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STRESS DISTRIBUTION IN A PRESSURISED THICK
WALLED TOROIDAL SHELL

by

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Stress Distribution in a Pressurised Thick Walled

Toroidal Shell

A Three Dimensional Analysis

- by -

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SUMMARY

An investigation is made into the state of stress in a closed toroidal shell of uniform thick-walled circular cross section, when loaded by uniform internal and external pressures.

The general equations of the classical theory of elasticity, expressed in terms of stress components, are solved approximately by expanding the solutions in power series in a small parameter $\left(\frac{a}{R}\right)$ (a is the external radius of the cross section and R the radius of the axis). This method was used (inter alia) by E. Gohner⁽¹⁾ in his investigation of the twist and pure bending of the sector of a circular ring.

The first approximation yields the known solution of the problem of Lamé for a thick walled cylinder. The equations for the higher approximations reduce to the problem of plane strain in a circular ring. Only the first three terms of the power series are calculated in this report. The convergence of the series is not investigated.

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1. Statement of the Problem. General Equations

It is convenient, because of the boundary conditions, to refer the torus to orthogonal curvilinear co-ordinates ρ , θ and ϕ (Fig. 1) defined as follows:

$$\begin{aligned} x &= (R + a\rho \sin \theta) \cos \phi \\ y &= (R + a\rho \sin \theta) \sin \phi \\ z &= a\rho \cos \theta \end{aligned} \quad (1)$$

where the symbols used are explained in Fig. 1. The non-dimensional radial co-ordinate, ρ , varies in the range

$$\alpha = \frac{b}{a} < \rho < 1,$$

while θ and ϕ vary between 0 and 2π .

The boundary surfaces $\rho = 1$ and $\rho = \alpha$ are loaded by constant pressures. The problem considered is thus an axisymmetrical one, which means that:

$$\frac{\partial u}{\partial \phi} = 0, \quad \frac{\partial v}{\partial \phi} = 0, \quad w = 0, \quad (2)$$

where u , v , w denote the components of the displacement vector in the directions ρ , θ , and ϕ respectively.

Making use of the basic equations of the theory of elasticity in general tensor form, or more conveniently in the form presented in Ref. 2 (pp 104-107), the following sets of equations are obtained for the axisymmetrical problems referred to co-ordinates ρ , θ and ϕ .

The strain-displacement relations are:

$$\begin{aligned} \epsilon_{\rho} &= \frac{1}{a} \frac{\partial u}{\partial \rho}, & \epsilon_t &= \frac{1}{a\rho} \left(\frac{\partial v}{\partial \theta} + u \right), & \epsilon_{\ell} &= \frac{1}{R\mu} (u \sin \theta + v \cos \theta), \\ \gamma_{\rho t} &= \gamma = \frac{1}{a\rho} \left(\frac{\partial u}{\partial \theta} - v + \rho \frac{\partial v}{\partial \rho} \right), & \gamma_{t\ell} &= 0, & \gamma_{\rho\ell} &= 0. \end{aligned} \quad (3)$$

where

$$\mu = 1 + \frac{a}{R} \rho \sin \theta \quad (4)$$

and the suffices ρ , t and ℓ correspond to the co-ordinates ρ , θ and ϕ respectively.

The stress-strain relations are

$$\begin{aligned} E \epsilon_{\rho} &= \sigma_{\rho} - \nu (\sigma_t + \sigma_{\ell}); & E \epsilon_t &= \sigma_t - \nu (\sigma_{\ell} + \sigma_{\rho}); & E \epsilon_{\ell} &= \sigma_{\ell} - \nu (\sigma_{\rho} + \sigma_t) \\ E \gamma &= 2(1 + \nu) \tau_{\rho t} = 2(1 + \nu) \tau; & \tau_{\rho\ell} &= 0; & \tau_{t\ell} &= 0. \end{aligned} \quad (5)$$

The differential equations of equilibrium are reduced in the axisymmetrical case to two only, namely:

$$\begin{aligned} \frac{\partial \sigma_\rho}{\partial \rho} + \frac{1}{\rho} \left(\sigma_\rho - \sigma_t + \frac{\partial \tau}{\partial \theta} \right) + \frac{a}{R\mu} \left[(\sigma_\rho - \sigma_\ell) \sin \theta + \tau \cos \theta \right] &= 0 \\ \frac{\partial \tau}{\partial \rho} + \frac{1}{\rho} \left(2\tau + \frac{\partial \sigma_t}{\partial \theta} \right) + \frac{a}{R\mu} \left[(\sigma_t - \sigma_\ell) \cos \theta + \tau \sin \theta \right] &= 0 \end{aligned} \quad (6)$$

while two of the six equations of compatibility are identically satisfied in view of (2), and the remaining four become

$$\begin{aligned} L(\sigma_\rho) - \frac{2}{\rho^2} \left(\sigma_\rho - \sigma_t + 2 \frac{\partial \tau}{\partial \theta} \right) - \frac{a}{R\mu} \frac{2 \cos \theta}{\rho} \tau - \left(\frac{a}{R\mu} \right)^2 \left[2 (\sigma_\rho - \sigma_\ell) \sin^2 \theta + \tau \sin 2\theta \right] \\ + \frac{1}{1+\nu} \frac{\partial^2 S}{\partial \rho^2} = 0, \\ L(\sigma_t) + \frac{2}{\rho^2} \left(\sigma_\rho - \sigma_t + 2 \frac{\partial \tau}{\partial \theta} \right) + \frac{a}{R\mu} \frac{2 \cos \theta}{\rho} \tau - \left(\frac{a}{R\mu} \right)^2 \left[2 (\sigma_t - \sigma_\ell) \cos^2 \theta + \tau \sin 2\theta \right] \\ + \frac{1}{1+\nu} \left[\frac{1}{\rho^2} \frac{\partial^2 S}{\partial \theta^2} + \frac{1}{\rho} \frac{\partial S}{\partial \rho} \right] = 0. \end{aligned} \quad (7)$$

$$\begin{aligned} L(\tau) + \frac{2}{\rho^2} \left[\frac{\partial}{\partial \theta} (\sigma_\rho - \sigma_t) - 2\tau \right] + \frac{a}{R\mu} \frac{\cos \theta}{\rho} (\sigma_\rho - \sigma_t) - \left(\frac{a}{R\mu} \right)^2 \left[(\sigma_\rho + \sigma_t - 2\sigma_\ell) \frac{\sin 2\theta}{2} \right. \\ \left. + \tau \right] + \frac{1}{1+\nu} \frac{\partial}{\partial \theta} \left(\frac{1}{\rho} \frac{\partial S}{\partial \rho} - \frac{S}{\rho^2} \right) = 0. \end{aligned}$$

$$L(\sigma_\ell) + 2 \left(\frac{a}{R\mu} \right)^2 (\sigma_\rho \sin^2 \theta + \sigma_t \cos^2 \theta + \tau \sin 2\theta - \sigma_\ell) + \frac{a}{R\mu} \frac{1}{1+\nu} \left(\sin \theta \frac{\partial S}{\partial \rho} + \frac{\cos \theta}{\rho} \frac{\partial S}{\partial \theta} \right) = 0.$$

where

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} \\ L &= \Delta + \frac{a}{R\mu} \left(\sin \theta \frac{\partial}{\partial \rho} + \frac{\cos \theta}{\rho} \frac{\partial}{\partial \theta} \right) \end{aligned} \quad (8)$$

$$S = \sigma_\rho + \sigma_t + \sigma_\ell$$

Our problem is thus to determine four functions σ_ρ , σ_t , σ_ℓ and τ , periodic in θ , which satisfy the six equations (6), (7) and the following boundary conditions

$$\begin{aligned} \sigma_\rho &= -p_0; \quad \text{and } \tau = 0 \quad \text{at } \rho = 1 \\ \sigma_\rho &= -p_1; \quad \text{and } \tau = 0 \quad \text{at } \rho = \alpha \end{aligned} \quad (9)$$

It should also be noted that the solution must be symmetric with respect to the plane $\theta = \pm \pi/2$.

2. Method of Solution

It is assumed that the solution can be represented in the form of a power series :

$$\begin{aligned} \sigma_k &= \sigma_{k,0} + \frac{a}{R} \sigma_{k,1} + \left(\frac{a}{R}\right)^2 \sigma_{k,2} + \dots \quad k = \rho, t, \ell \\ \tau &= \tau_0 + \frac{a}{R} \tau_1 + \left(\frac{a}{R}\right)^2 \tau_2 + \dots \end{aligned} \quad (10)$$

where the parameter $\frac{a}{R}$ is supposed to be small or

$$\frac{a}{R} \ll 1 \quad (11)$$

Introducing the series (10) into the basic equations (6), (7), expanding the term $\frac{1}{\mu}$ into power series and comparing terms with the same powers of the parameter $\frac{a}{R}$, gives the following equations for the first approximation :

$$\left. \begin{aligned} a_{(0)} &= \frac{\partial \sigma_{\rho,0}}{\partial \rho} + \frac{1}{\rho} \left(\sigma_{\rho,0} - \sigma_{t,0} + \frac{\partial \tau_0}{\partial \theta} \right) = 0, \\ b_{(0)} &= \frac{\partial \tau_0}{\partial \rho} + \frac{1}{\rho} \left(2\tau_0 + \frac{\partial \sigma_{t,0}}{\partial \theta} \right) = 0. \end{aligned} \right\} \quad (12)$$

$$\left. \begin{aligned} c_{(0)} &= \Delta(\sigma_{\rho,0}) - \frac{2}{\rho^2} \left(\sigma_{\rho,0} - \sigma_{t,0} + 2 \frac{\partial \tau_0}{\partial \theta} \right) + \frac{1}{1+\nu} \frac{\partial^2 S_0}{\partial \rho^2} = 0, \\ d_{(0)} &= \Delta(\sigma_{t,0}) + \frac{2}{\rho^2} \left(\sigma_{\rho,0} - \sigma_{t,0} + 2 \frac{\partial \tau_0}{\partial \theta} \right) \\ &\quad + \frac{1}{1+\nu} \left(\frac{1}{\rho^2} \frac{\partial^2 S_0}{\partial \theta^2} + \frac{1}{\rho} \frac{\partial S_0}{\partial \rho} \right) = 0, \\ e_{(0)} &= \Delta(\tau_0) + \frac{2}{\rho^2} \left[\frac{\partial}{\partial \theta} (\sigma_{\rho,0} - \sigma_{t,0}) - 2\tau_0 \right] \\ &\quad + \frac{1}{1+\nu} \frac{\partial}{\partial \theta} \left(\frac{1}{\rho} \frac{\partial S_0}{\partial \rho} - \frac{S_0}{\rho^2} \right) = 0, \\ \Delta(\sigma_{\ell,0}) &= 0 \end{aligned} \right\} \quad (13)$$

For the higher approximations, the system of equations (12), (13) becomes non-homogenous.

$$a_{(1)} = f_{(0)} \quad b_{(1)} = g_{(0)} \quad c_{(1)} = h_{(0)} \quad d_{(1)} = i_{(0)} \quad e_{(1)} = j_{(0)} \quad \Delta \sigma_{\ell,1} = k_{(0)}. \quad (14)$$

$$\left. \begin{aligned} a_{(2)} &= f_{(1)} - f_{(0)} \rho \sin \theta ; \quad b_{(2)} = g_{(1)} - g_{(0)} \rho \sin \theta ; \quad c_{(2)} = h_{(1)} - h_{(0)} \rho \sin \theta + l_{(0)} ; \\ d_{(2)} &= i_{(1)} - i_{(0)} \rho \sin \theta + p_{(0)} ; \quad e_{(2)} = j_{(1)} - j_{(0)} \rho \sin \theta + q_{(0)} ; \\ \Delta \sigma_{\ell,2} &= k_{(1)} - k_{(0)} \rho \sin \theta + r_{(0)} \end{aligned} \right\} \quad (15)$$

The formulae for $a_{(n)} \dots r_{(n)}$ are written down explicitly in Appendix I.

It is evident that the longitudinal stress component σ_ℓ cannot be calculated directly from the equations of equilibrium (12) but must be determined by integration of the compatibility equations (13). A more detailed analysis shows that this integration enables one to calculate σ_ℓ , to within an additive function

$$F(\rho, \theta) = A_1 + A_2 \rho \sin \theta + A_3 \rho \cos \theta$$

where A_1, A_2, A_3 , are arbitrary constants.

Symmetry implies $A_3 = 0$. The two remaining constants of integration, A_1, A_2 , will be chosen to make the resultant of the stress σ_ℓ satisfy the integral conditions of equilibrium, (16) below, which follow from (6) and (9). Rearranging equations (6), integrating them over the cross-section and keeping in mind the boundary conditions (9), one obtains the following expressions for the resultant force and bending moment of the longitudinal stresses*:

$$\left. \begin{aligned} \int_0^{2\pi} \int_\alpha^1 \sigma_\ell \rho \, d\rho \, d\theta &= \pi(\alpha^2 p_i - p_o), \\ \frac{a}{R} \int_0^{2\pi} \int_\alpha^1 \sigma_\ell \sin \theta \, \rho^2 \, d\rho \, d\theta &= 2\pi(\alpha^2 p_i - p_o) - \int_0^{2\pi} \int_\alpha^1 \rho \mu(\sigma_\rho + \sigma_t) \, d\rho \, d\theta. \end{aligned} \right\} (16)$$

Equations (16) will be used to determine A_1 and A_2 at each stage of the approximation. The right hand side integral of the second equation (16) is to be calculated in consistence with the order of approximation adopted in the differential equations of equilibrium and compatibility.

3. First approximation

It may be checked, by substitution, that the following solution satisfies all the differential equations (12), (13), the relations (16) and the boundary conditions (9):

$$\sigma_{\rho, o} = C_1 + \frac{C_2}{\rho^2}, \quad \sigma_{t, o} = C_1 - \frac{C_2}{\rho^2}, \quad \sigma_{\ell, o} = C_1, \quad \tau_o = 0 \quad (17)$$

where

$$C_1 = \frac{\alpha^2 p_i - p_o}{1 - \alpha^2}; \quad C_2 = \frac{\alpha^2(p_o - p_i)}{1 - \alpha^2}. \quad (18)$$

*

The bending moment lying in the equatorial plane $\theta = \pm \pi/2$ vanishes, i. e.

$$\int_0^{2\pi} \int_\alpha^1 \sigma_\ell \cos \theta \rho^2 \, d\rho \, d\theta = 0, \text{ in view of the symmetry of the solution.}$$

4. Second approximation

Equations (14), after substituting from formulae (17), give

$$\left. \begin{aligned} a_{(1)} &= -\frac{C_2}{\rho^2} \sin \theta, \\ b_{(1)} &= +\frac{C}{\rho^2} \cos \theta, \end{aligned} \right\} \quad (20)$$

$$\left. \begin{aligned} c_{(1)} &= +\frac{2C}{\rho^3} \sin \theta, \\ d_{(1)} &= -\frac{2C}{\rho^3} \sin \theta, \\ e_{(1)} &= -\frac{2C}{\rho^3} \cos \theta, \\ \Delta(\sigma_{\theta,1}) &= 0, \end{aligned} \right\} \quad (21)$$



where the $a_{(1)} \dots e_{(1)}$ expressions are defined by Appendix I.

In view of (9) and (17), the boundary conditions to be satisfied are

$$\sigma_{\rho,1} = 0 \text{ and } \tau_1 = 0 \text{ at } \rho = 1; \rho = \alpha \quad (22)$$

Using an Airy stress function, we may assume that the solution of the equilibrium equations (20) is

$$\begin{aligned} \sigma_{\rho,1} &= \frac{1}{\rho} \frac{\partial \Phi_1}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \Phi_1}{\partial \theta^2} - \frac{C_2}{2\rho} \sin \theta, \\ \sigma_{\theta,1} &= \frac{\partial^2 \Phi_1}{\partial \rho^2} + \frac{C_2}{2\rho} \sin \theta, \\ \tau_1 &= \frac{\partial}{\partial \theta} \left(\frac{1}{\rho^2} \Phi_1 - \frac{1}{\rho} \frac{\partial \Phi_1}{\partial \rho} \right) + \frac{C_2}{2\rho} \cos \theta. \end{aligned} \quad (23)$$

Introducing these relations into the compatibility equations (21) yields

$$\Delta \Delta \Phi_1 = 0 \quad (24)$$

Now, in view of the form of the particular integrals in (23), the biharmonic stress function Φ_1 should be taken in the form

$$\Phi_1 = \left(d_1 \rho^3 + \frac{d'_1}{\rho} + d''_1 \rho \ln \rho \right) \sin \theta \quad (25)$$

where the constants d_1 , d'_1 , and d''_1 must be chosen to satisfy the boundary conditions (22).

Substituting (25) into (23) and afterwards into (22) gives

$$\begin{aligned} 2d_1 + \frac{2d''_1 - C_2}{2} - 2d'_1 &= 0, \\ 2d_1 \alpha + \frac{2d''_1 - C_2}{2\alpha} - \frac{2d'_1}{\alpha^3} &= 0. \end{aligned} \quad (26)$$

Equations (26) are insufficient to determine the three unknowns. The third condition to be satisfied is that of the single-valuedness of the displacements* This condition yields (cf Appendix II)

$$d_1'' = \frac{C_2}{4} \frac{1 - 2\nu}{1 - \nu} \quad (27)$$

Now, equations (26) imply

$$d_1 = \frac{C_2}{8(1 - \nu)(1 + \alpha^2)}, \quad (28)$$

$$d_1' = - \frac{C_2 \alpha^2}{8(1 - \nu)(1 + \alpha^2)},$$

and after substitution into (25) and (23) one obtains

$$\sigma_{\rho,1} = \frac{C_2}{4(1 - \nu)} \left(\frac{\rho}{1 + \alpha^2} - \frac{1}{\rho} + \frac{\alpha^2}{1 + \alpha^2} \frac{1}{\rho^3} \right) \sin \theta,$$

$$\sigma_{t,1} = \frac{C_2}{4(1 - \nu)} \left(\frac{3\rho}{1 + \alpha^2} + \frac{3 - 4\nu}{\rho} - \frac{\alpha^2}{1 + \alpha^2} \frac{1}{\rho^3} \right) \sin \theta, \quad (29)$$

$$\tau_1 = \frac{C_2}{4(1 - \nu)} \left(\frac{\rho}{1 + \alpha^2} - \frac{1}{\rho} + \frac{\alpha^2}{1 + \alpha^2} \frac{1}{\rho^3} \right) \cos \theta.$$

Introducing formulae (29) into the compatibility relations (21) yields**

$$\frac{\partial^2 \sigma_{\ell,1}}{\partial \rho^2} = - \frac{C_2 (2 - \nu)}{1 - \nu} \frac{\sin \theta}{\rho^3},$$

$$\frac{1}{\rho} \frac{\partial^2 \sigma_{\ell,1}}{\partial \theta^2} + \frac{\partial \sigma_{\ell,1}}{\partial \rho} = \frac{C_2 (2 - \nu)}{1 - \nu} \frac{\sin \theta}{\rho^2}, \quad (30)$$

$$\frac{\partial}{\partial \theta} \left(\frac{\partial \sigma_{\ell,1}}{\partial \rho} - \frac{1}{\rho} \sigma_{\ell,1} \right) = \frac{C_2 (2 - \nu)}{1 - \nu} \frac{\cos \theta}{\rho^2}.$$

Integrating these and adjusting the constants to satisfy the integral conditions (16) of equilibrium, we find

$$\sigma_{\ell,1} = \frac{C_2 (2 - \nu)}{2(1 - \nu)} \left(\frac{2\rho}{1 + \alpha^2} - \frac{1}{\rho} \right) \sin \theta \quad (31)$$

* The general problem of single-valuedness of the displacements in the case of the state of plane strain in a circular (closed) ring is studied by Timpe in Ref. 4 and the results are quoted in Ref. 3 (pp 116-120). A different approach to the same problem by means of complex function theory is to be found in Ref. 5 (pp 218-223).

** See also Appendix II, equation (A.7).

5. Third approximation

Introducing the relations (17), (29) and (31) into the right hand sides of equations (15) one obtains

$$\left. \begin{aligned} a_{(2)} &= D_2 \rho \sin^2 \theta + \frac{C_2}{4(1-\nu)} \left\{ -\frac{\nu}{\rho} + \left(\frac{\rho}{1+\alpha^2} - \frac{1-\nu}{\rho} + \frac{\alpha^2}{1+\alpha^2} \cdot \frac{1}{\rho^3} \right) \cos 2\theta \right\} \\ b_{(2)} &= D_2 \rho \sin \theta \cos \theta + \frac{C_2}{4(1-\nu)} \left(-\frac{\rho}{1+\alpha^2} - \frac{6-5\nu}{\rho} + \frac{\alpha^2}{1+\alpha^2} \cdot \frac{1}{\rho^3} \right) \sin 2\theta \end{aligned} \right\} \quad (32)$$

$$\left. \begin{aligned} c_{(2)} &= \frac{C_2}{4(1-\nu)} \left[-\frac{2}{1+\alpha^2} + \frac{1}{\rho^2} + \left(-\frac{1}{1+\alpha^2} + \frac{2}{\rho^2} - \frac{3\alpha^2}{1+\alpha^2} \cdot \frac{1}{\rho^4} \right) \cos 2\theta \right] \\ d_{(2)} &= \frac{C_2}{4(1-\nu)} \left\{ -\frac{2}{1+\alpha^2} - \frac{1}{\rho^2} + \left[\frac{1}{1+\alpha^2} - \frac{12(1-\nu)}{\rho^2} + \frac{3\alpha^2}{1+\alpha^2} \cdot \frac{1}{\rho^4} \right] \cos 2\theta \right\} \\ e_{(2)} &= \frac{C_2}{4(1-\nu)} \left(\frac{1}{1+\alpha^2} + \frac{7-6\nu}{\rho^2} - \frac{3\alpha^2}{1+\alpha^2} \cdot \frac{1}{\rho^4} \right) \sin 2\theta \\ \Delta\sigma_{\ell,2} &= \frac{C_2}{1-\nu} \left[-\frac{5-\nu^2}{(1+\nu)(1+\alpha^2)} + \frac{7-5\nu}{2\rho^2} \cos 2\theta \right] \end{aligned} \right\} \quad (33)$$

where

$$D_2 = \frac{C_2 (2-\nu)}{(1-\nu)(1+\alpha^2)} \quad (34)$$

and the $a_{(2)} \dots e_{(2)}$ expressions are defined by Appendix I.

The boundary conditions to be satisfied are

$$\sigma_{\rho,2} = 0, \quad \tau_2 = 0 \quad \text{at } \rho = 1 \text{ and } \rho = \alpha \quad (35)$$

Applying, step by step, the same procedure as for the previous approximations, the stress components $\sigma_{\rho,2}$, $\sigma_{t,2}$, $\sigma_{\ell,2}$, and τ_2 can be calculated, satisfying equations (32) and (33) as well as boundary conditions (35) and integral relations (16). Substituting these stresses and the expressions (17), (29) and (31) into the power series (10), gives the final formulae :

$$\left. \begin{aligned} \sigma_{\rho} &= C_1 + \frac{C_2}{\rho^2} + \frac{C_2}{4(1-\nu)} \left\{ \left(\frac{a}{R} \right) \left(\frac{\rho}{1+\alpha^2} - \frac{1}{\rho} + \frac{\alpha^2}{1+\alpha^2} \cdot \frac{1}{\rho^3} \right) \sin \theta + \frac{1}{2} \left(\frac{a}{R} \right)^2 (f_r + g_r \cdot \cos 2\theta) \right\} \\ \sigma_t &= C_1 - \frac{C_2}{\rho^2} + \frac{C_2}{4(1-\nu)} \left\{ \left(\frac{a}{R} \right) \left(\frac{3\rho}{1+\alpha^2} + \frac{3-4\nu}{\rho} - \frac{\alpha^2}{1+\alpha^2} \cdot \frac{1}{\rho^3} \right) \sin \theta + \frac{1}{2} \left(\frac{a}{R} \right)^2 \right. \\ &\quad \left. (f_t + g_t \cdot \cos 2\theta) \right\} \\ \sigma_{\ell} &= C_1 + \frac{C_2}{2(1-\nu)} \left\{ \left(\frac{a}{R} \right) (2-\nu) \left(\frac{2\rho}{1+\alpha^2} - \frac{1}{\rho} \right) \sin \theta + \frac{1}{2} \left(\frac{a}{R} \right)^2 (f_{\ell} + g_{\ell} \cos 2\theta) \right\} \\ \tau &= \frac{C_2}{4(1-\nu)} \left\{ \left(\frac{a}{R} \right) \left(-\frac{\rho}{1+\alpha^2} + \frac{1}{\rho} - \frac{\alpha^2}{1+\alpha^2} \cdot \frac{1}{\rho^3} \right) \cos \theta + \frac{1}{4} \left(\frac{a}{R} \right)^2 f_s \sin 2\theta \right\} \end{aligned} \right\} \quad (36)$$

where $f_r \dots g_{\ell}$, f_s , are functions of the variable ρ alone and are written down in Appendix III.

In the limit, when the wall thickness becomes very small, (i. e. when $\rho \rightarrow 1$, $\alpha \rightarrow 1$), expressions (36) give

$$\left. \begin{aligned} \sigma_{\rho} &= 0, \quad \tau = 0, \\ \sigma_t &= \frac{pa}{2t} \left[2 - \frac{a}{R} \sin \theta + \left(\frac{a}{R} \right)^2 \sin^2 \theta \right] \\ \sigma_{\ell} &= \frac{pa}{2t} \left[1 + \frac{1}{2} \left(\frac{a}{R} \right)^2 \cos 2\theta \right], \end{aligned} \right\} \quad (37)$$

$$(38)$$

where $t = a - b$ denotes the wall thickness of the shell, and $p = p_i - p_o$.

Equations (37) coincide with corresponding formulae known from the membrane theory of toroidal shells; however, the axial stresses σ_{ℓ} , defined by (38), differ by the small underlined term from the membrane solution.

6. Numerical example

The results of calculations for the case $\nu = 0.3$, $\alpha = 0.5$, $a/R = 1/3$, $p_o = 0$, and $p_i = p$ are illustrated in Figs. 2-5. The stress components σ_t , σ_{ℓ} , σ_{ρ} and τ are plotted against ρ for $\theta = 0^\circ$, $\frac{\pi}{2}$, and $-\frac{\pi}{2}$. Some conclusions may be drawn from these figures :

- (a) The shearing stress, τ , is very small, relative to the other stress components.
- (b) The radial stress differs only slightly from its value for a cylindrical tube ($\frac{a}{R} = 0$). It reaches its maximum negative value at $\rho = \alpha$, where

$$\sigma_{\rho} = -p,$$

in accordance with the boundary conditions.

- (c) The hoop stress, σ_t , reaches its maximum value at points lying on the inner surface, ($\rho = \alpha$), with $\theta = -\pi/2$ (point A on Fig. 1). This is given (for $\nu = 0.3$) by

$$\sigma_t(\alpha, -\frac{\pi}{2}) = \frac{p}{1-\alpha^2} + \frac{p}{1-\alpha^2} \left[\alpha^2 + \left(\frac{a}{R} \right) \frac{1.714\alpha^3 + 0.286\alpha}{(1+\alpha^2)} + \left(\frac{a}{R} \right)^2 \lambda(\alpha) \right] \quad (39)$$

where the coefficient, $\lambda(\alpha)$, is plotted in Fig. 6.

- (d) The axial stress, σ_{ℓ} , seems to be somewhat sensitive to the curvature of the tube, a/R , and the convergence of its power series is rather poor. At the points $\rho = \alpha$, $\theta = -\pi/2$, this stress is given by

$$\sigma_{\ell}(\alpha, -\pi/2) = \frac{p}{1-\alpha^2} \left[\alpha^2 - \left(\frac{a}{R} \right) \frac{1.214(1-\alpha^2)\alpha}{1+\alpha^2} - \left(\frac{a}{R} \right)^2 \delta(\alpha) \right] \quad (40)$$

where the coefficient, $\delta(\alpha)$, is plotted in Fig. 6.

The equivalent stress will be critical at the points $\rho = \alpha$, $\theta = -\pi/2$.

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APPENDIX I

$$a_{(n)} = \frac{\partial \sigma_{\rho, n}}{\partial \rho} + \frac{1}{\rho} \left(\sigma_{\rho, n} - \sigma_{t, n} + \frac{\partial \tau_n}{\partial \theta} \right)$$

$$b_{(n)} = \frac{\partial \tau_n}{\partial \rho} + \frac{1}{\rho} \left(2 \tau_n + \frac{\partial \sigma_{t, n}}{\partial \theta} \right)$$

$$c_{(n)} = \Delta (\sigma_{\rho, n}) - \frac{2}{\rho^2} \left(\sigma_{\rho, n} - \sigma_{t, n} + 2 \frac{\partial \tau_n}{\partial \theta} \right) + \frac{1}{1 + \nu} \frac{\partial^2 S_n}{\partial \rho^2}$$

$$d_{(n)} = \Delta (\sigma_{t, n}) + \frac{2}{\rho^2} \left(\sigma_{\rho, n} - \sigma_{t, n} + 2 \frac{\partial \tau_n}{\partial \theta} \right) + \frac{1}{1 + \nu} \left[\frac{1}{\rho^2} \frac{\partial^2 S_n}{\partial \theta^2} + \frac{1}{\rho} \frac{\partial S_n}{\partial \rho} \right]$$

$$e_{(n)} = \Delta (\tau_n) + \frac{2}{\rho^2} \left[\frac{\partial}{\partial \theta} (\sigma_{\rho, n} - \sigma_{t, n}) - 2 \tau_n \right] + \frac{1}{1 + \nu} \left[\frac{1}{\rho} \frac{\partial^2 S_n}{\partial \rho \partial \theta} - \frac{1}{\rho^2} \frac{\partial S_n}{\partial \theta} \right]$$

$$f_{(n)} = - \left[(\sigma_{\rho, n} - \sigma_{\ell, n}) \sin \theta + \tau_n \cos \theta \right]$$

$$g_{(n)} = - \left[(\sigma_{t, n} - \sigma_{\ell, n}) \cos \theta + \tau_n \sin \theta \right]$$

$$h_{(n)} = - \left[\sin \theta \frac{\partial \sigma_{\rho, n}}{\partial \rho} + \frac{\cos \theta}{\rho} \frac{\partial \sigma_{\rho, n}}{\partial \theta} \right] + \frac{2 \cos \theta}{\rho} \tau_n$$

$$i_{(n)} = - \left[\sin \theta \frac{\partial \sigma_{t, n}}{\partial \rho} + \frac{\cos \theta}{\rho} \frac{\partial \sigma_{t, n}}{\partial \theta} \right] - \frac{2 \cos \theta}{\rho} \tau_n$$

$$j_{(n)} = - \left[\sin \theta \frac{\partial \tau_n}{\partial \rho} + \frac{\cos \theta}{\rho} \left(\frac{\partial \tau_n}{\partial \theta} + \sigma_{\rho, n} - \sigma_{t, n} \right) \right]$$

$$k_{(n)} = - \left[\sin \theta \frac{\partial \sigma_{\ell, n}}{\partial \rho} + \frac{\cos \theta}{\rho} \frac{\partial \sigma_{\ell, n}}{\partial \theta} + \frac{1}{1 + \nu} \left(\sin \theta \frac{\partial S_n}{\partial \rho} + \frac{\cos \theta}{\rho} \frac{\partial S_n}{\partial \theta} \right) \right]$$

$$l_{(0)} = 2 (\sigma_{\rho, 0} - \sigma_{\ell, 0}) \sin^2 \theta + \tau_0 \sin 2 \theta ,$$

$$p_{(0)} = 2 (\sigma_{t, 0} - \sigma_{\ell, 0}) \cos^2 \theta + \tau_0 \sin 2 \theta ,$$

$$q_{(0)} = (\sigma_{\rho, 0} + \sigma_{t, 0} - 2 \sigma_{\ell, 0}) \sin \theta \cos \theta + \tau_0 ,$$

$$r_{(0)} = - 2 \left[\sigma_{\rho, 0} \sin^2 \theta + \sigma_{t, 0} \cos^2 \theta + \tau_0 \sin 2 \theta - \sigma_{\ell, 0} \right] .$$

$n = 0, 1, 2.$

APPENDIX II

Relations (3) and (5) imply

$$E \frac{1}{a} \frac{\partial u}{\partial \rho} = (1 + \nu) \sigma_{\rho} - \nu S \quad (\text{A.1})$$

$$E \frac{1}{a\rho} \left(\frac{\partial v}{\partial \theta} + u \right) = (1 + \nu) \sigma_t - \nu S \quad (\text{A.2})$$

$$\frac{E}{a\rho} \left(\frac{\partial u}{\partial \theta} - v + \rho \frac{\partial v}{\partial \rho} \right) = 2(1 + \nu)\tau \quad (\text{A.3})$$

Integration of (A.1) gives

$$\frac{E}{a} u = (1 + \nu) \int \sigma_{\rho} d\rho - \nu \int S d\rho + f(\theta) \quad (\text{A.4})$$

where $f(\theta)$ is an arbitrary function of the variable θ .

Relation (A.2) yields, after integration,

$$\frac{E}{a} v = (1 + \nu)\rho \int \sigma_t d\theta - \nu\rho \int S d\theta - \frac{E}{a} \int u d\theta + g(\rho) \quad (\text{A.5})$$

where u is defined by (A.4) and $g(\rho)$ is an arbitrary function of the variable ρ alone.

On substituting equations (A.4), (A.5) and (23) into the relation (A.3), which must be identically satisfied, the functions $f(\theta)$ and $g(\rho)$ will be defined, and the single-valuedness of the displacements u and v checked.

To calculate S , use will be made of equations of compatibility (21). Since

$$\frac{2}{\rho^2} \left(\sigma_{\rho,1} - \sigma_{t,1} + 2 \frac{\partial \tau_1}{\partial \theta} \right) = \frac{\partial^2}{\partial \rho^2} (\Delta \Phi_1) - \Delta \left(\frac{\partial^2 \Phi_1}{\partial \rho^2} \right) - \frac{4C_2}{\rho^3} \sin \theta \quad (\text{A.6})$$

as can readily be checked, and since

$$\Delta(\sigma_{\rho,1}) = \Delta \left(\Delta \Phi_1 - \frac{\partial^2 \Phi_1}{\partial \rho^2} \right) = - \Delta \left(\frac{\partial^2 \Phi_1}{\partial \rho^2} \right)$$

in view of (24), the first equation (21) implies

$$\frac{\partial^2 S_1}{\partial \rho^2} = (1 + \nu) \left[\frac{\partial^2}{\partial \rho^2} (\Delta \Phi_1) - \frac{2C_2}{\rho^3} \sin \theta \right]. \quad (\text{A.7})$$

On integrating this relation one obtains

$$S_1 = (1 + \nu) \left[\Delta \Phi_1 - \frac{C_2}{\rho} \sin \theta \right] + f_1(\theta) + \rho f_2(\theta) \quad (\text{A.8})$$

Substituting (A.8) into the remaining equations of compatibility shows that the arbitrary functions $f_1(\theta)$ and $f_2(\theta)$ are

$$f_1(\theta) + \rho f_2(\theta) = A_1 + A_2 \rho \sin \theta ,$$

thus

$$S_1 = (1 + \nu) \left[\Delta \phi_1 - \frac{C_2}{\rho} \sin \theta \right] + A_1 + A_2 \rho \sin \theta \quad (\text{A.9})$$

where A_1 and A_2 are arbitrary constants.

Equations (A.9), (23), (25) and (A.4) imply

$$\frac{1}{1+\nu} \frac{E}{a} u_1 = \left[(1 - 4\nu)d_1 \rho^2 + \frac{d_1'}{\rho^2} + \frac{(2d_1'' - C_2)(1 - 2\nu)}{2} \ln \rho - \frac{\nu}{1+\nu} \frac{A_2}{2} \rho^2 \right] \sin \theta - \frac{\nu}{1+\nu} A_1 \rho + f(\theta) \quad (\text{A.10})$$

Equations (A.5), (23), (25) and (A.10) give :

$$\begin{aligned} \frac{1}{1+\nu} \frac{E}{a} v_1 = & - \left[(5 - 4\nu)d_1 \rho^2 + \frac{d_1'}{\rho^2} - \frac{(2d_1'' - C_2)(1 - 2\nu)}{2} \ln \rho + \frac{C_2(1 + 2\nu)}{2} \right. \\ & \left. + (1 - 2\nu)d_1'' - \frac{\nu}{1+\nu} \frac{A_2}{2} \rho^2 \right] \cos \theta - \int f(\theta) d\theta + g(\rho) \end{aligned} \quad (\text{A.11})$$

Substituting (A.10), (A.11), (23) and (25) into (A.3) yields :

$$\frac{df}{d\theta} + \int f(\theta) d\theta = 2 \left[\frac{C_2(1 - 2\nu)}{2} - 2 d_1''(1 - \nu) \right] \cos \theta$$

or

$$f(\theta) = \left[\frac{C_2(1 - 2\nu)}{2} - 2 d_1''(1 - \nu) \right] \theta \cos \theta .$$

The right hand side of the last expression must vanish, because of the single-valuedness of the displacements, and we obtain finally

$$d_1'' = \frac{C_2}{4} \frac{1 - 2\nu}{1 - \nu} \quad (\text{27})$$

APPENDIX III

$$\begin{aligned}
 f_r &= \frac{3-v^2}{(1+v)(1+\alpha^2)} \rho^2 & - \frac{3-v^2}{1+v} & + \frac{(3-v^2)\alpha^2}{(1+v)(1+\alpha^2)} \cdot \frac{1}{\rho^2} \\
 g_r &= \frac{-3+2v}{1+\alpha^2} \cdot \rho^2 & + \Delta + K + 3 - 2v & - \left[\Delta + (1+\alpha^2)K + \frac{(3-2v)\alpha^2}{1+\alpha^2} \right] \cdot \frac{1}{\rho^2} + \frac{K\alpha^2}{\rho^2} + (2-v)\ln\rho \\
 f_t &= \frac{1-4v+v^2}{(1+v)(1+\alpha^2)} \rho^2 & + \frac{-3+2v+3v^2}{1+v} & - \frac{(3-v^2)\alpha^2}{(1+v)(1+\alpha^2)} \cdot \frac{1}{\rho^2} \\
 g_t &= \left[3\Delta + (3-\alpha^2)K + \frac{3\alpha^2(2-v)+1+v}{1+\alpha^2} \right] \rho^2 & - \Delta - K + 3 - 4v & - \frac{\alpha^2}{1+\alpha^2} \cdot \frac{1}{\rho^2} - \frac{K\alpha^2}{\rho^4} - (2-v)\ln\rho \\
 f_\ell &= - \frac{5-v^2}{(1+v)(1+\alpha^2)} \rho^2 & - (1+v)\Delta + \frac{2+v^2+v^3}{1+v} & - (1-2v)(1+v)\ln\rho \\
 2g_\ell &= \left[3v\Delta + (3-\alpha^2)vK + \frac{3v\alpha^2(2-v)+5-v-v^2}{1+\alpha^2} \right] \rho^2 & + \left[-v\Delta - v(1+\alpha^2)K + \frac{(1-2v)(1-v)\alpha^2}{1+\alpha^2} \right] \cdot \frac{1}{\rho^2} \\
 f_s &= \left[3\Delta + (3-\alpha^2)K + \frac{3\alpha^2(2-v)+4-v}{1+\alpha^2} \right] \rho^2 & - 2\Delta - 2K - 4 + v & - \left[\Delta + (1+\alpha^2)K + \frac{2(1-v)\alpha^2}{1+\alpha^2} \right] \cdot \frac{1}{\rho^2} + \frac{2K\alpha^2}{\rho^4} - 2(2-v)\ln\rho
 \end{aligned}$$

Notations: $\Delta = \frac{(2-v)\alpha^2}{1-\alpha^2} \ln \alpha$; $K = \frac{3}{(1-\alpha^2)^2} [2-v+(1+\alpha^2)\Delta]$;

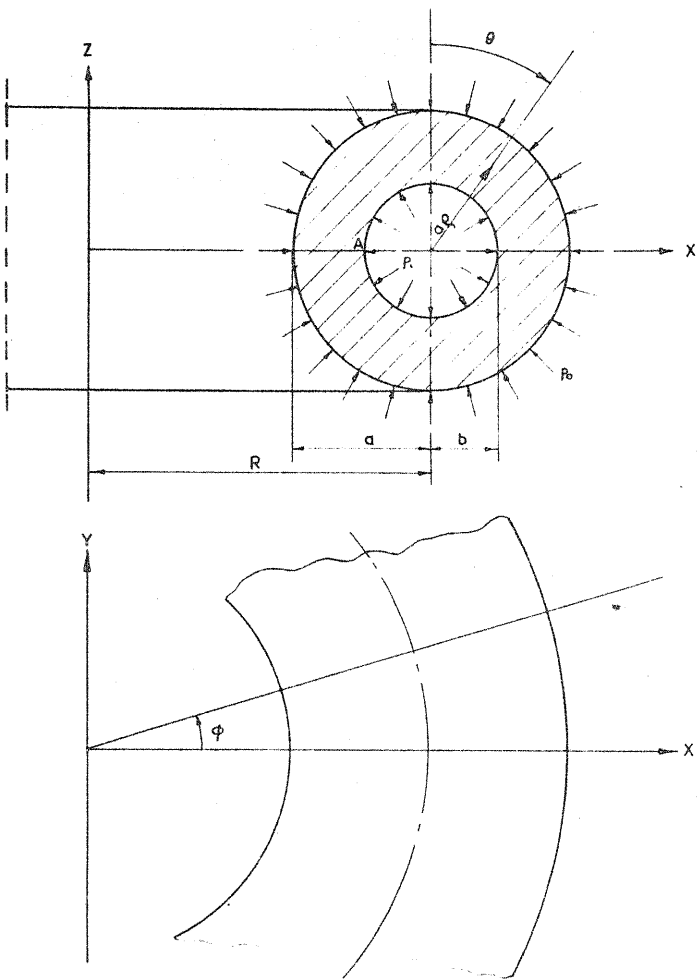


FIG. 1 THE SYSTEM OF CO-ORDINATES AND NOTATIONS.

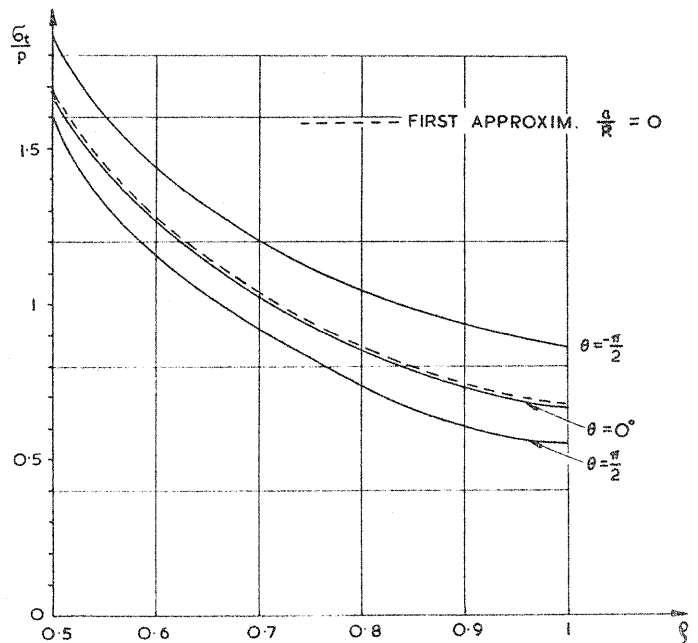


FIG. 2 THE DISTRIBUTION OF HOOP STRESSES (σ_t)

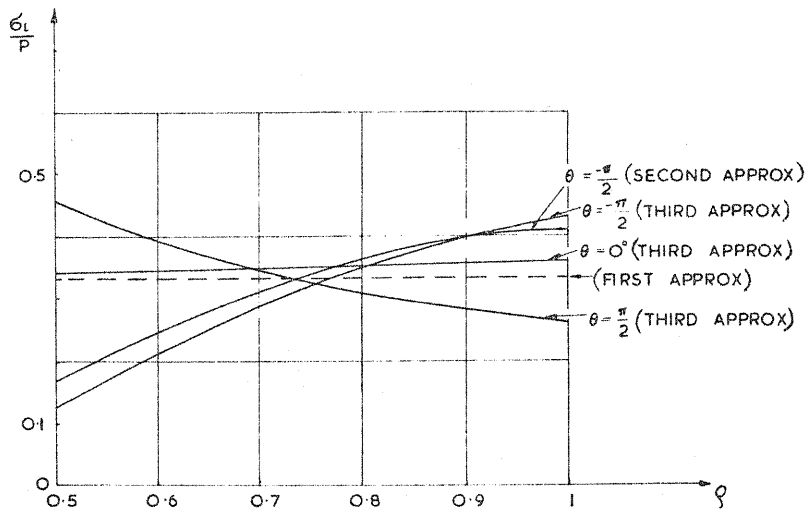


FIG 3 THE DISTRIBUTION OF AXIAL STRESSES (σ_l)

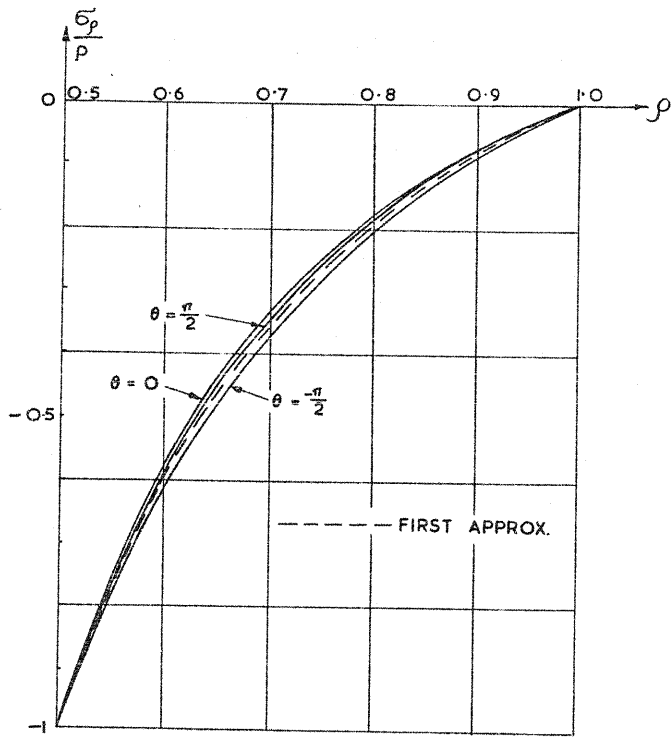


FIG. 4 THE DISTRIBUTION OF RADIAL STRESSES (σ_p)

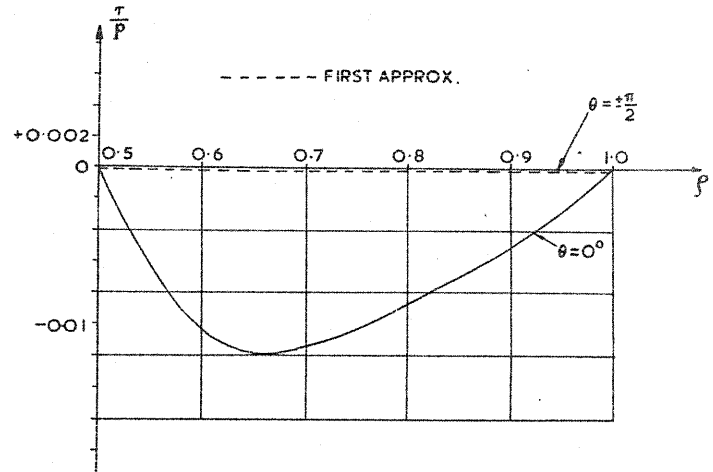


FIG. 5 THE DISTRIBUTION OF SHEARING STRESSES (τ)

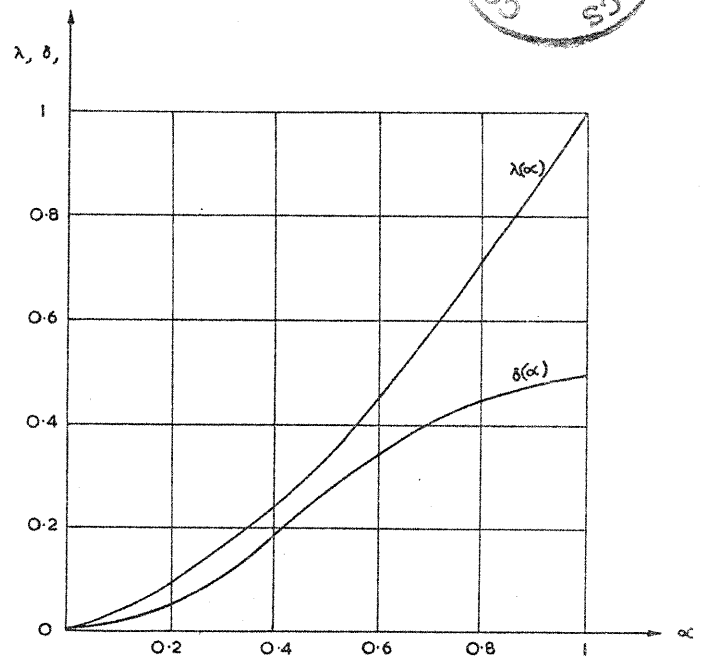


FIG. 6