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A condition-based maintenance policy for degrading systems with age- and state-dependent operating cost

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Abstract

Most of the maintenance policies in existing publications assume that no cost is incurred as long as the system can undertake missions while little consideration has been devoted to the operating cost during system operation. However, in practice, the operating cost increases while the system ages and degrades even if a system is in a functioning state. This paper proposes a maintenance policy for a degrading system with age- and state-dependent operating cost, which increases with system age and degradation levels. Under such a setting, a replacement model is first developed to investigate the optimal preventive replacement policy. The replacement model is then extended to a repair-replacement model, in which imperfect repair is assumed to restore the system to the operating condition. Particularly, the repair model with controllable and uncontrollable repair levels is considered separately. The paper proves that the optimal maintenance policy is actually a monotone control limit policy, where the optimal control limits decrease monotonically with system age. Finally, a numerical example along with sensitivity analysis is presented to illustrate the optimal maintenance policy. The proposed model implies a more conservative maintenance policy, compared with the traditional model without the age- and state-dependent operating cost.

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1. Introduction

Conventionally, time-dependent maintenance policies are widely studied, for which failure data are needed to estimate the reliability. However, recently, with the increasing improvement of product reliability and shorter product life cycle, obtaining failure data within a specific experimental horizon is becoming more difficult, which impedes the application of traditional reliability analysis methods. On the other hand, degradation models have gained significant attention for highly reliable products. Actually, many physical failures in engineering can be traced to an underlying degradation process (Liu et al, 2016). Most systems experience a degradation process before failure and the degradation indicators can be measured over time.

With the fast development of sensing techniques, system states can now be monitored or inspected at a much lower cost, which facilitates to characterise the deteriorating process of the system by a continuous-time stochastic model (Lam & Banjevic, 2015). For a system subject to a continuous-time degradation process, condition-based maintenance (CBM) has shown its effectiveness and predominance in preventing unexpected failures. As CBM utilises the real-time information of system health, it is usually more effective in reducing operating cost than the traditional age-based or block-based maintenance policies (Wu & Ryan, 2010). The procedure of CBM usually consists of three steps: condition data acquisition, reliability or remaining lifetime estimation/prediction, and optimal CBM decision making (Jardine et al, 2006). The ultimate goal of CBM is to reduce the risk of catastrophic failure and meanwhile reduce operating costs by eliminating unnecessary maintenance actions.

With respect to CBM, a lot of research effort has been devoted to the prognostic models, in terms of characterising the physical deterioration (Flory et al, 2015). The degradation process can be classified according to the degradation states: discrete or continuous (Liu et al, 2014).

For a system with discrete degradation states, Markov chain models are usually adopted to describe the deterioration process. The Markov chain models work well when the degradation states cannot be accurately detected. However, it suffers the disadvantage of arbitrary classification of the degradation states (Peng et al, 2013).

For a system with continuous degradation states, there has been a growing interest in stochastic-process-based models. Notably among them are the Wiener process, Gamma process and inverse Gaussian process (Ye & Chen, 2014). The Wiener process has gained popularity owing to its excellent mathematical property and physical interpretation. In addition, the Wiener process is a non-monotone stochastic process and has the ability to characterise the degradation variations for some system behaviors (Guo et al, 2013). Applications of the Wiener process can be found in many engineering settings, such as LET lights, heating cables and gyroscopes (Whitmore & Schenkelberg, 1997; Ye et al 2012; Si et al, 2013). Often, degradation data collected from sensors may contain noise and therefore do not conform to a monotone fashion. In those circumstances, the Wiener process is appropriate for describing such non-monotone degradation processes.

In traditional CBM models, the optimal decisions are usually obtained by minimizing the long-run cost rate for an infinite or a finite horizon. The cost items considered include the inspection, repair and replacement cost, and cost incurred by unexpected failures. It is assumed that during the operation period, no additional cost is incurred as long as the system stays in the functioning state, regardless of its degradation level. However, in reality, as the system/component degrades, its

performance would decrease as well, which results in increasing operating cost (Jeang, 1998; Hsu & Shu, 2010).

During system operation, the operating cost increases along with the degradation level, as shown in Fig 1. Generally, the age- and state-dependent operating cost can be modeled as a stochastic process, which depends on the degradation process. Decision-makers may have to take into account the two-dimensional degradation process when making maintenance policies. The operating cost may serve as a constraint on maintenance decision (for environmental issues) or an additional cost item (for economic concerns). However, little attention has been devoted to this important phenomenon in existing literature, which is further explained with the following two real-world examples.
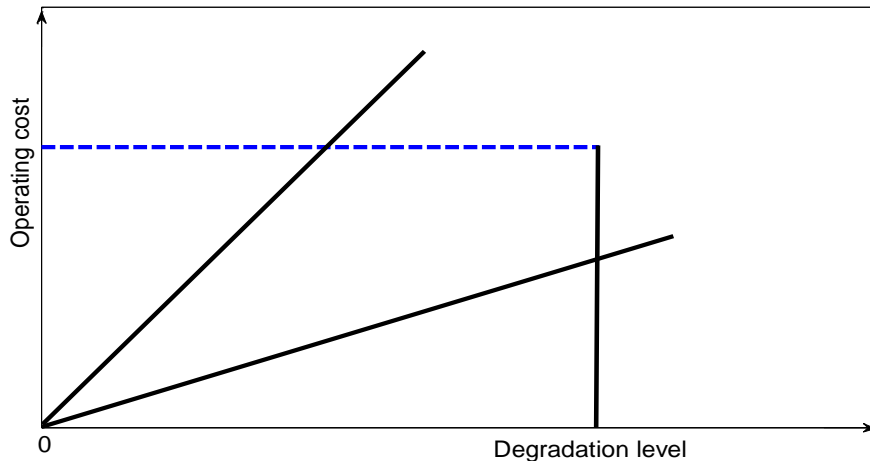
Fig. 1 Plot of degradation level and operating cost evolvement

Modern vehicles are equipped with more and more sophisticated exhaust after-treatment facilities to satisfy increasingly stringent emissions limits. After a moderate mileage, vehicles may produce pollutant rates which largely exceed certification levels, despite of the on-board electronic control equipment. The increased pollutant rate is highly dependent on how the vehicle is used and maintained. With the traditional maintenance policy, an aged automobile is maintained only when

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it is failed (corrective maintenance) or is about to fail (preventive maintenance), while neither the side effect of increased energy consumption nor exhaust gas emission is considered.

Another example is the production system. In a production system, if a tool deteriorates over time, the quality of the produced products would decrease accordingly, which increases the rate of spoiled products (Xu & Cao, 2015). Traditional maintenance strategies focus on the tool itself, with little effort to the product quality as a result of tool aging and degradation. For a company, retaining high quality and efficiency is essential in today's heavy competition. A more appropriate maintenance model is required to capture the operating cost due to aging and degradation.

From the above two examples, one can see that there is a need to take the side effects along with aging and degradation into consideration when making maintenance decisions. This paper aims to satisfy this need. It regards the side effects due to aging and degradation as the operating cost and then develops a condition-based maintenance policy for degrading systems with consideration of the age- and state-dependent operating cost. The age- and state-dependent operating cost occurs when the system is operating in a deteriorated state, even if the system is still functioning. In studies with respect to integrated preventive maintenance and quality control models, some researchers have considered the age- and state-dependent operating cost (Tagaras, 1988; Yeung et al., 2008; Xiang, 2013). Yes there exist two substantial differences between our model and the previous models. The existing models considered the operating cost associated with the system itself, while our model can describe the side effect of degradation process on the other systems or environments. In addition, the previous models use discrete Markov chains to characterize the evolution of system state, while we employ a continuous stochastic process.

The system is subject to a continuous-time degradation process. The Wiener process is used to characterise the evolution of the system states. By taking advantage of the independent increment

of the Wiener process, we model the CBM policy as a Markov decision process. The system is subject to periodic inspections. If the system has not failed at inspection, decision has to be selected among three maintenance actions: preventive replacement, repair and wait until the next inspection. The system is replaced when its degradation level exceeds a pre-specified threshold at inspection. The structural property of the optimal maintenance decision is investigated in depth and a monotone control limit policy is shown as the optimal strategy.

The novelty of this paper lies in the facts

- that the proposed maintenance model is able to capture the side effect due to system age and degradation, and

- that decisions made based on this model not only consider the economic benefits, but also the environmental and societal benefits.

We therefore claim that this is the first paper that incorporates side effect of degradation process into maintenance policies, which can be applied in various real systems to gain economic and societal benefits.

The remainder of this paper is organised as follows. Section 2 presents the degradation process and replacement model, where the maintenance cost function is derived and the structural properties of the optimal maintenance decision are further investigated. Section 3 formulates a more detailed maintenance decision with repair and replacement. Section 4 gives a numerical example to illustrate the maintenance decision and the associated managerial insights. Finally, Section 5 concludes the paper and suggests future research.

2. Degradation process and replacement model formulation

In this section, we develop an optimal replacement policy for a system subjected to the Wiener degradation process. The replacement model serves as the foundation for more sophisticated maintenance policies, e.g., the repair-replacement policy introduced in the next section.

2.1 Wiener degradation process

It is assumed that the system degrades according to the Wiener degradation process. The general form of the Wiener process is presented as

$$dX(t) = \mu(t)dt + \sigma(t)dB(t) \tag{1}$$

Taking integral of Eq (1), we have

$$X(t) = M(t) + \Lambda(t) + x_0 \tag{2}$$

where $M(t) = \int_0^t \mu(s)ds$ and $\Lambda(t) = \int_0^t \sigma(s)dB(s)$.

Usually, the initial degradation level at installation is 0, $x_0 = 0$. Further, if the system is assumed to go through a stationary Wiener process, which implies that μ and σ are constant, the degradation process is expressed as

$$X(t) = x_0 + \mu t + \sigma B(t) \tag{3}$$

Note that the Wiener process is not monotonously increasing in time t, but its expectation $E[X(t)]$ is linearly increasing in t, i.e., $E[X(t)] = x_0 + ut$. The Wiener degradation process exhibits identical independent increment properties (Guo et al, 2013; Ye & Xie, 2015). For any time sequence $\{t_i, i = 1, 2, ...\}$, the random increments $X(t_i) - X(t_{i-1})$ are independent and follows the normal distribution, $X(t_i) - X(t_{i-1}) \sim N\left(\mu\left(t_i - t_{i-1}\right), \sigma^2\left(t_i - t_{i-1}\right)\right)$.

The system is assumed to be failed when its degradation level hits a specific threshold 1, $X(t) > 1$. As the system state is fully revealed by the degradation level, we use the term "system state" and "degradation level" interchangeably in this paper.

2.2 Replacement model for system with operating cost

We consider the CBM policy in which periodic inspection is dedicated to monitor the degradation level of the system. We first consider the replacement model, where only preventive replacement is allowed to restore the system to a new state level. A sudden failure occurs when the degradation level exceeds the failure threshold 1. Correspondingly, corrective replacement is carried out immediately to replace the failed system, with the corrective replacement cost c_f.

During the operation horizon, inspection is implemented periodically. At each inspection time, if the degradation level does not exceed the failure threshold, decision has to be made whether or not to preventively replace the system or let it be until the next inspection. The decision is made based on the current degradation level at inspection. If the system is replaced preventively, a preventive replacement cost c_p is incurred. If one decides to wait until the next inspection, inspection cost c_i is paid and the system runs the risk of sudden failure before the next inspection. It is assumed that both preventive maintenance and corrective maintenance restore the system to the as-good-as-new state. Apparently, $c_f > c_p$, as c_f includes additional cost due to sudden failures, such as production loss and even life hazards.

Apart from the aforementioned cost items, another important cost is the operating cost during system operation. The operating cost is incurred due to system degradation, which is used to represent the efficiency decrement or quality loss. The operating cost depends not only on system degradation level but also on system age. This is due to the fact that a more aged system is more

likely to exhibit less efficiency and produce products with lower quality, taking a production line for example.

The system is inspected at equally spaced discrete time epochs, $\{\tau, 2\tau, 3\tau...\}$, where τ is the time unit between two consecutive inspections. At the kth inspection epoch, denote the system state as X_k and system age as $\tau_k = k\tau$, which is the time since the last maintenance action. At each inspection, when the degradation level is detected, the sequence $\{k, X_k; k = 1, 2,...\}$ constitutes the states of the sequential decision process. We follow the assumption of Elwany et al (2011): if the degradation level exceeds the failure threshold between inspections and then returns below the threshold at the next inspection, no failure is incurred.

The operating cost, $G(\bullet)$, is dependent on both the system age and the system state. We assume that the operating cost is incurred only after the system age exceeds specific threshold, $k_c\tau$, where k_c is a constant. Apparently the operating cost $G(t, x)$ exhibits the property that $G(t, x)$ is non-decreasing in system age t and system state x. The degradation process of the system may influence the randomness of the operating cost in a complicated mechanism. Here, for simplicity, we assume the operating cost $G(t, x)$ as a linear function of the degradation level. $G(t, x)$ can be denoted as $G(t, x(t)) = h(t)x(t)$, where $h(t) > 0$, is a non-decreasing function of t. The assumption is reasonable in reality because the operating cost may increase rapidly for an aged or seriously degraded system.

Denote $W(k, X_k)$ as the expected operating cost within one period transition from the kth inspection to the (k+1)th inspection. Given the system age and state (k, X_k), $W(k, X_k)$ can be obtained as

$$W\left(k, X_k\right) = E\left[\int_{k\tau}^{(k+1)\tau} G(t, x(t))dt\right] = \int_{k\tau}^{(k+1)\tau} \int_0^\infty f(x(t))G(t, x(t))dxdt$$

(4)

where $f(x(t))$ is the probability density function (pdf) of system state at time t.

Lemma 1. The expected operating cost within one period $W\left(k, X_k\right)$ is non-decreasing in the inspection epoch k and the observed system state X_k.

The proofs of the lemmas and theorems in this paper are provided in Appendix.

The incurred cost has a discounting factor $e^{-r\tau}$ during each inspection interval. Denote $V(k, X_k)$ as the infinite-horizon minimal total discounted cost with the initial state (k, X_k). The optimality decision satisfies the Bellman equation

$$V(k, X_k) = \begin{cases} c_f + V(0,0) & X_k > 1 \\ \min\{c_p + V(0,0), \\ \qquad e^{-r\tau}U\left(k, X_k\right) + e^{-r\tau}W\left(k, X_k\right)\}, & X_k \leq 1 \end{cases}$$

(5)

where $U(k, X_k) = E\left[V\left(k+1, X_{k+1}\right) \mid (k, X_k)\right]$ is the expected cost-to-go at the (k+1)th inspection.

Note that the inspection cost is treated as a separate cost item and not incorporated in Eq (5). The optimality equation follows the logic that (a) if the degradation level at the kth inspection exceeds the failure threshold [1], which implies that the system is failed, then the system has to be replaced correctively; (b) if the system still functions, the decision maker can choose either to preventively replace the system or to let it be till the next inspection, depending on which is more cost-effective.

2.3 structural properties of the replacement policy

We are now in a position to investigate the structural properties of the maintenance policy, as the structural properties facilitate to develop computationally efficient algorithms. In particular, we focus on the control limit policy. A control limit policy implies that the system is kept operating

until the observed degradation level exceeds specific threshold (Benyamini & Yechiali, 1999). Fig. 2 shows the maintenance procedure and the evolvement of system state under control limit policy.
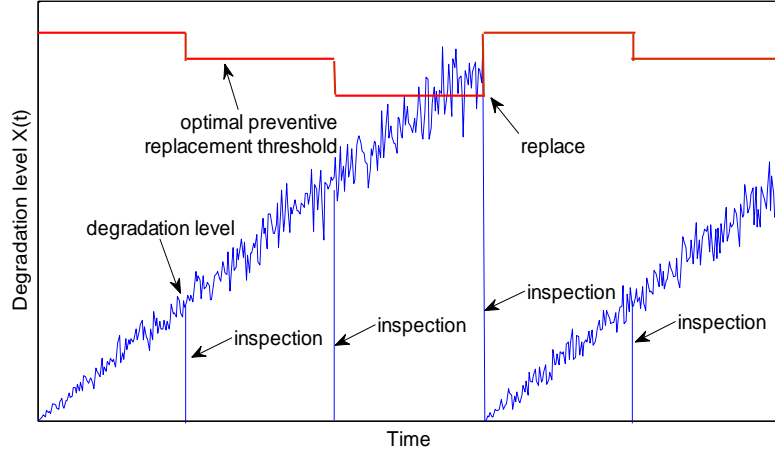

Fig. 2 Sketch of degradation process and maintenance actions

Some essential properties are presented as follows.

Theorem 1. The value function, $V(k, X_k)$, is non-decreasing in the inspection time k and system state X_k.

Theorem 2. The optimal maintenance policy that minimises the value function, $V(k, X_k)$, is a monotone control limit policy. At decision epoch k, preventive replacement is performed when the observed degradation level exceeds the optimal value δ_k. The sequence $\{\delta_k, k = 1, 2...\}$ is non-increasing in k.

The property of the monotone control limit policy can be incorporated into the optimisation algorithm to reduce the computational burden.

3. Replacement model with imperfect repair

In this section, we consider the repair-replacement model, where repair is allowed to restore the system, in addition to replacement. Repair can be achieved by technical maintenance operations such as oiling and adding more lubricants (Zhang et al, 2015). It is assumed that the repair is imperfect, that repair restores the system to a state between as-good-as-new and as-bad-ad-old.

At inspection, if the system still functions, three maintenance actions can be implemented: preventive replacement, repair, or wait until next inspection. Fig. 3 shows how the system state evolves under the repair-replacement strategy.
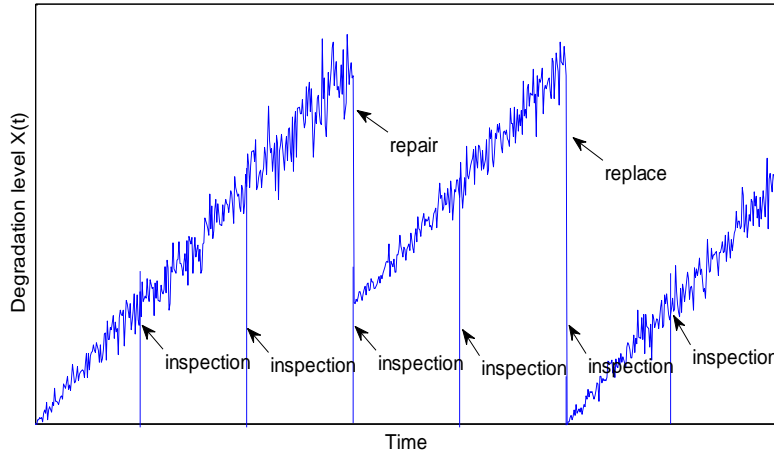

Fig. 3 Evolvement of system state under the repair-replacement model

3.1 Maintenance decision with controllable repair levels

At the decision epoch k with system state x, one can choose to repair the system to a lower level $y(k, x)$, where $y \in (0, x]$. $y = x$ indicates that the system is left as it be. $0 < y < x$ implies that the system is restored to a state between as-good-as-new and as-bad-as-old.

Denote the repair cost as $C(x, y)$. Of course it satisfies $C(x, y) = 0$, for $y = x$. We assume that the repair cost $C(x, y)$ is increasing in x and decreasing in y, which implies that more resources have to be devoted to achieve more improvement of the system state (Özekici, 1995). If the system

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has failed at inspection, then corrective replacement is carried out. The corrective replacement cost $c_f > C(x, y)$, $\forall x, y \geq 0$. The difference between repair and replacement (either preventive or corrective) is that repair can only restore the system state to a lower level, while replacement is able to restore both the system age and state to as-good-as-new condition. In addition, it is assumed that $c_p > C(x, y)$, $\forall x, y \geq 0$. This assumption is reasonable because preventive replacement is more effective than repair and therefore incurs higher cost. Otherwise, repair makes no sense.

Denote, again, $V(k, X_k)$ as the value function of the system starting with age k and state X_k. The Bellman equation is given as

$$
V(k, X_k) = \begin{cases} c_f + V(0,0) & X_k > 1 \\ \min\{c_p + V(0,0), \\ \quad \inf_{y < X_k} \{C(X_k, y) + e^{-r\tau}U(k, y) + e^{-r\tau}W(k, y)\}, \\ \quad e^{-r\tau}U(k, X_k) + e^{-r\tau}W(k, X_k)\}, & X_k \leq 1 \end{cases}
$$

(6)

Eq (6) implies that (a) if the system has failed at inspection, then corrective replacement is implemented; (b) if the system is operating, three maintenance actions can be selected: preventive replacement, repair to the optimal level, or wait until the next inspection, depending on which gives the minimal cost.

Lemma 2. The value function, $V(k, X_k)$, is non-decreasing in the inspection epoch k and non-decreasing in X_k.

The proof of Lemma 2 is analogous to that of Theorem 1. The details are thus omitted. Lemma 2 implies that the value function increases monotonically with the inspection epoch and system state; an increased inspection epoch and system state lead to higher maintenance cost. Based on Lemma 2, more results can be concluded, as shown in the following.

Denote $y_k^*(X_k)$ as the optimal decision at the kth inspection with the observed state X_k, which minimises the total expected discounted cost $V(k, X_k)$. More results can be obtained if restrictions are imposed on the repair cost $C(x, y)$.

Theorem 3. Suppose $C(x, y) < C(x, z) + C(z, y)$, for all $x < z < y$. (1) If the optimal decision at inspection is to repair the system, then $y^*(y^*(x)) = y^*(x)$ for all $x > 0$. (2) If $C(x, y) = c(y) - c(x)$, for $0 \le y \le x$, where $c(\bullet)$ is a non-increasing cost, then $y^*(x)$ is non-decreasing in x.

Theorem 3 implies that if repair is the optimal decision, then it is more preferable to repair the system directly to state w, rather than repair the system to an intermediate state z ($x > z > w$) then to state w. If the repair cost can be expressed as the difference of the current value and value after repair, the optimal repair level increases with the observed system state. This property can be used to reduce computational complexity of a search algorithm. The search interval of the optimal repair level $y_k^*(x)$ can be reduced by comparing the observed system state.

If the repair cost is only related to the repaired state y and independent of the current state x, we can conclude that the optimal maintenance policy between repair and doing nothing is a monotone control-limit policy.

Corollary 1. Suppose $C(x, y) = c(y)$, for $0 \le y \le x$, the optimal decision between repair and to do nothing is a control limit policy.

(1) At the kth inspection, there exists a critical value η_k, if the observed state $X_k > \eta_k$, then repair should be carried out. Otherwise, the system should be left as it is.

(2) There exist a critical value z_k^*, the optimal repair level is $y_k^*(X_k) = z_k^*$, for all $1 > X_k > \eta_k$.

(3) The sequence $\{\eta_k, k = 1, 2, ...\}$ is decreasing in k.

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3.2 Maintenance decision with uncontrollable repair levels

In the previous analyses, the repair level is assumed to be controllable in the sense that the decision maker can determine to what level the system state can be restored to. However, in many cases, especially for complex systems, the repair level may be uncontrollable. Maintenance crew may try to repair the degraded system as much as possible; yet to which level the system state is restored depends on the degraded state before repair. The repair cost c_{re} is assumed to be constant, independent of the system state. The system state after repair $y(x)$ satisfies the following properties: (1) $0 < y(x) < x$; (2) $y'(x) \geq 0$, for all $x \in (0,1)$. The first property implies that the repair is imperfect. The second property indicates that the restored system state increases with the current state. In the literature, a great number of models have been established to describe the repair effect (Pham & Wang, 1996), among which, the virtual-age reduction model is widely used (Doyen & Gaudoin, 2004). Similar to the virtual-age reduction model, we establish a repair model where the system state after repair is proportional to the current state. Simply, we have $y(x) = (1-\xi)x$, where ξ is a constant parameter, denoting the repair effect, $\xi \in (0,1)$. The optimal decision starting with the state (k, X_k) is given as

$$V(k, X_k) = \begin{cases} c_f + V(0, 0), & X_k > 1 \\ \min\{c_p + V(0,0), \\ \quad c_{re} + e^{-r\tau}U(k, y) + e^{-r\tau}W(k, y), \\ \quad e^{-r\tau}U(k, X_k) + e^{-r\tau}W(k, X_k)\}, & X_k \leq 1 \end{cases}$$

(7)

Lemma 3. The optimal decision between repair and preventive replacement is a control-limit type. At the kth inspection, there exists a critical value ε_k, if $X_k > \varepsilon_k$, then preventive replacement is preferred; otherwise, repair should be carried out.

Denote $\varphi_k(x) = e^{-r\tau}U(k, x) + e^{-r\tau}W(k, x)$, if $\varphi_k(x)$ is convex, it can be concluded that the optimal maintenance policy is of control-limit type.

Theorem 4. Suppose $\varphi_k(x)$ is a convex function, the optimal decision among preventive replacement, repair or to do nothing is a control-limit policy. At the kth inspection, the optimal decision π_k is

$$\pi_k = \begin{cases} \text{Preventive replacement, if } 1 > X_k > \varepsilon_k \\ \text{Repair, if } \varepsilon_k \geq X_k > \eta_k \\ \text{Do nothing, if } X_k \leq \eta_k \end{cases}$$

where ε_k and η_k are two critical values. In addition, the sequence $\{\varepsilon_k, k = 1, 2, ...\}$ and $\{\eta_k, k = 1, 2, ...\}$ are non-increasing in k.

Given the monotone property of the maintenance policy, the optimal policy can be obtained by many of the existing methods, e.g., value iteration algorithm and policy iteration algorithm (Puterman, 2014). Here, we adopt the monotone policy iteration to take advantage of the established structural properties. To make the solution feasible, we follow the existing practice to discretise the continuous state into finite intervals and use a time horizon long enough to approximate the infinite horizon. In particular, we define $k_{max}\tau$ as the maximum allowable system age for some integers $k_{max} < \infty$. k_{max} is set large enough so that the system will fail before $k_{max}\tau$ almost certainly. If the system survives beyond $k_{max}\tau$, a preventive replacement is carried out to restore the system. In addition, the continuous degradation process is discretised so as to reduce the state space of the sequential decision process. Details of the monotone policy iteration are omitted in this paper, as one can refer to Elwany et al (2011).

4. Numerical example

   In this section, a diesel vehicle is presented to illustrate the maintenance policy. During service, the engine of the diesel vehicle is assumed to follow a Wiener degradation process with the drift coefficient $\mu = 1$ and diffusion coefficient $\sigma = 1$. With the increase of usage and mileage, the engine may consume more diesels and produce more exhaust gas, which is modeled as the age- and state-dependent operating cost. Failures occur when the degradation level exceeds the failure threshold $l = 6$. The system is subject to periodic inspection, with the inspection interval $\tau = 1$ and the inspection cost $c_i = 0.05$. At the kth inspection epoch, if the system functions but its degradation level exceeds the control limit threshold, preventive replacement is carried out with cost $c_p = 4$. If the system has already failed, then corrective replacement is performed with cost $c_f = 10$. The incurred cost is discounted with the discounting factor $r = 0.02$. The total discounted inspection cost is calculated separately as

$$U_i = \sum_{k=0}^{\infty} c_i \exp(-rk\tau) = \frac{c_i}{1 - \exp(-r\tau)} = 2.52$$

(8)

4.1 The age- and state-dependent operating cost

 In the operation horizon, operating cost is incurred due to system aging and degradation. At the early stage of service, the engine may produce pollutant gas within certification levels. After running $k_c\tau$ years, where k_c is a pre-specified integer, the engine may no longer retain the pollutant rate and emit exhaust gas deviating from the standard level. The pollutant rate increases with age and degradation level of the engine. However, for an in-service vehicle, exact evaluation of exhaust gas emission and engine degradation is difficult to perform since many parameters may

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play a role (Pillot et al, 2014). Hence, we use a simple model to characterise the operating cost as a function of system age and degradation level. The operating cost of the system at age t and degradation level x is denoted as

$$G(t, x) = \begin{cases} 0, & t < k_c \tau \\ \alpha e^{\beta(t-k_c \tau)} x, & t \geq k_c \tau \end{cases} \tag{9}$$

where α and β are known parameters, scaling the effect of age and degradation level on the operating cost.

According to operating cost formulation, the cumulative operating cost between τ_k and τ_{k+1} is denoted as

$$Q(k, X_k) = \int_{\tau_k}^{\tau_{k+1}} G(t, x)dt = \int_{\tau_k}^{\tau_{k+1}} \alpha e^{(\beta-r)(t-k_c \tau)} x(t)dt \quad , \text{ for } \tau_k > k_c \tau \tag{10}$$

The expected operating cost within the interval (τ_k, τ_{k+1}) is given as

$$W(k, X_k) = E[Q(k, X_k)] = E\left[\int_{\tau_k}^{\tau_{k+1}} \alpha e^{\beta(t-k_c \tau)} x(t)dt\right]$$

$$= E\left[\int_{\tau_k}^{\tau_{k+1}} \alpha e^{\beta(t-k_c \tau)} \left(X_k + \mu(t-\tau_k) + \sigma B(t-\tau_k)\right)dt\right] \quad , \text{ for } \tau_k > k_c \tau \tag{11}$$

4.2 Optimal replacement policy and sensitivity analysis

The associated parameters are set as $k_c = 4$, $\alpha = 0.2$ and $\beta = 0.05$. We plot the survival probability of the degradation process in Fig. 4. As can be observed from Fig 4, when the system ages is older than 10, the system will fail with the probability of 0.9. In addition, the system will fail with the probability of 0.99 if it is aged over 15.
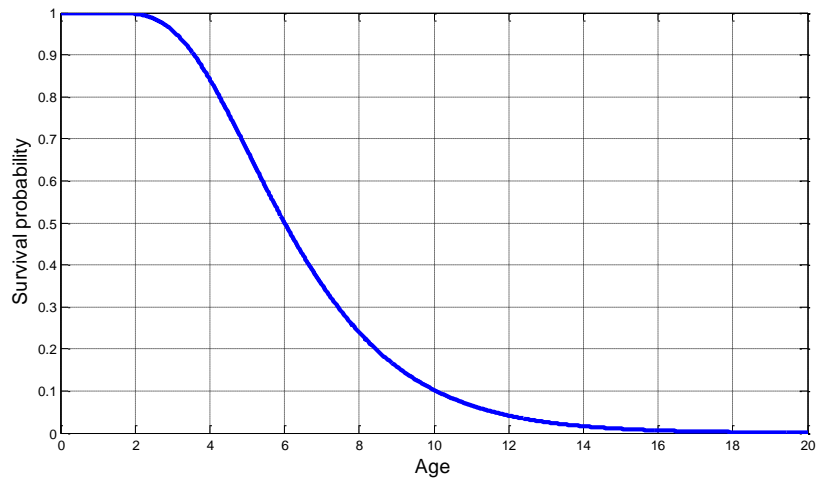
Fig. 4 Survival probability of degradation process

Fig. 5 plots the optimal maintenance policies with operating cost. It is clearly shown that the optimal maintenance policy is a monotone control limit policy, where the control limit shows a decreasing trend in system age. Whereas no operating cost is incorporated, as shown between age 0 and 4, the optimal maintenance policy is reduced to a constant control limit policy. This is due to the fact that the system degrades according to a homogeneous stochastic process and the system state increases independently after inspection. Compared with CBM policy where no age- and state-dependent operating cost is considered, the result shows a more conservative replacement threshold.
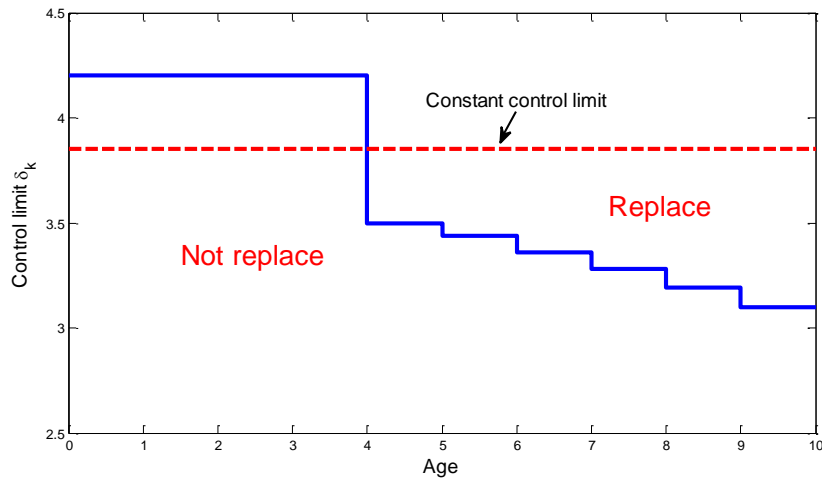
Fig. 5 Optimal control limit policies with operating cost

To investigate the effectiveness of the monotone control limit policy, we compare the monotone control limit policy with the traditional constant control limit policy. Fig. 6 shows how the total discounted cost varies under different preventive replacement thresholds. When the control limit is set as 3.85, the total discounted cost reaches its minimum 52.62. By comparison, the minimum total discount cost under monotone control limit policy is 48.45.
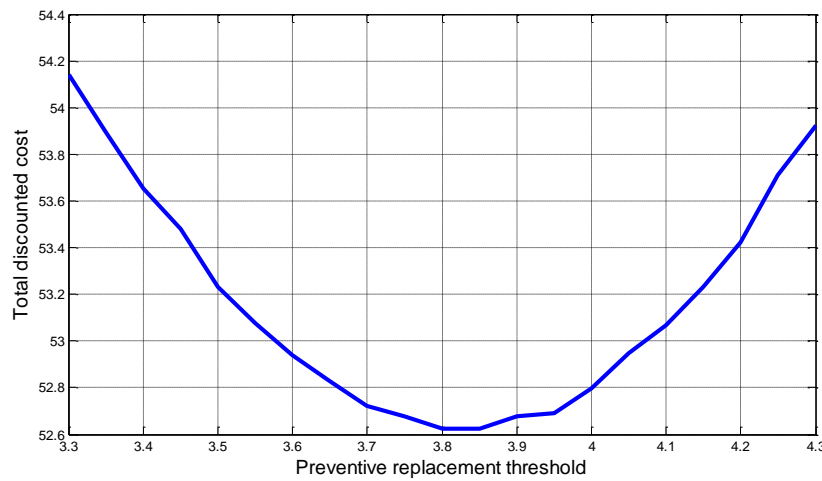

Fig. 6 Cost variation of constant control limit policy

We are interested in finding how the optimal control limit policy changes against various cost parameters. Sensitivity analysis is therefore conducted to investigate this relationship, as presented in Fig. 7 and Fig. 8. When the preventive replacement cost c_p becomes closer to the corrective replacement cost c_f, the optimal control limit policy is more tolerant to system failure. In other words, the optimal maintenance policy is less conservative. As shown in Fig. 7, with the increase of preventive replacement cost, the optimal control limit δ_k shows an increasing trend. The result is quite straightforward. It is less attractive to preventively replace the system when the preventive replacement cost increases. On the other hand, an increased corrective replacement cost c_f results in a more conservative control limit policy, as shown in Fig. 8. As can be observed from Fig. 7 and Fig. 8, the influence of different c_f is not so significant as that of c_p. This can be explained as that the optimal control limits are far from the failure threshold and sudden failure occurs infrequently. Therefore, the corrective replacement cost c_f has less influence on the optimal control limits.
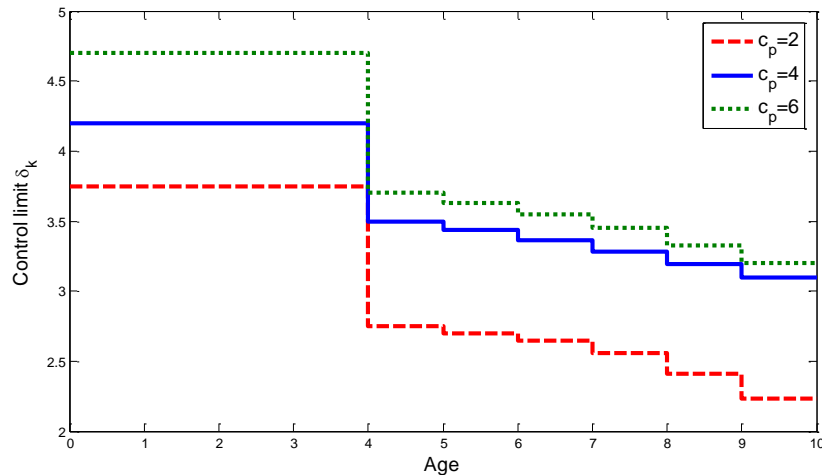

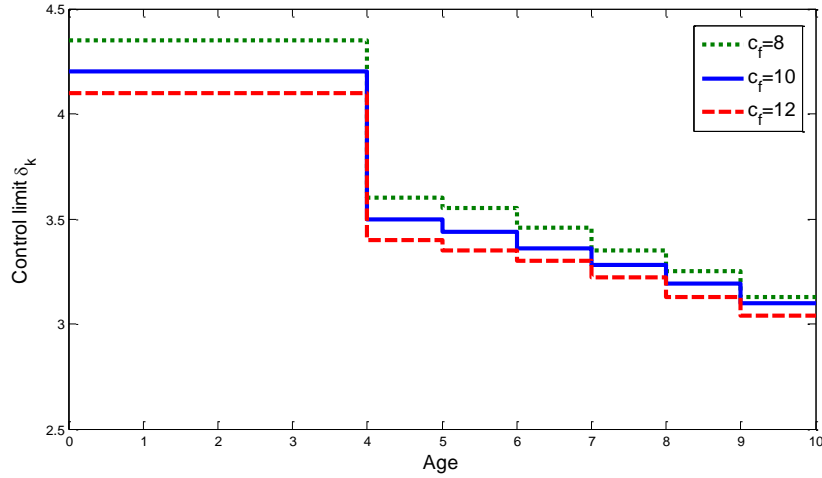Fig. 7 Sensitivity analysis for c_p

Fig. 8 Sensitivity analysis for c_f

5. Conclusion

This paper develops a condition-based maintenance policy for systems with age- and state-dependent operating cost. The system is subject to a continuous degradation process, characterised by a Wiener process with linear drift. The operating cost occurs during system operation, which increases with system age and the degradation level. Two models capable of arriving at maintenance decisions are developed. One is the replacement model and the other is the repair-replacement model. The structural properties of the optimal decision are investigated in depth. For the replacement model, this paper show that the optimal decision is actually a monotone control limit policy. For the repair-replacement model, it shows that, under mild assumptions of the repair cost, the optimal decision among preventive replacement, repair and doing nothing is also a monotone control limit policy. A numerical example is presented to illustrate the optimal maintenance decisions.

The proposed model can be applied in various systems. For example, in a power grid, the resistance of electric wire increases with the age and degradation level of power line, which

consumes more energy when transmitting electricity. The energy loss due to age and degradation can be modeled as the age- and state- dependent operating cost, and the proposed maintenance model can be used to capture the energy loss during power transmission.

Future extensions of this study can be conducted with respect to a non-stationary degradation process. The Wiener process with linear drift is only available to a limited number of systems, which fails to describe a complicated degradation process. Extension to a more general class of degradation process will remedy this disadvantage. Additionally, in the current work, we assume that inspection can fully reveal the underlying degradation level, which may be relaxed by considering imperfect inspection. In the imperfect inspection framework, the maintenance decision can be formulated as a partially observable Markov decision process.

Acknowledgement

Appendix

1. Proof of Lemma 1.

Proof. According to Eq. (4), we have

$$W(k, X_k) = \int_{k\tau}^{(k+1)\tau} \int_0^\infty f(x(t))G(t, x(t))dxdt$$

$$= \int_{k\tau}^{(k+1)\tau} \int_0^\infty f(x(t))h(t)x(t)dxdt$$

$$= X_k \int_{k\tau}^{(k+1)\tau} h(t)dt + \mu \int_{k\tau}^{(k+1)\tau} th(t)dt \tag{A1}$$

Define $H(t) = \int_0^t h(u)du$, so that $H'(t) = h(t) > 0$ and $H''(t) = h'(t) \geq 0$. As $h(t) > 0$, it can be easily observed that $W(k, X_k)$ is increasing in X_k. To prove that $W(k, X_k)$ is non-decreasing in k, we only need to prove that $\int_{k\tau}^{(k+1)\tau} h(t)dt$ and $\int_{k\tau}^{(k+1)\tau} th(t)dt$ are non-decreasing in k.

Let $\Gamma(k) = \int_{k\tau}^{(k+1)\tau} h(t)dt$ and $\Upsilon(k) = \int_{k\tau}^{(k+1)\tau} th(t)dt$. We have

$$\Delta\Gamma(k) = \Gamma(k+1) - \Gamma(k)$$

$$= H((k+2)\tau) + H(k\tau) - 2H((k+1)\tau)$$

As $H'(t) = h(t) > 0$ and $H''(t) = h'(t) \geq 0$, $H(t)$ is a convex function in t. The Jensen's inequality (Boyd & Vandenberghe, 2004) states that

$$\eta f(x_1) + (1-\eta) f(x_2) > f(\eta x_1 + (1-\eta)x_2)$$

for any convex function $f(\bullet)$ and $\eta \in (0,1)$. With the above inequality, we can readily obtain that

$$\Delta\Gamma(k) = H((k+2)\tau) + H(k\tau) - 2H((k+1)\tau) > 0 \tag{A2}$$

Let $q(t) = t \cdot h(t)$. Clearly $q(t)$ is increasing in t and $\int_0^t q(u)du$ is a convex function in t. Likewise, we can have

$$\Delta\Upsilon(k) = \Upsilon(k+1) - \Upsilon(k) > 0 \tag{A3}$$

Combining Eq. (A2) and Eq. (A3), we have $W(k, X_k)$ is non-decreasing in k, which completes the proof. □

2. Proof of Theorem 1.

Proof. We prove the theorem by mathematical induction. Denote $V^n(k, X_k)$ as the value function at the nth iteration of value iteration policy. At $n = 0$, we set $V^0(k, X_k) = 0$, which is a constant. Assume that the theorem holds for the nth iteration, i.e., $V^n(k, X_k)$ is non-decreasing in k and non-decreasing in X_k. Then according to Eq (5), we have

$$V^{n+1}(k, X_k) = \begin{cases} c_f + V^n(0,0), & X_k > 1 \\ \min\{c_p + V^n(0,0), \\ \quad e^{-r\tau}U^n(k, X_k) + e^{-r\tau}W(k, X_k)\}, & X_k \le 1 \end{cases}$$

Lemma 1 shows that $W(k, X_k)$ is non-decreasing in k and X_k. As $V^n(k, X_k)$ is non-decreasing in k and non-decreasing in X_k, its expectation, $U^n(k, X_k)$, holds this property as well. Because the terms of the right-hand side are non-decreasing in k and non-decreasing in X_k, $V^{n+1}(k, X_k)$ is also non-decreasing in k and X_k, which completes the proof. □

3. Proof of Theorem 2.

Proof. Preventive maintenance is optimal when $c_p + V(0,0) < e^{-r\tau}U(k, X_k) + e^{-r\tau}W(k, X_k)$. The left-side is a constant. Based on Theorem 1, $e^{-r\tau}U(k, X_k) + e^{-r\tau}W(k, X_k)$ is non-decreasing in X_k. Thus for any $X_k > \delta_k$, the inequality holds, which implies the monotone control limit policy. On the other hand, the term on the right-hand side is non-decreasing in k. Therefore, for the inspection epoch k, there exists an optimal replacement age k_δ, such that for any $k > k_\delta$, the optimal decision is to preventively replace the system, which indicates that δ_k is non-increasing in k. □

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4. Proof of Theorem 3.

Proof. (1) As repair is the optimal decision at the kth inspection, we have
$V(k, x) = C(x, y^*) + e^{-r\tau} U(k, y^*) + e^{-r\tau} W(k, y^*)$. If $y^*(x) = x$, then $y^*(y^*(x)) = y^*(x)$ obviously.

If $y^*(x) = z < x$, we need to prove that $y^*(z) = z$. Suppose not, then $y^*(z) = w < z$. Then we have

$$V(k, x) = C(x, z) + e^{-r\tau} U(k, z) + e^{-r\tau} W(k, z) \underset{<}{} C(x, w) + e^{-r\tau} U(k, w) + e^{-r\tau} W(k, w) \tag{B1}$$

This inequality holds because $y^*(x) = z$ is the optimal repair level, which restores the system

to state w and incurs more cost. In addition, we have
$$V(k, w) = C(z, w) + e^{-r\tau} U(k, w) + e^{-r\tau} W(k, w) < V(k, z)$$
$$= C(z, z) + e^{-r\tau} U(k, z) + e^{-r\tau} W(k, z) = e^{-r\tau} U(k, z) + e^{-r\tau} W(k, z) \tag{B2}$$

This inequality holds because repairing the system to state $y^*(z) = w$ is more cost-effective than

to do nothing until the next inspection, according to the assumption $y^*(z) = w < z$. Combining Eq

(B1) and Eq (B2), we have

$$C(z, w) < e^{-r\tau} U(k, z) + e^{-r\tau} W(k, z) - e^{-r\tau} U(k, w) - e^{-r\tau} W(k, w) < C(x, w) - C(x, z) \tag{B3}$$

so that we have $C(z, w) + C(x, z) < C(x, w)$, which is contradictory to the assumption

$C(x, y) < C(x, z) + C(z, y)$. Hence, it can be concluded that $y^*(z) = z$, and $y^*(y^*(x)) = y^*(x)$. □

(2) To prove this result, we need to show that $y^*(x+u) \geq y^*(x)$, for any $u > 0$. Consider the

case that repair is the optimal decision at inspection. We can have $V(k, x+u) = \inf_{y \leq x+u} \{c(y) - c(x+u) + e^{-r\tau} U(k, y) + e^{-r\tau} W(k, y)\}$

$$= \min \left\{ \begin{array}{l} \inf_{y \leq x} \{c(y) - c(x+u) + e^{-r\tau} U(k, y) + e^{-r\tau} W(k, y)\}, \\ \inf_{x < y \leq x+u} \{c(y) - c(x+u) + e^{-r\tau} U(k, y) + e^{-r\tau} W(k, y)\} \end{array} \right\}$$

If $y^*(x+u) \leq x$, then

$$V(k, x+u) = \inf_{y \le x} \left\{ c(y) + e^{-r\tau} U(k, y) + e^{-r\tau} W(k, y) \right\} - c(x+u)$$

In addition, we have

$$V(k, x) = \inf_{y \le x} \left\{ c(y) + e^{-r\tau} U(k, y) + e^{-r\tau} W(k, y) \right\} - c(x)$$

Combine the above two expressions, it can be readily obtained that $y^*(x+u) = y^*(x)$, for $y^*(x+u) \le x$.

If $x < y^*(x+u) \le x+u$, then obviously, $y^*(x+u) > y^*(x)$, as $y^*(x+u) > x \ge y^*(x)$. Therefore, $y^*(x)$ is non-decreasing in x, for all $0 \le y \le x$. □

5. Proof of Corollary 1.

Proof. (1) If preventive replacement is not a preferable option, we can have

$$V(k, X_k) = \begin{cases} \min \left\{ \inf_{y < X_k} \left\{ c(y) + e^{-r\tau} U(k, y) + e^{-r\tau} W(k, y) \right\}, \\ e^{-r\tau} U(k, X_k) + e^{-r\tau} W(k, X_k) \right\} \end{cases}$$
(C1)

$\inf_{y < X_k} \left\{ c(y) + e^{-r\tau} U(k, y) + e^{-r\tau} W(k, y) \right\}$ is non-increasing in X_k because the interval $(0, X_k)$

becomes wider, which relaxes the constraints of y. On the other hand, $e^{-r\tau} U(k, X_k) + e^{-r\tau} W(k, X_k)$ is increasing in X_k, as shown in Theorem 1. Thus, the minimum of

the two functions has to satisfy

$$V(k, X_k) = \begin{cases} \inf_{y < X_k} \left\{ c(y) + e^{-r\tau} U(k, y) + e^{-r\tau} W(k, y) \right\}, & \text{if } X_k > \eta_k \\ e^{-r\tau} U(k, X_k) + e^{-r\tau} W(k, X_k) \right\} & , \text{if } X_k \le \eta_k \end{cases}$$
(C2)

where $\eta_k = \inf \left\{ X_k \ge 0 : \inf_{y < X_k} \left\{ c(y) + e^{-r\tau} U(k, y) + e^{-r\tau} W(k, y) \right\} \le e^{-r\tau} U(k, X_k) + e^{-r\tau} W(k, X_k) \right\}$,

denotes the earliest time when repair is implemented.

(2) If $X_k \le \eta_k$, we can easily have $y_k^* = X_k$, implying that no repair is carried out if $X_k \le \eta_k$.

On the other hand, if $X_k > \eta_k$, then $V(k, X_k) = \inf\limits_{y < X_k} \left\{ c(y) + e^{-r\tau} U(k, y) + e^{-r\tau} W(k, y) \right\}$ according

to Eq (C2). Because $\inf\limits_{y < X_k} \left\{ c(y) + e^{-r\tau} U(k, y) + e^{-r\tau} W(k, y) \right\}$ is decreasing in $X_k \in [\eta_k, \infty)$ while

$V(k, X_k)$ is increasing in X_k, the only way they are equal is that $V(k, X_k)$ is a constant; $V(k, X_k)$

$= V(k, \eta_k)$ for all $X_k \in [\eta_k, \infty)$. Therefore, we can conclude if $y^*(\eta_k) = z_k^*$, then $y_k^*(X_k) = z_k^*$, for

all $X_k \in [\eta_k, \infty)$.

(3) The proof is analogous to Theorem 2. The details are not shown here to avoid repetition. □

6. Proof of Lemma 3

Proof. According to the assumption, the restored system state y increases with the current state

X_k. Meanwhile, both $U(k, y)$ and $W(k, y)$ are non-decreasing in y. Therefore, it can be

concluded that $U(k, y(X_k))$ and $W(k, y(X_k))$ are non-decreasing in X_k. On the other hand,

$c_p + V(0,0)$ is independent of X_k. Thus, there exists a critical value ε_k, such that

$C(X_k) + e^{-r\tau} U(k, y) + e^{-r\tau} W(k, y) > c_p + V(0,0)$, for $X_k > \varepsilon_k$. □

7. Proof of Theorem 4.

Proof. Lemma 3 shows that preventive replacement is preferred over repair for $X_k > \varepsilon_k$. Next,

we need to prove that repair is preferred over doing nothing for $\varepsilon_k \ge X_k > \eta_k$. Let

$\psi_k(x) = c_{re} + \varphi_k(y(x)) - \varphi_k(x)$. We have $\psi'_k(x) = \varphi'_k(y)y'(x) - \varphi'_k(x)$. According to the

assumption that $\varphi_k(x)$ is a convex function, $\varphi'_k(x)$ increases in x. As the repaired system state

$y < x$, we can have $\psi'_k(x) < 0$, which implies that $\psi_k(x)$ decreases in x. Because $\psi_k(0) = c_{re} > 0$, and $\psi_k(x) < 0$, for $x \rightarrow \infty$. Hence, we can conclude that there exists a critical value η_k, repair is preferred over doing nothing for $\varepsilon_k \geq X_k > \eta_k$. As all the right-hand items of Eq (8) increase with k, we can conclude that the sequence $\{\varepsilon_k, k = 1, 2, ...\}$ and $\{\eta_k, k = 1, 2, ...\}$ are non-increasing in k. $\square$

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