N}$ is represented in the $D_\infty$ version of $\mathbb{N}^+$, denoted by $\mathbb{N}^+_\infty$, by $K_{1\infty}(h^+)$.
Likewise, if we start with $D_0 = \mathbf{P}_R\omega$ for some $\bigcup$-closed regular set $R$ containing $\omega$ where the $D_\infty$ version of this is a $\mathbf{PCA}^+$ which thinks of natural numbers as the embedding of singleton sets of $\omega$, $\{n\} \in D_0$ into $D_\infty$ *i.e.* $\underline{n} = K_{0\infty}(\{n\})$ in [37], where indeed for all $n \in \mathbb{N}$ $\{n\} \in \mathbf{P}_R\omega$. However, in this case when we extend a function $h : \mathbb{N} \longrightarrow \mathbb{N}$ to a function $\tilde{h} : \mathbf{P}_R\omega \longrightarrow \mathbf{P}_R\omega$, $\tilde{h}$ is defined as

$$\tilde{h}(X) = \{\{h(x)\} : x \in X\}$$

This function is continuous because it is the restriction of a continuous function from $\mathbf{P}\omega$ to $\mathbf{P}\omega$ as was shown earlier in Section (5.3.2) and moreover, it can be viewed as a member of $D_\infty$ by the embedding $K_{1\infty}(\tilde{h})$. Thus every function $h : \mathbb{N} \longrightarrow \mathbb{N}$ is representable in $D_\infty$ when natural numbers are modelled by $K_{0\infty}(\{n\})$.
We conclude that when natural numbers are modelled as given in part ($i$) of definition (5.3.10), $\mathbf{FT_D}$ holds in the realizability structure $\mathtt{V}(D_\infty)$ provided that $\omega \subseteq D_\infty$.

**Theorem 5.3.14.** *If $D_\infty$ is the* Scott's $D_\infty$ *model containing $\omega$, say by taking $D_0 = \mathbb{N}^+$ with natural numbers modelled by its embedding into $D_\infty$ (i.e. $K_{0\infty}(n)$), then the following hold:*

*(i)* $\mathtt{V}(D_\infty) \models \mathbf{FT_D}$ *if* $\mathbf{FT_D}$ *holds in the background universe.*

*(ii)* $\mathtt{V}(D_\infty) \models \mathbf{BI_D}$ *if* $\mathbf{BI_D}$ *holds in the background universe.*

**Proof.** The proof is similar to that of Theorem (5.3.6). □

Secondly, consider part ($ii$) of definition (5.3.10). In investigating (5.13) where the natural numbers are modelled by the Church numerals, we claim that all functions from $\mathbb{N} \longrightarrow \mathbb{N}$ are representable.
To prove the claim we first need the following results:

**Definition 5.3.15** (**Lambda Definability**)**.** *Let $\phi : \mathbb{N}^k \longrightarrow \mathbb{N}$ be a function with $k$ arguments. $\phi$ is said to be $\lambda$-definable if for some term $F \in \Lambda$*

$$\forall n_1, n_2, ..., n_k \in \mathbb{N}\ F\underline{n}_1\underline{n}_2...\underline{n}_k =_\beta \underline{\phi(n_1, n_2, ..., n_k)} \qquad (*)$$

*where $\underline{n}$ is the $n-th$ Church numeral in a* **PCA**$^+$ *structure.*
*If (*) holds then, $\phi$ is $\lambda$-defined by $F$.*

**Proposition 5.3.16.** *All computable functions are $\lambda$-definable.*

**Proof.** see [5], proposition 6.3.11. □

**Theorem 5.3.17** ([6], Theorem 3.14)**.** *With respect to the Church numerals, all computable functions can be $\lambda$-defined.*

**Proof.** See [6], Theorem 3.14. □

**Proposition 5.3.18.** *Let $D_\infty$ be such that* $\mathbf{N} \subseteq D_\infty$ *is the set of the* Church numeral, *and let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a partial function. Then there is an element* $\mathbf{p} \in D_\infty$ *represents the function $f$.*

**Proof.** consider the following two functions:

$$Teq : \mathbb{N} \longrightarrow \mathbb{N} \longrightarrow \mathbb{N} \ \text{ and } \ Tzero : \mathbb{N} \longrightarrow \mathbb{N} \longrightarrow \mathbb{N} \longrightarrow \mathbb{N}$$

defined by:

$$Teq\, mn = \begin{cases} 0 & \text{if } m = n \\ 1 & \text{otherwise} \end{cases}$$

and

$$Tzero\, mnp = \begin{cases} n & \text{if } m = 0 \\ p & \text{otherwise} \end{cases}$$

Note that we are choosing to use 0 for *True* and 1 for *false*.

**Notation.** For readability, write **if** $m$ **then** $n$ **else** $p$. The functions $Teq$ and $Tzero$

are representable in $D_\infty$ by Theorem (5.3.17) and since $D_\infty$ is a $\lambda$-model.
Now for the given function $f : \mathbb{N} \longrightarrow \mathbb{N}$, write $d_n$ for the $f(n) - th$ Church numeral and consider the following infinite sequence of $\lambda$-terms:

$$
\begin{aligned}
p_0 &= \lambda x.\ \text{if } (Teq\, x0) \text{ then } d_0 \text{ else } \bot. \\
p_1 &= \lambda x.\ \text{if } (Teq\, x0) \text{ then } d_0 \text{ else ( if } (Teq\, x1) \text{ then } d_1 \text{ else } \bot). \\
p_2 &= \lambda x.\ \text{if } (Teq\, x0) \text{ then } d_0 \text{ else ( if } (Teq\, x1) \text{ then } d_1 \text{ else ( if } (Teq\, x2) \text{ then } d_2 \text{ else } \bot)). \\
&\dots \\
&\dots \\
&\dots \\
p_n &= \lambda x.\ \text{if } (Teq\, x0) \text{ then } d_0 \text{ else ( if } (Teq\, x1) \text{ then } d_1 \text{ else ...( if } (Teq\, xn) \\
&\quad \text{then } d_n \text{ else } \bot \underbrace{)))...)))}_{\text{n-times}} . \\
&\dots \\
&\dots \\
&\dots
\end{aligned}
$$

We can interpret these terms in $D_\infty$ which gives rise to an increasing sequence of elements $\mathbf{p}_0 \subseteq \mathbf{p}_1 \subseteq \mathbf{p}_2 \subseteq ...$ in $D_\infty$ because the term $p_n$ arises from the term $p_{n-1}$ by replacing a subterm $\bot$ by something else.
Now, let $\mathbf{p}$ be the least upper bound of this sequence within $D_\infty$. Moreover, we claim that $\mathbf{p}$ represents $f$, *i.e.* $\underline{d_n} = \mathbf{p} \bullet \underline{n}$.
To verify this, we have

$$
\begin{aligned}
\mathbf{p} \bullet \underline{n} &= (\underset{m}{sup}^{D_\infty}\, \mathbf{p}_m) \bullet \underline{n} && \text{by definition of } \mathbf{p}. \\
&= \underset{m \geq n}{sup}^{D_\infty}\, (\mathbf{p}_m \bullet \underline{n}) && \text{by continuity of application} \\
&= \underset{m \geq n}{sup}^{D_\infty}\, \underline{d_n} \\
&= \underline{d_n}.
\end{aligned}
$$

□

**Theorem 5.3.19.** *For any applicative structure, $D_\infty$ with natural numbers modelled by the Church Numerals (i.e. $\underline{n} = \lambda fx.f^n x$), the following hold:*

*(i)* $\mathtt{V}(D_\infty) \models \mathbf{FT_D}$ *if* $\mathbf{FT_D}$ *holds in the background universe.*

*(ii)* $\mathtt{V}(D_\infty) \models \mathbf{BI_D}$ *if* $\mathbf{BI_D}$ *holds in the background universe.*

**Proof.** The proof is similar to that of Theorem (5.3.6). □

# Chapter 6

# Some Properties of CZF and CZF + REA in the Universe of Truth Realizability

In this chapter, we investigate soundness of the theories **CZF** and **CZF + REA** in a different realizability structure, $\mathtt{V}^*(\mathcal{A})$ called *Realizability with Truth*. $\mathtt{V}^*(\mathcal{A})$ has been introduced by *Rathjen* in [30]. In [30], *Rathjen* showed that **CZF** and **CZF + REA** are sound for the special case of $\mathtt{V}^*(\mathcal{K}_1)$. He showed further that the disjunction and some other properties hold for **CZF**, however, the question of whether we get similar results when moving from $\mathtt{V}^*(\mathcal{K}_1)$ to $\mathtt{V}^*(\mathcal{A})$ for an arbitrary applicative structure, $\mathcal{A}$, was left open after [30]. The objective of this chapter is to scan the proofs in [30] and make a conclusion about this open question in the world $\mathtt{V}^*(\mathcal{A})$.

In what follows we first review some terminology that will be useful later on. This will be taken directly from [30] but we include it here for convenience as well as fixing notation.

## 6.1 Realizability with Truth

For an arbitrary applicative structure, $\mathcal{A}$, assume that such structure can be formalized in **CZF**, and that

$$\mathbf{CZF} \models \mathcal{A} \text{ is a model of } \mathbf{PCA}^{+}$$

**Notation.**

(i) For a class $C$, write $\mathcal{P}(C)$ for the class of all sets $b$ such that $b \subseteq C$.

(ii) If $x$ is the ordered pair $\langle a, b\rangle$, write $1^{st}(x)$ and $2^{nd}(x)$ for the first and second projections of $x$, respectively.

### 6.1.1 The general realizability structure

In this section we define the universe of truth realizability $\mathtt{V}^*(\mathcal{A})$ built on any applicative structure, $\mathcal{A}$.

**Definition 6.1.1.** Ordinals are transitive sets with transitive elements. As before, we use small Greek letters to range over ordinals.

$$\begin{aligned}
\mathtt{V}^*_\alpha(\mathcal{A}) &= \bigcup_{\beta\in\alpha}\big\{\langle a, b\rangle :\ a \in \mathtt{V}_\beta;\ b \subseteq |\mathcal{A}| \times \mathtt{V}^*_\beta(\mathcal{A});\ (\forall c \in b)\, 1^{st}(2^{nd}(c)) \in a\big\} \quad (6.1)\\
\mathtt{V}^*(\mathcal{A}) &= \bigcup_{\alpha} \mathtt{V}^*_\alpha(\mathcal{A})\\
\mathtt{V}_\alpha &= \bigcup_{\beta\in\alpha} \mathcal{P}(\mathtt{V}_\beta)\\
\mathtt{V} &= \bigcup_{\alpha} \mathtt{V}_\alpha.
\end{aligned}$$

In **CZF** it is not clear whether the classes $\mathtt{V}$ and $\mathtt{V}^*(\mathcal{A})$ can be formalized. However, using the fact that inductively defined classes is accommodated in **CZF**, this can be shown in simillar to [28], Lemma 3.4.

The definition of $\mathtt{V}^*_\alpha(\mathcal{A})$ in (6.1) is a bit involved. To make it clear we first note that all the elements of $\mathtt{V}^*(\mathcal{A})$ are ordered pairs $\langle a, b\rangle$ such that $b \subseteq |\mathcal{A}| \times \mathtt{V}^*(\mathcal{A})$. Next, for an ordered pair $\langle a, b\rangle$ to enter $\mathtt{V}^*_\alpha(\mathcal{A})$ the first requirement to be met is that for some $\beta \in \alpha$, $a \in \mathtt{V}_\beta$ and $b \subseteq |\mathcal{A}| \times \mathtt{V}^*_\beta(\mathcal{A})$. Moreover, it is required that $a$ contains enough elements from the transitive closure of $b$ such that whenever $\langle d, e\rangle \in b$ then $1^{st}(e) \in a$.

**Lemma 6.1.2.** (**CZF**).

*(i)* *For* $\beta \in \alpha$, *we have* $\mathtt{V}_\beta \subseteq \mathtt{V}_\alpha$ and $\mathtt{V}^*_\beta(\mathcal{A}) \subseteq \mathtt{V}^*_\alpha(\mathcal{A})$.

*(ii)* $a \in \mathtt{V}$, *for all sets* $a$, .

*(iii)* *Let* $a, b$ *be sets, such that* $b \subseteq |\mathcal{A}| \times \mathtt{V}^*(\mathcal{A})$ *and* $(\forall c \in b)\, 1^{st}(2^{nd}(c)) \in a$, *then* $\langle a, b\rangle \in \mathtt{V}^*(\mathcal{A})$.

**Proof.** See [30], Lemma 4.2. □

## 6.2 Defining realizability

Having the set up, we can now proceed to define a realizability semantics over $\mathtt{V}^*(\mathcal{A})$, where we use small gothic letters as variables $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f}, \mathfrak{g}, \ldots$ ranging over the elements of $\mathtt{V}^*(\mathcal{A})$ whereas $e, c, d, f, r, \ldots$ will be used to range over elements of the applicative structure $\mathcal{A}$. Note that for all elements $\mathfrak{a}$ of $\mathtt{V}^*(\mathcal{A})$, $\mathfrak{a}$ is an ordered pair $\langle x, y\rangle$, where $x$ in $\mathtt{V}$ and $y \subseteq |\mathcal{A}| \times \mathtt{V}^*(\mathcal{A})$; and we let $\mathfrak{a}^\circ$ and $\mathfrak{a}^*$ to be the components of $\mathfrak{a}$ defined by

$$\begin{aligned} \mathfrak{a}^\circ &:= 1^{st}(\mathfrak{a}) = x \\ \mathfrak{a}^* &:= 2^{nd}(\mathfrak{a}) = y. \end{aligned}$$

**Remark.** For each $\mathfrak{a} \in \mathtt{V}^*(\mathcal{A})$, if $\langle r, \mathfrak{c}\rangle \in \mathfrak{a}^*$ then $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$.

**Notation.** For a sentence $\varphi$ with parameters in $\mathtt{V}^*(\mathcal{A})$, write $\varphi^\circ$ for the result of replacing each parameter $\mathfrak{a}$ in $\varphi$ with $\mathfrak{a}^\circ$.

**Definition 6.2.1** ([30], definition 5.2)**.** In defining realizability, bounded and unbounded quantifiers will be treated as syntactically different quantifiers.

For $r \in |\mathcal{A}|$ and $\phi$ a sentence with parameters in $\mathtt{V}^*(\mathcal{A})$, define $r \Vdash^{\mathcal{A}}_{rt} \phi$ inductively as follows:

$$
\begin{array}{rcl}
r \Vdash^{\mathcal{A}}_{rt} \mathfrak{a} \in \mathfrak{b} & \text{iff} & \mathfrak{a}^\circ \in \mathfrak{b}^\circ \ \wedge \ \exists \mathfrak{c} \left[\langle (r)_0, \mathfrak{c}\rangle \in \mathfrak{b}^* \ \wedge \ (r)_1 \Vdash^{\mathcal{A}}_{rt} \mathfrak{a} = \mathfrak{c}\right] \\
r \Vdash^{\mathcal{A}}_{rt} \mathfrak{a} = \mathfrak{b} & \text{iff} & \mathfrak{a}^\circ = \mathfrak{b}^\circ \ \wedge \ \forall f \forall \mathfrak{c} \left[\langle f, \mathfrak{c}\rangle \in \mathfrak{a}^* \ \to \ (r)_0 f \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} \in \mathfrak{b}\right] \\
 & & \wedge \ \forall f \forall \mathfrak{c} \left[\langle f, \mathfrak{c}\rangle \in \mathfrak{b}^* \ \to \ (r)_1 f \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} \in \mathfrak{a}\right] \\
r \Vdash^{\mathcal{A}}_{rt} \phi \wedge \psi & \text{iff} & (r)_0 \Vdash^{\mathcal{A}}_{rt} \phi \ \wedge \ (r)_1 \Vdash^{\mathcal{A}}_{rt} \psi \\
r \Vdash^{\mathcal{A}}_{rt} \phi \vee \psi & \text{iff} & \left[(r)_0 = \mathbf{0} \ \wedge \ (r)_1 \Vdash^{\mathcal{A}}_{rt} \phi\right] \ \vee \ \left[(r)_0 \neq \mathbf{0} \ \wedge \ (r)_1 \Vdash^{\mathcal{A}}_{rt} \psi\right] \\
r \Vdash^{\mathcal{A}}_{rt} \neg\phi & \text{iff} & \neg\phi^\circ \ \wedge \ \forall f \ \neg f \Vdash^{\mathcal{A}}_{rt} \phi \\
r \Vdash^{\mathcal{A}}_{rt} \phi \to \psi & \text{iff} & (\phi^\circ \to \psi^\circ) \ \wedge \ \forall f \left[f \Vdash^{\mathcal{A}}_{rt} \phi \ \to \ rf \Vdash^{\mathcal{A}}_{rt} \psi\right] \\
r \Vdash^{\mathcal{A}}_{rt} (\forall x \in \mathfrak{a})\, \phi & \text{iff} & (\forall x \in \mathfrak{a}^\circ)\phi^\circ \ \wedge \\
 & & \forall f\, \forall \mathfrak{b} \big(\langle f, \mathfrak{b}\rangle \in \mathfrak{a}^* \ \to \ rf \Vdash^{\mathcal{A}}_{rt} \phi[x/\mathfrak{b}]\big) \\
r \Vdash^{\mathcal{A}}_{rt} (\exists x \in \mathfrak{a})\phi & \text{iff} & \exists \mathfrak{b} \left(\langle (r)_0, \mathfrak{b}\rangle \in \mathfrak{a}^* \ \wedge \ (r)_1 \Vdash^{\mathcal{A}}_{rt} \phi[x/\mathfrak{b}]\right) \\
r \Vdash^{\mathcal{A}}_{rt} \forall x \phi & \text{iff} & \forall \mathfrak{a}\, r \Vdash^{\mathcal{A}}_{rt} \phi[x/\mathfrak{a}] \\
r \Vdash^{\mathcal{A}}_{rt} \exists x \phi & \text{iff} & \exists \mathfrak{a}\, r \Vdash^{\mathcal{A}}_{rt} \phi[x/\mathfrak{a}]
\end{array}
$$

Observe that $r \Vdash^{\mathcal{A}}_{rt} x \in y$ and $r \Vdash^{\mathcal{A}}_{rt} x = y$ can be defined for any sets $x, y$, i.e., not just for $x, y \in \mathtt{V}^*(\mathcal{A})$. $r \Vdash^{\mathcal{A}}_{rt} x \in y$ and $r \Vdash^{\mathcal{A}}_{rt} x = y$ are defined by transfinite recursion. More precisely, the (class) functions

$$
\begin{array}{rcl}
F_{\in}(x, y) & = & \{r \in |\mathcal{A}| \ : \ r \Vdash^{\mathcal{A}}_{rt} x \in y\} \\
G_{=}(x, y) & = & \{r \in |\mathcal{A}| \ : \ r \Vdash^{\mathcal{A}}_{rt} x = y\}
\end{array}
$$

can be (simultaneously) defined on $\mathtt{V} \times \mathtt{V}$ by recursion on the relation

$$
\langle c, d\rangle \lhd \langle a, b\rangle \quad \Leftrightarrow \quad \big(c = a \ \wedge \ d \in \mathbf{TC}(b)\big) \ \vee \ \big(d = b \ \wedge \ c \in \mathbf{TC}(a)\big), \qquad (6.2)
$$

where $\mathbf{TC}(x)$ is the transitive closure of a set $x$. Moreover, It was shown in [34] Lemma 7.1 that **CZF** proves transfinite recursion on $\lhd$. More precisely, this principle is a consequence of Strong Collection (or Replacement) together with Set Induction.

**Definition 6.2.2.** Let $x$ be a set, using $\in$-recursion, define a set $x^{st}$ by:

$$x^{st} \quad = \quad \langle x, \{\langle \mathbf{0}, u^{st}\rangle \ : \ u \in x\}\rangle. \tag{6.3}$$

**Lemma 6.2.3.** *For any set $x$, $x^{st} \in \mathtt{V}^*(\mathcal{A})$. Moreover, $(x^{st})^\circ = x$.*

**Proof.** See [30], lemma 5.4. □

To show soundness for the theories **CZF** and **CZF** + **REA**, we require the following lemmas that we quote form [30] but we skip the proofs here.

**Lemma 6.2.4.** *If $\psi(\mathfrak{b}^\circ)$ holds for all $\mathfrak{b} \in \mathtt{V}^*(\mathcal{A})$ then $\forall x\, \psi(x)$.*

**Lemma 6.2.5.** *If $\mathfrak{a} \in \mathtt{V}^*(\mathcal{A})$ and $(\forall \mathfrak{b} \in \mathtt{V}^*(\mathcal{A}))[\mathfrak{b}^\circ \in \mathfrak{a}^\circ \to \psi(\mathfrak{b}^\circ)]$ then $(\forall x \in \mathfrak{a}^\circ)\psi(x)$.*

**Lemma 6.2.6.** *If $e \Vdash^{\mathcal{A}}_{rt} \phi$ then $\phi^\circ$.*

**Theorem 6.2.7.** *Let $\theta(v_1, \ldots, v_r)$ be a formula of* **CZF** *with free variables among $v_1, \ldots, v_r$; and let $\mathfrak{p}$ be a proof of $\theta(v_1, \ldots, v_r)$ in intuitionistic predicate logic with equality. Then there exists a closed application term $r_{\mathfrak{p}}$ such that*

$$\mathbf{CZF} \vdash \big(r_{\mathfrak{p}} \Vdash^{\mathcal{A}}_{rt} \forall v_1 \ldots \forall v_r\, \theta(v_1, \ldots, v_r)\big).$$

**Proof.** See [30] Theorem 5.13. □

### 6.2.1 Realizability for bounded formulae

In the following we shall often have occasion to employ the fact that for a bounded formula $\varphi(v)$ with parameters from $\mathtt{V}^*(\mathcal{A})$ and $x \subseteq \mathtt{V}^*(\mathcal{A})$,

$$\{\langle e, \mathfrak{c}\rangle : \ e \in \mid \mathcal{A} \mid \ \wedge \mathfrak{c} \in x \ \wedge \ e \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{c})\}$$

is a set. To prove this we shall consider an extended class of formulae.

**Definition 6.2.8** ([30], Definition 5.14.)**.** The *extended bounded formulae* are the smallest class of formulas containing the formulae of the form $x \in y$, $x = y$, $e \Vdash^{\mathcal{A}}_{rt} x \in y$, $e \Vdash^{\mathcal{A}}_{rt} x = y$ (where $x, y$ are variables or elements of $\mathtt{V}^*(\mathcal{A})$) which is closed under $\wedge, \vee, \neg, \to$ and bounded quantification.

**Lemma 6.2.9** ([30], Lemma 5.15.)**.** (**CZF**) *Separation holds for extended bounded formulae, i.e., for every extended bounded formula* $\varphi(v)$ *and set* $x$, $\{v \in x : \varphi(v)\}$ *is a set.*

**Proof.** See [30], Lemma 5.15. ☐

**Lemma 6.2.10** ([30], Lemma 5.16.)**.** (**CZF**) *Let* $\varphi(v, u_1, \ldots, u_r)$ *be a bounded formula of* **CZF** *all of whose free variables are among* $u_1, \ldots, u_r$*. Then there there is an extended bounded formula* $\widetilde{\varphi}(v, u_1, \ldots, u_r)$ *and* $f_\varphi \in\mid \mathcal{A} \mid$ *such that for all* $a_1, \ldots, a_r \in \mathtt{V}^*(\mathcal{A})$ *and* $e \in\mid \mathcal{A} \mid$,

$$e \Vdash_{rt}^{\mathcal{A}} \varphi(\vec{\mathfrak{a}}) \quad \textit{iff} \quad \widetilde{\varphi}(f_\varphi e, \vec{\mathfrak{a}}).$$

**Proof.** See [30], Lemma 5.16. ☐

**Corollary 6.2.11** ([30], Corollary 5.17.)**.** (**CZF**) *Let* $\varphi(v)$ *be a bounded formula with parameters from* $\mathtt{V}^*(\mathcal{A})$ *and* $x \subseteq \mathtt{V}^*(\mathcal{A})$*. Then*

$$\{\langle e, \mathfrak{c}\rangle : \ e \in\mid \mathcal{A} \mid \ \wedge \mathfrak{c} \in x \ \wedge \ e \Vdash_{rt}^{\mathcal{A}} \varphi(\mathfrak{c})\}$$

*is a set.*

**Proof.** The previous two lemmas ensures that this class is a set. ☐

## 6.3 The soundness theorem for CZF

**Theorem 6.3.1.** *Let* $\varphi$ *be a theorem of* **CZF***. Then, there is an application term* $t$ *such that*

$$\mathbf{CZF} \vdash \ (t \Vdash_{rt}^{\mathcal{A}} \varphi).$$

**Proof.** By Theorem 6.2.7 it enough to address the axioms of **CZF**, so that in what follows we shall address them one after the other.

(**Extensionality**): $\forall a \forall b[\forall x(x \in a \longleftrightarrow x \in b) \longrightarrow a = b]$.

Let $\mathfrak{a}, \mathfrak{b} \in \mathtt{V}^*(\mathcal{A})$ and suppose that there exists a realizer $e \in |\mathcal{A}|$ such that

$$e \Vdash_{rt}^{\mathcal{A}} \big[(\forall x \in \mathfrak{a})(x \in \mathfrak{b}) \ \wedge \ (\forall x \in \mathfrak{b})(x \in \mathfrak{a})\big]$$

We then have the following equivalences:

$$\begin{aligned}
& e \Vdash_{rt}^{\mathcal{A}} \big[(\forall x \in \mathfrak{a})(x \in \mathfrak{b}) \ \wedge \ (\forall x \in \mathfrak{b})(x \in \mathfrak{a})\big] \\
\Leftrightarrow \ & (e)_0 \Vdash_{rt}^{\mathcal{A}} (\forall x \in \mathfrak{a})(x \in \mathfrak{b}) \ \wedge \ (e)_1 \Vdash_{rt}^{\mathcal{A}} (\forall x \in \mathfrak{b})(x \in \mathfrak{a}) \\
\Leftrightarrow \ & [(\forall x \in \mathfrak{a}^\circ)(x \in \mathfrak{b}^\circ) \ \wedge \ \forall \mathfrak{c}(\langle f, \mathfrak{c}\rangle \in \mathfrak{a}^* \to (e)_0 f \Vdash_{rt}^{\mathcal{A}} \mathfrak{c} \in \mathfrak{b})] \ \wedge \\
& [(\forall x \in \mathfrak{b}^\circ)(x \in \mathfrak{a}^\circ) \ \wedge \ \forall \mathfrak{d}(\langle g, \mathfrak{d}\rangle \in \mathfrak{b}^* \to (e)_1 g \Vdash_{rt}^{\mathcal{A}} \mathfrak{d} \in \mathfrak{a})] \\
\Leftrightarrow \ & (\forall x \in \mathfrak{a}^\circ)(x \in \mathfrak{b}^\circ) \ \wedge \ (\forall x \in \mathfrak{b}^\circ)(x \in \mathfrak{a}^\circ) \ \wedge \ \theta(0, (e)_0, \mathfrak{a}, \mathfrak{b}) \ \wedge \ \theta(1, (e)_1, \mathfrak{b}, \mathfrak{a}) \\
\Leftrightarrow \ & \mathfrak{a}^\circ = \mathfrak{b}^\circ \ \wedge \ \theta(0, (e)_0, \mathfrak{a}, \mathfrak{b}) \ \wedge \ \theta(1, (e)_1, \mathfrak{b}, \mathfrak{a}) \\
\Leftrightarrow \ & e \Vdash_{rt}^{\mathcal{A}} \mathfrak{a} = \mathfrak{b}.
\end{aligned}$$

As a result, $\mathbf{p}(\lambda x.x)(\lambda x.x) \Vdash_{rt}^{\mathcal{A}}$ *Extensionality.*

(**Pairing**): $\forall x \forall y \exists z\, (x \in z \ \wedge \ y \in z)$.

We need to guarantee the existence of an $e \in |\mathcal{A}|$ such that

$$\forall \mathfrak{a}, \mathfrak{b} \in \mathtt{V}^*(\mathcal{A})\, \exists \mathfrak{c} \in \mathtt{V}^*(\mathcal{A})\ e \Vdash_{rt}^{\mathcal{A}} \mathfrak{a} \in \mathfrak{c} \ \wedge \ \mathfrak{b} \in \mathfrak{c}. \tag{6.4}$$

By the realizability definition, we have the following equivalences:

$$\begin{aligned}
& e \Vdash_{rt}^{\mathcal{A}} \mathfrak{a} \in \mathfrak{c} \wedge \mathfrak{b} \in \mathfrak{c} \\
\Leftrightarrow \ & (e)_0 \Vdash_{rt}^{\mathcal{A}} \mathfrak{a} \in \mathfrak{c} \wedge (e)_1 \Vdash_{rt}^{\mathcal{A}} \mathfrak{b} \in \mathfrak{c} \\
\Leftrightarrow \ & \mathfrak{a}^\circ \in \mathfrak{c}^\circ \ \wedge \ \exists \mathfrak{d}[\langle (e)_{00}, \mathfrak{d}\rangle \in \mathfrak{c}^* \ \wedge \ (e)_{01} \Vdash_{rt}^{\mathcal{A}} \mathfrak{a} = \mathfrak{d}] \wedge \\
& \mathfrak{b}^\circ \in \mathfrak{c}^\circ \ \wedge \ \exists \mathfrak{e}[\langle (e)_{10}, \mathfrak{e}\rangle \in \mathfrak{c}^* \ \wedge \ (e)_{11} \Vdash_{rt}^{\mathcal{A}} \mathfrak{b} = \mathfrak{e}]
\end{aligned}$$

Set $e = \mathbf{p}(\mathbf{p0i}_r)(\mathbf{p0i}_r)$ and let $\mathfrak{c} = \langle x, y\rangle$, with $x = \{\mathfrak{a}^\circ, \mathfrak{b}^\circ\}$ and $y = \{\langle \mathbf{0}, \mathfrak{a}\rangle, \langle \mathbf{0}, \mathfrak{b}\rangle\}$. Using 6.1.2, we see that $\mathfrak{c} \in \mathtt{V}^*(\mathcal{A})$. Thus, we have

$$\mathfrak{a}^\circ, \mathfrak{b}^\circ \in \mathfrak{c}^\circ$$

and

$$\langle \mathbf{0}, \mathfrak{a}\rangle, \langle \mathbf{0}, \mathfrak{b}\rangle \in \mathfrak{c}^*$$

Additionaly, we also have

$$\mathbf{i}_r \Vdash_{rt}^{\mathcal{A}} \mathfrak{a} = \mathfrak{a} \text{ and } \mathbf{i}_r \Vdash_{rt}^{\mathcal{A}} \mathfrak{b} = \mathfrak{b}$$

Therefore, $e \Vdash_{rt}^{\mathcal{A}} \mathfrak{a} \in \mathfrak{c} \wedge \mathfrak{b} \in \mathfrak{c}$ holds.

(**Union**): $\forall a \exists b \forall c \in a \forall u \in c\,(u \in b)$.

We define *Union* in the world of sets $\mathtt{V}^*(\mathcal{A})$ as follows:

For each $\mathfrak{a} \in \mathtt{V}^*(\mathcal{A})$, put

$$\begin{aligned} Un(\mathfrak{a}) &= \langle \bigcup \mathfrak{a}^\circ, A \rangle, \text{with} \\ A &= \{\langle h, \mathfrak{b} \rangle : \exists \langle f, \mathfrak{c} \rangle \in \mathfrak{a}^* \ \langle h, \mathfrak{b} \rangle \in \mathfrak{c}^*\}. \end{aligned}$$

Note that $\langle h, \mathfrak{b} \rangle \in A$ implies $\langle h, \mathfrak{b} \rangle \in \mathfrak{c}^*$ for some $\langle f, \mathfrak{c} \rangle \in \mathfrak{a}^*$, which yields $\mathfrak{b}^\circ \in \mathfrak{c}^\circ$ and $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$, and thus $\mathfrak{b}^\circ \in \bigcup \mathfrak{a}^\circ$. Then, Lemma 6.1.2, implies

$$Un(\mathfrak{a}) \in \mathtt{V}^*(\mathcal{A}).$$

We want to find a realizer $e \in |\mathcal{A}|$ such that

$$e \Vdash_{rt}^{\mathcal{A}} \forall w \in \mathfrak{a} \forall u \in w\,(u \in Un(\mathfrak{a}))$$

By the realizability definition, for any $\langle f, \mathfrak{c} \rangle \in \mathfrak{a}^* \ \wedge \ \mathfrak{c}^\circ \in \mathfrak{a}^\circ$.

$$\begin{aligned} ef \quad \Vdash_{rt}^{\mathcal{A}} \quad & (\forall u \in \mathfrak{c})(u \in Un(\mathfrak{a})) \\ \Longleftrightarrow \quad & \langle h, \mathfrak{b} \rangle \in \mathfrak{c}^* \ \wedge \ \mathfrak{b}^\circ \in \mathfrak{c}^\circ. \end{aligned}$$

Next, set

$$\mathfrak{q} := Un(\mathfrak{a}).$$

From $\mathfrak{c}^\circ \in \mathfrak{a}^\circ \ \wedge \ \mathfrak{b}^\circ \in \mathfrak{c}^\circ$ infer that $\mathfrak{b}^\circ \in \bigcup \mathfrak{a}^\circ$, so that $\mathfrak{b}^\circ \in \mathfrak{q}^\circ$. And by the realizability definition again, $(efh)_0 = h$, and we have $\langle (efh)_0, \mathfrak{b} \rangle \in \mathfrak{c}^*$, yielding $\langle (efh)_0, \mathfrak{b} \rangle \in A$ which implies $\langle (efh)_0, \mathfrak{b} \rangle \in \mathfrak{q}^*$. Moreover, as $\mathbf{i}_r \Vdash_{rt}^{\mathcal{A}} \mathfrak{b} = \mathfrak{b}$, we conclude that $efh \Vdash_{rt}^{\mathcal{A}} \mathfrak{b} \in Un(\mathfrak{a})$ verifying (6.5). From (6.5) we arrive at $e = \lambda u.\lambda v.\mathbf{p}v\mathbf{i}_r \Vdash_{rt}^{\mathcal{A}} \forall a \, \exists q\, (\forall w \in a)(\forall u \in w)(u \in q)$.

(**Bounded Separation**): $\forall a \exists b \forall x\, [x \in b \leftrightarrow x \in a \wedge \theta(x)]$.

Let $\theta(x)$ be a bounded formula with parameters in $\mathtt{V}^*(\mathcal{A})$. We are aiming to find realizers $r, r' \in |\mathcal{A}|$ such that for all $\mathfrak{a} \in \mathtt{V}^*(\mathcal{A})$ there exists a $\mathfrak{b} \in \mathtt{V}^*(\mathcal{A})$ such that

$$\big(r \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{b}\,[x \in \mathfrak{a} \,\wedge\, \theta(x)]\big) \,\wedge\, \big(r' \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{a}[\theta(x) \to x \in \mathfrak{b}]\big). \tag{6.5}$$

For $\mathfrak{a} \in \mathtt{V}^*(\mathcal{A})$, define

$$\begin{aligned} Sep(\mathfrak{a}, \theta) &= \{\langle \mathbf{p}fg, \mathfrak{c}\rangle :\ f, g \in |\mathcal{A}| \,\wedge\, \langle g, \mathfrak{c}\rangle \in \mathfrak{a}^* \,\wedge\, f \Vdash^{\mathcal{A}}_{rt} \theta[x/\mathfrak{c}]\}, \\ \mathfrak{b} &= \langle \{x \in \mathfrak{a}^\circ :\ \theta^\circ(x)\}, Sep(\mathfrak{a}, \theta)\rangle. \end{aligned}$$

By Corollary 6.2.11, $Sep(\mathfrak{a}, \theta)$ is a set, so that $\mathfrak{b}$ is a set. To show that $\mathfrak{b} \in \mathtt{V}^*(\mathcal{A})$ let $\langle h, \mathfrak{c}\rangle$ be in $Sep(\mathfrak{a}, \theta)$ which implies that $\langle g, \mathfrak{c}\rangle \in \mathfrak{a}^*$ and $f \Vdash^{\mathcal{A}}_{rt} \theta[x/\mathfrak{c}]$ for some $f, g \in |\mathcal{A}|$. Hence, $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$ and Lemma 6.2.6, implies $\theta^\circ[x/\mathfrak{c}^\circ]$ which entails that $\mathfrak{c}^\circ \in \{x \in \mathfrak{a}^\circ :\ \theta^\circ(x)\}$. Thus, by Lemma (6.1.2), we have $\mathfrak{b} \in \mathtt{V}^*(\mathcal{A})$.
To show the first part of (6.5), we assume that $\langle h, \mathfrak{c}\rangle \in \mathfrak{b}^*$ and $\mathfrak{c}^\circ \in \mathfrak{b}^\circ$. Then we have $h = \mathbf{p}fg$ for some $f, g \in |\mathcal{A}|$, $\langle g, \mathfrak{c}\rangle \in \mathfrak{a}^*$ and $f \Vdash^{\mathcal{A}}_{rt} \theta[x/\mathfrak{c}]$. As $\mathfrak{c}^\circ \in \mathfrak{b}^\circ$, we infer that $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$. In consequence, $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$ and $\langle g, \mathfrak{c}\rangle \in \mathfrak{a}^*$ with $\mathbf{i}_r \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} = \mathfrak{c}$, as a result $\mathbf{p}(h)_1\mathbf{i}_r \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} \in \mathfrak{a}$ and $(h)_0 \Vdash^{\mathcal{A}}_{rt} \theta[x/\mathfrak{c}]$. Furthermore, we have $(\forall x \in \mathfrak{b}^\circ)\big(x \in \mathfrak{a}^\circ \,\wedge\, \theta^\circ(x)\big)$ and thus we conclude that with $r = \mathbf{p}(\mathbf{p}(\lambda v.(v)_1)\mathbf{i}_r)(\lambda v.(v)_0)$, we obtain $r \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{b}\,[x \in \mathfrak{a} \,\wedge\, \theta(x)]$.

Now, for the second part of (6.5), suppose that $\langle g, \mathfrak{c}\rangle \in \mathfrak{a}^*$, $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$ and $f \Vdash^{\mathcal{A}}_{rt} \theta[x/\mathfrak{c}]$. Then, we have $\langle \mathbf{p}fg, \mathfrak{c}\rangle \in \mathfrak{b}^*$. Furthermore, by Lemma 6.2.6, $f \Vdash^{\mathcal{A}}_{rt} \theta[x/\mathfrak{c}]$ implies $\theta^\circ[x/\mathfrak{c}^\circ]$ so that $\mathfrak{c}^\circ \in \mathfrak{b}^\circ$. Therefore

$$\mathbf{p}(\mathbf{p}fg)\mathbf{i}_r \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} \in \mathfrak{b}.$$

Finally, it follows from the definition of $\mathfrak{b}$ that $(\forall x \in \mathfrak{a}^\circ)[\theta^\circ(x) \to x \in \mathfrak{b}^\circ]$, thus with

$$r' = \lambda u.\lambda v.\mathbf{p}(\mathbf{p}vu)\mathbf{i}_r$$

we see that $r' \Vdash^{\mathcal{A}}_{rt} (\forall x \in \mathfrak{a})[\theta(x) \to x \in \mathfrak{b}]$.

(**Set Induction**): For a formula, $\varphi(y)$, with parameters in $\mathtt{V}^*(\mathcal{A})$ and at most $y$ free. We would like to find an application term $t$ such that $t \Vdash^{\mathcal{A}}_{rt} \theta \to \psi$, with formulas $\theta$ and $\psi$ such that $\theta$ is the formula $\forall a\,[(\forall y \in a\,\phi(y)) \to \varphi(a)]$ and $\psi$ is the formula $\forall a\,\varphi(a)$. Note that $\theta^\circ \to \psi^\circ$ is an immediate instance of Set Induction, and therefore it is enough to construct a term $t$ with $tg \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{a})$ holds for all $\mathfrak{a} \in \mathtt{V}^*(\mathcal{A})$, whenever $g \Vdash^{\mathcal{A}}_{rt} \theta$.

Now, for all $\mathfrak{a} \in \mathtt{V}^*(\mathcal{A})$, suppose that

$$g \Vdash^{\mathcal{A}}_{rt} \big(\forall y \in \mathfrak{a}\, \varphi(y)\big) \;\rightarrow\; \varphi(\mathfrak{a}). \tag{6.6}$$

By Lemma 6.2.6, 6.2.4, (6.6) implies that

$$\forall a\, [(\forall y \in a\, \varphi^\circ(y)\big) \;\rightarrow\; \varphi^\circ(a)]$$

Thus, by Set Induction, we conclude for all $\mathfrak{b} \in \mathtt{V}^*(\mathcal{A})$,

$$\varphi^\circ(\mathfrak{b}^\circ). \tag{6.7}$$

Next, assume that $\mathfrak{a} \in \mathtt{V}^*_\alpha(\mathcal{A})$ and assume further that there exists a realizer $e$ such that for all $\mathfrak{b} \in \bigcup_{\beta\in\alpha} \mathtt{V}^*_\beta(\mathcal{A})$, $e \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{b})$.

From $g \Vdash^{\mathcal{A}}_{rt} \big(\forall y \in \mathfrak{a}\, \varphi(y)\big) \;\rightarrow\; \varphi(\mathfrak{a})$, we have the following equivalences:

$$\begin{aligned} & g \Vdash^{\mathcal{A}}_{rt} \big(\forall y \in \mathfrak{a}\, \varphi(y)\big) \;\rightarrow\; \varphi(\mathfrak{a}) \\ \Leftrightarrow\; & (\forall y \in \mathfrak{a}^\circ \varphi^\circ(y)) \rightarrow \varphi^\circ(\mathfrak{a}^\circ)) \;\wedge \\ & \forall e^*\, [e^* \Vdash^{\mathcal{A}}_{rt} \forall y \in \mathfrak{a}\varphi(y) \,\text{then}\, ge^* \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{a})] \end{aligned}$$

Now, $(\forall y \in \mathfrak{a}^\circ \varphi^\circ(y)) \rightarrow \varphi^\circ(\mathfrak{a}^\circ)$ is verified by (6.6) and $e^* \Vdash^{\mathcal{A}}_{rt} \forall y \in \mathfrak{a}\, \varphi(y)$ is equivalent to $\forall \langle f, \mathfrak{b}\rangle \in \mathfrak{a}^*$ then $e^* f \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{b})$ by the realizability definition. Thus, $\mathfrak{b} \in \bigcup_{\beta\in\alpha} \mathtt{V}^*_\beta(\mathcal{A})$, and hence $e \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{b})$, this $e$ is constructed from arbitrary $f \in |\mathcal{A}|$ so we may write $e$ as:

$$\lambda u.\mathbf{k}eu \Vdash^{\mathcal{A}}_{rt} \forall y \in \mathfrak{a}\varphi(y) \quad \text{and} \quad g(\lambda u.\mathbf{k}eu) \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{a}). \tag{6.8}$$

Using recursion theorem for applicative structures one can construct an application term $t$ with $tf \simeq f(\lambda u.\mathbf{k}(tf)u)$ holds for any $f \in |\mathcal{A}|$. In the above if we set $e := tg$ then, by induction on $\alpha$, we see that $tg \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{a})$ and therefore

$$t \Vdash^{\mathcal{A}}_{rt} \forall a \big[\big(\forall y \in a\varphi(y)\big) \;\rightarrow\; \varphi(a)\big] \;\rightarrow\; \forall a\, \varphi(a),$$

as required. (**Infinity**): To represent $\omega$ in $\mathtt{V}^*(\mathcal{A})$, the most obvious definition is given by an injection of $\omega$ into $\mathtt{V}^*(\mathcal{A})$, $\overline{\omega}$ such that for any $n \in \omega$, define

$$\overline{n} = \langle n, \{\langle \underline{k}, \overline{k} \rangle : k \in n\}\rangle \tag{6.9}$$

$$\overline{\omega} = \langle \omega, \{\langle \underline{n}, \overline{n} \rangle : n \in \omega\}\rangle. \tag{6.10}$$

Observe that $\overline{n}^\circ = n$ and $\overline{\omega}^\circ = \omega$. Using Lemma 6.1.2, one can clearly see that $\overline{n}, \overline{\omega} \in \mathtt{V}^*(\mathcal{A})$. To show realizability of the Infinity axiom, we need to write the axiom out in detail. Set $\bot_v$ to be the formula

$$\forall u \in v \, \neg\, u = u,$$

and put $SC(u, v)$ to be the formula

$$\forall y \in v \, [y = u \ \vee \ y \in u] \ \wedge \ [u \in v \ \wedge \ \forall y \in u \, y \in v].$$

Then the Infinity axiom is equivalent to

$$\exists x \left(\forall v \in x \, [\bot_v \ \vee \exists u \in x \, SC(u, v)] \ \wedge \ \forall v[(\bot_v \ \vee \exists u \in x \, SC(u, v)) \ \rightarrow \ v \in x]\right) \tag{6.11}$$

Let $\exists x \, \vartheta_{inf}(x)$ abbreviates the formula of (6.11) then clearly, $\vartheta^\circ_{inf}(\overline{\omega}^\circ)$.
Now, we are trying to find realizers $\mathbf{q}^+$ and $\mathbf{q}^{++}$ such that:

$$\mathbf{q}^+ \Vdash^{\mathcal{A}}_{rt} \forall v \in \overline{\omega} \left[\bot_v \ \vee \exists u \in \overline{\omega} \, SC(u, v)\right] \tag{6.12}$$

$$\mathbf{q}^{++} \ \Vdash^{\mathcal{A}}_{rt} \ \forall v[(\bot_v \ \vee \exists u \in \overline{\omega} \, SC(u, v)) \ \rightarrow \ v \in \overline{\omega}]. \tag{6.13}$$

For (6.12), by the realizability definition, we have the following equivalences:

$$\begin{aligned}
& \mathbf{q}^+ \Vdash^{\mathcal{A}}_{rt} \forall v \in \overline{\omega} \left[\bot_v \ \vee \exists u \in \overline{\omega} \, SC(u, v)\right] \\
\Leftrightarrow \ & \forall \langle f, \mathfrak{c} \rangle \in \overline{\omega}^* \ \rightarrow \ \mathbf{q}^+ f \Vdash^{\mathcal{A}}_{rt} \bot_{\mathfrak{c}} \ \vee \exists u \in \overline{\omega} \, SC(u, \mathfrak{c}) \\
\Leftrightarrow \ & \forall \langle f, \mathfrak{c} \rangle \in \overline{\omega}^* \ \rightarrow \ [((\mathbf{q}^+ f)_0 = \mathbf{0} \ \wedge \ (\mathbf{q}^+ f)_1 \Vdash^{\mathcal{A}}_{rt} \bot_{\mathfrak{c}})) \ \vee \\
& ((\mathbf{q}^+ f)_0 = \mathbf{1} \ \wedge \ (\mathbf{q}^+ f)_1 \Vdash^{\mathcal{A}}_{rt} \exists u \in \overline{\omega} \, SC(u, \mathfrak{c}))]
\end{aligned}$$

Suppose $\langle f, \mathfrak{c}\rangle \in \overline{\omega}^*$. Then $f = \underline{n}$ and $\mathfrak{c} = \overline{n}$ for some $n \in \omega$.

If $n = 0$ then $\overline{n} = \langle 0, 0\rangle$ and therefore $\mathbf{0} \Vdash^{\mathcal{A}}_{rt} \bot_{\mathfrak{c}}$.

Otherwise we have $n = k + 1$ for some $k \in \omega$. If $\langle \underline{m}, \overline{m}\rangle \in \overline{n}^*$ then $m = k$ or $m \in k$. $m = k$ yields $\mathbf{i}_r \Vdash^{\mathcal{A}}_{rt} \overline{m} = \overline{k}$ by (3.1.9). Now assume that $m \in k$. Note that $r \Vdash^{\mathcal{A}}_{rt} \overline{m} \in \overline{k}$ is equivalent to $\exists \mathfrak{c}[\langle (r)_0, \mathfrak{c}\rangle \in \overline{k}^* \wedge (r)_1 \Vdash^{\mathcal{A}}_{rt} \overline{m} = \mathfrak{c}]$. Thus $\mathbf{p}\underline{m}\mathbf{i}_r \Vdash^{\mathcal{A}}_{rt} \overline{m} \in \overline{k}$ provides the desired realizer. Whence we have

$$\mathbf{d}\underline{m}\,\underline{k}(\mathbf{p0i}_r)(\mathbf{p1}(\mathbf{p}\underline{m}\mathbf{i}_r)) \Vdash^{\mathcal{A}}_{rt} (\overline{m} = \overline{k} \vee \overline{m} \in \overline{k}).$$

As a consequence of the foregoing with $\ell(\underline{k}) := \lambda z.\mathbf{d}z\,\underline{k}(\mathbf{p0i}_r)(\mathbf{p1}(\mathbf{p}z\mathbf{i}_r))$, we have

$$\ell(\underline{k}) \Vdash^{\mathcal{A}}_{rt} \forall y \in \overline{n}\,(y = \overline{k} \vee y \in \overline{k})$$

Note that $\mathbf{p}\underline{k}\mathbf{i}_r \Vdash^{\mathcal{A}}_{rt} \overline{k} \in \overline{n}$ and $\lambda z.\mathbf{p}z\mathbf{i}_r \Vdash^{\mathcal{A}}_{rt} (\forall y \in \overline{k})\, y \in \overline{n}$, and hence $\wp(\underline{k}) \Vdash^{\mathcal{A}}_{rt} \overline{k} \in \overline{n} \wedge (\forall y \in \overline{k})\, y \in \overline{n}$, where $\wp(\underline{k}) := \mathbf{p}(\mathbf{p}\underline{k}\mathbf{i}_r)(\lambda z.\mathbf{p}z\mathbf{i}_r)$. We also have $\underline{k} = \mathbf{p}_N\,\underline{n}$. Put

$$t(\underline{n}) \quad := \quad \mathbf{p}(\mathbf{p}_N\underline{n})\big(\mathbf{p}(\ell(\mathbf{p}_N\,\underline{n}))(\wp(\mathbf{p}_N\,\underline{n}))\big)$$

we arrive at $t(\underline{n}) \Vdash^{\mathcal{A}}_{rt} \exists u \in \overline{\omega}\, SC(u, \overline{n})$.
As a result, since $n = 0$ or $n = k+1$ for some $k \in \omega$, $\underline{n} = f$ and $\overline{n} = \mathfrak{c}$ we conclude that

$$\mathbf{d}f\mathbf{0}(\mathbf{p00})(\mathbf{p1}t(f)) \Vdash^{\mathcal{A}}_{rt} [\bot_{\mathfrak{c}} \vee \exists u \in \overline{\omega}\; SC(u, \mathfrak{c})].$$

Therefore, with $\mathbf{q}^+ := \lambda f.\mathbf{d}f\mathbf{0}(\mathbf{p00})(\mathbf{p1}t(f))$, $\mathbf{q}^+$ is a realizer for $\forall v \in \overline{\omega}\,\big[\bot_v \vee \exists u \in \overline{\omega}\, SC(u, v)\big]$.

Conversely let $\mathfrak{a} \in \mathtt{V}^*(\mathcal{A})$ and suppose that

$$e \quad \Vdash^{\mathcal{A}}_{rt} \quad \bot_{\mathfrak{a}} \vee \exists u \in \overline{\omega}\, SC(u, \mathfrak{a}). \tag{6.14}$$

Then we have either

$$(e)_0 = \mathbf{0} \text{ and } (e)_1 \Vdash^{\mathcal{A}}_{rt} \bot_{\mathfrak{a}} \text{ or },$$

$$(e)_0 = \mathbf{1} \text{ and } (e)_1 \Vdash^{\mathcal{A}}_{rt} \exists u \in \overline{\omega}\, SC(u, \mathfrak{a})$$

.

The first sitauation entails $\bot_{\mathfrak{a}^\circ}$ by Lemma 6.2.6 and hence $\mathfrak{a}^\circ = 0$. Furthermore, it also entails that $\mathfrak{a} = \langle 0, 0\rangle$. To verify this assume $\langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*$. Then $(e)_1 f \Vdash^{\mathcal{A}}_{rt} \neg\, \mathfrak{c} = \mathfrak{c}$, or equivalently $\forall g \in |\mathcal{A}|\ \neg\, g \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} = \mathfrak{c}$. But, since $\mathbf{i}_r \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} = \mathfrak{c}$ this is absurd, so $\mathfrak{a}^* = 0$ which yields that $\mathbf{i}_r \Vdash^{\mathcal{A}}_{rt} \overline{0} = \mathfrak{a}$ and therefore

$$\mathbf{p}(e)_0\mathbf{i}_r \Vdash^{\mathcal{A}}_{rt} \mathfrak{a} \in \overline{\omega}. \tag{6.15}$$

The second situation yeilds that $((e)_1)_0 = \underline{n}$ for some $n \in \omega$ and $((e)_1)_1 \Vdash^{\mathcal{A}}_{rt} SC(\overline{n}, \mathfrak{a})$. Therefore, by the definition of realizability we have the following equivalences:

$$\begin{aligned}
& (e)_1 \Vdash^{\mathcal{A}}_{rt} \exists u \in \overline{\omega}\, SC(u, \mathfrak{a}) \\
\Leftrightarrow\ & \exists \mathfrak{c}\, \langle ((e)_1)_0, \mathfrak{c}\rangle \in \overline{\omega}^* \ \wedge\ ((e)_1)_1 \Vdash^{\mathcal{A}}_{rt} SC(\mathfrak{c}, \mathfrak{a}) \\
\Leftrightarrow\ & \exists \mathfrak{c}\, \langle ((e)_1)_0, \mathfrak{c}\rangle \in \overline{\omega}^* \ \wedge\ ((e)_1)_1 \Vdash^{\mathcal{A}}_{rt} \forall y \in \mathfrak{a}\, [y = \overline{n}\ \vee\ y \in \overline{n}]\ \wedge\ [\overline{n} \in \mathfrak{a}\ \wedge\ \forall y \in \overline{n}\, y \in \mathfrak{a}]
\end{aligned}$$

Put $s := ((e)_1)_1$ then, one concludes that $t_1 \Vdash^{\mathcal{A}}_{rt} \forall y \in \mathfrak{a}\, (y = \overline{n} \vee y \in \overline{n})$, $t_2 \Vdash^{\mathcal{A}}_{rt} \overline{n} \in \mathfrak{a}$, and $t_3 \Vdash^{\mathcal{A}}_{rt} \forall y \in \overline{n}\, y \in \mathfrak{a}$, with $t_1 := (s)_0$, $t_2 := ((s)_1)_0$ and $t_3 := ((s)_1)_1$.
The first aim is to cook up an application term $\mathbf{q}^{\#}$ where $\mathbf{q}^{\#} \Vdash^{\mathcal{A}}_{rt} \mathfrak{a} = \overline{(n+1)}$. To show this we first assume that $\langle f, \mathfrak{c}\rangle \in \mathfrak{a}^*$.
Then $t_1 f \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} = \overline{n} \vee \mathfrak{c} \in \overline{n}$ and $(t_1 f)_0 = \mathbf{0}$ or $(t_1 f)_0 = \mathbf{1}$. However, $(t_1 f)_0 = \mathbf{0}$ yeilds $(t_1 f)_1 \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} = \overline{n}$, and thus $\mathbf{p}\underline{n}(t_1 f)_1 \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} \in \overline{(n+1)}^*$. On the other hand, if $(t_1 f)_0 = \mathbf{1}$ then, $(t_1 f)_1 \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} \in \overline{n}$, which implies that $((t_1 f)_1)_0 = \underline{k}$ and $((t_1 f)_1)_1 \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} = \overline{k}$ for some $k \in n$. Hence,

$$\mathbf{p}r_0 r_1 \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} \in \overline{(n+1)}^*,$$

where $r_i := ((t_1 f)_1)_i$. Thus, we conclude that

$$\langle f, \mathfrak{c}\rangle \in \mathfrak{a}^* \ \ \rightarrow\ \ q_1(f) \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} \in \overline{n+1}^*, \tag{6.16}$$

such that $q_1(f) := \mathbf{d}(t_1 f)_0\, \mathbf{0}(\mathbf{p}\underline{n}(t_1 f)_1)(\mathbf{p}r_0 r_1)$.
Next, suppose that $\langle f, \mathfrak{c}\rangle \in \overline{(n+1)}^*$. Then for some $k \in n+1$, we have $f = \underline{k}$ and $\mathfrak{c} = \overline{k}$. So that $k = n\ \vee\ k \in n$.
However, $k = n$ implies $\overline{k} = \overline{n}$, which yields $t_2 \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} \in \mathfrak{a}^*$, whereas $k \in n$ implies $t_3\underline{k} \Vdash^{\mathcal{A}}_{rt} \overline{k} \in \mathfrak{a}$, and hence $t_3\underline{k} \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} \in \mathfrak{a}$. Therefore, as $f = \underline{k}$ we obtain $q_2(f) \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} \in \mathfrak{a}^*$

with $q_2(f) := \mathbf{d}f\underline{n}\,t_2(t_3f)$. To sumurize,

$$\langle f, \mathfrak{c}\rangle \in \overline{(n+1)}^{*} \;\;\rightarrow\;\; q_2(f) \Vdash_{rt}^{\mathcal{A}} \mathfrak{c} \in \mathfrak{a}^{*}. \tag{6.17}$$

With $\mathbf{q}^{\#} := \mathbf{p}(\lambda f.q_1(f))(\lambda f.q_2(f))$, (6.16) and (6.17) imply that $\mathbf{q}^{\#} \Vdash_{rt}^{\mathcal{A}} \mathfrak{a} = \overline{(n+1)}$. Now $r \Vdash_{rt}^{\mathcal{A}} \mathfrak{a} \in \overline{\omega}$ means $\exists \mathfrak{c}\,[\langle (r)_0, \mathfrak{c}\rangle \in \overline{\omega}^{*} \wedge (r)_1 \Vdash_{rt}^{\mathcal{A}} \mathfrak{a} = \mathfrak{c}]$, so we see that

$$\mathbf{p}(\underline{n+1})\mathbf{q}^{\#} \Vdash_{rt}^{\mathcal{A}} \mathfrak{a} \in \overline{\omega}.$$

The upshot of the foregoing is that from (6.14), we can conclude (6.15) if $(e)_0 = \mathbf{0}$ and if $(e)_0 = \mathbf{1}$ then, (6.18) holds. Note also that from (6.14) $(e)_0 = \mathbf{1}$ implies $\underline{n+1} = \mathbf{S_N}\,\underline{n} = \mathbf{S_N}\,((e)_1)_0$. Hence, we infer that $\ell^{\diamond}(e) \Vdash_{rt}^{\mathcal{A}} \mathfrak{a} \in \overline{\omega}$ where $\ell^{\diamond}(e) := \mathbf{d}(e)_0\mathbf{0}(\mathbf{p}(e)_0\mathbf{i}_r)(\mathbf{p}(\mathbf{S_N}\,((e)_1)_0)\mathbf{q}^{\#})$.
Therefore, we have

$$\mathbf{q}^{++} \;\; \Vdash_{rt}^{\mathcal{A}} \;\; \forall v[(\bot_v \;\vee \exists u \in \overline{\omega}\, SC(u,v)) \;\rightarrow\; v \in \overline{\omega}]. \tag{6.18}$$

with $\mathbf{q}^{++} := \lambda e.\ell^{\diamond}(e)$.
We deduce from (6.12) and (6.18)that $\mathbf{p}(\mathbf{q}^{+})(\mathbf{q}^{++})$ is a realizer for the axiom Infinity axiom.

(**Strong Collection**): For $\mathfrak{a} \in \mathtt{V}^{*}(\mathcal{A})$, suppose that $g \Vdash_{rt}^{\mathcal{A}} \forall x \in \mathfrak{a}\, \exists y\, \varphi(x,y)$. Then, by the realizability definition, we have

$$\forall x \in \mathfrak{a}^{\circ}\, \exists y\, \varphi^{\circ}(x,y) \tag{6.19}$$

Furthermore, for all $\langle f, \mathfrak{b}\rangle \in \mathfrak{a}^{*}$, $gf \Vdash_{rt}^{\mathcal{A}} \exists y\, \varphi(\mathfrak{b}, y)$, i.e., $\exists \mathfrak{c} \in \mathtt{V}^{*}(\mathcal{A})\ gf \Vdash_{rt}^{\mathcal{A}} \varphi(\mathfrak{b}, \mathfrak{c})$. Using Strong Collection in the background theory, there is a set $D$ such that the following hold

$$\forall \langle f, \mathfrak{b}\rangle \in \mathfrak{a}^{*}\, \exists \mathfrak{c} \in \mathtt{V}^{*}(\mathcal{A})\, \big[\langle \mathbf{p}(gf)f, \mathfrak{c}\rangle \in D \;\wedge\; gf \Vdash_{rt}^{\mathcal{A}} \varphi(\mathfrak{b}, \mathfrak{c})\big], \quad \text{and} \tag{6.20}$$

$$\forall z \in D\, \exists \langle f, \mathfrak{b}\rangle \in \mathfrak{a}^{*}\, \exists \mathfrak{c} \in \mathtt{V}^{*}(\mathcal{A})\, \big[z = \langle \mathbf{p}(gf)f, \mathfrak{c}\rangle \;\wedge\; gf \Vdash_{rt}^{\mathcal{A}} \varphi(\mathfrak{b}, \mathfrak{c})\big]. \tag{6.21}$$

In other words, $D \subseteq |\mathcal{A}| \times \mathtt{V}^{*}(\mathcal{A})$. Note that (6.20) also entails that

$$\forall \langle h, \mathfrak{c}\rangle \in D\, \exists \mathfrak{b}^{\circ} \in \mathfrak{a}^{\circ}\ \varphi^{\circ}(\mathfrak{b}^{\circ}, \mathfrak{c}^{\circ}). \tag{6.22}$$

Now, apply Strong Collection to (6.19), we infer the existence of a set $E$ such that $\forall x \in \mathfrak{a}^\circ\, \exists y \in E\, \varphi^\circ(x,y)\ \wedge\ \forall y \in E\, \exists x \in \mathfrak{a}^\circ\, \varphi^\circ(x,y)$. Setting

$$\begin{aligned} Y &= E \cup \{\mathfrak{c}^\circ :\ \exists k\ \langle k, \mathfrak{c}\rangle \in D\}, \\ \mathfrak{d} &= \langle Y, D\rangle. \end{aligned}$$

However, $\langle k, \mathfrak{c}\rangle \in D$ entails $\mathfrak{c}^\circ \in Y$ so 6.1.2 implies that $\mathfrak{d} \in \mathtt{V}^*(\mathcal{A})$ and by (6.22) we conculde

$$\forall x \in \mathfrak{a}^\circ\, \exists y \in Y\, \varphi^\circ(x,y)\ \wedge\ \forall y \in Y\, \exists x \in \mathfrak{a}^\circ\, \varphi^\circ(x,y). \tag{6.23}$$

Our aim is to cook up, from $g$, an application terms $e, e'$ with

$$e \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{a}\, \exists y \in \mathfrak{d}\, \varphi(x,y), \tag{6.24}$$

$$e' \Vdash^{\mathcal{A}}_{rt} \forall y \in \mathfrak{d}\, \exists x \in \mathfrak{a}\, \varphi(x,y). \tag{6.25}$$

Note that by the definition of realizability, form (6.24) we have the following equivalences:

$$\begin{aligned} (6.24) \ \Leftrightarrow\ & \forall x \in \mathfrak{a}^\circ\, \exists y \in \mathfrak{d}^\circ\, \varphi^\circ(x,y)\ \wedge\ \forall \langle f, \mathfrak{b}\rangle \in \mathfrak{a}^*\ ef \Vdash^{\mathcal{A}}_{rt} \exists y \in \mathfrak{d}\, \varphi(\mathfrak{b}, y) \\ \Leftrightarrow\ & \forall x \in \mathfrak{a}^\circ\, \exists y \in \mathfrak{d}^\circ\, \varphi^\circ(x,y) \\ & \wedge \forall \langle f, \mathfrak{b}\rangle \in \mathfrak{a}^*\ \exists \mathfrak{e}(\langle (ef)_0, \mathfrak{e}\rangle \in \mathfrak{d}^*\ \wedge\ (ef)_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{b}, \mathfrak{e})) \end{aligned}$$

So, let $\langle f, \mathfrak{b}\rangle \in \mathfrak{a}^*$. Then, using (6.20), one infers the existence of a $\mathfrak{c} \in \mathtt{V}^*(\mathcal{A})$ with $\langle \mathbf{p}(gf)f, \mathfrak{c}\rangle \in D\ \wedge\ gf \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{b}, \mathfrak{c})$. Therefore, $\mathbf{p}\big(\mathbf{p}(gf)f\big)(gf) \Vdash^{\mathcal{A}}_{rt} \exists y \in \mathfrak{d}\, \varphi(\mathfrak{b}, y)$. Let $e$ be such that

$$e = \lambda u.\mathbf{p}\big(\mathbf{p}(gu)u\big)(gu).$$

Taking into consideration (6.23), we get (6.24).

To show (6.25), we similarly have the following equivalences:

$$\begin{aligned} (6.25) \ \Leftrightarrow\ & \forall y \in \mathfrak{d}^\circ\, \exists x \in \mathfrak{a}^\circ\, \varphi^\circ(x,y)\ \wedge\ \forall \langle h, \mathfrak{c}\rangle \in \mathfrak{d}^*\ e'h \Vdash^{\mathcal{A}}_{rt} \exists x \in \mathfrak{a}\, \varphi(x, \mathfrak{c}) \\ \Leftrightarrow\ & \forall y \in \mathfrak{d}^\circ\, \exists x \in \mathfrak{a}^\circ\, \varphi^\circ(x,y) \\ & \wedge \forall \langle h, \mathfrak{c}\rangle \in \mathfrak{d}^*\ \exists \mathfrak{b}(\langle (e'h)_0, \mathfrak{b}\rangle \in \mathfrak{a}^*\ \wedge\ (e'h)_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{b}, \mathfrak{c})). \end{aligned}$$

So, let $\langle h, \mathfrak{c}\rangle \in \mathfrak{d}^*$, i.e., $\langle h, \mathfrak{c}\rangle \in D$. (6.21) entails the existence $\langle f, \mathfrak{b}\rangle \in \mathfrak{a}^*$ and $g \in |\mathcal{A}|$ with $h = \mathbf{p}(gf)f$ and $gf \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{b}, \mathfrak{c})$, and hence $\mathbf{p}(f)(gf) \Vdash^{\mathcal{A}}_{rt} \exists x \in \mathfrak{a}\, \varphi(x, \mathfrak{c})$. Therefore, putting $e' = \lambda v.\mathbf{p}(v)_1(v)_0$, and since by (6.23) we also have $\forall y \in \mathfrak{d}^\circ\, \exists x \in \mathfrak{a}^\circ\, \varphi^\circ(x, y)$ holds, so (6.25) is established.

Let $\vartheta(u, z)$ be the formula $\forall x \in u\, \exists y \in z\, \varphi(x, y) \;\wedge\; \forall y \in z\, \exists x \in u\, \varphi(x, y)$. Then by Strong Collection we also get

$$\forall x \in \mathfrak{a}^\circ\, \exists y\, \varphi^\circ(x, y) \;\rightarrow\; \exists z\, \vartheta^\circ(\mathfrak{a}^\circ, z) \tag{6.26}$$

Therefore, from (6.24), (6.25) and (6.26) we conclude that

$$\mathbf{p}(\lambda g.e)(\lambda g.e') \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{a}\, \exists y\, \varphi(x, y) \;\rightarrow\; \exists z\, \vartheta(\mathfrak{a}, z),$$

(**Subset Collection**): For $\mathfrak{a}, \mathfrak{b} \in \mathtt{V}^*(\mathcal{A})$, let $\varphi(x, y, u)$ be a formula with parameters from $\mathtt{V}^*(\mathcal{A})$ and at most the free variables shown. The aim is to construct a realizer $\mathbf{r}$ for which the following holds

$$\mathbf{r} \Vdash^{\mathcal{A}}_{rt} \exists q \forall u \big[\forall x \in \mathfrak{a}\, \exists y \in \mathfrak{b}\, \varphi(x, y, u) \rightarrow \exists v \in q\, \varphi'(\mathfrak{a}, v, u)\big], \tag{6.27}$$

and $\varphi'(\mathfrak{a}, v, u)$ is defined by the formula

$$\forall x \in \mathfrak{a}\, \exists y \in v\, \varphi(x, y, u) \;\wedge\; \forall y \in v\, \exists x \in \mathfrak{a}\, \varphi(x, y, u).$$

Let

$$e \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{a} \exists y \in \mathfrak{b}\, \varphi(x, y, u)$$

By the definition of realizability, we have the following equivalences:

$$\begin{aligned} e \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{a} \exists y \in \mathfrak{b}\, \varphi(x, y, u) \;\;\Leftrightarrow\;\; & \forall x \in \mathfrak{a}^\circ \exists y \in \mathfrak{b}^\circ\, \varphi^\circ(x, y, u) \;\wedge \\ & \forall \langle f, \mathfrak{c}\rangle \in \mathfrak{a}^* \exists \mathfrak{d}(\langle (ef)_0, \mathfrak{d}\rangle \in \mathfrak{b}^* \;\wedge\; (ef)_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{c}, \mathfrak{d}, u)) \end{aligned}$$

First, we define $B$ as follows

$$B \;=\; \{\langle \mathbf{p}ef, \mathfrak{d}\rangle \;:\; e, f \in |\mathcal{A}| \;\wedge\; ef\downarrow \;\wedge\; \langle (ef)_0, \mathfrak{d}\rangle \in \mathfrak{b}^*\}.$$

Observe that $B$ is a set.

Next, set $\psi(e,f,\mathfrak{c},u,z)$ to be the formula

$$u \in \mathtt{V}^*(\mathcal{A}) \wedge e,f \in |\mathcal{A}| \wedge ef\downarrow \;\wedge\; \exists \mathfrak{d}\,\big[\langle \mathbf{p}ef,\mathfrak{d}\rangle = z \wedge \langle (ef)_0,\mathfrak{d}\rangle \in \mathfrak{b}^* \wedge (ef)_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{c},\mathfrak{d},u)\big].$$

Using Subset Collection we infer the existence of a set $D$ such that

$$\forall u \forall e\, \big[\forall \langle f,\mathfrak{c}\rangle \in \mathfrak{a}^*\, \exists z \in B \psi(e,f,\mathfrak{c},u,z) \;\rightarrow\; \exists w \in D\, \psi'(\mathfrak{a}^*,e,u,w)\big], \tag{6.28}$$

where $\psi'(\mathfrak{a}^*,e,u,w)$ is the formula

$$\forall \langle f,\mathfrak{c}\rangle \in \mathfrak{a}^*\, \exists z \in w\, \psi(e,f,\mathfrak{c},u,z) \;\wedge\; forall z \in w\, \exists \langle f,\mathfrak{c}\rangle \in \mathfrak{a}^*\, \psi(e,f,\mathfrak{c},u,z).$$

Setting

$$\hat{D} := \{w \cap B :\; w \in D\},$$

(6.28) entails

$$\forall u \forall e\, \big[\forall \langle f,\mathfrak{c}\rangle \in \mathfrak{a}^*\, \exists z \in B \psi(e,f,\mathfrak{c},u,z) \;\rightarrow\; \exists w \in \hat{D}\, \psi'(\mathfrak{a}^*,e,u,w)\big]. \tag{6.29}$$

By invoking Subset Collection, there is a set $C$ with

$$\forall u \big[\forall x \in \mathfrak{a}^\circ\, \exists y \in \mathfrak{b}^\circ\, \varphi^\circ(x,y,u) \;\rightarrow\; \exists v \in C\, \vartheta(\mathfrak{a}^\circ,v,u)\big], \tag{6.30}$$

where $\vartheta(z,v,u)$ abbreviates the conjuction $\forall x \in z\, \exists y \in v\, \varphi^\circ(x,y,u) \;\wedge\; \forall y \in v\, \exists x \in z\, \varphi^\circ(x,y,u)$.

Now, for the existential quantifier $\exists q$ in (6.27), we need a witness in $\mathtt{V}^*(\mathcal{A})$. To define the witness, let

$$\begin{aligned} \mathcal{W} &:= \{\langle v \cup \{\mathfrak{c}^\circ :\; \exists h\, \langle h,\mathfrak{c}\rangle \in w\}, w\rangle :\; v \in C \;\wedge\; w \in \hat{D}\}, \\ E &:= C \cup \{\mathfrak{z}^\circ :\; \mathfrak{z} \in \mathcal{W}\}, \\ E^+ &:= \{\langle \mathbf{0},\mathfrak{z}\rangle :\; \mathfrak{z} \in \mathcal{W}\} \\ \mathfrak{e} &:= \langle E, E^+\rangle. \end{aligned} \tag{6.31}$$

Since $B \subseteq |\mathcal{A}| \times \mathtt{V}^*(\mathcal{A})$ and whenever $w \in \hat{D}$, we get $w \subseteq |\mathcal{A}| \times \mathtt{V}^*(\mathcal{A})$ so that using Lemma 6.1.2, for all $\mathfrak{z} \in \mathcal{W}$, $\mathfrak{z} \in \mathtt{V}^*(\mathcal{A})$ holds. Hence, for $\mathfrak{z} \in \mathcal{W}$, we have $\mathfrak{z}^\circ \in E$ and

$\langle \mathbf{0}, \mathfrak{z} \rangle \in |\mathcal{A}| \times \mathtt{V}^*(\mathcal{A})$ which by Lemma 6.1.2, implies $\mathfrak{e} \in \mathtt{V}^*(\mathcal{A})$.

Next, for $e \in |\mathcal{A}|$ and $\mathfrak{p} \in \mathtt{V}^*(\mathcal{A})$ satisfy

$$e \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{a}\, \exists y \in \mathfrak{b}\, \varphi(x, y, \mathfrak{p}). \tag{6.32}$$

We thus obtain,

$$\begin{aligned} &\forall x \in \mathfrak{a}^\circ\, \exists y \in \mathfrak{b}^\circ\, \varphi^\circ(x, y, \mathfrak{p}^\circ) \quad \text{and} \qquad (6.33)\\ &\forall \langle f, \mathfrak{c} \rangle \in \mathfrak{a}^*\, \exists \mathfrak{d} \left[ \langle (ef)_0, \mathfrak{d} \rangle \in \mathfrak{b}^* \;\wedge\; (ef)_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{c}, \mathfrak{d}, \mathfrak{p}) \right]. \end{aligned}$$

Therefore $\forall \langle f, \mathfrak{c} \rangle \in \mathfrak{a}^*\, \exists z \in B\, \psi(e, f, \mathfrak{c}, \mathfrak{p}, z)$ which by (6.29), entails the existence of $w \in \hat{D}$ that satisfies $\forall \langle f, \mathfrak{c} \rangle \in \mathfrak{a}^*\, \exists z \in w\, \psi(e, f, \mathfrak{c}, \mathfrak{p}, z)$ and $\forall z \in w\, \exists \langle f, \mathfrak{c} \rangle \in \mathfrak{a}^*\, \psi(e, f, \mathfrak{c}, \mathfrak{p}, z)$. Unravelling the definition of $\psi$ yeilds

$$\forall \langle f, \mathfrak{c} \rangle \in \mathfrak{a}^*\, \exists \mathfrak{d} \left[ \langle \mathbf{p}ef, \mathfrak{d} \rangle \in w \;\wedge\; \langle (ef)_0, \mathfrak{d} \rangle \in \mathfrak{b}^* \;\wedge\; (ef)_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{c}, \mathfrak{d}, \mathfrak{p}) \right] \tag{6.34}$$

$$\forall \langle g, \mathfrak{d} \rangle \in w\, \exists \mathfrak{c} \left[ \langle (g)_1, \mathfrak{c} \rangle \in \mathfrak{a}^* \;\wedge\; \langle (\hat{g})_0, \mathfrak{d} \rangle \in \mathfrak{b}^* \;\wedge\; (\hat{g})_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{c}, \mathfrak{d}, \mathfrak{p}) \right], \tag{6.35}$$

with $\hat{g} := (g)_0(g)_1$. Moreover, (6.32) entails that $\forall x \in \mathfrak{a}^\circ\, \exists y \in \mathfrak{b}^\circ\, \varphi^\circ(x, y, \mathfrak{p}^\circ)$ and hence, by (6.30), there is a $v \in C$ such that

$$\forall x \in \mathfrak{a}^\circ\, \exists y \in v\, \varphi^\circ(x, y, \mathfrak{p}^\circ) \text{ and } \forall y \in v\, \exists x \in \mathfrak{a}^\circ\, \varphi^\circ(x, y, \mathfrak{p}^\circ).$$

Set $\mathfrak{z} := \langle v \cup \{\mathfrak{d}^\circ :\; \exists h\, \langle h, \mathfrak{d} \rangle \in w\}, w \rangle$. Then we have $\langle \mathbf{0}, \mathfrak{z} \rangle \in \mathfrak{e}^*$ with $\mathfrak{z} \in \mathcal{W}$. By (6.35) and using Lemma 6.2.6, $\langle h, \mathfrak{d} \rangle \in w$ implies that for some $\mathfrak{c}^\circ \in \mathfrak{a}^\circ$ we have $\varphi^\circ(\mathfrak{c}^\circ, \mathfrak{d}^\circ, \mathfrak{p}^\circ)$. Thus,

$$\forall y \in \mathfrak{z}^\circ\, \exists x \in \mathfrak{a}^\circ\, \varphi^\circ(x, y, \mathfrak{p}^\circ).$$

Thus, one concludes that

$$\forall x \in \mathfrak{a}^\circ\, \exists y \in \mathfrak{z}^\circ\, \varphi^\circ(x, y, \mathfrak{p}^\circ) \;\wedge\; \forall y \in \mathfrak{z}^\circ\, \exists x \in \mathfrak{a}^\circ\, \varphi^\circ(x, y, \mathfrak{p}^\circ). \tag{6.36}$$

Moreover, from (6.34) and (6.35) we also infer that

$$\forall \langle f, \mathfrak{c} \rangle \in \mathfrak{a}^*\, \exists \mathfrak{d} \left[ \langle \mathbf{p}ef, \mathfrak{d} \rangle \in \mathfrak{z}^* \;\wedge\; (ef)_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{c}, \mathfrak{d}, \mathfrak{p}) \right], \tag{6.37}$$

$$\forall \langle g, \mathfrak{d} \rangle \in \mathfrak{z}^*\, \exists \mathfrak{c} \left[ \langle (g)_1, \mathfrak{c} \rangle \in \mathfrak{a}^* \;\wedge\; ((g)_0(g)_1)_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{c}, \mathfrak{d}, \mathfrak{p}) \right] \tag{6.38}$$

Setting

$$\mathbf{m}_0 := \lambda f.\mathbf{p}(\mathbf{p}ef)(ef)_1,$$
$$\mathbf{m}_1 := \lambda g.\mathbf{p}((g)_1)\big(\big((g)_0(g)_1\big)_1\big)$$

(6.36), (6.37) and (6.38) implies that

$$\mathbf{m}_0 \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{a}\, \exists y \in \mathfrak{z}\, \varphi(x, y, \mathfrak{p}),$$
$$\mathbf{m}_1 \Vdash^{\mathcal{A}}_{rt} \forall y \in \mathfrak{z}\, \exists x \in \mathfrak{a}\, \varphi(x, y, \mathfrak{p}).$$

Therefore, we have

$$\mathbf{p}\mathbf{m}_0\mathbf{m}_1 \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{a}\, \exists y \in \mathfrak{z}\, \varphi(x, y, \mathfrak{p}) \;\wedge\; \forall y \in \mathfrak{z}\, \exists x \in \mathfrak{a}\, \varphi(x, y, \mathfrak{p}). \tag{6.39}$$

To summorize, we have shown that (6.32) entails (6.39). Using this together with (6.30) and with the aid of the fact that $C \subseteq \mathfrak{e}^{\circ}$, we conclude that

$$\lambda e.\mathbf{p0}(\mathbf{p}\mathbf{m}_0\mathbf{m}_1) \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{a}\, \exists y \in \mathfrak{b}\, \varphi(x, y, u) \;\rightarrow\; \exists v \in \mathfrak{e}\, \varphi'(\mathfrak{a}, v, \mathfrak{p})$$

as $\langle \mathbf{0}, \mathfrak{z}\rangle \in \mathfrak{e}^*$.

Consequently, we obtain (6.27) with $\mathbf{r} := \lambda e.\mathbf{p0}(\mathbf{p}\mathbf{m}_0\mathbf{m}_1)$. □

## 6.4 The soundness theorem for CZF + REA

Next we show that the regular extension axiom holds in $\mathtt{V}^*(\mathcal{A})$ if it holds in the background universe.

**Lemma 6.4.1.** (**CZF**)

*(i) If $B$ is a regular set with $2 \in B$, then $B$ is closed under unordered and ordered pairs, i.e., whenever $x, y \in B$, then $\{x, y\}, \langle x, y\rangle \in B$.*

*(ii) If $B$ is a regular set, then $B \cap \mathtt{V}^*(\mathcal{A})$ is a set.*

**Proof.** See [30] Lemma 6.1. □

**Theorem 6.4.2.** *For every axiom $\theta$ of* **CZF**+**REA**, *there exists a closed application term $t$ such that*

$$\mathbf{CZF} + \mathbf{REA} \Vdash \ (t \Vdash^{\mathcal{A}}_{rt} \theta).$$

**Proof.** We proceed similarly to [30], Theorem 7.2. In view of Theorem 6.3.1, we need only find a realizer for the axiom **REA**. Let $\mathfrak{a} \in \mathtt{V}^*(\mathcal{A})$. Due to **REA** there exists a regular set $B$ such that $\mathfrak{a}, 2, |\mathcal{A}| \in B$. Let

$$\begin{aligned} A &:= B \cap \mathtt{V}^*(\mathcal{A}) \\ \mathfrak{c} &:= \langle B, \{\langle \mathbf{0}, \mathfrak{z}\rangle \ : \ \mathfrak{z} \in A\}\rangle. \end{aligned}$$

By Lemma 6.4.1(ii), $A$ is a set and hence $\mathfrak{c}$ is a set. Moreover, as $A \subseteq \mathtt{V}^*(\mathcal{A})$, it follows that $\{\langle \mathbf{0}, \mathfrak{z}\rangle \ : \ \mathfrak{z} \in A\} \subseteq |\mathcal{A}| \times \mathtt{V}^*(\mathcal{A})$ and we observe that $\mathfrak{z} \in A$ implies $\mathfrak{z} \in B$ and hence $\mathfrak{z}^\circ \in B$ by the transitivity of $B$. Therefore, by Lemma 6.1.2 (iii), $\mathfrak{c} \in \mathtt{V}^*(\mathcal{A})$. As $\mathfrak{a} \in B$ and $B$ is transitive it follows that $\mathfrak{a}^\circ \in B$, thus $\mathfrak{a}^\circ \in \mathfrak{c}^\circ$. Note also that $\mathfrak{a} \in A$ yielding $\langle \mathbf{0}, \mathfrak{a}\rangle \in \mathfrak{c}^*$. Thus we conclude that

$$\mathbf{p0i}_r \Vdash^{\mathcal{A}}_{rt} \mathfrak{a} \in \mathfrak{c}. \tag{6.40}$$

Let $\tilde{\mathbf{m}}$ and $\tilde{\mathbf{n}}$ be realizers for transitivity and inhabitedness of $\mathfrak{c}$, respectively, i.e.,

$$\begin{aligned} \tilde{\mathbf{m}} &\Vdash^{\mathcal{A}}_{rt} \ \forall u \in \mathfrak{c}\, \forall v \in u \ v \in \mathfrak{c} \\ \tilde{\mathbf{n}} &\Vdash^{\mathcal{A}}_{rt} \ \exists x \in \mathfrak{c} \ x \in \mathfrak{c} \end{aligned}$$

By the definition of realizability,

$$\begin{aligned} \tilde{\mathbf{m}} &\Vdash^{\mathcal{A}}_{rt} \ \forall u \in \mathfrak{c}\, \forall v \in u \ v \in \mathfrak{c} \\ &\Leftrightarrow \ \forall \langle x, \mathfrak{a}\rangle \in \mathfrak{c}^* \ \tilde{\mathbf{m}}x \Vdash^{\mathcal{A}}_{rt} \forall v \in \mathfrak{a} \ v \in \mathfrak{c} \\ &\Leftrightarrow \ \forall \langle x, \mathfrak{a}\rangle \in \mathfrak{c}^* \, \forall \langle y, \mathfrak{b}\rangle \in \mathfrak{a}^* (\tilde{\mathbf{m}}x)y \Vdash^{\mathcal{A}}_{rt} \mathfrak{b} \in \mathfrak{c} \end{aligned}$$

Thus, $(\tilde{\mathbf{m}}x)y = \mathbf{p0i}_r$ so $\tilde{\mathbf{m}} = \lambda x.\lambda y.\mathbf{p0i}_r$.

and

$$\begin{aligned}\tilde{\mathbf{n}} &\Vdash^{\mathcal{A}}_{rt} \exists x \in \mathfrak{c}\; x \in \mathfrak{c} \\ &\Leftrightarrow \exists \mathfrak{a}[\langle (\tilde{\mathbf{n}})_0, \mathfrak{a}\rangle \in \mathfrak{c}^* \;\wedge\; (\tilde{\mathbf{n}})_1 \Vdash^{\mathcal{A}}_{rt} \mathfrak{a} \in \mathfrak{c} \\ &\Leftrightarrow (\tilde{\mathbf{n}})_0 = \mathbf{0} \;\wedge\; (\tilde{\mathbf{n}})_1 \Vdash^{\mathcal{A}}_{rt} \mathfrak{a} \in \mathfrak{c}\,.\end{aligned}$$

Thus, $(\tilde{\mathbf{n}})_1 = \mathbf{p0i}_r$ so $\tilde{\mathbf{n}} = \mathbf{p0}(\mathbf{p0i}_r)$.
Therefore,

$$\mathbf{p}\tilde{\mathbf{m}}\tilde{\mathbf{n}} \Vdash^{\mathcal{A}}_{rt} \forall u \in \mathfrak{c}\, \forall v \in u\; v \in \mathfrak{c} \;\wedge\; \exists x \in \mathfrak{c}\, x \in \mathfrak{c}. \tag{6.41}$$

Since $\mathfrak{c}^\circ = B$, it is also the case that $\mathbf{Reg}(\mathfrak{c}^\circ)$ holds. Next we would like to find a realizer $\mathbf{q}$ such that

$$\mathbf{q} \Vdash^{\mathcal{A}}_{rt} \mathbf{Reg}(\mathfrak{c}). \tag{6.42}$$

To this end, suppose that $\langle \mathbf{0}, \mathfrak{b}\rangle \in \mathfrak{c}^*$, $f \in |\mathcal{A}|$, and $\varphi(x, y)$ is a formula with parameters in $\mathrm{V}^*(\mathcal{A})$ such that

$$f \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{b}\, \exists y \in \mathfrak{c}\, \varphi(x, y). \tag{6.43}$$

Note that all elements of $\mathfrak{c}^*$ are of the form $\langle \mathbf{0}, u\rangle$. As $B$ is transitive and $B$ is closed under taking pairs we have $\mathfrak{c}^* \subseteq B$, and thus(6.43) yields

$$\begin{aligned}&\forall \mathfrak{p} \forall e \big(\langle e, \mathfrak{p}\rangle \in \mathfrak{b}^* \;\rightarrow \\ &\qquad \exists z \in B\, \exists \mathfrak{q} \left[z = \langle e, \mathfrak{q}\rangle \;\wedge\; (fe)_0 = \mathbf{0} \;\wedge\; \langle \mathbf{0}, \mathfrak{q}\rangle \in \mathfrak{c}^* \;\wedge\; (fe)_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{p}, \mathfrak{q})\right]\big).\end{aligned} \tag{6.44}$$

Utilizing the regularity of $B$ and since $\mathfrak{b}^* \in B$, there exists $\hat{\mathbf{u}} \in B$ such that

$$\forall \mathfrak{p} \forall e \left[\langle e, \mathfrak{p}\rangle \in \mathfrak{b}^* \;\rightarrow\; \exists z \in \hat{\mathbf{u}}\, \exists \mathfrak{q}\big(z = \langle e, \mathfrak{q}\rangle \;\wedge\; \langle \mathbf{0}, \mathfrak{q}\rangle \in \mathfrak{c}^* \;\wedge\; (fe)_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{p}, \mathfrak{q})\big)\right]; \tag{6.45}$$

$$\wedge \forall z \in \hat{\mathbf{u}}\, \exists \mathfrak{p}, e \left[\langle e, \mathfrak{p}\rangle \in \mathfrak{b}^* \;\wedge\; \exists \mathfrak{q}\big(\langle \mathbf{0}, \mathfrak{q}\rangle \in \mathfrak{c}^* \;\wedge\; z = \langle e, \mathfrak{q}\rangle \;\wedge\; (fe)_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{p}, \mathfrak{q})\big)\right] \tag{6.46}$$

From (6.46) it follows that $\hat{\mathbf{u}} \subseteq |\mathcal{A}| \times A \subseteq |\mathcal{A}| \times \mathrm{V}^*(\mathcal{A})$, and thus with

$$\mathfrak{u} \;:=\; \langle \{\mathfrak{p}^\circ \mid \exists e \in |\mathcal{A}|\, \langle e, \mathfrak{p}\rangle \in \hat{\mathbf{u}}\}, \hat{\mathbf{u}}\rangle$$

we have $\mathfrak{u} \in \mathtt{V}^*(\mathcal{A})$ by 6.1.2. Moreover, the function $\langle e, \mathfrak{p}\rangle \mapsto \mathfrak{p}^\circ$ defined on $\hat{\mathbf{u}}$ maps into $B$, so that by the regularity of $B$ we have $\{\mathfrak{p}^\circ \mid \exists e \in |\mathcal{A}|\, \langle e, \mathfrak{p}\rangle \in \hat{\mathbf{u}}\} \in B$, thus as $B$ is closed under taking pairs we have $\mathfrak{u} \in B$ and hence $\mathfrak{u} \in B \cap \mathtt{V}^*(\mathcal{A}) = A$, which yields $\langle \mathbf{0}, \mathfrak{u}\rangle \in \mathfrak{c}^*$. So we get

$$\mathbf{p0i}_r \Vdash^{\mathcal{A}}_{rt} \mathfrak{u} \in \mathfrak{c}. \tag{6.47}$$

Let $r$ be a realizer for the the statement $\forall x \in \mathfrak{b}\, \exists y \in \mathfrak{u}\, \varphi(x, y)$. By the definition of realizability we have the following equivalences:

$$\begin{aligned} r \quad \Vdash^{\mathcal{A}}_{rt} \quad & \forall x \in \mathfrak{b}\, \exists y \in \mathfrak{u}\, \varphi(x, y) \\ \Leftrightarrow \quad & \forall e, \mathfrak{p}[\langle e, \mathfrak{p}\rangle \in \mathfrak{b}^* \to re \Vdash^{\mathcal{A}}_{rt} \exists y \in \mathfrak{u}\, \varphi(x, y)] \\ \Leftrightarrow \quad & \forall \langle e, \mathfrak{p}\rangle \in \mathfrak{b}^*\, \exists \mathfrak{q}\, [\langle (re)_0, \mathfrak{q}\rangle \in \mathfrak{u}^* \ \wedge \\ & (re)_1 \Vdash^{\mathcal{A}}_{rt} \phi(\mathfrak{p}, \mathfrak{q})]. \end{aligned}$$

and likewise we have

$$\begin{aligned} t \quad \Vdash^{\mathcal{A}}_{rt} \quad & \forall y \in \mathfrak{u}\, \exists x \in \mathfrak{b}\, \varphi(x, y) \\ \Leftrightarrow \quad & \forall e, \mathfrak{q}[\langle e, \mathfrak{q}\rangle \in \mathfrak{u}^* \to te \Vdash^{\mathcal{A}}_{rt} \exists x \in \mathfrak{b}\, \varphi(x, \mathfrak{q})] \\ \Leftrightarrow \quad & \forall \langle e, \mathfrak{q}\rangle \in \mathfrak{u}^*\, \exists \mathfrak{p}\, [\langle (te)_0, \mathfrak{p}\rangle \in \mathfrak{b}^* \ \wedge \\ & (te)_1 \Vdash^{\mathcal{A}}_{rt} \phi(\mathfrak{p}, \mathfrak{q})]. \end{aligned}$$

Letting $s(f) := \lambda e.\mathbf{p}e(fe)_1$, (6.45) and (6.46) yield

$$s(f) \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{b}\, \exists y \in \mathfrak{u}\, \varphi(x, y), \tag{6.48}$$

$$s(f) \Vdash^{\mathcal{A}}_{rt} \forall y \in \mathfrak{u}\, \exists x \in \mathfrak{b}\, \varphi(x, y). \tag{6.49}$$

As $\mathfrak{c}^\circ = B$ and $B$ is regular we also have

$$\begin{aligned} & \forall b \in \mathfrak{c}^\circ \big(\forall x \in b\, \exists y \in \mathfrak{c}^\circ\, \varphi^\circ(x, y) \ \to \\ & \quad \exists u \in \mathfrak{c}^\circ \big[\forall x \in b\, \exists y \in u\, \varphi^\circ(x, y) \ \wedge \ \forall y \in u\, \exists x \in b\, \varphi^\circ(x, y)\big]\big). \end{aligned} \tag{6.50}$$

We are looking for a realizer $r$ such that

$$r \Vdash_{rt}^{\mathcal{A}} \forall b \in \mathfrak{c} \left( \forall x \in b \, \exists y \in \mathfrak{c} \, \varphi(x,y) \ \rightarrow \right. \tag{6.51}$$

$$\left. \exists u \in \mathfrak{c} \left[ \forall x \in b \, \exists y \in u \, \varphi(x,y) \ \wedge \ \forall y \in u \, \exists x \in b \, \varphi(x,y) \right] \right)$$

$$\Leftrightarrow (6.50) \ \wedge \ \forall \langle \mathbf{0}, \mathfrak{b} \rangle \in \mathfrak{c}^* \rightarrow (r\mathbf{0}) \Vdash_{rt}^{\mathcal{A}} \forall x \in b \, \exists y \in \mathfrak{c} \, \varphi(x,y) \ \rightarrow$$

$$\left. \exists u \in \mathfrak{c} \left[ \forall x \in b \, \exists y \in u \, \varphi(x,y) \ \wedge \ \forall y \in u \, \exists x \in b \, \varphi(x,y) \right] \right)$$

$$\Leftrightarrow \quad (r\mathbf{0})f \Vdash_{rt}^{\mathcal{A}} \exists u \in \mathfrak{c} \left[ \forall x \in b \, \exists y \in u \, \varphi(x,y) \ \wedge \ \forall y \in u \, \exists x \in b \, \varphi(x,y) \right] \Big)$$

where $f$ here is from (6.43)

$$\Leftrightarrow \quad \exists \mathfrak{u} [\langle ((r\mathbf{0})f)_0, \mathfrak{u} \rangle \in \mathfrak{c}^* \ \wedge \ ((r\mathbf{0})f)_1 \Vdash_{rt}^{\mathcal{A}} \forall x \in b \, \exists y \in \mathfrak{u} \, \varphi(x,y) \ \wedge \ \forall y \in \mathfrak{u} \, \exists x \in b \, \varphi(x,y)$$

Therefore, letting $\tilde{\mathbf{q}} := \lambda h.\lambda f.\mathbf{p}(\mathbf{0})(\mathbf{p}s(f)s(f))$, (6.47), (6.48), (6.49) and (6.50) entail that

$$\tilde{\mathbf{q}} \Vdash_{rt}^{\mathcal{A}} \forall b \in \mathfrak{c} \left( \forall x \in b \, \exists y \in \mathfrak{c} \, \varphi(x,y) \ \rightarrow \right. \tag{6.52}$$

$$\left. \exists u \in \mathfrak{c} \left[ \forall x \in b \, \exists y \in u \, \varphi(x,y) \ \wedge \ \forall y \in u \, \exists x \in b \, \varphi(x,y) \right] \right).$$

Choosing $\varphi(x,y)$ to be the formula $\mathfrak{r} \subseteq b \times \mathfrak{c} \ \wedge \ \langle x,y \rangle \in \mathfrak{r}$, we deduce from (6.52) and (6.41) that

$$\mathbf{p}(\mathbf{p}\tilde{\mathbf{m}}\tilde{\mathbf{n}})\tilde{\mathbf{q}} \Vdash_{rt}^{\mathcal{A}} \mathbf{Reg}(\mathfrak{c}).$$

Thus, in view of (6.40), we conclude that

$$\mathbf{p}(\mathbf{p0i}_r)(\mathbf{p}(\mathbf{p}\tilde{\mathbf{m}}\tilde{\mathbf{n}})\tilde{\mathbf{q}}) \Vdash_{rt}^{\mathcal{A}} \forall a \, \exists c \left[ a \in c \ \wedge \ \mathbf{Reg}(c) \right].$$

□

# Chapter 7

# Preservation of Axioms of Choice AC Under Realizability with Truth

## 7.1 Truth Realizability for AC

On the basis of **CZF**, *Rathjen* showed in [33] that various choice principles including $\mathbf{AC}_\omega$, **DC**, **RDC**, and **PAx** hold true under truth realizability ($\Vdash_{rt}^{\mathcal{A}}$) defined on the realizability class structure $\mathtt{V}^*(\mathcal{A})$, for the special case of *Kleene's first model*, $\mathcal{K}_1$, providing that they hold in the background theory and it was left open whether this is also true when $\mathtt{V}^*(\mathcal{A})$ is the realizability structure built on an arbitrary $\mathbf{PCA}^+$ structure. The purpose of this chapter is to generalise the proofs in [33] concerned with $\mathtt{V}^*(\mathcal{K}_1)$ and show that they can adjusted to work for $\mathtt{V}^*(\mathcal{A})$, too.

For a formula $\varphi$ with parameters in $\mathtt{V}^*(\mathcal{A})$, write '$\mathtt{V}^*(\mathcal{A}) \models \varphi$' to mean that there is an application term $a$ with $a \Vdash_{rt}^{\mathcal{A}} \varphi$ holds and for a scheme of formulas $\mathcal{SC}$, we write $\mathtt{V}^*(\mathcal{A}) \models \mathcal{SC}$ to convey that for all instances $\varphi$ of $\mathcal{SC}$ there is an application term $a$ depending on $\varphi$ such that $a \Vdash_{rt}^{\mathcal{A}} \varphi$ is satisfied.

Since axioms of choice assure the existence of functions, the point of departure in investigating these axioms over $\mathtt{V}^*(\mathcal{A})$ would be the isolation of pairs and ordered pairs internal versions in the sense of $\mathtt{V}^*(\mathcal{A})$.

## 7.2 Internal pairing

**Definition 7.2.1.** *For* $\mathfrak{a}, \mathfrak{b} \in \mathtt{V}^*(\mathcal{A})$*, set*

$$\begin{aligned} \{\mathfrak{a}, \mathfrak{b}\}_{v^*} &:= \langle \{\mathfrak{a}^\circ, \mathfrak{b}^\circ\}, \{\langle \mathbf{0}, \mathfrak{a}\rangle, \langle \mathbf{1}, \mathfrak{b}\rangle\}\rangle, \\ \{\mathfrak{a}\}_{v^*} &:= \{\mathfrak{a}, \mathfrak{a}\}_{v^*}, \\ \langle \mathfrak{a}, \mathfrak{b}\rangle_{v^*} &:= \langle \langle \mathfrak{a}^\circ, \mathfrak{b}^\circ\rangle, \{\langle \mathbf{0}, \{\mathfrak{a}\}_{v^*}\rangle, \langle \mathbf{1}, \{\mathfrak{a}, \mathfrak{b}\}_{v^*}\rangle\}\rangle. \end{aligned}$$

**Lemma 7.2.2.** *(i)* $\{\mathfrak{a}, \mathfrak{b}\}^\circ_{v^*} = \{\mathfrak{a}^\circ, \mathfrak{b}^\circ\}$.

*(ii)* $\langle \mathfrak{a}, \mathfrak{b}\rangle^\circ_{v^*} = \langle \mathfrak{a}^\circ, \mathfrak{b}^\circ\rangle$.

*(iii)* $\{\mathfrak{a}, \mathfrak{b}\}_{v^*}, \langle \mathfrak{a}, \mathfrak{b}\rangle_{v^*} \in \mathtt{V}^*(\mathcal{A})$.

*(iv)* $\mathtt{V}^*(\mathcal{A}) \models \mathfrak{c} \in \{\mathfrak{a}, \mathfrak{b}\}_{v^*} \leftrightarrow [\mathfrak{c} = \mathfrak{a} \vee \mathfrak{c} = \mathfrak{b}]$.

*(v)* $\mathtt{V}^*(\mathcal{A}) \models \mathfrak{c} \in \langle \mathfrak{a}, \mathfrak{b}\rangle_{v^*} \leftrightarrow [\mathfrak{c} = \{\mathfrak{a}\}_{v^*} \vee \mathfrak{c} = \{\mathfrak{a}, \mathfrak{b}\}_{v^*}]$.

**Proof.** See [33], Lemma 7.2. □

## 7.3 Choice Axioms in $\mathtt{V}^*(\mathcal{A})$

**Theorem 7.3.1.** *(i)* $(\mathbf{CZF})$ $\quad \mathtt{V}^*(\mathcal{A}) \models \mathbf{AC}^{\omega,\omega}$.

*(ii)* $(\mathbf{CZF} + \mathbf{AC}_\omega)$ $\quad \mathtt{V}^*(\mathcal{A}) \models \mathbf{AC}_\omega$.

*(iii)* $(\mathbf{CZF} + \mathbf{DC})$ $\quad \mathtt{V}^*(\mathcal{A}) \models \mathbf{DC}$.

*(iv)* $(\mathbf{CZF} + \mathbf{RDC})$ $\quad \mathtt{V}^*(\mathcal{A}) \models \mathbf{RDC}$.

*(v)* $(\mathbf{CZF} + \mathbf{PAx})$ $\quad \mathtt{V}^*(\mathcal{A}) \models \mathbf{PAx}$.

**Proof.** In what follows we will repeatedly use the phrase that "$e^*$ *is constructed from* $e_1, \ldots, e_n$" by which we mean the existence of a closed application term $q$ such that $qe_1 \ldots e_n \simeq e^*$ holds in the $\mathbf{PCA}^+$, $\mathcal{A}$.

(i) From the proof of (6.3.1), recall that the set of natural numbers $\omega$ is modelled in $\mathtt{V}^*(\mathcal{A})$ by $\overline{\omega}$, with $\overline{\omega}$ given by an injection of $\omega$ into $\mathtt{V}^*(\mathcal{A})$:

$$\overline{n} = \langle n, \{\langle \underline{k}, \overline{k}\rangle : k \in n\}\rangle \tag{7.1}$$

$$\overline{\omega} = \langle \omega, \{\langle \underline{n}, \overline{n}\rangle : n \in \omega\}\rangle. \tag{7.2}$$

Now assume

$$e \Vdash_{rt}^{\mathcal{A}} \forall i \in \overline{\omega}\, \exists j \in \overline{\omega}\, \theta(i,j).$$

Then by the definition of realizability, we have

$$\forall i \in \overline{\omega}^\circ \exists j \in \overline{\omega}^\circ\, \theta(i,j) \wedge \forall \langle f, \mathfrak{a}\rangle \in \overline{\omega}^* \exists \mathfrak{b} \in \mathtt{V}^*(\mathcal{A})(\langle (ef)_0, \mathfrak{b}\rangle \in \overline{\omega}^* \wedge (ef)_1 \Vdash_{rt}^{\mathcal{A}} \theta(\mathfrak{a}, \mathfrak{b}))$$

Since, for $\langle s, t\rangle \in \overline{\omega}^*$, $t$ is uniquely determined by $s$ we conclude from the above the existence of a function $g : \omega \longrightarrow \omega$ such that for each $n \in \omega$

$$\langle (e\underline{n})_0, \overline{g(n)}\rangle \in \overline{\omega}^* \;\wedge\; (e\underline{n})_1 \Vdash_{rt}^{\mathcal{A}} \theta(\overline{n}, \overline{g(n)})$$

To internalize $g$ we first let $g_0 : \omega \longrightarrow V$ such that $g_0(n) = g(n)$ and $g_1 : \{\underline{n} : n \in \omega\} \longrightarrow \mathtt{V}^*(\mathcal{A})$ defined by $g_1(\underline{n}) = \langle \underline{n}, \langle \overline{n}, \overline{g(n)}\rangle_{v^*}\rangle$. Next, define

$$\mathfrak{h} := \langle g, g_1\rangle$$

Note that $g_1 = \{\langle \underline{n}, \langle \overline{n}, \overline{g(n)}\rangle_{v^*}\rangle : n \in \omega\} \subseteq |\mathcal{A}| \times \mathtt{V}^*(\mathcal{A})$. As $1^{st}(2^{nd}(\langle \underline{n}, \langle \overline{n}, \overline{g(n)}\rangle_{v^*}\rangle)) = 1^{st}(\langle \overline{n}, \overline{g(n)}\rangle_{v^*}) = \langle \overline{n}^\circ, \overline{g(n)}^\circ\rangle = \langle n, g(n)\rangle \in g$, it follows that $\mathfrak{h} \in \mathtt{V}^*(\mathcal{A})$.

Now, we first want to verify that $\mathfrak{h}$ is, internally, in $\mathtt{V}^*(\mathcal{A})$ is a binary functional relation whose domain is $\overline{\omega}$. Suppose that

$$k \Vdash_{rt}^{\mathcal{A}} \langle \mathfrak{a}, \mathfrak{b}\rangle_{v^*} \in \mathfrak{h} \text{ and } l \Vdash_{rt}^{\mathcal{A}} \langle \mathfrak{a}, \mathfrak{c}\rangle_{v^*} \in \mathfrak{h}$$

Then this is equivalent to the existence of $\mathfrak{d}, \mathfrak{q} \in \mathtt{V}^*(\mathcal{A})$ such that $\langle (k)_0, \mathfrak{d}\rangle \in \mathfrak{h}^* \;\wedge\; (k)_1 \Vdash_{rt}^{\mathcal{A}} \langle \mathfrak{a}, \mathfrak{b}\rangle_{v^*} = \mathfrak{d}$ and $\langle (l)_0, \mathfrak{q}\rangle \in \mathfrak{h}^* \;\wedge\; (l)_1 \Vdash_{rt}^{\mathcal{A}} \langle \mathfrak{a}, \mathfrak{c}\rangle_{v^*} = \mathfrak{q}$.

$\langle (k)_0, \mathfrak{d}\rangle \in \mathfrak{h}^*$ iff $\langle (k)_0, \mathfrak{d}\rangle \in \mathfrak{h}^*$ has the form $\langle \underline{n}, \langle \overline{n}, \overline{g(n)}\rangle_{v^*}\rangle$ and hence we have:

$$(k)_0 = \underline{n} \;\wedge\; (k)_1 \Vdash_{rt}^{\mathcal{A}} \langle \mathfrak{a}, \mathfrak{b}\rangle_{v^*} = \langle \overline{n}, \overline{g(n)}\rangle_{v^*}. \tag{7.3}$$

and $\langle (l)_0, \mathfrak{q}\rangle$ has the form $\langle \underline{m}, \langle \overline{m}, \overline{g(m)}\rangle_{v^*}\rangle$, so that

$$(l)_0 = \underline{m} \ \wedge \ (l)_1 \Vdash^{\mathcal{A}}_{rt} \langle \mathfrak{a}, \mathfrak{c}\rangle_{v^*} = \langle \overline{m}, \overline{g(m)}\rangle_{v^*}. \tag{7.4}$$

By Lemma (7.2.2), it follows that there exists a realizer $d \in |\mathcal{A}|$ such that $d \Vdash^{\mathcal{A}}_{rt} n = m$ which yields by (6.2.6) that $n = m$. Thus, a realizer $e^*$ can be constructed from $k$ and $l$ such that $e^* \Vdash^{\mathcal{A}}_{rt} \mathfrak{b} = \mathfrak{c}$.

Finally, we need to find a realizer for $\forall x \in \overline{\omega}\, \theta(x, \mathfrak{h}(x))$.

Since, $\forall n \in \omega\, (e\underline{n})_1 \Vdash^{\mathcal{A}}_{rt} \theta(\overline{n}, \overline{g(n)})$, (6.2.6) implies that $\forall n \in \omega\, \theta^\circ(\overline{n}^\circ, \overline{g(n)}^\circ)$, *i.e.* $\forall n \in \omega\, \theta^\circ(n, \mathfrak{h}^\circ(n))$. Moreover, as $\forall n \in \omega\, (e\underline{n})_1 \Vdash^{\mathcal{A}}_{rt} \theta(\overline{n}, \overline{g(n)})$ and $\mathfrak{h}^* = \{\langle \underline{n}, \langle \overline{n}, \overline{g(n)}\rangle_{v^*}\rangle : n \in \omega\}$, we can construct a term $\mathbf{q} \in |\mathcal{A}|$ with $\mathbf{q} := \lambda u.(eu)_1$ such that

$$(\mathbf{q}(e))(\underline{n}) \Vdash^{\mathcal{A}}_{rt} \theta(\overline{n}, \mathfrak{h}(\overline{n}))$$

and thus as $\mathbf{p}\underline{n}\mathbf{i}_r \Vdash^{\mathcal{A}}_{rt} \langle \overline{n}, \overline{g(n)}\rangle_{v^*} \in \mathfrak{h}$,

with $j := \lambda u.\mathbf{p}(\mathbf{p}(\mathbf{p}(eu)_0\mathbf{i}_r, \mathbf{p}u\mathbf{i}_r), (eu)_1)$, we obtain:

$$j\underline{n} \Vdash^{\mathcal{A}}_{rt} \exists y (y \in \overline{\omega} \ \wedge \ \langle \overline{n}, y\rangle_{v^*} \in \mathfrak{h} \ \wedge \ \forall \overline{n}\theta(\overline{n}, y)).$$

(ii) suppose

$$e \Vdash^{\mathcal{A}}_{rt} \forall i \in \overline{\omega}\, \exists y\, \theta(i, y).$$

By the realizability definition, this is equivalent to:

$$\forall i \in \overline{\omega}^\circ\, \exists y\, \theta^\circ(i, y) \ \wedge \ \forall \langle f, \mathfrak{a}\rangle \in \overline{\omega}^* \exists \mathfrak{b} \in \mathtt{V}^*(\mathcal{A})\, ef \Vdash^{\mathcal{A}}_{rt} \theta(\mathfrak{a}, \mathfrak{b})$$

A pair $\langle f, \mathfrak{a}\rangle \in \overline{\omega}^*$ must be of the form $\langle \underline{n}, \overline{n}\rangle$ and hence, it follows that for all $n$ in $\omega$

$$\exists \mathfrak{b}[e\underline{n}\downarrow \ \wedge e\underline{n} \Vdash^{\mathcal{A}}_{rt} \theta(\overline{n}, \mathfrak{b})]$$

Since $\mathbf{AC}_\omega$ holds in the background model, there exists a function $H : \omega \longrightarrow \mathtt{V}^*(\mathcal{A})$ such that

$$\forall n \in \omega\, e\underline{n} \Vdash^{\mathcal{A}}_{rt} \theta(\overline{n}, H(n))$$

Next, we define the internalization of $H$ in $\mathtt{V}^*(\mathcal{A})$ as follows:
Let $H_0 : \omega \longrightarrow V$ and $H_1 : \{\underline{n} : n \in \omega\} \longrightarrow \mathtt{V}^*(\mathcal{A})$ with

$$H_0(n) := (H(n))^\circ$$

and

$$H_1(\underline{n}) := \langle \overline{n}, H(n) \rangle_{v^*}$$

Define $\mathfrak{h}$ (the internalization of $H$) by

$$\mathfrak{h} := \langle H_0, H_1 \rangle.$$

$\mathfrak{h} \in \mathtt{V}^*(\mathcal{A})$ as for all $x$ in $\mathfrak{h}^*$, $x$ is of the form $\langle \underline{n}, \langle \overline{n}, H(n) \rangle_{v^*} \rangle$ and thus $1^{st}(2^{nd}(x)) = \langle n, (H(n))^\circ \rangle$.
To show that $\mathfrak{h}$ provides us with the function required for the validity of $\mathbf{AC}_\omega$ in $\mathtt{V}^*(\mathcal{A})$, we proceed similarly as in part (i) of this theorem.

(iii) Let $\mathfrak{a}, \mathfrak{u} \in \mathtt{V}^*(\mathcal{A})$ and assume

$$e \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{a}\, \exists y \in \mathfrak{a}\, \varphi(x, y) \tag{7.5}$$

and

$$e^* \Vdash^{\mathcal{A}}_{rt} \mathfrak{u} \in \mathfrak{a}. \tag{7.6}$$

Then (7.5) is equivalent to
$\forall f \forall \mathfrak{c}\big(\langle f, \mathfrak{c} \rangle \in \mathfrak{a}^* \longrightarrow \exists \mathfrak{d}[\langle (ef)_0, \mathfrak{d} \rangle \in \mathfrak{a}^* \ \wedge\ (ef)_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{c}, \mathfrak{d})]\big)$.
Moreover, (7.6) implies

$$\mathfrak{u}^\circ \in \mathfrak{a}^\circ \ \wedge\ \exists \mathfrak{c}_\mathfrak{u}[\langle (e^*)_0, \mathfrak{c}_\mathfrak{u} \rangle \in \mathfrak{a}^* \ \wedge\ (e^*)_1 \Vdash^{\mathcal{A}}_{rt} \mathfrak{u} = \mathfrak{c}_\mathfrak{u}]. \tag{7.7}$$

Consequently, (7.5) entails that

$$\forall \langle f, \mathfrak{c} \rangle \in \mathfrak{a}^*\, \exists \mathfrak{d} \in \mathtt{V}^*(\mathcal{A})\ \langle g, \mathfrak{d} \rangle \in \mathfrak{a}^*\, \varphi^{\Vdash^{\mathcal{A}}_{rt}}(\langle f, \mathfrak{c} \rangle, \langle g, \mathfrak{d} \rangle), \tag{7.8}$$

with $\varphi^{\Vdash^{\mathcal{A}}_{rt}}(\langle f, \mathfrak{c} \rangle, \langle g, \mathfrak{d} \rangle) \Leftrightarrow ef \downarrow \wedge\, g = (ef)_0 \ \wedge\ (ef)_1 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{c}, \mathfrak{d})$.
Invoking **DC** in $\mathtt{V}$, there exists a function $F : \omega \longrightarrow \mathfrak{a}^*$ with functions $F_0$ :

$\omega \to \mathcal{A}$ and $F_1 : \omega \to \mathtt{V}^*(\mathcal{A})$ such that $F_0(0) = (e^*)_0$, $F_1(0) = \mathfrak{c}_{\mathfrak{u}}$, $\forall n \in \omega \, \langle F_0(n), F_1(n) \rangle \in \mathfrak{a}^*$, and

$$\forall n \in \omega \, \varphi^{\Vdash^{\mathcal{A}}_{rt}}(\langle F_0(n), F_1(n)\rangle, \langle F_0(n+1), F_1(n+1)\rangle). \tag{7.9}$$

Thus, (7.9) entails that for all $n \in \omega$

$$e(F_0(n)) \downarrow \wedge F_0(n+1) \;=\; (e(F_0(n)))_0 \tag{7.10}$$
$$\wedge\; (e(F_0(n)))_1 \Vdash^{\mathcal{A}}_{rt} \varphi(F_1(n), F_1(n+1)). \tag{7.11}$$

Next, define

$$\begin{aligned} f &:= \{\langle n, (F_1(n))^\circ \rangle : n \in \omega\}, \\ g &:= \{\langle \underline{n}, \langle \overline{n}, F_1(n)\rangle_{v^*} \rangle : n \in \omega\}, \\ \mathfrak{h} &:= \langle f, g\rangle. \end{aligned}$$

$\mathfrak{h} \in \mathtt{V}^*(\mathcal{A})$. To verify this, we use (6.1.2) and the properties of internal pairing since $(\langle n, F_1(n)\rangle_{v^*})^\circ = \langle \underline{n}, (F_1(n))^\circ \rangle \in f$.

We claim that $\mathfrak{h}$ is a function that validates **DC** in $\mathtt{V}^*(\mathcal{A})$.

To verify the claim we first need to show that (in the sense of $\mathtt{V}^*(\mathcal{A})$) that $\mathfrak{h}$ is a functional binary relation with domain $\overline{\omega}$. Towards this goal, suppose that

$$k \Vdash^{\mathcal{A}}_{rt} \langle \mathfrak{a}, \mathfrak{b}\rangle_{v^*} \in \mathfrak{h} \tag{7.12}$$

and

$$l \Vdash^{\mathcal{A}}_{rt} \langle \mathfrak{a}, \mathfrak{c}\rangle_{v^*} \in \mathfrak{h}. \tag{7.13}$$

Then (7.12) implies that there exists an element $\mathfrak{d} \in \mathtt{V}^*(\mathcal{A})$ such that $\langle (k)_0, \mathfrak{d}\rangle \in \mathfrak{h}^* \;\wedge\; (k)_1 \Vdash^{\mathcal{A}}_{rt} \langle \mathfrak{a}, \mathfrak{b}\rangle_{v^*} = \mathfrak{d}$.

$\langle (k)_0, \mathfrak{d}\rangle \in \mathfrak{h}^*$ entails that $\langle (k)_0, \mathfrak{d}\rangle$ is of the form $\langle \underline{n}, \langle \overline{n}, F_1(n)\rangle_{v^*}\rangle$ for some $n \in \omega$. As a result,

$$(k)_1 \Vdash^{\mathcal{A}}_{rt} \langle \mathfrak{a}, \mathfrak{b}\rangle_{v^*} = \langle \overline{n}, F_1((n)\rangle_{v^*} \quad \text{and similarly} \tag{7.14}$$
$$(l)_1 \Vdash^{\mathcal{A}}_{rt} \langle \mathfrak{a}, \mathfrak{c}\rangle_{v^*} = \langle \overline{m}, F_1(m)\rangle_{v^*} \tag{7.15}$$

This implies (by the internal pairing properties) that $\mathtt{V}^*(\mathcal{A}) \models \overline{n} = \overline{m}$ which by (6.2.6) yields $n = m$. So that $F_1(n) = F_1(m)$ by functionality of $F_1$. Therefore, a realizer $\hat{e}$ can be constructed such that $\hat{e} \Vdash^{\mathcal{A}}_{rt} \mathfrak{b} = \mathfrak{c}$.

Finally, we need to show that a realizer $r$ constructible from $e$ and $e^*$ can be found such that

$$r(e, e^*) \Vdash^{\mathcal{A}}_{rt} \mathfrak{h}(0) = \mathfrak{u} \;\wedge\; \forall x \in \overline{\omega}\, \varphi(\mathfrak{h}(x), \mathfrak{h}(x+1)). \tag{7.16}$$

Let $s \Vdash^{\mathcal{A}}_{rt} \langle 0, \mathfrak{c}_\mathfrak{u}\rangle_{v^*} \in \mathfrak{h}$. Then this implies that there exists $\mathfrak{c} \in \mathtt{V}^*(\mathcal{A})$ such that $\langle (s)_0, \mathfrak{c}\rangle \in \mathfrak{h}^* \;\wedge\; (s)_1 \Vdash^{\mathcal{A}}_{rt} \langle 0, \mathfrak{c}_\mathfrak{u}\rangle_{v^*} = \mathfrak{c}$ . However, $\langle (s)_0, \mathfrak{c}\rangle \in \mathfrak{h}^*$ implies that $\langle (s)_0, \mathfrak{c}\rangle$ has the form $\langle \underline{n}, \langle \overline{n}, F_1(n)\rangle_{v^*}\rangle$ so that $(s)_0 = \underline{n}, \mathfrak{c} = \langle \overline{n}, F_1(n)\rangle_{v^*}$, and $(s)_1 \Vdash^{\mathcal{A}}_{rt} \langle 0, \mathfrak{c}_\mathfrak{u}\rangle_{v^*} = \langle \overline{n}, F_1(n)\rangle_{v^*}$ yeilding $\mathtt{V}^*(\mathcal{A}) \Vdash^{\mathcal{A}}_{rt} 0 = \overline{n}$ by the internal pairing properties. Applying (6.2.6) to the latter entails that $0 = n$. Thus, $\mathtt{V}^*(\mathcal{A}) \models \mathfrak{c}_\mathfrak{u} = F_1(0)$. In consequence of the foregoing and since $\mathfrak{h}(0) = \mathfrak{c}_\mathfrak{u} \wedge (e^*)_1 \Vdash^{\mathcal{A}}_{rt} \mathfrak{u} = \mathfrak{c}_\mathfrak{u}$, a realizer $(r)_0$ can be constructed from $e^*$ with $(r)_0 \Vdash^{\mathcal{A}}_{rt} \mathfrak{h}(0) = \mathfrak{u}$.

As for the realizability of $\forall x \in \overline{\omega}\, \varphi(\mathfrak{h}(x), \mathfrak{h}(x+1))$. Since (7.9) entails that there is an application term $\mathbf{q} \in |\mathcal{A}|$ constructible from $e$ using the recursion theorem for applicative structures such that

$$\mathbf{q}(\underline{0}) = (e^*)_0 \quad \text{and } \mathbf{q}(\underline{n+1}) = (e(\mathbf{q}(\underline{n})))_0$$

Set $\rho(\underline{n}) := (e(\mathbf{q}(\underline{n})))_1$, then

$$\begin{aligned} \mathbf{p}\underline{n}\mathbf{i}_r &\Vdash^{\mathcal{A}}_{rt} \langle \overline{n}, F_1(n)\rangle_{v^*} \in \mathfrak{h} \\ \rho(\underline{n}) &\Vdash^{\mathcal{A}}_{rt} \varphi(F_1(n), F_1(n+1)) \end{aligned}$$

This proves that we can construct a realizer $r(e, e^*)$ from $e$ and $e^*$ such that (7.16) holds.

(iv) Since we know from ([27], Lemma 3.4) that **RDC** entails **DC** and working in the theory $\mathbf{CZF}+\mathbf{DC}$, the $\mathbf{RDC}$ follows from the following scheme:

$$\forall x\left(\varphi(x) \rightarrow \exists y\left[\varphi(y) \wedge \psi(x,y)\right]\right) \wedge \varphi(\mathfrak{b}) \quad\rightarrow \qquad\qquad \exists s\left(\mathfrak{b}\in s \wedge \forall x\in s\,\exists y\in s\left[\varphi(y)\wedge\psi(x,y)\right]\right). \tag{7.17}$$

Therefore, with the aid of part (ii) of this theorem it is enough to prove that, in the basis of $\mathbf{CZF}+\mathbf{RDC}$, (7.17) holds in $\mathtt{V}^*(\mathcal{A})$.

Thus, let $\mathfrak{b}\in\mathtt{V}^*(\mathcal{A})$ and assume that the following holds for some $e, r\in|\mathcal{A}|$

$$e \Vdash^{\mathcal{A}}_{rt} \forall x\left(\varphi(x)\rightarrow \exists y\left[\varphi(y)\wedge\psi(x,y)\right]\right) \tag{7.18}$$

and

$$r \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{b}). \tag{7.19}$$

Then, (7.18) entails that

$\forall \mathfrak{a},\mathfrak{c}\in\mathtt{V}^*(\mathcal{A})(f \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{a})) \rightarrow \left[(ef)_0 \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{c}) \wedge (ef)_1 \Vdash^{\mathcal{A}}_{rt} \psi(\mathfrak{a},\mathfrak{c})\right]$. By the validity of $\mathbf{RDC}$ in the back ground theory applied to the above, one can find functions $u:\mathbf{N}\rightarrow\mathcal{A}$, $v:\mathbf{N}\rightarrow\mathcal{A}$, and $l:\omega\rightarrow\mathtt{V}^*(\mathcal{A})$ defined as follows:

$u(\underline{0})=r$, $l(0)=\mathfrak{b}$, and for every $n\in\omega$, we have:

$$u(\underline{n}) \Vdash^{\mathcal{A}}_{rt} \varphi(l(n)) \quad\text{and}\quad v(\underline{n}) \Vdash^{\mathcal{A}}_{rt} \psi(l(n), l(n+1)) \tag{7.20}$$

$$u(\underline{n+1}) = (e(u(\underline{n})))_0 \quad\text{and}\quad v(\underline{n}) = (e(u(\underline{n})))_1. \tag{7.21}$$

From (7.21), and using the recursion theorem for $\mathbf{PCA}^+$, we can conclude that there are application terms $\mathbf{q}_u$ and $\mathbf{q}_v$ can be constructed from $e$ and $r$ such that for all $n\in\omega$:

$$u(\underline{n}) = \mathbf{q}_u\underline{n} \text{ and } v(\underline{n}) = \mathbf{q}_v\underline{n}.$$

Define

$$\mathfrak{d} \;=\; \langle\{(l(n))^\circ : n\in\omega\}, \{\langle\underline{n}, l(n)\rangle : n\in\omega\}\rangle\,.$$

$\mathfrak{d}\in\mathtt{V}^*(\mathcal{A})$ is obvious. Next, let

$$t \Vdash^{\mathcal{A}}_{rt} \mathfrak{b}\in\mathfrak{d}$$

Then is equivalent to the existence of $\mathfrak{c} \in \mathtt{V}^*(\mathcal{A})$ such that

$$\langle (t)_0, \mathfrak{c} \rangle \in \mathfrak{d}^* \;\wedge\; (t)_1 \Vdash_{rt}^{\mathcal{A}} \mathfrak{b} = \mathfrak{c}$$

We have $\langle (t)_0, \mathfrak{c} \rangle \in \mathfrak{d}^*$ entails that $(t)_0$ is of the form $\underline{n}$ and $\mathfrak{c}$ is of the form $l(n)$ for some $n \in \omega$. Thus,

$$\mathbf{p0i}_r \Vdash_{rt}^{\mathcal{A}} \mathfrak{b} \in \mathfrak{d}. \tag{7.22}$$

(7.20) implies that for all $n$ in $\omega$

$$\mathbf{p}(u(\underline{n}))(v(\underline{n})) \Vdash_{rt}^{\mathcal{A}} \varphi(l(n)) \;\wedge\; \psi(\ell(n), \ell(n+1))$$

Consequently, for all $n$ in $\omega$

$$\mathbf{p}(\underline{n+1})\,(\mathbf{p}(u(\underline{n}))(v(\underline{n}))) \Vdash_{rt}^{\mathcal{A}} \exists y \in \mathfrak{d}\;[\varphi(l(n)) \;\wedge\; \psi(l(n), y)]\,.$$

Hence, choosing an application term $\mathbf{q}$ which can be constructed from $e$ and $r$ with the aid of recursion theorem for applicative structures satisfying $\mathbf{q}\underline{n} = \mathbf{p}(\underline{n+1})\,(\mathbf{p}(u(\underline{n}))(v(\underline{n})))$ we conclude that

$$\mathbf{q} \Vdash_{rt}^{\mathcal{A}} \forall x \in \mathfrak{d}\, \exists y \in \mathfrak{d}\;[\varphi(x) \;\wedge\; \psi(x, y)]\,. \tag{7.23}$$

Therefore, (7.22) and (7.23) yields that $\mathbf{p}(\mathbf{p0i}_r, \mathbf{q}\underline{n})$ realizes (7.17).

(v) Let $\mathfrak{a} \in \mathtt{V}^*(\mathcal{A})$. Since **PAx** holds in $\mathtt{V}$, there exists a base $Z$ and a surjection $f : Z \longrightarrow \mathfrak{a}$. Let $g : X \longrightarrow \mathfrak{a}^\circ$ and $h : Y \longrightarrow \mathfrak{a}^*$ and define

$$X' \;:=\; \{\langle \mathbf{0}, u \rangle : u \in X\}, \tag{7.24}$$

$$Y' \;:=\; \{\langle s_{\mathbf{N}}(h_0(v)), v \rangle :\, v \in Y\}, \tag{7.25}$$

where $h_0 : Y \to \mathcal{A}$ is defined by $h_0(v) := 1^{st}(h(v))$.

Note that $X'$ is in one-one correspondence with $X$ and $Y'$ is in one-one correspondence with $Y$, which entails that $X'$ and $Y'$ are also bases. Put,

$$B \;:=\; X' \cup Y' \tag{7.26}$$

The latter implies that $B$ is a base too, since $X'$ and $Y'$ have no element in common and for any $z \in B$ we know whether $z \in X'$ or $z \in Y'$ by testing $1^{st}(z)$ against $\mathbf{0}$ using definition by integer cases for a $\mathbf{PCA}^{+}$ if $\mathbf{N}(1^{st}(z))$ and decide whether $1^{st}(z) = \mathbf{0}$ or not. If $\neg\mathbf{N}(1^{st}(z))$ then ofcourse $z \in Y'$. Thus, we can define a function $G : B \to \mathfrak{a}^{\circ}$ such that

$$G(z) = \begin{cases} g(2^{nd}(z)) & \text{if} \quad z \in X' \\ \left(2^{nd}(h(2^{nd}(z)))\right)^{\circ} & \text{if} \quad z \in Y'. \end{cases} \tag{7.27}$$

Note that indeed $G$ takes its values in $\mathfrak{a}^{\circ}$ because $g(2^{nd}(z)) \in \mathfrak{a}^{\circ}$ and for all $v \in Y$, $\left(2^{nd}(h(2^{nd}(\langle s_{\mathbf{N}}(h_0(v)), v\rangle)))\right)^{\circ} = \left(2^{nd}(h(v))\right)^{\circ} \in \mathfrak{a}^{\circ}$. Moreover, as $g$ is surjective, $G$ is surjective as well.

Next, set

$$\begin{aligned} \wp(v) &:= \langle \left(S_{\mathbf{N}}(h_0(v))\right)^{st}, v^{st}\rangle_{v^*} \qquad \text{for } v \in Y, && (7.28) \\ B' &:= \{\langle\, h_0(v), \wp(v)\rangle \,:\, v \in Y\}, && (7.29) \\ \mathfrak{b} &:= \langle B, B'\rangle. && (7.30) \end{aligned}$$

$\mathfrak{b} \in \mathtt{V}^*(\mathcal{A})$ since by Lemma 7.2.2 and 6.2.3, we have:

$$\begin{aligned} (\wp(v))^{\circ} &= \left(\langle (s_{\mathbf{N}}(h_0(v)))^{st}, v^{st}\rangle_{v^*}\right)^{\circ} \\ &= \langle \left((S_{\mathbf{N}}(h_0(v)))^{st}\right)^{\circ}, \left(v^{st}\right)^{\circ}\rangle \\ &= \langle S_{\mathbf{N}}(h_0(v)), v\rangle \in Y' \\ &\in B \end{aligned}$$

So that, by (6.2.6) (iii), it follows that $\mathfrak{b} \in \mathtt{V}^*(\mathcal{A})$. Moreover, since $(\wp(v))^{\circ} = \langle s_{\mathbf{N}}(h_0(v)), v\rangle \in Y'$ and $Y'$ is in one-one correspondence with $Y$, which entails that $\wp$ is in one-one correspondence with $Y$ and hence the map $v \mapsto \langle h_0(v), \wp(v)\rangle$ is in a one-one correspondence between $Y$ and $B'$ proving that $B'$ is also a base.

We now want to show that $\mathfrak{b}$ is a base in the internal sense of $\mathtt{V}^*(\mathcal{A})$ and that $\mathfrak{a}$ is the surjective image of this $\mathfrak{b}$.

To show that we first need to define the surjection as follows:

For $v \in Y$, put

$$l(v) := \langle \wp(v), 2^{nd}(h(v)) \rangle_{v^*} \tag{7.31}$$

$$\mathfrak{h} := \{\langle h_0(v), l(v)\rangle : v \in Y\} \tag{7.32}$$

$$\mathfrak{f} := \langle G, \mathfrak{h} \rangle. \tag{7.33}$$

To verify that $\mathfrak{f} \in \mathtt{V}^*(\mathcal{A})$, let $x \in \mathfrak{f}^*$. Then $x \in \mathfrak{h}$ and hence $x$ is of the form $\langle h_0(v), l(v)\rangle$ for some $v \in Y$. Consequently,

$$\begin{aligned} 1^{st}(2^{nd}(x)) &= (l(v))^\circ = (\langle \wp(v), 2^{nd}(h(v))\rangle_{v^*})^\circ \\ &= \left\langle (\wp(v))^\circ, \left(2^{nd}(h(v))\right)^\circ \right\rangle \\ &= \left\langle \langle s_{\mathbf{N}}(h_0(v)), v\rangle, \left(2^{nd}(h(v))\right)^\circ \right\rangle \\ &\in G. \end{aligned}$$

First, we are aiming to prove that

$$\mathtt{V}^*(\mathcal{A}) \models \mathfrak{f} \text{ is a surjective function from } \mathfrak{b} \text{ to } \mathfrak{a}. \tag{7.34}$$

To show $\mathtt{V}^*(\mathcal{A}) \models \mathfrak{f} \subseteq \mathfrak{b} \times \mathfrak{a}$, assume that there is a realizer $e \in |\mathcal{A}|$ such that

$$e \Vdash_{rt}^{\mathcal{A}} \langle \mathfrak{c}, \mathfrak{d} \rangle_{v^*} \in \mathfrak{f}$$

Then by the definition of realizability, there exists $\mathfrak{e}$ such that $\langle (e)_0, \mathfrak{e}\rangle \in \mathfrak{f}^* \ \wedge$ $(e)_1 \Vdash_{rt}^{\mathcal{A}} \langle \mathfrak{c}, \mathfrak{d} \rangle_{v^*} = \mathfrak{e}$.

However, $\langle (e)_0, \mathfrak{e}\rangle \in \mathfrak{f}^*$ entails that $\langle (e)_0, \mathfrak{e}\rangle$ has the form $\langle h_0(v), l(v)\rangle$ and hence $(e)_0 = h_0(v)$ and $(e)_1 \Vdash_{rt}^{\mathcal{A}} \langle \mathfrak{c}, \mathfrak{d} \rangle_{v^*} = l(v) = \langle \wp(v), 2^{nd}(h(v))\rangle_{v^*}$ for some $v$ in $Y$. Now, from (7.31), (7.32) and (7.33), it follows that $\wp(v) \in \mathfrak{b}$ and we have a realizer $r \in |\mathcal{A}|$ with $r \Vdash_{rt}^{\mathcal{A}} 2^{nd}(h(v)) \in \mathfrak{a}$ iff $(2^{nd}(h(v)))^\circ \in \mathfrak{a}^\circ \ \wedge \ \exists \mathfrak{e}[\langle (r)_0, \mathfrak{e}\rangle \in \mathfrak{a}^* \ \wedge \ (r)_1 \Vdash_{rt}^{\mathcal{A}} 2^{nd}(h(v)) = \mathfrak{e}]$, and hence $(r)_0 = h_0(v)$ as $h$ is surjective and $\mathfrak{e} = 2^{nd}(h(v))$. As a result,

$$\mathbf{p}(h_0(v))\mathbf{i}_r \Vdash_{rt}^{\mathcal{A}} 2^{nd}(h(v)) \in \mathfrak{a}$$

Therefore, a realizer $e^*$ can be effectively constructed from $e$ such that $e^* \Vdash_{rt}^{\mathcal{A}}$

$\mathfrak{c} \in \mathfrak{b} \wedge \mathfrak{d} \in \mathfrak{a}$, establishing

$$\mathtt{V}^*(\mathcal{A}) \models \mathfrak{f} \subseteq \mathfrak{b} \times \mathfrak{a}. \tag{7.35}$$

Next, we aim at showing that $\mathfrak{f}$ is realizably total on $\mathfrak{b}$. So suppose that $e \Vdash_{rt}^{\mathcal{A}} \mathfrak{c} \in \mathfrak{b}$. Then by the definition of realizability, this is equivalent to the existence of $\mathfrak{e} \in \mathtt{V}^*(\mathcal{A})$ such that $\langle (e)_0, \mathfrak{e} \rangle \in \mathfrak{b}^*$ and $(e)_1 \Vdash_{rt}^{\mathcal{A}} \mathfrak{c} = \mathfrak{e}$. Hence by the definition of $\mathfrak{b}^*$, we conclude that there is a $v \in Y$ with $(e)_0 = h_0(v)$ and $\mathfrak{e} = \wp(v)$, and thus, $(e)_0 \mathbf{i}_r \Vdash_{rt}^{\mathcal{A}} \langle \mathfrak{e}, 2^{nd}(h(v)) \rangle_{v^*} \in \mathfrak{f}$ using the definition of $\mathfrak{f}$. In consequence, a realizer $\hat{e}$ can be constructed from $e$ with

$$\hat{e} \Vdash_{rt}^{\mathcal{A}} \mathfrak{c} \text{ is in the domain of } \mathfrak{f}$$

. Thus, there is a realizer $e^*$ such that

$$e^* \Vdash_{rt}^{\mathcal{A}} \mathfrak{b} \subseteq \mathbf{dom}(\mathfrak{f})$$

which with (7.35) entails that

$$\mathtt{V}^*(\mathcal{A}) \models \mathbf{dom}(\mathfrak{f}) = \mathfrak{b} \tag{7.36}$$

Next, we verify that $\mathfrak{f}$ is realizably functional. Suppose that $k \Vdash_{rt}^{\mathcal{A}} \langle \mathfrak{c}, \mathfrak{d} \rangle_{v^*} \in \mathfrak{f}$ and $l \Vdash_{rt}^{\mathcal{A}} \langle \mathfrak{c}, \mathfrak{e} \rangle_{v^*} \in \mathfrak{f}$. By definition of realizability the first implies that there exists $u \in Y$ such that $(k)_0 = h_0(u)$ and $(k)_1 \Vdash_{rt}^{\mathcal{A}} \langle \mathfrak{c}, \mathfrak{d} \rangle_{v^*} = \langle \wp(u), 2^{nd}(h(u)) \rangle_{v^*}$ and likewise we have for some $v \in Y$, $(l)_0 = h_0(v)$ and $(l)_1 \Vdash_{rt}^{\mathcal{A}} \langle \mathfrak{c}, \mathfrak{e} \rangle_{v^*} = \langle \wp(v), 2^{nd}(h(v)) \rangle_{v^*}$ consequently and by the use of internal pairing properties we arrive at $\mathtt{V}^*(\mathcal{A}) \models \wp(u) = \wp(v), i.e. \mathtt{V}^*(\mathcal{A}) \models \langle (S_{\mathbf{N}}(h_0(u)))^{st}, u^{st} \rangle_{v^*} = \langle (S_{\mathbf{N}}(h_0(v)))^{st}, v^{st} \rangle_{v^*}$, which by Lemma (7.2.2) yields $\mathtt{V}^*(\mathcal{A}) \models u^{st} = v^{st}$. Using (6.2.6), the latter implies $(u^{st})^\circ = (v^{st})^\circ i.e.\, u = v$. Therefore by functionality of $h$ there is a realizer $\mathbf{q}$ constructable from $k$ and $l$ such that $\mathbf{q} \Vdash_{rt}^{\mathcal{A}} \mathfrak{d} = \mathfrak{e}$ arriving at

$$\mathtt{V}^*(\mathcal{A}) \models \mathfrak{f} \text{ is a function }. \tag{7.37}$$

To establish (7.34), it remains to verify that $\mathfrak{f}$ realizably surjective.

Let $d \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} \in \mathfrak{a}$. By the realizability definition there exists $\mathfrak{e} \in \mathtt{V}^*(\mathcal{A})$ such that $\langle (d)_0, \mathfrak{e} \rangle \in \mathfrak{a}^*$ and $(d)_1 \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} = \mathfrak{e}$. Since $h$ is surjective, there is a $v \in Y$ with $h_0(v) = (d)_0$ and $\mathfrak{e} = 2^{nd}(h(v))$. Now, we need to find realizers $r, s \in |\mathcal{A}|$ such that

$$r \Vdash^{\mathcal{A}}_{rt} \wp(v) \in \mathfrak{b} \tag{7.38}$$

$$s \Vdash^{\mathcal{A}}_{rt} \langle \wp(v), \mathfrak{e} \rangle_{v^*} \in \mathfrak{f} \tag{7.39}$$

Then we have the following equivalences:

(7.38) $\Leftrightarrow (\wp(v))^\circ \in \mathfrak{b}^\circ \;\wedge\; \exists \mathfrak{d} \in \mathtt{V}^*(\mathcal{A})[\langle (r)_0, \mathfrak{d} \rangle \in \mathfrak{b}^* \;\wedge\; (r)_1 \Vdash^{\mathcal{A}}_{rt} \wp(v) = \mathfrak{d}]$

$\langle (r)_0, \mathfrak{d} \rangle \in \mathfrak{b}^*$ implies that $(r)_0 = h_0(v)$ and $\mathfrak{d} = \wp(v)$, so we conclude that

$$\mathbf{p} h_0(v) \mathbf{i}_r \Vdash^{\mathcal{A}}_{rt} \wp(v) \in \mathfrak{b}$$

And (7.39) $\Leftrightarrow (\langle \wp(v), \mathfrak{e} \rangle_{v^*})^\circ \in \mathfrak{f}^\circ \;\wedge\; \exists \mathfrak{d}[\langle (s)_0, \mathfrak{d} \rangle \in \mathfrak{f}^* \;\wedge\; (s)_1 \Vdash^{\mathcal{A}}_{rt} \langle \wp(v), \mathfrak{e} \rangle_{v^*} = \mathfrak{d}]$ with $\langle (s)_0, \mathfrak{d} \rangle \in \mathfrak{f}^*$ we conclude that $(s)_0 = h_0(v)$ and $\mathfrak{d} = l(v)$ and hence $\mathbf{p} h_0(v) \mathbf{i}_r \Vdash^{\mathcal{A}}_{rt} \langle \wp(v), \mathfrak{e} \rangle_{v^*} \in \mathfrak{f}$. So that, a realizer $d'$ can be constructed such that $d' \Vdash^{\mathcal{A}}_{rt} \mathfrak{c}$ is in the range of $\mathfrak{f}$.

Therefore, $\mathtt{V}^*(\mathcal{A}) \models \mathfrak{f}$ maps onto $\mathfrak{a}$ which together with (7.35), (7.36), and (7.37) implies (7.34).

Finally, we show that internally in $\mathtt{V}^*(\mathcal{A})$ $\mathfrak{b}$ is a base *i.e.*

$$\mathtt{V}^*(\mathcal{A}) \models \mathfrak{b} \text{ is a base }. \tag{7.40}$$

To establish this we first assume that there is a realizer $e \in |\mathcal{A}|$ such that for a formula $\varphi(x, y)$ the following holds:

$$e \Vdash^{\mathcal{A}}_{rt} \forall x \in \mathfrak{b}\, \exists y\, \varphi(x, y) \tag{7.41}$$

To verify (7.40) we need to show that a realizer $t$ can be obtained from $e$ such that

$$t \;\Vdash^{\mathcal{A}}_{rt}\; \exists H\; [\mathbf{fun}(H) \wedge \mathfrak{b}{=}\mathbf{dom}(H) \wedge \forall x \in \mathfrak{b}\, \varphi(x, H(x))]\,. \tag{7.42}$$

Now, (7.41) entails

$$\forall x \in \mathfrak{b}^\circ \, \exists y \, \varphi^\circ(x,y) \quad \wedge \quad \forall \langle a, \mathfrak{c} \rangle \in \mathfrak{b}^* \, ea \Vdash^{\mathcal{A}}_{rt} \exists y \, \varphi(\mathfrak{c}, y)$$
$$\Leftrightarrow \; \forall x \in \mathfrak{b}^\circ \, \exists y \, \varphi^\circ(x,y) \; \wedge \; \forall \langle a, \mathfrak{c} \rangle \in \mathfrak{b}^* \exists \mathfrak{d} \, ea \Vdash^{\mathcal{A}}_{rt} \varphi(\mathfrak{c}, \mathfrak{d})$$

As $\mathfrak{b}^\circ = B = X' \cup Y'$ by the truth part of the above we have

$$\forall x \in X' \, \exists y \, \varphi^\circ(x,y). \tag{7.43}$$

The second part yields the existence of $v \in Y$ and $\mathfrak{d} \in \mathtt{V}^*(\mathcal{A})$ with

$$e(h_0(v)) \Vdash^{\mathcal{A}}_{rt} \varphi(\wp(v), \mathfrak{d}). \tag{7.44}$$

As $X'$ and $Y$ are bases, we can find functions $I$ and $J$ such that $\mathbf{dom}(I) = X'$ and $J : Y \to \mathtt{V}^*(\mathcal{A})$ which satisfy the following

$$\forall x \in X' \, \varphi^\circ(x, I(x)), \tag{7.45}$$
$$\forall v \in Y \; e(h_0(v)) \Vdash^{\mathcal{A}}_{rt} \varphi(\wp(v), J(v)). \tag{7.46}$$

By (6.2.6), (7.46) entailes that $\forall v \in Y \, \varphi^\circ \left( (\langle (S_{\mathbf{N}}(h_0(v))^{st}, (v)^{st} \rangle_{v^*})^\circ, (J(v))^\circ \right) = \varphi^\circ \left( \langle s_{\mathbf{N}}(h_0(v)), v \rangle, (J(v))^\circ \right)$. Set $x = \langle s_{\mathbf{N}}(h_0(v)), v \rangle$ then $x \in Y'$ and thus $\varphi^\circ(x, (J(2^{nd}(x)))^\circ)$. For the same reasons explained in (7.27) and since $X' \cap Y' = \emptyset$, we can define a function $L$ whose domain is $B = X' \cup Y'$ such that

$$L(x) = \begin{cases} I(x) & \text{if} \quad x \in X' \\ \left(J(2^{nd}(x))\right)^\circ & \text{if} \quad x \in Y'. \end{cases} \tag{7.47}$$

Hence,

$$\forall x \in \mathfrak{b}^\circ \, \varphi^\circ(x, L(x)). \tag{7.48}$$

Next, an internalization of $L$ in $\mathtt{V}^*(\mathcal{A})$ is defined as follows: put

$$\mathcal{L} \;:=\; \{ \langle \, h_0(v), \langle \wp(v), J(v) \rangle_{v^*} \, \rangle \, : \, v \in Y \}, \tag{7.49}$$
$$\mathfrak{l} \;:=\; \langle L, \mathcal{L} \rangle. \tag{7.50}$$

$\mathfrak{l} \in \mathtt{V}^*(\mathcal{A})$ since for $x \in \mathfrak{l}^* = \mathcal{L}$, it follows that $x = \langle \, h_0(v), \langle \wp(v), J(v) \rangle_{v^*} \, \rangle$ for some $v \in Y$, and thus $1^{st}(2^{nd}(x)) = (\langle \wp(v), J(v)\rangle_{v^*})^\circ = \langle (\wp(v))^\circ, (J(v))^\circ \rangle = \langle \langle s_{\mathbf{N}}(h_0(v)), v\rangle, (J(v))^\circ \rangle$. Let $y := \langle s_{\mathbf{N}}(h_0(v)), v\rangle$ then, $y \in Y'$ and $(J(v))^\circ = \big(J(2^{nd}(y))\big)^\circ \in L$. As a result, $\mathfrak{l} \in \mathtt{V}^*(\mathcal{A})$.

Next, we aim at verifying that for some $e^* \in |\mathcal{A}|$ that can be extracted from $e$, the following holds:

$$e^* \quad \Vdash^{\mathcal{A}}_{rt} \quad \mathbf{fun}(\mathfrak{l}) \wedge \mathfrak{b} \subseteq \mathbf{dom}(\mathfrak{l}) \wedge \forall x \in \mathfrak{b}\, \varphi(x, \mathfrak{l}(x)) \tag{7.51}$$

We first show that $\mathfrak{l}$ is realizably functional, so assume that

$$s \Vdash^{\mathcal{A}}_{rt} \langle \mathfrak{c}, \mathfrak{d}\rangle_{v^*} \in \mathfrak{l} \quad \text{and } t \Vdash^{\mathcal{A}}_{rt} \langle \mathfrak{c}, \mathfrak{e}\rangle_{v^*} \in \mathfrak{l}. \tag{7.52}$$

Then there are $u, v \in Y$ with $(s)_0 = h_0(u)$, $(t)_0 = h_0(v)$, $(s)_1 \Vdash^{\mathcal{A}}_{rt} \langle \mathfrak{c}, \mathfrak{d}\rangle_{v^*} = \langle \wp(u), J(u)\rangle_{v^*}$, and $(t)_1 \Vdash^{\mathcal{A}}_{rt} \langle \mathfrak{c}, \mathfrak{e}\rangle_{v^*} = \langle \wp(v), J(v)\rangle_{v^*}$. Consequently, by the internal pairing properties $\mathtt{V}^*(\mathcal{A}) \models \wp(u) = \wp(v) \Leftrightarrow \mathtt{V}^*(\mathcal{A}) \models \langle (s_{\mathbf{N}}(h_0(u)))^{st}, u^{st}\rangle_{v^*} = \langle (s_{\mathbf{N}}(h_0(v)))^{st}, v^{st}\rangle_{v^*}$ and hence there exists a realizer $d$ such that $d \Vdash^{\mathcal{A}}_{rt} u^{st} = v^{st}$ which by (6.2.6) implies $(u^{st})^\circ = (v^{st})^\circ$ which by (6.2.3) entails that $u = v$. Thus, from $s$ and $t$ we can construct a realizer $d'$ such that $d' \Vdash^{\mathcal{A}}_{rt} \mathfrak{d} = \mathfrak{e}$.

Next, we verify that $\mathfrak{b}$ is in the domain of $\mathfrak{l}$. So suppose that

$$r \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} \in \mathfrak{b}$$

Then it follows from the realizability definition that there exists $\mathfrak{d} \in \mathtt{V}^*(\mathcal{A})$ with $\langle (r)_0, \mathfrak{d}\rangle \in \mathfrak{b}^* \wedge (r)_1 \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} = \mathfrak{d}$.

As $\langle (r)_0, \mathfrak{d}\rangle \in \mathfrak{b}^*$ has the form $\langle h_0(v), \wp(v)\rangle$ for some $v \in Y$, which implies that $(r)_0 = h_0(v)$ and $(r)_1 \Vdash^{\mathcal{A}}_{rt} \mathfrak{c} = \wp(v)$ and since $\langle h_0(v), \langle \wp(v), J(v)\rangle_{v^*}\rangle \in \mathfrak{l}^*$, it is clear that we can construct a realizer $r^*$ such that

$$r^*(h_0(v)) \Vdash^{\mathcal{A}}_{rt} \langle h_0(v), \wp(v)\rangle \in \mathbf{dom}(\mathfrak{l})$$

Therefore,

$$\mathtt{V}^*(\mathcal{A}) \models \mathfrak{b} = \mathbf{dom}(\mathfrak{l}). \tag{7.53}$$

Finally we show that

$$r' \Vdash_{rt}^{\mathcal{A}} \forall x \in \mathfrak{b}\, \varphi(x, \mathfrak{l}(x)) \tag{7.54}$$

for some $r' \in |\mathcal{A}|$ that can be obtained from $e$.

Note that (7.54) is equivalent to

$\forall x \in \mathfrak{b}^{\circ}\, \varphi^{\circ}(x, (\mathfrak{l}^{\circ}(x))) \;\wedge\; \forall f \forall \mathfrak{c}[\langle f, \mathfrak{c}\rangle \in \mathfrak{b}^{*}\; r'f \Vdash_{rt}^{\mathcal{A}} \varphi(\mathfrak{c}, \mathfrak{l}(\mathfrak{c}))]$.

By (7.45) the truth part is established. For the second part, every element $\langle f, \mathfrak{c}\rangle \in \mathfrak{b}^{*}$ has the form $\langle h_0(v), \wp(v)\rangle$ for some $v \in Y$ and hence $\forall \langle h_o(v), \wp(v)\rangle \in \mathfrak{b}^{*}\; r'h_0(v) \Vdash_{rt}^{\mathcal{A}} \varphi(\wp(v), \mathfrak{l}(\wp(v)))$. Now, since $\langle\, h_0(v), \langle \wp(v), J(v)\rangle_{v^*}\, \rangle \in \mathfrak{l}^{*}$ and $e(h_0(v)) \Vdash_{rt}^{\mathcal{A}} \varphi(\wp(v), J(v))$ is satisfied by (7.46), we see that a realizer $\tilde{r}$ can be constructed such that

$$(\tilde{r}e)(h_0(v)) \Vdash_{rt}^{\mathcal{A}} \varphi\left(\wp(v), \mathfrak{l}(\wp(v))\right)$$

As a result of the foregoing, we obtain the realizer $r'$ such that

$$r' \Vdash_{rt}^{\mathcal{A}} \forall x \in \mathfrak{b}\, \varphi(x, \mathfrak{l}(x))$$

which completes the proof of (7.51).

□

# Conclusion and Outlook

Chapter 3 closes a gap in the literature in that $\mathbf{AC}^{\omega,\omega}$ holds in all realizability universes $\mathtt{V}(\mathcal{A})$ for any applicative structure $\mathcal{A}$ and moreover countable choice, relativized dependent choice, and the presentation axiom hold in $\mathtt{V}(\mathcal{A})$ if they happen to hold in $\mathtt{V}$. Furthermore, it is shown that these preservation results can be established in the metatheory **CZF**.

Chapter 4 is exclusively concerned with the realizability universe $\mathtt{V}(D_\infty)$. It is shown that in this world there is an infinite set $A$ which is in 1-1 correspondence with its function space $A \to A$. Rathjen has used this structure to develop a model of set theory in which the equation $X = X \to X$ has a nontrivial solution (i.e. $X$ contains the naturals). The model is a model of intuitionistic **IZF** without set induction.

It is known that bar induction and the fan theorem hold in $\mathtt{V}(\mathcal{K}_2)$ if they hold in $\mathtt{V}$ [29],where $\mathcal{K}_2$ is the second Kleene algebra. In chapter 5 we showed that also to be the case for two other types of applicative structures, namely $\mathtt{V}(\mathcal{A})$ is a model of these Brouwerian principles when $\mathcal{A}$ is instantiated by the graph model and by Scott's $D_\infty$ models.

Chapters 6 and 7 address realizability with truth over $\mathtt{V}^*(\mathcal{A})$. Using the results about realizability with truth from chapters 6 and 7, Rathjen has established these derived rules for many intuitionistic set theories by employing Kleene's second algebra. T. Nemoto and Rathjen have also shown closure of many intuituionistic set theories under the independence of premise rules for the finite types over $\mathbb{N}$, where they use realizability with truth over graph models.

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