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# Multilayer PI Control for Consensus in Heterogeneous Multi-Agent Networks

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**Abstract**—In this paper, a distributed proportional-integral multilayer control strategy is proposed, for reaching consensus in networks of heterogeneous first-order linear node dynamics. The resulting closed-loop network can be seen as an instance of multiplex networks currently studied in network science. The strategy is able to guarantee consensus, even in the presence of constant disturbances and heterogeneity. The proportional and integral actions are deployed on two different layers across the network, each with its own topology. Explicit expressions for the consensus values are obtained together with sufficient conditions guaranteeing convergence. The effectiveness of the theoretical results are illustrated via numerical simulations using a power network example.

## I. INTRODUCTION

The problem of driving a group of interconnected agents towards a common desired state has become fundamental due to its potential application in many different areas of Science and Engineering [1], [2], [3]. This problem can be solved into two different manners via centralized or distributed control actions. Distributed control is more apt in all those situations where several constraints are present and cannot be avoided such as limited resources and energy, short wireless communication ranges, narrow bandwidths, etc [4].

One problem of particular interest is consensus, where the goal is for all agents in the network to asymptotically converge towards each other [5]. A large number of notable applications have made this problem of relevance. For instance, temperature regulation throughout an entire building or environment (e.g. a mine) [6]; distributed formation control in robotics [7], [8]; platooning of vehicles in intelligent transportation systems [9], [10]; and *frequency synchronization* [11], [12] in power grids and micro-grids. (For a more exhaustive list of applications see [13], [5] and references therein).

The existing literature on consensus is very rich and many extensions and sophisticated techniques have been studied. For example, solutions have been proposed where the distributed controller is nonlinear, or it is affected by switchings and delays [14], [15], [16], [5]; the network graph is directed, and/or the node dynamics is nonlinear (for a review see [17], [18]). As research on consensus has matured, more complicated and realistic scenarios must be addressed and analysed. Often, agents in a network do not

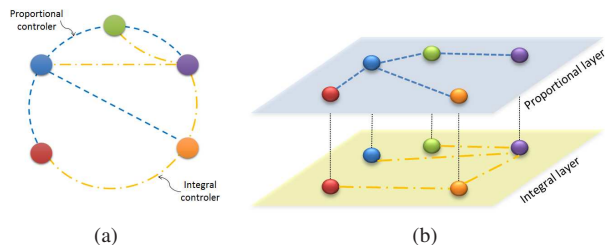


Fig. 1. (a) Network with two types of links representing proportional (blue line) and integral (yellow) distributed controllers, (b) Multiplex (Multilayer) network representing proportional and integral distributed controllers.

necessarily have the same intrinsic dynamics [19], [20], [21], [22], [23] and may be affected by noise and disturbances.

A pressing open problem is that of guaranteeing consensus in groups of agents in the presence of heterogeneity among their dynamics together with disturbances and noise. In [24], a distributed PID control is proposed for solving the consensus problem in the case where the nodes are heterogeneous and are affected by constant disturbances. Hence, the aim of this paper is to extend the result in [24] where a proportional (P) plus integral (I) actions are deployed for every link in the network, to the case where P and I actions can be added on different links as can be seen in Fig.1(a).

The resulting strategy is a multilayer distributed control architecture where each layer represents the proportional and integral control network respectively (see Fig.1(b)). The main advantage of the multilayer strategy is that both the control action and the convergence time can be substantially reduced by properly choosing the network structure in the proportional and integral layers.

Multiplex (or multilayer) networks are a collection of networks (called layers) which may interact with each other. They have been proposed as an effective modelling approach for representing and investigating network problems in many real and man-made networks where multiple types of interconnections are present [25]. In this paper we introduce the concept of multilayer control networks to model the action of different control layers with different structures. We focus on a PI multiplex strategy proving that it can effectively guarantee convergence in networks of linear systems despite the presence of heterogeneity in the node dynamics and

constant disturbances. We find conditions for tuning the gains of the distributed control actions in order to guarantee convergence that depend on the node dynamics.

## II. NOTATION AND MATHEMATICAL PRELIMINARIES

We denote by  $\mathbf{I}_N$  the identity matrix of dimension  $N \times N$ ; by  $\mathbf{0}_{M \times N}$  a matrix of zeros of dimension  $M \times N$ , and by  $\mathbf{1}_N$  a  $N \times 1$  vector with unitary elements. A diagonal matrix, say  $\mathbf{D}$ , with diagonal elements  $d_1, \dots, d_N$  is indicated by  $\mathbf{D} = \text{diag}\{d_1, \dots, d_N\}$ . The Frobenius norm is denoted by  $\|\cdot\|$  while the spectral norm by  $\|\cdot\|$ .  $\lambda_N(\cdot)$  denotes the  $N$ -th eigenvalue of a matrix. Given two vectors  $\zeta_1, \zeta_2 \in \mathbb{R}^{n \times 1}$  and a matrix  $\mathbf{Q} \in \mathbb{R}^{n \times n}$ , linear algebra implies

$$2\zeta_1^T \mathbf{Q}^T \zeta_2^T \leq \varepsilon \zeta_1^T \mathbf{Q}^T \mathbf{Q} \zeta_1 + \frac{1}{\varepsilon} \zeta_2^T \zeta_2, \forall \varepsilon > 0 \quad (1)$$

An *undirected graph*  $\mathcal{G}$  is a pair defined by  $\mathcal{G} = (\mathcal{N}, \mathcal{E})$  where  $\mathcal{N} = \{1, 2, \dots, N\}$  is a discrete set with  $N$  components representing the index of a node (vertex),  $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$  is the edge set containing  $M$  edges between nodes. Furthermore, we assume each edge has an associated weight denoted by  $w_{ij} \in \mathbb{R}^+$  for all  $i, j \in \mathcal{N}$ . The weighed adjacency matrix  $\mathcal{A}(\mathcal{G}) \in \mathbb{R}^{N \times N}$  with  $\mathcal{A}_{ij}$  entries, is defined as  $\mathcal{A}_{ij}(\mathcal{G}) = w_{ij}$  if there is an edge from node  $i$  to node  $j$  and zero otherwise.

Similarly, the *Laplacian matrix*  $\mathcal{L}(\mathcal{G}) \in \mathbb{R}^{N \times N}$  is defined as the matrix whose elements  $\mathcal{L}_{ij}(\mathcal{G}) = \sum_{j=1, j \neq i}^N w_{ij}$  if  $i = j$  and  $-w_{ij}$  otherwise. Thus, the Laplacian matrix can be recast in compact form as  $\mathcal{L}(\mathcal{G}) = \text{diag}\{\mathbf{1}_N^T \mathcal{A}(\mathcal{G})\} - \mathcal{A}(\mathcal{G})$ , where the matrix  $\text{diag}\{\mathbf{1}_N^T \mathcal{A}(\mathcal{G})\}$  is often called the degree matrix of the graph  $\mathcal{G}$ . Given two graphs sharing the same set of nodes  $\mathcal{G}_1 = (\mathcal{N}, \mathcal{E}_1)$  and  $\mathcal{G}_2 = (\mathcal{N}, \mathcal{E}_2)$ , we define the *projection graph* as the graph  $\text{proj}(\mathcal{G}_1, \mathcal{G}_2) := (\mathcal{N}, \mathcal{E}_p)$  with associate adjacency matrix  $\mathcal{A}_p := \mathcal{A}(\mathcal{G}_1) + \mathcal{A}(\mathcal{G}_2)$ .

**Definition II.1.** [26] We say that an  $n \times n$  matrix  $\mathcal{A} = [\mathcal{A}_{ij}]$ ,  $\forall i, j \in \mathcal{N}$  belongs to the set  $\Omega$  if it verifies the following properties

- 1)  $\mathcal{A}_{ij} \leq 0$ ,  $i \neq j$ , and  $\mathcal{A}_{ii} = - \sum_{j=1, j \neq i}^N \mathcal{A}_{ij}$ ,
- 2) its eigenvalues are such that  $\lambda_1(\mathcal{A}) = 0$  while all the others,  $\lambda_k(\mathcal{A})$ ,  $k \in \{2, \dots, N\}$ , are real and positive.

Note that the Laplacian matrix  $\mathcal{L}$  previously defined, satisfy the conditions of Definition II.1 if its associated graph  $\mathcal{G}$  is connected [5].

**Lemma II.1.** [24] Let  $\mathcal{A} \in \Omega$  be the Laplacian matrix of a generic undirected and connected graph  $\mathcal{G}$ , then  $\mathcal{A}$  can be written in block form as  $\mathcal{A} = \mathbf{R} \mathbf{A} \mathbf{R}^{-1}$ , where

$$\mathbf{R} = \begin{bmatrix} 1 & N \mathbf{R}_{21}^T \\ \mathbf{1}_{N-1} & N \mathbf{R}_{22}^T \end{bmatrix}, \mathbf{R}^{-1} = \begin{bmatrix} r_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix} \quad (2)$$

with  $\mathbf{R}_{21} \in \mathbb{R}^{(N-1) \times 1}$ ,  $\mathbf{R}_{22} \in \mathbb{R}^{(N-1) \times (N-1)}$  being blocks of appropriate dimensions, and

$$r_{11} = \frac{1}{N}, \quad \mathbf{R}_{12} = \frac{1}{N} \mathbf{1}_{N-1}^T, \quad (3)$$

$\mathbf{A} = \text{diag}\{0, \lambda_2(\mathcal{A}), \dots, \lambda_N(\mathcal{A})\}$  with  $0 = \lambda_1(\mathcal{A}) < \lambda_2(\mathcal{A}) \leq \dots \leq \lambda_N(\mathcal{A})$  being the eigenvalues of  $\mathcal{A}$ . Moreover, the blocks in  $\mathbf{R}$  and  $\mathbf{R}^{-1}$  must fulfill the following conditions

$$r_{11} + \mathbf{R}_{12} \mathbf{1}_{N-1} = 1 \quad (4)$$

$$\mathbf{R}_{21} + \mathbf{R}_{22} \mathbf{1}_{N-1} = \mathbf{0}_{(N-1) \times 1} \quad (5)$$

$$\mathbf{R}_{21} \mathbf{R}_{21}^T + \mathbf{R}_{22} \mathbf{R}_{22}^T = \frac{1}{N} \mathbf{I}_{N-1} \quad (6)$$

$$r_{11} \mathbf{R}_{21}^T + \mathbf{R}_{12} \mathbf{R}_{22}^T = \mathbf{0}_{1 \times (N-1)} \quad (7)$$

$$\mathbf{R}_{21} \mathbf{R}_{21}^T = \mathbf{R}_{22} \mathbf{1}_{N-1} \mathbf{1}_{N-1}^T \mathbf{R}_{22}^T \quad (8)$$

$$\|\mathbf{R}_{22}\| \leq \frac{1}{\sqrt{N}} \quad (9)$$

$$\|\mathbf{R}_{21}\| \leq \sqrt{N-1} \|\mathbf{R}_{22}\| \leq \sqrt{(N-1)/N} \quad (10)$$

$$N \mathbf{R}_{22}^T = (\mathbf{I}_{N-1} + \mathbf{1}_{N-1} \mathbf{1}_{N-1}^T)^{-1} \mathbf{R}_{22}^{-1} \quad (11)$$

**Definition II.2.** [27] A *multiplex graph*, is a pair  $\mathcal{M} = (\mathcal{G}, \mathcal{D})$  where  $\mathcal{G}$  is the set of  $M$  graphs  $\mathcal{G} := \{\mathcal{G}_1, \dots, \mathcal{G}_M\}$  called layers of  $\mathcal{M}$ , and  $\mathcal{D}$  is the set of edges representing interconnections between nodes of different layers.

## III. PROBLEM STATEMENT

We consider the problem of achieving *consensus* in a group of  $N$  agents (Multi-agent system), governed by heterogeneous linear dynamics of the form

$$\dot{x}_i(t) = \rho_i x_i(t) + \delta_i + u_i(t), i \in \mathcal{N} \quad (12)$$

where  $x_i(t) \in \mathbb{R}$  represents the state of the  $i$ -th agent,  $\rho_i \in \mathbb{R}$  is the agent pole,  $\delta_i \in \mathbb{R}$  is some constant disturbance (or constant external input) acting on each node, and  $u_i(t) \in \mathbb{R}$  is the distributed control input. Note that, assuming a null control input, the dynamics of the  $i$ -th node can be unstable ( $\rho_i > 0$ ) or stable ( $\rho_i < 0$ ) with  $-\delta_i/\rho_i$  as equilibrium point. Moreover,  $\delta_i$  can be used to represent different quantities in applications; such as, constant power injections in power grids [12] or a source of noise in minimal models of natural flocks of birds [28].

The problem is to find bounded and distributed control inputs  $u_i(t)$ , such that all states  $x_i(t)$  converge asymptotically towards each other.

**Definition III.1.** (Admissible consensus) [24]: The multi-agent system (12) is said to achieve *admissible consensus* if for any set of initial conditions  $x_i(0) = x_{i0}$ ,

$$\lim_{t \rightarrow \infty} |x_j(t) - x_i(t)| = 0, \quad |u_i(t)| < +\infty, \forall t \geq 0, i, j \in \mathcal{N}$$

To solve the problem of achieving admissible consensus in a network of heterogeneous linear agents with disturbances, we propose the use of a distributed PI protocol, obtained by setting:

$$u_i(t) = -\sigma_P \sum_{j=1, j \neq i}^N \alpha_{ij} (x_j(t) - x_i(t)) + z_i(t) \quad (13a)$$

$$\dot{z}_i(t) = -\sigma_I \sum_{j=1, j \neq i}^N \beta_{ij} (x_j(t) - x_i(t)) \quad (13b)$$

where the constant weights  $\alpha_{ij}, \beta_{ij} \in \mathbb{R}^+$  are local control strengths of the distributed proportional and integral contributions respectively. We consider undirected links, that is  $\alpha_{ij} = \alpha_{ji}$  and  $\beta_{ij} = \beta_{ji}$  without self-links  $\alpha_{ii} = \beta_{ii} = 0$ . Note that these control actions allow the deployment of proportional and integral actions on the network edges independently from each other ( $\alpha_{ij} = 0$  or  $\beta_{ij} = 0$  for some  $i, j \in \mathcal{N}$ ,  $i \neq j$ ). The constants  $\sigma_P, \sigma_I \in \mathbb{R}^+$  are additional parameters modulating globally the contribution of each control layer with respect to each other.

Equation (13) effectively defines two control layers each represented by a different weighted graph  $\mathcal{G}_P := (\mathcal{N}, \mathcal{E}_P)$  for the proportional layer and  $\mathcal{G}_I := (\mathcal{N}, \mathcal{E}_I)$  for the integral layer, where  $\mathcal{E}_P$  and  $\mathcal{E}_I$  are the set of edges with associated weights  $\alpha_{ij}$  and  $\beta_{ij}$  respectively. As depicted in Fig. 1 the resulting control strategy is therefore a multiplex distributed control network, with a multiplex graph  $\mathcal{M} = (\mathcal{G}, \mathcal{D})$  where  $\mathcal{G} := \{\mathcal{G}_P, \mathcal{G}_I\}$ , and since the layers do not interact between them,  $\mathcal{D}$  is an empty set. The closed-loop network is then given by

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{P} - \sigma_P \mathcal{P} & \mathbf{I}_N \\ -\sigma_I \mathcal{I} & \mathbf{0}_{N \times N} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix} + \begin{bmatrix} \Delta \\ \mathbf{0}_{N \times 1} \end{bmatrix} \quad (14)$$

where  $\mathbf{x}(t) = [x_1(t), \dots, x_N(t)]^T$ , and  $\mathbf{z}(t) = [z_1(t), \dots, z_N(t)]^T$  are the stack vectors of the node and integral states. Besides,  $\mathcal{P}, \mathcal{I} \in \Omega$  are two different Laplacian matrices associated to the graphs  $\mathcal{G}_P$  and  $\mathcal{G}_I$  that we assume are connected. Thus, the problem then becomes that of finding conditions on the global control gains  $\sigma_P, \sigma_I$  and the local control gains  $\alpha_{ij}$  and  $\beta_{ij}$  encoded by the matrices  $\mathcal{P}$  and  $\mathcal{I}$ , so as to guarantee convergence of all nodes in the closed-loop network (14) towards each other, i.e. admissible consensus.

#### IV. STABILITY ANALYSIS

The stability problem is that of finding conditions for the control gains  $\sigma_P, \sigma_I$ , and the Laplacians of the control layers,  $\mathcal{P}$  and  $\mathcal{I}$ , so as to guarantee convergence of all node states in the closed-loop network (14) towards each other. First we prove that the collective dynamics of the closed-loop network has a unique equilibrium which is a solution of the admissible consensus problem.

**Proposition IV.1.** *Network (14) has a unique equilibrium given by  $\mathbf{x}^* := x_\infty \mathbf{1}_N$ , and  $\mathbf{z}^* := -(x_\infty \mathbf{P} \mathbf{1}_N + \Delta)$  where  $x_\infty := -\sum_{k=1}^N \delta_k / \sum_{k=1}^N \rho_k$ .*

*Proof:* From Definition II.1, we know that  $\mathcal{P} \mathbf{1}_N = \mathcal{I} \mathbf{1}_N = \mathbf{0}_{N \times 1}$ . Thus, setting the left-hand side of (14) to zero one has that  $\mathbf{x}^* = a \mathbf{1}_N$ ,  $\forall a \in \mathbb{R}$  and  $\mathbf{z}^* = -(a \mathbf{P} \mathbf{1}_N + \Delta)$ . By definition, we also have that  $\mathbf{1}_N^T \mathbf{z}(t) = 0$ , then  $\mathbf{1}_N^T \mathbf{z}^* = 0$  and we obtain

$$a = -\mathbf{1}_N^T \Delta / \mathbf{1}_N^T \mathbf{P} \mathbf{1}_N = -\sum_{k=1}^N \delta_k / \sum_{k=1}^N \rho_k =: x_\infty$$

which completes the proof.  $\blacksquare$

**Remark IV.1.** *Note indeed that if controller (13) is able to render this equilibrium stable it is also able to guarantee consensus of all node states  $x(t)$  to a constant*

value  $x_\infty$  using bounded control energy. Also, note that the emergent behaviour from the collective dynamics of the network, follows a generic first-order "exo-system" given by  $\dot{s}(t) = (\sum_{k=1}^N \rho_k) s(t) + \sum_{k=1}^N \delta_k$ . This result for perturbed heterogeneous agents establishes a connection with [19] where internal models in networks are studied.

Now, shifting the origin via the further state transformation  $\mathbf{y}(t) := \mathbf{z}(t) + \Delta$  one has

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{y}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{P} - \sigma_P \mathcal{P} & \mathbf{I}_N \\ -\sigma_I \mathcal{I} & \mathbf{0}_{N \times N} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix} \quad (15)$$

Networks with different structures are considered in the two layers of the closed-loop network; therefore, we next provide a Lemma which associate them.

**Lemma IV.1.** *Let  $\mathbf{A} = \mathbf{R} \mathbf{\Lambda}_1 \mathbf{R}^{-1}$  and  $\mathbf{B} = \mathbf{Q} \mathbf{\Lambda}_2 \mathbf{Q}^{-1}$  be two Laplacian matrices belonging to the set  $\Omega$ , where  $\mathbf{R}$  and  $\mathbf{Q}$  are block matrices as in (2) and  $\mathbf{\Lambda}_k$ ,  $k \in \{1, 2\}$  are diagonal matrices containing the eigenvalues of  $\mathbf{A}$  and  $\mathbf{B}$  respectively. Then,*

$$\mathbf{R}^{-1} \mathbf{B} \mathbf{R} = \begin{bmatrix} 0 & \mathbf{0}_{1 \times (N-1)} \\ \mathbf{0}_{(N-1) \times 1} & \Xi \bar{\mathbf{\Lambda}}_2 \Xi^T \end{bmatrix} \quad (16)$$

where  $\Xi = N \mathbf{R}_{22} (\mathbf{1}_{N-1} \mathbf{1}_{N-1}^T + \mathbf{I}_{N-1}) \mathbf{Q}_{22}^T$  and  $\bar{\mathbf{\Lambda}}_2 = \text{diag} \{ \lambda_2(\mathbf{B}), \dots, \lambda_N(\mathbf{B}) \}$ . Moreover,  $\Xi \bar{\mathbf{\Lambda}}_2 \Xi^T$  is a symmetric matrix.

*Proof:* See Appendix A.  $\blacksquare$

#### A. Error Dynamics

Assuming the graphs in both proportional and integral layers are connected, using Lemma II.1 we can write  $\mathcal{P} = \mathbf{R} \mathbf{\Lambda}_1 \mathbf{R}^{-1}$  and  $\mathcal{I} = \mathbf{Q} \mathbf{\Lambda}_2 \mathbf{Q}^{-1}$ . We next define the error dynamics given by the state transformation  $\mathbf{x}^\perp(t) = \mathbf{R}^{-1} \mathbf{x}(t)$ ; therefore, using the block representation of  $\mathbf{R}^{-1}$  and letting  $\bar{\mathbf{x}}^\perp(t) = [x_2^\perp(t), \dots, x_N^\perp(t)]^T$ ,  $\bar{\mathbf{x}}(t) = [x_2(t), \dots, x_N(t)]^T$  we obtain

$$x_1^\perp(t) = r_{11} x_1(t) + \mathbf{R}_{12} \bar{\mathbf{x}}(t) \quad (17a)$$

$$\bar{\mathbf{x}}^\perp(t) = \mathbf{R}_{21} x_1(t) + \mathbf{R}_{22} \bar{\mathbf{x}}(t) \quad (17b)$$

Note that by adding and subtracting the term  $\mathbf{R}_{22} x_1(t) \mathbf{1}_{N-1}$  to (17b) and using property (4) one has

$$\bar{\mathbf{x}}^\perp(t) = \mathbf{R}_{22} (\bar{\mathbf{x}}(t) - x_1(t) \mathbf{1}_{N-1}) \quad (18)$$

Hence,  $\bar{\mathbf{x}}^\perp(t) = 0$  if and only if  $\bar{\mathbf{x}}(t) - x_1(t) \mathbf{1}_{N-1} = 0$  since  $\mathbf{R}_{22}$  is full rank [24]. Then, admissible consensus is achieved if  $\lim_{t \rightarrow \infty} \bar{\mathbf{x}}^\perp(t) = 0$  and  $\|\mathbf{y}(t)\| < +\infty, \forall t > 0$ . [This also implies that, when consensus is achieved all nodes must converge to  $x_1(t)$ .]

Now, recasting (15) in the new coordinates  $\mathbf{x}^\perp(t)$  and  $\mathbf{y}^\perp(t) = \mathbf{R}^{-1} \mathbf{y}(t)$ , we get

$$\begin{bmatrix} \dot{\mathbf{x}}^\perp(t) \\ \dot{\mathbf{y}}^\perp(t) \end{bmatrix} = \left( \Psi - \begin{bmatrix} 0 & \mathbf{0} \\ 0 & \sigma_P \bar{\mathbf{\Lambda}}_1 \end{bmatrix} \right) \mathbf{x}^\perp(t) + \begin{bmatrix} 0 \\ \bar{\mathbf{y}}^\perp(t) \end{bmatrix} \quad (19)$$

where  $\bar{\mathbf{y}}^\perp(t) := [y_2^\perp(t), \dots, y_N^\perp(t)]^T$ ,  $\bar{\mathbf{\Lambda}}_1 := \text{diag} \{ \lambda_2(\mathcal{P}), \dots, \lambda_N(\mathcal{P}) \}$  and  $\bar{\mathbf{\Lambda}}_2 :=$

$\text{diag}\{\lambda_2(\mathcal{I}), \dots, \lambda_N(\mathcal{I})\}$ . Note that the equation for  $y_1^\perp(t)$  can be neglected as it has trivial dynamics with null initial conditions and represents an uncontrollable and unobservable state. Moreover, it is important to note that matrix  $\Xi$  was obtained using Lemma IV.1 for  $\mathbf{R}^{-1}\mathcal{I}\mathbf{R}$ . Besides,  $\Psi$  is a block matrix defined as

$$\begin{aligned} \Psi &:= \mathbf{R}^{-1}\mathbf{P}\mathbf{R} = \begin{bmatrix} \psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix} \\ &= \mathbf{R}^{-1} \begin{bmatrix} \rho_1 & \mathbf{0}_{1 \times (N-1)} \\ \mathbf{0}_{(N-1) \times 1} & \mathbf{P} \end{bmatrix} \mathbf{R} \end{aligned} \quad (20)$$

with  $\bar{\mathbf{P}} := \text{diag}\{\rho_2, \dots, \rho_N\}$ . Now using properties (4)-(7) we can write [24]

$$\psi_{11} := (1/N) \sum_{k=1}^N \rho_k \quad (21)$$

$$\Psi_{12} := \bar{\rho}^T \mathbf{R}_{22}^T \quad (22)$$

$$\Psi_{21} := \mathbf{R}_{22} \bar{\rho} \quad (23)$$

$$\Psi_{22} := N\mathbf{R}_{22} (\bar{\mathbf{P}} + \rho_1 \mathbf{1}_{N-1} \mathbf{1}_{N-1}^T) \mathbf{R}_{22}^T \quad (24)$$

where

$$\bar{\rho} := [\rho_2 - \rho_1, \dots, \rho_N - \rho_1]^T \quad (25)$$

### B. Distributed-PI Multiplex Control

**Theorem IV.1.** *A group of  $N$  heterogeneous agents (12) controlled by the distributed multilayer PI strategy (13), achieves admissible consensus for any connected integral network  $\mathcal{G}_{\mathcal{I}}$  and  $\sigma_{\mathcal{I}} > 0$  if the following conditions hold*

$$\psi_{11} = (1/N) \sum_{k=1}^N \rho_k < 0, \quad (26a)$$

$$\sigma_P \lambda_2(\mathcal{P}) > \max_{i \in \mathcal{N}} \rho_i + \frac{\bar{\rho}^T \bar{\rho}}{N |\psi_{11}|} \quad (26b)$$

moreover, all node states converge to the equilibrium  $(\mathbf{x}^*, \mathbf{z}^*)$  defined in Prop.IV.1.

*Proof:* Consider the Lyapunov candidate function (in what follows we remove the time dependence of the state variables and the symbol  $\perp$  to simplify the notation)

$$V = \frac{1}{2} \left( x_1^2 + \mathbf{x}^T \mathbf{x} + \frac{1}{\sigma_{\mathcal{I}}} \mathbf{y}^T (\Xi \bar{\Lambda}_2 \Xi^T)^{-1} \mathbf{y} \right) \quad (27)$$

From Lemma IV.1 we know that  $\Xi \bar{\Lambda}_2 \Xi^T$  is an eigendecomposition of a symmetric matrix with positive eigenvalues, which are the diagonal entries of  $\bar{\Lambda}_2$ ; therefore, its inverse exist and it is also a positive definite matrix. Consequently, (27) is a positive definite function and differentiating it along the trajectories of (19) one has

$$\dot{V} = \psi_{11} x_1^2 + x_1 (\Psi_{12} + \Psi_{21}^T) \mathbf{x} + \mathbf{x}^T (\Psi_{22} - \sigma_P \bar{\Lambda}_1) \mathbf{x}$$

from (22) and (23) we have that  $\Psi_{12} = \Psi_{21}^T = \bar{\rho}^T \mathbf{R}_{22}^T$ , then one gets

$$\dot{V} = \psi_{11} x_1^2 + \mathbf{x}^T \Psi_{22} \mathbf{x} - \sigma_P \mathbf{x}^T \bar{\Lambda}_1 \mathbf{x} + 2x_1 \mathbf{x}^T \mathbf{R}_{22} \bar{\rho} \quad (28)$$

Now using relation (1) we can write the last term of (28) in a quadratic form. This is, letting  $\mathbf{Q}^T = \mathbf{R}_{22}$ ,  $\zeta_1 = \mathbf{x}$  and  $\zeta_2 = x_1 \bar{\rho}$  one has

$$\begin{aligned} \dot{V} &\leq (\psi_{11} + \frac{1}{\varepsilon} \bar{\rho}^T \bar{\rho}) x_1^2 + \mathbf{x}^T \Psi_{22} \mathbf{x} \\ &\quad - \sigma_P \mathbf{x}^T \bar{\Lambda}_1 \mathbf{x} + \varepsilon \mathbf{x}^T \mathbf{R}_{22} \mathbf{R}_{22}^T \mathbf{x} \end{aligned} \quad (29)$$

From linear algebra it is possible to obtain upper-bounds for each term of (29). Thus, by definition we know that  $\Psi = \mathbf{R}^{-1}\mathbf{P}\mathbf{R}$  is a symmetric matrix with eigenvalues  $\rho_i, \forall i \in \mathcal{N}$ , and  $\lambda_{\max}(\Psi_{22}) \leq \lambda_{\max}(\Psi) = \max_i \rho_i$  (see Theorem 8.4.5 in [29]). Furthermore,  $-\sigma_P \mathbf{x}^T \bar{\Lambda}_1 \mathbf{x} \leq -\sigma_P \lambda_2(\mathcal{P}) \mathbf{x}^T \mathbf{x}$ . Therefore, one gets

$$\begin{aligned} \dot{V} &\leq (\psi_{11} + \frac{1}{\varepsilon} \bar{\rho}^T \bar{\rho}) x_1^2 + \\ &\quad \left( \max_i \rho_i + \varepsilon \|\mathbf{R}_{22}\|^2 - \sigma_P \lambda_2(\mathcal{P}) \right) \mathbf{x}^T \mathbf{x} \end{aligned} \quad (30)$$

Now, using property (9) we have that  $\dot{V}$  is negative definite if  $\xi_1 := \psi_{11} + 1/\varepsilon \bar{\rho}^T \bar{\rho} < 0$  and  $\xi_2 := \max_i \rho_i + \varepsilon/N - \sigma_P \lambda_2(\mathcal{P}) < 0$ .

From the assumption, we have that  $\psi_{11} < 0$ , therefore  $\xi_1 < 0$  is ensured if  $\varepsilon > \bar{\rho}^T \bar{\rho} / |\psi_{11}|$ . Also,  $\xi_2 < 0$  if condition (26b) is fulfilled. Therefore, all agents in (12) achieve admissible consensus to the equilibrium  $(\mathbf{x}^*, \mathbf{z}^*)$  defined in Prop. IV.1.  $\blacksquare$

### C. Example

We consider a possible application of the distributed multiplex PI strategy introduced in this paper, to a simplified network inspired from the model of a power system introduced in [30] (for further information regarding control of power networks, we also encourage the reader to look up [31], [32] and references therein.) Specifically, we consider the problem of studying convergence in a network of  $N$  buses (nodes) with identical inertia constants, linearized around the synchronization manifold  $\omega_1(t) = \dots = \omega_N(t)$  given by [30]:

$$\dot{\omega}_i(t) = m(-d_i \omega_i(t) + p_i^* - p_i(t) + v_i(t)) \quad (31a)$$

$$\dot{p}_i(t) = \sum_{j=1, j \neq i}^N E_i E_j |Y_{ij}| (\omega_i - \omega_j), i \in \mathcal{N} \quad (31b)$$

where  $\omega_i(t)$  represent the frequency state of each bus,  $p_i^*$  is the power load at bus  $i$ ,  $v_i(t)$  is the mechanical input,  $d_i$  is damping coefficient and  $m$  is the inverse of the inertia parameter. Furthermore,  $E_i > 0$  is the nodal voltage,  $Y_{ij}$  is the admittance among buses  $i$  and  $j$ , and  $p_i(t)$  represents the active electrical power exchanged with the other buses.

We propose the use of the distributed control protocol given by

$$v_i(t) = \frac{1}{m} \left( k_i \omega_i(t) + \sigma_P \sum_{k=1}^N \mathcal{P}_{ij} \omega_k(t) \right) \quad (32)$$

with  $k_i$  being the local feedback gains,  $\sigma_P > 0$  and  $\mathcal{P} \in \Omega$  represents the Laplacian matrix of a certain graph  $\mathcal{G}_{\mathcal{P}}$  in the proportional layer with link weights  $\alpha_{ij}$ . From a practical viewpoint, the control strategy corresponds to enhance the



coupling between the agents in the open-loop network via proportional control network. This could be implemented via appropriate digital devices feeding the control action to the nodes.

Then, let  $\beta_{ij} := E_i E_j |Y_{ij}|$  be the weight on each edge of the power network and  $\mathcal{I} \in \Omega$  the associated Laplacian matrix describing the equivalent distributed integral action (31b), and setting  $z(t) = -mp_i(t)$ , the problem becomes that of proving convergence in the heterogeneous network given by

$$\dot{\omega}(t) = (\mathbf{H} - \sigma_P \mathcal{P}) \omega(t) + \mathbf{z}(t) + \Delta \quad (33a)$$

$$\dot{\mathbf{z}}(t) = -\mathcal{I} \omega(t) \quad (33b)$$

where  $\omega(t) := [\omega_1(t), \dots, \omega_N(t)]$ , and  $\mathbf{z}(t) := [z_1(t), \dots, z_N(t)]$  are the stack vectors of frequency and rescaled electrical power respectively. Besides,  $\mathbf{H} := \text{diag}\{k_1 - md_1, \dots, k_N - md_N\}$ ,  $\Delta := m \cdot [p_1^*, \dots, p_N^*]^T$ .

System (33) has the same structure as (14) with,  $\sigma_I = 1$ . Thus for suitable control parameters we have from Prop. IV.1 that (33) reaches consensus to  $\omega_\infty := -m \sum_{i=1}^N p_i^* / \sum_{i=1}^N (k_i - md_i)$ . It is important to highlight that the consensus value can be controlled varying the control gains  $k_i$ . As an illustration and without loss of generality, consider the network of six buses shown in Fig. 2(a), we assume  $m = 1$  and the following power loads in each node

$$\Delta = [\delta_1, \dots, \delta_6] = [100, 160, 200, 110, 100, 98]$$

with control and damping constants such that

$$\mathbf{H} = \text{diag}\{0.2, -0.86, -0.8, -0.9, -7, -9\}$$

Note that local feedback actions are only added to nodes  $\{1, 5, 6\}$  (self red-links in Fig. 2(a), all the other  $k_i$  are zeros). Then, Theorem IV.1 can be used to tune the control gains and guarantee convergence in the case of a fixed network structure. It follows that  $\psi_{11} = -3.06$  and then condition (26a) is fulfilled. Moreover,  $\bar{\rho}^T \bar{\rho} = 139.8136$ , and

$$\max_{i \in \mathcal{N}} \{k_i - d_i\} = \max_{i \in \mathcal{N}} \{\mathbf{H}_{ii}\} = 0.2$$

then, consensus is guaranteed if  $\sigma_P \lambda_2(\mathcal{P}) > 7.8151$ . Thus, considering the network control layer of proportional links shown in Fig.2(b) we have that  $\lambda_2(\mathcal{P}) = 3.3470$ ; therefore,  $\sigma_P > 2.3350$ . The resulting evolution of the node states and electrical power for  $\sigma_P = 2.34$  are shown in Fig. 2(c), where admissible consensus is reached as expected to the predicted value  $x_\infty = 50\text{Hz}$ . Note that the structure and distributed gains of the control layer can be changed so as to vary the threshold value of the gains above which consensus is attained. Indeed by varying  $\lambda_2(\mathcal{P})$  we can vary the value of the threshold value for  $\sigma_P$ . Also when compared to the approach in [24], the number of distributed proportional (or integral) actions used here is much smaller as it suffices for  $\mathcal{G}_P$  to be connected while in [24], a proportional (or integral) action had to be added for every link consider in the network.

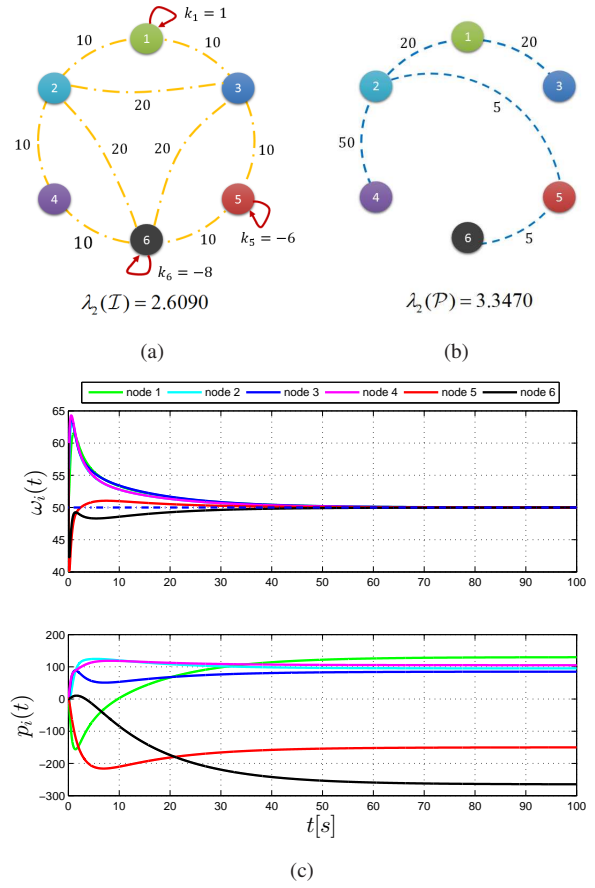


Fig. 2. (a) Schematic of the power network composed by six buses (nodes) with different damping coefficients. The red self-links represent local controllers acting on each node. (b) network architecture representing the proportional layer. (c) time response of the controlled power network. The blue dash-dot line represent the convergence point  $\omega_\infty$ .

## V. EXTENSION TO MORE GENERAL NODE DYNAMICS

Here we consider  $n$ -dimensional node dynamics of the form

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}_i \mathbf{x}_i(t) + \boldsymbol{\delta}_i + \mathbf{u}_i(t), i \in \mathcal{N} \quad (34)$$

where  $\mathbf{x}_i(t) \in \mathbb{R}^{n \times 1}$  represents the state of the  $i$ -th agent,  $\mathbf{A}_i \in \mathbb{R}^{n \times n}$  is the intrinsic node dynamic matrix,  $\boldsymbol{\delta}_i \in \mathbb{R}^{n \times 1}$  is some constant disturbance (or constant external input) acting on each node, and  $\mathbf{u}_i(t) \in \mathbb{R}^{n \times 1}$  is the distributed multilayer-PI controller given by

$$\begin{aligned} \mathbf{u}_i(t) = & -\sigma_P \sum_{j=1}^N \alpha_{ij} (\mathbf{x}_j(t) - \mathbf{x}_i(t)) \\ & - \sigma_I \sum_{j=1}^N \beta_{ij} \int_0^t (\mathbf{x}_j(\tau) - \mathbf{x}_i(\tau)) d\tau \end{aligned} \quad (35)$$

The main difference of this control strategy with the one reported in our previous work [24], is that the network structure of the proportional and integral layers, are not necessarily the same. Then, it is natural to wonder if this

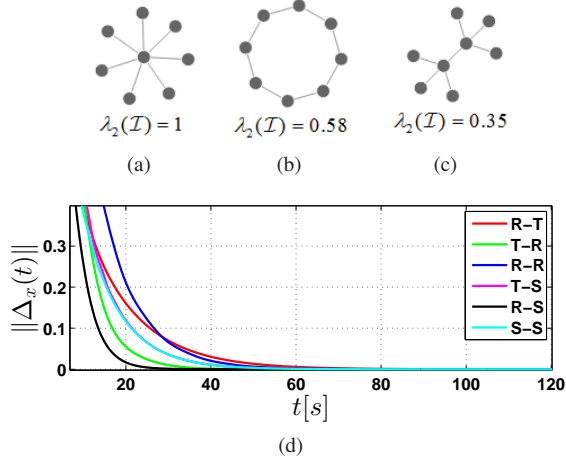


Fig. 3. Different network topologies with unitary weights: (a) all-to-all, (b) star, (c) ring, and (d) Tree. (e) time response of the consensus index  $\Delta_x$  when the network structure in each layer is varied. The **R** label denotes a ring topology, while **T** and **S** denote ring and star one.

extra degree of freedom can be exploited to improve the performance of the closed-loop network. In the next example, we explore the influence on the network behaviour, when the network topology of each layer  $\mathcal{G}_P$  and  $\mathcal{G}_I$  is varied.

#### A. Example

We first consider eight agents governed by (34), with three types of node dynamics; stable  $\mathbf{E}_1 := \text{diag}\{-1, -1.3\}$ , unstable  $\mathbf{E}_2 := \text{diag}\{1, 0.5\}$ , and oscillatory

$$\mathbf{E}_3 := \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$$

Then, the agents are described by  $\mathbf{A}_k = \mathbf{E}_1, k \in \{1, 2, 5, 6\}$ ,  $\mathbf{A}_k = \mathbf{E}_2, k \in \{4, 8\}$ , and  $\mathbf{A}_k = \mathbf{E}_3, k \in \{3, 7\}$  and the disturbances  $\delta_i \in \mathbb{R}^{2 \times 1}$  are given by  $\delta_i = [0, 10]^T$  for  $i=\{5,6\}$  and  $\delta_i = [0, 0]^T$  otherwise. Furthermore, four different network topologies with unitary weights as those shown in Fig. 3 are considered for each layer. Hence, setting  $\alpha = \beta = 2$ , we plot the evolution of the consensus index  $d_x(t) := \|\mathbf{x}(t) - (1/N)(\mathbf{1}_N \mathbf{1}_N^T \otimes \mathbf{I}_2) \mathbf{x}(t)\|$ , where  $d(t) = 0$  indicates that the closed-loop network has reached admissible consensus.

We consider six scenarios: a ring and tree topologies for the proportional and integral layers respectively (**R-T**), then a tree and a ring (**T-R**), ring and ring (**R-R**), tree and star (**T-S**), ring and star (**R-S**), and finally star-star configuration. Note that when the topologies are the same **R-R** and **S-S**, the network reaches consensus at  $t \approx 58s$ . While it is considerably reduced to  $30s$  by changing the topology to **R-S** star configuration. Surprisingly, the rate of convergence is not inversely proportional to the algebraic connectivity  $\lambda_2$  of each network.

## VI. CONCLUSIONS AND FUTURE WORK

We have proposed a novel control approach for controlling networks of heterogeneous nodes in the presence of constant disturbances. In particular, we proposed the

use of network control layers deploying different actions in a network, implementing a distributed proportional and integral multiplex strategy. We proved convergence of the strategy and derived conditions to select the control gains as a function of the open loop and control network structures and the node dynamics. Via numerical simulations on a representative example, we showed the effectiveness of the proposed strategy.

Several open problems are left for further study. First and foremost the effect of varying the structure of the network control layers should be studied in more details as preliminary results show the performance of the network evolution towards consensus can be affected by such variations. Also, it remains to be investigated if the additional degrees of freedom represented by the gains of the distributed P and I actions can be exploited to improve the performance of the closed-loop network, possibly in an optimal manner.

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## APPENDIX

Proof of Lemma IV.1 Multiplying both sides of  $\mathcal{B} = \mathbf{Q}\mathbf{\Lambda}_2\mathbf{Q}^{-1}$  by  $\mathbf{R}^{-1}$  and  $\mathbf{R}$ , yields  $\mathbf{R}^{-1}\mathcal{B}\mathbf{R} = \mathbf{R}^{-1}\mathbf{Q}\mathbf{\Lambda}_2\mathbf{Q}^{-1}\mathbf{R}$ . Now using the block form of  $\mathbf{R}$  and  $\mathbf{Q}$  as

shown in Lemma II.1 one gets

$$\mathbf{R}^{-1}\mathcal{W}\mathbf{R} = \begin{bmatrix} r_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix} \cdot \begin{bmatrix} 1 & N\mathbf{Q}_{21}^T \\ \mathbb{1}_{N-1} & N\mathbf{Q}_{22}^T \end{bmatrix} \cdot \begin{bmatrix} 0 & \mathbb{0}_{1 \times (N-1)} \\ \mathbb{0}_{(N-1) \times 1} & \bar{\mathbf{\Lambda}}_2 \end{bmatrix} \cdot \begin{bmatrix} q_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix} \cdot \begin{bmatrix} 1 & N\mathbf{R}_{21}^T \\ \mathbb{1}_{N-1} & N\mathbf{R}_{22}^T \end{bmatrix}$$

where  $\bar{\mathbf{\Lambda}}_2 = \text{diag}\{\lambda_2(\mathcal{B}), \dots, \lambda_N(\mathcal{B})\}$ . By definition  $q_{11} = r_{11}$  and  $\mathbf{Q}_{12} = \mathbf{R}_{12}$ , and then applying matrix multiplications we have

$$\mathbf{R}^{-1}\mathcal{B}\mathbf{R} = \begin{bmatrix} r_{11} + \mathbf{R}_{12}\mathbb{1}_{N-1} & N(q_{11}\mathbf{Q}_{21}^T + \mathbf{Q}_{12}\mathbf{Q}_{22}^T) \\ \mathbf{R}_{21} + \mathbf{R}_{22}\mathbb{1}_{N-1} & N(\mathbf{R}_{21}\mathbf{Q}_{21}^T + \mathbf{R}_{22}\mathbf{Q}_{22}^T) \end{bmatrix} \cdot \begin{bmatrix} 0 & \mathbb{0}_{1 \times (N-1)} \\ \mathbb{0}_{(N-1) \times 1} & \bar{\mathbf{\Lambda}}_2 \end{bmatrix} \cdot \begin{bmatrix} q_{11} + \mathbf{Q}_{12}\mathbb{1}_{N-1} & N(r_{11}\mathbf{R}_{21}^T + \mathbf{R}_{12}\mathbf{R}_{22}^T) \\ \mathbf{Q}_{21} + \mathbf{Q}_{22}\mathbb{1}_{N-1} & N(\mathbf{Q}_{21}\mathbf{R}_{21}^T + \mathbf{Q}_{22}\mathbf{R}_{22}^T) \end{bmatrix} \quad (36)$$

We next simplify each block of all matrices. Then, from (4) we have that  $r_{11} + \mathbf{R}_{12}\mathbb{1}_{N-1} = q_{11} + \mathbf{Q}_{12}\mathbb{1}_{N-1} = 1$ . While, from (5)

$$N(q_{11}\mathbf{Q}_{21}^T + \mathbf{Q}_{12}\mathbf{Q}_{22}^T) = N(r_{11}\mathbf{R}_{21}^T + \mathbf{R}_{12}\mathbf{R}_{22}^T) = 0$$

and using (6)

$$\mathbf{R}_{21} + \mathbf{R}_{22}\mathbb{1}_{N-1} = \mathbf{Q}_{21} + \mathbf{Q}_{22}\mathbb{1}_{N-1} = \mathbb{0}_{(N-1) \times 1}$$

Note also that  $\mathbf{R}_{21} = -\mathbf{R}_{22}\mathbb{1}_{N-1}$  and  $\mathbf{Q}_{21} = -\mathbf{Q}_{22}\mathbb{1}_{N-1}$ . Thus, the blocks

$$\begin{aligned} \Xi &:= N(\mathbf{R}_{21}\mathbf{Q}_{21}^T + \mathbf{R}_{22}\mathbf{Q}_{22}^T) \\ &= N\mathbf{R}_{22}(\mathbb{1}_{N-1}\mathbb{1}_{N-1}^T + \mathbf{I}_{N-1})\mathbf{Q}_{22}^T \end{aligned} \quad (37)$$

and,

$$\begin{aligned} \Xi^T &:= N(\mathbf{Q}_{21}\mathbf{R}_{21}^T + \mathbf{Q}_{22}\mathbf{R}_{22}^T) \\ &= N\mathbf{Q}_{22}(\mathbb{1}_{N-1}\mathbb{1}_{N-1}^T + \mathbf{I}_{N-1})\mathbf{R}_{22}^T \end{aligned} \quad (38)$$

Consequently we can rewrite (36) as in (16). Finally, to prove that  $\Xi\bar{\mathbf{\Lambda}}_2\Xi^T$  is a symmetric matrix we have to show that  $\Xi^T$  is an orthonormal matrix. Then, from (37) we have that

$$\Xi^{-1} = \frac{1}{N}(\mathbf{Q}_{22}^T)^{-1}(\mathbb{1}_{N-1}\mathbb{1}_{N-1}^T + \mathbf{I}_{N-1})^{-1}\mathbf{R}_{22}^{-1}$$

and using property (11) we obtain

$$\Xi^{-1} = N\mathbf{Q}_{22}(\mathbb{1}_{N-1}\mathbb{1}_{N-1}^T + \mathbf{I}_{N-1})\mathbf{R}_{22}^T = \Xi^T$$

which completes the proof.