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# Quantum Rate-Distortion Coding of Relevant Information

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Rate-distortion theory provides bounds for compressing data produced by an information source down to a specified encoding rate that is strictly less than the source's entropy. This necessarily entails some lossiness, or distortion, between the original source data and the best approximation after decompression. The so-called Information Bottleneck Method achieves a lossy compression rate below that of rate-distortion theory, by compressing only relevant information. Which information is relevant is determined by the correlation between the information being compressed and a variable of interest, so-called side information. In this paper an Information Bottleneck Method is introduced for the compression of quantum information. The channel communication picture is used for compression and decompression. A lower bound is derived for the rate of compression using an entanglement assisted quantum channel and the optimum channel achieving this rate for a given input state is characterised. The conceptual difficulties arising due to differences in the mathematical formalism between quantum and classical probability theory are discussed and solutions are presented.

## I. INTRODUCTION

One of the most central results in classical information theory is Shannon's data compression theorem [1] which gives a fundamental limit on lossless compressibility of information. Due to statistical redundancies, information can be compressed at a rate bounded below by the source entropy, such that after decompression the full information is recovered without loss. Rate-distortion theory (RDT) is the branch of information theory that compresses the data produced by an information source down to a specified encoding rate that is strictly less than the source entropy [2]. This necessarily entails some lossiness, or distortion, between the original source data and the best approximation after decompression, according to some distortion measure. RDT is frequently used in multimedia data compression where a large amount of information can be discarded without any noticeable change to a listener or viewer.

Whilst RDT is an important and widely used concept in information theory, there are cases where only part of the information to be compressed is relevant. For instance in speech signal processing one might be interested only in information about the spoken words in audio data. The Information Bottleneck Method (IBM), introduced by Tishby *et al.*, achieves a lossy compression rate even lower than the rate given by RDT by compressing only relevant information [3]. Which information is relevant is determined by the correlation between the information being compressed and a variable of interest. The information to be recovered after decoding is only the relevant part of the source data. For example, one might have access to the transcript of an audio recording

which has an entropy by orders of magnitude lower than the original audio data. This side-information can be used to compress the audio data further than what can be achieved by RDT, without increasing the distortion of the relevant information.

Loss of information in the context of a compression-decompression scheme is mathematically equivalent to transmission of information through a noisy channel. Rather than characterising the information lost by encoding, one characterises the information lost during transmission. The Information Bottleneck Method is formulated as a communication problem with the relevant variable acting as side information. Iterative algorithms to compute the optimum channel achieving the task are also provided in [3].

In this paper we extend the Information Bottleneck Method to the quantum case by considering the transmission of quantum information through a quantum channel with side information. We derive a bound for the rate at which quantum information can be sent through a lossy quantum channel with side information and given distortion. The optimum quantum channel that can achieve this rate is also characterised.

A quantum extension to RDT was introduced by Barnum [4]. However, the results were unsatisfactory, since the bound on the rate was given in terms of coherent information which can be negative. The results were improved and a complete characterisation of quantum channels achieving rate-distortion coding in a number of settings was given by Datta *et al.* [5]. Various settings of quantum RDT in the presence of auxiliary information were discussed in the work of Wilde *et al.* [6]. However, the specific question of transmitting relevant information

asked in the IBM with its choice of distance measure and the specifics of the optimisation problem have not been considered before the present work.

The paper is organised as follows. Section II provides preliminary information and discusses the difficulties arising due to the fundamental structure of quantum information theory as opposed to classical information theory, and how to overcome these difficulties. Section II A gives the lower bound on the rate of information transfer by a quantum channel in the Information Bottleneck setting. In Section III the optimum channel for this protocol is presented as a Lagrangian optimisation problem. We are concluding with some open questions. All proofs as well as numerical computations of the rate functions for some examples are given in the appendix.

## II. THE BASIC CONCEPTS, INTUITIONS AND CHALLENGES

The setting of the classical IBM is as follows. A correlated pair of random variables  $X$  and  $Y$  is given with a joint probability distribution  $P(x, y)$ . The task is to find the optimum channel with input  $X$  and output  $\tilde{X}$ , such that  $\tilde{X}$  retains a fixed amount of correlation,  $C$ , with variable  $Y$ . The amount of correlation is quantified by the Shannon mutual information  $I(\tilde{X}; Y) := H(\tilde{X}) + H(Y) - H(\tilde{X}Y)$ , where  $H(\cdot)$  is the Shannon entropy. For a successful completion of the IBM task it is required that  $I(\tilde{X}; Y) \geq C$ . Representing the channel by the conditional probability distribution  $P(\tilde{x}|x)$ , one can show that the classical rate of compression for a given minimum amount of correlation  $C$ ,  $R_{cls}(C)$ , is given by

$$R_{cls}(C) = \min_{P(\tilde{x}|x): I(\tilde{X}; Y) \geq C} I(X; \tilde{X}). \quad (1)$$

Notice that in the case of noiseless transmission the random variables  $X$  and  $\tilde{X}$  are identical, the mutual information reduces to the Shannon entropy, and the problem reduces to a rate distortion problem.

The IBM, however, is concerned not with an output distribution close to the input distribution but with an output characterised by its vicinity to some other variable  $Y$ . The task of the IBM is to find the lowest value of  $I(X; \tilde{X})$  such that  $I(\tilde{X}; Y)$  is still above some given threshold. The value of  $I(X; \tilde{X})$  can be reinterpreted as a communication rate, namely, the number of transmitted bits needed to specify an element of  $\tilde{X}$ , per element of  $X$  [3, Sec. 2]. Minimising the mutual information with respect to all channels that satisfy the threshold criterion achieves the task.<sup>1</sup>

The channel that achieves the rate in Eq. (1) can be found by the Lagrangian technique. The Lagrangian is defined as

$$\mathcal{L}_{cls} = I(X; \tilde{X}) - \beta I(\tilde{X}; Y) - \sum_{x, \tilde{x}} \lambda(x) P(\tilde{x}|x), \quad (2)$$

where  $\beta$  is the Lagrange multiplier for the information constraint and  $\lambda(x)$  are the Lagrange multipliers for the normalisation constraint of the conditional distribution  $P(\tilde{x}|x)$ . Taking the derivative of the Lagrangian with respect to the channel and setting it to 0 gives the expression for the channel as

$$P(\tilde{x}|x) = P(\tilde{x}) \frac{e^{-\beta D(P(y|x)||P(y|\tilde{x}))}}{Z}. \quad (3)$$

$D(\cdot||\cdot)$  in the exponent on the right hand side is the Kullback-Leibler divergence of two conditional probability distributions  $P(y|x)$  and  $P(y|\tilde{x})$  and  $Z$  is the normalising factor.

The setting of the quantum IBM is as follows. The input to the channel is the system  $X$  which is in an entangled state  $\rho_{xy}$  with the side information  $Y$ . The channel acts on the  $X$  part of this state,  $\rho_x$ . The output of the quantum channel is the system  $\tilde{X}$  which is also an entangled state with the side information  $Y$ ,  $\rho_{\tilde{x}y}$ . Entanglement of the state  $\rho_{\tilde{x}y}$  is measured by the von Neumann mutual information  $I(\tilde{X}; Y) := S(\tilde{X}) + S(Y) - S(\tilde{X}Y)$  where  $S(\cdot)$  is the von Neumann entropy to base  $e$ . The bottleneck constraint in the quantum case is  $I(\tilde{X}; Y)_{\rho_{\tilde{x}y}} \geq J$ , that is a minimum amount of entanglement  $J$  with system  $Y$ . In Secs. II A, II B and III the quantum equivalent of Eqs. (1)–(3) will be developed and an expression for a lower bound for compression rate of the quantum IBM will be given. Finding the relevant Lagrangian equations with respect to the quantum channel turns out to be non-trivial as discussed below. Details are given in the appendix.

### A. The Rate Function for the Bottleneck Method

In RDT, any rate-distortion  $(r, J)$  where  $r \geq 0$  is the rate and  $J \geq 0$  is the distortion, is called *achievable* if a lossy channel exists such that it can transmit a message, i.e., a state of the given input system, by sending  $r$  bits with at most  $J$  amount of distortion. The rate function,  $R(J)$ , is defined as

$$R(J) := \inf\{r : (r, J) \text{ is achievable}\}. \quad (4)$$

An expression for this rate was found using the quantum Reverse Shannon coding theorem [7]. This theorem states that it is possible to simulate the action of a quantum channel which causes non-zero distortion  $J$  by noiseless classical communication, in the presence of an unlimited resource of shared entanglement. It also provides a protocol that achieves the task. Therefore, if the

<sup>1</sup> Note that the analogy of IBM to RDT is only in spirit. The technical difference is the distortion measure, which is a function on the output alphabet in the case of RDT while it is a function on the probability distributions of the outputs in the case of IBM.

quantum Reverse Shannon theorem holds for a particular situation, it immediately gives an achievable rate. Bennett *et al.* in [7] provide a quantum Reverse Shannon theorem for sending the state of a system  $X$  through a lossy channel with a constant amount of distortion. They show that a rate of  $r = I(X'; \tilde{X})_{\tau_{x'\tilde{x}}}$  can be achieved, where the mutual information is computed using the state

$$\tau_{x'\tilde{x}} := (\mathcal{I}_{x'} \otimes \mathcal{N}_{x \rightarrow \tilde{x}})(\tau_{x'x}). \quad (5)$$

The input of the channel is described by the state  $\rho_x$  with a purification  $\tau_{x'x}$ . In Appendix A it is shown that their protocol satisfies the conditions of the IBM and that the rate  $I(X'; \tilde{X})$  is optimal up to a technical assumption. Therefore the rate function for the quantum IBM is given by

$$R(J) = \min_{\mathcal{N}_{x \rightarrow \tilde{x}}: I(\tilde{X}; Y)_{\rho_{\tilde{x}y}} \geq J} I(X'; \tilde{X})_{\tau_{x'\tilde{x}}}. \quad (6)$$

*Remark 1:* In Eq. (6) the quantum Reverse Shannon theorem is used, which in addition uses shared entanglement (“entanglement assistance”) to generate a protocol that achieves the rate. However, the requirements for the quantum Reverse Shannon theorem are much more stringent than those of the Bottleneck method. Therefore, it might be possible to find a rate function without entanglement assistance. This is still an open problem.

*Remark 2:* Generally, rate distortion functions obtained from the definition in (4) are nonincreasing functions of the distortion, whereas both Eqs. (6) and (1) are nondecreasing functions. This is not a fundamental difference between RDT and the Bottleneck method. It is merely due to the fact that in the IBM the constraint of minimisation is chosen to be the amount of correlation preserved,  $I(\tilde{X}; Y) \geq J$ ; while in RDT, the constraint of minimisation is the average loss of information,  $\langle d(x; \tilde{x}) \rangle \leq D$ , for some fixed  $D$  and some distortion measure  $d(x; \tilde{x})$ .

The formulation of the IBM can easily be changed to using a constraint on the loss of correlation such as  $I(X; Y) - I(\tilde{X}; Y) \leq D$ , in which case the rate function is a non-increasing function of  $D$ . Rather than changing it, the structure of the minimisation constraint is kept in line with the classical IBM. Next, the question of how to perform the channel optimisation using the Lagrangian method will be addressed.

## B. How to perform the channel optimisation

As discussed above, in a lossy compression-decompression protocol the minimisation is performed over all channels satisfying a certain criterion, see Eq. (6). In some cases we might be interested in computing the actual value of the channel, as it is crucial in various tasks in electrical engineering. In the quantum case there are several ways of representing a channel, e.g., the Kraus operators or the Choi-Jamiołkowski representation. It turns out that indeed the most compact

and convenient way to compute the derivatives of the Lagrangian is with respect to the Choi-Jamiołkowski representation defined as

$$\Psi_{x'\tilde{x}} := (\mathcal{I}_{x'} \otimes \mathcal{N}_{x \rightarrow \tilde{x}})(\Phi_{x'x}), \quad (7)$$

where  $\Phi_{x'x} := \sum_{i,j=0}^{d-1} |i\rangle\langle j|_{x'} \otimes |i\rangle\langle j|_x$ . For the rate function given in Eq. (6), one can write the Lagrangian

$$\begin{aligned} \mathcal{L} := & I(X'; \tilde{X})_{\tau_{x'\tilde{x}}} - \beta I(\tilde{X}; Y)_{\rho_{\tilde{x}y}} \\ & - \text{Tr}_{x\tilde{x}}(\Psi_{x\tilde{x}}^{T_x}(\Lambda_x \otimes I_{\tilde{x}})), \end{aligned} \quad (8)$$

where  $\beta$  is the Lagrange multiplier for the constraint of minimisation and the Hermitian operator  $\Lambda_x$  is the Lagrange multiplier to guarantee the normalisation of the channel. The states in Eq. (8) can be written as functions of the Choi-Jamiołkowski state of the channel  $\Psi_{x\tilde{x}}$ . The joint state  $\tau_{x'x}$  in Eq. (8) can be written as

$$\tau_{x'x} = \text{Tr}_x \{ \Psi_{x\tilde{x}}^{T_x} \tau_{x'x} \} = (\rho_{x'}^x \otimes I_{\tilde{x}})^{1/2} \Psi_{x'\tilde{x}}^{T_{x'}} (\rho_{x'}^x \otimes I_{\tilde{x}})^{1/2}, \quad (9)$$

where  $\rho_{x'}^x$  is the same state as  $\rho_x$  acting on the Hilbert space  $\mathcal{H}_{x'}$  of the system  $X'$ .

By similar consideration, one can show that the joint state

$$\rho_{\tilde{x}y} := (\mathcal{N}_{x \rightarrow \tilde{x}} \otimes \mathcal{I}_y)(\rho_{xy}) \quad (10)$$

can be written as

$$\begin{aligned} \rho_{\tilde{x}y} &= \text{Tr}_x \{ \Psi_{x\tilde{x}}^{T_x} \rho_{xy} \} \\ &= \text{Tr}_{x'y'} \left\{ (\rho_{x'y'}^{xy} \otimes I_{\tilde{x}y})^{1/2} (\Psi_{x'\tilde{x}}^{T_{x'}} \otimes \Phi_{y'y'}^{T_{y'}}) (\rho_{x'y'}^{xy} \otimes I_{\tilde{x}y})^{1/2} \right\}. \end{aligned} \quad (11)$$

In the third term of (8) the dependence on the channel state is already explicit. With all the terms in the Lagrangian expressed as functions of the channel state  $\Psi_{x\tilde{x}}$ , the derivative  $\delta\mathcal{L}/\delta\Psi_{x\tilde{x}}^{T_x}$  can be performed using standard techniques of matrix calculus, as shown in Appendix D.

With the technical issues and the proof methods specified, we will now give a formal account of the quantum IBM problem and its solution.

## III. THE OPTIMUM CHANNEL FOR THE BOTTLENECK METHOD

The general protocol of the quantum IBM is illustrated in Fig. 1. The information to be compressed is many independently and identically distributed (i.i.d) copies,  $\rho_x^{\otimes n}$ , of the state of the system  $X$ . Furthermore, every copy  $\rho_x$  is entangled with a system  $Y$  and the entangled state is described by the density operator  $\rho_{xy}$ . The input and output of the protocol share an entangled state  $\Phi_{T_X T_{\tilde{X}}}$ , where the system  $T_X$  is with the input and the system  $T_{\tilde{X}}$  is with the output. As mentioned above, the noisy communication picture is mathematically equivalent to a protocol which first compresses the data, then

transmits them via noiseless channels, and then decompresses the data. The protocol in Fig. 1 is written in the latter picture. Here, the input state  $\rho_x^{\otimes n}$  and the state of half of the entangled pair,  $T_X$ , are acted on by the map  $\mathcal{E}_n := \mathcal{E}_{X^n T_X \rightarrow W}$ , where  $W$  is the output of the noiseless channels, a classical system of size  $\approx e^{nr}$ , where  $r$  is the communication rate. The output of  $W$  and the state of the other half of the entangled pair,  $T_{\tilde{X}}$ , is acted on with the decompression channel  $\mathcal{D}_n := \mathcal{D}_{W T_{\tilde{X}} \rightarrow \tilde{X}^n}$ . Recall that the task is to compress the state of the system  $X$ , such that the *relevant* information in that system is recovered to a good degree after decompression. Defining the relevant information in  $X$  as the information that this system provides about the system  $Y$ , the mutual information between the two systems is a measure of the relevant information contained in  $X$ . Therefore, the output of the compression-decompression scheme has to be such that the correlation of system  $X$  with the system  $Y$  does not decrease below a certain threshold.

As discussed in Section II A and shown in Appendix A the rate at which information can be sent through this compression-decompression channel with input  $X$  and output  $\tilde{X}$  is given by

$$R(J) = \min_{\mathcal{N}_{x \rightarrow \tilde{x}}: I(\tilde{X}; Y)_{\rho_{\tilde{x}y}} \geq J} I(X'; \tilde{X})_{\tau_{x'\tilde{x}}}. \quad (12)$$

The minimisation runs over all channels  $\mathcal{N}_{x \rightarrow \tilde{x}}$  which meet the constraint  $I(\tilde{X}; Y)_{\rho_{\tilde{x}y}} \geq J$ . Eq. (12) will be proven under the assumption that the right hand side is a convex function of  $J$ , for which we present numerical evidence in Appendix C.

Having found the compression rate for the quantum IBM, an important task left is to give a method for computing the channel that achieves this rate. It can be found using the Lagrangian method (for details, see Appendix D): Define

$$D_{x\tilde{x}}^{\beta y} := \beta \log \rho_{\tilde{x}} \otimes I_x - \beta \text{Tr}_y \{ (\rho_x^{-1/2} \rho_{xy} \rho_x^{-1/2} (\log \rho_{\tilde{x}y} \otimes I_x)) \}, \quad (13)$$

and

$$\tilde{\Lambda}_x := \rho_x^{-1/2} \Lambda_x \rho_x^{-1/2}. \quad (14)$$

Taking the derivative of the Lagrangian with respect to the channel and setting it to zero gives the channel as

$$\Psi_{x\tilde{x}}^{T_x} = (\rho_x \otimes I_{\tilde{x}})^{-1/2} e^{\log \rho_{\tilde{x}} \otimes I_x - D_{x\tilde{x}}^{\beta y} + \tilde{\Lambda}_x \otimes I_{\tilde{x}}} (\rho_x \otimes I_{\tilde{x}})^{-1/2}. \quad (15)$$

Note that this determines  $\Psi_{x\tilde{x}}$  implicitly since it also appears on the right and side of this equation in  $\rho_{\tilde{x}}$  and in the definition of  $D_{x\tilde{x}}^{\beta y}$  (for details, see Appendix C).

Eq. (15) reduces to its classical counterpart in Eq. (3) in the case of diagonal density operators. To see this, consider the diagonal case where the density operators reduce to probability distributions. From Eq. (15) it fol-

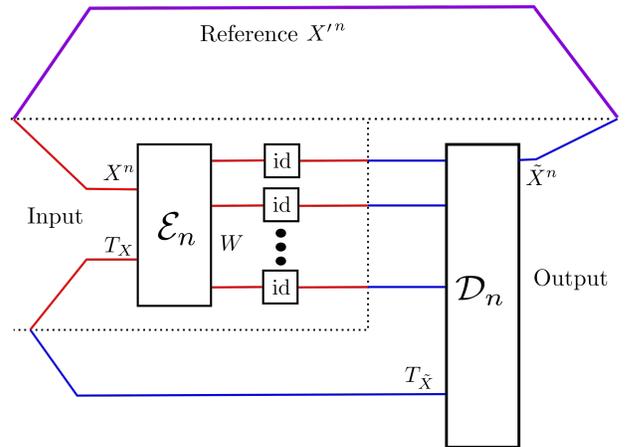


FIG. 1: The protocol for entanglement assisted quantum information Bottleneck method. The compression encoding  $\mathcal{E}_n$  acts on  $n$  copies of the state of the system  $X$ . The  $n$  copies of the state of the system  $Y$  are used as the *relevance variable* and are entangled with the system  $X$ . However, as the system  $Y$  is not transmitted through the channel, we do not depict the system in the protocol.  $T_X$  and  $T_{\tilde{X}}$  are the two entangled systems that the input and the output of the protocol share to assist the transmission. The compressed states are transferred to the output section for decompression,  $\mathcal{D}_n$ , via the noiseless channels “id”. The reference, labeled  $X'$ , is what purifies the state  $\rho_x$ . The state  $\rho_x$  is the reduced density operator of the given initial state  $\rho_{xy}$ .

lows that

$$P(\tilde{x}|x) = \frac{1}{P(x)} \exp \left\{ \log P(\tilde{x}) - \beta (\log P(\tilde{x}) - \sum_y P(y|x) \log P(\tilde{x}y)) + \frac{\lambda(x)}{P(x)} \right\}, \quad (16)$$

with  $\lambda(x)$  being the same normalisation Lagrange multiplier as in Eq. (2). Notice that since  $H(Y|X = x) = \sum_y P(y|x) \log P(y|x)$  depends only on  $x$ , but not on  $\tilde{x}$ , it can be absorbed into  $\lambda(x)$ . Defining

$$\tilde{\lambda}(x) := \frac{\lambda(x)}{P(x)} - \beta H(Y|X = x) - \log P(x), \quad (17)$$

Eq. (16) becomes

$$P(\tilde{x}|x) = P(\tilde{x}) e^{-\beta D(P(y|x)||P(y|\tilde{x})) + \tilde{\lambda}(x)}, \quad (18)$$

which is the same classical channel as Eq. (3), with all the extra terms being absorbed into the normalisation factor. This also shows that  $D_{x\tilde{x}}^{\beta y}$  is a quantum operator corresponding to the distance measure in the classical Bottleneck method. The idea of distance operators has been used in a number of quantum information processing tasks [8, 9], however the  $D_{x\tilde{x}}^{\beta y}$  is particular to the present setting. Eq. (15) can be used in principle to compute numerical values of quantum channels using iterative algorithms, akin to their classical counterparts by methods introduced by Blahut and Arimoto [10, 11].

#### IV. CONCLUSION AND OUTLOOK

This paper introduced the quantum extension of the Information Bottleneck Method. This method compresses only the relevant information with respect to some given variable. We derive a lower bound to the compression rate of relevant quantum information. The problem was formulated as a communication channel problem and the rate was shown to be achievable by explicitly constructing a channel achieving it. Just like in the classical case, the compression rate of the quantum Information Bottleneck Method is lower than that given by quantum rate distortion theory. Several conceptual issues arose from the structural differences between the mathematical formalism of quantum theory and classical probability theory which were discussed and solutions were presented.

Some open questions remain. Our proof of Eq. (12) relied on a technical assumption (convexity of the expression on the right hand side of (12) in  $J$ ). While this seems to be fulfilled in examples (cf. Appendix C), a proof of this property is currently missing.

In Appendix C a simple algorithm is used to compute the optimum channel and thus the rate function  $R(J)$  in

low dimensional systems; but for systems of realistic size a more efficient algorithm would be required.

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#### Appendix A: Proof of the Compression Rate

Here we prove that the rate-distortion function for the Bottleneck method is given by the expression in Eq. (6). Consider the general protocol illustrated in Fig. 1. The information to be compressed in system  $X$  is represented by many independently and identically distributed (i.i.d.) copies,  $\rho_x^{\otimes n}$ , of the density operator  $\rho_x$ , with a purification  $\tau_{x'x}$ . The input, however, is entangled with a system,  $Y$ , which contains our relevant information and the entangled state is denoted  $\rho_{xy}$ . The input and output share an entangled state  $\Phi_{T_X T_{\tilde{X}}}$ , where the system  $T_X$  is with the input and the system  $T_{\tilde{X}}$  is with the output. We act on the input state,  $\rho_x^{\otimes n}$ , and the state of half of the entangled pair,  $T_X$ , with the compression map  $\mathcal{E}_n := \mathcal{E}_{X^n T_X \rightarrow W}$ , where  $W$  is the output of the noiseless quantum channels, a classical system of size  $\approx e^{nr}$ , where  $r$  is the communication rate. Then, we act on  $W$  and the state of the other half of the entangled pair,  $T_{\tilde{X}}$ , with the decompression channel  $\mathcal{D}_n := \mathcal{D}_{W T_{\tilde{X}} \rightarrow \tilde{X}^n}$ . Consider the overall action of the compression-decompression channel  $\mathcal{F}_n := \mathcal{D}_n \circ \mathcal{E}_n$ , and the marginal operation defined by

$$\begin{aligned} \mathcal{F}_n^{(i)}(\xi_x) := & \\ & Tr_{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{i-1}, \tilde{x}_{i+1}, \dots, \tilde{x}_n} [\mathcal{F}_n(\rho_x^{\otimes(i-1)} \otimes \xi_x \otimes \rho_x^{\otimes(n-i)})]. \end{aligned} \quad (\text{A1})$$

Then for any  $i$  we can define

$$\sigma_{\tilde{x}_i y_i} := \mathcal{F}_n^{(i)} \otimes \mathcal{I}_y(\rho_{xy}), \quad (\text{A2})$$

where  $\mathcal{I}_y$  is the identity channel acting on the system  $Y$ , and its partial traces are

$$\sigma_{\tilde{x}_i} = \mathcal{F}_n^{(i)}(\rho_x), \quad (\text{A3})$$

$$\sigma_{y_i} = \rho_y. \quad (\text{A4})$$

Using Eqs. (A2)–(A4) we can define

$$I_i(\tilde{X}; Y) := S(\sigma_{\tilde{x}_i}) + S(\sigma_{y_i}) - S(\sigma_{\tilde{x}_i y_i}). \quad (\text{A5})$$

The mutual information in Eq. (A5), averaged over the many uses of the channel is

$$\overline{I_i(\tilde{X}; Y)} = \frac{1}{n} \sum_{i=1}^n \left( S(\sigma_{\tilde{x}_i}) + S(\sigma_{y_i}) - S(\sigma_{\tilde{x}_i y_i}) \right), \quad (\text{A6})$$

where the bar denotes averaging over  $i$ . The quantum operation  $\mathcal{F}_n$  defines an  $(n, r)$  quantum rate-distortion code. In the case of the Bottleneck method, for any  $r, J \geq 0$ , the pair  $(r, J)$  is called an *achievable* rate-distortion pair, if there exists a sequence of  $(n, r)$  quantum rate-distortion codes such that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} I_i(\tilde{X}; Y) > J. \quad (\text{A7})$$

Notice that Eq. (A7) is the quantum counterpart of the optimization constraint in the classical Bottleneck method given in Eq. (1). The rate function is then defined as

$$R(J) := \inf\{r : (r, J) \text{ is achievable}\}. \quad (\text{A8})$$

Given the situation described above, we want to prove that the rate function  $R(J)$  for the Bottleneck method is given by Eq. (6). To that end, we temporarily denote the right-hand side as

$$M(J) := \min_{\mathcal{N}_{x \rightarrow \tilde{x} : I(\tilde{X}; Y)_{\rho_{\tilde{x}y}} \geq J}} I(X'; \tilde{X})_{\tau_{x'\tilde{x}}}. \quad (\text{A9})$$

We conjecture that this function is convex in  $J$ . While we are currently unable to prove this property in general, we have verified it in numerical examples, which are discussed in Appendix C. We will base our following result on this assumption.

We need to show achievability of the rate ( $R(J) \leq M(J)$ ) as well as optimality ( $R(J) \geq M(J)$ ). We will consider optimality first. Let  $(r, J)$  be an achievable rate-distortion pair and  $\mathcal{F}_n$  a corresponding sequence of codes.

We have for large  $n$ ,

$$\begin{aligned} nr &\geq S(W)_{\tau_{x'x}} \\ &\geq S(W|T_{\tilde{X}})_{\tau_{x'x}} \\ &\geq S(W|T_{\tilde{X}})_{\tau_{x'x}} - S(W|X'^n T_{\tilde{X}})_{\tau_{x'x}} \\ &= I(W; X'^n | T_{\tilde{X}})_{\tau_{x'x}} \\ &= I(W; X'^n | T_{\tilde{X}})_{\tau_{x'x}} + I(X'^n; T_{\tilde{X}})_{\tau_{x'x}} \\ &= I(W T_{\tilde{X}}; X'^n)_{\tau_{x'x}} \\ &\geq I(\tilde{X}^n; X'^n)_{\tau_{x'\tilde{x}}} \\ &\geq \sum_{i=1}^n I(\tilde{X}_i; X'_i)_{\tau_{x'\tilde{x}}} \\ &\geq \sum_{i=1}^n M(I_i(\tilde{X}; Y))_{\rho_{\tilde{x}y}} \\ &= n \sum_{i=1}^n \frac{1}{n} M(I_i(\tilde{X}; Y))_{\rho_{\tilde{x}y}} \\ &\geq n M\left(\sum_{i=1}^n \frac{1}{n} I_i(\tilde{X}; Y)\right)_{\rho_{\tilde{x}y}} \\ &\geq n M(J). \end{aligned} \quad (\text{A10})$$

The first inequality follows from the fact that the entropy of uniform distribution,  $nr$ , is the upper bound of  $S(W)$ . The second inequality follows because entropy is nondecreasing under conditioning. The third inequality follows because the state of the system  $W X'^n T_{\tilde{X}}$  is separable with respect to the classical system  $W$  and therefore  $S(W|X'^n T_{\tilde{X}}) \geq 0$  [13, footnote 10]. The first equality follows from the definition of mutual information. The second equality follows since the state  $\Phi_{T_X T_{\tilde{X}}}$  is in a tensor product with the state of the remaining input and therefore  $I(X'^n; T_{\tilde{X}}) = 0$ . In the third equality we use the chain rule for mutual information. The fourth inequality follows from the data processing inequality. The fifth inequality follows from superadditivity of quantum mutual information. The sixth inequality follows from the definition of  $M(J)$ , where we use the channel  $\mathcal{N}_{x \rightarrow \tilde{x}} = \mathcal{F}_n^{(i)}$ . In the seventh inequality we have used convexity of  $M$ . Finally the last inequality follows for large  $n$  from Eq. (A7), using that  $M(J)$  is a nondecreasing function of  $J$ . The rate function is nondecreasing in  $J$ , because for any  $J' > J$  the domain of minimisation in Eq. (6) becomes smaller, which implies that the rate function can only become larger.—Now, since  $(r, J)$  was arbitrary, (A10) implies  $R(J) \geq M(J)$ .

Achievability follows from an application of the quantum reverse Shannon theorem. Specifically, fix  $J > 0$  and let  $\mathcal{N}_{x \rightarrow \tilde{x}}$  be the optimum channel at which the minimum in Eq. (A9) is attained. For a given  $\epsilon > 0$ , we use the quantum reverse Shannon theorem in the form of [7, Theorem 3(a)] to construct a sequence of channels  $\mathcal{F}_n = \mathcal{D}_n \circ \mathcal{E}_n$  with the following properties:

- (a) They are close to an  $n$ -fold application of  $\mathcal{N}_{x \rightarrow \tilde{x}}$ , in the sense that

$$\|\sigma_{x'^n \tilde{x}^n} - \tau_{x'\tilde{x}}^{\otimes n}\|_1 \leq \epsilon, \quad (\text{A11})$$

where

$$\tau_{x'\bar{x}} := (\mathcal{I}_{x'} \otimes \mathcal{N}_{x \rightarrow \bar{x}})(\tau_{x'x}) \quad (\text{A12})$$

is the effect of the channel  $\mathcal{N}_{x \rightarrow \bar{x}}$  and

$$\sigma_{x'^n \bar{x}^n} := \mathcal{I}_{x'^n} \otimes \mathcal{F}_n(\tau_{x'x}^{\otimes n}). \quad (\text{A13})$$

(b) They use a classical communication rate of (at most)  $r = I(X'; \tilde{X})_{\mathcal{N}_{x \rightarrow \bar{x}}, \tau_{x'\bar{x}}}$ , cf. the resource inequality in Eq. (17) and Fig. 2 in [7].

From (a) and the fact that  $I(\tilde{X}; Y) \geq J$  for the channel  $\mathcal{N}_{x \rightarrow \bar{x}}$ , one can show (see Lemma 2 in Appendix B) that Eq. (A7) is fulfilled with  $J - \delta$  instead of  $J$ , where  $\delta \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Hence,  $(r, J - \delta)$  is achievable and  $R(J - \delta) \leq M(J)$ .

We have now shown that

$$\forall \delta > 0 : M(J - \delta) \leq R(J - \delta) \leq M(J). \quad (\text{A14})$$

From this it follows that

$$\lim_{\delta \searrow 0} R(J - \delta) = M(J). \quad (\text{A15})$$

Since  $R$  is nondecreasing by (A7) and (A8) and  $M$  is continuous (a property that follows from convexity), then  $M(J) = R(J)$  for all  $J$ . Let us summarize the result:

**Proposition 1.** *Suppose that for a given input state  $\rho_{xy}$ , the function  $M(J)$  is convex in  $J$ . Then  $M(J) = R(J)$ .*

### Appendix B: Lemmas for the proof of the Bottleneck rate function

The following lemma is relevant for the proof of achievability of the communication rate. It has the same application as in Theorem 19 of [14] and Lemma 1 of [5], but has been adapted to the distortion criterion for the quantum Bottleneck method.

**Lemma 2.** *Let  $\eta(\lambda) := -\lambda \log \lambda$ . There exists a constant  $k > 0$  depending only on the dimension of  $\mathcal{H}_{\bar{x}y}$  such that the following holds:*

*Let  $0 < J \leq I(X; Y)$  be fixed. Let a quantum channel  $\mathcal{N}_{x \rightarrow \bar{x}}$  be such that if we apply the channel to the system  $X$  and an identity channel  $\mathcal{I}_y$  on the system  $Y$  the effect will meet the condition  $I(\tilde{X}; Y)_{\rho_{\bar{x}y}} \geq J$ , where  $\rho_{\bar{x}y}$  is given by (10). Further, let  $\mathcal{F}_n$  be a sequence of quantum channels from the space of density matrices  $\mathfrak{D}(\mathcal{H}_{x'x}^{\otimes n})$  to  $\mathfrak{D}(\mathcal{H}_{x'\bar{x}}^{\otimes n})$  such that*

$$\|\sigma_{x'^n \bar{x}^n} - \tau_{x'\bar{x}}^{\otimes n}\|_1 \leq \epsilon \quad (\text{B1})$$

*for some  $0 \leq \epsilon < \frac{1}{e}$  and large enough  $n$ , where  $\tau_{x'\bar{x}}$  and  $\sigma_{x'^n \bar{x}^n}$  are given by (A12) and (A13), respectively.*

*Then, for large enough  $n$  and  $\delta := k\eta(\epsilon)$  we have*

$$\overline{I_i(\tilde{X}; Y)} \geq J - \delta. \quad (\text{B2})$$

*Proof.* Adding and subtracting  $I(\tilde{X}; Y)$  to the left hand side of Eq. (B2) and using the triangle inequality, we obtain

$$\begin{aligned} \overline{I_i(\tilde{X}; Y)} &= |I(\tilde{X}; Y) - (I(\tilde{X}; Y) - \overline{I_i(\tilde{X}; Y)})| \\ &\geq |I(\tilde{X}; Y)| - |I(\tilde{X}; Y) - \overline{I_i(\tilde{X}; Y)}|. \end{aligned} \quad (\text{B3})$$

Since we assumed that  $I(\tilde{X}; Y) \geq J$ , what we need to show is that

$$|I(\tilde{X}; Y) - \overline{I_i(\tilde{X}; Y)}| \leq k\eta(\epsilon) =: \delta \quad (\text{B4})$$

with some  $k > 0$  and  $\eta$  as above. Inserting  $\bar{I}_i$  given in Eq. (A6), we have

$$\begin{aligned} |I(\tilde{X}; Y) - \overline{I_i(\tilde{X}; Y)}| &= \\ &\left| \frac{1}{n} n \left( S(\rho_{\bar{x}}) + S(\rho_y) - S(\rho_{\bar{x}y}) \right) \right. \\ &\left. - \frac{1}{n} \sum_{i=1}^n \left( S(\sigma_{\bar{x}_i}) + S(\sigma_{y_i}) - S(\sigma_{\bar{x}_i y_i}) \right) \right|. \end{aligned} \quad (\text{B5})$$

Hence, it suffices to show that for all  $1 \leq i \leq n$ ,

$$\begin{aligned} |S(\rho_{\bar{x}}) + S(\rho_y) - S(\rho_{\bar{x}y}) - S(\sigma_{\bar{x}_i}) \\ - S(\sigma_{y_i}) + S(\sigma_{\bar{x}_i y_i})| \leq k\eta(\epsilon). \end{aligned} \quad (\text{B6})$$

In particular, we will prove bounds of the above type on  $|S(\rho_{\bar{x}}) - S(\sigma_{\bar{x}_i})|$ ,  $|S(\rho_{\bar{x}y}) - S(\sigma_{\bar{x}_i y_i})|$  and  $|S(\rho_y) - S(\sigma_{y_i})|$  for all  $i$ . We start with the first of these. To prove this inequality, we recall that  $\tau_{x'x}$  is a purification of  $\rho_x$ . Now let  $\rho_{x'y'xy}$  be a purification of  $\rho_{xy}$ , then there is a Hilbert space  $\hat{\mathcal{H}}$  and a unitary  $U : \mathcal{H}_{x'} \oplus \hat{\mathcal{H}} \rightarrow \mathcal{H}_{x'y'xy}$  such that

$$(I_x \otimes U)\tau_{x'x}(I_x \otimes U)^\dagger = \rho_{x'y'xy}, \quad (\text{B7})$$

where  $\tau_{x'x}$  is extended to the orthogonal complement of  $\mathcal{H}_{x'}$  by zeros. Then, (B1) implies

$$\|\rho_{x'y'xy}^{\otimes n} - (I_x \otimes U^\dagger)^{\otimes n} \sigma_{x'^n \bar{x}^n} (I_x \otimes U)^{\otimes n}\|_1 \leq \epsilon, \quad (\text{B8})$$

where  $\rho_{x'y'xy} = (\mathcal{N}_{x \rightarrow \bar{x}} \otimes \mathcal{I}_{x'y'}) (\rho_{x'y'xy})$ . Further, one computes that

$$(I_x \otimes U^\dagger)^{\otimes n} \sigma_{x'^n \bar{x}^n} (I_x \otimes U)^{\otimes n} = \sigma_{x'^n y'^n \bar{x}^n y^n}, \quad (\text{B9})$$

$$\sigma_{x'^n y'^n \bar{x}^n y^n} := (\mathcal{I}_{x'y'^n y^n} \otimes \mathcal{F}_n)(\rho_{x'y'xy}^{\otimes n}). \quad (\text{B10})$$

To summarize, we found that

$$\|\rho_{x'y'xy}^{\otimes n} - \sigma_{x'^n y'^n \bar{x}^n y^n}\|_1 \leq \epsilon. \quad (\text{B11})$$

Using monotonicity of the trace norm under partial trace, we find that

$$\|\rho_{x'y'xy} - \sigma_{x'_i y'_i \bar{x}_i y_i}\|_1 \leq \|\rho_{x'y'xy}^{\otimes n} - \sigma_{x'^n y'^n \bar{x}^n y^n}\|_1. \quad (\text{B12})$$

Moreover,

$$\|\rho_{\bar{x}} - \sigma_{\bar{x}_i}\|_1 \leq \|\rho_{x'y'xy} - \sigma_{x'_i y'_i \bar{x}_i y_i}\|_1. \quad (\text{B13})$$

again using the monotonicity of the trace norm under partial trace. This implies that  $\|\rho_{\tilde{x}} - \sigma_{\tilde{x}_i}\|_1 \leq \epsilon$ .

Now, by Fannes Inequality the following bound holds for all  $1 \leq i \leq n$ :

$$|S(\rho_{\tilde{x}}) - S(\sigma_{\tilde{x}_i})| \leq \log(k') \|\rho_{\tilde{x}} - \sigma_{\tilde{x}_i}\|_1 + \frac{1}{\log(2)} \eta(\|\rho_{\tilde{x}} - \sigma_{\tilde{x}_i}\|_1), \quad (\text{B14})$$

where  $k'$  is the dimension of  $\mathcal{H}_{\tilde{x}}$ . Then, using the bound  $\|\rho_{\tilde{x}} - \sigma_{\tilde{x}_i}\|_1 \leq \epsilon$ , we find for all  $1 \leq i \leq n$ ,

$$|S(\rho_{\tilde{x}}) - S(\sigma_{\tilde{x}_i})| \leq \epsilon \log(k') + \frac{\eta(\epsilon)}{\log(2)} \leq \hat{k}' \eta(\epsilon), \quad (\text{B15})$$

where the last inequality uses the fact that  $\eta(\epsilon) \geq \epsilon$  for  $0 \leq \epsilon < \frac{1}{e}$ , and where  $\hat{k}'$  is defined in terms of the constants in the first inequality, including  $k'$ .

With a similar method, one can prove that for all  $1 \leq i \leq n$ ,

$$|S(\rho_{\tilde{x}y}) - S(\sigma_{\tilde{x}_i y_i})| \leq \hat{k}'' \eta(\epsilon), \quad (\text{B16})$$

$$|S(\rho_y) - S(\sigma_{y_i})| \leq \hat{k}''' \eta(\epsilon), \quad (\text{B17})$$

where  $\hat{k}''$ ,  $\hat{k}'''$  also depend on the dimensions of  $\mathcal{H}_{\tilde{x}y}$  and  $\mathcal{H}_y$ , respectively.

Combining Eq. (B15), (B16) and (B17) we find for all  $1 \leq i \leq n$  that Eq. (B6) holds for a constant  $k$  that includes the constants in the three estimates above and depends only on the dimension of  $\mathcal{H}_{\tilde{x}y}$ .

Hence, we obtain for large enough  $n$  that  $I_i(\tilde{X}; Y) \geq J - k\eta(\epsilon)$ .  $\square$

### Appendix C: Numerical examples

The aim of this appendix is to compute the communication rate as a function of  $J$  for some examples, using a numerical optimisation algorithm for evaluating the right-hand side of Eq. (12). In particular, in all these examples the rate function turns out to be convex in  $J$ . Consider the following normalized version of the rate

$$\hat{R}(J) := \min_{\mathcal{N}_{x \rightarrow \tilde{x}: \frac{I(\tilde{X}; Y) \rho_{\tilde{x}y}}{I(X; Y) \rho_{xy}} \geq J}} \frac{I(X'; \tilde{X})_{\tau_{x' \tilde{x}}}}{I(X'; X)_{\tau_{x' x}}}, \quad (\text{C1})$$

where now  $0 < J < 1$ . In the following examples the systems  $X$  and  $Y$  are described by two-dimensional Hilbert spaces spanned by the basis  $|\uparrow\rangle, |\downarrow\rangle$ .

To find the optimum numerically, a simple random search algorithm is used [12]. It initially chooses a number of channels at random (in terms of their Krauss operators) and computes the related mutual information, then randomly varies those channels with the lowest  $I(X', \tilde{X})$  further until a stable optimum is reached.

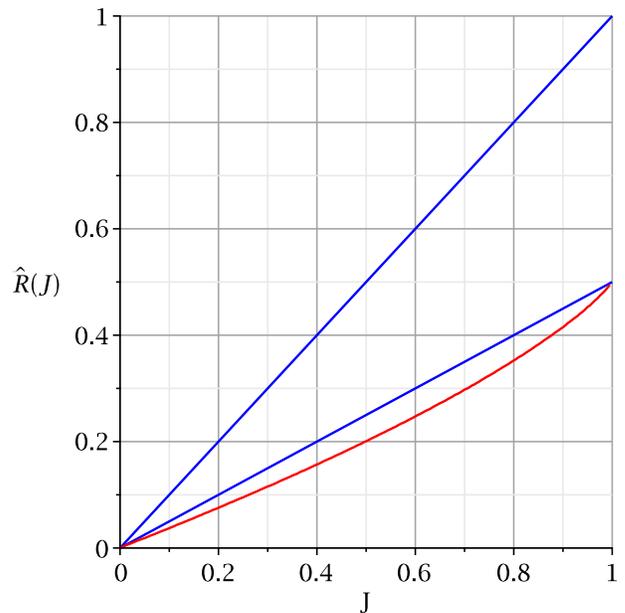


FIG. 2: The function  $\hat{R}(J)$  (in red) for the initial state  $\rho_{xy}^{(1)}$  with  $(p_1, p_2, p_3, p_4) = (0.1, 0.2, 0.3, 0.4)$ .

This algorithm is applied to three classes of input states  $\rho_{xy}$ : The first example is a “classical” state, i.e., a state without entanglement between the systems  $X$  and  $Y$ , given by the density matrix

$$\rho_{xy}^{(1)} := p_1 |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + p_2 |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + p_3 |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + p_4 |\downarrow\downarrow\rangle\langle\downarrow\downarrow|,$$

where  $p_1, p_2, p_3, p_4$  are nonnegative numbers with  $p_1 + p_2 + p_3 + p_4 = 1$ . The second example is a state with entanglement between  $X$  and  $Y$ , namely,

$$\rho_{xy}^{(2)} = \frac{1}{2} |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{4} |\uparrow\uparrow\rangle\langle\downarrow\downarrow| + \frac{1}{4} |\downarrow\downarrow\rangle\langle\uparrow\uparrow| + \frac{1}{2} |\downarrow\downarrow\rangle\langle\downarrow\downarrow|.$$

Finally, the third example is again a state with entanglement defined as

$$\rho_{xy}^{(3)} = p_1 |v\rangle\langle v| + p_2 |w\rangle\langle w|$$

with the normalized vectors

$$v = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle), \quad w = |\downarrow\downarrow\rangle$$

and nonnegative numbers  $p_1, p_2$  with  $p_1 + p_2 = 1$ . The plots presented in Figs. 2–5 show the rate as a function of distortion. The blue lines correspond to the curves  $\hat{R}(J) = J$  and  $\hat{R}(J) = \frac{1}{2}J$ , and are introduced for comparison with the actual result (red line).

The four plots show that the function  $\hat{R}(J)$  given by (C1) is indeed a convex function in  $J$  for the specific choices of initial state  $\rho_{xy}$  made, and within the limits of the numerical approximation. Note that in Figs. 3–5, the graph does not appear to be differentiable at the point  $\hat{R} = \frac{1}{2}$ . This seems to be a common feature of

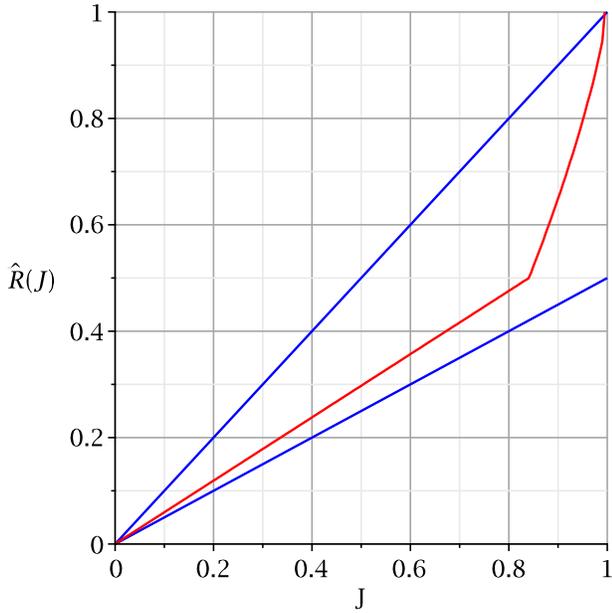


FIG. 3: The function  $\hat{R}(J)$  (in red) for the initial state  $\rho_{xy}^{(2)}$ .

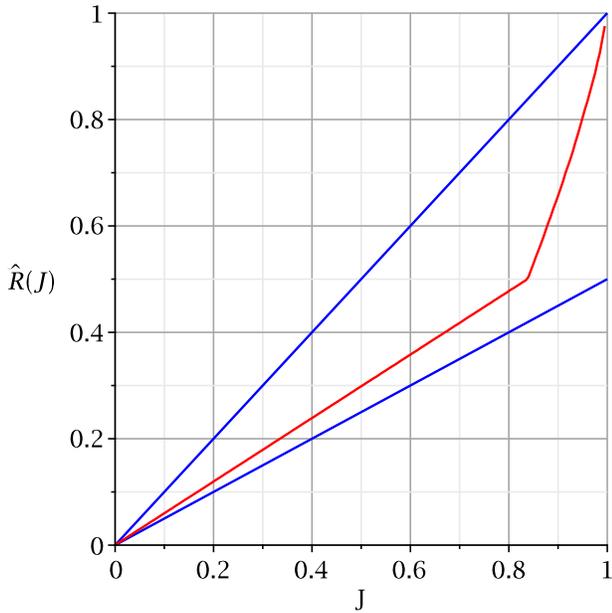


FIG. 4: The function  $\hat{R}(J)$  (in red) for the initial state  $\rho_{xy}^{(3)}$  with  $(p_1, p_2) = (0.4, 0.6)$ .

the examples  $\rho_{xy}^{(2)}, \rho_{xy}^{(3)}$ , but we do not currently have an analytic explanation for this behaviour. Note that in Fig. 2, one has  $M(1) = \frac{1}{2}$ , while in Figs. 3–5, one has  $M(1) = 1$ . In other words, in the case of a “classical” (non entangled) state  $\rho_{xy}$ , there is a channel such that  $I(\tilde{X}; Y) = I(X; Y)$  and  $I(\tilde{X}; X') = \frac{1}{2}I(X; X')$ . To obtain an analytic expression of this channel we proceed as follows.

The initial state is  $\rho_{xy}^{(1)}$ . Then,  $\rho_x^{(1)} = \text{Tr}_y \rho_{xy}^{(1)} = (p_1 + p_3)|\uparrow\rangle\langle\uparrow| + (p_2 + p_4)|\downarrow\rangle\langle\downarrow|$  and a purification is given by

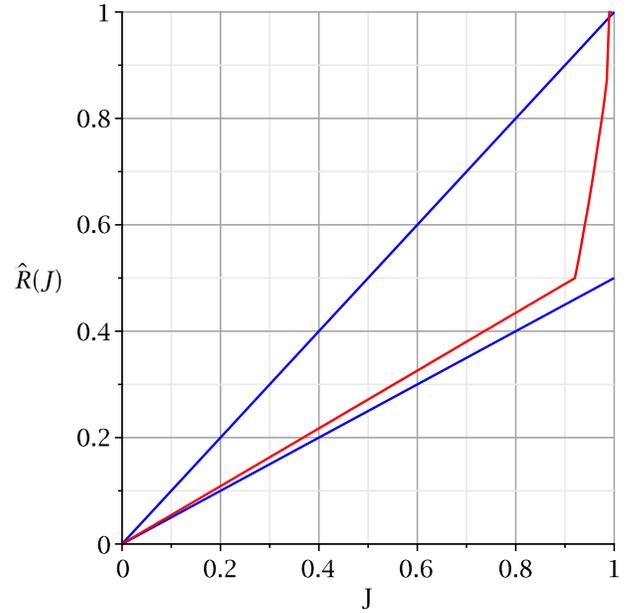


FIG. 5: The function  $\hat{R}(J)$  (in red) for the initial state  $\rho_{xy}^{(3)}$  with  $(p_1, p_2) = (0.2, 0.8)$ .

$\tau_{x'x}^{(1)} = |w\rangle\langle w|$  with  $|w\rangle = \sqrt{p_1 + p_3}|\uparrow\uparrow\rangle + \sqrt{p_2 + p_4}|\downarrow\downarrow\rangle$ . Our ansatz for the channel is

$$\mathcal{N}_{x \rightarrow \tilde{x}}(\rho) = \frac{1}{2} \sum_{i=1}^2 K_i \rho K_i^\dagger \quad (\text{C2})$$

with  $K_1 := |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$  and  $K_2 := |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$ . Applying this channel to  $\rho_{xy}^{(1)}$  and  $\tau_{x'x}^{(1)}$ , we obtain  $\rho_{\tilde{x}y}^{(1)} = \rho_{xy}^{(1)}$  and

$$\tau_{x'\tilde{x}}^{(1)} = (p_1 + p_3)|\uparrow\uparrow\rangle\langle\uparrow\uparrow|_{x'\tilde{x}} + (p_2 + p_4)|\downarrow\downarrow\rangle\langle\downarrow\downarrow|_{x'\tilde{x}}.$$

Note that here  $\tau_{\tilde{x}}^{(1)} = \tau_{x'}^{(1)} = \tau_x^{(1)} = \rho_x^{(1)}$  as matrices, but they act on different Hilbert spaces. With this choice of channel and initial state it is clear that  $I(\tilde{X}; Y)_{\rho_{\tilde{x}y}^{(1)}} = I(X; Y)_{\rho_{xy}^{(1)}}$ . In order to show that  $I(\tilde{X}; X')_{\tau_{x'\tilde{x}}^{(1)}} = \frac{1}{2}I(X; X')_{\tau_{x'x}^{(1)}}$  we compute the von Neumann entropies in  $I(\tilde{X}; X') = S(\tilde{X}) + S(X') - S(\tilde{X}X')$  and in  $I(X; X') = S(X) + S(X') - S(XX')$  using the fact that  $S(XX') = 0$ , since  $\tau_{x'x}^{(1)}$  is pure, and that  $\rho_x^{(1)}, \tau_{x'\tilde{x}}^{(1)}$  are diagonal. After evaluating the matrix functions, we obtain

$$\begin{aligned} I(\tilde{X}; X') &= -(p_1 + p_3) \log(p_1 + p_3) - (p_2 + p_4) \log(p_2 + p_4) \\ &= \frac{1}{2} I(X; X') \end{aligned}$$

as desired. Therefore,  $\hat{R}(1) \leq 1/2$  for this class of states, consistent with the graph.

This somewhat unexpected feature may be understood as follows: In order to transmit the  $X$  part of the non-entangled state  $\rho_{xy}^{(1)}$  perfectly, a *classical* channel of 1

bit capacity is sufficient. By the usual quantum teleportation result, this corresponds to a quantum channel of only  $\frac{1}{2}$  qbit capacity, if shared entanglement is available in abundance.

In this appendix a simple algorithm was used to compute the optimum channel and thus the rate function  $R(J)$  in low dimensional systems; but for systems of realistic size a more efficient algorithm would be required. One way forward is as follows. To find the optimum channel, Eq. (15) needs to be solved iteratively for  $\Psi_{x\tilde{x}}$ . Note that the unknown Lagrange multiplier  $\Lambda_x$ , which is associated with the normalization constraint, is still contained in this equation. An algorithm that recursively computes the channel might work as follows. Starting with a guess for the channel  $\Psi_{x\tilde{x}}$  and normalizing it, this guess is inserted into Eq. (D15) to compute a self-consistent value for  $\Lambda_x$ . This would allow to compute  $\rho_{\tilde{x}}$  and  $\rho_{\tilde{x}y}$  from the channel, and hence give an approximation for all quantities that enter the right hand side of Eq. (15). Thus a new approximation for the left-hand side is obtained, i.e., the channel  $\Psi_{x\tilde{x}}$ . When this procedure is repeated iteratively, an optimal channel  $\Psi_{x\tilde{x}}$  is obtained at a given value for  $\beta$ . Repeating this procedure for different values of  $\beta$  and optimizing under the constraint  $I(\tilde{X}; Y)_{\rho_{\tilde{x}y}} \geq J$  yields the minimum in (6).

#### Appendix D: The Optimal Map

In this appendix we compute the derivative of the following Lagrangian with respect to the channel that transforms system  $X$  to  $\tilde{X}$ , described in Section II. Setting the derivative equal to zero, we find the optimal channel. We use the Lagrangian in Eq. (8),

$$\begin{aligned} \mathcal{L} := & I(X'; \tilde{X})_{\tau_{x'\tilde{x}}} - \beta I(\tilde{X}; Y)_{\rho_{\tilde{x}y}} \\ & - \text{Tr}_{x\tilde{x}}(\Psi_{x\tilde{x}}^{T_x}(\Lambda_x \otimes I_{\tilde{x}})). \end{aligned} \quad (\text{D1})$$

We use the Choi-Jamiołkowski representation

$$\Psi_{x'\tilde{x}} := (\mathcal{I}_{x'} \otimes \mathcal{N}_{x \rightarrow \tilde{x}})(\Phi_{x'x}) \quad (\text{D2})$$

of the channel in order to compute the derivative of the Lagrangian, where  $\Phi_{x'x} := \sum_{i,j=0}^{d-1} |i\rangle\langle j|_{x'} \otimes |i\rangle\langle j|_x$  is the Choi-Jamiołkowski matrix corresponding to the identity channel from the Hilbert space  $\mathcal{H}_{x'}$  to  $\mathcal{H}_x$ , and  $\mathcal{N}_{x \rightarrow \tilde{x}}$  is the channel that simulates the compression-decompression process.  $\Lambda_x$ , an operator on the Hilbert space  $\mathcal{H}_x$ , is the Lagrange multiplier introduced for the normalisation of the channel  $\Psi_{x\tilde{x}}^{T_x}$ . Considering the definition of the mutual information, to compute the derivative  $\frac{\delta \mathcal{L}}{\delta \Psi_{x\tilde{x}}^{T_x}}$  of the Lagrangian, we need to compute the follow-

ing derivatives,

$$\frac{\delta S(X')_{\tau_{x'\tilde{x}}}}{\delta \Psi_{x\tilde{x}}^{T_x}}, \quad \frac{\delta S(Y)_{\rho_{\tilde{x}y}}}{\delta \Psi_{x\tilde{x}}^{T_x}}, \quad (\text{D3})$$

$$\begin{aligned} & \frac{\delta S(\tilde{X})_{\tau_{x'\tilde{x}}}}{\delta \Psi_{x\tilde{x}}^{T_x}}, \quad \frac{\delta S(\tilde{X})_{\rho_{\tilde{x}y}}}{\delta \Psi_{x\tilde{x}}^{T_x}}, \quad \frac{\delta S(X'\tilde{X})_{\tau_{x'\tilde{x}}}}{\delta \Psi_{x\tilde{x}}^{T_x}}, \\ & \frac{\delta S(\tilde{X}Y)_{\rho_{\tilde{x}y}}}{\delta \Psi_{x\tilde{x}}^{T_x}}, \quad \frac{\delta \text{Tr}_{x\tilde{x}}(\Psi_{x\tilde{x}}(\Lambda_x \otimes I_{\tilde{x}}))}{\delta \Psi_{x\tilde{x}}^{T_x}}. \end{aligned} \quad (\text{D4})$$

Notice that the functions in the numerator of the expressions in (D3) are independent of the channel and hence the derivatives are zero. For the five remaining ones, we note that for an Hermitian operator,  $A$ , and a function,  $f$ , which is analytic on the spectrum of  $A$ , the directional derivative of  $\text{Tr}[f(A)]$  is given by

$$\frac{\delta \text{Tr}[f(A)]}{\delta A} [B] = \text{Tr}[f'(A)B], \quad (\text{D5})$$

with the direction given by the operator  $B$  and  $f'$  being the first derivative of the function  $f$ . (This follows from analytic functional calculus, expanding  $\text{Tr} f(A + \epsilon B)$  in a Taylor series around  $\epsilon = 0$ .) Specifically, let us define  $f(z) := z \log(z)$ . Since the derivative of our function  $f$  is given by  $f'(\cdot) = (1 + \log)(\cdot)$ , using (D5) and (9), we have

$$\frac{\delta S(\tilde{X})_{\tau_{x'\tilde{x}}}}{\delta \Psi_{x\tilde{x}}^{T_x}} [B_{x\tilde{x}}] = - \text{Tr}_{x'\tilde{x}} \{ [(I_{\tilde{x}} + \log \tau_{\tilde{x}}) \otimes I_{x'}] \mathbb{E} \}, \quad (\text{D6})$$

where

$$\mathbb{E} := \text{Tr}_x (B_{x\tilde{x}} \tau_{x'x}). \quad (\text{D7})$$

Likewise, we can compute

$$\frac{\delta S(X'\tilde{X})_{\tau_{x'\tilde{x}}}}{\delta \Psi_{x\tilde{x}}^{T_x}} [B_{x\tilde{x}}] = - \text{Tr}_{x'\tilde{x}} \{ [(I_{x'\tilde{x}} + \log \tau_{x'\tilde{x}})] \mathbb{E} \}. \quad (\text{D8})$$

In order to compute the derivative for  $S(\tilde{X})_{\rho_{\tilde{x}y}}$  and  $S(\tilde{X}Y)_{\rho_{\tilde{x}y}}$  we use (11) and (D5), we find

$$\frac{\delta S(\tilde{X})_{\rho_{\tilde{x}y}}}{\delta \Psi_{x\tilde{x}}^{T_x}} [B_{x\tilde{x}}] = - \text{Tr}_{\tilde{x}y} \{ [(I_{\tilde{x}} + \log \rho_{\tilde{x}}) \otimes I_y] \mathbb{G} \} \quad (\text{D9})$$

and

$$\frac{\delta S(\tilde{X}Y)_{\rho_{\tilde{x}y}}}{\delta \Psi_{x\tilde{x}}^{T_x}} [B_{x\tilde{x}}] = - \text{Tr}_{\tilde{x}y} \{ (I_{\tilde{x}y} + \log \rho_{\tilde{x}y}) \mathbb{G} \}, \quad (\text{D10})$$

where

$$\mathbb{G} := \text{Tr}_x (B_{x\tilde{x}} \rho_{xy}). \quad (\text{D11})$$

For the last derivative we have

$$\begin{aligned} & \frac{\delta \text{Tr}_{x\tilde{x}} \{ \Psi_{x\tilde{x}}^{T_x}(\Lambda_x \otimes I_{\tilde{x}}) \}}{\delta \Psi_{x\tilde{x}}^{T_x}} [B_{x\tilde{x}}] = \\ & \text{Tr}_{x\tilde{x}} \{ (\Lambda_x \otimes I_{\tilde{x}}) B_{x\tilde{x}} \}. \end{aligned} \quad (\text{D12})$$

Putting all the terms together we have

$$\begin{aligned}
\frac{\delta \mathcal{L}}{\delta \Psi_{x\bar{x}}^{T_x}}[B_{x\bar{x}}] &= \frac{\delta S(\tilde{X})_{\tau_{x'\bar{x}}} [B_{x\bar{x}}]}{\delta \Psi_{x\bar{x}}^{T_x}} - \frac{\delta S(X'\tilde{X})_{\tau_{x'\bar{x}}} [B_{x\bar{x}}]}{\delta \Psi_{x\bar{x}}^{T_x}} \\
&\quad - \beta \frac{\delta S(\tilde{X})_{\rho_{\bar{x}y}} [B_{x\bar{x}}]}{\delta \Psi_{x\bar{x}}^{T_x}} + \beta \frac{\delta S(\tilde{X}\tilde{Y})_{\rho_{\bar{x}y}} [B_{x\bar{x}}]}{\delta \Psi_{x\bar{x}}^{T_x}} \\
&\quad - \frac{\delta \text{Tr}_{x\bar{x}}(\Psi_{x\bar{x}}^{T_x}(\Lambda_x \otimes I_{\bar{x}}))}{\delta \Psi_{x\bar{x}}^{T_x}} [B_{x\bar{x}}] \\
&= \text{Tr}_{x'\bar{x}} \left\{ [-I_{x'} \otimes (I_{\bar{x}} + \log \tau_{\bar{x}}) \right. \\
&\quad \left. + (I_{x'\bar{x}} + \log \tau_{x'\bar{x}}) + \beta I_{x'} \otimes (I_{\bar{x}} + \log \tau_{\bar{x}})] \mathbb{E} \right\} \\
&\quad - \beta \text{Tr}_{\bar{x}y} \left\{ (I_{\bar{x}y} + \log \rho_{\bar{x}y}) \mathbb{G} \right\} \\
&\quad - \text{Tr}_{x\bar{x}} \left\{ (\Lambda_x \otimes I_{\bar{x}}) B_{x\bar{x}} \right\}.
\end{aligned}$$

Setting this expression to zero ( $\frac{\delta \mathcal{L}}{\delta \Psi_{x\bar{x}}^{T_x}}[B_{x\bar{x}}] = 0$ ), we find

$$\begin{aligned}
\text{Tr}_{x'\bar{x}} \left\{ \log \tau_{x'\bar{x}} \mathbb{E} \right\} &= \text{Tr}_{x'\bar{x}} \left\{ [I_{x'} \otimes \log \tau_{\bar{x}} \right. \\
&\quad \left. - \beta I_{x'} \otimes \log \tau_{\bar{x}}] \mathbb{E} \right\} \\
&\quad + \beta \text{Tr}_{\bar{x}y} \left\{ \log \rho_{\bar{x}y} \mathbb{G} \right\} \\
&\quad + \text{Tr}_{x\bar{x}} \left\{ (\Lambda_x \otimes I_{\bar{x}}) B_{x\bar{x}} \right\}.
\end{aligned}$$

Rearranging left and right hand sides of this equation, we find

$$\begin{aligned}
&\text{Tr}_{x\bar{x}} \left\{ B_{x\bar{x}} \text{Tr}_{x'} \left\{ \tau_{x'\bar{x}} (\log \tau_{x'\bar{x}} \otimes I_x) \right\} \right\} = \\
&\text{Tr}_{x\bar{x}} \left\{ B_{x\bar{x}} \text{Tr}_{x'} \left\{ \tau_{x'\bar{x}} [(I_{x'} \otimes \log \tau_{\bar{x}} - \beta I_{x'} \otimes \log \tau_{\bar{x}}) \otimes I_x] \right\} \right\} \\
&+ \beta \text{Tr}_{x\bar{x}} \left\{ B_{x\bar{x}} \text{Tr}_y \left\{ \rho_{xy} (\log \rho_{\bar{x}y} \otimes I_x) \right\} \right\} \\
&+ \text{Tr}_{x\bar{x}} \left\{ B_{x\bar{x}} (\Lambda_x \otimes I_{\bar{x}}) \right\}. \tag{D13}
\end{aligned}$$

This holds for all directions  $B_{x\bar{x}}$ , which implies

$$\begin{aligned}
&\text{Tr}_{x'} \left\{ \tau_{x'\bar{x}} (\log \tau_{x'\bar{x}} \otimes I_x) \right\} \\
&= \text{Tr}_{x'} \left\{ \tau_{x'\bar{x}} [(I_{x'} \otimes \log \tau_{\bar{x}} - \beta I_{x'} \otimes \log \tau_{\bar{x}}) \otimes I_x] \right\} \\
&+ \beta \text{Tr}_y \left\{ \rho_{xy} (\log \rho_{\bar{x}y} \otimes I_x) \right\} \\
&+ \Lambda_x \otimes I_{\bar{x}}. \tag{D14}
\end{aligned}$$

By performing the partial trace on the left hand side of this expression, we obtain

$$\begin{aligned}
\rho_x^{1/2} (\log \tau_{x\bar{x}})^{T_x} \rho_x^{1/2} &= \tau_x (\log \tau_{\bar{x}} - \beta \log \tau_{\bar{x}}) \\
+ \beta \text{Tr}_y \left\{ \rho_{xy} (\log \rho_{\bar{x}y} \otimes I_x) \right\} &+ \Lambda_x \otimes I_{\bar{x}}. \tag{D15}
\end{aligned}$$

Simplifying this expression further, we find

$$\begin{aligned}
(\log \tau_{x\bar{x}})^{T_x} &= I_x \otimes (\log \tau_{\bar{x}} - \beta \log \tau_{\bar{x}}) \\
+ \beta \text{Tr}_y \left\{ \rho_x^{-1/2} \rho_{xy} \rho_x^{-1/2} (\log \rho_{\bar{x}y} \otimes I_x) \right\} \\
+ \rho_x^{-1/2} \Lambda_x \rho_x^{-1/2} \otimes I_{\bar{x}}. \tag{D16}
\end{aligned}$$

Let us denote

$$D_{x\bar{x}}^{\beta y} := \beta I_x \otimes \log \tau_{\bar{x}} - \beta \text{Tr}_y \left\{ \rho_x^{-1/2} \rho_{xy} \rho_x^{-1/2} (\log \rho_{\bar{x}y} \otimes I_x) \right\}, \tag{D17}$$

and the normalisation term  $\tilde{\Lambda}_x := \rho_x^{-1/2} \Lambda_x \rho_x^{-1/2}$ . Exponentiating both sides of (D16), we obtain

$$\tau_{x\bar{x}}^{T_x} = e^{\log \tau_{\bar{x}} \otimes I_x - D_{x\bar{x}}^{\beta y} + \tilde{\Lambda}_x \otimes I_{\bar{x}}}. \tag{D18}$$

Using Eq. (9), we arrive at the expression for the Choi-Jamiolkowski matrix corresponding to the channel,

$$\Psi_{x\bar{x}}^{T_x} = (\rho_x \otimes I_{\bar{x}})^{-1/2} e^{\log \tau_{\bar{x}} \otimes I_x - D_{x\bar{x}}^{\beta y} + \tilde{\Lambda}_x \otimes I_{\bar{x}}} (\rho_x \otimes I_{\bar{x}})^{-1/2}. \tag{D19}$$