CoA Note No. 105

THE COLLEGE OF AERONAUTICS

CRANFIELD

THERMAL BUCKLING OF A FREE CIRCULAR PLATE

by

J. P. H. WEBBER and D. S. HOUGHTON

NOTE NO. 105
August 1960

THE COLLEGE OF AERONAUTICS

CRANFIELD

Thermal Buckling of a Free Circular Plate*

- by -

J. P. H. Webber, D. C. Ae.,

and

D. S. Houghton, M. Sc. (Eng.),
A. F. R. Ae. S., A. M. I. Mech. E.

SUMMARY

The buckling of a free circular plate subjected to a temperature field, which varies only in the radial direction, is considered. The problem is first investigated experimentally and the mode of deflection and form of the temperature distribution are measured. Expressions are developed for the thermal stresses and the deflection mode of the plate, which are used for a theoretical small deflection energy analysis. This solution is found to compare favourably with the experimental results.

---

* This work was begun by the first author, under the supervision of the second, in part fulfilment for the award of the Diploma of the College of Aeronautics.

CONTENTS

NOTATION

A	constant in deflection mode of plate
b	radius of plate
D	flexural rigidity
E	Young's modulus
h	plate thickness
M_r, $M_{r\theta}$	moment and torque resultants
n	index
Q_r	shear resultant
r	radial co-ordinate
ds	element al surface area of plate
T	temperature rise above ambient
T_o	temperature at centre of plate
ΔT	temperature difference between centre and edge of plate
ΔT_c	value of ΔT to produce buckling
t	time
w	lateral displacement of median surface of plate
α	coefficient of thermal expansion
β_1, β_2	constants in deflection mode of plate
ζ	coefficient = $E\alpha$
ξ	the ratio $\frac{r}{b}$
σ_r, σ_θ	radial and hoop membrane stresses

Notation (Continued)

$\tau_{r\theta}$	membrane shear stress
θ	angular co-ordinate
ϕ	stress function
∇^2	Laplace's operator $= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$
∇^4	operator $= \left[\nabla^2\right]^2$

1. Introduction

A free circular plate may buckle if temperature gradients are set up in the material.

The case considered here is when the temperature field varies only in the radial direction, giving rise to compressive stresses which may be large enough to cause buckling. This problem has been investigated by Cotterell[1] who found a large discrepancy between his experimental and theoretical results. The difference was not explained and it was stated that further experimental work should be carried out.

It has been suggested by Hoff[2], that the accepted methods of stability analysis might not be applicable to the case of thermal stresses because these stresses are influenced by the deformations which the body undergoes when it buckles. To clarify his point, he investigated both theoretically and experimentally, the behaviour of a column subjected to a uniform temperature rise while placed between two fixed platens[3]. He found that at the moment of buckling, first order small displacements caused second order small changes in the thermal stresses, and that the results of a classical analysis agreed with those obtained by experiment. It was then concluded that the thermal buckling of plates and shells could be predicted using the accepted methods of analysis, but that further studies of their reliability should be undertaken.

2. The experimental investigation

A photograph of the circular plate specimen, with the experimental apparatus is shown in Fig. 1.

2.1. Specimen

The circular plate was cut from a nominal 16 s.w.g. rolled sheet of commercial mild steel, and measured 10.78 in. dia. x .060 in. thick. Tensile tests on the material gave a value of Young's modulus of 29.7×10^6 lb/in^2., a yield stress of 22,300 lb/in^2. and a 0.1 per cent proof stress of 31,000 lb/in^2.

In order to support the specimen it was found necessary to drill a $^1/8$ in. diameter hole in the centre of the plate which was then screwed to a tripod stand. This method of attachment allows dial gauge measurements to be taken, and is assumed to have a negligible constraining effect on the plate.

2.2. The experimental apparatus

A photograph of the experimental apparatus is shown in Fig. 1(a). A Calor gas burning ring was used as a heating source, and an overhead Dexion structure gave support to the gas supply pipe which was connected to a four way junction piece, giving four inlets to the copper burning ring. Provision was made for a shield to be lowered between the edge of the plate and burning ring to dissipate the flames.

It was hoped to induce radial temperature gradients in the plate by applying a uniform heat input to the edge of the plate at a high rate of heating. Although a high rate could easily be obtained it was not possible, at the same time, to provide uniformity of heating around the edge of the plate. Recourse was made to a very slow heating rate, in which the plate was brought to a near uniform temperature rise, and a radial temperature gradient was induced by cooling the plate in the centre by compressed air. This method was found to give satisfactory results.

2.3. Instrumentation

The buckling of the plate was investigated under transient cooling conditions, and in order to obtain a time history of the temperature distribution and deflection in the plate, recording equipment had to be used. This equipment consisted of a 12-channel mirror galvonometer trace recording set, made by New Electronics Products Ltd., Type 1000, and a Shackman Auto camera.

40 s.w.g. Chrome-Alumel thermocouples, of equal resistances, were used to obtain the overall temperature distribution in the plate. These were connected to the recording set which had been previously calibrated using a representative thermocouple.

The buckling point and mode of deflection were measured by taking photographs of the dial gauges during cooling of the plate, with a Shackman Auto camera. Since the speed of the trace recording and camera are known, the temperature distribution and mode of deflection could be collated at any instant. The buckling point was obtained by examining the dial gauge readings in order to establish the time at which large plate deflections began.

3. Experimental results

Fig. 2 shows the measured temperature distribution in the plate at different time intervals during the test, and in particular the critical radial temperature distribution. This is, in fact, a mean distribution corresponding to the temperature mid-way through the plate thickness.

The mode of deflection is shown in Figs. 3(a) and 3(b) where the actual experimental points are indicated. These are seen to agree well with the theoretical mode used below in paragraph 4.

For the form of temperature distribution shown in Fig. 2 the critical temperature differential between the centre and edge of the plate is seen to be 39^{o}C. having an estimated accuracy of $\pm$ 3.0^{o} C.

4. Theoretical analysis

4.1. General remarks

The results obtained from experiment enable expressions to be derived representing the form of temperature distribution in the plate and the mode of deformation. The radial and hoop stresses are determined, and the work done by the internal forces are calculated. The strain energy of bending of the plate is determined and on equating these two expressions, the point of neutral equilibrium is found directly; a variational procedure is not required.

4.2. Assumptions

The following assumptions are made :-

(i) The plate is considered to be free.
(ii) The mechanical properties of the material are independent of temperature.
(iii) The plate material is isotropic.
(iv) The temperature distribution is axi-symmetrical.
(v) The thermal stressing problem may be considered as one of plane stress.

The following remarks concerning the above assumptions may be made.

(i) The presence of a $^1/8$ in. diameter attachment hole at the centre of the plate is assumed to affect only the stress distribution in the immediate locality of the hole, leaving the stresses over the outer region (which are the cause of buckling), identical to the case of the free plate.

(ii) The maximum uniform temperature rise attained in the plate was of the order of 150°C. and since the material used was steel, it is felt that this assumption is not unreasonable.

(iii) The value of Young's modulus was found to be the same in the direction of the grain of the material and in the direction across the grain, by experiment. Other mechanical properties may be taken to be the same in both directions.

(iv) This assumption is justified by experimental data.

(v) The theoretical buckling stresses are found to be well below the yield stress of the material. (See Fig. 6).

(vi) In a recent article by Gatewood[4], thermal strssses in moderately thick plates are investigated, and it is concluded that if the plate is thin, then the temperature gradients must be extremely large to have any effect on the plane stress solution.

4.3. Solution

In plane stress, the thermal problem reduces to the solution of the following differential equation subject to certain boundary conditions[5].

$$\nabla^4 \phi + \nabla^2 \zeta T = 0, \qquad (1)$$

where ϕ is a stress function and the stress components are given by

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2},$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2},$$

and

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r\,\partial \theta}. \qquad (2a.b.c.)$$

A thin disc with a temperature distribution which is a function of the radial distance only, is in a state of plane stress with rotational symmetry. For a free disc, the stress components can be shown to be[6]

$$\sigma_r = \zeta\left(\frac{1}{b^2}\int_o^b Trdr - \frac{1}{r^2}\int_o^r Trdr\right),$$

and

$$\sigma_\theta - \sigma_r = \zeta\left(-T + \frac{2}{r^2}\int_o^r Trdr\right) \quad . \qquad (3a.b)$$

For the problem under consideration, it is found that the form of the temperature distribution can be represented by

$$T = T_o + \Delta T \xi^n , \qquad (4)$$

and the corresponding stress components then become

$$\sigma_r = \frac{\zeta \Delta T}{n+2}\left[1 - \xi^n\right],$$

and

$$\sigma_\theta = \frac{\zeta \Delta T}{n+2}\left[1 - \xi^n (n+1)\right] . \qquad (5a.b)$$

The energy equation for buckling in polar co-ordinates becomes[7] for $\tau_{r\theta} = 0$

$$\frac{D}{2}\iint_S\left\{\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right]^2\right.$$

$$\left. - 2(1-\nu)\left[\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\right) - \left(\frac{1}{r^2}\frac{\partial w}{\partial \theta} - \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta}\right)^2\right]\right\} ds$$

$$= -\frac{h}{2}\iint_S\left[\sigma_r\left(\frac{\partial w}{\partial r}\right)^2 + \frac{\sigma_\theta}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right] ds. \qquad (6)$$

A suitable displacement mode may be represented by the equation

$$w = A\xi^2\left[1 + \beta_1 \xi^2 + \beta_2 \xi^4\right]\sin 2\theta , \qquad (7a)$$

where β_1 and β_2 are suitably chosen constants which satisfy the conditions at the free edge[8] that

$$Q_r - \frac{\partial M_{r\theta}}{r d\theta} = 0 ,$$

and (7b.c)

$$M_r = 0 .$$

Substitution of equations (5) and (7a) into equation (6) yields an equation for the critical temperature differential. This becomes for the case when $\nu = .287$

$$\Delta T_c = \frac{1.607\, D}{b^2\, h \zeta F(n)} , \tag{8}$$

where F(n) is given by

$$F(n) = \frac{n}{n+2} \left[\frac{1.25}{n+4} + \frac{1.25}{n+8}\beta_1^2 + \frac{1.084}{n+12}\beta_2^2 + \frac{2.67}{n+6}\beta_1 + \frac{2.75}{n+8}\beta_2 + \frac{2.4}{n+10}\beta_1\beta_2 \right] , \tag{9}$$

and where $\beta_1 = -.279$ and $\beta_2 = .063$.

These coefficients give an expression for w which coincides with the experimentally determined mode of deflection, see Figs. 3a, b.

The condition $\frac{d(\Delta T_c)}{dn} = 0$ gives a value of $n = 2.6$ and a corresponding value of $\Delta T_{c\,min} = 23.3^{o}C.$ for the dimensions and material of the test specimen.

Fig. 4 shows the variation of the critical temperature differential ΔT_c with n, and this solution is generally applicable provided that the form of temperature distribution may be represented by a suitable choice of n, and that the buckling mode remains unchanged for various temperature distributions.

4.4. Comparison with experiment

For the specimen under consideration $h = .06$ in., $b = 5.39$ in., and α and ν are taken equal to $11 \times 10^{-6}/^{o}C.$, and .287 respectively.

Fig. 5 shows that a good approximation to the measured temperature distribution, at the point of buckling, may be obtained by putting $n = 0.6$ in equation (4). The corresponding value of ΔT_c is $38.0^{o}C.$, from Fig. 4. This value compares well with $\Delta T_c = 39 \pm 3.0^{o}C.$ from experiment.

The theoretical buckling stress distribution for this case (equ. 5) is shown in Fig. 6.

5. References

1. Cotterell, B. — Thermal buckling of a free circular plate. R.A.E. internal memo Structures 378. November, 1957.

2. Hoff, N.J. — Buckling at high temperature. Journal of the Royal Aeronautical Society. Vol.61, 1957, pp 756-77.

3. Hoff, N.J. — Effects thermiques dans le calcul de la resistance des structures d'avions et d'engins. Organisation Du Traite de l'Atlantique Nord, Palais de Chaillot, Paris 16. Janvier 1956. pp 43.

4. Gatewood, B.E. — Thermal stresses in moderately thick elastic plates. Journal of Applied Mechanics, Vol.26, Series C, 1959. pp 432-436.

5. Gatewood, B.E. — Thermal stresses. McGraw-Hill Book Co. Inc., 1957.

6. Goodier, J.N. — Thermal stress and deformation. Journal of Applied Mechanics, Vol.24, 1957.

7. Timoshenko, S. — Theory of elastic stability. McGraw-Hill Book Co. Inc., First Edition, 1936.

8. Wang, C. — Applied elasticity. McGraw-Hill Publishing Co. Ltd., 1953.

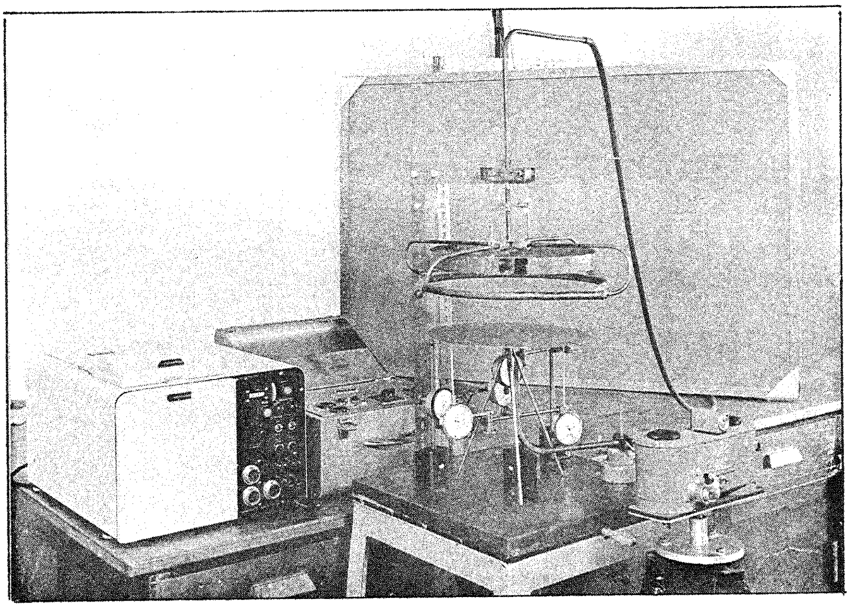

FIG. 1 (a). PHOTOGRAPH OF THE TEST APPARATUS SHOWING AN EXPLODED VIEW OF THE SPECIMEN

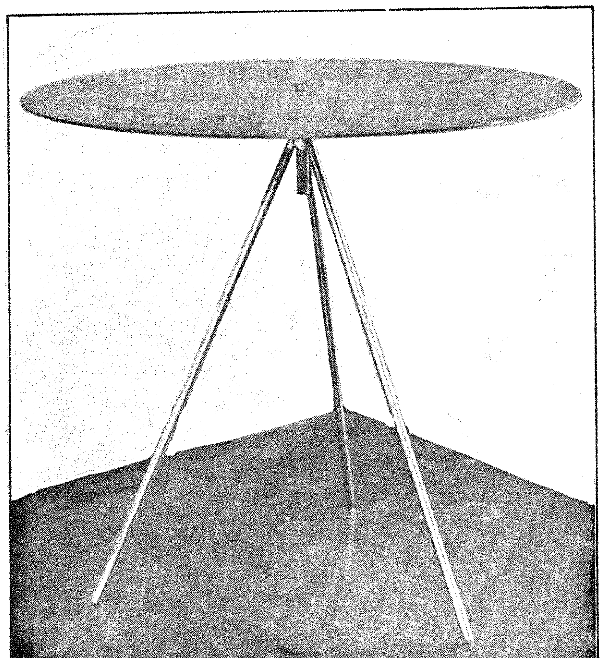

FIG. 1 (b). CIRCULAR PLATE SPECIMEN WITH TRIPOD MOUNTING

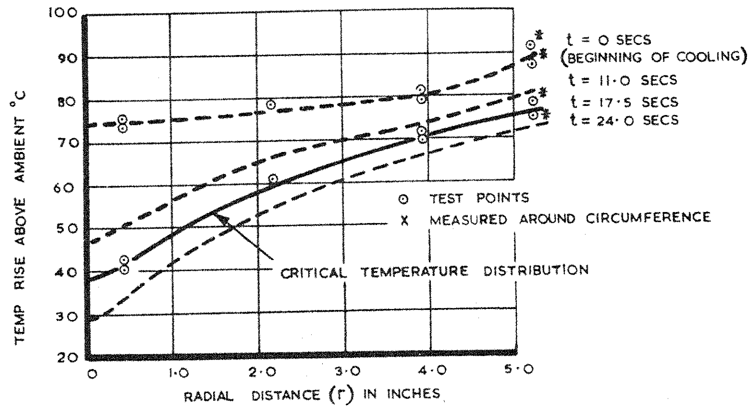

FIG. 2. TEMPERATURE DISTRIBUTION DURING COOLING.

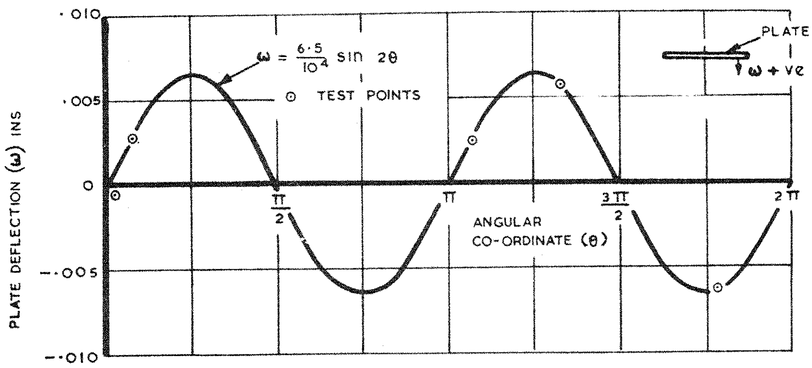

FIG 3a. DEFLECTION MODE AT GIVEN RADIUS.

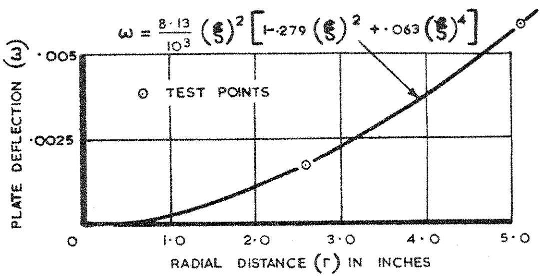

FIG. 3b. DEFLECTION MODE AT GIVEN 'θ'.

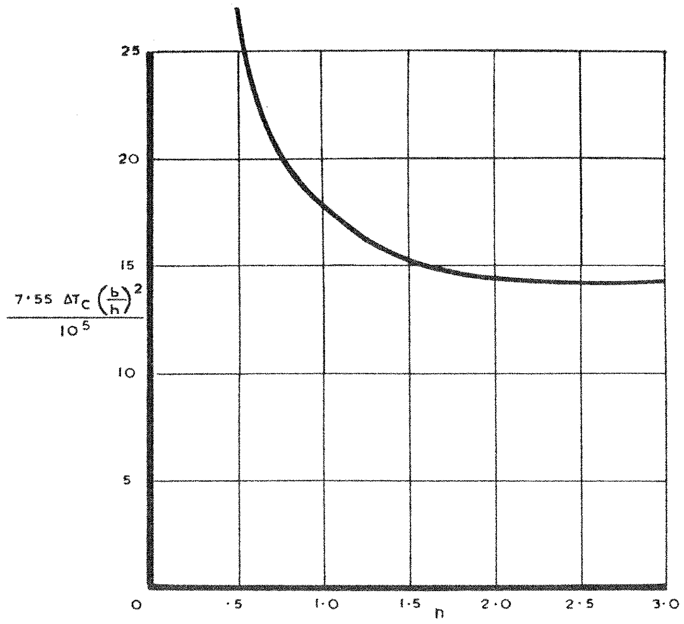

FIG. 4. VARIATION OF ΔT_c WITH n.

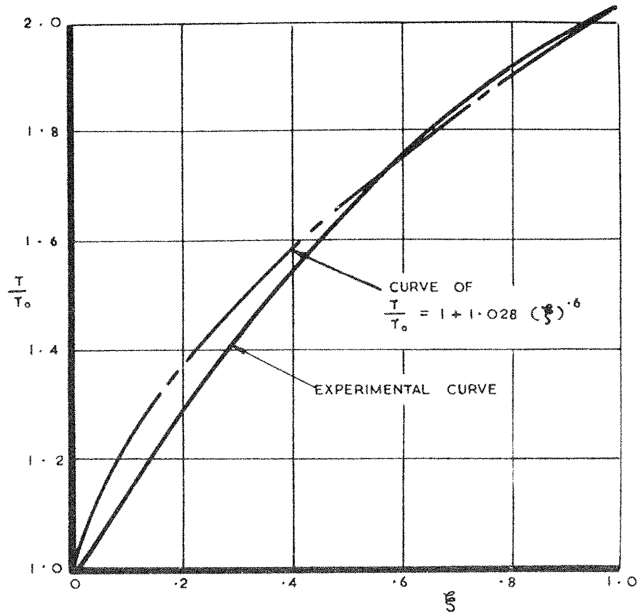

FIG. 5. RADIAL TEMPERATURE DISTRIBUTION AT BUCKLING POINT.

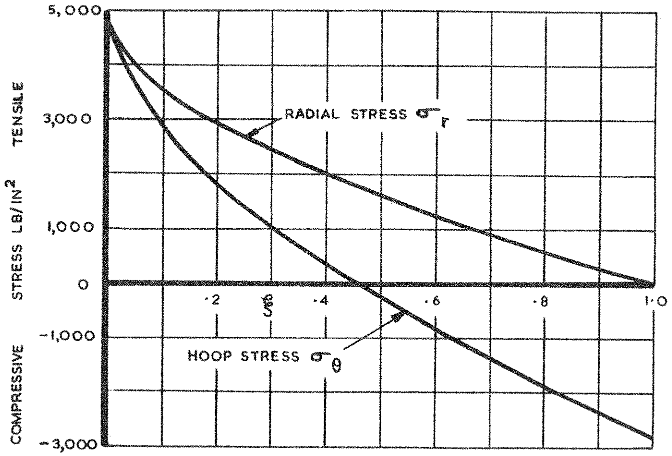

FIG. 6. THEORETICAL STRESS DISTRIBUTION AT BUCKLING POINT