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Characterizations of Perfect Recall

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Abstract This paper considers the condition of perfect recall for the class of arbitrarily large discrete extensive form games. The known definitions of perfect recall are shown to be equivalent even beyond finite games. Further, a qualitatively new characterization in terms of choices is obtained. In particular, an extensive form game satisfies perfect recall if and only if the set of choices, viewed as sets of ultimate outcomes, fulfill the "Trivial Intersection" property, that is, any two choices with nonempty intersection are ordered by set inclusion.

Keywords Perfect Recall · Large Extensive Form Games · Non-Cooperative Games

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1 Introduction

Extensive form games are the key tool to analyze multi-person sequential decision making by rational agents. The ingredients for this basic representation of a game are a tree that describes an omniscient observer's view of the interaction and a specification of what players may do and under which informational restrictions. The formalizations of these vary, but invariably an additional assumption is imposed, which ensures that the player's informational restrictions are consistent with rationality: perfect recall.

This condition was introduced by Kuhn (1953), who explained it as a memory requirement. Subsequently alternative definitions of the same concept were proposed, e.g., by Selten (1975) and by Osborne and Rubinstein (1994). Somewhat surprisingly, little effort has gone into showing that these definitions are indeed equivalent. The present note fills this gap by showing that they are.

Additionally, a further equivalent definition of perfect recall is proposed. The latter is of interest, because it characterizes perfect recall by a tree-property of the players' choices and/or information sets. In particular, if a play (maximal path) passes through two choices and/or information sets of the same player, then all plays passing trough one of those must also pass through the other. This is analogous to a basic property satisfied by any game tree: If a play passes through two *nodes* of the tree, then every play passing through one must also pass through the other. We refer to this property as "Trivial Intersection." Hence, the new characterization boils down to the condition that each player's choices and/or information sets "look like" a tree, though possibly one without a root.

The original definition of perfect recall (Kuhn, 1953) was stated for finite games only. In this paper we also extend all three definitions to games with a potentially infinite horizon and possibly large action sets. That is, in the present framework an extensive form game may not end, and players may at times choose from a continuum, or from even larger sets. This extension is important for applications, as those often work with large games.

The paper concludes with a discussion of the weaker condition of "no-absent-mindedness," which was part of the seminal definitions of (finite) extensive forms by Kuhn (1953) and Selten (1975) and is known to be implied by perfect recall. It requires that each play passes through a choice, or an information set, at most once. In the present framework, though, such an additional requirement is not needed—it holds automatically.

The reason is that we work with the generalization of the formalization of extensive form games by von Neumann and Morgenstern (1944), as proposed by Alós-Ferrer and Ritzberger (2005, 2008, 2013). In that formalization the relevant objects, choices or nodes in the tree, are sets of plays, hence events in the sense of statistics—if one thinks of the set of all plays as the state space. At an information set the nodes contained in it count as the events that the decision maker regards as possible. Hence, the plays passing through some node in the information set are the states that the player regards as possible. If a play would pass through two distinct choices that are simultaneously available at the same information set, then the decision maker could—by not taking one of the choices—rule out a state that she regards as possible. But why should I regard a state as possible if I can prevent it from materializing?

The paper proceeds as follows. Section 2 lays out definitions and notation. Section 3 states the equivalence of the known definitions of perfect recall, and Section 4 gives the novel characterization. Section 5 discusses no-absent-mindedness and shows that even without perfect recall mixed strategies are at least as powerful as behavior strategies. Section 6 concludes.

2 Large Games: Discrete Extensive Forms

Extensive form games are defined on a tree capturing the order of players' decisions. Alós-Ferrer and Ritzberger (2005, 2008, 2013) have developed the concept of game trees and shown how to define arbitrarily large extensive game forms on the resulting objects (see also Alós-Ferrer and Ritzberger 2016, forthcoming-a, forthcoming-b). We rely on this research and directly present here the appropriate concepts.

The basic approach, originally advocated by von Neumann and Morgenstern (1944, Section 8), relies on the idea that nodes in a tree are subsets of a given set of possible ultimate outcomes, i.e., a node is the set of outcomes that may still occur conditional on the node having been reached. Hence a tree is a collection of nodes partially ordered by set inclusion, that is, a node x precedes a node y if $x \supseteq y$. A simple way to think about this approach is to start with the set of all potential outcomes that might occur (which is the space on which eventually players' preferences should be defined). The root is identified with the whole set. As the game proceeds, some outcomes are excluded, and each node is the set of outcomes which have not been discarded yet, and which might still be the ultimate outcome of play when the game proceeds through that node.

The relation to the traditional "graph-approach" of Kuhn (1953) and Selten (1975) is intuitive. In that approach, a node is an abstract decision point. One can simply consider the set of all "plays", that is, maximal chains of nodes from the root to the end of the game (if there is an end). Each such play corresponds to one and only one ultimate outcome. Identify then each node with the set of plays which pass through the node, i.e., the set of plays which have not been discarded yet. The resulting object is a set representation which corresponds to a tree viewed as a set of sets, as given above.

Our previous work has synthesized the approach of von Neumann and Morgenstern (1944, Section 8) with the graph-approach of Kuhn (1953) and Selten (1975). In particular Alós-Ferrer and Ritzberger (2005, Theorem 3) formally demonstrate the equivalence between *game trees*, which take outcomes as primitives, and trees as partially ordered sets, like graphs, which take nodes as primitives. Further, due to this equivalence, it is always possible to identify outcomes with plays.

Formally, we work with *discrete* game trees as introduced in Alós-Ferrer and Ritzberger (2013) (and used in Alós-Ferrer and Ritzberger 2016, forthcoming-a, forthcoming-b), and present a condensed definition here. Recall that in a partially ordered set a *chain* is defined as a subset that is completely ordered. Henceforth, maximum, minimum, and infimum of a chain are with respect to set inclusion. The symbol $\subseteq$ denotes weak set inclusion and $\subset$ denotes proper inclusion.

Definition 1 A **discrete** (rooted and complete) **game tree** $(N, \supseteq)$ is a collection of nonempty subsets $x \in N$ (the nodes) of a given set W (of outcomes) partially ordered by set inclusion such that $W \in N$, $\{w\} \in N$ for all $w \in W$, and the following two additional conditions hold
(GT1) $h \subseteq N$ is a chain if and only if there is $w \in W$ such that $w \in \cap_{x \in h} x$,
(GT2) every chain in the set $X = N \setminus \{\{w\}\}_{w \in W}$ (of *moves*) has a maximum, and it either has an infimum in the set $E = \{\{w\}\}_{w \in W}$ (of terminal nodes) or it has a minimum.

Property (GT1) requires that if two nodes have a nonempty intersection then one contains the other ("Trivial Intersection"), and that all chains have a nonempty intersection ("Boundedness"). Property (GT2) is discreteness. Its first part, "up-discreteness," is necessary for every pure strategy combination to induce a unique play/outcome (Alós-Ferrer and Ritzberger, 2008, Theorems 3 and 6). Its second part, "down-discreteness," excludes e.g. continuous time but still allows large action spaces and infinite horizon (Alós-Ferrer and Ritzberger, 2013, Definition 5).[1]

For each node $x \in N$ define the *up-set* ("the past") $\uparrow x$ and the *down-set* ("the future") $\downarrow x$ by

$$\uparrow x = \{y \in N \,|\, y \supseteq x\} \text{ and } \downarrow x = \{y \in N \,|\, x \supseteq y\}. \tag{1}$$

By the if-part of (GT1) $\uparrow x$ is a chain for all $x \in N$. A *play* is a chain of nodes $h \subseteq N$ that is maximal in N, i.e., there is no $x \in N \setminus h$ such that $h \cup \{x\}$ is a chain. Intuitively, a play is a complete history of all events along the tree, from the beginning (the root $W \in N$) to the "end"—if there is an end: Since infinite histories are allowed, plays need not be finite.

The advantage of game trees is that the set of plays can be one-to-one identified with the underlying set W (Alós-Ferrer and Ritzberger, 2005, Theorem 3(c)). A node then consists of the plays passing through it, and the underlying set W represents all plays. An element $w \in W$ can thus be seen either as a possible outcome (element of some node) or as a play (maximal chain of nodes). If h is a play, there exists a unique outcome $w \in W$ such that $\cap_{x \in h} x = \{w\}$, or, equivalently, $\uparrow \{w\} = h$. Henceforth we will not distinguish between plays and outcomes.

[1] Definition 1 is equivalent to the concept of discrete game tree in Definition 5 of Alós-Ferrer and Ritzberger (2013) plus the property that $\{w\} \in N$ for all $w \in W$, which is called *completeness* in that work and can be assumed without loss of generality (Alós-Ferrer and Ritzberger, 2013, Proposition 4).

For a discrete game tree $(N, \supseteq)$ a node $x \in N$ is *terminal* if $\downarrow x = \{x\}$. Players in an extensive form game, however, decide at non-terminal nodes, called *moves*. It can be shown (see Alós-Ferrer and Ritzberger, 2008, Lemma 1, and Alós-Ferrer and Ritzberger, 2013, Lemma 3(b)) that a node $x \in N$ in a discrete game tree is terminal if and only if there is $w \in W$ such that $x = \{w\}$. Hence the set $E = \{\{w\}\}_{w \in W}$ introduced in (GT2) coincides with the set of terminal nodes. Likewise, the set $X = N \setminus E$ is the set of moves.

The possibility of infinite horizon yields a further classification of nodes. A node $x \in N \setminus \{W\}$ is *finite* if $\uparrow x \setminus \{x\}$ has a minimum and *infinite* if $x = \inf \uparrow x \setminus \{x\}$. In a discrete game tree every node is either finite or infinite, and every move is a finite node, $X \subseteq F(N)$ or, in other words, every infinite node is terminal (Alós-Ferrer and Ritzberger, 2013, Proposition 3 and Theorem 1(c)). Denote by $F(N)$ the set of finite nodes together with the root $W \in N$. On this set a function $p : F(N) \to X$ can be defined that assigns to every finite node its *immediate predecessor*. Namely, for each $x \in F(N) \setminus \{W\}$ let

$$p(x) = \min \uparrow x \setminus \{x\} \tag{2}$$

and $p(W) = W$ by convention. Hence, $x \subset p(x) = \cap \{y \,|\, y \in \uparrow x \setminus \{x\}\}$ for all $x \in F(N) \setminus \{W\}$.

For a discrete game tree $(N, \supseteq)$ let (by a slight abuse of notation) $W : N \twoheadrightarrow W$ denote the correspondence[2] that assigns to every node, viewed as an element of the tree, the set of its constituent plays, that is, the node itself viewed as a set of plays, i.e. $W(x) = x$ for all $x \in N$. For a set $Y \subseteq N$ of nodes write $W(Y) = \cup_{x \in Y} x \subseteq W(x)$ for the union, and refer to $W(Y)$ as the *plays passing through* Y.

At each move, one or more players will choose certain sets of plays. In order to model such decisions, one needs a notion of which sets of plays are available at which moves. For a set $a \subseteq W$ of plays let $\downarrow a = \{x \in N \,|\, x \subseteq a\}$ be its *down-set* and define the set of *immediate predecessors* of a as

$$P(a) = \{x \in N \,|\, \exists y \in \downarrow a : \uparrow x = \uparrow y \setminus \downarrow a\}, \tag{3}$$

Say that a set a of plays is *available at* the move $x \in X$ if $x \in P(a)$.

The idea behind the definition is as follows. Nodes are sets of plays, and the objects that are chosen at nodes are also certain sets of plays (to be specified below). Let x be a node and a a set of plays which intersects x. The node x is a predecessor of a (and hence a is available at x) whenever there exists some other node y, a successor of x contained in a, such that x is the minimum among those predecessors of y that are not contained in a. That is, there is a chain of nodes containing x whose members are eventually contained in the set of plays a, but the node x is the "last" one along the chain which is *not* contained in a. Intuitively, the set of plays a "leads towards" a certain chain of successors of x. Clearly, if $x \in P(a)$, then there is a play $w \in x \cap a$ that

[2] Even though the same symbol serves for the map and its codomain, no confusion can arise, because the argument will always be specified.

passes through x and a and another $w' \in x \setminus a$ that passes through x but not through a, hence a represents some "choice".

Effectively, $P(a)$ is the collection of nodes that are minimal with respect to a nontrivial intersection with $a \subseteq W$. For, if $x \cap a \neq \emptyset \neq x \setminus a$ and $y \subseteq a$ for all $y \in \downarrow x \setminus \{x\}$, then for any $y \in \downarrow x \setminus \{x\}$ it holds that $y \in \downarrow a$ and $\uparrow x = \uparrow y \setminus \downarrow a$, hence $x \in P(a)$. Conversely, if $x \in P(a)$, then for any $y \in \downarrow x \setminus \{x\}$ it holds that $y \subseteq a$, hence $x \cap a \neq \emptyset$, and $x \notin \downarrow a$ implies $x \setminus a \neq \emptyset$.

With the appropriate concept of a tree at hand the definition of an extensive form is fairly direct compared with other approaches in the literature (see Alós-Ferrer and Ritzberger, 2013, for further details).

Definition 2 A **discrete extensive form** (DEF) with player set I is a pair (T, C), where $T = (N, \supseteq)$ is a (rooted, complete) discrete game tree with set of plays W and $C = (C_i)_{i \in I}$ a system consisting of collections C_i (the sets of players' choices) of nonempty unions of nodes (hence, sets of plays) for all $i \in I$, such that

(DEF1) if $P(c) \cap P(c') \neq \emptyset$ and $c \neq c'$, then $P(c) = P(c')$ and $c \cap c' = \emptyset$, for all $c, c' \in C_i$ for all $i \in I$;

(DEF2) $p^{-1}(x) = \left\{ x \cap \left(\cap_{i \in J(x)} c_i \right) \middle| (c_i)_{i \in J(x)} \in A(x) \right\}$, for all $x \in X$;

where $A(x) = \times_{i \in J(x)} A_i(x)$, $A_i(x) = \{c \in C_i \,| x \in P(c)\}$ are the choices available to $i \in I$ at $x \in X$, and $J(x) = \{i \in I \,| A_i(x) \neq \emptyset\}$ is the set of decision makers at x, which is required to be nonempty for all $x \in X$.

(DEF1) is the "information set property" that players cannot deduce from the available choices at which move in the information set they are. Note that information sets are not primitive objects in this formulation but rather derived from choices: they correspond to the sets $P(c)$. Formally, the collection of *information sets* of player $i \in I$ is given by $\{P(c) \subseteq X | c \in C_i\}$. That is, an information set is the set of nodes $P(c)$ where c is a choice. The role of information sets is the same as in the framework of Kuhn (1953) and Selten (1975): If a move in an information set h is reached, the player controlling it is asked to choose among the available choices (those choices c such that $P(c) = h$)—no other information than the menu of available choices is released to her.

The second property is also fairly intuitive. (DEF2) says that, at any given move, the combined decisions of the relevant players lead to a new node, and that any (immediate) successor of the move can be selected by an appropriate combination of the players' decisions. (The intersection with the move $x \in X$ is needed because of potentially large information sets.)

As shown in Alós-Ferrer and Ritzberger (2013), the definition of discrete extensive form corresponds to the general definition of *extensive decision problem* when discreteness is assumed. That definition (Alós-Ferrer and Ritzberger, 2013, Definition 3 or Alós-Ferrer and Ritzberger, 2005, Definition 7) employs four properties, (EDP.i-iv). (EDP.i) is identical to (DEF1). Under (GT2), however, the three independent conditions (EDP.ii-iv) collapse to (DEF2), in the sense that (DEF2) holds if and only if those three conditions hold simultaneously. One of the three conditions summarized by (DEF2), namely (EDP.iv), excludes absent-mindedness (see Alós-Ferrer and Ritzberger, 2005, Proposition 13), i.e. that a play intersects an information set more than once.

A detailed translation between a DEF and the graph-based approach by Kuhn (1953) and Selten (1975) has been provided in Alós-Ferrer and Ritzberger (2013, Section 4). There it is shown that one can go back and forth between a DEF and a "simple extensive form" that takes nodes as primitives (Alós-Ferrer and Ritzberger, 2013, Proposition 6).[3]

Remark 1 Choices essentially partition the set of plays passing through the corresponding information set. Hence, without loss of generality it can be assumed that $c \subseteq W(P(c))$ for all $c \in C_i$ and all $i \in I$. This assumption will be maintained throughout. There is a subtle (though inconsequential) point regarding this assumption. If it is not made, a choice could include an infinite terminal node corresponding to a play that does not pass through the information set at which the choice is available. In fact, there is $w \in c \setminus W(P(c))$ if and only if $\{w\} \in E$ is an infinite terminal node. It is easy to see that the DEF is not affected at all by removing such irrelevant plays from choices.

Henceforth the set I of players' names is fixed and treated as a finite set $I = \{1, ..., n\}$ for some $n \in \mathbb{N}$. If chance is present, it will be treated as one of the players in I.

Associated with any DEF there are two derived objects: Pure strategies for the players and a surjection from pure strategy combinations into plays.

Definition 3 For a DEF (T, C) and a player $i \in I$ the set S_i of **pure strategies** for player i is the set of all functions $s_i : X_i \equiv \{x \in X \,| i \in J(x)\} \to C_i$ that satisfy

$$s_i^{-1}(c) = P(c) \text{ for all } c \in s_i(X_i) \equiv \{s_i(x) \,| x \in X_i\}. \tag{4}$$

That is, the function s_i assigns to every move $x \in X_i$ a choice $c \in C_i$ such that (a) choice c is *available* at x, i.e. $s_i(x) = c \Rightarrow x \in P(c)$ or $s_i^{-1}(c) \subseteq P(c)$, and (b) to every move x in an information set $P(c)$ the same choice gets assigned, i.e. $x \in P(c) \Rightarrow s_i(x) = c$ or $P(c) \subseteq s_i^{-1}(c)$, for all $c \in C_i$ that are chosen somewhere, viz. $c \in s_i(X_i)$. Let S_i denote the set of all pure strategies for player $i \in I$. A *pure strategy combination* is an element $s = (s_i)_{i \in I}$ of the set $S \equiv \times_{i \in I} S_i$ of all pure strategy combinations.

Furthermore, for every DEF there is a surjection $\phi : S \to W$ that assigns to every pure strategy combination the play that it induces (Alós-Ferrer and Ritzberger, 2013, Theorems 4 and 6). Given a pure strategy combination s, the outcome $\phi(s)$ is defined as the unique fixed point of the map

$$R_s(w) = \cap \{s_i(x) \,| w \in x \in X_i\}. \tag{5}$$

[3] Example 10 in Alós-Ferrer and Ritzberger (2013) shows that the sequence-based definition by Osborne and Rubinstein (1994) is also captured.

Lemma 1 *$\phi(s) = w$ if and only if $w \in x \in X_i \Rightarrow w \in s_i(x)$ for all $i \in I$.*

Proof By (5), $\phi(s) = w$ if and only if $\{w\} = R_s(w)$. Hence, if $w \in x \in X_i \Rightarrow w \in s_i(x)$ for all $i \in I$, then $w = \phi(s)$. To see the converse, assume that for some $i \in I$ there is $x \in X_i \cap \uparrow\{w\}$ such that $w \notin s_i(x)$. Then $w \notin R_s(w)$, hence, $w \neq \phi(s)$. □

3 Perfect Recall

A property of extensive forms that is crucial for applications yet is not implied by the definition of a DEF is *perfect recall*. This property was introduced by Kuhn and, according to him, "... is equivalent to the assertion that each player is allowed by the rules of the game to remember everything he knew at previous moves and all of his choices at those moves." (Kuhn, 1953, p. 213) As a matter of fact, perfect recall does a little more. It can be shown (see Ritzberger, 1999) that it is equivalent to the simultaneous fulfillment of three independent properties: Players never forget what they did; they never forget what they knew; and (for a given player) past, present, and future have an unambiguous meaning.

In a sense, an extensive form that fails perfect recall does not properly capture rational behavior on the part of the players. Therefore, it appears mandatory to include perfect recall among the assumptions.

Given a DEF (T, C) with tree $T = (N, \supseteq)$ and a move $x \in X_i$ of player $i \in I$, say that x is *possible* under $s_i \in S_i$ for player i, denoted $x \in \text{Poss}(s_i)$, if there is $s_{-i} \in \times_{j \neq i} S_j \equiv S_{-i}$ such that $\phi(s_i, s_{-i}) \in x$. Similarly, an information set $P(c)$ for $c \in C_i$ for player $i \in I$ is *relevant* under $s_i \in S_i$ for player i, denoted $P(c) \in \text{Rel}(s_i)$, if $P(c) \cap \text{Poss}(s_i) \neq \emptyset$. The following definition is from Kuhn (1953).

Definition 4 A DEF (T, C) satisfies **perfect recall** if, for each $i \in I$, each $s_i \in S_i$, and each $c \in C_i$, it holds that $P(c) \in \text{Rel}(s_i) \Rightarrow P(c) \subseteq \text{Poss}(s_i)$.

Clearly, perfect recall could be defined for each player separately by dropping the first quantifier. Furthermore, the definition may also be rewritten in terms of the function ϕ as follows: A DEF satisfies *perfect recall* if, for each $i \in I$, $s_i \in S_i$, $c \in C_i$, and $x \in P(c)$,

$$x \cap \phi(s_i, S_{-i}) \neq \emptyset \Rightarrow y \cap \phi(s_i, S_{-i}) \neq \emptyset, \ \forall y \in P(c), \tag{6}$$

with $\phi(s_i, S_{-i}) = \{\phi(s_i, s_{-i}) \mid s_{-i} \in S_{-i}\}$. This follows from

$$\begin{aligned} \text{Poss}(s_i) &= \{x \in X_i \mid x \cap \phi(s_i, S_{-i}) \neq \emptyset\} \\ \text{Rel}(s_i) &= \{P(c) \mid c \in C_i, \exists x \in P(c) : x \cap \phi(s_i, S_{-i}) \neq \emptyset\} \end{aligned}$$

for all $s_i \in S_i$. The rewriting (6) is particularly useful when the normal form is under scrutiny.

A drawback of Definition 4 is that it refers to pure strategies, that is, to derived objects, rather than primitives (the tree and choices). To develop a characterization of perfect recall in terms of primitives an auxiliary result is needed.

Lemma 2 *For a DEF (T, C) and any player $i \in I$: If $x \in P(c)$ for some $c \in C_i$ and $x \subseteq c' \in C_i$, then there is $x' \in P(c')$ with $x \subset x'$.*

Proof Suppose that $x \in P(c)$ for some $c \in C_i$ and $x \subseteq c' \in C_i$. The chain $\uparrow x \setminus \downarrow c'$ (where $\downarrow c' = \{y \in N \mid y \subseteq c'\}$) is contained in the set X of moves, the move x does not belong to it, and $x \subset z$ for all $z \in \uparrow x \setminus \downarrow c'$, that is, x provides a lower bound for the chain. Therefore, the chain $\uparrow x \setminus \downarrow c'$ cannot have an infimum in the set E of terminal nodes, as all terminal nodes are singletons and a singleton cannot contain a move. It follows from (GT2) that it has a minimum $x' = \min \uparrow x \setminus \downarrow c'$. By (3) it follows that $x' \in P(c')$ with $x \subset x'$, as desired. □

The following result provides three characterizations of perfect recall for a DEF. The first is based on the correspondence $\phi_i : S_i \twoheadrightarrow W$ defined by $\phi_i(s_i) = \phi(s_i, S_{-i})$ for all $s_i \in S_i$ and its lower (or weak) inverse $\phi_i^-(V) = \{s_i \in S_i \mid V \cap \phi_i(s_i) \neq \emptyset\}$ for any subset $V \subseteq W$. The second is the definition of perfect recall proposed by Selten (1975) and the third the one proposed by Osborne and Rubinstein (1994).

Theorem 1 *For a DEF (T, C) each of the following statements is equivalent to perfect recall: For all players $i \in I$ and all choices $c, c' \in C_i$ of player i,*[4]
(a) $\phi_i^-(x) = \phi_i^-(y)$ for all $x, y \in P(c)$;
(b) if there is $x \in P(c)$ such that $x \subseteq c'$, then $y \subseteq c'$ for all $y \in P(c)$;
(c) if there are $x \in P(c)$ and $x' \in P(c')$ with $x \subset x'$, then there is a unique $c'' \in A_i(x')$ such that $W(P(c)) \subseteq c''$.

Proof "Perfect recall implies (a):" If $x \in P(c)$ for $c \in C_i$ and $s_i \in \phi_i^-(x)$, then $x \cap \phi(s_i, S_{-i}) \neq \emptyset$ by the definition of ϕ_i^-. Perfect recall (6) then implies that $y \cap \phi(s_i, S_{-i}) \neq \emptyset$ for all $y \in P(c)$, hence, $s_i \in \phi_i^-(y)$. Since $x, y \in P(c)$ enter symmetrically in this argument, the statement follows.

"(a) implies (b):" Suppose for $c \in C_i$ there is $x \in P(c)$ such that $x \subseteq c' \in C_i$. By Theorem 4 of Alós-Ferrer and Ritzberger (2008) there is $s \in S$ such that $\phi(s) \in x$. Then, first, $s_i \in \phi_i^-(x)$ implies $s_i \in \phi_i^-(y)$, that is, $y \cap \phi(s_i, S_{-i}) \neq \emptyset$, for all $y \in P(c)$ by (a). Second, because $x \subseteq c' \in C_i$ by hypothesis, there is $x' \in P(c')$ with $x \subset x'$ by Lemma 2. Third, $s_i(x') = c'$, because otherwise $s_i(x') \cap c' = \emptyset$ by (DEF1) would contradict $\phi(s) \in x$ by Lemma 1.

Consider any $y \in P(c)$, and suppose $y \setminus c' \neq \emptyset$. By Theorem 4 of Alós-Ferrer and Ritzberger (2008) there is $s' \in S$ such that $\phi(s') \in y \setminus c'$. There are two possibilities.

[4] If perfect recall were defined as a property of player i's choice set alone, as it is possible, the first quantifier could be dropped.

First, if there were $y' \in P(c')$ with $\phi(s') \in y'$, then $\phi(s') \in s'_i(y')$ by Lemma 1. On the other hand, by the definition of a pure strategy, (4), $s'_i(y') = s'_i(x') \neq c'$, because $\phi(s') \notin c'$. This yields $s'_i(x') \cap c' = \emptyset$ by (DEF1), which implies from $x \subseteq c'$ that $x \cap \phi(s'_i, S_{-i}) = \emptyset$, that is, $s'_i \notin \phi_i^-(x)$, even though $\phi(s') \in y \in P(c)$, in contradiction to (a).

This leaves the second possibility, that $P(c') \cap \uparrow \{\phi(s')\} = \emptyset$. In this case let $s''_i \in S_i$ be such that $s''_i(z) = s'_i(z)$ for all $z \in X_i \setminus P(c')$ but $s''_i(z) \neq c'$ for all $z \in P(c')$. From $P(c') \cap \uparrow \{\phi(s')\} = \emptyset$ it follows that $\phi\left(s''_i, s'_{-i}\right) = \phi(s') \in y \in P(c)$, hence, $s''_i \in \phi_i^-(y)$. But $x \cap \phi(s''_i, S_{-i}) = \emptyset$, because $s''_i(x') \cap c' = \emptyset$ by (DEF1) and $x \subseteq c'$, which once again contradicts (a).

Therefore, $y \setminus c' = \emptyset$ or, equivalently, $y \subseteq c'$. Since $y \in P(c)$ was arbitrary, statement (b) follows.

"(b) implies (c):" Suppose there are $x \in P(c)$ and $x' \in P(c')$ with $x \subset x'$. By (GT2) the chain $\uparrow x \setminus \uparrow x'$ has a maximum $z = \max \uparrow x \setminus \uparrow x' \in X$. Since $z \in p^{-1}(x')$, by (DEF2) there is $c'' \in A_i(x')$ such that $z = x' \cap c'' \cap \left[\cap_{j \in J(x') \setminus \{i\}} c_j\right]$ for some choice combination $(c_j)_{j \in J(x') \setminus \{i\}} \in \times_{j \in J(x') \setminus \{i\}} A_j(x')$. It follows from $x \subseteq z \subset x'$ and $z \subseteq c''$ that $x \subseteq c''$. By (b) this implies that $y \subseteq c''$ for all $y \in P(c)$, hence, $W(P(c)) \subseteq c''$. That $c'' \in A_i(x')$ is unique follows from (DEF1), because any $\hat{c} \in A_i(x') \setminus \{c''\}$ is disjoint from c'' and, therefore, cannot cover $x \in P(c)$.

"(c) implies perfect recall:" Suppose for $s_i \in S_i$ there is $s_{-i} \in S_{-i}$ such that $\phi(s_i, s_{-i}) \in x \in P(c)$ for some $c \in C_i$, that is, $P(c) \in \mathrm{Rel}(s_i)$. There are two cases to consider. Either player i has a decision point that properly contains x or not.

Suppose first that there is $x' \in \uparrow x \setminus \{x\}$ with $x' \in P(c')$ for some $c' \in C_i$. Then by (c) there is a unique $c'' \in A_i(x')$ such that $W(P(c)) \subseteq c''$. It follows from Lemma 1 and the definition of a pure strategy that $s_i(y') = c''$ for all $y' \in P(c')$ and, therefore, $y \cap \phi(s_i, S_{-i}) \neq \emptyset$. Since this argument holds for all nodes in $X_i \cap (\uparrow x \setminus \{x\})$, it follows that $y \in \mathrm{Poss}(s_i)$ for all $y \in P(c)$, as required by perfect recall.

Otherwise, if $X_i \cap (\uparrow x \setminus \{x\}) = \emptyset$, then $X_i \cap (\uparrow y \setminus \{y\}) = \emptyset$ for all $y \in P(c)$. For, if there were $y \in P(c)$ and $y' \in X_i$ with $y \subset y'$, then by (c) there would by a unique $c'' \in A_i(y')$ such that $W(P(c)) \subseteq c''$, in particular, $x \subseteq c''$. By Lemma 2 there would be $x' \in P(c'')$ with $x \subset x'$, in contradiction to the hypothesis. Hence, indeed $X_i \cap (\uparrow y \setminus \{y\}) = \emptyset$ for all $y \in P(c)$. But then $y \cap \phi(s_i, S_{-i}) \neq \emptyset$ for all $y \in P(c)$, that is, $P(c) \subseteq \mathrm{Poss}(s_i)$, as required by perfect recall (Definition 4). □

It follows that any one of the three definitions of perfect recall from the literature can be used in applications. This also holds if the game under scrutiny has an infinite horizon and/or large action sets.

Perea (2001, Definition 2.1.2) gives a very intuitive definition of perfect recall for the case of finite games which also extends to the infinite case in a straightforward way.

Corollary 1 *A DEF (T, C) satisfies perfect recall if and only if for each $i \in I$, each $c \in C_i$, and each $x, y \in P(c)$, the path from the root to x implies the same collection of player i choices as the path from the root to y.*

Proof The collection of choices implied by a path of play from the root to x is

$$Ch(x) = \{c' \in C_i \mid c' \supseteq x \text{ and } \exists y \in X_i \text{ with } x \subset y, y \in P(c')\}.$$

It is then straightforward that $Ch(x) = Ch(y)$ if and only if condition (a) in Theorem 1 holds. □

4 A Characterization by Trivial Intersection of Choices

Theorem 1 essentially extends known definitions of perfect recall to potentially large games and proves their equivalence. This section presents a qualitatively new characterization that refers to a basic tree structure of the set of choices. Recall that one of the components of (GT1) is "Trivial Intersection," stating that any two nodes are either disjoint or ordered by set inclusion. The natural extrapolation of this property to the set of choices characterizes perfect recall.

Theorem 2 *A DEF (T, C) has perfect recall if and only if the set of choices C_i satisfies Trivial Intersection, i.e., for all $i \in I$ and all $c, c' \in C_i$,*

$$\text{if } c \cap c' \neq \emptyset \text{ then either } c \subset c' \text{ or } c' \subseteq c. \tag{7}$$

Proof "if:" It suffices to demonstrate that statement (b) from Theorem 1 holds under the hypothesis. Hence, suppose that, for $c, c' \in C_i$ and $i \in I$, there is $x \in P(c)$ such that $x \subseteq c'$. Since $x \subseteq c'$ and $x \setminus c \neq \emptyset$ by (3), there is $w' \in c' \setminus c$. Therefore, $c \subset c'$ by $\emptyset \neq x \cap c \subseteq c' \cap c$ and the hypothesis of Trivial Intersection for C_i. Hence, $c' \in C_i$ cannot be available at moves in $P(c)$ by (DEF1). Since $x' \cap c \neq \emptyset$ for all $x' \in P(c)$ by (3), it follows that $x' \cap c' \neq \emptyset$ for all $x' \in P(c)$. Suppose there were $x' \in P(c)$ such that $x' \setminus c' \neq \emptyset$. Because there is a node $y' \in p^{-1}(x')$ with $y' \subseteq c$ by (DEF2) and by $c \subset c'$ this node would satisfy $y' \in \downarrow c'$ and it would follow from (3) that $x' \in P(c')$, a contradiction. Therefore, $x' \subseteq c'$ for all $x' \in P(c)$, as required.

"only if:" Assume perfect recall. Suppose $c \cap c' \neq \emptyset$ for $c, c' \in C_i$ with $c \neq c'$. Then, the two choices must be available at different information sets by (DEF1). Let $w \in c \cap c'$. Then, $w \in W(P(c)) \cap W(P(c'))$ by $\hat{c} \subseteq W(P(\hat{c}))$ for all $\hat{c} \in C_i$ (as explained in Remark 1). Hence, there are moves $x \in P(c)$ and $x' \in P(c')$ such that $x, x' \in \uparrow \{w\}$ (and $x \neq x'$). By (GT1), either $x \subset x'$ or $x' \subset x$. Say $x \subset x'$. It follows from Theorem 1(c) that $W(P(c)) \subseteq c'$ and from $c \subset W(P(c))$ that $c \subset c'$. □

The proof of the only-if part of this latter characterization in fact demonstrates a stronger property than Trivial Intersection for choices: Perfect recall implies that choices and information sets *jointly* satisfy Trivial Intersection, that is, for all $b, b' \in B_i \equiv C_i \cup \{W(P(c)) \,|\, c \in C_i\}$

$$\text{if } b \cap b' \neq \emptyset \text{ then either } b \subset b' \text{ or } b' \subseteq b \tag{8}$$

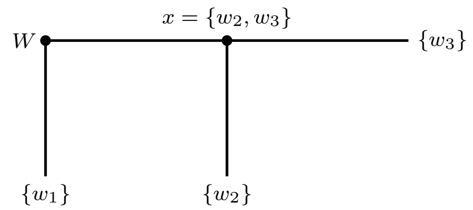

Fig. 1 The tree of the absent-minded driver example. A choice to stop must include both w_2 and w_3. But a single choice to continue must contain x, and hence contain w_2. Thus, it is impossible to have just two choices to continue and stop in this game.

for all $i \in I$. Since the latter implies that choices alone satisfy Trivial Intersection, (7), it is again equivalent to perfect recall by the if part of Theorem 2. Furthermore, under perfect recall information sets alone, $\{W(P(c)) \,|c \in C_i\}$, clearly also satisfy Trivial Intersection. Yet, the proof of the if part of Theorem 2 reveals that the equivalence of both (7) and (8) to perfect recall is due to the tree structure.

Corollary 2 *If (T, C) is a DEF with perfect recall, then for each choice $c \in C_i$ the set $\{c' \in C_i \,|c \subseteq c'\}$ of i's choices that come before c is a finite chain, for all players $i \in I$.*

Proof By the only-if part of Theorem 2 the set $\{c' \in C_i \,|c \subseteq c'\}$ is a chain. Since each $x \in P(c)$ is a finite node by Theorem 1(b) of Alós-Ferrer and Ritzberger (2013), x has only finitely many predecessors $x' \in P(c') \cap \uparrow x$ with $c \subseteq c' \in C_i$. As these predecessors along a play $w \in c$ correspond one-to-one to choices $c' \in C_i$, the latter also form a finite chain. □

5 Absent-Mindedness and Randomized Strategies

In the traditional definitions of (finite) extensive form games by Kuhn (1953) and Selten (1975), an implication of perfect recall was explicitly incorporated: that each play passes through an information set at most once. This has been dubbed "no-absent-mindedness." As shown by Piccione and Rubinstein (1997), violations of no-absent-mindedness generate ill-behaved examples where, e.g., not all outcomes are reachable by pure strategy combinations.

In the present setting, unlike in Kuhn's or Selten's formalism, the exclusion of absent-mindedness need not be assumed, but is an implication of the model. Specifically in a DEF (and, actually, even in the non-discrete case covered in Alós-Ferrer and Ritzberger, 2005, 2008), every play can pass through an information set at most once (Alós-Ferrer and Ritzberger, 2005, Proposition 13). In other words, the issues related to absent-mindedness cannot occur in the present framework, *even if perfect recall is not assumed.*

To understand why this is the case, consider the "absent-minded driver" of Piccione and Rubinstein (1997), whose tree of this game is presented in Figure

1. There are three possible outcomes, w_1, w_2, and w_3. A single player decides at the root, $W = \{w_1, w_2, w_3\}$ and at an intermediate node $x = \{w_2, w_3\}$. At W, the player can either stop, leading to the terminal node $\{w_1\}$, or continue, leading to x. At x, the player can either stop or continue, leading to the terminal nodes $\{w_2\}$ or $\{w_3\}$, respectively.

Piccione and Rubinstein (1997) posed the example where both decisions to stop were identified into a single choice, and analogously for both decisions to continue. Such a violation of perfect recall is simply not feasible in the current setting. Identifying both decisions to stop requires a choice c such that $w_1, w_2 \in c$. Identifying both decisions to continue requires a different choice c' such that $x \subseteq c'$ and $w_3 \in c'$. But $x \subseteq c'$ implies that $w_2 \in c'$. Since c and c' are available at the same nodes (formally, $P(c) = P(c') = \{W, x\}$), this contradicts (DEF1).

The reason why the absent-minded driver problem cannot arise in the present framework is that our approach, as developed in Alós-Ferrer and Ritzberger (2005, 2008, 2013), is built on the concept of play (or outcome) as a primitive. Nodes and choices are sets of plays capturing the possibilities still available before and after a decision, respectively. When the decision to continue is made at the root, the choice allows both w_2 and w_3 as ultimate outcomes. The textbook approach would implicitly rely on a "choice" which has two elements, the play w_3, and the *node* x. Hence, the node x would be taken as a mere intermediate step which can however only lead to the ultimate outcome w_3, in spite of the fact that $w_2 \in x$. In a sense, such an approach would not clearly distinguish between nodes and outcomes.[5]

Under different formalizations of games it is known that for finite games no-absent-mindedness is characterized by the statement that, for every behavior strategy, there is an equivalent mixed strategy (Ritzberger, 2001, Theorem 3.2, p. 122). Hence, we proceed to show that this property also holds in our setting without additional assumptions. This requires discussing how to model randomizations in large games.

To obtain mixed strategies one may want to define probability measures on pure strategies. Yet, pure strategies are functions, so the set of all pure strategies of a player is a function space, hence potentially large. Still, one would want that every mixed strategy profile induces a well-defined probability measure on plays, that is, on the set W endowed with a σ-algebra $\mathscr{W}$. A minimal requirement on the measurable space $(W, \mathscr{W})$ is, of course, that $\mathscr{W}$ contains at least all nodes $x \in N$. Such a construction may fail, though. Aumann (1964) observed that, if the set of pure strategies is "too large," it is by no means true that a randomization among an arbitrary subset of pure strategies does induce a well-defined probability distribution on outcomes (plays $w \in W$). Indeed, Aumann (1961) provides a characterization showing that probability measures on large enough spaces of strategies will *not* induce measures on outcomes. The following example illustrates some of the difficulties.

[5] Not all violations of no-absent-mindedness contradict the basic idea of choice, i.e. condition (DEF1). Example 15 of Alós-Ferrer and Ritzberger (2005) gives a two-player game violating no-absent-mindedness which fulfills (DEF1) and fails (DEF2) instead.

Example 1 There are two players, 1 and 2, engaged in ultimatum bargaining. Player 1 can propose a split of a surplus, which is any number from the interval $S_1 = [0,1]$. The set S_1 hence coincides with the set of pure strategies of player 1. Player 2 observes the proposal and responds by either accepting (1) or rejecting (0) the proposed split. Thus, the set of pure strategies of player 2 is the set S_2 of all functions of the form $s_2 : [0,1] \to \{0,1\}$. The set of plays is $W = [0,1] \times \{0,1\}$. The requirement on the σ-algebra $\mathscr{W}$ on W is that singletons and sets of the form $\{r\} \times \{0,1\}$ are measurable. This is fulfilled if one takes, for instance, the product of the Borel σ-algebra on $[0,1]$ and the discrete σ-algebra on $\{0,1\}$. The outcome function ϕ is explicitly given by $\phi(s_1, s_2) = (s_1, s_2(s_1))$.

Take a non-Borel set A of $[0,1]$ and consider the indicator function $1_A \in S_2$. This is a pure strategy for player 2. Thus, for a pure decision theorist, no restriction can rule it out. Now suppose player 1 randomizes uniformly over $S_1 = [0,1]$. What is the induced distribution over outcomes? Clearly, the set $[0,1] \times \{1\}$ should be measurable for any reasonable model of the game. But $\phi_1^{-1}([0,1] \times \{1\}) = A$, which is by construction not measurable. Thus the uniform randomization of player 1 (which is a well-defined random variable) does not induce a distribution over the set of outcomes.

Aumann (1964) proposes to bypass this problem by working with an extraneous probability space and viewing randomized strategies as random variables, rather than as distributions. Following this approach, fix a "standard" probability space $(\Omega, \Sigma, \lambda)$, i.e., such that Ω is either finite or countable with the discrete σ-algebra or isomorphic to the unit interval. Endow the product space S of pure strategy profiles with a σ-algebra $\mathscr{S}$ such that the outcome function $\phi : S \to W$ is $(\mathscr{S}, \mathscr{W})$-measurable. Finally, for each player $i \in I$ endow the space S_i of pure strategies with the σ-algebra $\mathscr{S}_i$ given by the projection σ-algebra onto S_i.

A *mixed strategy* for player $i \in I$ then is a $(\Sigma, \mathscr{S}_i)$-measurable function $\sigma_i : \Omega \to S_i$. Denote by M_i the set of all mixed strategies of player $i \in I$ and by $M = \times_{i \in I} M_i$ the space of all mixed strategy profiles. The interpretation of a mixed strategy of player $i \in I$ is that i picks the set $\vartheta \in \mathscr{S}_i$ of pure strategies with probability $\lambda\left(\sigma_i^{-1}(\vartheta)\right)$. By varying the function σ_i the player chooses this probability. Of course, the same caveat as before applies. If S_i is too large as compared to Ω, only "few" pure strategies can be chosen with positive probability.

Another, piecemeal approach to randomized strategies is to allow players to randomize among available choices, independently, at each information set. Formally, for each player $i \in I$ endow the set C_i of choices with a σ-algebra $\mathscr{C}_i$ and let B_i denote the set of all $(\Sigma, \mathscr{C}_i)$-measurable functions $b : \Omega \to C_i$. A *behavior strategy* for player $i \in I$ is a function $\beta_i : X_i \to B_i$, whose values are denoted $b_{ix} = \beta_i(x) : \Omega \to C_i$ for all $x \in X_i$, such that, for all $x, y \in X_i$, (a) $b_{ix}(\Omega) \subseteq A_i(x)$, (b) if $y \in P(b_{ix}(\omega))$ for some $\omega \in \Omega$, then $b_{iy} = b_{ix}$, and (c) if there is no $c \in C_i$ with $x, y \in P(c)$, then $\lambda\left(b_{ix}^{-1}(\vartheta) \cap b_{iy}^{-1}(\vartheta')\right) = \lambda\left(b_{ix}^{-1}(\vartheta)\right) \lambda\left(b_{iy}^{-1}(\vartheta')\right)$ for all $\vartheta, \vartheta' \in \mathscr{C}_i$.

Condition (a) states that if $c \in b_{ix}(\Omega) = \{b_{ix}(\omega) \in C_i \,|\omega \in \Omega\}$, then $c \in A_i(x) = \{c \in C_i \,|x \in P(c)\}$, i.e. $x \in P(c)$, for all $x \in X_i$; that is, it ensures that the random variable b_{ix} is supported on choices that are available at $x \in X_i$. Condition (b) demands that the same random variable b_{ix} is assigned to all moves y in the information set that contains x; hence, the behavior strategy ρ_i does not use more information than what the player has. Finally, condition (c) imposes independence on decisions at distinct information sets. Denote by $\mathcal{B}_i$ the set of all behavior strategies of player $i \in I$, and by $\mathcal{B} = \times_{i\in I}\mathcal{B}_i$ the space of all behavior strategy profiles.

The interpretation of the probability $\lambda\left(b_{ix}^{-1}(\vartheta)\right)$ is as the *conditional* probability that player i takes a choice in the set $\vartheta \in \mathscr{C}_i$ *given* that move $x \in X_i$ has materialized. Player i decides on this conditional probability by choosing the function (random variable) $b_{ix} \in B_i$. By condition (b) these decisions are perfectly correlated across all moves in the information set that contains x, but independent across different information sets by condition (c). Hence, while mixed strategies pick functions from decision points to choices potentially at random, behavior strategies pick choices at each decision point (again potentially at random), independently across different information sets. Because of the independence inherent in behavior strategies, they are, in general, less powerful than mixed strategies are.

Proposition 1 *Let (T, C) be a DEF. If the behavior strategy profile $\beta \in \mathcal{B}$ induces the probability measure $\mu : \mathscr{W} \to [0, 1]$ on the measurable space $(W, \mathscr{W})$, then there exists a mixed strategy combination $\sigma \in M$ that also induces μ.*

Proof Given the behavior strategy profile $\beta \in \mathcal{B}$ define for each player $i \in I$ the function $f_i : X_i \times \Omega \to C_i$ by $f_i(x,\omega) = b_{ix}(\omega) = \beta_i(x)(\omega)$ for all $(x,\omega) \in X_i \times \Omega$, and let $f = (f_i)_{i\in I}$ denote the associated profile. Let $\phi : S \to W$ be the surjection that assigns to each pure strategy combination $s \in S$ the play that it induces, as in (5). This function exists by Theorems 4 and 6 of Alós-Ferrer and Ritzberger (2008). Observe that for each fixed $\omega \in \Omega$ the function $f_i(\cdot,\omega) : X_i \to C_i$ is a pure strategy of player i, i.e. $f(\cdot,\omega) = (f_i(\cdot,\omega)) \in S$. For, (a) guarantees that $f_i(\cdot,\omega)^{-1}(c) \subseteq P(c)$ and (b) ensures that $f_i(\cdot,\omega)^{-1}(c) \supseteq P(c)$ for all $c \in C_i$ and all $i \in I$. Hence, β induces μ if $\mu(V) = \lambda(\{\omega \in \Omega \,|\phi(f(\cdot,\omega)) \in V\})$ for all $V \in \mathscr{W}$. For fixed $\omega \in \Omega$ a mixed strategy profile $\sigma \in M$ is a pure strategy combination, $\sigma(\omega) \in S$. Hence, σ induces μ if $\mu(V) = \lambda(\{\omega \in \Omega \,|\phi(\sigma(\omega)) \in V\})$ for all $V \in \mathscr{W}$.

Given the behavior strategy profile $\beta \in \mathcal{B}$, construct a mixed strategy profile $\sigma \in M$ by setting $\sigma(\omega) = f(\cdot,\omega) \in S$ for each $\omega \in \Omega$. By construction, if β induces μ, then $\sigma \in M$ also does, and the statement is verified. □

This result is made possible by the fact that no-absent-mindedness is an integral part of the approach at hand. Specifically, the proof of Proposition 1 implicitly relies on the fact that in a DEF every play can pass through an information set at most once, viz. no-absent-mindedness.

Proposition 1 is the generalization to large games of one of the implications contained in Kuhn's theorem (Kuhn, 1953), which in its most general form states that behavior strategies and mixed strategies are equivalent if and only if the game satisfies perfect recall (for an alternative approach to the proof of Kuhn's theorem, see von Stengel, 2002, Corollary 4.2). Without perfect recall, the converse of Proposition 1 (i.e., that behavior strategies are as powerful as mixed strategies), does not hold, because of the independence (condition (c)) in the definition of behavior strategies. This can lead to problems in applications, where one often determines optimal choices locally, at each information set separately, and pastes together a solution to the overall game from these local solutions. For instance, an "equilibrium in behavior strategies" may not be an equilibrium at all due to a profitable deviation in mixed strategies, or there may be equilibria in mixed strategies which cannot be reproduced in behavior strategies (see, e.g., the remarkable example by Wichardt, 2008).

This explains the importance of the characterization embodied in Kuhn's theorem. Kuhn (1953) established this result for the finite case only. This was because the general case poses the technical difficulties sketched above. It took over ten years until Aumann (1964) proved a weaker version of Kuhn's theorem for the case of games with perfect recall and an infinite horizon, but restricting the action sets to be homeomorphic to the unit interval. This version states only the sufficiency part of Kuhn's theorem, and it does so only for a given strategy of the opponents. The latter appears acceptable for practical purposes. The former, the necessity of perfect recall, was proved another ten years later by Schwarz (1974) under some additional (mild) measurability assumptions.

6 Conclusions

This paper provides five characterizations of perfect recall. First, it is shown that the known definitions, by Kuhn (1953), Selten (1975), Osborne and Rubinstein (1994), and Perea (2001) are equivalent irrespective of how large the extensive form game is. Second, a new characterization of perfect recall is obtained that relates it to a basic tree-property, called "Trivial Intersection." Finally, we show that even without perfect recall mixed strategies always do at least as well as behavior strategies. This is because in the present set-up, that follows von Neumann and Morgenstern (1944), the weaker condition of "no-absent-mindedness" always holds.

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