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# AN ANALYSIS OF AN UNSTIFFENED CYLINDRICAL SHELL SUBJECTED TO INTERNAL PRESSURE AND AXIAL LOADING

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An Analysis of an Unstiffened Cylindrical Shell
Subjected to Internal Pressure and Axial Loading

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## SUMMARY

General equations are obtained for the deflections and stresses in long thin unreinforced cylinders, which are subjected to an axial load and internal pressure. By making suitable simplifying assumptions, results are presented which show the variation of the structural weight parameter with the structural axial loading index, for both pressurised and unpressurised shells. An allowance is made for the effects of shell initial eccentricities on the buckling stress coefficient K, in accordance with R.Ae.S. data sheet 04.01.01.

Extreme cases are considered, in which the shell is assumed to be either fully effective (K = 0.6), or completely ineffective (K = 0), in resisting axial compressive loads. For this latter case, complete pressure stabilisation of the shell is considered, and it is shown that the weight penalty involved in using this design philosophy, is negligible for a certain range of the structural loading index.

A simple modification to the analysis for this case, i.e. K = 0, is made to allow for the effect of an external longitudinal bending moment.

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# NOTATION

| | |
|---|---|
| $A_1$ , $A_2$ | constants in equation (1) |
| $D$ | flexural rigidity |
| $E$ | Young's modulus |
| $F$ | axial compressive force |
| $f_a$ | allowable stress |
| $f_b$ | buckling stress |
| $K$ | buckling coefficient |
| $\ell$ | head of liquid in vertical shell |
| $M$ | longitudinal bending moment |
| $n$ | longitudinal acceleration (in g's) |
| $p$ | internal pressure |
| $p_s$ | stabilising pressure |
| $R$ | radius of cylindrical shell |
| $T_1$ | longitudinal force/in. |
| $T_2$ | circumferential force/in. |
| $t$ | shell thickness |
| $w$ | radial displacement |
| $x$ | axial co-ordinate along length of shell measured from the lower support point |
| $\alpha$ | weight parameter defined in equation (14) |
| $\sigma$ | density of shell material |
| $\nu$ | Poisson's ratio |
| $\rho$ | density of liquid in the shell |

Notation (Continued)

$\lambda$ structural index, defined after equation (12)

$\psi$ $\left[\frac{n \rho \ell}{E} + \frac{\lambda}{E}\right]$, and $\psi' = \psi + \frac{2M}{\pi R^3 E}$

$\delta$ initial shell irregularity

$\mu$ $\left(\frac{Et}{4 DR^2}\right)^{\frac{1}{4}}$

## 1. Introduction

This paper is concerned with the design problem of thin cylindrical shells, which are primarily subjected to axial loading and internal pressure. Such a problem is typical of ballistic missile shell structures, in which the axial loading is due mainly to axial inertia forces, and the internal pressure could arise from the hydrostatic effect of the fuel in the tanks, or may be introduced for the purposes of shell stabilisation.

An attempt is made to assess the structural efficiencies of various configurations, making due allowance for the effects of shell initial irregularities.

A simple modification to the analysis is later considered to allow for an external longitudinal bending moment on the shell.

## 2. Theory

The fourth order differential equation defining the axi-symmetric radial deformation of a vertical, circular shell subjected to internal pressure and axial loading may be written as

$$\frac{d^4 w}{dx^4} + 4\mu^4 w = \frac{1}{D}\left[ p(1 - \frac{\nu}{2}) + n\rho(\ell - x) + \frac{F\nu}{2\pi R^2} \right] .$$

The various terms in the bracket above correspond to a constant gas pressure, a linear hydrostatic pressure, and a term containing the axial compressive force.

For most missile structures, the fuel tanks are sufficiently long for the solution to the above equation to be written as

$$w = e^{-\mu x}\left[ A_1 \cos \mu x + A_2 \sin \mu x \right] + \frac{nR^2}{Et}\left[ p' + \rho(\ell - x) \right] , \quad \ldots\ldots \quad (1)$$

where $p' = \frac{p}{n}\left[ 1 - \frac{\nu}{2} + \frac{F\nu}{2\pi R^2 p} \right]$, and $4\mu^4 = \frac{Et}{DR^2}$ .

Solution of equation (1) depends upon the edge boundary conditions which have been taken to be either fully clamped or simply supported.

The corresponding expressions for w are, for the fully clamped edge,

$$w = \frac{nR^2}{Et}\left[p' + \rho(\ell - x) - (p' + \rho\ell)\theta_{(\mu x)} - (p' + \rho\ell - \frac{\rho}{\mu})\zeta_{(\mu x)}\right], \quad \ldots\ldots \quad (2)$$

and for the simply supported edge,

$$w = \frac{nR^2}{Et}\left[p' + \rho(\ell - x) - (p' + \rho\ell)\theta_{(\mu x)}\right] . \quad (3)$$

The relationship between the radial deflection w, and the tank hoop force/in. $T_2$ , is

$$T_2 = \frac{Etw}{R} + \frac{pR\nu}{2} - \frac{\nu F}{2\pi R} .$$

The functions $\theta_{(\mu x)} = e^{-\mu x}\cos\mu x$ ; $\zeta_{(\mu x)} = e^{-\mu x}\sin\mu x$.

$$\phi_{(\mu x)} = \theta + \zeta \ ; \quad \psi_{(\mu x)} = \theta - \zeta ,$$

are tabulated in Ref. 1, page 394.

Under the assumed loading conditions, the axial force/in., $T_1$ , may be written

$$T_1 = \frac{pR}{2} - \frac{F}{2\pi R} , \quad (4)$$

and the buckling design criterion for the shell in compression is given by R.Ae.S. Data Sheet 04.01.01 as

$$f_b = \frac{-T_1}{t} = KE\left(\frac{t}{R}\right) . \quad (5)$$

Substitution of equation (4) into (5) yields

$$p_s = \frac{F}{\pi R^2} - 2KE\left(\frac{t}{R}\right)^2 , \quad (6)$$

where $p_s$ is the internal gas pressure which stabilises the shell under the compression loading F.

Clearly the introduction of the internal pressure gives rise to a tensile hoop stress $\frac{T_2}{t}$ . If this stress must not exceed the allowable hoop stress $f_a$ then the condition for $f_a$ is

$$f_a = \frac{Ew}{R} + \frac{pR\nu}{2t} - \frac{\nu F}{2\pi Rt} \quad . \tag{7}$$

It can be shown that when x is small equations (2) and (3) may be simplified, with very little error, to give respectively

$$w = \frac{nR^2}{Et}\left[p' + \rho\ell\right](1 - \phi) , \tag{8}$$

and

$$w = \frac{nR^2}{Et}\left[p' + \rho\ell\right](1 - \theta). \tag{9}$$

Substitution of equations (8) and (9) into (7) yields

$$f_a = \frac{nR}{t}\rho\ell(1 - \phi) + \frac{pR}{t}\left[1 - \phi(1 - \frac{\nu}{2})\right] - \frac{F\nu}{2\pi Rt}\phi , \tag{10}$$

for clamped edge conditions, and a similar expression for the simply supported edge with $\phi$ replaced by $\theta$.

When equation (6) applies, i.e. the shell is just stabilised by internal pressure, $p = p_s$ in equation (10) which becomes,

$$f_a = \left[\frac{nR}{t}\rho\ell + \frac{F}{\pi Rt}\right](1 - \phi) - \frac{2KEt}{R}\left[1 - \phi(1 - \frac{\nu}{2})\right]. \tag{11}$$

Inspection of this result shows that the maximum $f_a$ occurs at $\mu x = \pi$, when $(1 - \phi) = 1.0432$. The corresponding maximum result for the simply supported edge occurs at $\mu x = \frac{3\pi}{4}$, when $(1 - \theta) = 1.067$.

These results are for a long shell, but an investigation of the effects of finite length of shell suggests that the most conservative factor is about 1.09. This factor will be applied in the subsequent analysis, and will be assumed to apply for either fully clamped or simply supported edges. Hence if $\nu = \frac{1}{3}$ equation (11) can be written as

$$\frac{f_a}{E} = 1.09\,\frac{R}{t}(\psi) - 2.15K\,(\frac{t}{R}) , \tag{12}$$

where $\psi = \left[\frac{n\rho\ell}{E} + \frac{\lambda}{E}\right]$, and $\lambda = \frac{F}{\pi R^2}$, is the structural index in compression.

## 3. Discussion of the parameter K

The results which are presented in the R.Ae.S. Data Sheet 04.01.01 may be used to determine the buckling stress coefficient K for thin walled unstiffened circular cylinders under combined axial compression and internal pressure. It is generally considered that the depth of initial irregularity of the shell wall ($\delta$) plays a dominant part in predicting the buckling stress of circular cylinders, and this fact is considered in the R.Ae.S. Data Sheet 04.01.01, where K is plotted against $\frac{\delta}{t}$ for various values of the parameter $\frac{p}{E}\left(\frac{R}{t}\right)^2$. The results of a large number of experiments have been correlated in determining these curves, which show the dependence of $\delta/t$ on $R/t$.

For the purpose of this paper, comparisons are drawn between the results obtained using either $\delta/t = 0$ or 1, i.e. independent of R/t, or in accordance with the data sheet for the more critical uppermost line B.B. This line represents the limiting maximum values of $\delta/t$ for over 80 per cent of all experimental results which were used in the correlation.

## 4. Results

### 4.1. The unpressurised shell

Fig. 1 shows the structural efficiency of the unpressurised shell in compression, compared with that of the pressure stabilised shell, for a shell material having the properties $E = 28 \times 10^6$ lb/in.$^2$, $\sigma$ = .273 lb/in.$^3$ and allowable tensile stress $f_a$ = 170,000 lb/in$^2$. The curve for the unpressurised shell was obtained by using equation (6) with $p_s = 0$, in which case the structural index becomes

$$\lambda = 2KE\left(\frac{t}{R}\right)^2 . \tag{13}$$

By assuming values for $\frac{R}{t}$ in the range 100 to 3000, corresponding values of $\delta/t$ and hence K were found from the R.Ae.S. Data Sheet 04.01.01 line BB. The structural index was found from equation (13) and the weight parameter $\alpha$ which is plotted in Fig. 1 is defined by

$$\alpha = \frac{\sigma t}{R} . \tag{14}$$

## 4.2. The pressurised shell

By specifying the allowable tensile stress to be $f_a = 170{,}000$ lb/in$^2$ equation (12) is presented in Fig. 1, for the pressure stabilised shell, with the hydrostatic effect neglected, i.e.

$$\frac{f_a}{E} = 1.09 \; \frac{R}{t}\left(\frac{\lambda}{E}\right) - 2.15K\left(\frac{t}{R}\right) .$$

The fullline shown is in accordance with R.Ae.S. Data Sheet 04.01.01, and the analysis is similar to that indicated above for the unpressurised shell. The dotted line in Fig. 1 shows the effect of assuming $K = 0$, i.e. the pressure to be such that no compressive stress is set up in the shell, in which case

$$\frac{f_a}{E} = 1.09 \left(\frac{R}{t}\right)\left(\frac{\lambda}{E}\right) .$$

Comparison of the solid and dotted curves for the pressurised shell shows that for values of the weight parameter $\alpha < 2 \times 10^{-4}$ lb/in$^3$ i.e. $\frac{R}{t} > 1500$ or $\lambda < 150$, the assumption that $K = 0$ gives negligible error to the weight parameter.

From Fig. 1 it may be concluded that for low values of the structural index (low loads, large dimensions), the pressure stabilised structure has a significant weight advantage. This result was previously noted in Ref. 2 where curves which correspond almost exactly to the solid lines in Fig. 1, were derived using Ref. 3. This reference provided some of the experimental results considered in deriving the R.Ae.S. Data Sheet 04.01.01. In Ref. 2, it was also shown that for high values of the structural index, conventional stiffening is optional in terms of structural efficiency.

The above results prompted an investigation into the effect of the parameter $\frac{\delta}{t}$ and hence K, on the structural efficiency, for various values of $\psi$. The results are presented in Fig. 2, which was obtained using equation (12), and values of $\frac{\delta}{t} = 0$ $(K = 0.6)$ and $K = 0$ $(\frac{\delta}{t} \to \infty)$.

The value $K = 0$ corresponds to the case when there is no compressive stress in the shell, in which case equation (12) becomes

$$\frac{f_a}{E} = 1.09 \frac{R}{t} (\psi) . \tag{15}$$

The results indicate that the difference between the corresponding curves, for the extreme values K = 0 and K = 0.6, decreases as $\frac{R}{t}$ increases. This also implies that the effect of initial irregularities decreases as $\frac{R}{t}$ increases. This point is further emphasised by the curve plotted for $\frac{\delta}{t} = 1.0$ and $\psi = 2 \times 10^{-6}$.

5. The effect of an external longitudinal bending moment

If an external longitudinal bending moment M is applied to the shell together with the axial load F, then it may be shown that, for the case when K = 0, equation (15) is still applicable provided that the parameter ( $\psi$) is defined as

$$\psi' = \frac{n \rho \ell}{E} + \frac{\lambda}{E} + \frac{2M}{\pi R^3 E} .$$

In this case Fig. 2 still applies when $\psi'$ is read for $\psi$ .

## 6. References


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3. Harris, L., and others — The stability of thin walled unstiffened circular cylinders under axial compression including the effects of internal pressure. Journal of the Aeronautical Sciences, vol.24, 1959, pp 587-596.

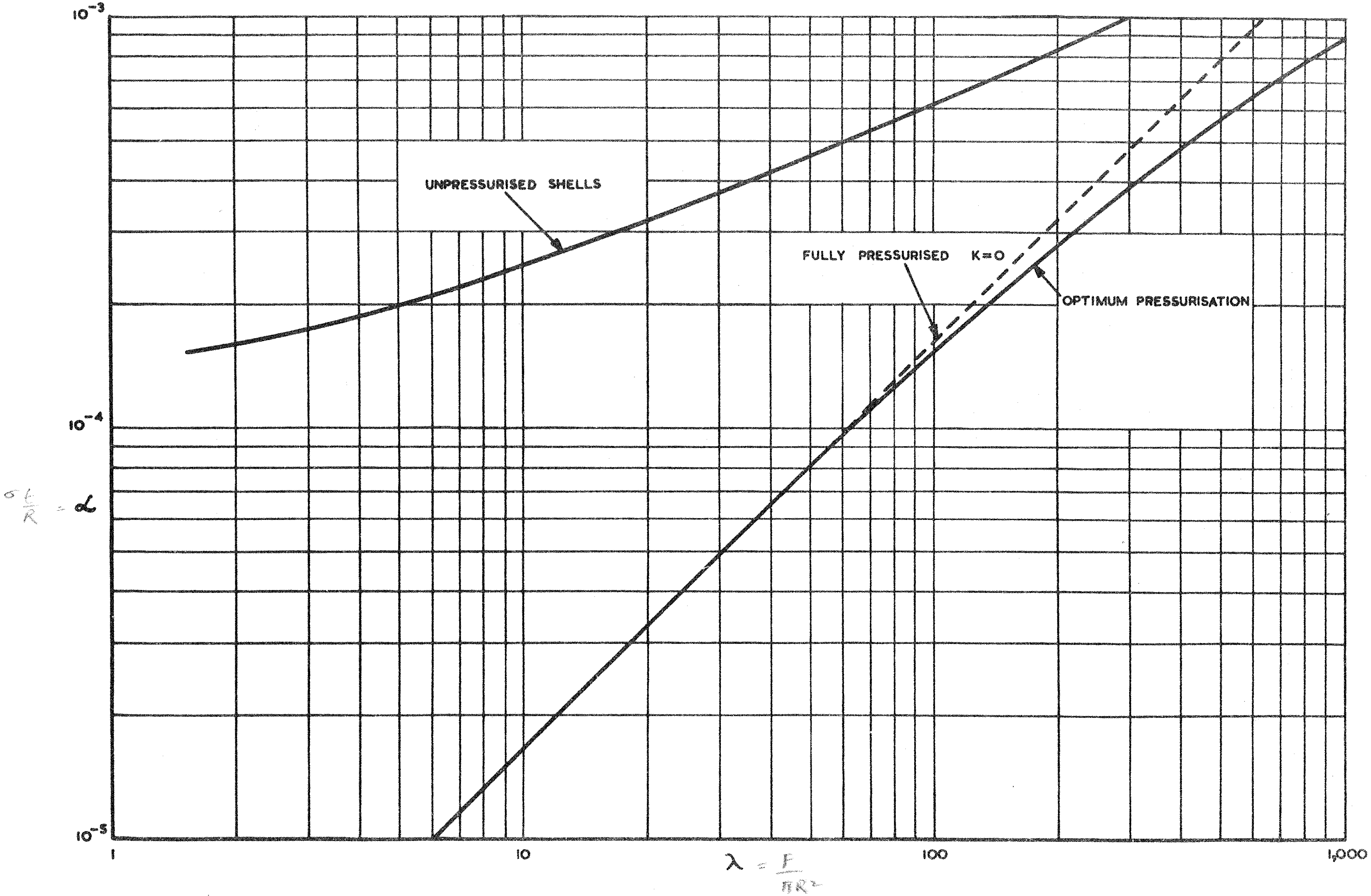


FIG. 1. STRUCTURAL EFFICIENCY OF UNSTIFFENED SHELLS IN COMPRESSION.

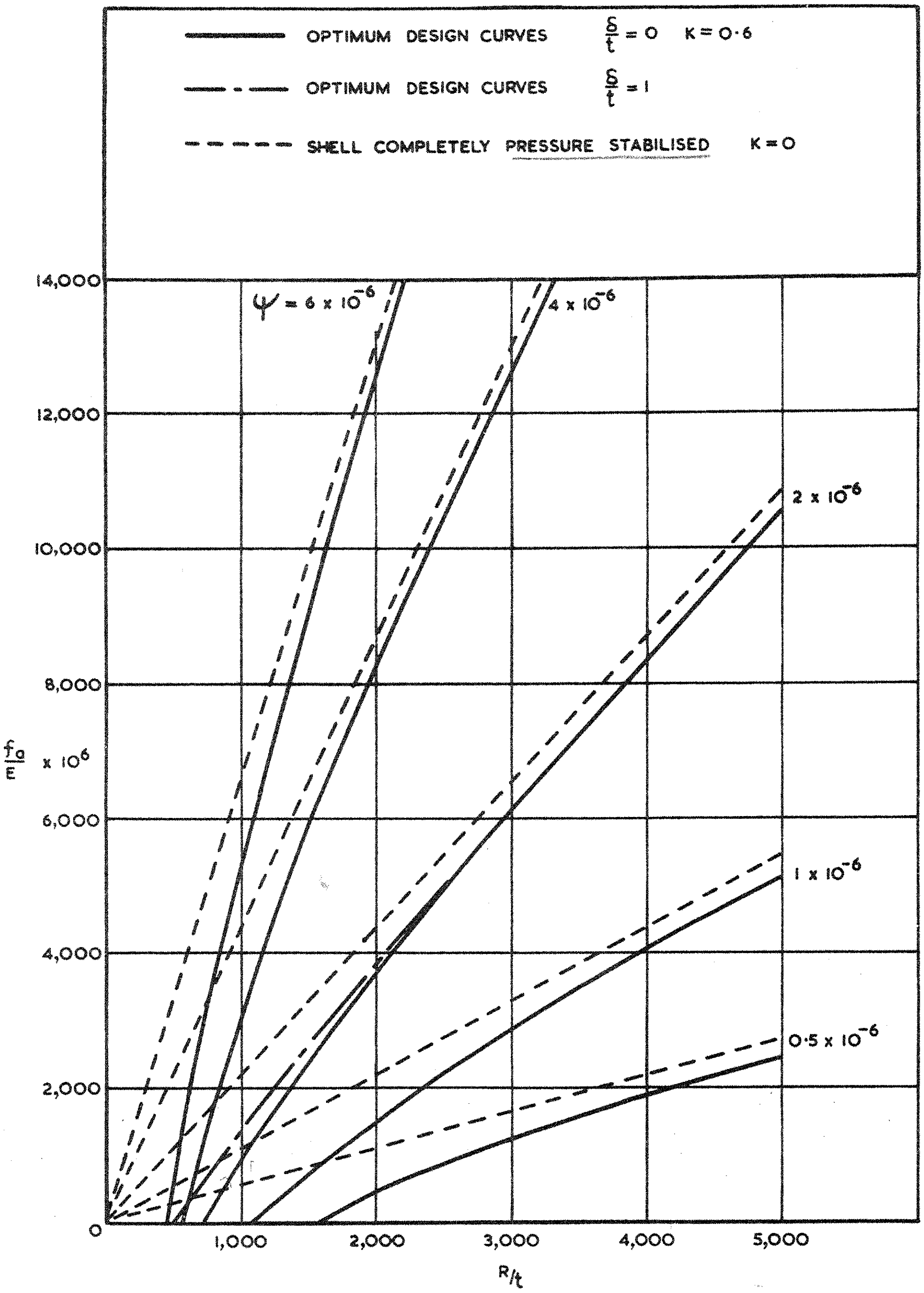


FIG. 2. OPTIMUM SHELL $R/t$ RATIOS FOR GIVEN LOADING $\psi$ AND ALLOWABLE HOOP STRESS $f_a$.