SEARCHING FOR KNIGHTS AND SPIES: A MAJORITY/MINORITY GAME

MARK WILDON

ABSTRACT. There are n people, each of whom is either a knight or a spy. It is known that at least k knights are present, where $n/2 < k < n$. Knights always tell the truth. We consider both spies who always lie and spies who answer as they see fit. This paper determines the minimum number of questions required to find a spy or prove that everyone in the room is a knight. We also determine the minimum number of questions needed to find at least one person's identity, or a nominated person's identity, or to find a spy (under the assumption that a spy is present). For spies who always lie, we prove that these searching problems, and the problem of finding a knight, can be solved by a simultaneous optimal strategy. We also give some computational results on the problem of finding all identities when spies always lie, which show that a plausible suggestion made by Aigner is false. We end by stating some open problems. The questioning strategies in the paper have applications to fault finding in distributed computing networks.

1. INTRODUCTION

In a room there are n people, numbered from 1 up to n. Each person is either a *knight* or a *spy*, and will answer any question of the form

'Person x, is Person y a knight?'

Knights always answer truthfully. We consider two types of spy: *liars*, who always lie, and *moles*, who lie or tell the truth as they see fit. We work in the adaptive model in which future questions may be chosen in the light of the answers to earlier questions. We always assume that knights are in a strict majority, since otherwise, even if every permitted question is asked, it may be impossible to be certain of anyone's identity.

In this paper we determine the minimum number of questions that are necessary and sufficient to find a spy, or to find at least one person's identity, or to find an identity of a specific person, nominated in advance. These quantitative results are complemented by two more qualitative theorems that determine the extent to which the problems considered in this paper,

Date: May 7, 2016.

2010 *Mathematics Subject Classification.* 05C57; Secondary 91A43, 91A46

Email: mark.wildon@rhul.ac.uk, *Tel:* +44(0)1784414021.

Key words and phrases. Majority game, minority game, knight, spy, combinatorial searching.

and the already solved problems of finding a knight, or finding everyone's identity, admit a common optimal solution. In the final section we state some open problems suggested by the five main theorems. We also present some computational results on the problem of finding all identities when the spies are liars. A recurring theme is that an early answer of 'no' (claiming that Person y is a spy) is very helpful when searching for spies, since either Person x or Person y must be a spy.

Problems of this type arise in multiprocessor computer systems, and in distributed computing; in either case the reliability of a computational unit must be discovered by queries made by other, potentially unreliable, computational units. This problem was first considered for non-adaptive queries in [9], and for adaptive queries in [8]; for some recent results on the problem where the network graph is incomplete, and further references, see [7]. The questioning strategies used in this paper, in particular, the Binary Spy Hunt in §3, have obvious applications in this setting. Another intriguing extension permits self-referential questions: this is considered in [6].

We work in the general setting, also considered in [1], where it is known that at least k of the n people are knights, where $n/2 < k < n$. Throughout this paper n and k have these meanings. When all spies are liars, let

- $T_L(n, k)$ be the minimum number of questions that are necessary and sufficient either to identify a liar, or to make a correct claim that everyone in the room is a knight;
- $T^\star_L(n, k)$ be the minimum number of questions that are necessary and sufficient to identify a liar, if it is known that at least one liar is present.

Our first main theorem is proved in §3.

Theorem 1. Let $n = q(n - k + 1) + r$ where $0 \le r \le n - k$. Then

$$T_L(n,k) = \begin{cases} n-q+1 & \text{if } r = 0 \\ n-q & \text{if } r = 1 \\ n-q & \text{if } r \ge 2 \end{cases}$$

and

$$T^\star_L(n,k) = \begin{cases} n-q & \text{if } r = 0 \\ n-q & \text{if } r = 1 \\ n-q-1 & \text{if } r \ge 2. \end{cases}$$

with the single exception that $T^\star_L(5, 3) = 4$.

In particular, we have $T_L(n, k) = T^\star_L(n, k) + 1$ except when $(n, k) = (5, 3)$ or $n = q(n - k + 1) + 1$ for some $q \in \mathbf{N}$; in these cases equality holds. The proof of the lower bound needed for Theorem 1 has some features in common with Theorem 4 in [1]: we connect these results in §8.1.

Let $T^\star_S(n,k)$ and $T_S(n,k)$ be the analogously defined numbers if all spies are moles. We prove the following theorem in §6.

Theorem 2. We have $T^\star_S(n,k) = n-1$ and $T_S(n,k) = n$.

Note that, in contrast to $T^\star_L(n,k)$ and $T_L(n,k)$, the numbers $T^\star_S(n,k)$ and $T_S(n,k)$ are independent of k.

To state the third main theorem we must introduce eight further numbers. When all spies are liars, let

- $K_L(n,k)$ be the minimum number of questions that are necessary and sufficient to find a knight;
- $E_L(n,k)$ be the minimum number of questions that are necessary and sufficient to find at least one person's identity;
- $N_L(n,k)$ be the minimum number of questions that are necessary and sufficient to identify Person 1.

Let $K_S(n,k)$, $E_S(n,k)$ and $N_S(n,k)$ be the analogously defined numbers when all spies are moles. Let $N^\star_L(n,k)$ and $N^\star_S(n,k)$ be the analogously defined numbers to $N_L(n,k)$ and $N_S(n,k)$ when it is known that a spy is present. Let $B(s)$ be the number of 1s in the binary expansion of $s \in \mathbf{N}$. In §4 we prove the following theorem.

Theorem 3. We have

$$K_S(n,k) = K_L(n,k) = E_S(n,k) = E_L(n,k) = 2(n-k) - B(n-k)$$

and the same holds for the analogous numbers defined on the assumption that a spy is present. Moreover

$$N_S(n,k) = N_L(n,k) = N^\star_S(n,k) = N^\star_L(n,k) = 2(n-k) - B(n-k) + 1.$$

with the exception that $N^\star_L(n,k) = 2(n-k) - B(n-k) = n-2$ when $n = 2^{e+1}+1$ and $k = 2^e + 1$ for some $e \in \mathbf{N}$.

The numbers $K_S(n,k)$ and $K_L(n,k)$ have already been studied. If all spies are liars then one person supports another if and only if they are of the same type and accuses if and only if they are of different types. Therefore finding a knight is equivalent to the *majority game* of identifying a ball of a majority colour in a collection of n balls coloured with two colours, using only binary comparisons between pairs of balls that result in the information 'same colour' or 'different colours'. For an odd number of balls, the relevant part of Theorem 3 is that $K_L(2k-1,k) = 2(k-1) - B(k-1)$. This result was first proved by Saks and Werman in [10]. A particularly elegant proof was later given by Alonso, Reingold and Schott in [3]. In Theorem 6 of [1], Aigner adapts the questioning strategy introduced in [10] to show that $K_S(n,k) \le 2(n-k) - B(n-k)$. We recall Aigner's questioning strategy and the proof of this result in §2 below. Aigner also claims a proof, based on Lemma 5.1 in [11], that $K_L(n,k) \ge 2(n-k) - B(n-k)$. A flaw in these proofs was pointed out in [5], and a correct proof was given. It is obvious that $K_S(n,k) \geq K_L(n,k)$, so it follows that $K_S(n,k) = K_L(n,k) = 2(n-k) - B(n-k)$, giving part of Theorem 3.

It is natural to ask whether when there are questioning strategies that solve the searching problems considered in Theorems 1, 2 and 3 simultaneously, using the minimum number of questions for each. In §5 we prove that, perhaps surprisingly, there is such a strategy when all spies are liars. Define $K(n,k) = 2(n-k) - B(n-k)$.

Theorem 4. Suppose that spies always lie. There is a questioning strategy that will find a knight by question $K(n,k)$, find Person 1's identity by question $K(n,k)+1$ and by question $T_L(n,k)$ either find a spy or prove that everyone in the room is a knight. Moreover if a spy is known to be present then a spy will be found by question $T_L^\star(n,k)$.

We also show in §5 that by asking further questions it is possible to determine all identities by question $n-1$. In the important special case where $n = 2k-1$, so all that is known is that knights are in a strict majority, there are 2^{n-1} possible sets of spies, and so $n-1$ questions are obviously necessary to determine all identities. In this case, all five problems admit a simultaneous optimal solution. We make some further remarks on finding all identities when all spies are liars, and ask a natural question suggested by Theorem 4, in the final section of this paper.

When all spies are moles it is impossible in general to solve the four problems by a single strategy. The following theorem, proved in §7, shows one obstruction.

Theorem 5. Suppose that all spies are moles. There is a questioning strategy that will find a knight by question $K(n,k)+1$, find Person 1's identity by question $K(n,k)+2$, and by question $T_S(n,k) = n$ either find a spy, or prove that everyone in the room is a knight. Moreover, if a spy is known to be present, then a spy will be found by question $T_S^\star(n,k) = n-1$. When $n = 7$ and $k = 4$ and a spy is known to be present there is no questioning strategy that will both find a knight by question $K_S(7,4) = 4$ and find a spy by question $T_S^\star(7,4) = 6$.

There is an adversarial game associated to each of our theorems, in which questions are put by an *Interrogator* and answers are decided by a *Spy Master*, whose task is to ensure that the Interrogator asks at least the minimum number of questions claimed to be necessary. We shall use this game-playing setup without further comment. We represent positions part-way through a game by a *question graph*, with vertex set $\{1,2,\ldots,n\}$, in which there is a directed edge from x to y if Person x has been asked about Person y, labelled by Person x's reply. If Person x answers 'yes', claiming that Person y is a knight, we say that Person x *supports* Person y. Otherwise Person x answers 'no' and we say that Person x *accuses* Person y. In figures, accusations are shown by dashed arrows and supportive statements by solid arrows. We rule out loops by making the simplifying assumption that no-one is ever asked for his own identity: such questions are clearly pointless.

Finally we make an important general observation that will be used many times below: each question (in any questioning strategy) creates one new edge in the question graph, and so reduces the number of components of the question graph by at most one; hence at least $n - c$ questions are required to form a question graph having c or fewer components. Moreover, if the question graph is a forest, then it has c components if and only if exactly $n - c$ questions have been asked.

Outline. We remind the reader of the structure of the paper: §2 gives a basic strategy for finding a knight. In §3, §4 and §5 we prove Theorems 1, 3 and 4. In Theorems 1 and 4 all spies are liars, and this is also the most important case for Theorem 3. In §6 and §7 we prove Theorems 2 and 5 on the case when all spies are moles. In §8 we give some computational results and state some open problems.

2. Binary Knight Hunt

This questioning strategy was introduced in Theorem 6 of [1]. (The present name is the author's invention.) We shall use variants of it in the proofs of Theorems 1, 3, 4 and 5. Figure 1 overleaf gives an example of this strategy. See also Example 5.1 for its use in the strategy used to prove Theorem 4 and [2, page 24] for an alternative exposition.

Strategy (Binary Knight Hunt). The starting position is a set P of people, none of whom has been asked a question or asked about. After each question, the subgraph of the question graph in P is a forest, and every component C of the question graph that is contained in P has a unique *sink vertex* which can be reached by a directed path from any other vertex in C.

- If the components in P in which no accusation has been made all have different sizes, the strategy terminates.
- Otherwise, the Interrogator chooses two components C and C' in P of equal size in which no accusation has been made. If C has sink vertex x and C' has sink vertex x', then he asks Person x about Person x', forming a new component with sink vertex x'.

We call components in which an accusation has been made *accusatory*. Since anyone who supports a spy (either directly, or via a directed path of supportive edges) is a spy, and each accusatory component is formed by connecting two sink vertices in components of equal size having no accusations, each accusatory component contains at least as many spies as knights.

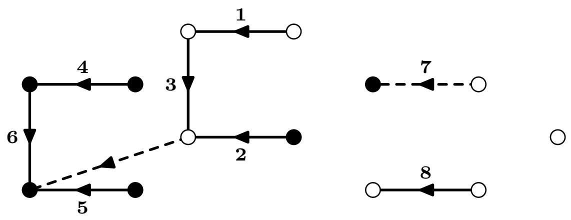

Figure 1. An example of the Binary Knight Hunt in a room of 7 knights, shown by white dots, and 6 moles, shown by black dots. Questions are shown by numbered edges using the convention described at the end of the introduction.

The Binary Knight Hunt is immediately effective when knights are in a strict majority in P. Note that after each question, each component in P has size a power of two. Suppose that when the strategy terminates, there are non-accusatory components of distinct sizes $2^{b_1}, 2^{b_2}, \ldots, 2^{b_u}$ where $b_1 < \ldots < b_u$. Since there are at least as many spies as knights in each accusatory component, and $2^{b_u} > 2^{b_1} + \cdots + 2^{b_{u-1}}$, the person corresponding to the sink vertex of the component of size 2^{b_u} must be a knight. If the accusatory components have sizes $2^{a_1}, \ldots, 2^{a_t}$ and $m = |P|$ then

$$m = 2^{b_1} + \cdots + 2^{b_u} + 2^{a_1} + \cdots + 2^{a_t}.$$

Hence $t + u \geq B(m)$. Since the subgraph of the question graph in P is a forest, the number of questions asked is $n-(t+u) \leq m-B(m)$. In the usual room of n people known to contain at least k knights, any set of $2(n-k)+1$ people has a strict majority of knights. Thus, as proved by Aigner in [1, Theorem 6], $2(n-k)-B(n-k)$ questions suffice to find a knight, even when all spies are moles.

3. Proof of Theorem 1

Throughout this section we suppose that all spies are liars. Let $s = n-k$. By hypothesis there are at most s liars in the room and $n = q(s+1) + r$, where $0 \leq r \leq s$. Each component C of the question graph has a partition Y, Z, unique up to the order of the parts, such that the people in Y and Z have opposite identities. Choosing Y and Z so that $|Y| \geq |Z|$, we define the *weight* of C to be $|Y| - |Z|$. The multiset of component weights then encodes exactly the same information as the 'state vector' in [1, page 5] or the 'game position' in [5, Section 2], [10, page 384] and [11, Section 3]. The following lemma also follows from any of these papers, and is proved here only for completeness.

Lemma 3.1. *Suppose that all spies are liars. Let C and C' be components in a question graph of weights c, c' respectively, where $c \geq c' \geq 1$. Let Persons v and v' be in the larger parts of the partitions C and C', respectively. Suppose that Person v is asked about Person v', forming a new component $C \cup C'$.*

If Person v supports Person v' then the weight of $C \cup C'$ is $c + c'$ and if Person v accuses Person v' then the weight of $C \cup C'$ is $c - c'$.

Proof. Let Y, Z and Y', Z' be the unique partitions of C and C' respectively such that the people in Y and Z have opposite identities, the people in Y' and Z' have opposite identities, and $|Y| - |Z| = c$, $|Y'| - |Z'| = c'$. By assumption $v \in Y$ and $v' \in Y'$. The unique partition of $C \cup C'$ into people of opposite identities is $Y \cup Y'$, $Z \cup Z'$ if Person v supports Person v', and $Y \cup Z'$, $Z \cup Y'$ if Person v accuses Person v'. The lemma follows. □

3.1. Lower bounds. It will be convenient to say that a component in the question graph is *small* if it contains at most s people. If no accusations have been made, then the identities of people in a small component are ambiguous.

Suppose that it is not known whether a liar is present. The Spy Master should answer the first $n - q - 1$ questions asked by the Interrogator with supportive statements. After question $n - q - 1$ there are at least $q + 1$ components in the question graph, of which at least one is small. Hence $T_L(n, k) \geq n - q$. Moreover, if $r = 0$ then, after question $n - q - 1$, there are at least two small components, say C and C'. If question $n - q$ connects C and C' then the Spy Master should accuse; it is then ambiguous which of C and C' consists of liars. Otherwise he supports, and a small component remains. In either case at least one more question is required, and so $T_L(n, k) \geq n - q + 1$ when $r = 0$.

The proof is similar if it is known that a liar is present. The Spy Master answers the first $n-q-2$ questions with supportive statements. This leaves at least $q + 2$ components in the question graph. If $r = 0$ or $r = 1$ then at least three of these components are small, and otherwise at least two are small. The Interrogator is unable to find a liar after $n - q - 2$ questions. Hence $T^\star_L(n, k) \geq n - q - 1$. Now suppose that $r = 0$ or $r = 1$. If question $n - q - 1$ is between two small components then the Spy Master should accuse; otherwise he supports. As before, at least one more question is needed. Hence $T^\star_L(n, k) \geq n - q$ in these cases.

3.2. Upper bound when $n \neq 2s+1$. We start with a questioning strategy which allows the Interrogator to find a knight while keeping the components in the question graph small. See Example 5.1 for an example of the strategy in this context of Theorem 4.

Strategy (Switching Knight Hunt). Let $d \in \mathbf{N}$. Let $c_1, \ldots, c_d \in \mathbf{N}$ be such that $c_{j+1} > c_1 + \cdots + c_j$ for each $j \in \{1, \ldots, d-1\}$. The starting position for a Switching Knight Hunt is a question graph G having distinguished components $C_1, C_2, \ldots, C_d$ and $C'_2, \ldots, C'_d$ such that

(a) C_1 has weight c_1,
(b) both C_i and C'_i have weight c_i for all $i \in \{2, \ldots, d\}$,

(c) each C_i has a vertex p_i and each C'_i has a vertex p'_i, both in the larger part of the partitions defining the weights of these components.

Set $b = 1$.

Step 1. If $b = d$ then terminate. Otherwise, ask Person p_{b+1} about Person p_b, then Person p_{b+2} about Person p_{b+1}, and so on, stopping either when an accusation is made, or when Person p_d supports Person p_{d-1}. In the latter case the strategy terminates. If Person p_{b+j} accuses Person p_{b+j-1} then replace b with $b + j$ and go to Step $1'$.

Step $1'$. The specification of the algorithm for this step is obtained from Step 1 by replacing all appearances of p_i with p'_i, and instead going to Step 1 after an accusation.

Lemma 3.2. *Let P be a set of knights and liars in which knights are in a strict majority. Suppose that P has components $C_1, \ldots, C_d$ and $C'_2, \ldots, C'_d$ satisfying the conditions for a Switching Knight Hunt. Let $X = C_1 \cup \cdots \cup C_d$ and $X' = P \backslash X$. Suppose that X' is a union of components of the question graph and that the components in X' other than $C'_2, \ldots, C'_d$ have total weight at most $c_1 - 1$. Let G be the question graph when a Switching Knight Hunt terminates. If the strategy terminates in Step* 1 *then Person p_d is a knight, and if the strategy terminates in Step* $1'$ *then Person p'_d is a knight. Moreover each component in G is either contained in X or contained in X'.*

Proof. We suppose that the Switching Knight Hunt terminates in Step 1. (This happens when either $d = 1$, or Person p_d supports Person p_{d-1}, or Person p'_d accuses Person p'_{d-1} and there is a final switch.) The proof in the other case is symmetric. Suppose that Persons $p_{u_1}, p_{u'_2}, \ldots, p_{u_{2t-1}}, p_{u'_{2t}}$ make accusations, where $u_1 < u'_2 < \ldots < u_{2t-1} < u'_{2t}$. Set $u'_0 = 1$. After the final question, the component of G containing Person p_{u_i} is contained in X and, by Lemma 3.1, has weight $c_{u_i} - (c_{u_i-1} + \cdots + c_{u'_{i-1}})$. Similarly the component of G containing Person $p_{u'_i}$ is contained in X' and has weight $c_{u'_i} - (c_{u'_i-1} + \cdots + c_{u_{i-1}})$.

Fix $i \in \{1, \ldots, t\}$ and let $u'_{2i-2} = \alpha$, $u_{2i-1} = \beta$ and $u'_{2i} = \gamma$. The difference between the number of knights and the number of liars in the components $C_{\gamma-1}, \ldots, C_\alpha$ and $C'_\gamma, \ldots, C'_{\alpha+1}$ of the original question graph is at most

$$\begin{aligned} &c_{\gamma-1} + \cdots + c_{\beta+1} + \bigl(c_\beta - (c_{\beta-1} + \cdots + c_{\alpha+1} + c_\alpha)\bigr) \\ &\qquad + \bigl(c_\gamma - (c_{\gamma-1} + \cdots + c_{\beta+1} + c_\beta)\bigr) + c_{\beta-1} + \cdots + c_{\alpha+1}, \end{aligned}$$

where the top line shows contributions from components in X, and the bottom line contributions from components in X'. This expression simplifies to $c_\gamma - c_\alpha$. Hence the difference between the number of knights and the number of liars in all components of G contained in P *except* for the component containing Person p_d, is at most

$$\sum_{i=1}^{t}(c_{u'_{2i}} - c_{u'_{2i-2}}) + (c_d + \cdots + c_{u'_{2t}+1}) + (c_1 - 1) = c_d + \cdots + c_{u'_{2t+1}} + c_{u'_{2t}} - 1$$

where the second two summands on the left-hand side come from components in X'. The component containing Person p_d has weight $c_d + \cdots + c_{u'_{2t}}$. If the people in the larger part of this component are liars then liars strictly outnumber knights in P, a contradiction. Hence Person p_d is a knight. $\square$

We remark that in some cases, depending on the structure of the components C_i and C'_i, and provided Persons p_i and p'_i are chosen appropriately, the Switching Knight Hunt may be effective even when spies are moles. For example, this is the case in Example 5.1.

We are now ready to give a questioning strategy that meets the targets set for $T^\star_L(n,k)$ and $T_L(n,k)$ in Theorem 1. In outline: the Interrogator finds a knight in $K(n,k)$ questions while also attempting to create q components of size $s+1$ or more, each with no accusatory edges. If he fails in creating these components it is because of an earlier accusation; asking the knight about the accuser then identifies a liar. Remarks needed to show that the strategy is well-defined are given in square brackets.

Strategy (Binary Spy Hunt). Take a room of n people known to contain at most s liars where $2(s+1) \le n$. Let $2^{a_1}+2^{a_2}+\cdots+2^{a_d}$ where $d = B(s+1)$ and $a_1 < \ldots < a_d$ be the binary expansion of $s+1$.

Phase 1. Choose disjoint subsets X, $X' \subseteq \{1, 2, \ldots, n\}$ such that $|X| = s+1$ and $|X'| = s$. Perform a Binary Knight Hunt in X and then perform a Binary Knight Hunt in X'. [The questions asked are consistent with an incomplete Binary Knight Hunt in $X \cup X'$.]

(i) If an accusation has been made, complete a Binary Knight Hunt in $X \cup X'$. Then choose any person, say Person z, who made an accusation in Phase 1, and terminate after asking the knight just found about Person z.

(ii) If all answers so far have been supportive, go to Phase 2.

Phase 2. [The components of the question graph in X have sizes $2^{a_1}, \ldots, 2^{a_d}$. Since $s = (1 + \cdots + 2^{a_1-1}) + 2^{a_2} + \cdots + 2^{a_d}$, there are components in X' of sizes $2^{a_2}, \ldots, 2^{a_d}$. No accusations have been made so far, hence the size of each component is equal to its weight.] Perform a Switching Knight Hunt in $X \cup X'$ and go to Phase 3.

Phase 3. [By Lemma 3.2, each component of the question graph is either a singleton, or contained in X or contained in X'. There are $(q-2)(s+1)+r+1$ singleton components not contained in $X \cup X'$.] Let Person w be the knight found at the end of Phase 2. Ask questions to create $q-1$ components of size $s+1$, one component of size s, and $r+1$ singleton components. At the first accusation, stop building components and ask Person w about the person who made the accusation. Then terminate. (Thus Phase 3 ends after one question if there was an accusation in Phase 2.) If no accusations are made, go to Phase 4.

Phase 4. Let Persons $x_1, \ldots, x_{r+1}$ be the singleton components of the graph. Let Person y be in the component of size s.

- If $r = 0$ then ask Person w about Person x_1. If he accuses, Person x_1 is a liar. If he supports, and a liar is known to be present, then Person y is a liar. Otherwise asking Person w about Person y either shows that Person y is a liar, or proves that no liars are present.
- If $r \geq 1$ then ask Person y about Person x_{r+1}. If he accuses then asking Person w about Person y identifies a liar. Otherwise ask Person w about Persons $x_1, \ldots, x_{r-1}$. Any accusation identifies a liar. Suppose that all these people turn out to be knights. If a liar is known to be present, then Person x_r is a liar; otherwise asking Person w about Person x_r either shows that Person x_r is a liar, or proves that no liars are present.

In Example 5.1 the Binary Spy Hunt is shown ending in Phase 4.

Lemma 3.3. *Let $s = n - k$ and suppose that $n \geq 2(s+1)$. Assume that all spies are liars. Suppose that a Binary Spy Hunt is performed in the room of n people. If a liar is known to be present, then a liar is found after at most $T^\star_L(n,k)$ questions. Otherwise, after $T_L(n,k)$ questions, either a liar is found, or it is clear that no liar is present.*

Proof. At the beginning of Phase 4 the question graph has $q + r + 1$ components. Hence if an accusation is made in an earlier phase then, after the accusation, the question graph has at least $q+r+1$ components. Therefore, after Person w is used to identify the accuser, there are at least $q + r$ components. This question identifies either the accuser or the person accused as a liar, and so a liar is found after at most $n - q - r$ questions. This meets the targets in Theorem 1.

Suppose the strategy enters Phase 4. If a liar is known to be present then the Interrogator asks 1 question if $r = 0$, at most 2 questions if $r = 1$, and at most r questions if $r \geq 2$. The final numbers of components are at least q, q and $q + 1$, respectively. If a liar is not known to be present then the Interrogator asks at most 2 questions if $r = 0$, exactly 2 questions if $r = 1$, and at most $r+1$ questions if $r \geq 1$. The final numbers of components are at least $q - 1$, q and q, respectively. This meets the targets in Theorem 1. □

This completes the proof of Theorem 1 in the case $n \neq 2s + 1$. For later use in the proof of Theorem 4 in §5 we record the following result on the Binary Spy Hunt.

Proposition 3.4. *Suppose that all spies are liars and that a Binary Spy Hunt enters Phase 2. A knight is found at the end of Phase 2 after exactly $K(n,k)$ questions.*

Proof. Let $s = n - k$ and suppose as before that $s+1 = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_d}$ where $d = B(s+1)$ and $a_1 < \ldots < a_d$. Let X and X' be the subsets of size $s+1$ and s, respectively, chosen in the strategy. After the end of Phase 2 of the Binary Spy Hunt, in the quotient of the question graph obtained by identifying Persons p_i and p_i' for $i \in \{2, \ldots, d\}$, the images of the vertices $p_d, \ldots, p_1$ form a directed path of length d. No edge in this path comes from Phase 1. Hence exactly $d-1$ questions are asked in Phase 2. The numbers of components after Phase 1 in X and X' are d and $d-1+a_1$, respectively; thus after Phase 2, the number of components in $X \cup X'$ is $d + a_1$. Hence the number of questions asked in Phases 1 and 2 is

$$2s + 1 - (d + a_1) = 2s - B(s)$$

which equals $K(n,k)$, as required. □

3.3. Upper bound when $n = 2s+1$. The remaining case when $n = 2s+1$ has a number of exceptional features. When $n = 3$, it is clear that a single question cannot identify a liar, while any two distinct questions will, so $T_L(3,2) = T_L^\star(3,2) = 2$, as required. When $n = 5$ and $s = 2$, the Spy Master should support on his first answer. He may then choose his remaining answers so that the (undirected) question graph after three questions appears in Figure 2 below. In each case a liar must be present, it is consistent that the spies lied in every answer, and no liar can be identified without asking one more question. Hence $T_L^\star(5,3) = 4$ and $T_L(5,3) = 4$.

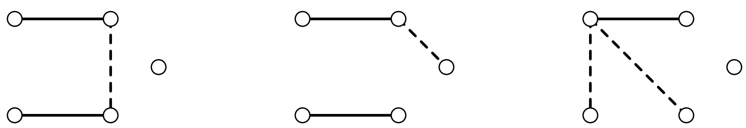

Figure 2. Undirected question graphs after four questions when $n = 5$ and $k = 3$ with optimal play by the Spy Master.

Now suppose that $s \geq 3$. The lower bound proved in §3.1 shows that $T_L^\star(2s+1, s+1) \geq 2s - 1$ and $T_L(2s+1, s+1) \geq 2s$.

We saw in §2 that a Binary Knight Hunt will find a knight, say Person w, after at most $2s - B(s)$ questions. At this point the question graph is a forest. If s is not a power of two then, since $B(s) \geq 2$, the Interrogator can ask Person w further questions until exactly $2s-2$ questions have been asked, choosing questions so that the question graph remains a forest. Suppose that after question $2s-2$ the components in the question graph are X, Y and Z, where X is the component containing Person w. If an accusation has been made by someone in X, then a liar is known. Moreover, if an accusation has been made by someone in Y or Z, then asking Person w about this person will identify a liar. Suppose that no accusations have been made. If it is known that a liar is present then Person w will support a person in component Y if and only if everyone in component Z is a liar, and so one further question suffices to find a liar. Otherwise, two questions asked to Person w about people in components Y and Z will find all identities.

The remaining case is when $s = 2^e$ where $e \geq 2$. It now requires $2s - B(s) = 2s - 1$ questions to find a knight using a Binary Knight Hunt. One further question will connect the two remaining components in the question graph, finding all identities in $T_L(2s+1, s+1) = 2s$ questions. Suppose now that a liar is known to be present. Then the danger is that, as in the question graphs shown in Figure 2, after asking the target number of $2s-1$ questions, the Interrogator succeeds in identifying a knight, but not a spy. When $s = 4$ this trap may be avoided using the following lemma.

Lemma 3.5. $T_L^\star(9, 5) \leq 7$.

Proof. The table in Figure 3 below shows the sequence of questions the Interrogator should ask, together with an optimal sequence of replies from the Spy Master. The final column gives the continuation if the Spy Master gives the opposite answer to the one anticipated in the main line. (The further questions in these cases are left to the reader.) It is routine to check that in every case the Interrogator finds a liar after at most seven questions. □

Components of question graph	Question	Anticipated answer	Continuation for opposite answer
$\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$	$(1,2)$	Support*	$(3,4)$ BKH
$\{1,2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$	$(1,3)$	Support	$(4,5)$ BKH
$\{1,2,3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$	$(4,5)$	Support	$(1,6)$
$\{1,2,3\}$, $\{4,5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$	$(4,6)$	Accuse	$(1,4)$
$\{1,2,3\}$, $\{4,5\} \mid \{6\}$, $\{7\}$, $\{8\}$, $\{9\}$	$(4,7)$	Support*	$(1,5)$
$\{1,2,3\}$, $\{4,5,7\} \mid \{6\}$, $\{8\}$, $\{9\}$	$(1,8)$	Accuse	$(1,4)$
$\{1,2,3\} \mid \{8\}$, $\{4,5,7\} \mid \{6\}$, $\{9\}$	$(1,9)$	Support: Person 8 is a spy Accuse: Person 6 is a spy	

FIGURE 3. In a room of nine people, each either a knight or a liar, seven questions suffice to find a liar. The question 'Person x, is Person y a knight?' is shown by (x, y). Components of the question graph known to contain a spy are shown by $X \mid Y$ where the people in X and Y have opposite identities. Answers marked $\star$ are the unique optimal replies by the Spy Master. The abbreviation BKH indicates that the continuation is a Binary Knight Hunt. (If the second question results in an accusation, regard the component $\{1, 2, 3\}$ of weight 1 as the singleton $\{2\}$.)

Now suppose that $s = 2^e$ where $e \geq 3$. Let

$$\{1, 2, \ldots, 2s + 1\} = X_1 \cup X_2 \cup \cdots \cup X_8 \cup \{2s + 1\}$$

where the union is disjoint and $|X_i| = 2^{e-2}$ for all i. The Interrogator should start by performing a separate Binary Knight Hunt in each X_i. Suppose that an accusation is made, say when two components both of size 2^f are connected. Let Y be the set of people not in either of these components. The questions asked so far in Y form an incomplete Binary Knight Hunt in Y. Since $|Y| = 2(2^e - 2^f) + 1$, a knight may be found after

$$2(2^e - 2^f) - B(2^e - 2^f)$$

further questions. Asking this knight about a person in the accusatory component of size 2^{f+1} identifies a liar. The total number of questions asked is $2^{f+1} - 1 + 2(2^e - 2^f) - B(2^e - 2^f) + 1 = 2^{e+1} - B(2^e - 2^f) = 2s - B(2^e - 2^f)$. Since $f \leq e - 3$, this is strictly less than $2s - 1$.

If no accusations are made then, after the eight Binary Knight Hunts are performed, each X_i is a connected component of the question graph containing 2^{e-2} people of the same identity. There is also a final singleton component containing Person $2s + 1$. Let Person p_i belong to X_i for each i, and let $p_9 = 2s + 1$. It is routine to check that replacing i with p_i in the question strategy shown in Figure 3 will now find a liar in at most 7 more questions, leaving a final question graph with at least two components.

4. Proof of Theorem 3

Suppose that all spies are liars and that G is a question graph in which the Interrogator can correctly claim that Person x is a liar. Let C be the component containing Person x and let X, Y be the unique partition of C into people of different types, chosen so that $x \in X$. If $|X| \geq |Y|$ then, given any assignment of identities to the people in the room that makes the people in X liars, we can switch knights and liars in component C to get a new consistent assignment of identities. Hence $|X| \leq |Y|$ and the people in Y must be knights. Thus $E_L(n, k) = K_L(n, k)$. Since

$$E_L(n, k) \leq E_S(n, k) \leq K_S(n, k)$$

is obvious and $K_S(n, k) = K_L(n, k)$ was seen in the introduction, it follows that $E_S(n, k) = K_S(n, k) = E_L(n, k) = K_L(n, k)$. Since $K(n, k) = 2(n - k) - B(n - k) \leq n - 2$, there is a person not involved in any question by question $K(n, k)$. Therefore the same result holds for the corresponding quantities defined on the assumption that a liar is present.

For the next part of Theorem 3 we must recall a basic result on the reduction of the majority game to multisets of weights.

Lemma 4.1. *Suppose that a question graph has components $C_1, \ldots, C_d$. Let c_i be the weight of C_i and let $c_1 + \cdots + c_d = 2s + e$ where $e = k - (n - k)$*

and $s \in \mathbf{N}_0$. The identities of the people in component C_i are unambiguous if and only if $c_i \geq s+1$.

Proof. See [1, Equation (14)] or [5, Section 2]. □

Let $t = K(n,k)$. It is clear that $N_L(n,k) \leq N_S(n,k) \leq K_S(n,k)+1$, so to show that $N_L(n,k) = N_S(n,k) = t+1$, it suffices to show that $N_L(n,k) \geq t+1$. The Spy Master can ensure that after $t-1$ questions the Interrogator is unable to identify a knight. Suppose one of the first t questions forms a cycle in the question graph. By the remarks on the majority game following the statement of Theorem 3, this question is redundant from the point of view of finding a knight. The results already proved in this section show that no identities can be found until a knight is found. We may therefore assume that the question graph after question t is a forest.

Let $e = k-(n-k)$. Define $s \in \mathbf{N}$ so that the sum of the weights of the components of the question graph after question $t-1$ is $2s+e$. (Thus $s = n-k$ if and only if there are no accusations in the first $t-1$ answers.) By Lemma 4.1 each component has weight at most s. Suppose that on question t the Interrogator asks a person in component C about a person in component C'. Let c be the weight of C and let c' be the weight of C'. By the reduction to the majority game, we may assume that $c \geq c'$.

(i) If $1 \not\in C \cup C'$ then the Spy Master supports. The weight of the component containing Person 1 is unchanged, so by Lemma 4.1, the identity of Person 1 is still ambiguous.
(ii) If $1 \in C \cup C'$ then the Spy Master accuses. By Lemma 3.1, the weight of the new component containing Person 1 is $c-c'$. The sum of all component weights is now $2(s-c')+e$, and we have $c-c' \leq s-c'$. By Lemma 4.1 the identity of Person 1 is still ambiguous.

Hence $N_L(n,k) = N_S(n,k) = t+1$, as required. In case (ii) a spy is clearly present. In case (i), the question graph after question t has a component $C \cup C'$ not containing Person 1. If all spies are moles then a source vertex in this component may be a spy. Hence $N^\star_S(n,k) = t+1$. Moreover, unless $n = 2^{e+1}+1$ and $k = 2^e+1$ for some $e \in \mathbf{N}$ we have $t+1 \leq n-2$, and so after question $t+1$ there is person not yet involved in any question, implying that $N^\star_L(n,k) = t+1$. The proof of Theorem 3 is completed by the following lemma which deals with the exceptional case when $t = n-2$.

Lemma 4.2. *If $n = 2^{e+1}+1$ and $k = 2^e+1$ then $N^\star_L(n,k) = n-2$.*

Proof. The Interrogator performs a Binary Knight Hunt using Persons 2 up to n. If there is no accusation on or before question $n-2$ then Person 1 is a liar. Suppose that the first accusation occurs when two components of size 2^f are connected. If $f = e$ then Person 1 is identified as a knight after $n-2$ questions. Otherwise, ignoring the new component of weight 0, the new multiset of component weights is consistent with a Binary Knight Hunt in a room of $2^{e+1} - 2^{f+1} + 1$ people, known to contain at least $2^e - 2^f + 1$ knights. A knight may therefore be found after at most

$$(2^{f+1} - 1) + 2(2^e - 2^f) - B(2^e - 2^f) = 2^{e+1} - B(2^e - 2^f) - 1 \leq n - 3$$

questions, and Person 1's identity found by question $n - 2$. □

5. Proof of Theorem 4

Let $s = n - k$. We must deal with the cases $n \geq 2(s + 1)$ and $n = 2s + 1$ separately.

Proof when $n \geq 2(s+1)$. Let 2^a be the greatest power of two such that $2^a \leq s + 1$. Perform a Binary Spy Hunt, as described in §3.2, choosing the sets X and X' of sizes $s + 1$ and s, respectively, so that $1 \in X$. Whenever permitted in the Binary Knight Hunt in Phase 1, ask questions to Person 1, or failing that, within X. Suppose there is an accusation in Phase 1; then the Binary Spy Hunt is completed in $X \cup X'$ and a knight, say Person w, is found after at most $K(n, k)$ questions. If the first accusation is in X then an easy inductive argument shows that after question $K(n, k)$ either Person 1 is in an accusatory component, or in the same component as Person w. If the first accusation is in X' then, before this accusation, Person 1 is in a non-accusatory component in X of size 2^a; since at most one other component of this size can be formed in X', the same conclusion holds. If Person 1 is in an accusatory component after question $K(n, k)$ then asking Person w about Person 1 in question $K(n, k) + 1$ both determines the identity of Person 1 and finds a liar; in the other case case Person 1's identity is known, and question $K(n, k) + 1$ may be used to find a liar. Let $n = q(s + 1) + r$ and note that $K(n, k) + 1 = 2s - B(s) + 1$. If $r \leq 1$ then

$$T_L^\star(n, k) = n - q = qs + r \geq 2s - B(s) + 1$$

with equality if and only if $q = 2$, $r = 0$ and $B(s) = 1$. If $r \geq 2$ then

$$T_L^\star(n, k) = n - q - 1 = qs + r - 1 > 2s - B(s) + 1.$$

Thus the targets for finding a liar are met.

Now suppose the Binary Spy Hunt enters Phase 2. By Proposition 3.4 a knight, say Person w, is found at the end of Phase 2 after $K(n, k)$ questions. If after Phase 2, Persons 1 and w are in the same component of the question graph, then the identity of Person 1 is known, and the strategy continues as usual, either finding a liar or proving that no liar is present. If they are in different components then at least one switch from X to X' occurred in the Switching Knight Hunt in Phase 2, and so there is an accusatory edge in the component of Person 1, and the earlier argument applies. □

Proof when $n = 2s + 1$. The case $s = 1$ is easily dealt with. If $s > 1$ then $K(2s{+}1, s{+}1) = 2s{-}B(s)$, $T_L^\star(2s{+}1, s{+}1) = 2s{-}1$ and $T_L(2s{+}1, s{+}1) = 2s$. If $B(s) \geq 2$ then perform a Binary Knight Hunt, always asking questions to Person 1 whenever permitted. This finds a knight, say Person w, in at most $2s-2$ questions. Again either Person 1 is in a component with an accusatory edge, or in the same component as Person w. The former case is as earlier. In the latter case asking further questions to Person w, while keeping the question graph a forest, meets the targets for finding a liar.

Suppose that s is a power of two. If $s \geq 4$ then $K(2s+1, s+1) = 2s-1$. As shown in §3.3, it is possible to find a knight by question $2s-1$ and all identities by question $2s$. Suppose that a liar is known to be present. Then the strategy in §3.3 finds both a knight and a liar by question $2s-1$. This leaves one further question to determine the identity of Person 1. The case $s=1$, is easily dealt with, as is the case $s=2$ when $T_L^\star(5,3)=4$. $\square$

In all cases, after the question when a liar is identified, or, in the case of $T_L(n,k)$, when it becomes clear that no liars are present, the question graph is a forest. Hence all identities may be obtained in $n-1$ questions, as claimed in the introduction.

Example 5.1. Figure 4 overleaf shows an example of the Binary Spy Hunt used to prove Theorem 4 in which all four phases of the strategy are required.

6. Proof of Theorem 2

We now turn to the first of the two main theorems dealing with moles, who may answer as they see fit. Suppose it is known that a mole is present. After $n-2$ questions have been asked, there are two people in the room who have never been asked about. If no accusations have been made then it is consistent that exactly one of these people is a mole. Hence $T_S^\star(n,k) \geq n-1$. A similar argument shows that $T_S(n,k) \geq n$.

To establish the upper bounds in Theorem 2 we use a modified version of the questioning strategy used in [4] and [12] to find everyone's identity. We show in §7 below that a suitable modification of the Binary Knight Hunt can also be used, provided $k-1$ is not a power of two. It is worth noting that the unmodified Binary Knight Hunt is ineffective, since 2^{a-1} questions are required to rule out the presence of moles in a component of size 2^a whose sink vertex is a knight.

Strategy (Extended Spider Interrogation Strategy).

Phase 1. Ask Person 1 about Person 2, then Person 2 about Person 3, until either there is an accusation, or Person $n-1$ supports Person n. If there is an accusation, say when Person p accuses Person $p+1$, then set $\ell = n-k$ and go to Phase 2, treating Person p as a candidate who has been supported by $p-1$ people and accused by one person. Otherwise terminate.

Phase 2. Let $U \subseteq \{1,\dots,n\}$ be the set of numbers of people who have not yet been asked a question, or asked about. Ask people in U about the chosen candidate until either

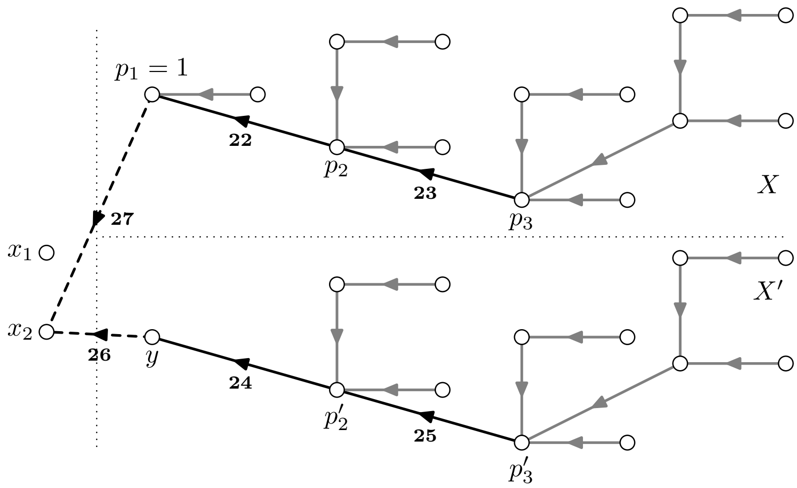

FIGURE 4. Example of the Binary Spy Hunt used to prove Theorem 4. We take $s = 13$ and $n = 29$. Notation is as in the description of this strategy in §3. The set X consists of the 14 vertices above the dotted line, and X' consists of the 13 vertices other than x_1 and x_2 below the dotted line. Phase 1 lasts 21 questions (indicated by grey arrows). After Phase 1 there are components in X of sizes 2, 4 and 8 and components in X' of sizes 1, 4 and 8. Later questions are numbered. Phase 2 consists of a Switching Knight Hunt with components of sizes 2, 4, 4, 8, 8. It ends after question $K(29, 13) = 23$ with Person 1 identified as a knight. Phase 3 ends after question 25. Person x_2 is identified as a liar at the end of Phase 4 after $T^\star_L(29, 13) = 27$ questions. (Questions 26 and 27 are the only that result in accusations.) One further question will find the identity of Person x_1, determining all identities in $n - 1 = 28$ questions.

To give a more interesting example of the Switching Knight Hunt in Phase 2, we remark that if p_2 had accused p_1 on question 22 then the strategy would have switched to X'. Suppose that p'_3 then supports p'_2. Then p'_2 is identified as a knight after 23 questions, and asking p'_2 about 1 both finds a liar and determines the identity of Person 1 after 24 questions.

(a) strictly more people have accused the candidate than have supported him, *or*
(b) at least ℓ people have supported the candidate.

If Phase 2 ends in (a) then replace ℓ with $\ell - m$, where m is the number of people accusing the candidate, choose as a new candidate someone who has not yet been asked a question or asked about, and perform Phase 2 again from the start. If Phase 2 ends in (b) then the strategy terminates. (Thus Phase 2 ends immediately if and only if $p > \ell$.)

If the strategy terminates in Phase 1 then $n-1$ questions have been asked. If a mole is known to be present then Person 1 is a mole; otherwise asking Person n about Person 1 will decide whether any moles are present, using n questions in total.

Whenever a candidate is discarded in Phase 2, the connected component of the question graph containing him contains at least as many moles as knights. This shows that it is always possible to pick a new candidate when required by Phase 2, and that the candidate when the strategy terminates is a knight. Let this knight be Person w. If Person w has been accused by anyone then a mole is known. Otherwise asking Person w about Person p from Phase 1 finds a mole in one more question. In either case the question graph remains a forest, and so at most $n-1$ questions are asked.

This shows that $T_S(n,k) \le n$ and $T^\star_S(n,k) \le n-1$, completing the proof of Theorem 2.

7. Proof of Theorem 5

We first deal with the case $n = 7$ and $k = 4$ since this shows the obstacle addressed by the questioning strategy used in the main part of the proof. For a conditional generalization of the following lemma, see Corollary 8.4.

Lemma 7.1. *Let $n = 7$ and $k = 4$. Suppose that all spies are moles and that a spy is known to be present. There is no questioning strategy that will both find a knight by question $K(7,4) = 4$ and find a mole by question $T^\star_S(7,4) = 6$.*

Proof. It is easily shown that, even if spies always lie, the Interrogator has only two questioning sequences that find a knight by question 4 against best play by the Spy Master. Representing positions by multisets of weights with multiplicities indicated by exponents, they are

$$\begin{aligned}\{1^7\} \to \{2,1^5\} \to \{2^2,1^3\} &\to \{1^3,0\} \to \{2,1,0\}\\ \ldots &\to \{2^3,1\} \to \{4,2,1\}.\end{aligned}$$

In either case the Interrogator must ask a question that connects two components of size 2. If the Interrogator's question creates an edge into a source vertex, the Spy Master should accuse. The Interrogator is then unable to find a knight by question 4. If the new edge is into a sink vertex, the Spy Master should support. This may reveal a knight by question 3, but in all cases the Interrogator is unable to find a mole by question 6. □

A similar argument shows that there is no questioning strategy that will both find a knight by question $K(7,4) = 4$ and either find a mole or prove that no moles are present by question $T_S(7,4) = 7$.

For the main part of Theorem 5 we need the following strategy which finds a knight after $K(n,k)+1$ questions. The comment in square brackets shows that the strategy is well-defined.

Strategy (Modified Binary Knight Hunt). Let P be a subset of $2s+1$ people in which at most s spies are present and let X be a subset of P of size 2^{a+1}, where 2^a is the greatest power of two such that $2^a \le s$.

Phase 1. [Each component in X is a directed path. There are distinct components in X of equal size unless X is connected.] If X is connected then terminate. Otherwise choose two components C and C' in X of equal size and ask the sink vertex in C about the source vertex in C'. If there is an accusation go to Phase 2, otherwise continue in Phase 1.

Phase 2. Disregard the accusation ending Phase 1 and complete a Binary Knight in P, now connecting sink vertices as usual.

Proof of Theorem 5. Let $s = n-k$. Choose a subset P of $2s+1$ people such that $1 \in P$ and choose a subset X of P, as in the Modified Binary Knight Hunt, so that $1 \notin X$. Perform a Modified Binary Knight Hunt in P.

Suppose this ends in Phase 1 after $2^{a+1}-1$ questions. Since $2^{a+1} > s$ the sink vertex in X is a knight. Since $s \geq 2^a + B(s) - 1$ we have $2^{a+1} - 1 \leq 2s+1-2B(s) \leq 2s-B(s) = K(n,k)$. Let Person w be the knight just found. Ask Person w about Person 1, and then about each of the other people in singleton components of the question graph. If there are no accusations after $n-1$ questions then either no moles are present, or the unique source vertex in the component of Person w is a mole: if necessary this can be decided in one more question.

Suppose the Modified Binary Knight Hunt ends in Phase 2 after $K(n,k)+1$ questions. Ask Person w about Person 1, and then about the first person to make an accusation. This identifies a mole in at most $K(n,k)+2 = 2s-B(s)+2$ questions. Unless $n = 2s+1$ and s is a power of two, this meets the target $T^\star_S(n,k) = n-1$, and in any case meets the target $T_S(n,k) = n$.

In the exceptional case when s is a power of two and $n = 2s+1$, the target for finding a knight is $K(2s+1, s+1)+1 = 2s - B(s) + 1 = 2s$. To motivate Problem 8.6 we prove a slightly stronger result in this case, using the Extended Spider Interrogation Strategy from §6. If the strategy is in Phase 1 after $2s-1$ questions then a knight is known. Suppose the strategy enters Phase 2. The first candidate can be rejected no later than question $2s-1$ (after being supported by $s-1$ people and accused by s people), so if the first candidate is accepted then a knight is known by question $2s-1$. If the first candidate is rejected then a knight is found when the question graph has at least two components, so by question $2s-1$ at the latest. In all cases a mole is found by question $T^\star_S(2s+1, s+1) = 2s$. Question $2s+1$ may be used to find Person 1's identity, if necessary. □

8. Further results and open problems

8.1. **Finding all identities.** A related searching problem asks for the identity of every person in the room. Blecher proved in [4] that, when all spies are moles, $n+(n-k)-1$ questions are necessary and sufficient to find everyone's identity. This result was proved independently by the author in [12] using similar arguments. It follows from [12, §3.3] that $n+(n-k)-1$

questions may be required even if moles lie in every answer (but cannot be assumed to do so), and the first question is answered with an accusation, thereby guaranteeing that at least one mole is present.

Let $A_L(n,k)$ be the minimum number of questions necessary and sufficient to determine all identities when all spies are liars. Let $n = q(n-k+1)+r$ where $0 \le r \le n-k$, as in Theorem 2. Aigner proved in [1, Theorem 4] that

$$A_L(n,k) = \begin{cases} n-q+1 & \text{if } 0 \le r \le 1 \\ n-q+\varepsilon_{(n,k)} & \text{if } 2 \le r < n-k \\ n-q & \text{if } r = n-k. \end{cases}$$

where $\varepsilon_{(n,k)} \in \{0,1\}$. It is notable that if $r=0$ or $r=n-k$ then $A_L(n,k) = T_L(n,k)$, and so, in these cases, it is no harder to find a spy or to prove that everyone in the room is a knight than it is to find all identities. When $r=1$ we have $A_L(n,k) = T_L(n,q)+1$. Inspection of the proof of Theorem 4 in [1] shows that Aigner's result holds unchanged if it is known that a liar is present.

Aigner makes the plausible suggestion that $A_L(n,k) = n-q+1$ whenever $r < n-k$. However this is not the case. In fact, if $n \le 30$ and $r < n-k$ then $A_L(n,k) = n-q$, contrary to Aigner's suggestion, if and only if

$$(n,k) \in \left\{ \begin{matrix} (13,9),(16,11),(18,14),(19,13),(21,14),(22,15),(22,17), \\ (23,19),(24,16),(25,17),(25,19),(26,17),(26,20),(27,18), \\ (28,19),(28,23),(28,24),(29,19),(29,22),(30,20),(30,23) \end{matrix} \right\}.$$

This can be checked by an exhaustive search of the game tree, using the program `MajorityGame.hs` available from the author's website[1]. The following problem therefore appears to be unexpectedly deep.

Problem 8.1. *Determine $A_L(n,k)$ when $n = q(n-k+1)+r$ and $2 \le r < n-k$.*

8.2. **The majority game.** The values of $K_L(n,k)$ were found for all n and k in [5], but many natural questions about the majority game remain open. Given a multiset M of component weights and $e \in \mathbf{N}$ such that the sum of the weights in M has the same parity as e, let $n - V_e(M)$ be the minimum number of questions that are necessary and sufficient to find a knight starting from the position M, when the excess of knights over liars is at least e. Thus $V_e(M)$ is the number of components in the final position, assuming optimal play.

Problem 8.2. *Give an algorithm for computing $V_e(M)$ that is qualitatively faster than searching the game tree.*

For multisets M all of whose elements are powers of two, the Binary Knight Hunt gives a lower bound on $V_e(M)$. The Switching Knight Hunt

[1] See `www.ma.rhul.ac.uk/~uvah099/`

gives a lower bound in some of the remaining cases. In [5] a family of statistics $SW_e(M)$ were defined, generalizing the statistic $\Phi(M) = SW_1(M)$ used in [10]. In [5, Section 5] it was shown that $V_e(M) \leq SW_e(M)$. However, Lemma 7 in [5] shows that the difference may be arbitrarily large. It therefore seems that fundamentally new ideas will be needed for Problem 8.2.

One natural special case occurs when $e = 1$.

Conjecture 8.3. *Let k, $a \in \mathbf{N}$ be such that $a < k$. Then $V_1(\{2^a, 1^{2k-2a-1}\}) = B(k-1) + 1$.*

The conjecture is true when $a = 0$ since $V_1(\{1^{2k-1}\}) = 2k - 1 - K(2k - 1, k) = B(k-1) + 1$. Moreover, since the position $\{2^a, 1^{2k-2a-1}\}$ may arise in a Binary Knight Hunt, we have $V_1(\{2^a, 1^{2k-2a-1}\}) \geq B(k-1) + 1$ for all a. The conjecture has been checked for $k \leq 20$ using the program `MajorityGame.hs` already mentioned. One motivation for the conjecture is the following corollary which strengthens part of Theorem 5.

Corollary 8.4 (Conditional on Conjecture 8.3). *Let k be even and let $n = 2k - 1$. Suppose that all spies are moles and that a mole is known to be present. If $k \geq 4$ then there is no questioning strategy that will both find a knight by question $K(2k-1, k) = 2(k-1) - B(k-1)$ and find a mole by question $T^\star_S(2k-1, k) = 2(k-1)$.*

Proof. The Spy Master should support until the Interrogator asks a question that does not connect two singleton components. When this happens the multiset of component sizes is $\{2^a, 1^{2k-2a-1}\}$ for some $a \in \mathbf{N}$. If the question connects a component of size 2 with a component of size 1 then the Spy Master accuses, and then promises the Interrogator that spies lie in all their answers. The multiset of component weights in the resulting majority game is $\{2^{a-1}, 1^{2k-2a-1}\}$. Since k is even, $B(k-2) = B(k-1) - 1$ and so, by the conjecture, $V_1(\{2^{a-1}, 1^{2k-2a-1}\}) = V_1(\{2^a, 1^{2k-2a-1}\}) - 1$. The Interrogator is therefore unable to find a knight by question $K(2k-1, k)$.

If the question connects two components of size 2 then, as in the proof of Lemma 7.1, the Spy Master accuses if the new edge is into a source vertex, and supports if the new edge is into a sink vertex. In the former case, the Spy Master can promise the Interrogator that the component just created has exactly three knights and one spy, and so corresponds to the position $\{2^{a-1}, 1^{2k-2a-1}\}$ in the majority game. The argument in the previous paragraph then applies. In the latter case, let v and v' be the source vertices in the new component. The Spy Master should support on all further questions. The Interrogator must, in some later question, ask a knight, say Person w, for the identity of either v or v'. Suppose without loss of generality that v is identified. If w and v are in the same component this creates a cycle in the question graph. Otherwise the component C of w has a source vertex (other than w itself, since there are no accusations in the question graph), which the Interrogator must identify. Again this creates a cycle. Hence the final question graph has at least $2k-1$ edges. □

8.3. **Combined games.** In the following problem we change the victory condition in Theorem 4 to combine two of the searching games considered in this paper in a different way. The analogous problem replacing $T_L^\star(n,k)$ with $T_L(n,k)$ is also of interest.

Problem 8.5. *Suppose that all spies are liars and that a liar is known to be present. Consider the searching game where the Interrogator wins if he either finds a knight by question $K(n,k)-1$, or a liar by question $T_L^\star(n,k)-1$. When is this game winning for the Interrogator?*

Theorem 5 and the conditional Corollary 8.4 invite the following question. Again the analogous problem replacing $T_S^\star(n,k)$ with $T_S(n,k)$ is also of interest.

Problem 8.6. *Suppose that all spies are moles and that a spy is known to be present. When is there a questioning strategy that finds a knight by question $K(n,k)$ and a spy by question $T_S^\star(n,k) = n-1$?*

The final paragraph of the proof of Theorem 5 in §7 shows that there is such a questioning strategy when $n = 2^e + 1$ and $k = 2^{e-1} + 1$, for any $e \in \mathbf{N}$. This reflects the ease with which the Interrogator may meet the target $K(n,k) = n-2$ for finding a knight. We remark that Example 5.1 shows one situation in which the Switching Knight Hunt is effective even when all spies are moles; it might be useful in this problem.

Acknowledgements

The author thanks two anonymous referees for their helpful critical comments on an earlier version of this paper.

References

[1] Martin Aigner, *Variants of the majority problem*, Discrete Appl. Math. **137** (2004), no. 1, 3–25, 1st International Workshop on Combinatorics of Searching, Sorting, and Coding (COSSAC '01) (Ischia).

[2] Martin Aigner, *Two colors and more*, in *Entropy, search, complexity*, Bolyai Soc. Math. Stud. 16, Springer (2007), pages 9–26.

[3] Laurent Alonso, Edward M. Reingold, and René Schott, *Determining the majority*, Inform. Process. Lett. **47** (1993), 253–255.

[4] Pavel M. Blecher, *On a logical problem*, Discrete Math. **43** (1983), 107–110.

[5] John R. Britnell and Mark Wildon, *The majority game with an arbitrary majority*, To appear in Disc. Appl. Math (accepted September 2014).

[6] Robert Cowen, *Adaptive fault diagnosis using self-referential reasoning*, arXiv:1407.6255 (2014), 10 pages.

[7] Evangelos Kranakis, Andrzej Pelc, and Anthony Spatharis, *Optimal adaptive fault diagnosis for simple multiprocessor systems*, Networks **34** (1999), 206–214.

[8] K. Nakajima, *A new approach to system diagnosis*, Proc. 19th Allerton Conf. Commun. Contr. and Computing, 697–706 (1981).

[9] Franco P. Preparata, G. Metze, and Robert T. Chien, *On the connection assignment problem of diagnosable systems*, IEEE Transactions on Electronic Computers **16** (1967), 848–854.

[10] Michael E. Saks and Michael Werman, *On computing majority by comparisons*, Combinatorica **11** (1991), 383–387.

[11] Gábor Wiener, *Search for a majority element*, J. Statist. Plann. Inference **100** (2002), no. 2, 313–318, International Conference (of the Forum for Interdisciplinary Mathematics) on Combinatorics, Information Theory and Statistics (Portland, ME, 1997).

[12] Mark Wildon, *Knights, spies, games and ballot sequences*, Discrete Math. **310** (2010), 2974–2983.

Mark Wildon, Royal Holloway, University of London, Egham TW20 0EX, UK, +44(0)1784414021

E-mail address: mark.wildon@rhul.ac.uk