View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Explore Bristol Research

OPEN ACCESS

University of BRISTOL

Booker, A. R., & Krishnamurthy, M. (2016). A converse theorem for GL(n). Advances in Mathematics, 296, 154-180. DOI: 10.1016/j.aim.2016.03.041

Peer reviewed version

License (if available):
CC BY-NC-ND

Link to published version (if available):
10.1016/j.aim.2016.03.041

Link to publication record in Explore Bristol Research
PDF-document

This is the author accepted manuscript (AAM). The final published version (version of record) is available online via Elsevier at 10.1016/j.aim.2016.03.041.

University of Bristol - Explore Bristol Research

General rights

This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available: http://www.bristol.ac.uk/pure/about/ebr-terms.html

A CONVERSE THEOREM FOR $\mathrm{GL}(n)$

ANDREW R. BOOKER AND M. KRISHNAMURTHY

Abstract. We complete the work of Cogdell and Piatetski-Shapiro [3] to prove, for $n \geq 3$, a converse theorem for automorphic representations of GL_n over a number field, with analytic data from twists by *unramified* representations of GL_{n-1}.

1. Introduction

In this paper, we complete the work of Cogdell and Piatetski-Shapiro [3] to prove the following.

Theorem 1.1. *Let F be a number field with adèle ring $\mathbb{A}_F$. Fix an integer $n \geq 3$, and let $\pi = \bigotimes_v \pi_v$ be an irreducible, admissible representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with automorphic central character. For every unitary, isobaric, automorphic representation $\tau = \bigotimes_v \tau_v$ of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$ which is unramified at all finite places, assume that the complete Rankin–Selberg L-functions*

$$\Lambda(s, \pi \times \tau) = \prod_v L(s, \pi_v \times \tau_v)$$

converge absolutely in some right half plane, continue to entire functions of finite order, and satisfy the functional equation

$$\Lambda(s, \pi \times \tau) = \epsilon(s, \pi \times \tau)\Lambda(1 - s, \widetilde{\pi} \times \widetilde{\tau}), \tag{1.1}$$

where $\epsilon(s, \pi\times\tau)$ is the product of the corresponding local ϵ-factors defined in [7, Thm. 2.7] *and* [12]. *Then π is quasiautomorphic, in the sense that there is a unique isobaric automorphic representation $\Pi = \bigotimes_v \Pi_v$ of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\pi_v \cong \Pi_v$ for all non-archimedean places v where π_v is unramified.*

Remarks.

(1) We recall the notion of an isobaric automorphic representation [14]: Given a partition $n_1, \ldots, n_k$ of n and cuspidal representations σ_i of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$, let P be the corresponding standard parabolic subgroup of GL_n and let ω_i denote the central character of σ_i. Then there is a real number t_i such that $|\omega_i(z)| = \|z\|^{t_i}$ for z in the center of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$. By re-ordering, if necessary, we may assume $t_1 \geq \ldots \geq t_k$, and form globally the induced representation $\Upsilon = \mathrm{Ind}_{P(\mathbb{A}_F)}^{\mathrm{GL}_n(\mathbb{A}_F)}(\sigma_1 \otimes \cdots \otimes \sigma_k)$. On the other hand, if $\sigma_i = \bigotimes_v \sigma_{i,v}$ (here and throughout the paper, the symbol $\bigotimes$ means a restricted tensor product with respect to a distinguished set of spherical vectors for almost all places), we may also form locally the induced representations $\Upsilon_v = \mathrm{Ind}_{P(F_v)}^{\mathrm{GL}_n(F_v)}(\sigma_{1,v}\otimes\cdots\otimes\sigma_{k,v})$ for each v. (For archimedean v, one has to pass to the smooth completion of $\sigma_{i,v}$ in

A. R. B. was supported by EPSRC Grants EP/H005188/1, EP/K004581/1, EP/L001454/1 and EP/K034383/1.

M. K. was supported by NSA Grant H98230-12-1-0218.

order to form the induced representation Υ_v; see §3.1 and the references therein for details.) Then, by definition, $\sigma_{1,v} \boxplus \cdots \boxplus \sigma_{k,v}$ is the $(\mathfrak{g}_v, K_v)$-module associated to the Langlands quotient of Υ_v. We may also form their (restricted) tensor product to obtain an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. This representation, denoted $\sigma_1 \boxplus \cdots \boxplus \sigma_k$, is called *isobaric*, and it satisfies the following properties:

(a) *Strong multiplicity one* [8]. If $n'_1, \ldots, n'_l$ is another partition of n and σ'_i, $1 \le i \le l$, are cuspidal representations of $\mathrm{GL}_{n'_i}(\mathbb{A}_F)$ such that

$$\sigma_1 \boxplus \cdots \boxplus \sigma_k \cong \sigma'_1 \boxplus \cdots \boxplus \sigma'_l$$

then $l = k$ and there is a permutation ϕ of $\{1, \ldots, k\}$ such that $n'_i = n_{\phi(i)}$ and $\sigma'_i \cong \sigma_{\phi(i)}$.

(b) *Multiplicativity of local factors.* The Rankin–Selberg method [7, 9] of associating local factors is bi-additive with respect to isobaric sums. In particular, for any automorphic representation $\tau = \bigotimes_v \tau_v$ of $\mathrm{GL}_n(\mathbb{A}_F)$, one has

$$L(s, (\sigma_{1,v} \boxplus \cdots \boxplus \sigma_{k,v}) \times \tau_v) = \prod_{i=1}^{k} L(s, \sigma_{i,v} \times \tau_v),$$
$$\epsilon(s, (\sigma_{1,v} \boxplus \cdots \boxplus \sigma_{k,v}) \times \tau_v, \psi_v) = \prod_{i=1}^{k} \epsilon(s, \sigma_{i,v} \times \tau_v, \psi_v)$$

for each v.

We call an isobaric automorphic representation $\sigma = \sigma_1 \boxplus \cdots \boxplus \sigma_k$ *unitary* if each cuspidal representation σ_i has unitary central character. In this case, for each v, it follows from the description of the unitary dual of $\mathrm{GL}_n(F_v)$ [17, 16] that the corresponding parabolically induced representation $\mathrm{Ind}_{P(F_v)}^{\mathrm{GL}_n(F_v)}(\sigma_{1,v} \otimes \cdots \otimes \sigma_{k,v})$ is irreducible, unitary and generic; therefore, σ is the full induced representation $\mathrm{Ind}_{P(\mathbb{A}_F)}^{\mathrm{GL}_n(\mathbb{A}_F)}(\sigma_1 \otimes \cdots \otimes \sigma_k)$.

(2) The results in [3] are stated in terms of analytic properties of $\Lambda(s, \pi \times \tau)$ for *cuspidal* representations τ of $\mathrm{GL}_r(\mathbb{A}_F)$ for all $r < n$. Our statement in terms of unitary isobaric representations of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$ generalizes this slightly (since one can pass from cuspidal twists to isobaric twists by multiplying), though as we show in Proposition 3.1 below, one can derive the properties of the twists by cuspidal representations of $\mathrm{GL}_r(\mathbb{A}_F)$ from our hypotheses.

Alternatively, one could state the theorem in terms of twists by all unramified, generic, automorphic subrepresentations τ of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$, and in fact the proof proceeds along these lines, i.e. we first derive the properties of such twists from those for unitary isobaric representations in Proposition 3.1. (Here and throughout the paper, we say that a global representation or idèle class character is *unramified* if it is unramified at every finite place.)

(3) Note that the local constituents π_v are not assumed to be generic. If they happen to be generic for all v then, following the proof of [3, §7, Cor. 2], one can modify the argument to produce a unique generic (but not necessarily isobaric) automorphic representation Π such that $\Pi_v \cong \pi_v$ for all unramified v. In this case, we also obtain $\Pi_v \cong \pi_v$ for all archimedean v.

Our proof closely follows the method of Cogdell and Piatetski-Shapiro, who established a version of Theorem 1.1 (cf. [3, Thm. 3]) under the assumption that F has class number 1. In fact, in full generality, their method exhibits a classical automorphic form (i.e. at the archimedean places) with the expected properties, but they encountered some combinatorial difficulties in relating it back to the representation π (via Hecke eigenvalues), and were only able to overcome them under the class number assumption. Our proof avoids attacking the combinatorics directly; rather, we rely on the classical fact, due to Harish-Chandra, that any K-finite, $\mathcal{Z}$-finite, automorphic form is a finite linear combination of Hecke eigenforms, from which we realize the L-function of π as a linear combination of automorphic L-functions. The final ingredient is multiplicativity—since each of the L-functions in question is given by an Euler product, only the trivial linear relation is possible. To make this precise, we adapt work of Kaczorowski, Molteni and Perelli [13] on linear independence in the Selberg class, generalizing it to number fields.

Acknowledgements. This work began during visits by boths authors to the Research Institute for Mathematical Sciences (Kyoto, Japan), and the second author to the Tata Institute of Fundamental Research (Mumbai, India). We thank these institutions and our hosts, Professors Akio Tamagawa and Dipendra Prasad, for their generous hospitality. The second author would also like to thank C. S. Rajan for useful discussions on converse theorems during his stay at the Tata Institute of Fundamental Research in Fall 2014. We are grateful to James Cogdell for helpful comments on drafts at several stages of this work, as well as enlightening discussions regarding converse theorems on many occasions in the past. We thank Miodrag Iovanov for helpful discussions related to Lemma 3.3, and Dinakar Ramakrishnan for suggesting the use of isobaric representations. Finally, we thank the anonymous referee for valuable corrections and suggestions for improving the exposition.

2. Preliminaries

Suppose F is a number field with ring of integers $\mathfrak{o}_F$. For each place v of F, let F_v denote the completion of F at v. For finite v, let $\mathfrak{o}_v$ denote the ring of integers in F_v, $\mathfrak{p}_v$ the unique maximal ideal in $\mathfrak{o}_v$, q_v the cardinality of $\mathfrak{o}_v/\mathfrak{p}_v$, and ϖ_v a generator of $\mathfrak{p}_v$ with absolute value $\|\varpi_v\|_v = q_v^{-1}$. Put $F_\infty = \prod_{v|\infty} F_v$, and let $\mathbb{A}_F = F_\infty \times \mathbb{A}_{F,f}$ denote the ring of adèles of F.

Recall that a *Größencharakter* of conductor $\mathfrak{q}$ is a multiplicative function χ of non-zero integral ideals satisfying $\chi(a\mathfrak{o}_F) = \chi_f(a)\chi_\infty(a)$ for associated characters $\chi_f : (\mathfrak{o}_F/\mathfrak{q})^\times \to \mathbb{C}^\times$ and $\chi_\infty : F_\infty^\times \to \mathbb{C}^\times$, with χ_f primitive and χ_∞ continuous, and all $a \in \mathfrak{o}_F$ relatively prime to $\mathfrak{q}$. By convention we set $\chi(\mathfrak{a}) = 0$ for any ideal $\mathfrak{a}$ with $(\mathfrak{a}, \mathfrak{q}) \neq 1$. The Größencharakters are in one-to-one correspondence with idèle class characters $\omega : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$, and the correspondence is such that $\chi_\infty = \omega_\infty^{-1}$ and $\chi(\mathfrak{p}_v \cap \mathfrak{o}_F) = \omega(\varpi_v)$ at each finite place v co-prime to $\mathfrak{q}$. For any idèle class character ω, we write χ_ω to denote the associated Größencharakter.

For any $r > 1$ and any commutative ring R, let $B_r(R) = T_r(R)U_r(R) \subset \mathrm{GL}_r(R)$ be the Borel subgroup of upper triangular matrices, $P_r'(R)$ the parabolic subgroup of type $(r-1, 1)$, and $N_r(R)$ its unipotent radical. Let $P_r(R) \subset P_r'(R)$ denote the mirabolic subgroup consisting of matrices whose last row is of the form $(0, \dots, 0, 1)$, i.e.

$$P_r(R) = \left\{ \begin{pmatrix} h & y \\ & 1 \end{pmatrix} : h \in \mathrm{GL}_{r-1}(R), y \in R^{r-1} \right\} \cong \mathrm{GL}_{r-1}(R) \ltimes N_r(R).$$

Let w_r denote the long Weyl element in $\mathrm{GL}_r(R)$, and put $\alpha_r = \left(\begin{smallmatrix} w_{r-1} & \\ & 1 \end{smallmatrix}\right)$.

From now on, we fix an integer $n \geq 3$ and consider GL_n along with certain distinguished subgroups. For each $v < \infty$, we will consider certain compact open subgroups of $\mathrm{GL}_n(F_v)$; namely, let $K_v = \mathrm{GL}_n(\mathfrak{o}_v)$, and for any integer $m \geq 0$, set

$$K_{1,v}(\mathfrak{p}_v^m) = \left\{ g \in \mathrm{GL}_n(\mathfrak{o}_v) : g \equiv \begin{pmatrix} & & * \\ & * & \vdots \\ & & * \\ 0 & \cdots\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}_v^m} \right\},$$

$$K_{0,v}(\mathfrak{p}_v^m) = \left\{ g \in \mathrm{GL}_n(\mathfrak{o}_v) : g \equiv \begin{pmatrix} & & * \\ & * & \vdots \\ & & * \\ 0 & \cdots\ 0 & * \end{pmatrix} \pmod{\mathfrak{p}_v^m} \right\},$$

so that $K_{1,v}(\mathfrak{p}_v^m)$ is a normal subgroup of $K_{0,v}(\mathfrak{p}_v^m)$, with quotient $K_{0,v}(\mathfrak{p}_v^m)/K_{1,v}(\mathfrak{p}_v^m) \cong (\mathfrak{o}_v/\mathfrak{p}_v^m)^\times$. Next, define $K_f = \prod_{v<\infty} K_v$, and for an integral ideal $\mathfrak{a}$ of F, set

$$K_i(\mathfrak{a}) = \prod_{v<\infty} K_{i,v}(\mathfrak{p}_v^{m_v}) \quad \text{for } i = 0, 1,$$

where m_v are the unique non-negative integers such that $\mathfrak{a} = \prod_v (\mathfrak{p}_v \cap \mathfrak{o}_F)^{m_v}$. Then $K_1(\mathfrak{a}) \subseteq K_0(\mathfrak{a}) \subseteq K_f$ are compact open subgroups of $\mathrm{GL}_n(\mathbb{A}_{F,f})$. We consider also the corresponding principal congruence subgroups of $\mathrm{GL}_n(F_\infty)$, embedded diagonally, namely,

$$\Gamma_i(\mathfrak{a}) = \{\gamma \in \mathrm{GL}_n(F) : \gamma_f \in K_i(\mathfrak{a})\} \subset \mathrm{GL}_n(F_\infty) \quad \text{for } i = 0, 1,$$

where γ_f denotes the image of γ in $\mathrm{GL}_n(\mathbb{A}_{F,f})$.

From strong approximation for GL_n, one knows that $\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/\mathrm{GL}_n(F_\infty)K_1(\mathfrak{a})$ is finite, with cardinality h, the class number of F. Let us write

$$\mathrm{GL}_n(\mathbb{A}_F) = \coprod_{j=1}^{h} \mathrm{GL}_n(F) g_j \,\mathrm{GL}_n(F_\infty) K_1(\mathfrak{a}),$$

where each $g_j \in \mathrm{GL}_n(\mathbb{A}_{F,f})$. In particular,

$$\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/K_1(\mathfrak{a}) \cong \coprod_{j=1}^{h} \Gamma_{1,j}(\mathfrak{a})\backslash \mathrm{GL}_n(F_\infty),$$

where $\Gamma_{1,j}(\mathfrak{a}) = \{\gamma \in \mathrm{GL}_n(F) : \gamma_f \in g_j\, K_1(\mathfrak{a}) g_j^{-1}\} \subset \mathrm{GL}_n(F_\infty)$, embedded diagonally. Replacing $K_1(\mathfrak{a})$ by $K_0(\mathfrak{a})$ in this definition, we get the corresponding groups $\Gamma_{0,j}(\mathfrak{a})$.

For groups $H \subseteq G$, let $\mathcal{F}(H\backslash G)$ denote the vector space of all complex-valued functions $f : G \to \mathbb{C}$ that are left invariant under H. Further, for any subgroup $L \subseteq G$, let $\mathcal{F}(H\backslash G)^L$ denote the subspace of right L-invariant functions in $\mathcal{F}(H\backslash G)$. Then we have an isomorphism of vector spaces

$$\mathcal{F}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F))^{K_1(\mathfrak{a})} \cong \coprod_{j=1}^{h} \mathcal{F}(\Gamma_{1,j}(\mathfrak{a})\backslash \mathrm{GL}_n(F_\infty)) \tag{2.1}$$

given by $f \mapsto (f_j)$, where $f_j(x) = f(xg_j)$ for $x \in \mathrm{GL}_n(F_\infty)$.

3. The method of Cogdell and Piatetski-Shapiro

3.1. **Initial setup.** For convenience, we write G_∞ to denote the group $\mathrm{GL}_n(F_\infty)$. Let $\mathfrak{g}_\infty$ be the real Lie algebra of G_∞, and let $\mathcal{U}$ denote the universal enveloping algebra of its complexification, $\mathfrak{g}_\infty^{\mathbb{C}}$. Let $K_\infty = \prod_{v|\infty} K_v$, which is a maximal compact subgroup of G_∞. If

S is a finite set of places of F, we write $\mathbb{A}_F^S$ to denote the restricted product $\prod'_{v\notin S} F_v$ and $G^S = \mathrm{GL}_n(\mathbb{A}_F^S) = \prod'_{v\notin S} \mathrm{GL}_n(F_v)$.

Let us recall the notion of a (smooth) $\mathrm{GL}_n(\mathbb{A}_F)$-*module*, which is a complex vector space X equipped with an action of $\mathcal{U}$, K_∞, and $\mathrm{GL}_n(\mathbb{A}_{F,f})$ satisfying the following usual conditions:

(1) the actions of $\mathcal{U}$ and K_∞ commute with that of $\mathrm{GL}_n(\mathbb{A}_{F,f})$;
(2) each $u \in X$ is fixed by some compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_{F,f})$;
(3) X has the structure of a $(\mathfrak{g}_\infty, K_\infty)$-module under the actions of $\mathcal{U}$ and K_∞.

Suppose δ is an irreducible representation of K_∞ and H is any compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_{F,f})$. Let $X(\delta, H)$ denote the subspace consisting of elements in X which are fixed by H and of isotypic type δ. For Δ a finite collection of irreducible representations of K_∞, let $X(\Delta, H) = \sum_{\delta\in\Delta} X(\delta, H)$. Then X is said to be *admissible* if $X(\Delta, H)$ is finite dimensional for every Δ and H. We will also use the notation X^H to denote the subspace of H-fixed vectors in X.

For each place v, let $\mathcal{H}_v$ denote the Hecke algebra of $\mathrm{GL}_n(F_v)$ (defined with respect to K_v for $v \mid \infty$), and let $*$ denote the multiplication operation in $\mathcal{H}_v$. Set $\mathcal{H}_\infty = \bigotimes_{v\mid\infty} \mathcal{H}_v$ and $\mathcal{H}_f = \bigotimes_{v<\infty} \mathcal{H}_v$, so that the global Hecke algebra $\mathcal{H}$ satisfies $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_f$. Given a smooth $\mathrm{GL}_n(\mathbb{A}_F)$-module X as above, it inherits an action of the Hecke algebra $\mathcal{H}_v$ for each v and hence becomes a module for $\mathcal{H}$. It is well known that a smooth irreducible admissible $\mathrm{GL}_n(\mathbb{A}_F)$-module is factorizable in the sense of [5, Thm. 3]. As is customary, by a smooth irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_F)$ we mean a smooth irreducible admissible $\mathrm{GL}_n(\mathbb{A}_F)$-module.

Now, let $\pi = \bigotimes_v \pi_v$ be an irreducible, admissible representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with automorphic central character ω_π, as in the statement of Theorem 1.1. We fix an additive character $\psi = \bigotimes_v \psi_v$ of $F\backslash\mathbb{A}_F$ whose conductor is the inverse different $\mathfrak{d}^{-1}$ of F. For each v, let Ξ_v be the induced representation of “Langlands type” having π_v as the unique irreducible quotient [7, 9]. For $v \mid \infty$, by definition, Ξ_v is actually a smooth admissible representation of $\mathrm{GL}_n(F_v)$ of moderate growth and π_v is the underlying $(\mathfrak{g}_v, K_v)$-module, also known as the Harish-Chandra module, of the unique irreducible quotient of Ξ_v. Each Ξ_v is an *induced representation of Whittaker type* in the sense of [3, p. 159], also called a *generic induced representation* [9, p. 4]. In particular, Ξ_v is admissible of finite type and admits a non-zero ψ_v-Whittaker form which is unique up to scalar factor. In general, such a representation is said to be of *Whittaker type*. It should be noted that any constituent of Ξ_v has the same central character as that of π_v.

For any representation τ_v of Whittaker type, we write $\mathcal{W}(\tau_v, \psi_v)$ to denote its Whittaker model with respect to ψ_v [7, 9]. An important feature of the Ξ_v defined in the previous paragraph is that, although it may not be irreducible, the usual map $f \mapsto W_f$ from its space to $\mathcal{W}(\Xi_v, \psi_v)$ is bijective [11, 9]. The local factors associated to π_v (not necessarily generic) are then defined via integral representations using the Whittaker model of Ξ_v. More precisely, by definition (cf. [7, 9]), for every representation τ_v of $\mathrm{GL}_r(F_v)$ that is induced of Whittaker type one has

$$L(s, \pi_v \times \tau_v) = L(s, \Xi_v \times \tau_v),$$

with a similar equality for the local ϵ-factors.

In the following two paragraphs, we rely heavily on [9] and refer the reader to that paper for any unexplained notation or terminology. For a fixed $v \mid \infty$, suppose V is the representation space of Ξ_v, let $V_0 \subset V$ be the unique minimal invariant subspace which is generic, and let π'_v

be the corresponding underlying Harish-Chandra module. By [9, Lemma 2.4], any non-zero ψ_v-form λ on V will restrict to a non-zero ψ_v-form on V_0. Therefore, for any generic induced representation τ_v of $\mathrm{GL}_{n-1}(F_v)$, the family of integrals defining $L(s, \pi'_v \times \tau_v)$ is a subspace of those defining $L(s, \Xi_v \times \tau_v)$. By passing to the projective tensor product and applying [9, Thm. 2.6], we see that the factor $L(s, \pi'_v \times \tau_v)$ is equal to an integral of the form

$$\int W\left[\left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right), g\right] \| \det g \|_v^{s-\frac{1}{2}} \, dg,$$

where W corresponds to a smooth vector in the projective completion of the representation $\pi'_v \otimes \tau_v$.

Then, by continuity and using the extension of [9, Prop. 11.1] to the complete tensor product (see §12.3 of loc. cit.), we see that $L(s, \pi'_v \times \tau_v)$ must be a holomorphic multiple of $L(s, \pi_v \times \tau_v)$. Let us realize π'_v (in the above sense) as the Langlands quotient of an induced representation (Ξ'_v, V') of Langlands type. Then, it follows from [9, Lemma 2.5] that Ξ'_v is irreducible and consequently π'_v is the Harish-Chandra module of Ξ'_v. Now, as in [9, §11], (Ξ_v, V) (and hence (Ξ'_v, V')) is a subrepresentation of a principal series representation $I_{\mu,t}$, where μ is an n-tuple of characters and t is an n-tuple of complex numbers. Similarly, τ_v is a subrepresentation of an $I_{\mu',t'}$. Then, from the proof of Proposition 11.1 of loc. cit., it follows that both $L(s, \pi'_v \times \tau_v)$ and $L(s, \pi_v \times \tau_v)$ are polynomial multiples of

$$\prod_{i,j} L(s + t_i + t'_j, \mu_i \mu'_j).$$

Thus $L(s, \pi'_v \times \tau_v)$ is a rational multiple of $L(s, \pi_v \times \tau_v)$. Since it is a holomorphic multiple as well, we conclude that

$$L(s, \pi'_v \times \tau_v) = f(s) L(s, \pi_v \times \tau_v), \tag{3.1}$$

where $f(s)$ is a polynomial. Finally, it also follows from loc. cit. that the γ-factors associated with the pairs (π_v, τ_v) and (π'_v, τ_v), respectively, are the same.

Now, let $\pi' = \bigotimes_{v|\infty} \pi'_v \otimes \bigotimes_{v<\infty} \pi_v$, which is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with the same central character as π. Suppose τ is a unitary isobaric automorphic representation of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$. As noted in the remarks following Theorem 1.1, τ_v is a generic induced representation for each $v \mid \infty$, so it follows from the above that $\Lambda(s, \pi' \times \tau) = P(s)\Lambda(s, \pi \times \tau)$ for some polynomial $P(s)$. Further,

$$\begin{aligned}
\gamma(s, \pi' \times \tau) &:= \frac{\epsilon(s, \pi' \times \tau)\Lambda(1-s, \widetilde{\pi}' \times \widetilde{\tau})}{\Lambda(s, \pi' \times \tau)} \\
&= \prod_{v|\infty} \gamma(s, \pi'_v \times \tau_v, \psi_v) \prod_{v<\infty} \epsilon(s, \pi'_v \times \tau_v, \psi_v) \cdot \frac{L(1-s, \widetilde{\pi}' \times \widetilde{\tau})}{L(s, \pi' \times \tau)} \\
&= \prod_{v|\infty} \gamma(s, \pi_v \times \tau_v, \psi_v) \prod_{v<\infty} \epsilon(s, \pi_v \times \tau_v, \psi_v) \cdot \frac{L(1-s, \widetilde{\pi} \times \widetilde{\tau})}{L(s, \pi \times \tau)} \\
&= \gamma(s, \pi \times \tau).
\end{aligned}$$

Since $\gamma(s, \pi \times \tau) = 1$ identically if and only if $\Lambda(s, \pi \times \tau)$ satisfies the functional equation (1.1), the functional equation for $\Lambda(s, \pi' \times \tau)$ is equivalent to that of $\Lambda(s, \pi \times \tau)$. In summary, π' satisfies the hypotheses of Theorem 1.1 if π does. The representation π' has the added feature that its archimedean local components have a Whittaker model, and if π_v is generic for all $v \mid \infty$ to begin with then $\pi = \pi'$.

In general, we may replace π by π', and assume without loss of generality that π_v is generic for all archimedean v, at the expense of losing compatibility between π_v and Π_v for those v. In [3], the authors work with the full induced representation Ξ_v instead of π_v at archimedean places, but we have made the above modification in order to preserve irreducibility, which is essential in Lemma 3.4 (see §3.4 below).

Next, for $v < \infty$, choose ξ_v^0 in the space of Ξ_v as in [3, p. 203]. In particular, for $v < \infty$ where π_v (and hence Ξ_v) is unramified, ξ_v^0 is the unique K_v-fixed vector that projects onto the distinguished spherical vector of π_v. For $v < \infty$ where π_v is ramified, the choice of ξ_v^0 is such that it is fixed by $K_{1,v}(\mathfrak{p}_v^{m_v})$ for some $m_v > 0$. Set $\mathfrak{n} = \prod_{v<\infty}(\mathfrak{p}_v \cap \mathfrak{o}_F)^{m_v}$. We note that when π is generic, we may choose each ξ_v^0 to be the *essential vector* [6, 10] and $\mathfrak{n}$ to be the conductor of π. At any rate, for $v < \infty$ where π_v is unramified, and τ_v any unramified representation of $\mathrm{GL}_{n-1}(F_v)$ of Langlands type with normalized spherical function $W_{\tau_v}^0 \in \mathcal{W}(\tau_v, \psi_v^{-1})$, one has (cf. [8, §1, (3)])

$$\int_{U_{n-1}(F_v)\backslash \mathrm{GL}_{n-1}(F_v)} W_{\xi_v^0}\begin{pmatrix} g & \\ & 1 \end{pmatrix} W_{\tau_v}^0(g) \|\det g\|_v^{s-\frac{1}{2}}\, dg = L(s, \pi_v \times \tau_v). \tag{3.2}$$

As pointed out in [10, Remark 2] as well as in [15, §1.5], the above equality is derived for generic unramified representations in [8] but the proof extends verbatim to unramified representations of Langlands type.

3.2. **The functions U_ξ and V_ξ.** In this subsection, we summarize the construction in [3] of the functions U_ξ and V_ξ associated to π, and describe their properties, culminating in the identity given in Proposition 3.1; we defer to [3] for detailed proofs.

First note that if we take τ to be the isobaric sum of $n-1$ copies of the trivial character then, by hypothesis, the product $\prod_v L(s, \pi_v \times \tau_v)$ converges absolutely for s in a right half plane. For each v, let $\pi_v \boxtimes \tau_v$ denote the functorial tensor product defined via the local Langlands correspondence. For an unramified finite place v, let $\alpha_{v,1}, \ldots, \alpha_{v,n}$ denote the Satake parameters of π_v. Then $\pi_v \boxtimes \tau_v$ has the same parameters, repeated with multiplicity $n-1$. Applying [3, Lemma 2.2] to the representation $\pi \boxtimes \tau = \bigotimes_v(\pi_v \boxtimes \tau_v)$, we obtain an estimate of the form $\alpha_{v,i} = O(q_v^\sigma)$ for some $\sigma \in \mathbb{R}$. Hence, the Euler product defining the standard L-function, $\prod_v L(s, \pi_v)$, also converges absolutely for s in a right half plane, which is the form that this hypothesis takes in [3].

Next, as discussed in [3, §8], ω_π determines a character $\chi_\pi = \bigotimes_v \chi_{\pi_v}$ of $K_0(\mathfrak{n})$ which is trivial on $K_1(\mathfrak{n})$. Moreover, it follows from loc. cit. that $K_0(\mathfrak{n})$ acts on the space of $K_1(\mathfrak{n})$-fixed vectors via χ_π. In particular, for $v < \infty$ and $g \in K_{0,v}(\mathfrak{p}_v^{m_v})$, we have

$$\Xi_v(g)\xi_v^0 = \chi_{\pi_v}(g)\xi_v^0.$$

Let V_{π_∞} denote the space of π_∞, and fix a $\xi_\infty = \bigotimes_{v|\infty} \xi_v \in V_{\pi_\infty}$. Let $\xi = \xi_\infty \otimes \xi_f^0$, where $\xi_f^0 = \bigotimes_{v<\infty} \xi_v^0$, and consider

$$U_\xi(g) = \sum_{\gamma \in U_n(F)\backslash P_n(F)} W_\xi(\gamma g) = \sum_{\gamma \in U_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\xi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right).$$

This sum converges absolutely and uniformly on compact subsets to a continuous function on $\mathrm{GL}_n(\mathbb{A}_F)$ which is cuspidal along the unipotent radical of any standard maximal parabolic subgroup of $\mathrm{GL}_n(\mathbb{A}_F)$. Since ω_π is assumed to be automorphic, as a function of $\mathrm{GL}_n(\mathbb{A}_F)$, U_ξ is left invariant under both $P_n(F)$ and the center $Z_n(F)$.

We also consider a second function V_ξ attached to ξ, which will be related to U_ξ through the functional equation. Namely, let $\widetilde{W}_\xi(g) = W_\xi(w_n\, {}^tg^{-1})$, put

$$\widetilde{U}_\xi(g) = \sum_{\gamma \in U_n(F)\backslash P_n(F)} \widetilde{W}_\xi(\gamma g),$$

and define $V_\xi(g) = \widetilde{U}_\xi(\alpha_n\, {}^tg^{-1})$, where $\alpha_n = \left(\begin{smallmatrix} w_{n-1} & \\ & 1\end{smallmatrix}\right)$, as defined in §2. In other words,

$$V_\xi(g) = \sum_{\gamma\in U_n(F)\backslash P_n(F)} \widetilde{W}_\xi(\gamma\alpha_n\, {}^tg^{-1}) = \sum_{\gamma \in U_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} \widetilde{W}_\xi\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}\alpha_n\, {}^tg^{-1}\right).$$

Hence, if $Q_n = {}^tP_n^{-1}$, then $V_\xi(g)$ is invariant on the left by both $Q_n(F)$ and $Z_n(F)$.

We record another formula for $V_\xi(g)$ which shows that it agrees with the definition given in [3]. By definition, since ${}^t\alpha_n^{-1} = \alpha_n$, we have

$$\begin{aligned} V_\xi(g) = \sum_{\gamma\in U_n(F)\backslash P_n(F)} \widetilde{W}_\xi(\gamma\alpha_n\, {}^tg^{-1}) &= \sum_{\gamma\in U_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\xi\left(w_n\begin{pmatrix}{}^t\gamma^{-1} & \\ & 1\end{pmatrix}\alpha_n g\right)\\ &= \sum_{\gamma\in U_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\xi\left(w_n\alpha_n\begin{pmatrix}w_{n-1}^{-1}\,{}^t\gamma^{-1}w_{n-1} & \\ & 1\end{pmatrix} g\right)\\ &= \sum_{\gamma\in U_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\xi\left(w_n\alpha_n\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right).\end{aligned}$$

Here, the last equality follows since the transformation $\gamma \mapsto w_{n-1}{}^t\gamma^{-1}w_{n-1}^{-1}$ permutes the set of right cosets $U_{n-1}(F)\gamma$ in $\mathrm{GL}_{n-1}(F)$. Thus,

$$V_\xi(g) = \sum_{\gamma\in U_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\xi\left(\alpha_n'\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right),$$

where $\alpha_n' = \left(\begin{smallmatrix} & 1\\ I_{n-1} & \end{smallmatrix}\right)$.

Now, let τ be an automorphic subrepresentation of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$, and let ϕ be an automorphic form in the space of τ. Suppose $\tau \cong \bigotimes_v \tau_v$ and ϕ corresponds to a pure tensor $\bigotimes_v \phi_v$ under this isomorphism. Let

$$I(s; U_\xi, \phi) = \int_{\mathrm{GL}_{n-1}(F)\backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)} U_\xi\begin{pmatrix}h & \\ & 1\end{pmatrix}\phi(h)\|\det h\|^{s-\frac12}\, dh.$$

The above integral is absolutely convergent for $\Re(s) \gg 1$ and if τ is cuspidal, it converges for all s. Further, the integral unfolds to give

$$\begin{aligned} I(s; U_\xi, \phi) &= \int_{U_{n-1}(\mathbb{A}_F)\backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)} W_\xi\begin{pmatrix}h & \\ & 1\end{pmatrix} W_\phi(h)\|\det h\|^{s-\frac12}\, dh\\ &= \prod_v \int_{U_{n-1}(F_v)\backslash \mathrm{GL}_{n-1}(F_v)} W_{\xi_v}\begin{pmatrix}h_v & \\ & 1\end{pmatrix} W_{\phi_v}(h_v)\|\det h_v\|_v^{s-\frac12}\, dh_v \qquad (3.3)\\ &= \prod_v \Psi_v(s; W_{\xi_v}, W_{\phi_v}),\end{aligned}$$

where $W_\phi(h) = \int_{U_{n-1}(F)\backslash U_{n-1}(\mathbb{A}_F)} \phi(nh)\psi(n)\,dn$, i.e. $W_\phi \in \mathcal{W}(\tau, \psi^{-1})$. In particular, the integral vanishes unless τ is generic. Further, from the theory of local L-functions [7, 9],

$$E_v(s) = \frac{\Psi_v(s; W_{\xi_v}, W_{\phi_v})}{L(s, \pi_v \times \tau_v)}$$

is entire for all v. If v is non-archimedean then $E_v(s) \in \mathbb{C}[q_v^s, q_v^{-s}]$, and for almost all such v we have $E_v(s) = 1$. Thus, setting $E(s) = \prod_v E_v(s)$, we have

$$I(s; U_\xi, \phi) = E(s) \prod_v L(s, \pi_v \times \tau_v).$$

Similarly, we define the integral

$$I(s; V_\xi, \phi) = \int_{\mathrm{GL}_{n-1}(F)\backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)} V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \phi(h) \| \det h\|^{s-\frac{1}{2}}\, dh,$$

which converges for $-\Re(s) \gg 1$. If we unfold this integral, we get

$$\begin{aligned} I(s; V_\xi, \phi) &= \int_{U_{n-1}(\mathbb{A}_F)\backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)} \widetilde{W}_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \widetilde{W}_\phi(h) \| \det h\|^{\frac{1}{2}-s}\, dh \\ &= \prod_v \int_{U_{n-1}(F_v)\backslash \mathrm{GL}_{n-1}(F_v)} \widetilde{W}_{\xi_v} \begin{pmatrix} h_v & \\ & 1 \end{pmatrix} \widetilde{W}_{\phi_v}(h_v) \| \det h_v\|_v^{\frac{1}{2}-s}\, dh_v \\ &= \prod_v \Psi_v(1-s; \widetilde{W}_{\xi_v}, \widetilde{W}_{\phi_v}), \end{aligned}$$

where $\widetilde{W}_\phi(h) = W_\phi(w_{n-1}\,{}^t h^{-1})$. In passing, we mention that this is the ψ-Whittaker coefficient of the dual function $\widetilde{\phi}(h) = \phi(w_{n-1}\,{}^t h^{-1})$, i.e. $\widetilde{W}_\phi(h) = \int_{U_{n-1}(F)\backslash U_{n-1}(\mathbb{A}_F)} \widetilde{\phi}(uh)\psi^{-1}(u)\,du$. Now, for every v, just as we defined $E_v(s)$, let

$$\widetilde{E}_v(s) = \frac{\Psi_v(s; \widetilde{W}_{\xi_v}, \widetilde{W}_{\phi_v})}{L(s, \widetilde{\pi}_v \times \widetilde{\tau}_v)}$$

denote the corresponding entire function attached to the pair of dual representations $(\widetilde{\pi}_v, \widetilde{\tau}_v)$. Then

$$I(s; V_\xi, \phi) = \widetilde{E}(1-s) \prod_v L(1-s, \widetilde{\pi}_v \times \widetilde{\tau}_v),$$

where $\widetilde{E}(s) = \prod_v \widetilde{E}_v(s)$.

Hence the two integrals $I(s; U_\xi, \phi)$ and $I(s; V_\xi, \phi)$ continue to meromorphic or analytic functions of s if the respective L-functions $\Lambda(s, \pi \times \tau)$ and $\Lambda(1-s, \widetilde{\pi} \times \widetilde{\tau})$ do. In addition, if these L-functions satisfy the standard functional equation, together with the local functional equation (cf. [3, p. 169]), it follows that the two analytically-continued integrals are in fact equal.

In what follows, we abuse notation and identify GL_{n-1} with its image in P_n via the embedding $h \mapsto \left(\begin{smallmatrix} h & \\ & 1 \end{smallmatrix}\right)$. We now prove a slight generalization of [3, Prop. 10.2]:

Proposition 3.1. *We have $U_\xi(g) = V_\xi(g)$ for all $g \in \mathrm{GL}_n(F_\infty)\,\mathrm{GL}_{n-1}(\mathbb{A}_F) Z_n(\mathbb{A}_F) K_0(\mathfrak{n})$.*

Proof. Since $K_0(\mathfrak{n})$ (resp. $Z_n(\mathbb{A}_F)$) acts on ξ_f^0 (resp. $\xi_\infty \otimes \xi_f^0$) through the central character, it is sufficient to prove the identity for $g \in \mathrm{GL}_n(F_\infty)\,\mathrm{GL}_{n-1}(\mathbb{A}_F)$. First, we prove it for $g \in \mathrm{GL}_{n-1}(\mathbb{A}_F)$. As in the proof of [3, Prop. 10.2], this follows from the Langlands spectral theory, provided that one knows the expected analytic properties of $\Lambda(s, \pi \times \tau)$ for all unramified, generic, automorphic subrepresentations τ of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$. Here we only assume these for unitary isobaric representations. Our proof proceeds by passing from unitary to non-unitary isobaric representations, and then to generic subrepresentations.

To that end, we first take τ_1 to be the isobaric sum of $n-1$ trivial characters, so that $\Lambda(s, \tau_1) = \xi_F(s)^{n-1}$, where $\xi_F(s)$ is the complete Dedekind zeta-function, and $\Lambda(s, \pi \times \tau_1) = \Lambda(s, \pi)^{n-1}$. (This follows from the multiplicativity property in Remark 1, which will be put to repeated use throughout this proof without further mention.) Thus, by our hypotheses, $\Lambda(s, \pi)^{n-1}$ continues to an entire function of finite order and does not vanish identically. Hence,

$$\frac{\Lambda'}{\Lambda}(s, \pi) = \frac{1}{n-1}\frac{d}{ds}\log\bigl(\Lambda(s, \pi)^{n-1}\bigr)$$

has meromorphic continuation to $\mathbb{C}$.

Next, we take τ_2 to be the isobaric sum of one trivial character and $n-2$ copies of the character $\|\cdot\|^{it}$ for a fixed $t \in \mathbb{R}$, so that $\Lambda(s, \tau_2) = \xi_F(s)\xi_F(s+it)^{n-2}$ and $\Lambda(s, \pi \times \tau_2) = \Lambda(s, \pi)\Lambda(s+it, \pi)^{n-2}$. Then as above we find that

$$\frac{\Lambda'}{\Lambda}(s, \pi) + (n-2)\frac{\Lambda'}{\Lambda}(s+it, \pi)$$

has meromorphic continuation to $\mathbb{C}$ and has non-negative integral residues at every point. Now, since $\frac{\Lambda'}{\Lambda}(s, \pi)$ has at most countably many poles, there exists $t \in \mathbb{R}$ such that $\frac{\Lambda'}{\Lambda}(s+it, \pi)$ and $\frac{\Lambda'}{\Lambda}(s, \pi)$ have no poles in common. Hence, $\frac{\Lambda'}{\Lambda}(s, \pi)$ has non-negative integral residues, and therefore $\Lambda(s, \pi)$ continues to an entire function. Moreover, from the functional equation

$$\Lambda(s, \pi)^{n-1} = \Lambda(s, \pi \times \tau_1) = \epsilon(s, \pi \times \tau_1)\Lambda(1-s, \widetilde{\pi} \times \widetilde{\tau}_1) = \epsilon(s, \pi)^{n-1}\Lambda(1-s, \widetilde{\pi})^{n-1},$$

we derive

$$\Lambda(s, \pi) = \mu\epsilon(s, \pi)\Lambda(1-s, \widetilde{\pi}),$$

where μ is an $(n-1)$st root of unity (which may depend on π), and similarly the finite order of $\Lambda(s, \pi)$ follows from that of $\Lambda(s, \pi)^{n-1}$.

Now, let σ be an unramified unitary cuspidal representation of $\mathrm{GL}_r(\mathbb{A}_F)$ for some $r < n$, and let τ_3 be the isobaric sum of σ and $n-1-r$ copies of $\|\cdot\|^{it}$ for $t \in \mathbb{R}$. Then arguing as above we see that $\Lambda(s, \pi \times \sigma)$ continues to an entire function of finite order, and from the functional equations for $\Lambda(s, \pi \times \tau_3)$ and $\Lambda(s, \pi)$, we derive

$$\Lambda(s, \pi \times \sigma) = \mu^r\epsilon(s, \pi \times \sigma)\Lambda(1-s, \widetilde{\pi} \times \widetilde{\sigma}).$$

Since every cuspidal representation is unitary up to twisting by a power of the determinant, by shifting s in this equation by a real displacement, we conclude the same properties of $\Lambda(s, \pi \times \sigma)$ for every cuspidal representation σ, not necessarily unitary.

For $i = 1, \ldots, k$, let σ_i be an unramified (not necessarily unitary) cuspidal representation of $\mathrm{GL}_{r_i}(\mathbb{A}_F)$, assume that $r_1 + \ldots + r_k = n-1$, and put $\tau = \sigma_1 \boxplus \cdots \boxplus \sigma_k$. Then by the above we see that $\Lambda(s, \pi \times \tau)$ continues to an entire function of finite order and satisfies the functional equation

$$\Lambda(s, \pi \times \tau) = \mu^{r_1+\ldots+r_k}\epsilon(s, \pi \times \sigma_1)\cdots\epsilon(s, \pi \times \sigma_k)\Lambda(1-s, \widetilde{\pi} \times \widetilde{\tau}) = \epsilon(s, \pi \times \tau)\Lambda(1-s, \widetilde{\pi} \times \widetilde{\tau}).$$

Now let τ' be an unramified, generic, automorphic subrepresentation of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$. By Langlands' classification, it can be realized as a subquotient of an induced (parabolic) representation of the form $\mathrm{Ind}_P^{\mathrm{GL}_{n-1}(\mathbb{A}_F)}(\sigma_1 \otimes \cdots \otimes \sigma_k)$ for some cuspidal automorphic representations σ_i. Since τ' is a subrepresentation to begin with, as explained in the proof of [3, Prop. 6.1], it is in fact a subrepresentation of the induced representation. Now, at a finite place v where τ'_v is unramified, by [3, p. 201, proof of Prop. 10.5] τ'_v is the full induced representation; in particular, $\tau'_v = \tau_v$, where τ is the isobaric sum $\sigma_1 \boxplus \cdots \boxplus \sigma_k$. This need not be the case at archimedean places, but by the argument leading up to (3.1) (with the roles of π and τ reversed), we see that $L(s, \pi_v \times \tau'_v)$ is a polynomial multiple of $L(s, \pi_v \times \tau_v)$ and that the corresponding local γ-factors are the same. Thus, the analytic properties of $\Lambda(s, \pi \times \tau')$ follow from those of $\Lambda(s, \pi \times \tau)$.

It remains to prove the assertion for any $g = (g_\infty, g_f)$ with $g_\infty \in \mathrm{GL}_n(F_\infty)$, $g_f = \begin{pmatrix} g'_f & \\ & 1 \end{pmatrix}$, $g'_f \in \mathrm{GL}_{n-1}(\mathbb{A}_{F,f})$. To this end, let $F_{\xi_\infty} = U_\xi - V_\xi$, where $\xi = \xi_\infty \otimes \xi_f^0$. Then we have $F_{\xi_\infty}((1, g_f)) = 0$ from the above conclusion. In other words, the linear functional on V_{π_∞} given by

$$\xi_\infty \mapsto F_{\xi_\infty}((1, g_f))$$

is trivial. Since $\xi_\infty \mapsto W_{\xi_\infty}$ is continuous on the Casselman–Wallach completion of V_{π_∞} and the K_∞-finite vectors are dense in this completion, it follows that $F_{\xi_\infty}((1, g_f)) = 0$ for all smooth vectors ξ_∞. Finally, fixing a pure tensor ξ_∞ in V_{π_∞} as in the statement of the Proposition, we have $F_{\xi_\infty}(g) = F_{g_\infty \cdot \xi_\infty}((1, g_f)) = 0$. □

3.3. **Congruence subgroups and classical automorphic forms.** Let $\{t_1, \ldots, t_h\} \subset \mathbb{A}_{F,f}^\times$ be a set of representatives for the ideal class group of F, with $t_1 = 1$, and let $\mathfrak{a}_j$ denote the ideal generated by t_j, which we assume to be integral. Put $g_j = \mathrm{diag}(t_j, 1, \ldots, 1) \in \mathrm{GL}_n(\mathbb{A}_{F,f})$. For $\xi_\infty \in V_{\pi_\infty}$, we associate the h-tuple of functions $(\Phi_{\xi_\infty,1}, \ldots, \Phi_{\xi_\infty,h})$ given by

$$\Phi_{\xi_\infty,j}(g) = U_{\xi_\infty \otimes \xi_f^0}((g, g_j)) = V_{\xi_\infty \otimes \xi_f^0}((g, g_j)) \quad \text{for } g \in \mathrm{GL}_n(F_\infty).$$

In the notation of §2, for $j = 1, \ldots, h$, let us set $G_j = \Gamma_{1,j}(\mathfrak{o}_F) = \Gamma_{0,j}(\mathfrak{o}_F)$. In concrete terms,

$$G_j = \left\{ \gamma \in \begin{pmatrix} * & \mathfrak{a}_j & \cdots & \mathfrak{a}_j \\ \mathfrak{a}_j^{-1} & & & \\ \vdots & & * & \\ \mathfrak{a}_j^{-1} & & & \end{pmatrix} : \det \gamma \in \mathfrak{o}_F^\times \right\}.$$

For each j, $\Gamma_{1,j}(\mathfrak{n}) \subseteq \Gamma_{0,j}(\mathfrak{n})$ are then subgroups of G_j. For instance, if $\gamma = (\gamma_{kl}) \in \Gamma_{1,j}(\mathfrak{n})$, then the congruence condition on its last row is given by

$$\gamma_{n1} \in \mathfrak{n}\mathfrak{a}_j^{-1}, \gamma_{n2} \in \mathfrak{n}, \ldots, \gamma_{nn} - 1 \in \mathfrak{n}.$$

Now, for $i = 0, 1$, $j = 1, \ldots, h$, let

$$\Gamma_{i,j}^P(\mathfrak{n}) = Z_n(F)P_n(F) \cap \mathrm{GL}_n(F_\infty) g_j K_i(\mathfrak{n}) g_j^{-1},$$
$$\Gamma_{i,j}^Q(\mathfrak{n}) = Z_n(F)Q_n(F) \cap \mathrm{GL}_n(F_\infty) g_j K_i(\mathfrak{n}) g_j^{-1},$$

which are subgroups of $\Gamma_{i,j}(\mathfrak{n}) \subseteq G_j \subset \mathrm{GL}_n(F_\infty)$. Since the functions $U_{\xi_\infty \otimes \xi_f^0}$ and $V_{\xi_\infty \otimes \xi_f^0}$ are left invariant under $Z_n(F)P_n(F)$ and $Z_n(F)Q_n(F)$, respectively, and ξ_f^0 is fixed by $K_1(\mathfrak{n})$, we see that $\Phi_{\xi_\infty,j}$ is invariant on the left by both $\Gamma_{1,j}^P(\mathfrak{n})$ and $\Gamma_{1,j}^Q(\mathfrak{n})$ for $j = 1, \ldots, h$. We now need the following result, generalizing [3, Prop. 9.1].

Proposition 3.2. *For $j = 1, \dots, h$, the groups $\Gamma^P_{1,j}(\mathfrak{n})$ and $\Gamma^Q_{1,j}(\mathfrak{n})$ together generate the congruence subgroup $\Gamma_{1,j}(\mathfrak{n})$.*

Proof. Let $\gamma \in \Gamma_{1,j}(\mathfrak{n})$ be a typical element, and let $(a_1, \dots, a_n)$ be its bottom row. Recall that $\det \gamma$ is a unit in $\mathfrak{o}_F$. Expanding the determinant along the bottom row, we find that

$$1 = \sum_{i=1}^{n} c_i a_i$$

for some $c_1 \in \mathfrak{a}_j$, $c_2, \dots, c_n \in \mathfrak{o}_F$. In particular, $(c_1 a_1, a_2, \dots, a_n) \in \mathfrak{o}_F^n$ is unimodular. Since $n \geq 3$, it follows from the Bass stable range theorem [1, Thm. 11.1] that there are $b_2, \dots, b_n \in \mathfrak{o}_F$ such that if $a'_i = a_i + b_i c_1 a_1$ then $(a'_2, \dots, a'_n)$ is unimodular. Put

$$\sigma = \begin{pmatrix} 1 & b_2 c_1 & \cdots & b_n c_1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

so that $\sigma \in \Gamma^P_{1,j}(\mathfrak{n})$ and $(a_1, \dots, a_n)\sigma = (a_1, a'_2, \dots, a'_n)$. Hence, replacing γ by $\gamma\sigma$ if necessary, we may assume without loss of generality that $(a_2, \dots, a_n)$ is unimodular.

Next let $\mathfrak{m} = a_2 \mathfrak{o}_F + \dots + a_{n-1}\mathfrak{o}_F$, so that $\mathfrak{m} + a_n \mathfrak{o}_F = \mathfrak{o}_F$. In particular, if $\mathfrak{m}$ is the zero ideal then a_n is a unit; but then right-multiplying by the matrix

$$\tau = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ -a_1 a_n^{-1} & 0 & \cdots & 0 & a_n^{-1} \end{pmatrix} \in \Gamma^Q_{1,j}(\mathfrak{n})$$

reduces the bottom row of γ to $(0, \dots, 0, 1)$, so that $\gamma\tau \in \Gamma^P_{1,j}(\mathfrak{n})$, and we are finished. Hence, we may assume that $\mathfrak{m}$ is non-zero.

Choose $y_1 \in \mathfrak{a}_j \setminus \bigcup_{\mathfrak{p} \supseteq \mathfrak{m}} \mathfrak{p}\mathfrak{a}_j$ and $z \in \mathfrak{a}_j^{-1} \setminus \bigcup_{\mathfrak{p} \supseteq \mathfrak{m}} \mathfrak{p}\mathfrak{a}_j^{-1}$. Then $z y_1 \in \mathfrak{o}_F$ is invertible modulo $\mathfrak{m}$. Let $z' \in \mathfrak{o}_F$ be a multiplicative inverse of $z y_1 \pmod{\mathfrak{m}}$, and set $u_1 = z z' \in \mathfrak{a}_j^{-1}$, so that $u_1 y_1 \equiv 1 \pmod{\mathfrak{m}}$. Further, since $\mathfrak{a}_j^{-1}\mathfrak{a}_j = \mathfrak{o}_F$, there are elements $u_2, \dots, u_K \in \mathfrak{a}_j^{-1}$ and $v_2, \dots, v_K \in \mathfrak{a}_j$ such that $1 = \sum_{k=2}^{K} u_k v_k$. Setting $y_k = (1 - u_1 y_1) v_k \in \mathfrak{a}_j \mathfrak{m}$ for $k \geq 2$, we have

$$1 = \sum_{k=1}^{K} u_k y_k \quad \text{and} \quad a_n + (1 - a_n) \sum_{k=1}^{K'} u_k y_k \equiv 1 \pmod{\mathfrak{m}}$$

for $K' = 1, \dots, K$. Next we set $x_k = (a_n - 1) u_k \in \mathfrak{a}_j^{-1}\mathfrak{n}$ so that

$$a_n - 1 = \sum_{k=1}^{K} x_k y_k,$$

and $a_n - \sum_{k=1}^{K'} x_k y_k$ is invertible modulo $\mathfrak{m}$ for $K' = 0, \dots, K$.

By unimodularity there exist $d_2, \dots, d_n \in \mathfrak{o}_F$ such that $d_2 a_2 + \dots + d_n a_n = 1$. If we put

$$\tau_1 = \begin{pmatrix} 1 & & & \\ d_2(x_1 - a_1) & 1 & & \\ \vdots & & \ddots & \\ d_n(x_1 - a_1) & & & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & & -y_1 \\ & \ddots & \\ & & 1 \end{pmatrix},$$

then $\tau_1 \in \Gamma^Q_{1,j}(\mathfrak{n})$, $\sigma_1 \in \Gamma^P_{1,j}(\mathfrak{n})$, and $(a_1, \dots, a_n)\tau_1\sigma_1 = (x_1, a_2, \dots, a_{n-1}, a_n - x_1 y_1)$. Moreover, since $a_n - x_1 y_1$ is invertible modulo $\mathfrak{m}$, we have $\mathfrak{m} + (a_n - x_1 y_1)\mathfrak{o}_F = \mathfrak{o}_F$, so that $(a_2, \dots, a_{n-1}, a_n - x_1 y_1)$ is again unimodular. Thus, we may repeat the construction with

a_n replaced by $a_n - \sum_{k=1}^{K'} x_k y_k$ for $K' = 1, \ldots, K-1$, obtaining matrices $\tau_2, \sigma_2, \ldots, \tau_K, \sigma_K$ such that

$$(a_1, \ldots, a_n)\tau_1\sigma_1 \cdots \tau_K\sigma_K = (x_K, a_2, \ldots, a_{n-1}, 1).$$

Finally, applying the matrix

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ -x_K & -a_2 & \cdots & -a_{n-1} & 1 \end{pmatrix} \in \Gamma_{1,j}^Q(\mathfrak{n})$$

reduces the bottom row to $(0, \ldots, 0, 1)$, and we are finished. $\square$

Thus, for each j, the function $\Phi_{\xi_\infty,j}$ is left invariant under $\Gamma_{1,j}(\mathfrak{n})$. Indeed, these are classical automorphic forms. To be precise, observe that for each j, χ_π also determines a character of $\Gamma_{0,j}(\mathfrak{n})$ that is trivial on $\Gamma_{1,j}(\mathfrak{n})$, which we continue to denote by χ_π. Let $\mathcal{A}(\Gamma_{0,j}(\mathfrak{n})\backslash \mathrm{GL}_n(F_\infty); \omega_{\pi_\infty}, \chi_\pi^{-1})$ denote the space of classical automorphic forms f on $\mathrm{GL}_n(F_\infty)$ satisfying

$$\begin{aligned} f(\gamma g) &= \chi_\pi^{-1}(\gamma) f(g) \quad \text{for all } \gamma \in \Gamma_{0,j}(\mathfrak{n}) \subset \mathrm{GL}_n(F_\infty), \\ f(zg) &= \omega_{\pi_\infty}(z) f(g) \quad \text{for all } z \in Z_n(F_\infty). \end{aligned}$$

Then it follows that $\Phi_{\xi_\infty,j}$ belongs to $\mathcal{A}(\Gamma_{0,j}(\mathfrak{n})\backslash \mathrm{GL}_n(F_\infty); \omega_{\pi_\infty}, \chi_\pi^{-1})$. (The relevant growth properties follow from [4].) The character χ_π^{-1} is usually referred to as the Nebentypus character.

Now, let $\mathcal{A}(\omega_\pi)$ denote the space of automorphic forms on $\mathrm{GL}_n(\mathbb{A}_F)$ which transform under the central character ω_π. Then the isomorphism (2.1) induces a topological isomorphism

$$\mathcal{A}(\omega_\pi)^{K_1(\mathfrak{n})} \cong \coprod_{j=1}^{h} \mathcal{A}(\Gamma_{0,j}(\mathfrak{n})\backslash \mathrm{GL}_n(F_\infty); \omega_{\pi_\infty}, \chi_\pi^{-1}). \tag{3.4}$$

In particular, the family of functions $\{\Phi_{\xi_\infty,j}\}_{j=1}^h$ determine a global automorphic form Φ_{ξ_∞} through this isomorphism. Explicitly, given $g \in \mathrm{GL}_n(\mathbb{A}_F)$, choose j (which is uniquely determined) and $\gamma \in \mathrm{GL}_n(F)$ so that $\gamma g \in g_j \mathrm{GL}_n(F_\infty) K_1(\mathfrak{n})$; then $\Phi_{\xi_\infty}(g) = \Phi_{\xi_\infty,j}(\gamma_\infty g_\infty)$. One checks that this is well defined, in the sense that it is independent of the choice of γ. In the reverse direction, as mentioned above, we have

$$\Phi_{\xi_\infty,j}(g) = \Phi_{\xi_\infty}(g, g_j) \quad \text{for all } g \in \mathrm{GL}_n(F_\infty),\ j = 1, \ldots, h.$$

For later reference, we note that Φ_{ξ_∞} satisfies the relation

$$\Phi_{\xi_\infty}(g) = U_{\xi_\infty \otimes \xi_f^0}(g) \quad \text{for } g \in P_n(\mathbb{A}_F), \tag{3.5}$$

which readily follows from its construction.

3.4. **Hecke eigenforms and automorphic representations.** We continue with the notation of §3.1. It is clear that (3.4) is an isomorphism of $(\mathfrak{g}_\infty, K_\infty)$-modules. Then the mapping $\xi_\infty \mapsto \Phi_{\xi_\infty}$ gives us a canonical embedding $\iota : V_{\pi_\infty} \to \mathcal{A}(\omega_\pi)$ of $(\mathfrak{g}_\infty, K_\infty)$-modules. Moreover, since π_∞ is irreducible, the center $\mathcal{Z}$ of $\mathcal{U}$ acts on V_{π_∞} through a character, say λ. Let (Π, W) be the smallest $\mathrm{GL}_n(\mathbb{A}_F)$-submodule of $\mathcal{A}(\omega_\pi)$ containing $\iota(V_{\pi_\infty})$, which is admissible according to [2, Prop. 4.5]. Indeed, for a fixed $\xi_\infty \neq 0$, $W = \mathcal{H} \star \Phi_{\xi_\infty}$. In [3, Prop. 10.4], Cogdell and Piatetski-Shapiro show that Π consists of Hecke eigenforms for an appropriate Hecke algebra, under the assumption that $h = 1$. Our proof departs from their approach in what follows.

For each v, as mentioned in §3.1, π_v induces an action of $\mathcal{H}_v$, which we continue to denote by π_v. For Δ a finite collection of irreducible representations of K_∞, let $e_\Delta \in \mathcal{H}_\infty$ be the corresponding idempotent. We write $V_{\pi_\infty}(\Delta)$ to denote the image of the operator $\pi_\infty(e_\Delta)$, i.e. $V_{\pi_\infty}(\Delta)$ is the sum of the δ-isotopic components $V_{\pi_\infty}(\delta)$ for $\delta \in \Delta$. This is a finite-dimensional vector space since π_∞ is admissible. Further, if $V_{\pi_\infty}(\Delta) \neq 0$, then it is an irreducible module for the (unital) subalgebra $\mathcal{H}_\infty(\Delta) = e_\Delta * \mathcal{H}_\infty * e_\Delta$. Before we proceed further, we need the following basic result from the theory of finite-dimensional representations of unital algebras.

Lemma 3.3. *Suppose A and B are unital algebras over $\mathbb{C}$. Set $C = A \otimes B$, and suppose (ρ, E) is a finite-dimensional representation of C. Let M be a simple A-module, and set $M' = \mathrm{Hom}_A(M, E)$. Consider the left B-module structure of M' coming from that of the left B-module E, and regard $M \otimes M'$ as a C-module. Then the natural map $\alpha : M \otimes M' \to E$ induced by $v \otimes f \mapsto f(v)$ is a monomorphism of C-modules.*

Proof. First, it is straightforward to see that α is a morphism of C-modules. Let $I = \mathrm{Ann}_A(M)$; then I is cofinite and, since M is a simple A-module, it follows that $A/I \cong M_n(\mathbb{C})$ for some n. Thus, by reducing to $A/I \otimes B$, we may assume that A is simple Artinian. Now, let $\{v_1, \dots, v_n\}$ be a basis for M, and let $e_{kl} \in A = M_n(\mathbb{C})$ be the matrix units with respect to that basis. If $x = \sum_{i=1}^n v_i \otimes f_i \in M \otimes M'$ is such that $\alpha(x) = 0$, then $0 = e_{kl} \cdot \alpha(x) = \alpha(e_{kl} \cdot \sum_i v_i \otimes f_i) = \alpha(v_k \otimes f_l) = f_l(v_k)$, for all k, l. Thus, $f_l = 0$ for all l, i.e. $x = 0$, and hence α is injective. □

Next we show that one can split π_∞ off from Π in the following sense.

Lemma 3.4. *There exists a smooth admissible $\mathrm{GL}_n(\mathbb{A}_{F,f})$-module (Π_f, U) such that*

$$\pi_\infty \otimes \Pi_f \cong \Pi$$

as $\mathrm{GL}_n(\mathbb{A}_F)$-modules. Further, there exists $\Phi_f \in U$ such that for any $\xi_\infty \in V_{\pi_\infty}$, $\xi_\infty \otimes \Phi_f \mapsto \Phi_{\xi_\infty}$ under this isomorphism.

Proof. Let (Π_f, U) denote the $\mathrm{GL}_n(\mathbb{A}_{F,f})$-module $\mathrm{Hom}_{\mathcal{H}_\infty}(V_{\pi_\infty}, W)$ and let $\Phi_f \in U$ be the element $\xi_\infty \mapsto \Phi_{\xi_\infty}$. Then $\Phi_f \in U^{K_1(\mathfrak{n})}$ as every Φ_{ξ_∞} is right $K_1(\mathfrak{n})$-invariant. Since W is smooth and admissible, and V_{π_∞} is a cyclic $\mathcal{H}_\infty$-module, it follows that U is a smooth, admissible $\mathrm{GL}_n(\mathbb{A}_{F,f})$-module. There is a natural homomorphism of $\mathrm{GL}_n(\mathbb{A}_F)$-modules from

$$V_{\pi_\infty} \otimes \mathrm{Hom}_{\mathcal{H}_\infty}(V_{\pi_\infty}, W) \longrightarrow W, \tag{3.6}$$

and it is clear that $\xi_\infty \otimes \Phi_f \mapsto \Phi_{\xi_\infty}$, for $\xi_\infty \in V_{\pi_\infty}$, under this morphism. It is surjective since W is cyclic and generated by any nonzero Φ_{ξ_∞}. Finally, in order to show that (3.6) is injective, it is sufficient to do so after fixing a level $H = \prod_{v<\infty} H_v$, a compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_{F,f})$, and an infinity type Δ. To this end, take $A = \mathcal{H}_\infty(\Delta)$, $B = \mathcal{H}(\mathrm{GL}_n(\mathbb{A}_{F,f}), H) = \bigotimes_{v<\infty} \mathcal{H}(\mathrm{GL}_n(F_v), H_v)$, $M = V_{\pi_\infty}(\Delta)$, and $E = W(\Delta, H)$, and apply Lemma 3.3. □

Now, let T be the smallest finite set of places of F containing the archimedean places such that π_v is unramified at all $v \notin T$. By construction Π is also unramified at every $v \notin T$. Let $\mathcal{H}^T = \bigotimes_{v \notin T} \mathcal{H}(\mathrm{GL}_n(F_v), K_v)$ denote the spherical Hecke algebra, which is known to be commutative. Then, as explained in [3, §2, Appendix], $\mathcal{H}^T$ is naturally a subalgebra of

$\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_{F,f}), K_1(\mathfrak{n}))$, and hence acts on the space of $K_1(\mathfrak{n})$-fixed vectors $U^{K_1(\mathfrak{n})}$. Therefore, $U^{K_1(\mathfrak{n})}$ has a basis consisting of Hecke eigenvectors for the action of the algebra $\mathcal{H}^T$. In particular, we may write

$$\Phi_f = \sum_{i=1}^{m} \eta_i, \tag{3.7}$$

where η_i is a Hecke eigenvector with eigencharacter Λ_i, say.

For $1 \le i \le m$, put $V_i = \mathbb{C}\eta_i$, and let U_i be the smallest $\mathrm{GL}_n(\mathbb{A}_{F,f})$-submodule of U that contains η_i, viz. $U_i = \mathcal{H}_f \star V_i$. Then U_i is admissible and $U_i^{K^T} = V_i$, where $K^T = \prod_{v \notin T} K_v$ is the maximal compact open subgroup of G^T. Now, let (π_i', U_i') denote the unique spherical representation of G^T associated to Λ_i. Then, by an argument identical to that in Lemma 3.4, it follows that there is an admissible representation U_i'' of $\prod_{\substack{v\in T\\ v<\infty}} \mathrm{GL}_n(F_v)$ such that $U_i \cong U_i' \otimes U_i''$ as $\mathcal{H}_f$-modules, or equivalently, as representations of $\mathrm{GL}_n(\mathbb{A}_{F,f})$. Therefore, we may assume that each η_i is of the form $\eta_{i,T} \otimes \bigotimes_{v\notin T} \eta_{i,v}^0$, where $\eta_{i,v}^0$ is the spherical vector at v (normalized to give the correct local L-factor, as in (3.2)), and $\eta_{i,T}$ is a vector belonging to the space of an admissible representation of $\prod_{\substack{v\in T\\ v<\infty}} \mathrm{GL}_n(F_v)$.

Lemma 3.5. *There exists a unique isobaric automorphic representation π_i such that $\pi_i^T = \bigotimes_{v\notin T} \pi_{i,v}$ is the unique irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_F^T)$ associated to the character Λ_i.*

Proof. By [3, Thm. A], there exists an irreducible (but not necessarily isobaric) automorphic representation Π_i with the required property and also satisfying $\Pi_{i,\infty} \cong \pi_\infty$. We may realize Π_i as a component of an induced representation $\Xi = \mathrm{Ind}(\sigma_1 \otimes \cdots \otimes \sigma_k)$ of Langlands type, where the σ_j are cuspidal representations of $\mathrm{GL}_{r_j}(\mathbb{A}_F)$ with $r_1 + \ldots + r_k = n$. Since $\Pi_{i,v}$ is unramified for $v \notin T$, it follows that the representation $\Xi_v = \mathrm{Ind}(\sigma_{1,v} \otimes \cdots \otimes \sigma_{k,v})$ is also unramified and that $\Pi_{i,v}$ is the unique spherical constituent of Ξ_v. Let π_i be the isobaric representation $\sigma_1 \boxplus \cdots \boxplus \sigma_k$. Then, since the Langlands quotient of Ξ_v is the same as the unique spherical constituent for an unramified place v, it follows that $\pi_{i,v} \cong \Pi_{i,v}$ for $v \notin T$. The uniqueness of π_i follows from the strong form of multiplicity one (cf. Remark 1). $\square$

3.5. **A linear relation of L-functions.** Next, given any automorphic form ϕ on $\mathrm{GL}_n(\mathbb{A}_F)$, we recall that its Whittaker–Fourier coefficient W_ϕ (for a fixed ψ) is defined as

$$W_\phi(g) = \int_{U_n(F)\backslash U_n(\mathbb{A}_F)} \phi(ug)\overline{\psi}(u)\, du.$$

Lemma 3.6. *Assume the measure is normalized so that $F\backslash\mathbb{A}_F$ has unit volume. For $\xi_\infty \in V_{\pi_\infty}$, let Φ_{ξ_∞} be as defined after* (3.4)*. Then its Whittaker–Fourier coefficient satisfies*

$$W_{\Phi_{\xi_\infty}}(g) = W_\xi(g) \quad \text{for } g \in P_n(\mathbb{A}_F),$$

where $\xi = \xi_\infty \otimes \xi_f^0$.

Proof. For any subgroup $N \subseteq U_n$, we write $[N]$ to denote the adelic quotient $N(F)\backslash N(\mathbb{A}_F)$. Also, for $m < n$, we view GL_m as a subgroup of GL_n via the diagonal embedding $h \mapsto$

$\left(\begin{smallmatrix} h & \\ & I_{n-m}\end{smallmatrix}\right)$. It follows from (3.5) that for $g \in P_n(\mathbb{A}_F)$,

$$\begin{aligned} W_{\Phi_{\xi_\infty}}(g) = \int_{[U_n]} U_\xi(ug)\overline{\psi}(u)\,du &= \int_{[U_n]} \sum_{\gamma \in U_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} W_\xi\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} ug\right)\overline{\psi}(u)\,du, \\ &= \sum_{\gamma \in U_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} \int_{[U_n]} W_\xi\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} ug\right)\overline{\psi}(u)\,du. \end{aligned}$$

Let us write $u = u_1u_2$, $u_1 \in N_n(\mathbb{A}_F)$, $u_2 \in U_{n-1}(\mathbb{A}_F) \subset \mathrm{GL}_{n-1}(\mathbb{A}_F) \subset \mathrm{GL}_n(\mathbb{A}_F)$. We further write elements of N_n as $u(X) = \left(\begin{smallmatrix} I_{n-1} & X \\ & 1\end{smallmatrix}\right)$, where X is a column vector. Since $\left(\begin{smallmatrix} \gamma & \\ & 1\end{smallmatrix}\right)$ normalizes N_n, we get

$$W_{\Phi_{\xi_\infty}}(g) = \sum_{\gamma \in U_{n-1}(F)\backslash \mathrm{GL}_{n-1}(F)} \int_{(F\backslash \mathbb{A}_F)^{n-1}} \psi(u(\gamma X - X))\,dX \int_{[U_{n-1}]} W_\xi\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} u_2 g\right)\overline{\psi}(u_2)\,du_2.$$

It is straightforward to check that the first integral in the above expression vanishes unless $\gamma \in P_{n-1}(F)$. Since $U_{n-1}(F)\backslash P_{n-1}(F)$ may be identified with $U_{n-2}(F)\backslash \mathrm{GL}_{n-2}(F)$ via $\mu \mapsto \left(\begin{smallmatrix} \mu & \\ & 1\end{smallmatrix}\right)$, we finally obtain

$$W_{\Phi_{\xi_\infty}}(g) = \sum_{\gamma \in U_{n-2}(F)\backslash \mathrm{GL}_{n-2}(F)} \int_{[U_{n-1}]} W_\xi\left(\begin{pmatrix}\gamma & \\ & I_2\end{pmatrix} ug\right)\overline{\psi}(u)\,du.$$

We may now argue inductively to obtain the desired conclusion. $\square$

Let $\Phi_i \in W$ be the automorphic form corresponding to $\xi_\infty \otimes \eta_i$. Then it follows from the above lemma and (3.7) that

$$W_\xi\begin{pmatrix} h & \\ & 1\end{pmatrix} = \sum_{i=1}^m W_{\Phi_i}\begin{pmatrix} h & \\ & 1\end{pmatrix} \quad \text{for } h \in \mathrm{GL}_{n-1}(\mathbb{A}_F).$$

Choosing ξ_∞ so that $W_{\xi_\infty}(I_n) \neq 0$, we evaluate this at (I_{n-1}, h) for $h \in \mathrm{GL}_{n-1}(\mathbb{A}_{F,f})$ and cancel the factor of $W_{\xi_\infty}(I_n)$ on both sides to get

$$\prod_{v<\infty} W_{\xi_v^0}\begin{pmatrix} h_v & \\ & 1\end{pmatrix} = \sum_{i=1}^m W_{\eta_i}\begin{pmatrix} h & \\ & 1\end{pmatrix} \quad \text{for } h \in \mathrm{GL}_{n-1}(\mathbb{A}_{F,f}).$$

For a finite $v \in T$, according to [11, Prop. 3.2], we can choose $h_v^0 \in \mathrm{GL}_{n-1}(F_v)$ so that $W_{\xi_v^0}\left(\begin{smallmatrix} h_v^0 & \\ & 1\end{smallmatrix}\right) \neq 0$. Then since η_i is of the form $\eta_{i,T} \otimes \bigotimes_{v\notin T} \eta_{i,v}^0$, we may evaluate the above at $\left(\prod_{\substack{v\in T\\ v<\infty}} h_v^0, h\right)$ for $h \in \mathrm{GL}_{n-1}(\mathbb{A}_F^T)$ and divide by $\prod_{\substack{v\in T\\ v<\infty}} W_{\xi_v^0}\left(\begin{smallmatrix} h_v^0 & \\ & 1\end{smallmatrix}\right)$, to obtain

$$\prod_{v\notin T} W_{\xi_v^0}\begin{pmatrix} h_v & \\ & 1\end{pmatrix} = \sum_{i=1}^m c_i \prod_{v\notin T} W_{\eta_{i,v}^0}\begin{pmatrix} h_v & \\ & 1\end{pmatrix} \quad \text{for } h \in \mathrm{GL}_{n-1}(\mathbb{A}_F^T), \tag{3.8}$$

for some constants $c_i \in \mathbb{C}$.

Note that $\mathrm{GL}_1(\mathbb{A}_F) = \mathbb{A}_F^\times$ is a subgroup of $\mathrm{GL}_{n-1}(\mathbb{A}_F)$ as described at the beginning of the proof of Lemma 3.6. So, for any unramified idèle class character ω of F, we may multiply (3.8) by $\omega(h)\|\det h\|^{s-\frac{n-1}{2}}$ for $h \in (\mathbb{A}_F^T)^\times$, integrate over h and use (3.2) to get

$$L^T(s, \pi\otimes\omega) = \sum_{i=1}^m c_i L^T(s, \pi_i \otimes \omega). \tag{3.9}$$

Here L^T denotes the partial Euler product over all places $v \notin T$, viz.
(3.10)

$$L^T(s, \pi \otimes \omega) = \prod_{v \notin T} \prod_{j=1}^{n} \frac{1}{1 - \alpha_{0,v,j}\omega_v(\varpi_v)q_v^{-s}}, \quad L^T(s, \pi_i \otimes \omega) = \prod_{v \notin T} \prod_{j=1}^{n} \frac{1}{1 - \alpha_{i,v,j}\omega_v(\varpi_v)q_v^{-s}},$$

where $\alpha_{0,v,j}$ (resp. $\alpha_{i,v,j}$) are the Satake parameters of π_v (resp. $\pi_{i,v}$).

4. Some multiplicative number theory

In this section, we extend some of the basic notions of multiplicative number theory to number fields. First, let $\mathcal{I}_F$ denote the set of non-zero integral ideals of $\mathfrak{o}_F$ and $\mathcal{P}_F \subset \mathcal{I}_F$ the set of prime ideals. Let D_F denote the set of all functions $f : \mathcal{I}_F \to \mathbb{C}$. Given $f, g \in D_F$, we define their *Dirichlet convolution* $f * g \in D_F$ via

$$f * g(\mathfrak{a}) = \sum_{\substack{\mathfrak{b} \in \mathcal{I}_F \\ \mathfrak{b} \supseteq \mathfrak{a}}} f(\mathfrak{b}) g(\mathfrak{a}\mathfrak{b}^{-1}), \quad \forall \mathfrak{a} \in \mathcal{I}_F.$$

This gives D_F the structure of a commutative ring with multiplicative identity

$$1_{D_F}(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{a} = \mathfrak{o}_F, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, one can show that D_F is an integral domain, though we will not need that in what follows.

For any $f \in D_F$, we say that

(i) f is *multiplicative* if $f(\mathfrak{o}_F) = 1$ and $f(\mathfrak{a}\mathfrak{b}) = f(\mathfrak{a})f(\mathfrak{b})$ for every $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_F$ satisfying $\mathfrak{a} + \mathfrak{b} = \mathfrak{o}_F$;

(ii) f has *polynomial growth* if there exists $\sigma \in \mathbb{R}$ such that $f(\mathfrak{a}) = O(N(\mathfrak{a})^\sigma)$ for all $\mathfrak{a} \in \mathcal{I}_F$;

(iii) f is $\mathfrak{p}$*-finite* if there is a finite set $S \subset \mathcal{P}_F$ such that $f(\mathfrak{a}) = 0$ for every $\mathfrak{a} \in \mathcal{I}_F$ which is contained in a prime ideal $\mathfrak{p} \in \mathcal{P}_F \setminus S$.

Further, we call multiplicative functions $f, g \in D_F$ *equivalent* if there is a finite set $S \subset \mathcal{P}_F$ such that $f(\mathfrak{p}^k) = g(\mathfrak{p}^k)$ for all $\mathfrak{p} \in \mathcal{P}_F \setminus S$ and all $k \geq 1$, and *inequivalent* otherwise.

Finally, let $M_F \subset D_F$ and $R_F \subset D_F$ denote the subsets of multiplicative and $\mathfrak{p}$-finite elements, respectively. It is easy to verify that R_F is a subring of D_F and M_F is a subgroup of the unit group $D_F^\times$.

Lemma 4.1 (adapted from [13], Thm. 2)**.** *Let m be a positive integer and let $f_1, \ldots, f_m \in M_F$ be pairwise inequivalent, multiplicative functions. Then $f_1, \ldots, f_m$ are linearly independent over R_F, i.e. if $c_1, \ldots, c_m \in R_F$ satisfy $\sum_{j=1}^m c_j * f_j = 0$, then $c_1 = \ldots = c_m = 0$ identically.*

Proof. Suppose otherwise, and let $f_1, \ldots, f_m \in M_F$ and $c_1, \ldots, c_m \in R_F$ be a counterexample with m minimal; in particular, none of $c_1, \ldots, c_m$ vanishes identically. Since all elements of M_F are units in D_F, we must have $m > 1$. Let $S \subset \mathcal{P}_F$ be a finite set of primes such that $c_j(\mathfrak{a}) = 0$ for $j = 1, \ldots, m$ whenever $\mathfrak{a}$ has a prime factor outside of S. Since f_1 and f_2 are inequivalent, there exists $\mathfrak{p} \in \mathcal{P}_F \setminus S$ and $k \in \mathbb{Z}_{>0}$ such that $f_1(\mathfrak{p}^k) \neq f_2(\mathfrak{p}^k)$.

We consider the equation

$$\sum_{j=1}^{m} c_j * f_j(\mathfrak{a}) = 0 \tag{4.1}$$

with $\mathfrak{a} = \mathfrak{p}^k\mathfrak{b}$ for all $\mathfrak{b}$ co-prime to $\mathfrak{p}$, obtaining

$$\sum_{j=1}^{m} f_j(\mathfrak{p}^k) \sum_{\mathfrak{n}\supseteq\mathfrak{b}} c_j(\mathfrak{n}) f_j(\mathfrak{b}\mathfrak{n}^{-1}) = 0. \tag{4.2}$$

Next we replace $\mathfrak{a}$ by $\mathfrak{b}$ in (4.1), multiply by $f_1(\mathfrak{p}^k)$, and subtract (4.2) to get

$$\sum_{j=2}^{m} (f_1(\mathfrak{p}^k) - f_j(\mathfrak{p}^k)) \sum_{\mathfrak{n}\supseteq\mathfrak{b}} c_j(\mathfrak{n}) f_j(\mathfrak{b}\mathfrak{n}^{-1}) = 0.$$

Finally, for $j = 2, \ldots, m$ we define

$$\tilde{c}_j(\mathfrak{a}) = (f_1(\mathfrak{p}^k) - f_j(\mathfrak{p}^k))c_j(\mathfrak{a}) \quad \text{and} \quad \tilde{f}_j(\mathfrak{a}) = \begin{cases} 0 & \text{if } \mathfrak{a} \subseteq \mathfrak{p}, \\ f_j(\mathfrak{a}) & \text{otherwise,} \end{cases}$$

so that $\tilde{f}_j \in M_F$, $\tilde{c}_j \in R_F$, $\tilde{c}_2$ is not identically 0, and

$$\sum_{j=2}^{m} \tilde{c}_j * \tilde{f}_j = 0.$$

This contradicts the minimality of m and completes the proof. □

When $F = \mathbb{Q}$, it is well known that one can identify any $f \in D_F$ of polynomial growth with its Dirichlet series $\sum_{n=1}^{\infty} f(n\mathbb{Z})n^{-s}$, which defines a holomorphic function in a right half-plane. When $F \neq \mathbb{Q}$, the map $f \mapsto \sum_{\mathfrak{a}\in\mathcal{I}_F} f(\mathfrak{a})N(\mathfrak{a})^{-s}$ is still a ring homomorphism, but it is no longer injective since there may be multiple ideals with the same norm. However, we recover a one-to-one correspondence if we include the twists by unramified Größencharakters, as the following lemma shows.

Lemma 4.2. *Let $f \in D_F$ be a function of polynomial growth. Suppose that there exists $\sigma \in \mathbb{R}$ such that*

$$\sum_{\mathfrak{a}\in\mathcal{I}_F} f(\mathfrak{a})\chi_\omega(\mathfrak{a})N(\mathfrak{a})^{-s} = 0$$

for every unramified, unitary, idèle class character ω and all $s \in \mathbb{C}$ with $\Re(s) > \sigma$. Then $f = 0$ identically.

Proof. Collecting the terms with a common value of $N(\mathfrak{a})$, we have

$$\sum_{n=1}^{\infty} n^{-s} \sum_{\substack{\mathfrak{a}\in\mathcal{I}_F \\ N(\mathfrak{a})=n}} f(\mathfrak{a})\chi_\omega(\mathfrak{a}) = 0$$

for all unramified unitary characters ω and all s with $\Re(s)$ sufficiently large. Considering the asymptotic behavior as $s \to \infty$, we find that $\sum_{\substack{\mathfrak{a}\in\mathcal{I}_F \\ N(\mathfrak{a})=n}} f(\mathfrak{a})\chi_\omega(\mathfrak{a})$ vanishes for all unramified ω.

Fix a choice of n, and let $\mathfrak{a}_1, \ldots, \mathfrak{a}_m \in \mathcal{I}_F$ be the ideals of norm n. It suffices to show that $\mathbb{C}^m$ is spanned by the vectors $(\chi_\omega(\mathfrak{a}_1), \ldots, \chi_\omega(\mathfrak{a}_m))$, with ω running through all unramified characters. If that is not the case then there exist $c_1, \ldots, c_m \in \mathbb{C}$, not all zero, such that $c_1\chi_\omega(\mathfrak{a}_1) + \ldots + c_m\chi_\omega(\mathfrak{a}_m) = 0$ for all such ω. Reordering if necessary, we may assume that $c_j \neq 0$ for $1 \le j \le k$ and $c_j = 0$ for $k < j \le m$. Further, by scaling we may assume that $c_1 = 1$, so that

$$1 + c_2\chi_\omega(\mathfrak{a}_2\mathfrak{a}_1^{-1}) + \ldots + c_k\chi_\omega(\mathfrak{a}_k\mathfrak{a}_1^{-1}) = 0.$$

Since this holds for all unramified characters ω, we are free to replace ω by any unramified twist $\omega\omega'$. In particular, letting ω' run through all characters of the class group and taking the average, all terms for which $\mathfrak{a}_j\mathfrak{a}_1^{-1}$ is not a principal fractional ideal vanish. Thus, we may assume without loss of generality that $\mathfrak{a}_j\mathfrak{a}_1^{-1}$ is principal for each $j = 2, \ldots, k$, so that $\mathfrak{a}_j\mathfrak{a}_1^{-1} = (\gamma_j)$ for some $\gamma_j \in F^\times$. Since ω is unramified, we have $\chi_\omega(\mathfrak{a}_j\mathfrak{a}_1^{-1}) = \chi_\omega((\gamma_j)) = \omega_\infty(\gamma_j)^{-1}$, so the above becomes

$$1 + c_2\omega_\infty(\gamma_2)^{-1} + \ldots + c_k\omega_\infty(\gamma_k)^{-1} = 0.$$

Next we replace ω by the twist $\omega(\omega')^\ell$, take the average over $\ell \in \{0, 1, \ldots, L-1\}$, and let $L \to \infty$. Recall that any idèle class character is unitary up to a power of $\|\cdot\|$; since $\|\gamma_j\|_\infty = N(\mathfrak{a}_j\mathfrak{a}_1^{-1}) = 1$, it follows that $\omega'_\infty(\gamma_j)$ is a complex number of modulus 1, so that

$$\lim_{L\to\infty} \frac{1}{L}\sum_{\ell=0}^{L-1} \omega'_\infty(\gamma_j)^{-\ell} = \begin{cases} 1 & \text{if } \omega'_\infty(\gamma_j) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now, for any $j = 2, \ldots, k$, $\mathfrak{a}_1$ and $\mathfrak{a}_j$ are distinct ideals, so γ_j is not an element of $\mathfrak{o}_F^\times$. Since we are free to choose any ω'_∞ in the dual of $\mathfrak{o}_F^\times\backslash\{y \in F_\infty^\times : \|y\|_\infty = 1\}$, we may always arrange it so that $\omega'_\infty(\gamma_j) \neq 1$ for a particular j. Thus, by repeating the above averaging procedure, all of the terms for $j = 2, \ldots, k$ vanish, so we are left with the absurd conclusion $1 = 0$. This completes the proof. □

We conclude this section with two consequences of the above for automorphic L-functions that may be of independent interest.

Corollary 4.3 (Linear independence of automorphic L-functions)**.** *Let $n_1, \ldots, n_m$ be positive integers, and for each $i = 1, \ldots, m$, let $\pi_i = \bigotimes_v \pi_{i,v}$ be an irreducible automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$. For each pair $i \neq j$, assume that there is a finite place v such that $\pi_{i,v}$ and $\pi_{j,v}$ are both unramified and $\pi_{i,v} \not\cong \pi_{j,v}$. Let S be a finite set of places containing all archimedean places, and consider the partial L-functions*

$$L^S(s, \pi_i \otimes \omega) = \prod_{v \notin S} L(s, \pi_{i,v} \otimes \omega_v),$$

where ω is an unramified idèle class character. Then, if $c_0, \ldots, c_m \in \mathbb{C}$ are such that

$$c_0 + c_1L^S(s, \pi_1 \otimes \omega) + \ldots + c_mL^S(s, \pi_m \otimes \omega) = 0 \tag{4.3}$$

for every unramified ω, then $c_0 = \ldots = c_m = 0$.

Proof. Let $\lambda_{\pi_i}(\mathfrak{a})$ denote the Dirichlet coefficients of $L(s, \pi_i)$, so that

$$L(s, \pi_i \otimes \omega) = \sum_{\mathfrak{a}} \lambda_{\pi_i}(\mathfrak{a})\chi_\omega(\mathfrak{a})N(\mathfrak{a})^{-s}$$

for $\Re(s) > 0$ sufficiently large. Next let $\mathfrak{m}$ be the product of the prime ideals corresponding to the finite places in S, and define

$$\lambda^S_{\pi_i}(\mathfrak{a}) = \begin{cases} \lambda_{\pi_i}(\mathfrak{a}) & \text{if } \mathfrak{a} + \mathfrak{m} = \mathfrak{o}_F, \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$L^S(s, \pi_i \otimes \omega) = \sum_{\mathfrak{a}} \lambda^S_{\pi_i}(\mathfrak{a}) \chi_\omega(\mathfrak{a}) N(\mathfrak{a})^{-s}.$$

By Lemma 4.2, the linear relation (4.3) implies that $c_0 1_{D_F}(\mathfrak{a}) + c_1 \lambda^S_{\pi_1}(\mathfrak{a}) + \ldots + c_m \lambda^S_{\pi_m}(\mathfrak{a}) = 0$ identically. Moreover, by restricting $\mathfrak{a}$ in this equality to the ideals co-prime to a fixed modulus, we are free to replace S by any larger finite set of places. In particular, we may assume without loss of generality that S contains all finite places of ramification of $\pi_1, \ldots, \pi_m$.

Next, following the proof of Lemma 3.5, for each i there is a unique isobaric automorphic representation $\Pi_i = \bigotimes_v \Pi_{i,v}$ of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$ such that $\Pi_{i,v} \cong \pi_{i,v}$ for every finite place v at which $\pi_{i,v}$ is unramified. Then for any $i \neq j$, by hypothesis there is an unramified finite place v for which $\Pi_{i,v} \cong \pi_{i,v} \not\cong \pi_{j,v} \cong \Pi_{j,v}$, so that $\Pi_i \not\cong \Pi_j$.

Finally, since S contains all ramified finite places, we are free to replace π_i by Π_i, so we may assume without loss of generality that π_i is isobaric. Then, by strong multiplicity one for isobaric representations, $\lambda^S_{\pi_1}, \ldots, \lambda^S_{\pi_m}$ are pairwise inequivalent, multiplicative elements of D_F. Moreover, for every unramified place v, $L(s, \pi_{i,v})$ is not identically 1, so each $\lambda^S_{\pi_i}$ is also inequivalent to the identity 1_{D_F}. The conclusion now follows from Lemma 4.1. $\square$

Corollary 4.4 (Algebraic independence of cuspidal automorphic L-functions)**.** *Assume the hypotheses of Corollary 4.3, and suppose that $\pi_1, \ldots, \pi_m$ are cuspidal. Then, if $P \in \mathbb{C}[x_1, \ldots, x_m]$ is such that*

$$P(L^S(s, \pi_1 \otimes \omega), \ldots, L^S(s, \pi_m \otimes \omega)) = 0$$

for every unramified ω, then $P = 0$ identically.

Proof. We may write $P = \sum_{e_1,\ldots,e_m} c_{e_1,\ldots,e_m} x_1^{e_1} \cdots x_m^{e_m}$ as a linear combination of monomials. For each non-zero m-tuple $(e_1, \ldots, e_m)$, we may define

$$\Pi_{e_1,\ldots,e_m} = \underbrace{\pi_1 \boxplus \cdots \boxplus \pi_1}_{e_1 \text{ times}} \boxplus \cdots \boxplus \underbrace{\pi_m \boxplus \cdots \boxplus \pi_m}_{e_m \text{ times}}.$$

Then, by [8], the $\Pi_{e_1,\ldots,e_m}$ are pairwise non-isomorphic isobaric representations satisfying

$$L^S(s, \Pi_{e_1,\ldots,e_m} \otimes \omega) = L^S(s, \pi_1 \otimes \omega)^{e_1} \cdots L^S(s, \pi_m \otimes \omega)^{e_m},$$

and the conclusion follows from Corollary 4.3 applied to these. $\square$

Remark. This result should be compared to that of Jacquet and Shalika [8], who proved the multiplicative independence of cuspidal L-functions, and thus showed the existence of the class of isobaric representations. In our notation, this means that for any solution to

$$L^S(s, \pi_1 \otimes \omega)^{c_1} \cdots L^S(s, \pi_m \otimes \omega)^{c_m} = 1,$$

where the π_i are pairwise non-isomorphic cuspidal representations and $c_1, \ldots, c_m \in \mathbb{C}$, one has $c_1 = \ldots = c_m = 0$. (Here we interpret $L^S(s, \pi_i \otimes \omega)^{c_i}$ to mean $\exp(c_i \log L^S(s, \pi_i \otimes \omega))$, where $\log L^S(s, \pi_i \otimes \omega)$ is the unique logarithm with zero constant term in its expansion as a Dirichlet series.) In particular, taking the c_i to be integers, one sees that for any isobaric representation π, $L^S(s, \pi\otimes\omega)$ has a unique factorization into products of cuspidal L-functions $L^S(s, \pi_i \otimes \omega)$. Note that Corollary 4.4 constitutes a strengthening of this particular case, from multiplicative independence to algebraic independence.

5. Conclusion of the proof

By Lemma 4.2 and (3.9)–(3.10), there are unique functions $f_0, f_1, \ldots, f_m \in M_F$ such that

$$\sum_{\mathfrak{a}\in\mathcal{I}_F} f_0(\mathfrak{a})\chi_\omega(\mathfrak{a})N(\mathfrak{a})^{-s} = L^T(s, \pi \otimes \omega)$$

and

$$\sum_{\mathfrak{a}\in\mathcal{I}_F} f_j(\mathfrak{a})\chi_\omega(\mathfrak{a})N(\mathfrak{a})^{-s} = L^T(s, \pi_j \otimes \omega)$$

for $j = 1, \ldots, m$ and all unramified idèle class characters ω, and they are related by the identity

$$f_0 = \sum_{j=1}^{m} c_j f_j. \tag{5.1}$$

By collecting common terms of (3.9) if necessary, we may assume without loss of generality that the π_j are pairwise non-isomorphic. Then strong multiplicity one for isobaric representations (see Remark 1) implies that, for any $i \neq j$, the local L-factors $L(s, \pi_{i,v})$ and $L(s, \pi_{j,v})$ differ at infinitely many places, and it follows that $f_1, \ldots, f_m$ are pairwise inequivalent. Thus, by Lemma 4.1, f_0 must be equivalent to f_j for some $j \in \{1\ldots, m\}$, and by reordering if necessary we may assume that f_0 is equivalent to f_1.

Let $S \subset \mathcal{P}_F$ be the finite set of primes $\mathfrak{p}$ for which $f_0(\mathfrak{p}^k) \neq f_1(\mathfrak{p}^k)$ for some $k \geq 1$. For $j = 0, 1$ we factor f_j as $f_j^\flat * f_j^\sharp$, where $f_j^\flat, f_j^\sharp \in M_F$ are the unique multiplicative functions satisfying

$$f_j^\flat(\mathfrak{p}^k) = \begin{cases} f_j(\mathfrak{p}^k) & \text{if } \mathfrak{p} \in S, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_j^\sharp(\mathfrak{p}^k) = \begin{cases} 0 & \text{if } \mathfrak{p} \in S, \\ f_j(\mathfrak{p}^k) & \text{otherwise.} \end{cases}$$

Note that $f_0^\flat$ and $f_1^\flat$ are $\mathfrak{p}$-finite, and $f_0^\sharp = f_1^\sharp$, so we may rewrite (5.1) in the form

$$(c_1 f_1^\flat - f_0^\flat) * f_1^\sharp + \sum_{j=2}^{m} c_j f_j = 0.$$

Invoking Lemma 4.1 again, we see that $c_j = 0$ for $j = 2, \ldots, m$, and thus (5.1) becomes

$$f_0 = c_1 f_1.$$

Evaluating both sides at $\mathfrak{o}_F$, we find that $c_1 = 1$, and it follows that $L(s, \pi_v) = L(s, \pi_{1,v})$ for all places $v \notin T$. Since π_v is unramifed for all $v \notin T$, we conclude that $\pi_v \cong \pi_{1,v}$, as desired.

References

1. H. Bass, *K-theory and stable algebra*, Inst. Hautes Études Sci. Publ. Math. (1964), no. 22, 5–60. MR 0174604 (30 #4805)
2. A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, With a supplement "On the notion of an automorphic representation" by R. P. Langlands, pp. 189–207. MR 546598 (81m:10055)
3. J. W. Cogdell and I. I. Piatetski-Shapiro, *Converse theorems for* GL_n, Inst. Hautes Études Sci. Publ. Math. (1994), no. 79, 157–214. MR 1307299 (95m:22009)
4. ———, *On partial Poincaré series*, Automorphic forms and L-functions I. Global aspects, Contemp. Math., vol. 488, Amer. Math. Soc., Providence, RI, 2009, pp. 83–93. MR 2522028 (2010m:22015)
5. D. Flath, *Decomposition of representations into tensor products*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 179–183.
6. H. Jacquet, I. I. Piatetski-Shapiro, and J. Shalika, *Conducteur des représentations du groupe linéaire*, Math. Ann. **256** (1981), no. 2, 199–214. MR 620708 (83c:22025)
7. H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), no. 2, 367–464. MR 701565 (85g:11044)
8. H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic forms. II*, Amer. J. Math. **103** (1981), no. 4, 777–815.
9. Hervé Jacquet, *Archimedean Rankin-Selberg integrals*, Automorphic forms and L-functions II. Local aspects, Contemp. Math., vol. 489, Amer. Math. Soc., Providence, RI, 2009, pp. 57–172. MR 2533003 (2011a:11103)
10. ———, *A correction to conducteur des représentations du groupe linéaire [mr620708]*, Pacific J. Math. **260** (2012), no. 2, 515–525. MR 3001803
11. Hervé Jacquet and Joseph Shalika, *The Whittaker models of induced representations*, Pacific J. Math. **109** (1983), no. 1, 107–120.
12. ———, *Rankin-Selberg convolutions: Archimedean theory*, Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989), Israel Math. Conf. Proc., vol. 2, Weizmann, Jerusalem, 1990, pp. 125–207. MR 1159102 (93d:22022)
13. J. Kaczorowski, G. Molteni, and A. Perelli, *Linear independence in the Selberg class*, C. R. Math. Acad. Sci. Soc. R. Can. **21** (1999), no. 1, 28–32. MR 1669479 (2000h:11094)
14. R. P. Langlands, *Automorphic representations, Shimura varieties, and motives. Ein Märchen*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 205–246.
15. Nadir Matringe, *Essential Whittaker functions for* $GL(n)$, Doc. Math. **18** (2013), 1191–1214.
16. Marko Tadić, *Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case)*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 3, 335–382.
17. ———, $GL_n(\mathbb{C})\check{}$ *and* $GL_n(\mathbb{R})\check{}$, Automorphic forms and L-functions II. Local aspects, Contemp. Math., vol. 489, Amer. Math. Soc., Providence, RI, 2009, pp. 285–313.

School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, United Kingdom

E-mail address: andrew.booker@bristol.ac.uk

Department of Mathematics, University of Iowa, 14 MacLean Hall, Iowa City, IA 52242-1419, USA

E-mail address: muthu-krishnamurthy@uiowa.edu