ST. NO. R 25737/A
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CRANFIELD

WALL PRESSURE FLUCTUATIONS UNDER TURBULENT BOUNDARY LAYERS AT SUBSONIC AND SUPERSONIC SPEEDS

by

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March, 1963.

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Wall Pressure Fluctuations under Turbulent Boundary Layers at Subsonic and Supersonic Speeds

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SUMMARY

The problem of pressure fluctuations at a rigid wall under a turbulent boundary layer has attracted much attention in the past decade. At low Mach numbers the theory is well established from the work of Kraichnan and Lilley, and reasonable agreement is obtained with the experiments of Willmarth, Hodgson and others. At high Mach numbers, measurements exist due to the work of Kistler and Chen but so far no theory is available, apart from that due to Phillips, which is however related to the noise radiated from supersonic turbulent shear flows.

The present paper reviews the theory of wall pressure fluctuations in incompressible flow, and shows how the character of the pressure fluctuations changes in passing from the flow to the wall. Attention is drawn to the more important interactions giving rise to the pressure fluctuations, as well as to the region of the boundary layer mainly responsible for the wall pressure fluctuations.

The work is extended to include the effects of compressibility. It is found that an analysis similar to that of Phillips is appropriate, although, unlike the latter work, this new treatment is not restricted to the case of very high supersonic Mach numbers. The analysis makes use of the ratio a_w/u_τ as a large parameter, where a_w is the speed of sound at the wall and u_τ is the shear velocity. This is certainly true for a very wide range of Mach numbers provided that the wall is not subjected to large rates of heat transfer. It is shown that the wall pressure fluctuations are now the result of fluctuations in both the vorticity and sound modes. At high Mach numbers, the latter contribution is in the form of eddy Mach waves, as suggested by Phillips. On making certain assumptions regarding the dominant interactions, estimates of the magnitude and spectrum of the wall pressure fluctuations are made which show similar trends to the measurements of Kistler and Chen.

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CONTENTS

NOTATION

$\bar{a} \equiv$	a/a_w
a	speed of sound
$A(\underline{x}, t)$	source function
a_i	amplitude of covariance
c_f	skin friction coefficient
F	function (equ. 117)
$f, \bar{f}$	functions (equ. 80)
G_o	Green function for flow without boundaries
G_i	Green function for image flow
$h(o)$	coefficient in approximation to mean shear
$H(y)$	source function (equ. 48)
$J(x)$	function tabulated by Goodwin and Staton
$\underset{\sim}{k}$	two dimensional wave number
$\ell_1, \ell_2, \ell_3, \ell$	scales of turbulence
L	function (equ. 83)
P	pressure covariance
p	pressure
q	(see equ. 47)
$\underline{r}$	spatial separation
R	Reynolds number
R_{p_2}	pressure - u_2 velocity covariance
S	independent variable
t	time
T	temperature
T_s	sub-layer temperature (equ. 74)

$\underset{\sim}{U}$	mean flow velocity
$\underset{\sim}{u}$	velocity (turbulent)
U_c	convection velocity
$X(y)$, $X(o)$	functions (equ. 80)
y	independent variable
dz_2	Fourier coefficient of u_2
τ	delay time; mean shear (dU_1/dx_2)
τ_w	wall shear stress
ω	frequency
α_i	inverse length scale of turbulence
α	function of y
β	function of y ; coefficient in approximation to mean shear
α^*	function of y
β^*	function of y
γ	ratio of specific heats
$d\gamma$	Fourier coefficient
δ	boundary layer thickness
δ_1	displacement thickness
$\zeta \equiv a/a_w \cdot d\tilde{\omega}$	modified Fourier coefficient (pressure)
η	independent variable
κ_2	wave number
λ	a_w/u_τ
μ	viscosity
$d\tilde{\omega}$	Fourier coefficient of pressure
ρ	density
θ	angle
$\Pi(\underline{k}, \omega)$	spectrum function (pressure)
$u_\tau \equiv \sqrt{\tau_w/\rho_w}$	

Φ_{22}	spectrum function (velocity component u_2)
$\phi(s)$	source function
Ω	dependent variable (equ. 65)

Subscripts

w	denotes wall value
∞	denotes value external to the boundary layer
o	incompressible value
i, j	tensor notation
x_1	streamwise direction
x_2	normal to the wall
x_3	transverse direction
bar	denotes an incompressible value unless otherwise stated

1. Introduction

The work of Heisenberg[1], Obukhov[2] and Batchelor[3] has shown that in isotropic turbulence, the root mean square fluctuating pressure is given by

$$\sqrt{\overline{p^2}} = 0 \cdot 58 \rho \overline{u^2} \qquad (1)$$

where $\overline{u^2}$ is the mean square value of any fluctuating component of the turbulent velocity. Uberoi[4] has shown that a relation of this form exists in grid turbulence, but the constant in equation (1) was approximately 0.65 over a wide range of Reynolds numbers.

The first attempt to calculate the fluctuating pressure field in a turbulent shear flow was made by Kraichnan[5] who found that at the wall (subscript w)

$$\sqrt{\overline{p_w^2}} / \tau_w = 2 \text{ to } 12 \qquad (2)$$

where τ_w is the local wall mean shear stress.

Experiments in pipe flow by Willmarth[17] showed that

$$\sqrt{\overline{p_w^2}} \simeq 0 \cdot 006 \tfrac{1}{2} \rho U_\infty^2 \qquad (3)$$

and that the pressure field was convected past the wall at an average speed of 0.82 U_∞, where U_∞ is the speed of the uniform flow external to the boundary layer. However, in spite of care to reduce the extraneous noise in the air supply leading to the pipe, Willmarth was unable to obtain accurate readings of the power spectral density in the lower frequencies. Other investigators have obtained similar results but only recently has an attempt been made to check the constant in Kraichnan's formula. (A more complete review of the experimental work on wall pressure fluctuations will be given in a paper by Hodgson[6] which is to be published shortly).

An additional problem noted by Willmarth was the correction necessary to allow for the effects of the finite size of the pressure transducer on both the root mean square and power spectral density measurements. With this correction applied, most of the available measurements suggest

$$\sqrt{\overline{p_w^2}} / \tau_w = 1 \cdot 5 \text{ to } 3 \qquad (4)$$

over a moderate range of Reynolds numbers at low Mach numbers, or

$$\sqrt{\overline{p_w^2}} / \tau_w = a(R) \qquad (5)$$

where a(R) is a slowly varying function of Reynolds number at sufficiently high Reynolds numbers.

Recent work by Kistler and Chen[7] has extended the measurements to high Mach numbers and their work shows that a(R) increases progressively with Mach number, reaching a value between 5 and 6 at a freestream Mach number of 5, at least for the case of zero heat transfer. Their results suggest that, at a Mach number of 5, the function a(R) has nearly reached its asymptotic value for very high Mach numbers.

The work of Kraichnan (loc. cit.) has been reviewed and extended by Lilley[8] and by Lilley and Hodgson[9]. The latter work showed that the lower estimate of a(R) obtained by Kraichnan was more correct, and this work also went some way towards confirming that the pressure fluctuations in a turbulent shear flow are dominated by the mean shear. The calculated spectrum function for the wall pressure fluctuations showed moderate agreement with the measured spectra at high frequencies, but at lower frequencies, the calculated fall was not observed in the measurements made in pipes, wind tunnels etc. The corresponding two-point pressure covariances showed marked differences between longitudinal and transverse separations, while the area under the longitudinal pressure covariance and the related autocorrelation was exactly zero. In fact, the theory showed, in agreement with the work of Phillips[10], a vanishing surface integral of the two-point pressure covariance taken over the wall. The differences between theory and the measurements of Willmarth and others have been investigated by Hodgson[6]. He showed that the ill-defined strong negative loop in both the measured longitudinal pressure covariance and the autocorrelation, and the non-vanishing transverse pressure covariance at large spearations, were the result of extraneous disturbances external to the boundary layer. (The effects of extraneous disturbances were also known to Willmarth and are also discussed at some length in the recent work of Willmarth and Wooldridge[11]). The measurements made by Hodgson (loc. cit.), on the wing of a glider in flight, which were free from extraneous disturbances, confirmed the relation

$$\sqrt{\overline{p_w^2}} \simeq 2{\cdot}2\,\tau_w \qquad (6)$$

and showed the falling spectrum in the lower frequencies, together with

$$\int_{-\infty}^{\infty} P(o;\tau)\,d\tau = 0 \qquad (7)$$

where $P(o;\,\tau)$ is the autocorrelation of the pressure at the wall. On applying the convected hypothesis, which is supported by all the measurements, Hodgson finds that equation (7) is equivalent to

$$\iint_{-\infty}^{\infty} P(\xi_1, o, \xi_3)\,d\xi_1 d\xi_3 = 0 \qquad (8)$$

which is, an experimental confirmation of Phillips' result.

If we return to the problem of wall pressure fluctuations at supersonic Mach numbers, we find that no theory exists, apart from the work by Phillips[12], on the related problem of sound generation by supersonic turbulent shear layers.

Phillips has shown that the radiated sound arises from eddy Mach waves which are generated by some wave-numbers of the turbulence in those layers of the shear zero for which the difference between the mean velocity of the fluid outside and the local eddy convection velocity is greater than the speed of sound outside the zone. Phillips does not include the case of a wall shear flow, although clearly this must present an analogous problem, and indeed Phillips argues that his model should be qualitatively correct in this case. However measurements by Laufer(13) of radiated sound from supersonic turbulent boundary layers are not in good numerical agreement with Phillips' theory, although undoubtedly some aspects of the phenomenon described by Phillips, such as the production of eddy Mach waves, do exist and have been observed by many workers. However, as Laufer points out, the experimental Mach numbers may not be high enough for Phillips' asymptotic theory to be applicable in the range of freestream Mach numbers up to 6. The more general problem of the sound radiated from shear flows at supersonic speeds has been treated by Ffowcs Williams(14), and Lighthill(15).

The present paper sets out to extend the theory of pressure fluctuations in turbulent boundary layers in incompressible flow to that at higher speeds, and to provide a basis for comparison with the measured results of Kistler and Chen (loc. cit.), Williams, D(16) and Willmarth(17).

2. Incompressible flow theory

2.1. The pressure covariance

It has been shown by Lilley and Hodgson (loc. cit.) that the pressure at the wall is dominated by contributions from the turbulence in the 'inner' region of the boundary layer, extending up to about 1.6 δ_1 , where δ_1 is the boundary layer displacement thickness. In this region, the typical length and velocity scales of the flow are

$$\mu_0/\rho_0 u_\tau \qquad \text{and} \qquad u_\tau = \sqrt{\tau_w/\rho_0} \quad \text{respectively.}$$

Measurements in this region indicate that all except the larger wave numbers of the turbulence are being convected at a mean speed of near 0.8 U_∞, and a theory of the sub-layer of the 'inner' region, based on this hypothesis, is given by Sternberg(18). Hence we might expect that the pressure at the wall is also dominated by eddies having this convection speed of near 0.8 U_∞. Since the correlation lengths for the wall pressure are of the order of a boundary layer thickness, it would seem reasonable to neglect the rate of growth of the boundary layer in calculations of the wall pressure fluctuations. We will assume, therefore, that the mean flow field is given by $[U_1(x_2), 0, 0]$ where x_1 is measured in the direction of the mainstream and x_2 is normal to the wall. If all terms in the equations of motion are made non-dimensional with respect to u_τ and $\mu_0/\rho_0 u_\tau$, we find

$$\frac{\partial u_i}{\partial t} + U_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{dU_1}{dx_2}\delta_{i1} + \frac{\partial(u_i u_j - \overline{u_i u_j})}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nabla^2 u_i \qquad (9)$$

where $\quad t \equiv t \rho_o u_T^2/\mu_o \quad ; \quad x_i \equiv \rho_o x_i u_T/\mu_o \quad ; \quad p \equiv p/\rho_o u_T^2$

$$u_i \equiv u_i/u_T \quad ; \quad U_1 \equiv U_1/u_T$$

Since the equation of continuity for the turbulence is $\partial u_i/\partial x_i = 0$, we find, on taking the divergence of (9), that the equation for the pressure* is

$$\nabla^2 p = -2\frac{dU_1}{dx_2}\frac{\partial u_2}{\partial x_1} - \frac{\partial^2 (u_i u_j - \overline{u_i u_j})}{\partial x_i \partial x_j}$$

$$= A(\underset{\sim}{x}, t) \qquad (10)$$

The two terms contributing to $A(\underset{\sim}{x}, t)$, which defines the velocity field, can be referred to as the mean shear - turbulence interaction (M - T) and the turbulence-turbulence interaction (T - T) respectively.

The solution of (10) can be put in the form

$$p(\underset{\sim}{x}, t) = -\frac{1}{4\pi}\int_0^\infty dx_2' \iint_{-\infty}^{\infty} dx_1' \, dx_3' \; A(\underset{\sim}{x}', t)\,[G_o + G_i]$$

$$-\frac{1}{2\pi}\iint_{-\infty}^{\infty} dx_1' \, dx_3' \; G_o \frac{\partial p}{\partial x_2'} \qquad (11)$$

where the surface integral is taken over the wall at $x_2' = 0$. The Green functions G_o and G_i are given respectively by

$$\left.\begin{aligned} G_o &= |\underset{\sim}{x} - \underset{\sim}{x}'|^{-1} \\ G_i &= |\underset{\sim}{x} - \underset{\sim}{x}'^{*}|^{-1} \end{aligned}\right\} \qquad (12)$$

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* It is incorrect to argue that the second term on the right hand side of (10) is small by comparison with the first term. However, integrals involving $A(\underset{\sim}{x}, t)$ are usually dominated by the (M - T) terms unless this contribution is identically zero.

where $\underset{\sim}{x}'^{*} \equiv (x_1', -x_2', x_3')$ is the 'image' point. However, from the equation of motion (9), we see that at the wall

$$\partial p/\partial x_2 = \partial^2 u_2/\partial x_2^2 \qquad (13)$$

since both U_i and u_i vanish at the wall, and then (11) becomes

$$p(\underset{\sim}{x}, t) = -\frac{1}{4\pi}\int_0^\infty dx_2' \iint_{-\infty}^{\infty} dx_1'\, dx_3' \left(G_o + G_i\right) A(\underset{\sim}{x}', t)$$

$$- \frac{1}{2\pi}\iint_{-\infty}^{\infty} dx_1'\, dx_3' \left(G_o\, \partial^2 u_2/\partial x_2'^2\right)_{x_2'=0} \qquad (14)$$

showing that the pressure fluctuations can be determined once the velocity field is known. The covariance between the pressure at any point ($\underset{\sim}{x}$, t) and a random function $q''(\underset{\sim}{x}'', t)$, such as the pressure or a components of the turbulent velocity at ($\underset{\sim}{x}''$, t), is therefore given by

$$\overline{p(\underset{\sim}{x}, t)\, q''(\underset{\sim}{x}'', t)} = -\frac{1}{4\pi}\int_0^\infty dx_2' \iint_{-\infty}^{\infty} dx_1'\, dx_3' \left(G_o + G_i\right) \overline{A(\underset{\sim}{x}')\, q''(\underset{\sim}{x}'')}$$

$$- \frac{1}{2\pi}\iint_{-\infty}^{\infty} dx_1'\, dx_3' \left(G_o \frac{\partial^2}{\partial x_2'^2} \overline{u_2(\underset{\sim}{x}')\, q''(\underset{\sim}{x}'')}\right)_{x_2'=0} \qquad (15)$$

where the bars indicate time means. Thus in order to determine the pressure covariance $\overline{p(\underset{\sim}{x})\, p''(\underset{\sim}{x}'')}$ anywhere in the shear layer we need values of

$$\overline{A(\underset{\sim}{x}')\, p''(\underset{\sim}{x}'')} \equiv -2\frac{dU_1}{dx_2}(x_2')\frac{\partial}{\partial x_1'}\overline{u_2(\underset{\sim}{x}')\, p''(\underset{\sim}{x}'')} - \frac{\partial^2}{\partial x_i'\partial x_j'}\overline{u_i u_j(\underset{\sim}{x}')\, p''(\underset{\sim}{x}'')} \qquad (16)$$

But $\overline{p'' u_2}$ can be determined from (15) by replacing q'' by u_2'' so that

$$\overline{p u_2''} = \frac{1}{4\pi}\int_0^\infty dx_2' \iint_{-\infty}^{\infty} dx_1'\, dx_3' \left(G_o + G_i\right)\left\{2\frac{dU_1}{dx_2'}\frac{\partial}{\partial x_1'}\overline{u_2' u_2''} + \frac{\partial^2}{\partial x_i'\partial x_j'}\overline{u_i' u_j' u_2''}\right\}$$

$$- \frac{1}{2\pi}\iint_{-\infty}^{\infty} dx_1'\, dx_3' \left(G_o \frac{\partial^2}{\partial x_2'^2}\overline{u_2' u_2''}\right)_{x_2'=0} \qquad (17)$$

while $\overline{p u_i'' u_j''}$ is similarly obtained by replacing q'' in (15) by $u_i''\, u_j''$. Hence the determination of the pressure covariance formally involves the evaluation of integrals of the form of (15) and (17) over the entire flow field. However this cannot be performed with any great precision since the second-order velocity correlations are incompletely known and little is known of the third-order and fourth-order velocity correlations. The mean square of the pressure at the wall, $\overline{p(0)^2}$, and the pressure-velocity covariance, $\overline{p(0)\, u_2''(\underline{x}'')}$ are just two particular results which can be obtained from the general relations (15) - (17).

On the assumption that the (M - T) and (T - T) terms are independent, Hodgson (loc. cit.) has shown, on using the best available data for the mean and turbulent velocity flow fields and making extensive numerical calculations, that the contributions

$$\text{to} \quad \sqrt{\overline{p(0)^2}} \equiv \left(\sqrt{\overline{p_W^2}} \,/\, \tau_W \right) \qquad \text{from the (M - T)}$$

and (T - T) terms are respectively 2.6 and 0.5. This shows that the contribution of the (T - T) terms to the mean square of the wall pressure is only 4%, and in view of the approximate nature of the calculations can be assumed negligible. (The difference between these calculations and the measured values is not considered to be of major importance, in view of the fact that on each occasion a velocity flow field closer to the experimental one was used, a value of $\sqrt{\overline{p(0)^2}}$ nearer to 2.2 was obtained).

If we turn next to the evaluation of $\overline{p(0)u_2''} \Big/ \sqrt{\overline{p(0)^2}}\sqrt{\overline{u_2''^2}}$ we find that the contribution from the surface integral is negligible if $x_2'' \gg 1$ (Note that in our notation $x_2 \equiv \rho_0 u_\tau x_2 / \mu_0$). The contribution from the (M - T) term can be obtained on choosing a suitable form for $Q_{22} \equiv \overline{u_2' u_2''}$. (The contribution from the (T - T) term cannot be obtained, even approximately, since values of $\overline{u_i' u_j' u_2''}$ have not been measured except for zero separation). It is found that the contribution from the (M - T) term gives

$$\overline{p(0)\, u_2''} \quad \sim \quad \frac{x_1'' x_2''}{\delta_1^2} \; \exp\left(-\frac{1}{2} \, \frac{x_1''^2 + x_2''^2}{\delta_1^2} \right) \qquad \text{———— (18)}$$

and a comparison with the measurements of Willmarth and Wooldridge[27] is shown in Figs. 1 and 2. The agreement is reasonable, qualitatively, except at large separations. A slightly modified form of (18) is shown to represent closer the asymmetry in the polar distribution (see Fig. 3) and so we can represent R_{p2} for moderate values of x by

$$R_{p2}(\underline{x}) \sqrt{\overline{u_2^2}} \simeq a_2 x_1 \sqrt{x_2} \; \exp\left(-\alpha_i^2 x_i^2\right) \qquad \text{———— (19)}$$

with $\quad a_2 / \alpha_1^{3/2} = 1 \quad ; \quad \alpha_1 \delta_1 = 0{\cdot}35 \quad ; \quad \alpha_3^2 / \alpha_1^2 = 4$

and α_2^2 / α_1^2 between 0.75 and 1.

The evaluation of $\overline{p(0)^2}$ can now be attempted by using R_{p2} defined by (19). It can be shown that the contribution from the surface integral can be neglected, while again it would appear not unreasonable to assume that the (T - T) contribution, involving as it does third-order covariances, can be neglected. It is found that $\sqrt{\overline{p(0)^2}}$ has a value of the order of 2.3 if the values of α_1, α_2 and α_3 are chosen as given above. The integral on which this value of $\sqrt{\overline{p^2(0)}}$ is based is

$$\sqrt{\overline{p(0)^2}} = \frac{4}{\pi K}\int_0^{\infty} x^{1/2}\exp(-\alpha_2^2 x^2/\alpha_1^2)\,dx \int_0^{\pi/2}\left(I_0(u) + I_1(u)\right)\frac{\sin^3\theta}{\cos^{1/2}\theta}\exp(-bx^2\sin^2\theta)\,d\theta \qquad (20)$$

where

$$b = \tfrac{1}{2}\left(\alpha_3^2/\alpha_1^2 + 1\right) - \alpha_2^2/\alpha_1^2$$

$$u = x^2\left(\alpha_3^2/\alpha_1^2 - 1\right)\sin^2\theta$$

and I_0 and I_1 are the modified Bessel functions of order zero and unity respectively. K is the con Karman constant used in defining the mean velocity shear. The major contribution to $\sqrt{\overline{p_w^2}}$ arises from x/δ_1 near 1.43, and, if we take δ_1 = 1610 (corresponding to the value used in one of Wooldridge and Willmarth's experiments), this occurs at x= 2300. A small variation in the choice of α_2/α_1 could place this major contribution to $\sqrt{\overline{p_w^2}}$ in the range x = 2300 to 2700, which straddles the value of x_2 = 2550, at which U_1/U_∞ = 0.83. It must not be assumed, however, that the major contribution to $\overline{p(0)^2}$ occurs over a small region of the shear layer. In fact the contributions are spread diffusely over a fairly large range of x_2, with the maximum at about $x_2 = 1.5\,\delta_1$ as shown also in the experiments of Willmarth and Wooldridge and in the analysis of Hodgson.

The extension of this work to include the evaluation of $\overline{p^2(x_2)}$ involves many difficulties, although formally it can be obtained from (15) - (17). In view of the fact that $\partial p/\partial x_2$ is nearly zero at the wall, together with the result that the major contribution to $\overline{p(0)^2}$ arises from near $x_2 = 1.5\,\delta_1$, leads us to suggest that $\overline{p^2}$ is nearly constant varying at most by a factor of 2 over most of the 'inner' region. Since over most of the constant stress layer $\overline{u_2^2}$ is of order unity and $-\overline{pu_2} \equiv -\overline{pu_2}/\rho_c u_\tau^3$ is of order 4, we find some confirmation in this suggestion.* An alternative suggestion by Remenyik and Kovasznay[19] is that the fluctuating pressure falls rapidly outside the 'laminar sub-layer', but this is not in agreement with our results. However the results in § 3 do show that the major contribution to the wall pressure comes from layers closer to the wall as the Mach number is increased, and it is probably this effect which might have some bearing on the results obtained by Remenyik and Kovasznay.

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* In the case of the constant stress layer if we put

$$\overline{q^2 u_2} = \overline{q^2}\,v$$

and

$$|\overline{pu_2}| = \sqrt{\overline{p^2}}\,v$$

then if $v = 1$, $\sqrt{\overline{p^2}} = 2\sqrt{\overline{p_w^2}}$

We will now discuss the pressure-velocity product $\overline{pu_2}$. This term, which vanishes at the wall, is known to play an important role in the energy transfer across the boundary layer, and has roughly a constant value across the entire 'inner' region of the boundary layer outside the viscous layer. Clearly the pressure-velocity covariance $\overline{p(x_2)u_2''}$ must have an essentially different form from $\overline{p(o)u_2''}$, since, as we have shown above, $\overline{p(o)u_2''(o, x_2'', o)}$ is zero, whereas $\overline{pu_2(x_2)}$ is clearly finite. In addition, the surface integral of $\overline{pu_2''}$ taken over a plane parallel with the wall, must vanish if there is no disturbance outside the boundary layer. The results of Wooldridge and Willmarth are in agreement with this boundary condition for $\overline{p(o)u_2''}$.

The modifications to $\overline{pu_2''}$, as the pressure measuring station is moved away from the wall, can be shown to depend on contributions from the surface integral and the (T - T) term in (17), both of which have been shown to give negligible contributions to $\overline{p(o)u_2''}$. Indeed if $Q_{22}(x_2;r)$ is symmetrical about $r_i = o$, the contribution of the (M - T) term to $\overline{pu_2}$ is zero, as noted by Corcos[20], and so $\overline{pu_2}$ depends entirely on the surface integral in (17) together with the contribution from the third-order velocity covariance. Outside the viscous layer, the dominant contribution to $\overline{pu_2}$ arises from the (T - T) terms. This result is made obvious by noting that if structural similarity exists in the constant stress region of the boundary layer and outside the viscous layer

$$\overline{q^2 u_2}/2 = a_3(R)\,\overline{q^2}^{3/2}$$

$$-\overline{pu_2} = a_2(R)\,\overline{q^2}^{3/2}$$

and so

$$-\overline{pu_2} = (a_2/2a_3)\,\overline{q^2 u_2} \qquad (21)$$

where $\overline{q^2}$ is the mean square of the turbulent velocity, and $a_2(R)$ and $a_3(R)$ are slowly varying functions of Reynolds number.

2.2. The structure of the big eddies

The work of Townsend[21] and Grant[22] has suggested that the structure of the big eddies in a turbulent boundary layer have the form of 'mixing jets' which erupt near the surface and spread into the outer regions of the boundary layer. The available experimental evidence in support of this hypothesis is scanty, although it is consistent with Grant's own measurements of nine second-order velocity correlations, and the more recent work of Wooldridge and Willmarth (loc. cit.) in which they report extensive measurements of wall pressure-velocity covariances. Since the big eddies play such a vital role in the determination of the pressure at the wall, it is of interest to discuss the work of the present author, in which an attempt has been made to put the 'mixing jet' hypothesis on a more quantitative basis with results in agreement with the measurements both of Grant and Wooldridge and Willmarth. Only the essential details of this work will be given here.

When the rate of turbulent energy production exceeds its equilibrium level, it is followed by increased dissipation and an increased rate of diffusion of turbulent energy both towards and away from the wall. The outward flux of energy can be presented roughly by

$$v \frac{d}{dx_2} \Delta q^2/2$$

where v is the mixing jet velocity and $\Delta q^2/2$ is the excess of turbulent energy. This release of energy from regions near the wall is followed by an energy deficiency, and so the outward 'mixing jet' must be followed by a return flow towards the surface. Such a return flow is also required from considerations of continuity. It might be expected that the outward 'mixing jets' should have a relatively high turbulent intensity and a relatively small scale, while the return flow should have a somewhat lower turbulen intensity and a larger scale. An overall scale of the 'mixing jets', extending to about a boundary layer thickness, is clearly shown in the measurements of Wooldridge and Willmarth, as well as those of Grant.

Since the 'mixing jet' is essentially a turbulent flow away from the surface, we might extect the major part of the correlation $\overline{R}_{22}(r,0,0)$ to be the result of the relatively simple structure of the big eddy. The results shown in Fig. 4 confirm this. However a fair fit with the remaining correlations measured by Grant could only be obtained if, in addition to the mixing jets, we superpose larger eddies rotating in planes parallel with the wall and uncorrelated with the 'mixing jets'. The presence of these big eddies can be demonstrated in Fig. 4, where the separate contributions from the 'mixing jets' and the larger scale eddies are given for $\overline{R}_{11}(r,0,0)$.The figure also shows the measure of agreement between the model and the measurements but with a relatively free choice in the values of so many length scales defining the three part structure of the big eddies, the agreement in many of the examples shown is probably fortuitous. Briefly the length scales of the eddies have been found to be as follows :-

(i) Eddies rotating in planes parallel with the wall have a scale of order δ, where δ is the boundary layer thickness, and have a structure similar to the simple form suggested by Townsend as being representative of the big eddies produced at random in the boundary layer.

(ii) The outward 'mixing jet' has a scale of order $\delta/10$.

(iii) The scale of the return flow is of order $\delta/3$.

(iv) Secondary motions in planes parallel to the wall accompanying the 'mixing jet' have scales of order $\delta/10$.

A diagrammatic representation of the big eddy structure is shown in Fig. 5.

2.3. The pressure spectrum

If we define the three-dimensional Fourier-Stieltjes transforms of $p(\underset{\sim}{x}, t)$ and $A(\underset{\sim}{x}, t)$ respectively as

$$p(\underset{\sim}{x},t) = \int e^{i(k_1 x_1 + k_3 x_3 + \omega t)} \, d\tilde{\omega}(x_2; \underset{\sim}{k}, \omega) \qquad (22)$$

and

$$A(\underset{\sim}{x},t) = \int e^{i(k_1 x_1 + k_3 x_3 + \omega t)} \, d\gamma(x_2; \underset{\sim}{k}, \omega)$$

where $\underset{\sim}{k} \equiv (k_1, k_3)$ is the wave number vector in the (x_1, x_3) plane and ω is the frequency, then the equation for the Fourier coefficient $d\tilde{\omega}$ is

$$d\tilde{\omega}'' - k^2 \, d\tilde{\omega} = d\gamma \qquad (23)$$

where primes denote differentiation with respect to x_2. If we assume that the disturbance outside the boundary layer is zero, $d\tilde{\omega}(\infty) = d\gamma(\infty) = 0$. From the equation of motion

$$d\tilde{\omega}'(0) = d\tilde{z}_2''(0) \qquad (24)$$

where $d\tilde{z}_2$ is the Fourier coefficient for u_2.

Hence the solution to (23), satisfying the boundary condition (24), is

$$\begin{aligned} d\tilde{\omega}(x_2) = {} & \frac{d\tilde{\omega}(0)}{2}\left(e^{kx_2} + e^{-kx_2}\right) \\ & + \frac{d\tilde{\omega}'(0)}{2k}\left(e^{kx_2} - e^{-kx_2}\right) \\ & + \frac{e^{kx_2}}{2k}\int_0^{x_2} e^{-ky} \, d\gamma(y) \, dy \\ & - \frac{e^{-kx_2}}{2k}\int_0^{x_2} e^{ky} \, d\gamma(y) \, dy \end{aligned} \qquad (25)$$

and for the Fourier coefficient of the pressure at the wall

$$d\tilde{\omega}(0) = -\, d\tilde{\omega}'(0)/k - (1/k)\int_0^{\infty} e^{-ky} \, d\gamma(y) \, dy \qquad (26)$$

If we assume that we can neglect $d\tilde{\omega}'(0)$ then the pressure spectrum function at the wall is given by

$$\Pi(0;\underset{\sim}{k},\omega) = \frac{1}{k^2}\int_0^\infty e^{-2ky}\,dy\int_{-y}^\infty e^{-kz}\,\overline{d\gamma(y)\,d\gamma^*(y+z)}\,dz \qquad (27)$$

where $$\Pi(0;\underset{\sim}{k},\omega) = \frac{1}{8\pi^3}\iint \overline{p(0;\underset{\sim}{x},t)\,p(0;\underset{\sim}{x}+\underset{\sim}{r},t+t')}\,e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}+\omega t')}\,d\underset{\sim}{r}\,dt'$$

$$= \frac{\overline{d\tilde{\omega}(0;\underset{\sim}{k},\omega)\,d\tilde{\omega}^*(0;\underset{\sim}{k},\omega)}}{dk_1\,dk_3\,d\omega} \qquad (28)$$

and $\underset{\sim}{r} \equiv (r_1, r_3)$ and the asterisk denotes the complex conjugate. The mean square of the pressure at the wall is

$$\overline{p^2(0)} = \iiint_{-\infty}^{\infty} d\underset{\sim}{k}\,d\omega\,\Pi(0;\underset{\sim}{k},\omega) \qquad (29)$$

Since all wave-numbers, apart from the highest, are convected at a constant speed, U_c, the frequency spectrum can be obtained, relative to co-ordinates fixed in the wall, from the integrated wave number spectrum function. Thus

$$\Pi(0;k_1) = \int_{-\infty}^{\infty}\Pi(0;\underset{\sim}{k})\,dk_3 = \Pi\left(0;-\frac{\omega}{U_c}\right) \qquad (30)$$

where $$\Pi(0;\underset{\sim}{k}) = \int_{-\infty}^{\infty}\Pi(0;\underset{\sim}{k},\omega)\,d\omega.$$

As a result of the vanishing surface integral of the pressure covariance

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$$\Pi(0;0) = 0 \qquad (31)$$

In these results k and ω are dimensionless, and are given by the relations

$$k = k\delta_1 \Big/ \frac{\rho_0 u_\tau \delta_1}{\mu_0}$$

and

$$\omega = \frac{\omega\delta_1}{u_\tau}\Big/\frac{\rho_0 u_\tau \delta_1}{\mu_0} \qquad (32)$$

If $A(\underset{\sim}{x}, t)$ in (10) is independent of Reynolds number, then $\Pi(0;\underset{\sim}{k},\omega)$ should be a universal function of k and ω (Hence spectra plotted as functions of $\omega\delta_1/U_\infty$, say, will not be universal functions).

If we now replace $d\gamma(x_2;\underset{\sim}{k},\omega)$ by the (M - T) term only

$$d\gamma(x_2;\underset{\sim}{k},\omega) = -2ik_1\frac{dU_1}{dx_2}\,dZ_2(x_2;\underset{\sim}{k},\omega) \qquad (33)$$

and substitute into equation (27)

$$\Pi(0; \underset{\sim}{k}, \omega) = (4k_1^2/k^2)\int_0^\infty e^{-2ky}\,\tau(y)\,dy \cdot \int_{-\infty}^{\infty} \Phi_{22}(y;\underset{\sim}{\kappa},\omega)\,d\kappa_2 \cdot \int_{-y}^{\infty} e^{-kz}\,e^{i\kappa_2 z}\,\tau(y+z)\,dz \qquad (34)$$

where Φ_{22} is the four-dimensional spectrum function, $\underset{\sim}{\kappa} \equiv (\kappa_1, \kappa_2, \kappa_3)$, $\kappa_1 \equiv k_1$, $\kappa_3 \equiv k_3$ and $\tau = dU_1/dx_2$.

If we assume that

$$\frac{\Phi_{22}(y;\underset{\sim}{\kappa},\omega)}{\sqrt{\overline{u_2^2}(y)}\sqrt{\overline{u_2^2}(y+z)}} = \frac{\ell_1^2\ell_2\ell_3\;\ell^2k^2\left(1+2\ell_1^2\omega^2/U_c^2\right)e^{-\left(\ell^2k^2+\ell_1^2\omega^2/U_c^2\right)}}{2\pi^{5/2}\,U_c\left(1+\ell_2^2\kappa_2^2\right)} \qquad (35)$$

where $\ell^2k^2 \equiv \ell_1^2k_1^2 + \ell_3^2k_3^2$

as shown in appendix 1, and

$$\tau\sqrt{\overline{u_2^2}} = h(0)\,e^{-\beta y} \quad ; \quad \ell_2 = y \quad ; \quad \ell_1 = \ell_3 = \ell$$

we find

$$\Pi(0;\underset{\sim}{k},\omega) = \frac{\left[e^{-1} + e^{2}\,Ei(-2) + e^{-2}\,\overline{Ei}(2) - e^{-2}\,\overline{Ei}(1)\right]}{\pi^{3/2}\,(k+\beta)^3\,U_c} \cdot 2k_1^2\,h^2(0)\,\ell^5\left(1+2\ell^2\omega^2/U_c^2\right)e^{-\left(\ell^2k^2+\ell^2\omega^2/U_c^2\right)} \qquad (36)$$

where the term in square brackets is equal to 0.42 approximately. (The contribution to $\Pi(0;\underset{\sim}{k},\omega)$) from the sub-layer can be shown to be negligible and since the major contribution is found to come from layers between $\frac{1}{2}$ to $2\delta_1$, we are justified in putting h(o) as finite). The integration over all frequencies can be performed and the wave number spectrum is then given by

$$\Pi(0;\underset{\sim}{k}) = \frac{1.68\,k_1^2\,\ell^4\,h^2(0)\,e^{-\ell^2k^2}}{\pi\,(k+\beta)^2} \qquad (37)$$

and $$\overline{p^2}(0) = 1 \cdot 68\, h^2(0)\, \ell^2 \int_0^\infty \frac{\bar{k}^3 e^{-\bar{k}^2} d\bar{k}}{(\bar{k} + \beta\ell)^2} \qquad (38)$$

An analysis of the contributions to $\Pi(0, \underset{\sim}{k})$ from different values of y shows that the larger contributions come from layers of order δ_1 from the wall, at least for the energy containing eddies. If we then choose values for h(o) and β to give a good fit to h(y) near y = δ_1, we find

$$h(0) = 5 \cdot 4/\delta_1 \quad ; \quad \ell = \delta_1 \quad ; \quad \beta\delta_1 = 1$$

The integral* can be evaluated in terms of

$$J(x) = \int_0^\infty \frac{e^{-k^2} dk}{k + x}$$

which is tabulated by Goodwin and Staton[23].

We find, finally, that

$$\sqrt{\overline{p_w^2}} = 2 \cdot 2 \qquad (39)$$

which shows good agreement with the experimental results.

3. Compressible flow theory

3.1. The pressure disturbance equation (zero heat transfer)

Since the experimental results of the wall pressure fluctuations by Kistler and Chen (loc. cit.) at high Mach numbers show only a relatively small divergence from the linear relation between $\sqrt{\overline{p_w^2}}$ and τ_w we might expect that the dominant terms contributing to the wall pressure in incompressible flow also play a dominant role in compressible flow. Thus, if we neglect the diffusive terms and write our equations in dimensionless form, we find, following Phillips (loc. cit.), if the mean values of the density and viscosity at the wall are constant, that

$$-\frac{u_\tau^2}{a_\infty^2}\frac{D^2 p}{Dt^2} + \frac{a^2}{a_\infty^2}\nabla^2 p + \frac{d\, a^2/a_\infty^2}{dx_2} \cdot \frac{\partial p}{\partial x_2} = \frac{\rho_\infty}{\rho_w} A(\underset{\sim}{x}, t) \qquad (40)$$

where $p = p/\rho_w u_\tau^2$; $\underset{\sim}{x} = \rho_w u_\tau \underset{\sim}{x}/\mu_w$; $t = t u_\tau^2 \rho_w/\mu_w$; $u_\tau = \sqrt{\frac{\tau_w}{\rho_w}}$

$$A(\underset{\sim}{x}, t) = -\left(2\tau \frac{\partial u_1}{\partial x_1} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right) \qquad (41)$$

---

* $$\int_0^\infty \frac{k^3 e^{-k^2}}{(k + \beta\ell)^2} dk = (1/2)\left(1 - 2\sqrt{\pi}\,\beta\ell\right) + 3(\beta\ell)^2 J(\beta\ell) + (\beta\ell)^3 J'(\beta\ell)$$

and $$\frac{D}{Dt} = \frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x_1} \;, \quad \text{where} \quad U_1 \equiv U_1(x_2) \;.$$

The speed of sound, a, is also a function of x_2 only.

As Phillips remarks, the left hand side of this equation neglects convection and scattering of the sound by the turbulence and by fluctuations in the speed of sound. The diffusive terms can be added to $A(\underset{\sim}{x}, t)$ and hence, by using (40) as the basic equation defining pressure fluctuations in a turbulent compressible flow, there is little loss of generality. We see that the effect of including fluctuations in density (sound waves) has given rise to additional terms as compared with the equation in incompressible flow. Since diffusive effects, however, have been neglected, we find that the pressure fluctuations are the result of the fluctuating vorticity and sound modes, where in general the vorticity mode is the larger. If this were not so, it would imply that the mean properties of the turbulence in a compressible shear flow could not be derived from a transformation of the results in incompressible flow. But both Morkovin[24] and Coles[25] have shown that this scaling up of the incompressible data gives fair agreement with the limited measurements made in supersonic flows. If the sound mode could be ignored, we could write (40) in the form

$$\nabla^2 \dot{p} = (\rho/\rho_w)\, A(\underset{\sim}{x}, t) \qquad \text{——— (40a)}$$

and its solution would then follow on the lines given above for incompressible flow. However, at this stage, we will continue to retain (40) in full and investigate more fully the terms giving rise to the sound mode.

The equation for the Fourier coefficient $d\tilde{\omega}$, as defined in (22), is, if $k^2 = k_1^2 + k_3^2$

$$d\tilde{\omega}'' + \frac{d \ln a^2/a_\infty^2}{d x_2} d\tilde{\omega}' - \left[k^2 - \frac{u_T^2}{a^2}(\omega + U_1 k_1)^2\right] d\tilde{\omega}$$
$$= (\rho/\rho_w)\, d\gamma \qquad \text{——— (42)}$$

since $\quad a^2/a_\infty^2 = \rho_\infty/\rho \quad$ for constant mean pressure over the entire shear layer, and primes denote differentiation with respect to x_2.
If, following Phillips (loc. cit.), the first derivative is eliminated by the use of the new dependent variable $\quad \zeta = (a/a_w)\, d\tilde{\omega}$

then

$$\zeta'' - \left[k^2 - \frac{u_T^2}{a^2}(\omega + U_1 k_1)^2 + \frac{a''}{a}\right] \zeta = (\rho a/\rho_w a_w)\, d\gamma \qquad \text{——— (43)}$$

with $\zeta(0; \underset{\sim}{k}, \omega) = d\tilde{\omega}(0; \underset{\sim}{k}, \omega)$. This equation should be compared with its corresponding equation in incompressible flow (23), which is obtained from (43), when $a = a_w \longrightarrow \infty$.

For the case of zero heat transfer, we have

$$a^2 = a_w^2 - \frac{\gamma-1}{2} u_\tau^2 U_1^2 \qquad ; \qquad a_w' = 0$$

and

$$a''/a = -\frac{\gamma-1}{2}(u_\tau^2/a^2)\left(U_1 U_1'' + \frac{a_w^2}{a^2} U_1'^2\right) \qquad (44)$$

It can therefore be seen that for small wave numbers, a''/a is a not unimportant term in the left hand side of (43). However, except at the wall, it is small compared with the source terms on the right hand side and so in general can be neglected.

Now in Phillips' analysis, he chose non-dimensional co-ordinates such that the width of the shear layer was unity and the Mach number of the external flow was very high. He found, finally, the radiation of sound from the shear layer by a solution which neglected terms of order $1/M_\infty$.

In our problem we have chosen boundary layer co-ordinates such that $U_1 \longrightarrow U_\infty$ as $x_2 \longrightarrow \infty$. We cannot strictly estimate the radiation of sound from the boundary layer since we cannot enter the far field outside the boundary layer. However we can estimate the disturbance in the outer region of the layer if we find the solution to (43) satisfying appropriate boundary conditions. By neglecting diffusive effects, our boundary conditions are

$$d\tilde{\omega}'(0) = \xi'(0) = 0 \qquad (45)$$

and either $\quad d\tilde{\omega}(\infty) = \xi(\infty) = 0$

or, $\quad \xi \sim e^{-i\lambda x_2} \qquad x_2 \rightarrow \infty \qquad (46)$

where we differentiate between the cases of zero disturbance outside the boundary layer and that of outward propagation waves.

In our problem it is too restrictive to find only a solution for large values of M_∞ and it is desirable to choose some other parameter which defines the flow. In (43) we noted the existence of the term u_τ^2/a^2 and we find that for zero heat transfer

$$u_\tau^2/a_w^2 = (c_f/2)\, M_\infty^2 \ll 1$$

for all M_∞*. It follows that a solution to (43) is required for large values of the

---

* If following Coles, we put
$$c_f/c_{f_i} = \left[1 + \frac{\gamma-1}{2} M_\infty^2 (1 - K c_{f_i})\right]^{-1}$$
for the case of zero heat transfer, where $K \simeq 153$, we see that
$$u_\tau^2/a_w^2 = 5 c_{f_i} / 2(1 - K c_{f_i}) \quad \text{as} \quad M_\infty \rightarrow \infty.$$

parameter $\lambda = a_w / u_T$. (This is true also for most cases with heat transfer).

Let us now define the new independent variable

$$y = x_2 u_T^2 / a_w^2$$

Then with

$$q(y) = k^2 a_w^2 / u_T^2 - \left(\frac{\omega + U_1 k_1}{\bar{a}}\right)^2 + \frac{\bar{a}_{yy}}{\bar{a}} u_T^2 / a_w^2 \quad \text{——} (47)$$

where $\bar{a} = a/a_w$, we find (43) becomes

$$\frac{d^2 \zeta}{d y^2} - \lambda^2 q(y) \zeta = H(y) \quad \text{——} (48)$$

where

$$H(y) = (\rho a / \rho_w a_w) (a_w / u_T)^4 \, d\gamma .$$

and the term in λ^4 in H(y) is absorbed finally in the velocity derivatives and wave number terms.

If we now follow the arguments used in the incompressible flow theory, we see that H(y) will be negligible over the outer three quarters of the boundary layer. Also, in this same region, $\bar{a}_{yy}/\bar{a}$ will be small, owing to the small gradients in the mean velocity. Thus a good approximation to (43) in the outer region of the boundary layer will be to replace U_1 by U_∞, $\bar{a}$ by a_∞/a_w and then

$$\frac{d^2 \zeta}{d y^2} - \lambda^2 \left[k^2 a_w^2 / u_T^2 - \left(\omega a_w / a_\infty + k_1 M_\infty a_w / u_T\right)^2 \right] \zeta = 0 \quad \text{——} (49)$$

But if the convection speed of the turbulence is U_c, the frequency ω is given by

$$\omega = - k_1 U_c \quad \text{——} (50)$$

and (49) becomes

$$\frac{d^2 \zeta}{d y^2} - \lambda^2 \left[k^2 a_w^2 / u_T^2 - (1 - U_c/U_\infty)^2 k_1^2 M_\infty^2 a_w^2 / u_T^2 \right] \zeta = 0 \quad \text{——} (51)$$

having solutions of the form

$$\left.\begin{aligned} \zeta &\sim \exp\left(\pm \lambda q_1^{1/2} y\right) \quad \text{for} \quad q_1 > 0 \\ &\text{and} \\ \zeta &\sim \exp\left(\pm i \lambda |q_1|^{1/2} y\right) \quad \text{for} \quad q_1 < 0 \end{aligned}\right\} \quad \text{——} (52)$$

where $$q_1 = q(\infty) = \lambda^2\left[k^2 - M_\infty^2 k_1^2\left(1 - U_c/U_\infty\right)^2\right] \quad (53)$$

The interpretation of the second type of solution was given by Phillips, who showed that it is equivalent to inward and outward propagating eddy Mach waves respectively having wave numbers such that, with $k_1 = k\cos\theta$,

$$\cos^2\theta > \left[M_\infty\left(1 - U_c/U_\infty\right)\right]^{-2} \quad (54)$$

Hence eddy Mach waves are generated over wave numbers in the plane for which

$$\pi - \theta_m < \theta < \pi + \theta_m \qquad \text{and} \qquad -\theta_m < \theta < \theta_m$$

where

$$\theta_m = \cos^{-1}\left(1 \Big/ \left(1 - \frac{U_c}{U_\infty}\right) M_\infty\right) \quad (55)$$

when $U_c/U_\infty = 0.8$, outward radiating Mach waves occur when M exceeds 1.25. The region near $|\underset{\sim}{k}| = 0$ is excluded since eddies beyond a certain size do not exist.

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Although these considerations give us some idea of the solution to (48), they really only help us define conditions which have to be satisfied near the outer edge of the boundary layer. It is shown, therefore, that the first boundary condition of (46) is applicable for wave numbers defined by

$$\pi - \theta_m < \theta < \theta_m \qquad \text{and} \qquad \pi + \theta_m < \theta < 2\pi - \theta_m$$

while the second boundary condition applies for all other values of θ.

In regions closer to the wall, q(y) changes sign when*

$$a^2/u_T^2(U_c - U_1)^2 = 1 + \left(\frac{\gamma-1}{2}\right)\left(U_1 U_1'' + U_1'^2 a_w/a\right) \Big/ k^2(U_c - U_1)^2 \quad (56)$$

and this occurs at the wall when

$$\frac{\left(1 + \frac{\gamma-1}{2} M_\infty^2\right)}{M_\infty^2} \left(U_\infty/U_c\right)^2 \left(1 - \frac{\gamma-1}{2} \Big/ k^2\lambda^2\right) = 1 \quad (57)$$

---

* The terms involving a''/a are included for convenience only.

As already stated, it is preferable to put them on the right hand side of (48) and to regard them as source terms.

Hence for $\gamma = 1.4$ and $U_c/U_\infty = 0.8$ to 0.7, this limiting value of M_∞ ranges from 1.5 to 1.8 approximately, provided that the bracket term is O(1). At low supersonic Mach numbers there still exists an extensive region of the boundary layer where the flow is subsonic relative to the speed of sound at the wall, even though the local flow is partly supersonic, and eddy Mach waves do not exist. At higher Mach numbers, as shown by Kistler and Chen, the convection velocity is supersonic relative to the speed of sound at the wall and at each region distance y from the wall, there is a range of wave numbers from which eddy Mach waves are generated. The remaining wave numbers in the turbulence produce disturbances near the wall of the exponential type, just as in incompressible flow. The condition for the generation of eddy Mach waves is

$$|\cos\theta| > \frac{(a/a_\infty)\left(1 + \bar{a}_{yy}/\bar{a}\, k^2\lambda^4\right)}{M_\infty \left|(U_c - U_1)/U_\infty\right|} \qquad (58)$$

Even so except at very high Mach numbers, there is a large range of wave numbers for which the effective speed of the disturbances is subsonic with respect to the wall.

3.2. The solution of the pressure equation

We have shown above that the pressure disturbance equation can be written in the form

$$\frac{d^2\xi}{dy^2} - \lambda^2 q(y)\,\xi = H(y) \qquad (48)$$

where q(y) and its derivatives are continuous functions of y. This equation has a transition point at $y = y_0$, where $q(y_0) = 0$. If $y = Y$ when $\omega + U_1 k_1 = 0$ where $\omega = -U_c k_1$, then q(y) will be positive in the range $y_0 < y < Y$, provided that $\bar{a}_{yy}/\bar{a}\lambda^2$ can be neglected. If a second transition point occurs at $y = y_1$ where $q(y_1) = 0$ and $y_1 > y_0$, then q(y) will be positive in the range $y_0 < y < y_1$, but q(y) will be negative for $0 < y < y_0$ and $y > y_1$.

Case I $\quad q(y) > 0 \qquad 0 \leq y \leq y_2$

Since λ is a large parameter, we can find an asymptotic solution to (48). On neglecting terms of $O(1/\lambda)$ and inserting the boundary condition $d\xi/dy = 0$ at y = 0 we find

$$\xi(y) = \frac{2A'}{q^{1/4}} \cosh\left(\lambda \int_0^y q^{1/2}\, dy'\right) + \frac{1}{\lambda q^{1/4}} \int_0^y \frac{H(y')}{q(y')^{1/4}} \sinh\left(\lambda \int_{y'}^{y} q^{1/2}\, dy''\right) dy' \qquad (59)$$

where A' is a constant given by

$$\xi(0) = d\tilde{\omega}(0) = 2A'/q(0)^{1/4} \qquad (60)$$

but can only be determined when some other boundary condition is inserted, for instance at $y = y_2$. If we argue, however, that the region surrounding y = Y provides the dominant contribution to the disturbance at the wall and for $y > y_2$, $\xi(y) \ll \xi(0)$

we find that

$$d\tilde{\omega}(0) \sim -\left(1/\lambda\, q(0)^{1/4}\right) \int_0^{y_2} \frac{H(y')}{q(y')^{1/4}} \exp\left(-\lambda \int_0^{y'} q^{1/2}\, dy''\right) dy' \qquad (61)$$

which reduces to (26) in incompressible flow when $\lambda q^{1/2} = \frac{a_w^2 k}{U_T^2}$ and $y_2 = \infty$.

Case II $\quad q < 0 \qquad y_1 < y < \infty$

If a second transition point occurs at $y = y_1$, then $q(y_1) = 0$ and q is negative in the region $y_1 < y \leqslant \infty$

Let us put $q^*(y) = -q(y)$

$$= \left(\frac{\omega + U_1 k_1}{\bar{a}}\right)^2 - k^2 a_w^2/U_T^2 - \frac{\bar{a}_{yy}}{\bar{a}}\, U_T^2/a_w^2 \qquad (62)$$

then a solution is required of

$$\frac{d^2 \zeta}{dy^2} + \lambda^2 q^*(y)\, \zeta = H(y) \qquad (63)$$

where $q^*(y) > 0$ in $y_1 < y < \infty$ and $q^*(y_1) = 0$.

Near $y = y_1$ we have

$$q^* \sim q^{*\prime}(y_1)(y - y_1) \qquad (64)$$

where $q^{*\prime}(y_1) > 0$.

If we introduce the new dependent variable $\Omega(\xi)$ where

$$\zeta = \Omega(\xi) \Big/ \sqrt{\frac{d\xi}{dy}} \qquad (65)$$

and

$$\left(d\xi/dy\right)^2 = q^*/\xi \qquad (66)$$

then from (63)

$$\frac{d^2\Omega}{d\xi^2} + \lambda^2 \xi\, \Omega = H\Big/(\xi')^{3/2} + \text{small terms} \qquad (67)$$

with

$$\xi = \left[(3/2) \int_{y_1}^{y} q^{*\,1/2}\, dy'\right]^{2/3} \qquad (68)$$

The range of ξ is $0 \leqslant \xi < \infty$ and we take

$$\frac{d\xi}{dy} = \left(q^*/\xi\right)^{1/2} \qquad (69)$$

If we assume that only outgoing Mach waves exist for $y \gg y_1$, then

$$\xi(y) \sim \exp\left(-i\lambda \int_{y_1}^{y} \sqrt{q^*}\, dy'\right) \qquad (70)$$

The solution of the homogeneous part of (67) can be written down in terms of the functions

$$\mu^{*\,1/3}\, J_{\pm\frac{1}{3}}(\mu^*)$$

where $\mu^* = \frac{2}{3}\lambda \xi^{3/2}$ and $0 \leq \mu^* < \infty$
and the solution for $\xi(y)$, which can be continued analytically into the region $y \leq y_1$, follows apart from one undetermined constant.

Case III $\quad q > 0 \qquad y_0 < y < y_1$

As stated above, the solution obtained in the region $y_1 < y < \infty$ can be continued analytically into the region $y_0 < y < y_1$ and therefore involves terms of the form

$$\mu^{1/3}\, I_{\frac{1}{3}}(\mu) \qquad \text{and} \qquad \mu^{1/3}\, K_{\frac{1}{3}}(\mu)$$

where
$$\mu = \frac{2}{3}\lambda t^{3/2}$$
and
$$t = \left[(3/2)\int_{y}^{y_1} \sqrt{q}\; dy'\right]^{2/3} .$$

The solution for $\xi(y)$ is therefore given in this region also, apart from one undetermined constant, the same constant as in Case II above. This solution cannot, however, be extended to values of y near y_0 where $q(y_0) = 0$. A solution can, however, be found around $y = y_0$ involving the functions

$$\eta^{1/3}\, I_{\frac{1}{3}}(\eta) \qquad \text{and} \qquad \eta^{\frac{1}{3}}\, K_{\frac{1}{3}}(\eta)$$

where
$$\eta = \frac{2}{3}\lambda s^{3/2}$$
and
$$s = \left[(3/2)\int_{y_0}^{y} \sqrt{q}\; dy'\right]^{2/3}$$

The solution contains two undetermined constant.

Case IV $\quad q < 0 \qquad 0 < y < y_0$

The solution obtained in III can be continued analytically into the region $0 < y < y_0$, and hence it is given apart from the two constants. Boundary conditions at the wall provide one relation and two more relations are obtained by patching the solutions obtained in III at some convenient value of y in $y_0 < y < y_1$. The three unknown constants can now be evaluated and the solution for $\xi(y)$ throughout the entire boundary layer has been obtained, and in particular the value at the wall. (Full details of this solution are given in a separate paper which will be published shortly).

First of all let us find the changes that result in $d\tilde{\omega}(o)$ as a result of compressibility effects at low supersonic and high subsonic Mach numbers and these can be demonstrated on evaluation of (61).

If we assume, as in incompressible flow, that the dominant contribution to H(y) arises from the (M - T) term then

$$H(y) = (\rho a/\rho_w a_w)(a_w^4/u_\tau^4)\, d\gamma = -(\rho a/\rho_w a_w)(a_w/u_\tau)^4 2ik_1\tau(dz_2) \qquad (71)$$

where $\tau = dU_1/dx_2$; $q(y) \approx k^2 a_w^2/u_\tau^2 - (\omega + U_1 k_1)^2/\bar{a}^2$

and

$$d\tilde{\omega}(o) = \frac{2ik_1}{(k^2 - \omega^2 u_\tau^2/a_w^2)^{1/4}} \int \frac{\rho a}{\rho_w a_w} \frac{dU_1}{dy} \frac{e^{-\lambda\int_0^y q^{\frac{1}{2}} dy'}}{\sqrt{\frac{u_\tau}{a_w}}\; q^{1/4}} (dz_2)\, dy \qquad (72)$$

for $k^2 > \omega^2 u_\tau^2/a_w^2$.

We now require values of dU_1/dx_2 and $\overline{u_2^2}$ as functions of the freestream Mach number, and these can be obtained from their so-called equivalent incompressible counterparts using Coles transformation formulae.

If quantities with a bar represent incompressible values

$$\bar{x}_2 = \int_0^{x_2} \frac{\rho}{\rho_w} \sqrt{\frac{\mu_{w}}{\mu_s}}\, dx_2 \qquad (73)$$

where μ_s is the sub-layer viscosity which, for zero heat transfer and $\gamma = 1{\cdot}4$, is evaluated at the temperature T_s given by

$$T_s/T_w = (1 + 0{\cdot}11 M_\infty^2)/(1 + 0{\cdot}2 M_\infty^2) \qquad (74)$$

Also

$$\bar{U}_1 = U_1 \sqrt{\frac{\mu_w}{\mu_s}} \qquad (75)$$

where $\bar{U}_1$ is the non-dimensional mean velocity in incompressible flow and U_1 is its corresponding value in compressible flow. If further we assume that the relation between T and U_1 is given by the Crocco energy integral, we can perform the integration in (73) and so find the relation between x_2 and $\bar{x}_2$ in the form

$$x_2 = \bar{x}_2 \sqrt{\frac{\mu_s}{\mu_w}} \left[1 - \frac{\frac{\gamma-1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2} M_\infty^2} \cdot \frac{\bar{u}_\tau^2}{\bar{U}_\infty^2} \left\{ \bar{U}_1(\bar{U}_1 - 2/K) + 2/K^2 \right\} \right] \qquad (76)$$

where K is the von Kármán constant. In addition it is found that

$$\frac{\rho a}{\rho_w a_w} \cdot \frac{dU_1}{dx_2} = \frac{d\bar{U}_1/d\bar{x}_2}{\left[1 - \bar{U}_1^2 \frac{\frac{\gamma-1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2} M_\infty^2} \frac{\bar{u}_\tau^2}{\bar{U}_\infty^2} \right]^{3/2}} \qquad (77)$$

for the case of zero heat transfer.

If we assume that $\overline{u_2^2}$ changes with compressibility in the same way as the mean flow*

$$\overline{u_2^2}\,(x_2)\,\frac{\mu_W}{\mu_S} = \overline{\bar{u}_2^2}\,(\bar{x}_2) \qquad (78)$$

and a similar relation is assumed to exist between the Fourier coefficients such that

$$dZ_2\,(x_2) = d\bar{Z}_2\,(\bar{x}_2)\sqrt{\frac{\mu_S}{\mu_W}} \qquad (79)$$

After some reduction we find, on substituting (76), (77) and (79) into (72), that

$$d\tilde{\omega}(0;\underset{\sim}{k},\omega) = \frac{2ik_1\,\mu_S/\mu_W}{k\,X(0)^{1/2}}\int_0 \frac{d\bar{U}_1}{d\bar{x}_2}\,\frac{d\bar{Z}_2\,e^{-kf(\bar{x}_2)}}{X(\bar{x}_2)^{1/2}\left(T(\bar{x}_2)/T_W\right)^{1/2}}\,d\bar{x}_2 \qquad (80)$$

where

$$k^2X(\bar{x}_2)^2 = k^2 - \frac{\mu_W}{\mu_S}\frac{\bar{u}_T^2}{\bar{U}_\infty^2}\frac{M_\infty^2}{1+\frac{\gamma-1}{2}M_\infty^2}\frac{\left(\omega + k_1\bar{U}_1\sqrt{\mu_W/\mu_S}\right)^2}{\left(1-\frac{\bar{U}_1^2\,\bar{u}_T^2}{\bar{U}_\infty^2}\frac{\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_\infty^2}\right)}$$

and

$$kf(\bar{x}_2) = k\int_0^{\bar{x}_2} X(\bar{x}_2')\sqrt{\frac{\mu_S}{\mu_W}}\left(1-\frac{\bar{U}_1^2\,\bar{u}_T^2}{\bar{U}_\infty^2}\frac{\frac{\gamma-1}{2}M_\infty^2}{\left(1+\frac{\gamma-1}{2}M_\infty^2\right)}\right)d\bar{x}_2'$$

The spectrum function is therefore given by

$$\Pi(0;\underset{\sim}{k},\omega) = \frac{4k_1^2\left(\mu_S/\mu_W\right)^2}{k^2X(0)}\int_0^\infty \frac{\tau(y)\,e^{-2kf(y)}}{X(y)^{1/2}\left(T(y)/T_W\right)^{1/2}}\,dy$$
$$\cdot\int_{-\infty}^{\infty}\Phi_{22}(y,z;\underset{\sim}{k},\kappa_2,\omega)\,d\kappa_2 \qquad (81)$$
$$\cdot\int_{-y}^{\infty}\frac{\tau(y+z)\,e^{i\kappa_2 z}\,e^{-k\left(f(y+z)-f(y)\right)}}{X(y+z)^{1/2}\left(T(y+z)/T_W\right)^{1/2}}\,dz$$

where the upper limit in (80) has been replaced by infinity. We see that (81) is identical with (34), apart from the term in μ_S/μ_W, T/T_W, $kX(y)$ in place of k, and $kf(y)$ in place of ky.

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* This relation between $\overline{u_2^2}$ and $\overline{\bar{u}_2^2}$ differs from that used by Morkovin (loc. cit.) but neither relation is in good agreement with the available experimental data. All that can be said of (79) is that it qualitatively has the right trend with increase in Mach number.

A rough approximation to (81) is found by replacing f(y) with $\bar{f}y$ and X(y) with $\bar{X}$, where $\bar{f}$ and $\bar{X}$ are independent of y. We then find that the integrals have the same form as in incompressible flow and have the values obtained previously. Hence, on reference to (35) and (36), we find

$$\Pi(0; \underset{\sim}{k}) \approx \frac{1{\cdot}68\, k_1^2\, \ell^4\, h^2(0)\,(\mu_s/\mu_w)^2}{\pi\ (k\bar{f}+\beta)^2}\, e^{-\ell^2 k^2} \qquad \text{———} (82)$$

where the integration over all frequencies has been performed on the assumption that, except at very small wave numbers,

$$X(y) \approx \bar{X} \approx X(x_2 \text{ at } U_1 = U_c) = 1;\ h(y) = \tau\Big/\sqrt{\overline{u_2^2}}\sqrt{\frac{T_W}{T}} = h(0)e^{-\beta y}.$$

The integration over all wave numbers k_1 and k_3 can next be obtained and leads to

$$\overline{p^2(0)} = 1{\cdot}68\,(\mu_s/\mu_w)^2\, h^2(0)\, L(M_\infty) \qquad \text{———} (83)$$

where

$$L(M_\infty) = (1/\bar{f}^2)\int_0^\infty \frac{k^3 e^{-k^2}\, dk}{(k+\beta\ell/\bar{f})^2} \qquad \text{———} (84)$$

which can be evaluated in terms of the function tabulated by Goodwin and Staton (loc. cit.).

All that remains is to choose suitable values for h(o), $\beta\ell$ and $\bar{f}$. If we follow the arguments used in the incompressible flow analysis, we must choose h(o) such that h(y) is a good approximation to its corresponding value in compressible flow near $x_2 = \delta_1$.

It follows that some adjustment to $\beta\bar{\delta}_1$ with Mach number is necessary and we put $\beta\ell = 1$,

$$h(0) = \frac{2e^{\beta\bar{\delta}_1}}{\bar{\delta}_1\ (T/T_W)^{1/2}_{\bar{x}_2=\bar{\delta}_1}} \qquad \text{———} (85)$$

with

$$\beta\bar{\delta}_1 = \left[1 - 0{\cdot}6\left(\frac{\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_\infty^2}\right)\frac{\bar{u}_\tau^2}{\bar{U}_\infty^2}\right]\Bigg/\left(T/T_W\right)^{1/2}_{\bar{x}_2=\bar{\delta}_1} \qquad \text{———} (86)$$

and

$$\bar{f} = \sqrt{\mu_s/\mu_w}\left[1 - \left(\frac{\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_\infty^2}\right)\frac{\bar{u}_\tau^2}{\bar{U}_\infty^2}\left(\bar{U}_1(\bar{U}_1 - 2/K) + 2/K^2\right)\right]_{\bar{x}_2=\bar{\delta}_1} = \sqrt{\frac{\mu_s}{\mu_w}}\left(\frac{1+0{\cdot}1M_\infty^2}{1+0{\cdot}2M_\infty^2}\right)$$

approximately. ——— (87)

If we use the asymptotic expansion for the function J(x) given by Goodwin and Staton we find

$$\sqrt{\overline{p^2(0)}} \longrightarrow 6{\cdot}3 \quad \text{as} \quad M_\infty \longrightarrow \infty.$$

Fig. 6 shows results for $\sqrt{\overline{p^2(o)}}$ evaluated from (83), together with (84) to (87), and the experimental results of Kistler and Chen (loc. cit.) and Willmarth, Hodgson and, Mull and Algranti[26] at low speeds. As already stated, these theoretical results should only be applicable for low supersonic Mach numbers but it is here that we have the greatest divergence with the experimental results. There does not appear to be any justification for reducing the experimental results of Kistler and Chen, by a constant factor, but if this is done, they fit both the low speed data of other workers and the theoretical curve throughout the entire range of Mach number.

However on the assumption that the difference between the present theory and experiment is real, clearly we must find what is wrong with regard to the theory. It is difficult to see how it is wrong at low supersonic Mach numbers and high subsonic Mach numbers, where no eddy Mach waves can exist and where the compressibility effects on the mean flow field and the turbulence are known to be small.

If we ignore the experimental point of Kistler and Chen at $M_\infty = 0.6$, and assume that at some value of M_∞ above 1.25 a large increase in pressure level occurs as a result of the generation of eddy Mach waves, then it is surprising that an ever increasing divergence between our theory and experiment does not exist as M_∞ is increased. The fact that both sets of results appear to have the same asymptotic behaviour at high Mach numbers seems to suggest that eddy Mach waves do not contribute greatly to the wall pressure fluctuations. The significance of this will be explored in the next section.

Before closing this section we note that the changes in the frequency spectrum with increase in M_∞ will not be large and the peak should occur at near $\omega\delta_1/U_\infty = 0{\cdot}3$, its value in incompressible flow. In fact the results of Kistler and Chen are in good agreement with the low speed results of Willmarth. Also, the main effects of Mach number on $\sqrt{\overline{p_w^2}}$ appear to be a reduction due to the decrease in with increase in M_∞, and an increase due to the shift of the dominant source region nearer to the wall with increase in M_∞.

3.3. The pressure equation at high Mach numbers

We have shown above that the solution given in (61) is restricted to the case $q(y) > 0$. If, therefore, a transition point exists between the wall and the station where $U_1 = U_c$, we must turn to the solution outlined in Case III above. The solution follows the approach used by Phillips, although we find it necessary to modify that treatment when applied to our problem. We will, however, still not consider the full solution which must include radiation outwards from the boundary layer.

The solution to (48) in the region around $y = y_o$ where $q(y_o) = 0$ is found to be

$$\xi(y) = (s/q)^{1/4}\left\{\eta^{\frac{1}{3}} I_{\frac{1}{3}}(\eta)\,\alpha(y) + \eta^{\frac{1}{3}} K_{\frac{1}{3}}(\eta)\,\beta(y)\right\} \qquad (88)$$

where

$$s = \left((3/2)\int_{y_o}^{y} \sqrt{q}\,dy\right)^{2/3} \qquad \text{for } q \geq 0$$

and $$q(y_0) = 0 \quad , \qquad \eta = (2/3)\,\lambda\, s^{3/2}$$

$$\alpha(y) = A + \frac{(2/3)^{1/3}}{\lambda^{2/3}} \int_0^s \phi(s)\, \eta^{\frac{1}{3}} K_{\frac{1}{3}}(\eta)\, ds \qquad \text{—————(89)}$$

$$\beta(y) = B - \frac{(2/3)^{1/3}}{\lambda^{2/3}} \int_0^s \phi(s)\, \eta^{\frac{1}{3}} I_{\frac{1}{3}}(\eta)\, ds \qquad \text{—————(90)}$$

and $$\phi(s) = \frac{H(y)}{(q/s)^{3/4}} + \xi(y)\,(ds/dy)^{1/2}\left(\frac{s'''}{2s^3} - \frac{3}{4}\left(\frac{s''}{s'^2}\right)^2\right)$$

and since $\phi(s)$ contains $\xi(y)$ (88) is an integral equation for $\xi(y)$.

Near $y = y_0$, since $$(ds/dy)^2 = q/s \quad ,$$

$$s^{3/2} = \pm \sqrt{q'(y_0)}\,(y - y_0)^{3/2}$$

or $$s = q'(y_0)^{1/3}(y - y_0) \quad , \quad q'(y_0)^{1/3}(y - y_0)e^{2\pi i/3}$$

corresponding to the + or - signs respectively. However, because we have put

$$s = \left((3/2)\int_{y_0}^{y} \sqrt{q}\; dy\right)^{2/3} \quad ; \quad ds/dy = \sqrt{q/s}$$

we take the plus sign, and so in the region $0 \leqslant y < y_0$

$$s = s^* e^{i\pi} \quad ; \quad q = q^* e^{i\pi}$$

with

$$s^* = \left((3/2)\int_{y_0}^{y} -\sqrt{q^*}\; dy\right)^{2/3}$$

where both s^* and q^* are real and positive.

We also have

$$\eta = \eta^* e^{3i\pi/2}$$

where

$$\eta^* = (2/3)\,\lambda\, s^{*\,3/2} = \lambda \int_{y_0}^{y} -\sqrt{q^*}\; dy \,.$$

Hence in the region $0 \leqslant y \leqslant y_o$

$$s/q = s^*/q^*$$

$$\eta^{\frac{1}{3}} I_{\frac{1}{3}}(\eta) = - \eta^{*\frac{1}{3}} J_{\frac{1}{3}}(\eta^*)$$

and
$$\eta^{\frac{1}{3}} K_{\frac{1}{3}}(\eta) = - \eta^{*\frac{1}{3}} \frac{i\pi}{2} e^{i\pi/3}\left[H^{(2)}_{\frac{1}{3}}(\eta^*) + e^{i\pi/3} H^{(1)}_{\frac{1}{3}}(\eta^*)\right]$$

$$= - \eta^{*\frac{1}{3}} \frac{i\pi}{2} e^{i\pi/3} \mathcal{J}_{\frac{1}{3}}(\eta^*) \qquad \text{say,}$$

where
$$i e^{i\pi/3} \mathcal{J}_{\frac{1}{3}}(\eta^*) = - \frac{\left(J_{\frac{1}{3}}(\eta^*) + J_{-\frac{1}{3}}(\eta^*)\right)}{\sin \pi/3}$$

Thus the analytical continuation of the function $\zeta(y)$, in the region $0 \leqslant y < y_o$ is

$$\zeta(y) = -\left(s^*/q^*\right)^{\frac{1}{4}}\left[\eta^{*\frac{1}{3}} J_{\frac{1}{3}}(\eta^*)\,\alpha^*(y) + \frac{i\pi}{2} e^{i\pi/3}\eta^{*\frac{1}{3}}\mathcal{J}_{\frac{1}{3}}(\eta^*)\,\beta^*(y)\right] \qquad (91)$$

where

$$\alpha^*(y) = A - \frac{\pi\,(2/3)^{1/3}}{2\sin\pi/3 \cdot \lambda^{2/3}}\int_0^{s^*}\phi(s^*)\,\eta^{*\frac{1}{3}}\left(J_{\frac{1}{3}}(\eta^*) + J_{-\frac{1}{3}}(\eta^*)\right)ds^* \qquad (92)$$

$$\beta^*(y) = B - \frac{(2/3)^{1/3}}{\lambda^{2/3}}\int_0^{s^*}\phi(s^*)\,\eta^{*\frac{1}{3}} J_{\frac{1}{3}}(\eta^*)\,ds^* \qquad (93)$$

with

$$\phi(s^*) = \frac{H(y)}{(q^*/s^*)^{3/4}} + \zeta(y)\,(q^*/s^*)^{\frac{1}{4}}\left[\frac{s^{*\,\prime\prime\prime}}{2 s^{*\,3}} - \frac{3}{4}\left(\frac{s^{*\,\prime\prime}}{s^{*\,\prime 2}}\right)^2\right]$$

If we apply the boundary condition $d\zeta/dy$ at y = 0 and write subscript w to denote conditions at the wall, we find, for large values of λ, that

$$\alpha^*(0) = \frac{\pi}{\sqrt{3}}\,\beta^*(0)\left(1 - J_{\frac{2}{3}}(\eta^*_w)/J_{-\frac{2}{3}}(\eta^*_w)\right) \qquad (94)$$

and

$$\zeta(0) = \zeta_w = d\tilde{\omega}(0) = \frac{(3/2)^{1/6}\quad\beta^*(0)}{\lambda^{1/6}\, q_w^{*\,1/4}\,\eta_w^{*\,1/2}\, J_{-2/3}(\eta^*_w)} \qquad (95)$$

on using the identity

$$J_{\frac{1}{3}}(x)\, J_{\frac{2}{3}}(x) \;+\; J_{-\frac{1}{3}}(x)\, J_{-\frac{2}{3}}(x) \;=\; \frac{2 \sin \pi/3}{\pi x}$$

where

$$\beta^*(o) = B - \frac{(2/3)^{1/3}}{\lambda^{2/3}} \int_0^{s_w^*} \phi(s^*)\, \eta^{*\,\frac{1}{3}}\, J_{\frac{1}{3}}(\eta^*)\, ds^*$$

and

$$\alpha^*(o) = A - \frac{\pi (2/3)^{1/3}}{\sqrt{3}\ \lambda^{2/3}} \int_0^{s_w^*} \phi(s^*)\, \eta^{*\,\frac{1}{3}} \left(J_{\frac{1}{3}}(\eta^*) + J_{-\frac{1}{3}}(\eta^*) \right) ds^* \qquad (96)$$

It is convenient to approximate to these integrals by assuming $\phi(s^*)$ = constant = $\phi(o)$ then

$$\alpha^*(o) \sim A - \frac{2\pi (2/3)^{2/3}}{\sqrt{3}\ \lambda^{4/3}} \cdot \frac{H(y_o)}{\sqrt{q_h'(y_o)}} \qquad (97)$$

and

$$\beta^*(o) \sim B - \frac{(2/3)^{2/3}}{\lambda^{4/3}} \cdot \frac{H(y_o)}{\sqrt{q'(y_o)}} \qquad (98)$$

noting that $q(y_o) = 0$, and that for the value of η_w^* given by $1 = J_{\frac{2}{3}}(\eta_w^*) / J_{-\frac{2}{3}}(\eta_w^*)$ we must put $\alpha^*(o) = 0$ in order to satisfy (94), and then $\beta^*(o)$ is not determined. But clearly this only arises because the approximation given by (94) is inadequate i.e. higher order terms must be retained, and therefore in what follows η_w^* must be less than unity.

Now the value of q_w^* is found from (47) to be

$$q_w^* = \omega^2 - k^2\lambda^2 + \frac{\gamma - 1}{2}$$

and since $\omega = -U_c k_1$ with $k_1 = k \cos\theta$

$$q_w^* > 0 \qquad \text{for} \qquad 0 < \theta < \theta_m$$

where

$$|\cos \theta_m| = \left(\frac{a_w/u_T}{U_c} \right) \sqrt{1 - (\gamma-1)/2k^2\lambda^2}$$

which shows that at $\theta = \theta_m$ the speed along the normal to the wave fronts is roughly equal to the speed of sound of the gas at the wall. Thus the position of the transition layer at which $y = y_0$, $q(y_0) = 0$, changes with frequency and coincides with the wall when $q_w^* = 0$. In the range $0 < \theta < \theta_m$, $q_w < 0$, while in the range $\theta_m < \theta < \pi/2$, $q_w > 0$. For the latter region we can make use of the solution obtained in (60)

or

$$d\tilde{\omega}(o) = \frac{2A'}{q_w^{1/4}} \qquad \theta_m < \theta < \pi/2 \qquad (60)$$

The value of ζ_w when $q_w^* = 0$ is found from (91) and is

$$d\tilde{\omega}(o) = \frac{\pi\, 2^{1/3}}{\sqrt{3}\ \Gamma(2/3)} \frac{B}{(q'_w)^{1/6}} \qquad \theta = \theta_m \qquad \text{———}(99)$$

Similarly, from (95)

$$d\tilde{\omega}(o) = \frac{(3/2\lambda)^{1/6}\ \beta^*(o)}{q'^{1/4}_w\ \eta^{*\,1/2}_w\ J_{-\frac{2}{3}}(\eta^*_w)} \qquad 0 \leq \theta < \theta_m \text{ ———}(100)$$

So far we have applied only the boundary condition that the normal pressure gradient vanishes at the wall when diffusive effects can be neglected. However one further condition is required in order to determine the constants in the formulae for $\xi(o)$. This further boundary condition results from the disturbance level near or beyond the outer edge of the boundary layer. If we assume that the bulk of the pressure disturbance near the wall arises from the region around $y = Y$, it would seem not unreasonable to assume that the level of the pressure fluctuations near $y = y_1$, where $y_1 > Y$ and $q(y_1) = 0$, must be very much smaller than at the wall, in spite of the fact that radiation outwards is taking place. As already stated, the proper boundary conditions at the edge of the boundary layer can only be applied to the region $y_1 < y < \infty$, where, for sufficiently high Mach numbers, there will exist a range of wave numbers for which q will be negative. However, for our purpose, it will be sufficient to assume that y_o and y_1 are well separated so that we can put

$$\xi = 0 \qquad \text{for} \qquad y \gg Y.$$

(Of course the radiation outwards can only be determined by proper matching of the solutions around $y = y_o$ and $y = y_1$ as previously discussed). We therefore find that $\alpha(y) = 0$ for $y \gg Y$ or from (89)

$$A = -\frac{(2/3)^{1/3}}{\lambda^{2/3}} \int_0^\infty \phi(s)\ \eta^{1/3}\ K_{\frac{1}{3}}(\eta)\ ds \qquad \text{————}(101)$$

Now, for $y_o = 0$, we find $H(o) = 0$ and so, according to the approximations used above, $\phi(o) = 0$ at $y_o = 0$. Hence according to (98)

$$\beta^*(o) = B \qquad \text{at} \qquad y_o = 0$$

and from (94) and (97) on putting $\eta^*_w = 0$

$$A = (\pi/\sqrt{3})\, B$$

or

$$B = -\frac{(2/3)^{1/3}\sqrt{3}}{\pi\ \lambda^{2/3}} \int_0^\infty \phi(s)\ \eta^{\frac{1}{3}}\ K_{\frac{1}{3}}(\eta)\ ds \text{ ——}(102)$$

which determines ζ_w in (99) when $\theta = \theta_m$.

For $y_o \neq 0$, $H(y_o)$ is finite and then making an approximation to A in (97) we find from (94) that

$$\beta^*(o) = -\frac{6^{1/3}}{\lambda^{4/3}} \cdot \frac{H(y_o)}{\sqrt{q'(y_o)}} \Big/ \left[1 - J_{\frac{2}{3}}(\eta^*_w)/J_{-\frac{2}{3}}(\eta^*_w)\right] \text{ ——}(103)$$

since
$$\int_0^\infty K_{\frac{1}{3}}(\eta)\,d\eta = \frac{\Gamma(\frac{1}{3})\,\Gamma(\frac{2}{3})}{2} = \pi/\sqrt{3}$$

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and therefore we can find ζ_w from (100) in the range $0 < \theta < \theta_m$. We note, further, as stated previously that $\beta^*(0)$ is undetermined for that value of η_w^* which makes the denominator of (103) vanish and hence we must only use (103) for $\eta_w^* < 1$ approximately.

For other values of η_w^* it will be sufficient to put $\beta^*(0)$ = B, where B is given by (102).

The values of the constants are completed by putting in the range $\theta_m < \theta < \pi/2$

$$A' = -(1/2\lambda)\int_0^\infty \frac{H(y)}{q_h^{1/4}}\exp\left(-\lambda\int_0^y \sqrt{q_h}\,dy'\right)dy \quad \text{——(104)}$$

from (59). On collecting our results together we have

$$d\tilde{\omega}(0) = -\frac{1}{q_w^{1/4}}\int_0^\infty \frac{H(y)}{q_h^{1/4}}\,e^{-\lambda\int_0^y \sqrt{q_h}\,dy'}dy\ , \quad \theta_m < \theta < \pi/2 \quad \text{——(105)}$$

$$d\tilde{\omega}(0) = -\frac{2^{2/3}}{3^{1/3}\,\Gamma(2/3)\,(q_w')^{1/6}}\int_0^\infty \frac{H(y)}{(q/s)^{3/4}}\,\eta^{\frac{1}{3}}K_{\frac{1}{3}}(\eta)\,ds\ , \quad \theta = \theta_m \quad \text{——(106)}$$

and
$$d\tilde{\omega}(0) = -\frac{3^{1/2}\,2^{1/6}\,H(y_0)}{\lambda^{4/3}\,q'(y_0)^{1/3}\,q_w^{*\,1/4}\,\eta_w^{*\,1/2}\left(J_{-\frac{2}{3}}(\eta_w^*) - J_{\frac{2}{3}}(\eta_w^*)\right)}\ , \quad 0 < \theta < \theta_m \quad \text{——(107)}$$

for values of $\eta_w^* < 1$ and by (105) with q_w replaced by q_w^* for higher values of η_w^*.

We can approximate to (106) by putting

$$\eta^{1/2}K_{\frac{1}{3}}(\eta) \sim \sqrt{\frac{\pi}{2}}\,e^{-\eta}\ , \quad \text{where} \quad \eta = \lambda\int_0^y \sqrt{q_h}\,dy'\ ,$$

and hence
$$d\tilde{\omega}(0) \sim -\frac{\sqrt{\pi}}{3^{1/6}\,\Gamma(\frac{2}{3})\,\lambda^{1/6}(q_w')^{1/6}}\int_0^\infty \frac{H(y)}{q_h^{1/4}}\,e^{-\eta}\,dy \qquad \theta = \theta_m \quad \text{——(106')}$$

having a similar form to that given in the range $\theta_m < \theta < \pi/2$.

The values of q_w' and $q'(y_0)$ can be obtained from (47) and if we ignore derivatives of the speed of sound, noting that this is a poor approximation* near y = 0,

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* The full expression for $|q_w|$ is given by

$$|q_w| = \lambda^2 k^2\left[1 - \omega^2/\lambda^2 k^2 - (\gamma-1)/2\,\lambda^2 k^2\right]$$

but note our previous remarks about the term a''/a.

we find,

$$q_w' = 2\lambda^2 |\omega k_1| \qquad (108)$$

since $dU_1/dx_2 = 1$ at $y = 0$

and

$$q'(y_0) = (2\lambda^2/\bar{a}^2) \left| k_1(\omega + U_1 k_1) \right| \qquad (109)$$

where U_1 and $\bar{a}$ are evaluated at $y = y_0$.

In (109) we find, on approximating to $q(y_0) = 0$, that

$$q'(y_0) = (2\lambda^3 k/\bar{a}) |k_1| \qquad (110)$$

and from (108)

$$q_w' = 2\lambda^3 k |k_1| \qquad (111)$$

Hence, with

$$H(y) = -(\rho a/\rho_w a_w)(a_w/u_\tau)^4 \, 2 i k_1 \tau \, dz_y$$

as given by (71), we can find expressions for $d\tilde{\omega}$, and the corresponding values of the spectrum functions are

$$\Pi(0; \underset{\sim}{k}, \omega) = \frac{4 k_1^2 \lambda^2}{q_w^{1/2}} \int_0^\infty \Phi(y, z; \underset{\sim}{k}, \omega) \frac{dU_1}{dy} \frac{\bar{\rho}\bar{a} e^{-\eta}}{q_1^{1/4}} dy \int_{-y}^{\infty} \frac{dU_1}{dz} \frac{\bar{\rho}\bar{a} e^{-\eta'}}{q_1^{1/4}} dz$$

$$\theta_m < \theta < \pi/2 \qquad (112)$$

$$\Pi(0; \underset{\sim}{k}, \omega) = \frac{4\pi k_1^2 \lambda^2}{3^{1/3} \Gamma(\frac{2}{3})^2 \lambda^{1/3} (q_w')^{1/3}} \int_0^\infty \Phi(y, z; \underset{\sim}{k}, \omega) \frac{dU_1}{dy} \frac{\bar{\rho}\bar{a}}{q^{1/4}} e^{-\eta} dy \int_{-y}^{\infty} \frac{dU_1}{dz} \frac{\bar{\rho}\bar{a}}{q^{1/4}} e^{-\eta'} dz$$

$$\theta = \theta_m \qquad (113)$$

and

$$\Pi(0; \underset{\sim}{k}, \omega) = \frac{12 . 2^{1/3} \lambda^{4/3} k_1^2}{q'(y_0)\, q_w^{*1/2}\, \eta_w^*} \; \frac{\Phi(y_0; \underset{\sim}{k}, \omega)}{\left(J_{-\frac{2}{3}}(\eta_w^*) - J_{\frac{2}{3}}(\eta_w^*)\right)} \left(\frac{dU_1}{dy} \bar{\rho}\bar{a}\right)^2_{y=y_0}$$

$$0 < \theta < \theta_m \qquad (114)$$

where throughout (112) - (114) $\bar{\rho} = \rho/\rho_w$; $\bar{a} = a/a_w$, and (114) only applies when $\eta_w^* < 1$ and for larger values of η_w^* we replace it by (112) with q_w^* in place of q_w.

Now, according to our approximation,

$$q^* = (\omega + U_1 k_1)^2 / \bar{a}^2 - k^2 \lambda^2$$

and so at a fixed wave number $\underset{\sim}{k}$, $d\omega = -k_1 \frac{dU_1}{dy} dy$ at $y = y_o$ corresponding to $q(y_o) = 0$. Thus for $0 < \theta < \theta_m$ we find that the integration over frequencies is given in terms of an integration over y_o, provided $q^* > 0$. However we find that for small values of η_w^* the contribution from wavenumbers in the range $0 < \theta < \theta_m$ is negligible, while at larger values of η_w^* the contribution is of the same order as that for the range $\theta_m < \theta < \pi/2$. Since the contribution from around $\theta = \theta_m$ is finite, we can represent $\Pi(o; \underset{\sim}{k}, \omega)$ for all θ by

$$\Pi(o; \underset{\sim}{k}, \omega) = \frac{4 k_1^2 \lambda^2}{|q_w|^{1/2}} \int_0^\infty \Phi(y, z; \underset{\sim}{k}, \omega) \frac{dU_1}{dy} \frac{\bar{\rho}\bar{a}\, e^{-\eta}}{q^{1/4}} dy \cdot \int_y^\infty \frac{dU_1}{dz} \frac{\bar{\rho}\bar{a}\, e^{-\eta'}}{q^{1/4}} dz \qquad (115)$$

On replacing the compressible flow quantities by the Coles' equivalent incompressible values, we then find, as previously,

$$\Pi(o; \underset{\sim}{k}, \omega) = \frac{4 \lambda k_1^2 \left(\frac{\mu_s}{\mu_w}\right)^2}{k \left[1 - (\omega/k\lambda)^2 - (\gamma-1)/2(k\lambda)^2\right]^{1/2}} \int_0^\infty \frac{d\bar{U}_1}{d\bar{x}_2} \frac{\sqrt{\bar{u}_2^2}}{\bar{a}^{1/2}} \frac{e^{-\eta}}{q^{1/4}} d\bar{x}_2$$

$$\cdot \int_{-\infty}^{\infty} \Phi(\bar{x}_2, \bar{x}_2'; \underset{\sim}{k}, \kappa_2, \omega)\, e^{i\kappa_2 \bar{x}_2'} d\kappa_2$$

$$\cdot \int_{-\bar{x}_2}^{\infty} \frac{d\bar{U}_1}{d\bar{x}_2} \frac{\sqrt{\bar{u}_2^2}}{\bar{a}^{1/2}} \frac{e^{-\eta'}}{q^{1/4}} d\bar{x}_2' \qquad (116)$$

We find, on making similar approximations to those used above in evaluating (81), that the integration over frequency gives

$$\Pi(o; \underset{\sim}{k}) \sim (0{\cdot}84/\pi^{3/2}) \frac{k_1^2 h^2(o)\, \ell^4 (\mu_s/\mu_w)^2}{(k\bar{F} + \beta)^2} e^{-\ell^2 k^2} F\left(\frac{\ell k \lambda}{U_c}\right) \qquad (117)$$

where

$$F\left(\frac{\ell k \lambda}{U_c}\right) = \int_{-\infty}^{\infty} \frac{(1 + 2\omega^2)\, e^{-\omega^2} d\omega}{\left| 1 - \frac{U_c^2 \omega^2}{\ell^2 k^2 \lambda^2} - \frac{(\gamma-1)}{2k^2\lambda^2} \right|^{1/2}}$$

On approximating to this integral, we find that

$$\overline{p^2(0)} \sim 1{\cdot}68 \left(\mu_s/\mu_w\right)^2 h^2(0)\, \ell^2 \; L\left(M_\infty\right) \quad \text{———} (118)$$

just as in the previous case. (83). Thus, although we have taken account of the contribution due to the eddy Mach waves, we see that, according to the approximations made, they do not contribute more than those eddies which travel at subsonic speeds relative to the wall.

4. Conclusions

It has been shown that theory and experiment are in fairly good agreement in the prediction of wall pressure fluctuations in a turbulent incompressible boundary layer. On the other hand, theory and experiment, in the case of supersonic flows, show some divergence, although both appear to tend to a similar asymptotic value and both demonstrate the presence of eddy Mach waves above a certain Mach number. However the theory does not show any marked increase in pressure level at the wall due to eddy Mach waves. In fact, a simple extension to the incompressible flow theory is shown to give similar results to the more elaborate theory in which eddy Mach waves are approximately taken into account. This result is perhaps not surprising when we note that the dominant region associated with the pressure at the wall, even in the incompressible case, is displaced away from the wall and the level of pressure is roughly constant over an appreciable distance normal to the wall.

It is clear that further experimental results are required to explain the differences between theory and experiment, as well as to obtain further information in the case of flows with pressure gradient and with heat transfer.

5. References

1. Heisenberg, W. — Zur Statistischen Theorie der Turbulenz. Z. Physik, Vol.124, 1948, p.628.

2. Obukhov, A.M. — Pressure fluctuations in a turbulent flow. C.R.Acad.Sc. URSS, Vol. 66, 1949, p.17.

3. Batchelor, G.K. — Pressure fluctuations in isotropic turbulence. Proc. Cambridge Phil. Soc., Vol.47, 1951, p.359.

4. Uberoi, M.S. — Quadruple velocity correlations and pressure fluctuations in isotropic turbulence. J.Aero.Sci. Vol.20, 1953, p.197.

5. Kraichnan, R.H. — Pressure fluctuations in turbulent flow over a flat plate. J.Acoust.Soc. Amer. Vol.28, 1956, p.378.

6. Hodgson, T.H. — Pressure fluctuations in shear flow turbulence. C. of A. Note 129, 1962 (unpublished).

7. Kistler, A.L., Chen, W.S. — The fluctuating pressure field in a supersonic turbulent boundary layer. Jet. Prop.Lab. Tech.Rep. No.32-277, 1962.

8. Lilley, G.M. — Pressure fluctuations in an incompressible turbulent boundary layer. C. of A. Report 133, 1960.

9. Lilley, G.M., Hodgson, T.H. — On surface pressure fluctuations in turbulent boundary layers. AGARD Report 276, 1960.

10. Phillips, O.M. — On the aerodynamic surface sound from a plane turbulent boundary layer. Proc. Roy.Soc. London, Ser.A., Vol.234, 1956, p.327.

11. Wooldridge, C.E., Willmarth, W.W. — Measurements of the fluctuating pressure at the wall beneath a thick turbulent boundary layer. U. of M. Tech.Report ORA project 02920, April 1962.

12. Phillips, O.M. — On the generation of sound by supersonic turbulent shear layers. J.Fluid Mech. Vol.9, 1960, pp 1-28.

13. Laufer, J. — Aerodynamic noise in supersonic wind tunnels. J.Aero/Space Sci. Vol.28, 1961, pp 685-692.

14. Ffowcs Williams, J.E. — The noise from turbulence convected at high speed. (To appear in Proc.Roy.Soc.London, Ser.A, 1962).

15. Lighthill, M.J. — Jet noise. The Wright Brothers Memorial Lecture, Jan. 1963.

References (Continued)

16. Williams, D.J.M. — Measurements of the surface pressure fluctuations in a turbulent boundary layer at supersonic speeds. Southampton Univ. Rep. A.A.S.U.128, 1960.

17. Willmarth, W.W — Space-time correlation and spectra of wall pressure in a turbulent boundary layer. NASA Memo 3-17-59W, 1959.

18. Sternberg, J. — A theory for the laminar sublayer of a turbulent flow. J. Fluid Mech. Vol.13, 1962, p.241.

19. Remenyik, C.J., Kovasznay, L.S.G. — The orifice-hot-wire probe and measurements of wall pressure fluctuations. Proc. 1962 Heat Transfer and Fluid Mech. Inst., 1962.

20. Corcos, G.M. — Pressure fluctuations in shear flows. U. of California Rep. Series 183/2, 1962.

21. Townsend, A.A. — The structure of turbulent shear flow. C.U.P. 1956.

22. Grant, H.L. — The large eddies of turbulent motion. J. Fluid Mech. Vol.4, 1958, p.149.

23. Goodwin, E.T., Staton, J. — The evaluation of an integral. Quart. J. Mech. Appl. Math., Vol.1, 1948, p.319.

24. Morkovin, M.V. — Effects of compressibility in turbulent shear flows. Mechanique de la turbulence. CNRS, Paris, France, 1962.

25. Coles, D.E. — The turbulent boundary layer in a compressible fluid. Rand Corp. Rep. P-2417, 1961.

26. Mull, H.R., Algranti, J.S. — Flight measurements of wall pressure fluctuations and boundary layer turbulence. NASA TN. D-280, 1960.

27. Wooldridge, C.E., Willmarth, W.W. — Measurements of the correlation between the fluctuating velocities and the fluctuating wall pressure in a thick turbulent boundary layer. U. of M. Tech. Rep. ORA project 02920, 1962.

APPENDIX 1

The velocity spectrum function for u

In order to obtain numerical values for the wall pressure, the following form has been chosen for the velocity spectrum function in fixed co-ordinates

$$\Phi_{22}\left(x_2, x_2+x_2'; \underset{\sim}{k}, \kappa_2, \omega\right) = \frac{\sqrt{\overline{u_2^2}(x_2)}\sqrt{\overline{u_2^2}(x_2+x_2')}\; \ell_1^2 \ell_2 \ell_3 \ell^2 k^2}{2\pi^{5/2}\, U_c\, (1+\ell_2^2\kappa_2^2)} \cdot$$

$$\cdot \left(1 + 2\ell_1^2\omega^2/U_c^2\right) e^{-(\ell^2 k^2 + \ell_1^2\omega^2/U_c^2)}$$

where U_c is the convection speed and $\ell^2 k^2 = \ell_1^2 k_1^2 + \ell_3^2 k_3^2$
It is assumed that ℓ_1 and ℓ_3 are constant, while $\ell_2 = x_2$, thus an allowance is made for a change in scale of the turbulence, with increase in distance from the wall.

The integration over all values of ω leads to

$$\frac{\Phi_{22}(x_2, x_2+x_2'; \underset{\sim}{k}, \kappa_2)}{\sqrt{\overline{u_2^2}(x_2)}\sqrt{\overline{u_2^2}(x_2+x_2')}} = \frac{\ell_1\ell_3\, \ell^2 k^2\, x_2\, e^{-\ell^2 k^2}}{\pi^2\left(1+x_2^2\kappa_2^2\right)}$$

and its Fourier transform, with respect to κ_2 gives the two-dimensional wave number spectrum function

$$\Phi_{22}\left(x_2, x_2+x_2'; \underset{\sim}{k}\right) = \int_{-\infty}^{\infty} \Phi_{22}\left(x_2, x_2+x_2'; \underset{\sim}{k}, \kappa_2\right) e^{i\kappa_2 x_2'}\, d\kappa_2$$

$$= \frac{\sqrt{\overline{u_2^2}(x_2)}\sqrt{\overline{u_2^2}(x_2+x_2')}\; \ell_1\ell_3\, \ell^2 k^2\, e^{-\ell^2 k^2}\, e^{-|x_2'/x_2|}}{\pi}$$

which clearly displays qualitatively the correct physical properties across the boundary layers, even though it fails to demonstrate the true anisotropic wave number distribution.

The integration over all k_3 leads to

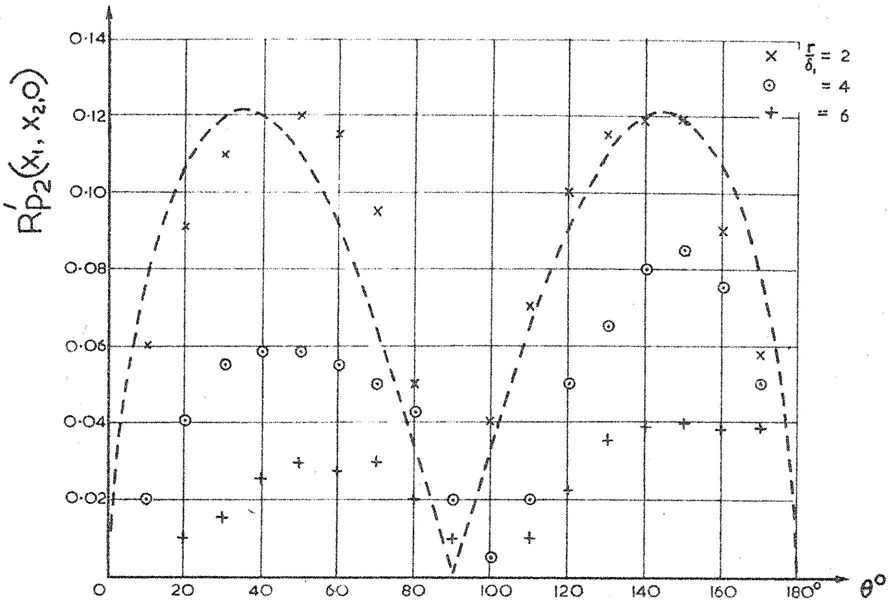
$$\Phi_{22}\left(x_2, x_2+x_2'; k_1\right) = \frac{\sqrt{\overline{u_2^2}(x_2)}\sqrt{\overline{u_2^2}(x_2+x_2')}\; \ell_1\left(1+2\ell_1^2 k_1^2\right) e^{-\ell_1^2 k_1^2 - |x_2'/x_2|}}{2\pi^{1/2}}$$

which is also equal to $\Phi_{22}\left(x_2, x_2+x_2'; -\omega/U_c\right)$ in agreement with Taylor's hypothesis.

Finally the integration over all k_1 gives

$$R_{22}\left(0, x_2, 0; 0, x_2+x_2', 0\right) = \frac{\overline{u_2(x_2)\, u_2(x_2+x_2')}}{\sqrt{\overline{u_2^2}(x_2)}\sqrt{\overline{u_2^2}(x_2+x_2')}} = e^{-|x_2'/x_2|}$$

a form for the velocity correlation which is in fair agreement with experiment over most of the boundary layer. It is therefore reasonable to expect that our calculated value for the mean pressure which is based on this assumed form for Φ_{22} will be in fair agreement with its exact value. Finally it is worth noting that we have a complete freedom of choice with respect to the longitudinal and transverse scales, ℓ_1 and ℓ_3 respectively, as well as the convection speed U_c, except that in the latter case we have assumed that it is independent both of the distance from the wall and of the wave number in the turbulence.



POLAR DISTRIBUTION OF R'_{p_2} FROM MEASUREMENTS OF WOOLDRIDGE AND WILLMARTH (1962).

FIG. 3.

$X_1 = r \cos\theta$; $X_2 = r \sin\theta$; $X_3 = O$.

------ const. $\cos\theta$. $\sin^{1/2}\theta$.

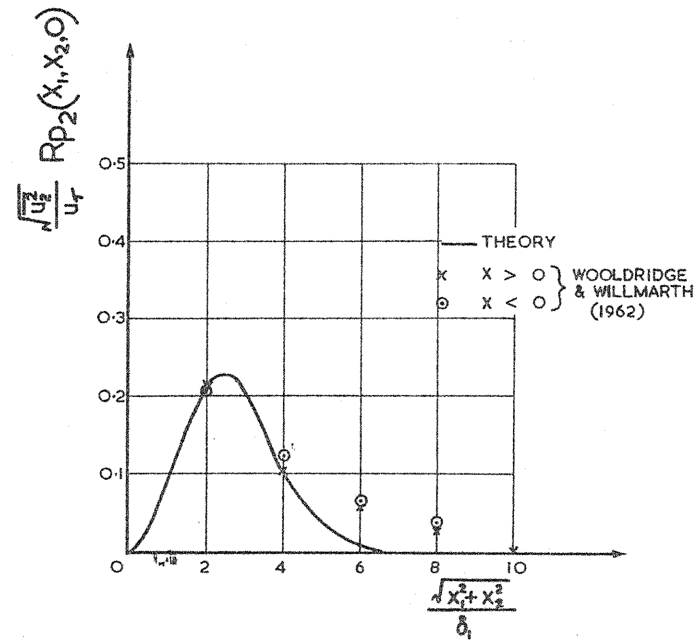


FIG. 1.

CORRELATION COEFFICIENT R_{p_2} VARIATION IN X_1, X_2, PLANE.

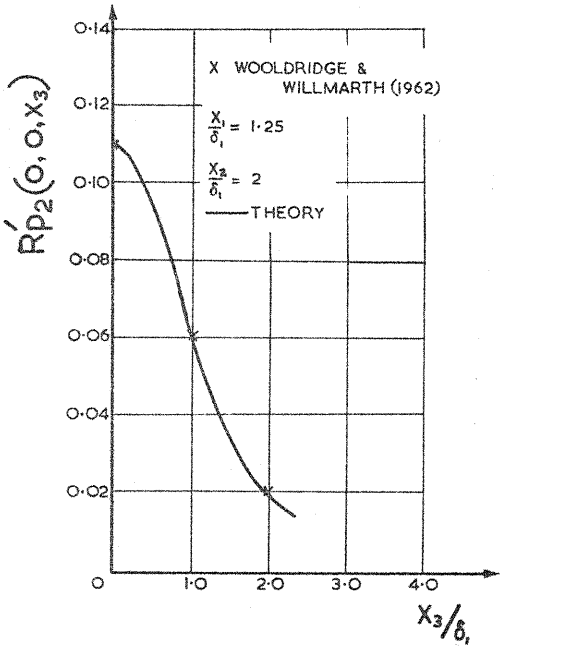


FIG. 2.

CORRELATION COEFFICIENT R'_{p_2} VARIATION WITH X_3/δ_1.

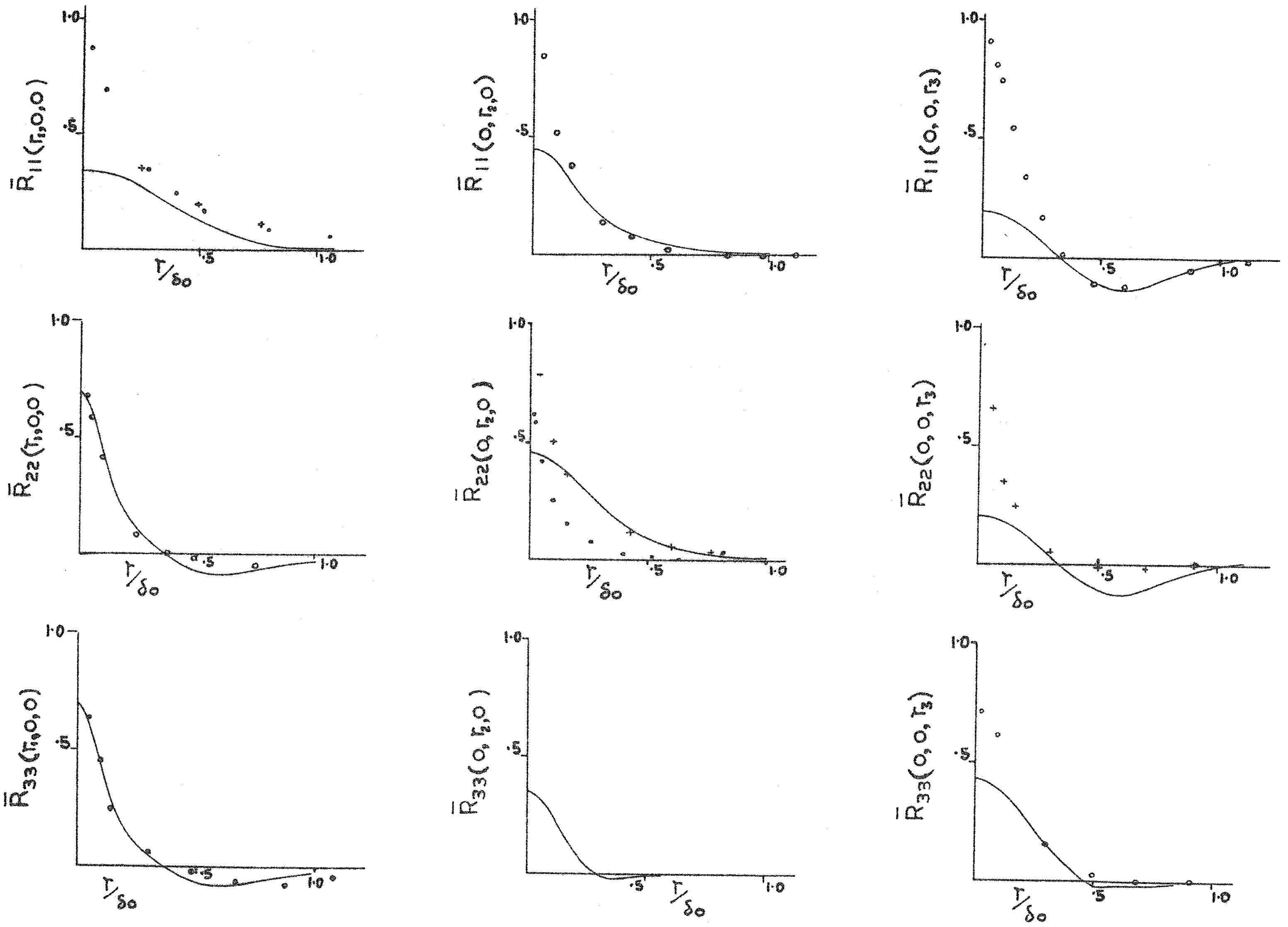


FIG. 4. COMPARISON BETWEEN THEORETICAL AND EXPERIMENTAL VELOCITY CORRELATIONS
(Experimental points, Grant (1958) — big eddy model).

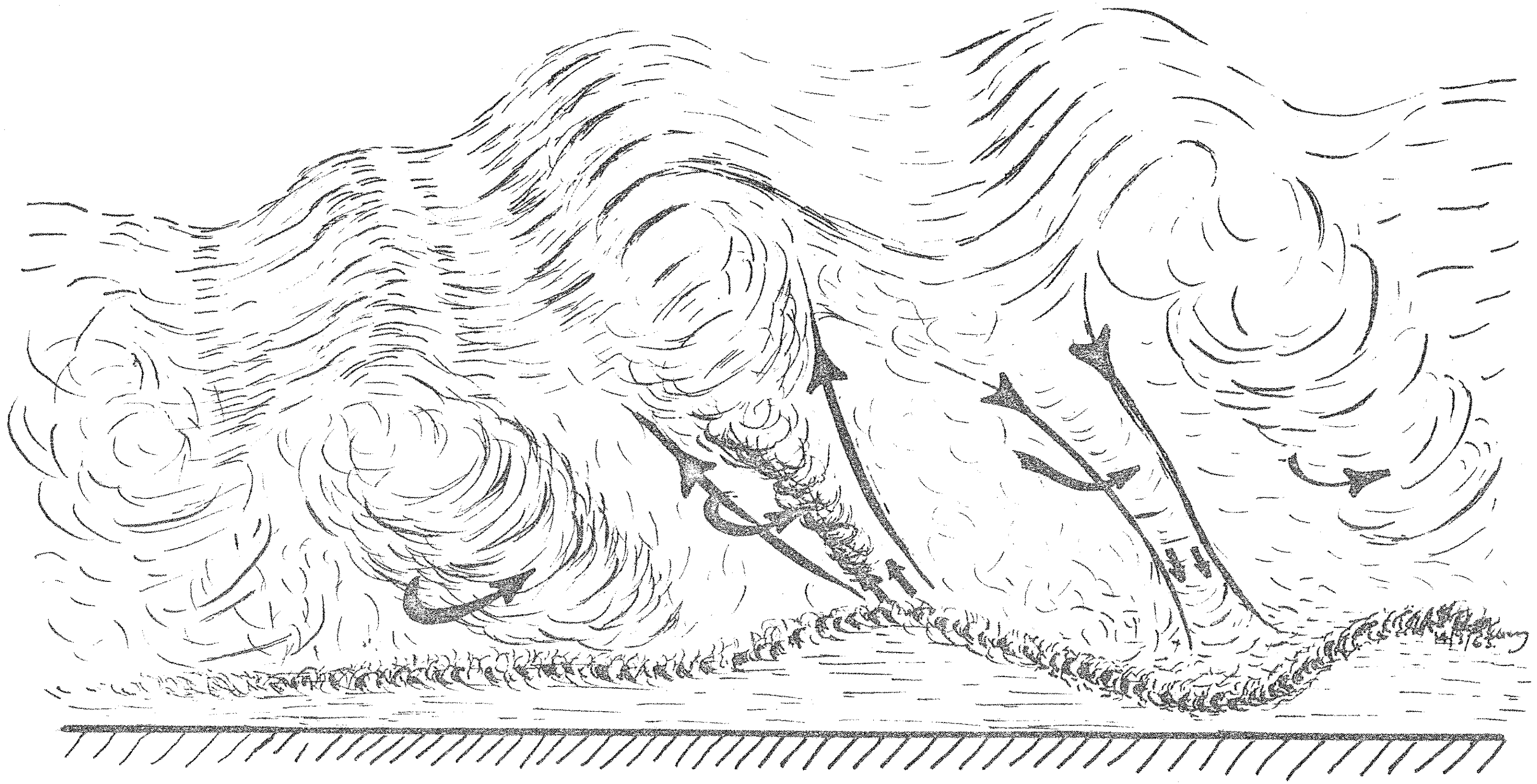


FIG. 5. DIAGRAMMATIC ARRANGEMENT OF BIG EDDIES RELATIVE TO AN OBSERVER MOVING WITH THE FLOW.

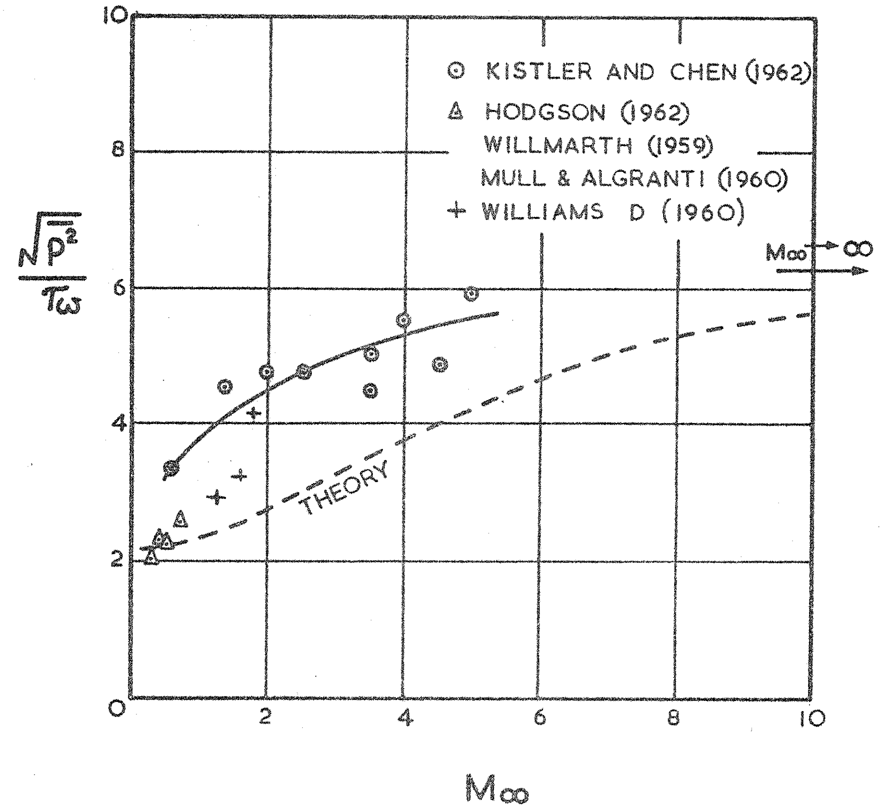


FIG. 6. WALL PRESSURE FLUCTUATIONS IN A TURBULENT BOUNDARY LAYER.

M_∞ = FREESTREAM. MACH N° 5