Sharing the proceeds from a hierarchical venture[*]

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Abstract

We consider the problem of distributing the proceeds generated from a joint venture in which the participating agents are hierarchically organized. We introduce and characterize a family of allocation rules where revenue 'bubbles up' in the hierarchy. The family is flexible enough to accommodate a *no-transfer* rule (where no revenue bubbles up) and a *full-transfer* rule (where all the revenues bubble up to the top of the hierarchy). Intermediate rules within the family are reminiscent of popular incentive mechanisms for social mobilization or multi-level marketing. Our benchmark model refers to the case of linear hierarchies, but we also extend the analysis to the case in which hierarchies may convey a general tree structure and include joint ownerships.

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1 Introduction

Agents often organize themselves into hierarchies when involved in joint ventures (e.g., Mookherjee, 2006). There exist numerous reasons to explain this fact. The hierarchical form, in which workers deal with the routine problems and managers deal with the exceptions, arises as an optimal way to structure the organization of knowledge (e.g., Garicano, 2000). Ownership or power structures generate natural hierarchies with related chains of command and responsibility (e.g., Ichniowski and Shaw, 2003). It is also argued that workplace structures that are rich in sequentiality are desirable from the point of view of incentives (e.g., Winter, 2010). Demange (2004) further shows that hierarchies yield stable cooperation structures when it comes to allocating resources. Hierarchies may also relate to crowdsourcing and social mobilization systems (e.g., Pickard et al., 2011), as well as multi-level marketing (e.g., Emek et al., 2011), task solving systems such as Amazon Mechanical Turk (e.g., Rand, 2012), or financial systems such as BitCoin (Babaioff et al. 2012).

In this paper, we are concerned with the problem of sharing the collective proceeds generated from hierarchical ventures. To analyze this problem, we consider a stylized model in which a group of agents are involved in a joint venture. The group is structured in several layers, each reflecting a different degree of responsibility, command, or even seniority. Thus, an agent located at a given layer is in command of (or, at least, held accountable for) all agents located at a lower layer. In such a hierarchy, agents are characterized by their degree of responsibility (location in the hierarchy), and the individual revenue they produce for the joint venture. Based on that information, the issue is how to allocate the overall produced revenue among the agents. Our stylized model is flexible enough to accommodate various forms of organizations that are frequent in different professional sectors. Instances are law firms (e.g., Galanter and Palay, 1990), physicians' practice arrangements (e.g., Kletke et al., 1996) as well as renowned architectural practices (e.g., Winch and Schneider, 1993).

Two focal, and somewhat polar, allocation rules can be considered for the setting described above. On the one hand, the *no-transfer rule*, in which each agent keeps her revenue (thus, ignoring the command structure conveyed by the hierarchy). On the other hand, the *full-transfer rule*, in which the agent at the top of the hierarchy (the *boss*, or *venture capitalist*) gets all the proceeds (thus, ignoring individual contributions to the joint proceeds). A compromise between these two polar rules, in which certain upward transfers are allowed, can be formalized, and we do so in this paper. The resulting family of *transfer rules* is close in spirit to the *MIT*

strategy (e.g., Pickard et al., 2011), the winning strategy for the so-called DARPA Network Challenge.[1] It is also reminiscent of multi-level marketing strategies (e.g., Emek et al., 2011) in which individuals are compensated not only for the sales they generate, but also for the sales of those they recruited.[2] These strategies can be seen as specific *geometric (incentive tree) mechanisms* (e.g., Lv and Moscibroda, 2013) that are usually considered in the computer science literature.[3] An *incentive tree* models the participation of people in crowdsourcing or human tasking systems. An incentive tree mechanism is an algorithm that determines how much reward each individual participant receives based on all the participants' contributions, as well as the structure of the solicitation tree. In geometric (incentive tree) mechanisms, a certain fraction α 'bubbles up' from one agent to the immediate superior, a fraction α^2 bubbles up to the immediate superior of the immediate superior, and so forth. In our case, a transfer rule imposes that the lowest-ranked agent gets a share λ of her revenue, her immediate superior gets a share λ of her revenue, and of any remaining 'surplus' from the lowest-ranked agent, etc. Thus, there is an obvious connection between the transfer rules we consider here and geometric incentive tree mechanisms. It is worth emphasizing, nevertheless, that the latter are typically not budget balanced and, thus, cannot be considered as sharing rules.

We provide normative foundations for the family of transfer rules described above. In the benchmark case of linear hierarchies, we show that the family is characterized by four simple and intuitive axioms (Lowest Rank Consistency, Highest Rank Revenue Independence, Highest Rank Splitting Neutrality, and Scale Invariance). If we add an additional axiom, referring to two-agent problems in which the boss is not productive, the intermediate transfer rule for which $\lambda = 0.5$ is singled out within the family. If, instead, axioms modeling order preservation (with respect to either individual revenues, or the command structure) are added, the two polar rules are singled out.

The member of the family arising when $\lambda = 0.5$, which translates to our context the MIT strategy mentioned above, is also singled out as an optimal rule, when we enrich our framework to deal with endogenous hierarchies. More precisely, suppose the aim is to maximize the

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[1]This is a social network mobilization experiment, conducted by the Defense Advanced Research Projects Agency, to identify distributed mobilization strategies and demonstrate how quickly a challenging geolocation problem could be solved by crowdsourcing.

[2]Famous cases include companies such as Avon Products, Inc., or Herbalife International.

[3]Computer scientists have been concerned with mechanisms that are immune to *sybil attacks* in which a reputation system is subverted by forging identities in peer-to-peer networks (e.g., Drucker and Fleischer, 2012). This is actually a type of manipulation to which the mechanism arising from the *MIT strategy* is susceptible.

expected revenues of the agent at the top of the hierarchy (the boss), when the process to get subordinates is probabilistic and based on the upward transfers the rules allow. The boss, while selecting a transfer rule, would face a tradeoff: high upward transfers vs. weak incentives for subordinates to join the hierarchy voluntarily. We show that the optimal rule to deal with such a tradeoff is precisely the intermediate transfer rule for which $\lambda = 0.5$. This occurs, not only when (possible) subordinates are *myopic*, but also when they are *farsighted* and take into account their ability to hire further subordinates themselves.

Our contribution is also related to the sizable literature on fair division in networks. This literature mostly organizes itself into two strands.

On the one hand, the strand in which the networks give rise to cooperative games and where the structure of the network is exploited in order to define fair allocation among agents connected in the graph. The canonical case is that of cost sharing within a rooted tree, which can be traced back to Claus and Kleitman (1973) and Bird (1976). For fixed trees, the so-called Bird rule, which can be seen as a counterpart to the no-transfer rule, and the so-called serial rules, which convey a different form of transfers to the ones described above, are prominent. This is, for instance, the case in the problem of sharing a polluted river (e.g., Ni and Wang, 2007; Dong et al., 2012) which is reminiscent of the problem considered here (with the modification of considering negative revenues, and thus interpreting them as costs). Another specific (and well-known) instance of this strand of the literature is the so-called *airport problem* (e.g., Littlechild and Owen, 1973), in which the runway cost has to be shared among different types of airplanes with a linear graph representing the runway. The rules (and some of the axioms) highlighted in our work will also be reminiscent of some of the rules considered for airport problems. A common feature for the models within this strand of the literature is that the cheapest connection (minimal distance) to the root becomes a crucial element, as it represents the stand-alone option for the agents. This is not the case in our model, where the crucial feature is the combination of the agents' revenues and the location in the hierarchy. Consequently, stand-alone options are not naturally specified.

On the other hand, there is a strand of the literature where networks restrict cooperative games. Myerson (1977, 1980) pioneered this approach by using graphs to represent communication structures in cooperative games. A central result within this approach is that if agents are allowed to cooperate in tree structures, the original TU-game need only be superadditive to guarantee that the graph-restricted game has a non-empty core (e.g., Le Breton et al., 1992, Demange, 1994). In particular, our analysis can be related to the case of TU-games restricted

by a permission structure (e.g., Gilles et al., 1992, van den Brink and Gilles, 1996), prece-dence structure (e.g., Faigle and Kern, 1992, Grabisch and Sudhölter, 2016) and peer groups (e.g., Branzei et al., 2002). For instance, if hierarchies are interpreted as permission structures, and these are restricting a predefined additive game where the worth of a coalition equals the sum of revenues for agents in that coalition, our revenue sharing problem can be construed as a game with permission structure (Gilles et al., 1992) where agents need permission from all their superiors before they can cooperate (the conjunctive approach). For such restricted games, a generalization of the Shapley value, dubbed *the permission value* (characterized in van den Brink and Gilles, 1996, and van den Brink, 1997), coincides with the result of using the serial principle on linear and branch hierarchies. In other words, it shares the revenue of each agent equally among this agent and all of her superiors. As we shall see later, this is a radically different solution from the family of transfer rules presented here.

It is important to stress that, in our model formulation, there is no predefined cooperative game where the hierarchies are restricting cooperation. Instead, we relate fairness requirements directly to the hierarchical network structure.

The rest of the paper is organized as follows. In Section 2, we analyze the basic model in which the hierarchy can be expressed as a line. We introduce and characterize our family of transfer rules for such a setting. We also consider a setting of endogenous hierarchies, which allows us to link further the transfer rules to incentive tree mechanisms, as well as to single out the intermediate rule of the family as optimal. In Section 3, we generalize the analysis to the case of branch hierarchies (not necessarily linear) and show how the results from the linear case generalize to such a setting with minimal adjustments. In Section 4, we generalize the analysis further to be able to account for the case of joint ownerships. We conclude in Section 5.

2   Linear hierarchies

We present in this section our benchmark model dealing with linear hierarchies. Suppose there exists a set of potential **agents**, identified with the set of natural numbers. Let $\mathcal{M}$ be the class of finite subsets of natural numbers, with generic element M. Each set $M \in \mathcal{M}$ will represent a **linear hierarchy**, with the convention that lower numbers in M refer to lower positions in the hierarchy. For instance, if $M = \{1, \ldots, m\}$, then 1 is representing the agent with the lowest rank in the hierarchy, whereas m is representing the agent with the highest rank. Often we refer to agent m as the *boss*.

Agents in each linear hierarchy will be involved in a joint venture to which all of them contribute. Formally, for each $i \in M$, let $r_i \in \mathbb{R}_+$ be the **revenue** that agent i generates, and $r = (r_i)_{i \in M}$ the profile of revenues.[4] We imagine the boss as the sole owner (e.g., patent owner or venture capitalist) so without the participation of the boss there is no possibility of obtaining revenue. Furthermore the hierarchical structure can be seen as a chain of command in the sense that every agent i refers to her direct superior in hierarchy.

A *linear hierarchy revenue sharing problem*, or simply, a **problem** is a duplet consisting of a linear hierarchy $M \in \mathcal{M}$ and a profile of revenues $r \in \mathbb{R}_+^{|M|}$. Let $\mathcal{R}^M$ be the set of problems involving the hierarchy M and $\mathcal{R} = \bigcup_{M \in \mathcal{M}} \mathcal{R}^M$.

Given a problem $(M, r) \in \mathcal{R}$, an **allocation** is a vector $x \in \mathbb{R}_+^{|M|}$ satisfying **balance**, i.e., $\sum_{i \in M} x_i = \sum_{i \in M} r_i$.

An **allocation rule** is a mapping ϕ assigning to each problem $(M, r) \in \mathcal{R}$ an allocation $\phi(M, r)$. We assume from the outset that rules are **anonymous**, i.e., for each problem $(M, r) \in \mathcal{R}$, and for each strictly monotonic function $g : M \to M'$, $\phi_{g(i)}(M', r') = \phi_i(M, r)$, where $r'_{g(i)} = r_i$, for each $i \in M$. Thus, in what follows for this section, we assume, without loss of generality, that $M = \{1, \ldots, m\}$.

2.1 Transfer rules

Two (polar) examples of rules are those capturing the minimal and maximal possible revenue transfers from subordinates to their superiors in the hierarchy.

More precisely, the first one imposes that each agent in the linear hierarchy keeps her own revenue and transfers nothing to her superiors. Formally,

No-Transfer rule, $\phi^{\mathbf{NT}}$: For each $(M, r) \in \mathcal{R}$,

$$\phi^{NT}(M, r) = r.$$

Its polar rule imposes that the boss receives all revenues. Formally,

Full-Transfer rule, $\phi^{\mathbf{FT}}$: For each $(M, r) \in \mathcal{R}$,

$$\phi^{FT}(M, r) = \left(0, \ldots, 0, \sum_{i \in M} r_i \right).$$

In between the two extreme rules presented above a vast number of rules can be imagined. Instead of endorsing a specific rule directly, we take an axiomatic approach and propose first

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[4]For each $M \in \mathcal{M}$, each $S \subseteq M$, and each $z \in \mathbb{R}^m$, let $z_S = (z_i)_{i \in S}$. For each $i \in M$, let $z_{-i} = z_{M \setminus \{i\}}$.

several axioms reflecting principles that we find normatively appealing in the context of these problems. Ultimately, our goal will be to single out rules as a result of combining those axioms.

We start with the principle of consistency, an operational notion that has played an instrumental role in axiomatic analyses of diverse problems, and for which normative underpins have also been provided (e.g., Thomson, 2012). The principle refers to the way in which rules react to agents leaving the scene with their awarded amounts. Here we concentrate on a minimalistic version of the principle referring only to the case in which the agent with the lowest rank leaves the hierarchy after the allocation took place. It seems natural to assume that the lowest ranked agent refer to his immediate superior in the linear hierarchy to terminate his relationship. Thus, we assume that, after leaving, a new problem arises in which the agent with the second-lowest rank in the original problem becomes the lowest-ranked agent, but now also generating the eventual revenue that the leaving agent generated in the original problem, and did not take in the allocation. The axiom then states that the solution of the new problem agrees with the solution of the original problem for all the standing agents in the hierarchy.[5] Formally,

Lowest Rank Consistency: For $|M| \geq 2$, where $(M, r) \in \mathcal{R}$, and $(M \setminus \{1\}, (r_2 + r_1 - \phi_1(M, r), r_{M \setminus \{1,2\}})) \in \mathcal{R}$, we have,

$$\phi_{M \setminus \{1\}}(M, r) = \phi\left(M \setminus \{1\}, (r_2 + r_1 - \phi_1(M, r), r_{M \setminus \{1,2\}})\right).$$

The next two properties focus on the opposite end of the hierarchy. As we imagine that the boss can monitor and veto changes in generation of revenue and changes in the hierarchy we need conditions that protect the subordinates from changes and manipulation of revenues beyond their control.

The first one says that the size of the revenue generated by the highest-ranked agent (i.e., the boss) is irrelevant for the allocation of all the subordinates: A plausible rationale for this axiom is that, in a linear hierarchy, subordinates have no influence on the revenue generated by the boss and, thus, should not be affected by its size. Formally,

Highest Rank Revenue Independence: For each $(M, r) \in \mathcal{R}$, and each $\hat{r}_m \in \mathbb{R}_+$,

$$\phi_{M \setminus \{m\}}(M, r) = \phi_{M \setminus \{m\}}\left(M, (r_{-m}, \hat{r}_m)\right).$$

The second one avoids certain strategic manipulations of the allocation by the highest-ranked agent. More precisely, it says that if the boss splits her revenue into two amounts,

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[5]This axiom is actually reminiscent of the so-called "first agent consistency" axiom proposed by Potters and Sudhölter (1999) for airport problems.

represented by two agents ranked highest in the new hierarchy, the remaining agents should not be affected.[6] Formally,

Highest Rank Splitting Neutrality: For each $(M, r) \in \mathcal{R}$, let $(M', r') \in \mathcal{R}$ be such that $M' = M \cup \{k\}$, $k > m$, $r_m = r'_k + r'_m$, and $r'_{M \setminus \{m\}} = r_{M \setminus \{m\}}$. Then,

$$\phi_{M \setminus \{m\}}(M', r') = \phi_{M \setminus \{m\}}(M, r).$$

In a sense, the previous two axioms convey certain rights and obligations for the boss, as well as for the remaining members of the hierarchy. On the one hand, *Highest Rank Revenue Independence* says that the boss is entitled to her whole generated revenue, whereas the remaining agents must respect that. On the other hand, *Highest Rank Splitting Neutrality* says that the boss is entitled to bring a newcomer to the top of the hierarchy with whom to split her revenue, but the remaining agents are entitled to preserve their allocations intact after such a move.

Finally, we consider a technical property stating that if revenues are scaled by a factor α, so is the solution. In particular, the axiom says that the currency in which we measure revenue is irrelevant for the allocation process.[7]

Scale Invariance: For each $(M, r) \in \mathcal{R}$, and each $\alpha > 0$,

$$\phi(M, \alpha r) = \alpha \phi(M, r).$$

The no-transfer rule and the full-transfer rule presented above satisfy the previous four axioms. Both rules are extreme in an obvious sense, which suggests that the set of all rules satisfying the axioms should consist of those resulting from a compromise between them. It turns out that this compromise can be described as follows:

Suppose the lowest-ranked agent gets a share $\lambda \in [0, 1]$ of her revenue, her immediate superior gets a share λ of her revenue, as well as any remaining 'surplus' from the lowest-ranked agent, etc., and the highest-ranked agent gets the residual. Hence, if $M = \{1, \ldots, m\}$, payment shares are determined recursively as

$$x_i^\lambda = \lambda r_i + (1 - \lambda) x_{i-1}^\lambda, \tag{1}$$

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[6]Axioms of this sort have been widely explored in various models of resource allocation (e.g., Ju, 2013). Note that our axiom only requires "splitting-proofness" in a specific situation, which makes it weaker than the standard counterpart axioms in such a literature.

[7]This axiom appears frequently in axiomatic studies (e.g., Aumann and Serrano, 2008).

for each $i \in M \setminus \{m\}$, with the notational convention that $x_0^\lambda = 0$. Furthermore,

$$x_m^\lambda = \sum_{i=1}^m r_i - \sum_{i=1}^{m-1} x_i^\lambda. \tag{2}$$

Note that we may rewrite (1) and (2) in the closed-form expressions

$$x_i^\lambda = \lambda \left(r_i + (1-\lambda)r_{i-1} + \cdots + (1-\lambda)^{i-1} r_1 \right),$$

for $i = 1, \ldots, m-1$ and

$$x_m^\lambda = r_m + (1-\lambda)r_{m-1} + \cdots + (1-\lambda)^{m-1} r_1.$$

Denote the corresponding family of allocation rules, so defined, which we call **transfer rules**, by $\{\phi^\lambda\}_{\lambda \in [0,1]}$. Note that the rule ϕ^1 corresponds to the no-transfer allocation rule, whereas ϕ^0 corresponds to the full-transfer allocation rule.

Example 1: Consider the problem $(\{1, 2, 3\}, (12, 6, 12))$, i.e., a linear hierarchy made of three agents, 1, 2, and 3, in which agent 1 generates a revenue of 12, agent 2 a revenue of 6, and agent 3 a revenue of 12. Figure 1 below illustrates the situation.
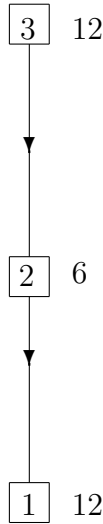
Figure 1: A linear hierarchy.

It is straightforward to see that the no-transfer rule selects the allocation $(12, 6, 12)$ for this example, whereas the full-transfer rule selects the allocation $(0, 0, 30)$. In general, the transfer rules select the allocations

$$(12\lambda, (18 - 12\lambda)\lambda, 30(1 - \lambda) + 12\lambda^2),$$

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for each $\lambda \in [0,1]$. In particular, for $\lambda = 0.5$, the corresponding transfer rule selects the allocation $(6, 6, 18)$. Thus, in such a case, agent 2 receives the same as agent 1, despite the fact that agent 1 is generating twice the revenue. ∎

When agents generate equal revenues, the transfer rules can be fully ranked by means of the *Lorenz criterion* $\succsim_L$, according to the parameter describing the family.[8] More precisely, for each $(M, r) \in \mathcal{R}$, such that $r_i = r_j$ for each pair $i, j \in N$, it follows that $\phi^\lambda(M, r) \succsim_L \phi^{\lambda'}(M, r)$ if and only if $\lambda \geq \lambda'$.

Our main result, stated next, shows that the family of transfer rules is characterized by the combination of the axioms introduced above.

Theorem 1 *A rule ϕ satisfies Lowest Rank Consistency, Highest Rank Revenue Independence, Highest Rank Splitting Neutrality, and Scale Invariance if and only if it is a transfer rule, i.e., $\phi \in \{\phi^\lambda\}_{\lambda \in [0,1]}$.*

Proof: It is not difficult to see that the transfer rules satisfy all the axioms in the statement of the theorem. As an illustration, we show that they satisfy *Lowest Rank Consistency*. To do so, let $\lambda \in [0,1]$ and $(M, r) \in \mathcal{R}$ be given. For each $i \in M$, let $x_i = \phi_i^\lambda(M, r)$ and $\widetilde{x}_i = \phi_i^\lambda \left(M \setminus \{1\}, (r_2 + r_1 - x_1, r_{M \setminus \{1,2\}}) \right)$. Then, $\widetilde{x}_2 = \lambda(r_2 + r_1 - x_1) = x_2$. For each $j \neq m$, $\widetilde{x}_j = \lambda r_j + (1 - \lambda)\widetilde{x}_{j-1}$. Thus, by induction, $\widetilde{x}_j = x_j$ and $\widetilde{x}_m = r_m + r_{m-1} - x_1 - \sum_{k=2}^{n-1} \widetilde{x}_k = x_m$.

We now suppose that ϕ is a rule satisfying all the axioms in the statement of the theorem. First, let $M = \{1\}$ and $r = r_1$. By balance, $\phi_1(M, r) = r_1 = \phi_1^\lambda(M, r)$, for each $\lambda \in [0,1]$. Next, add a superior agent 2 with revenue r_2. Let $M' = \{1, 2\}$ and $r' = (r_1, r_2)$. Now we claim that, $\phi_1(M', r') \in [0, r_1]$, so $\phi_1(M', r') = \lambda r_1 = \phi_1^\lambda(M', r')$ for some $\lambda \in [0,1]$. Indeed, assume that $\phi_1(M', r') > r_1$; then by Highest Rank Revenue Independence $\phi_1(M', r') = \phi_1(M', (r_1', 0))$ so by balance $\phi_1(M', r') \leq r_1$ - a contradiction.

By *Highest Rank Revenue Independence*, λ is independent of r_2. Moreover, λ is independent of r_1. To see this, suppose, by contradiction, that we have $\tilde{r} = (\tilde{r}_1, \tilde{r}_2)$ with $r_2 = \tilde{r}_2$ and $\phi_1(M', r') = \lambda r_1$ and $\phi_1(M', \tilde{r}) = \widetilde{\lambda} \tilde{r}_1$ with $\lambda \neq \widetilde{\lambda}$. Then, by *Scale Invariance*, $\phi_1(M', \frac{\tilde{r}_1}{r_1} r) = \frac{\tilde{r}_1}{r_1} \lambda r_1 = \lambda \tilde{r}_1 \neq \widetilde{\lambda} \tilde{r}_1$, contradicting that λ is independent of r_2. Now, by balance, $\phi_2(M', r') = r_2 - r_1 - \phi_1(M', r') = \phi_2^\lambda(M', r')$.

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[8]Given two vectors, we say that the former Lorenz dominates the latter if its smallest coordinate is at least as large as the smallest coordinate of the second vector, the sum of its two smallest coordinates is at least as large as the corresponding sum for the second vector, and so on.

Next, suppose there is λ such that $\phi = \phi^\lambda$ for all problems with up to k agents, $k \geq 2$. Now, consider the problem (M^k, r^k) with $M^k = \{1, \ldots, k\}$ and $r^k = (r_1, \ldots, r_k)$ and add an agent $k + 1$. By *Highest Rank Revenue Independence*, and *Highest Rank Splitting Neutrality*, $\phi_i(M^{k+1}, r^{k+1}) = \phi_i(M^k, r^k) = \phi_i^\lambda(M^k, r^k)$ for all $i \leq k - 1$. By *Lowest Rank Consistency*, $\phi_k(M^{k+1}, r^{k+1}) = \phi_k(M^{k+1} \setminus \{1\}, r_2^{k+1} + r_1^{k+1} - \phi_1(M^{k+1}, r^{k+1}), r_{M^{k+1} \setminus \{1,2\}})$. As, by the induction hypothesis, $\phi_1(M^{k+1}, r^{k+1}) = \phi_1^\lambda(M^k, r^k) = \lambda r_1$, it follows by *Lowest Rank Consistency* that $\phi_k(M^{k+1}, r^{k+1}) = \phi_k^\lambda(M^{k+1}, r^{k+1})$. Finally, by balance,

$$\phi_{k+1}(M^{k+1}, r^{k+1}) = r_{k+1} - \sum_{i=1}^{k} \phi_i^\lambda(M^{k+1}, r^{k+1}) = \phi_{k+1}^\lambda(M^{k+1}, r^{k+1}). \qquad \blacksquare$$

The axioms from Theorem 1 are logically independent:

- The classical *serial rule* (e.g., Moulin and Shenker, 1992) imposes that each agent's revenue is split equally among her superiors and herself. In Example 1, it would yield the allocation $(4, 7, 19)$. The serial rule violates *Highest Rank Splitting Neutrality*, while satisfying all the remaining axioms at the statement of Theorem 1.

- Another natural rule is the one in which all agents keep a fraction λ of their own revenue and the boss receives the residual. In Example 1, it would yield the allocation $(6, 3, 21)$, for the case with $\lambda = 0.5$. This rule violates *Lowest Rank Consistency*, while satisfying all the remaining axioms at the statement of Theorem 1.

- The hybrid rule obtained while using the full-transfer rule if $\sum_{j=2}^{m} r_j < r_1$, and the no-transfer rule otherwise, is another well-defined rule for our setting. This rule violates *Highest Rank Revenue Independence*, while satisfying all the remaining axioms at the statement of Theorem 1.

- Finally, a rule defined as a transfer rule, but in which λ depends on the sum of the revenues, violates *Scale Invariance*, while satisfying all the remaining axioms at the statement of Theorem 1.

It is worth noting that the parameter defining each transfer rule is independent of individual revenues. This fact follows from the axiom of *Scale Invariance*. As such, what just appears to be a requirement of independence of monetary units, actually has normative impact in our setting.

In what follows, we complement the above characterization result by adding several new axioms, which will single out focal members of our family.

We start with an axiom referring to canonical two-agent problems in which the boss is not productive. For those settings, one might find appealing to allocate revenues equally. A plausible rationale is that, although the lowest-ranked agent is the only productive one, the boss is also necessary for the production to take place. Formally,[9]

Canonical Fairness: For each $x \in \mathbb{R}_+$, $\phi(\{1,2\}, (x,0)) = (\frac{x}{2}, \frac{x}{2})$.

The intermediate transfer rule for which $\lambda = 0.5$ is the only transfer rule satisfying the previous axiom. As a matter of fact, and as shown by the next result, the rule is characterized when replacing scale invariance in Theorem 1 by this new axiom.[10]

Theorem 2 *A rule ϕ satisfies Lowest Rank Consistency, Highest Rank Revenue Independence, Highest Rank Splitting Neutrality, and Canonical Fairness if and only if it is the intermediate transfer rule for which $\lambda = 0.5$, i.e., $\phi \equiv \phi^{0.5}$.*

Proof: By Theorem 1, we know that $\phi^{0.5}$ satisfies the first three axioms of the statement. It is straightforward to see that it also satisfies *Canonical Fairness*. Conversely, suppose that ϕ is a rule satisfying all the axioms in the statement of the theorem. First, let $M = \{1,2\}$ and $r = (r_1, r_2)$. By *Highest Rank Revenue Independence*, $\phi_1(M, r) = \phi_1(M, (r_1, 0))$. By *Canonical Fairness*, $\phi_1(M, (r_1, 0)) = \frac{r_1}{2}$. Then, by balance, $\phi(M, r) = (\frac{r_1}{2}, r_2 + \frac{r_1}{2}) = \phi^{0.5}(M, r)$.

Next, suppose that $\phi \equiv \phi^{0.5}$ for all problems with up to k agents, $k \geq 2$. Now, consider the problem (M^k, r^k) with $M^k = \{1, \ldots, k\}$ and $r^k = \{r_1, \ldots, r_k\}$ and add an agent $k+1$. By *Highest Rank Revenue Independence*, and *Highest Rank Splitting Neutrality*, $\phi_i(M^{k+1}, r^{k+1}) = \phi_i(M^k, r^k) = \phi_i^{0.5}(M^k, r^k)$ for all $i \leq k-1$. By *Lowest Rank Consistency*, $\phi_k(M^{k+1}, r^{k+1}) = \phi_k(M^{k+1} \setminus \{1\}, r_2^{k+1} + r_1^{k+1} - \phi_1(M^{k+1}, r^{k+1}), r_{M^{k+1} \setminus \{1,2\}})$ and thus, by the induction hypothesis, $\phi_k(M^{k+1}, r^{k+1}) = \phi_k^{0.5}(M^{k+1}, r^{k+1})$. Finally, by balance,

$$\phi_{k+1}(M^{k+1}, r^{k+1}) = r_{k+1} - \sum_{i=1}^{k} \phi_i^{0.5}(M^{k+1}, r^{k+1}) = \phi_{k+1}^{0.5}(M^{k+1}, r^{k+1}). \qquad \blacksquare$$

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[9]Canonical Fairness follows from Structural Monotonicity and the Necessary Player Property applied to two-player games with permission structures (van den Brink and Gilles, 1996).

[10]If instead of *Canonical Fairness*, one considers the axiom stating that those (canonical) two-agent problems are solved in a specific (inegalitarian) way, say $(\lambda x, (1 - \lambda)x)$, then the corresponding transfer rule (ϕ^λ) would be characterized.

It is worth mentioning that $\phi^{0.5}$ satisfies a stronger version of *Canonical Fairness*, which indicates that in a hierarchy in which only the lowest-ranked agent is productive, each agent gets one half of the incoming revenue and bubbles up the remainder. More precisely, if the revenue of the lowest-ranked agent is x, this agent keeps $x/2$, her immediate superior gets $x/4$, the immediate superior to the latter gets $x/8$, etc. This is precisely the rationale behind the so-called MIT strategy, which is described in more detail below.

We now consider two new axioms formalizing two polar forms of order preservation.

The first axiom states that agents producing higher revenues should be awarded more. The second states that agents located higher in the hierarchy should be awarded more. Formally,

Revenue Order Preservation: For each $(M, r) \in \mathcal{R}$, and each pair $i, j \in M$ such that $r_i \geq r_j$, $\phi_i(M, r) \geq \phi_j(M, r)$.

Hierarchical Order Preservation: For each $(M, r) \in \mathcal{R}$, and each pair $i, j \in M$, where $i \geq j$, $\phi_i(M, r) \geq \phi_j(M, r)$.[11]

The full-transfer rule is the only transfer rule satisfying the first axiom, whereas the no-transfer rule is the only transfer rule satisfying the second axiom. More interestingly, and as shown by the next result, each of the rules is characterized by the corresponding axiom, in combination with either *Highest Rank Splitting Neutrality* or *Highest Rank Revenue Independence*.

Theorem 3 *The following statements hold:*

- *A rule satisfies Highest Rank Revenue Independence and Revenue Order Preservation if and only if it is the no-transfer rule.*

- *A rule satisfies Highest Rank Splitting Neutrality and Hierarchical Order Preservation if and only if it is the full-transfer rule.*

Proof: We concentrate on the non-trivial implication of each statement. First, let ϕ be a rule satisfying *Highest Rank Revenue Independence* and *Revenue Order Preservation*. Let $(M, r) \in \mathcal{R}$ be given. We claim first that $\sum_{j=1}^{m-1} \phi_j(M, r) \leq \sum_{j=1}^{m-1} r_j$. By contradiction, assume otherwise. Then, by *Highest Rank Revenue Independence* we can vary r_m without affecting the shares of the other agents $(i = 1, \ldots, m-1)$. Thus, let $r_m = 0$, which contradicts balance. As

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[11]Hierarchical Order Preservation corresponds to Structural Monotonicity (see van den Brink and Gilles, 1996) when applied to additive games with linear permission structure.

$\sum_{j=1}^{m-1} \phi_j(M, r) \leq \sum_{j=1}^{m-1} r_j$, balance implies that $\phi_m(M, r) \geq r_m$. Now, for each $i \neq m$ letting $r_m = r_i$ we get, by *Revenue Order Preservation*, that $\phi_i(M, r) = \phi_m(M, r) \geq r_i$. Now, balance gives $\phi_i(M, r) = r_i$ for all $i \in M$.

Let now ϕ be a rule satisfying *Highest Rank Splitting Neutrality* and *Hierarchical Order Preservation*. By contradiction, suppose that there exists a problem $(M, r) \in \mathcal{R}$ and an agent $i \neq m$, such that $\phi_i(M, r) = \epsilon > 0$. Consider a new problem (M', r'), where $M' = \{1, \ldots, m+x\}$, $r'_i = r_i$ for all $i < m$ and $\sum_{j=m}^{m+x} r'_j = r_m$. By *Highest Rank Splitting Neutrality* $\phi_i(M', r') = \phi_i(M, r)$ for all $i < m$. Now, choose $x > \frac{\sum_{j=1}^{m+x} \phi_j(M', r')}{\epsilon}$. By *Hierarchical Order Preservation*, $\phi_j(M', r') \geq \epsilon$ for all $j = m, \ldots, m+x$, which contradicts balance. ∎

2.2 Optimal transfer rules

We have considered so far a purely normative approach. We singled out the parametric family of transfer rules satisfying compelling incentive and fairness constraints. Now, we assume that the boss has the power to implement a member of this parametric family, and the question is which one would maximize the boss' payoff. It is clear that, for a given fixed hierarchy, the boss will maximize her payoff by choosing the full-transfer rule. However, a full-transfer rule leaves no incentive to join the hierarchy in the first place. So, what if the boss starts on her own and needs to build a hierarchy from the start? It is natural to expect that the probability of recruiting a (revenue-generating) subordinate is connected to her potential earnings.

A dilemma thus emerges: a transfer rule associated with a low λ yields a high upward transfer, but also reduces the subordinates' (expected) payoff and, thus, the incentive to join the hierarchy voluntarily. We aim to address such a dilemma in this subsection. For ease of exposition, we assume in what follows that all revenues are normalized to unity.

One approach is to assume that agents are somewhat myopic, which would translate into stating that the probability of getting a subordinate is equal to λ itself. Another approach is to assume that subordinates are farsighted, which would imply that they take into account their ability to hire further subordinates from whom revenues will bubble up. In this latter case, the probability of getting a subordinate would be represented by the payoff of the subordinate. In what follows, we analyze both cases.

We first consider the case in which λ is the probability that any agent in the hierarchy gets a subordinate. That is, if the boss selects the full-transfer rule the probability of having agents to join the hierarchy as subordinates is 0, as all their revenues are transferred to the boss.

Likewise, using the no-transfer rule the probability is 1, as agents keep their own revenue. In general, using a transfer rule, the boss' expected revenue is given by

$$\sum_{t=0}^{\infty}(1-\lambda)^t$$

Now, if the boss aims to maximize total revenue in expected terms, when λ denotes the probability that an agent within a linear hierarchy gets a subordinate, the following problem should be solved:

$$\max_{\lambda}\sum_{t=1}^{\infty}((1-\lambda)\lambda)^t.$$

It is straightforward to see that the previous problem is equivalent to the following one:

$$\max_{\lambda}(1-\lambda)\lambda,$$

with solution $\lambda = 0.5$.

As an illustration, note that the expected transfer from subordinates to the boss at (optimal) $\lambda = 0.5$ is $\sum_{t=1}^{\infty}(1/4)^t = 1/3$.

We now assume that (possible) subordinates are farsighted and, thus, take into account their ability to hire further subordinates (once in the hierarchy) from whom revenues would bubble up.

Let δ denote the probability of getting a subordinate. In this (farsighted) case, the boss would solve the problem

$$\sup_{\lambda,\delta}\sum_{t=1}^{\infty}((1-\lambda)\delta)^t, \tag{3}$$

under the constraint that the probability δ equals the payoff of any non-boss agent,

$$\delta = \lambda + \lambda\sum_{t=1}^{\infty}((1-\lambda)\delta)^t. \tag{4}$$

It is not difficult to show that the only two possible solutions of (4) are $\delta = 1$ and $\lambda > 0$, or $\delta = \frac{\lambda}{1-\lambda}$.

In the former case it is therefore profit maximizing for the boss to set $\lambda \to 0$. In the latter case, it follows that (3) is solved when $\lambda \to 0.5$ and $\delta \to 1$.

As shown above, $\lambda = 0.5$ crops up in different settings of endogenous hierarchies as the optimal parameter choice for the boss. Thus, it is interesting to note that the transfer rule with $\lambda = 0.5$ has a close relation to the so-called *MIT strategy* (e.g., Pickard et al., 2011), a specific

mechanism for solving a task via linear recruitment graphs. More precisely, suppose solving the task amounts to a benefit of B dollars. Then, the *MIT strategy* imposes the following payment scheme: the agent who solves the task keeps $B/2$, then her recruiter gets $B/4$, the recruiter's recruiter gets $B/8$, and, so forth.[12] Such a strategy corresponds exactly to the transfer rule with $\lambda = 0.5$, in a situation where the revenue of the lowest-ranked agent is B and all other agents have revenue 0, provided the boss gets to keep the residual (due to the balance condition of our rules).

3 Branch hierarchies

In this section, we extend the linear-hierarchy case considered above to account for *branch hierarchies*, i.e., situations in which a given agent can have more than one immediate subordinate. We represent a branch hierarchy as a tree, where each agent is connected to the (unique) boss via a unique rank path consisting of all her superiors (see Figure 2).

A *branch hierarchy revenue sharing problem*, or simply, a **b-problem** is a triple (N, r, s), where N is a non-empty finite set of agents, r is a revenue profile specifying the revenue of each agent in N, and s is a function mapping each agent $i \in N$ to her immediate superior agent $j = s(i)$ (with the convention that $s(i) = \emptyset$ if i is the boss), such that the graph induced by s has no cycles.[13] Let $\mathcal{B}$ denote the set of b-problems.

Given a b-problem (N, r, s), a **b-allocation** is a vector $x \in \mathbb{R}_+^{|N|}$ satisfying **balance**, i.e., $\sum_{i \in N} x_i = \sum_{i \in N} r_i$. A **b-allocation rule** is a mapping β assigning to each problem (N, r, s) an allocation $\beta(N, r, s) = x$. We also impose, as in the linear case, that rules are **anonymous**, i.e., for each bijective function $g : N \to N'$, $\beta_{g(i)}(N', r', s') = \beta_i(N, r, s)$, where $r'_{g(i)} = r_i$, and $s'(g(i)) = g(s(i))$ for each $i \in N$.

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[12]Note that this mechanism is never in deficit, i.e., the residual from B, after obeying this payment scheme, is always non-negative.

[13]Note the deliberate change in notation from M (in the linear case) to N, as "places" in the hierarchy do not make sense for non-linear hierarchies.
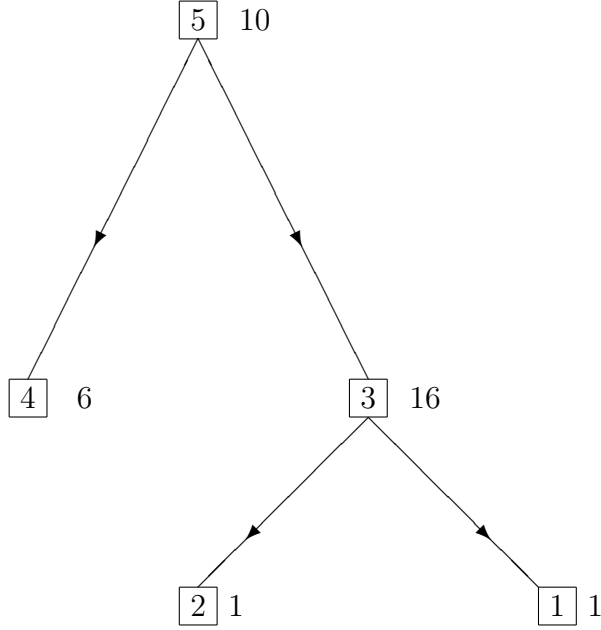
Figure 2: A branch hierarchy. This figure illustrates a branch hierarchy involving five agents, with agent 5 denoting the boss, agents 3 and 4 her direct subordinates and agents 1 and 2 being the subordinates of agent 3. Each of the two agents at the third layer generate a revenue of 1. Agent 4 yields a revenue of 6, whereas agent 3 yields a revenue of 16. Finally, agent 5 yields a revenue of 10. In summary, the hierarchy so illustrated is $(N, r, s) = (\{1, 2, 3, 4, 5\}, (1, 1, 16, 6, 10), s)$, where $s(1) = s(2) = 3$, $s(3) = s(4) = 5$ and $s(5) = \emptyset$.

The transfer rules have a simple generalization to branch hierarchies. Formally, let i be an agent at the bottom of the hierarchy, somewhere in the tree. Then,

$$x_i^\lambda = \lambda r_i.$$

Her immediate superior $k = s(i)$ gets

$$x_k^\lambda = \lambda \left(r_k + \sum_{j \in N:\ k=s(j)} (1 - \lambda) r_j \right),$$

and so forth. Denote the corresponding family of b-allocation rules by $\{\beta^\lambda\}_{\lambda \in [0,1]}$.

Our axioms from the linear hierarchy model also have a natural extension to the branch hierarchy model. Formally,

b-Lowest Rank Consistency: For each $(N, r, s) \in \mathcal{B}$, with $|N| \geq 2$, and each $i \in N$ without subordinates, such that $(N \setminus \{i\}, (r_{s(i)} + r_i - \beta_i(N, r, s), r_{N \setminus \{i, s(i)\}}), s_{N \setminus \{i\}}) \in \mathcal{B}$,[14]

$$\beta_{N \setminus \{i\}}(N, r, s) = \beta \left(N \setminus \{i\}, (r_{s(i)} + r_i - \beta_i(N, r, s), r_{N \setminus \{i, s(i)\}}), s_{N \setminus \{i\}} \right).$$

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[14]By $s_{N \setminus \{i\}}$ we denote the restriction of the function s to the domain $N \setminus \{i\}$.

b-Highest Rank Revenue Independence: For each $(N, r, s) \in \mathcal{B}$, each $i \in N$ such that $s(i) = \emptyset$, and each $\hat{r}_i \in \mathbb{R}_+$

$$\beta_{N \setminus \{i\}}(N, r, s) = \beta_{N \setminus \{i\}}\left(N, (r_{-i}, \hat{r}_i), s\right).$$

b-Highest Rank Splitting Neutrality: For each $(N, r, s) \in \mathcal{B}$, and each $i \in N$ such that $s(i) = \emptyset$, let (N', r', s'), be such that $N' = N \cup \{k\}$, $s'(i) = k$, $s = s'$ otherwise, $r_i = r'_k + r'_i$, and $r'_{N \setminus \{i,k\}} = r_{N \setminus \{i\}}$. Then,

$$\beta_{N \setminus \{i,k\}}(N', r', s') = \beta_{N \setminus \{i\}}\left(N, r, s\right).$$

b-Scale Invariance: For each $(N, r, s) \in \mathcal{B}$, and each $\alpha > 0$,

$$\beta(N, \alpha r, s) = \alpha \beta(N, r, s).$$

With these extended axioms in place we can now extend Theorem 1 to branch hierarchies.[15]

Theorem 4 *A b-rule β satisfies b-Lowest Rank Consistency, b-Highest Rank Revenue Independence, b-Highest Rank Splitting Neutrality, and b-Scale Invariance if and only if it is a b-transfer rule, i.e., $\beta \in \{\beta^\lambda\}_{\lambda \in [0,1]}$.*

Proof: It is not difficult to see that the b-transfer rules satisfy all the axioms at the statement of the theorem. Conversely, let β be a rule satisfying all the axioms at the statement of the theorem. Let $(N, r, s) \in \mathcal{B}$. We distinguish two cases.

Case 1: (N, r, s) is a linear hierarchy.

In this case, the branch hierarchy $(N, r, s) \in \mathcal{B}$ consists of a line, and thus we use the abbreviated notation $(N, r) \in \mathcal{R}$. Then, by Theorem 1, there exists $\lambda \in [0, 1]$, such that $\beta(N, r) = \beta^\lambda(N, r)$.

Case 2: (N, r, s) is not a linear hierarchy.

Let i denote an agent without subordinates in the branch hierarchy (N, r, s). Then, $x_i = \beta_i(N, r, s) = \delta r_i$ for some $\delta \in [0, 1]$.

Iteratively, we can apply *b-Lowest Rank Consistency* to all agents not located on the direct path of superiors from i to the boss, in order to reduce the branch hierarchy to a line. For each

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[15]Note that replacing scale invariance by canonical fairness singles out the intermediate transfer rule for which $\lambda = 0.5$.

iteration, the payment remains unchanged for agent i and we, ultimately, end up with a linear hierachy. It then follows from Case 1 that $\delta = \lambda$.

The previous argument can be repeated for any agent without subordinates, which shows that δ is not agent-specific. Thus, $x_j = \beta_j^\lambda(N, r, s)$, for each agent j without subordinates. Now, consider an agent h who is the immediate superior of an agent without subordinates. By using *b-Lowest Rank Consistency*, for each subordinate of h, we obtain a new problem in which agent h has no subordinates, and in which the revenue of agent h corresponds to her original revenue, plus the surplus from all the subordinates of h. Applying the same argument as above, it follows that $x_h = \beta_h^\lambda(N, r, s)$. The proof easily concludes from here. ∎

4 Joint ownerships

An important limitation of the previous analysis is that hierarchies contain a single boss. It is often the case that a given agent has more than one superior, in which case we talk about *joint ownerships*. For instance, two firms may jointly own an entity on an equal partnership basis and that entity may again own other entities, either alone or as joint ventures. Similarly, for social mobilization schemes, an agent may be approached by several recruiters and may solve tasks for all of them. The aim of this section is to extend the previous analysis to account for the case of joint ownerships. As we shall see, a generalized version of our family of transfer rules will also arise in this setting.

A *joint ownership revenue sharing problem*, or simply, a **j-problem** is a triple (N, r, S), where N is a non-empty finite set of agents, r is a revenue profile specifying the revenue of each agent in N, and S is a correspondence, mapping each agent $i \in N$ to her immediate superior agents $S(i) \subset N$ (with the convention that $S(i) = \emptyset$ if i is a boss), such that the graph induced by S is connected and cycle free (in the undirected sense: that is, for instance, $S(1) = \{2, 3\}, S(2) = \{3\}, S(3) = \emptyset$ is considered a cycle). Let $\mathcal{J}$ denote the set of j-problems.

Note that, as the graph induced by S has no cycles, deleting any link ij leads to two components of such a graph, dubbed the i- and the j-component, and denoted by G_{ij}^i and G_{ij}^j respectively.

Given a j-problem (N, r, S), a **j-allocation** is a vector $x \in \mathbb{R}_+^{|N|}$ satisfying **balance**, i.e., $\sum_{i \in N} x_i = \sum_{i \in N} r_i$.

A **j-allocation rule** is a mapping ζ assigning to each problem (N, r, S) an allocation $\zeta(N, r, S) = x$. We also impose from the outset, as in the linear case, that rules are **anony-**

mous, i.e., for each bijective function $g : N \to N'$, $\zeta_{g(i)}(N', r', S') = \zeta_i(N, r, S)$, where, for each $i \in N$, $r'_{g(i)} = r_i$, and $S'(g(i)) = g(S(i)) = \{g(s) : s \in S(i)\}$. Our family of transfer rules generalizes easily to the joint ownership setting by transferring an equal split of the accumulated surplus of a given agent i to each of her immediate superiors.
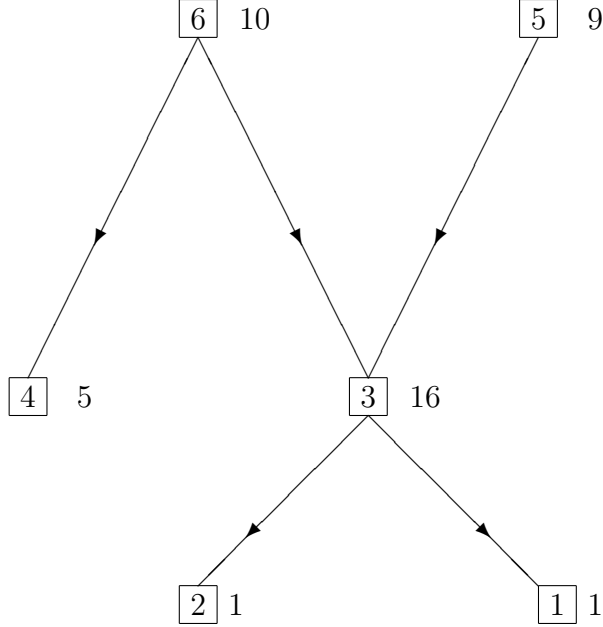
Figure 3: A joint ownership. This figure illustrates a joint ownership involving six agents, with agents 5 and 6 denoting the bosses, agent 3 being direct subordinate of both, agent 4 direct subordinate of 6, and agents 1 and 2 being the subordinates of agent 3. Each of the two agents at the third layer generate a revenue of 1. Agent 3 yields a revenue of 16, whereas agent 4 yields a revenue of 5. Finally, agent 5 yields a revenue of 9, and agent 6 yields a revenue of 10. In summary, the problem so illustrated is $(N, r, S) = (\{1, 2, 3, 4, 5, 6\}, (1, 1, 16, 5, 9, 10), S)$, where $S(1) = S(2) = \{3\}$, $S(3) = \{5, 6\}$, $S(4) = \{6\}$ and $S(5) = S(6) = \emptyset$.

Formally, let i be an agent at the bottom of the hierarchy, somewhere in the tree. Then,

$$x_i^\lambda = \lambda r_i.$$

Each of her immediate superiors $k \in S(i)$ gets

$$x_k^\lambda = \lambda \left(r_k + \sum_{j \in N: \ k \in S(j)} \frac{1}{|S(j)|} (1 - \lambda) r_j \right),$$

and so forth. Denote the corresponding family of j-allocation rules by $\{\zeta^\lambda\}_{\lambda \in [0,1]}$.

Three of our axioms from the linear hierarchy model have a natural extension to the joint ownership model. Formally,

j-Highest Rank Revenue Independence: For each $(N, r, S) \in \mathcal{J}$, each $i \in N$ such that $S(i) = \emptyset$, and each $\hat{r}_i \in \mathbb{R}_+$,

$$\zeta_{N \setminus \{i\}}(N, r, S) = \zeta_{N \setminus \{i\}}\left(N, (r_{-i}, \hat{r}_i), S\right).$$

j-Highest Rank Splitting Neutrality: For each $(N, r, S) \in \mathcal{J}$ and each $i \in N$ such that $S(i) = \emptyset$, let (N', r', S'), be such that $N' = N \cup \{k\}$, $S'(i) = \{k\}$, $S' = S$ otherwise, $r_i = r'_k + r'_i$, and $r'_{N \setminus \{i,k\}} = r_{N \setminus \{i\}}$. Then,

$$\zeta_{N \setminus \{i,k\}}(N', r', S') = \zeta_{N \setminus \{i\}}\left(N, r, S\right).$$

j-Scale Invariance: For each $(N, r, S) \in \mathcal{J}$, and each $\alpha > 0$,

$$\zeta(N, \alpha r, S) = \alpha \zeta(N, r, S).$$

On the other hand, the fact that a joint ownership might involve several bosses, as well as several superiors for the lowest ranked agents, calls for adjustments of the remaining axioms, as well as for new axioms.

We first strengthen lowest rank consistency. To do so, consider an agent i and one of her immediate subordinates j. It seems normatively appealing to state that deleting the j-component, and transferring any surplus from that component to i, should leave the payoffs of all agents in the i-component unchanged. Formally,

Component Consistency: For each $(N, r, S) \in \mathcal{J}$, and each pair $i, j \in N$ such that $i \in S(j)$, let

- $N' = G^i_{ij}$,

- $r'_i = r_i + \sum_{k \in G^j_{ij}} (r_k - \zeta_k(N, r, S))$,

- $r'_h = r_h$ for each $h \in G^i_{ij} \setminus \{i\}$,

- $S'(k) = S(k)$, for each $k \in G^i_{ij}$.

Then, $(N', r', S') \in \mathcal{J}$ and, for each $h \in N'$,

$$\zeta_h(N', r', S') = \zeta_h(N, r, S).$$

Clearly, component consistency implies lowest rank consistency, as the j-component may consist of agent j alone.

The following axiom is new. It refers to the case in which a given agent has several bosses that are not bosses for any other agents. In such situations, merging the bosses will not change the payoff of the remaining agents. Formally,

Top Merger: For each $(N, r, S) \in \mathcal{J}$ and each $j \in N$ such that $|S(j)| \geq 2$, $S(k) = \emptyset$ for each $k \in S(j)$, and $S(h) \cap S(j) = \emptyset$, for each $h \in N \setminus \{j\}$, let $(N', r', S') \in \mathcal{J}$ be such that

- $N' = (N \setminus S(j)) \cup \{k'\}$,

- $r'_{k'} = \sum_{k \in S(j)} r_k$, and $r'_h = r_h$ for each $h \in N \setminus S(j)$,

- $S'(k') = \emptyset$, $S'(j) = k'$, and $S'(k) = S(k)$, for each $k \in N \setminus (S(j) \cup \{j\})$.

Then, for each $h \in N \setminus S(j)$,

$$\zeta_h(N', r', S') = \zeta_h(N, r, S).$$

We are now ready to extend Theorem 1 to joint ownership problems.[16]

Theorem 5 *A j-rule ζ satisfies j-Highest Rank Revenue Independence, j-Highest Rank Splitting Neutrality, j-Scale Invariance, Component Consistency, and Top Merger if and only if it is a j-transfer rule, i.e., $\zeta \in \{\zeta^\lambda\}_{\lambda \in [0,1]}$.*

Proof: It is not difficult to see that the j-transfer rules satisfy all the axioms at the statement of the theorem. Conversely, let ζ be a rule satisfying all the axioms at the statement of the theorem. We prove this implication by induction. First, by Theorem 1, there exists λ such that $\zeta = \zeta^\lambda$ for two-agent problems. Suppose there is λ such that $\zeta = \zeta^\lambda$ for all problems with up to $k \geq 2$ agents and consider the subfamily of problems with $k + 1$ agents. Let $(N, r, S) \in \mathcal{J}$ be one of those problems and let $i \in N$ be such that $S^{-1}(i) = \emptyset$. We now claim that $\zeta_i(N, r, S) = \lambda r_i$. Indeed, by repeated use of *Component Consistency*, we can construct a new problem for which all other agents (different from i) have a unique linear path to i, such that i's payoff is unchanged. Now, by repeated use of *Top Merger* and *j-Highest Rank Splitting Neutrality*, we obtain a new (two-agent) problem for which agent i gets $\zeta_i(N, r, S) = \lambda r_i$.

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[16]Note that replacing scale invariance by canonical fairness singles out the intermediate transfer rule for which $\lambda = 0.5$.

Now, let $j \in S(i)$. We claim that $\sum_{h \in G_{ij}^j} \zeta_h(N,r,S) = \sum_{h \in G_{ij}^j} r_h + \frac{(1-\lambda)r_i}{|S(i)|}$. Indeed, by repeated use of *Component Consistency*, *Top Merger* and *j-Highest Rank Splitting Neutrality* we can reduce the j-components to a single agent, where this agent receives the same payoff as the entire j-component did before. By *j-Highest Rank Revenue Independence* and anonymity of ζ the claim follows.

Consequently, $\sum_{h \in G_{ij}^i} \zeta_h(N,r,S) = \sum_{h \in G_{ij}^i} \zeta_h^\lambda(N,r,S)$.

Now, for a given $j \in S(i)$, by *Component Consistency* we can add the surplus of the i-component, i.e., $\sum_{h \in G_{ij}^i}(r_h - \zeta_h(N,r,S))$, to j and then eliminate the i-component. By our induction hypothesis, the payoff of an arbitrary agent $h \in G_{ij}^j$ is $\zeta_h(N,r,S) = \zeta_h^\lambda(N,r,S)$, which concludes the proof. ∎

5 Conclusion

Priorities among agents is a compelling way to express asymmetric rights. We have dealt in this paper with a resource allocation problem arising when every agent is arranged in a priority structure and generates a collective profit. More precisely, we have considered a stylized model in which participating agents, who are hierarchically organized, contribute with (possibly different) individual revenues to the collective proceeds. The canonical application of our model is the case of multi-level marketing. Another interesting application is the allocation of profit in companies, as our model is flexible enough to accommodate several forms of professional organizations and practices in real life.

We have introduced a family of allocation rules for our model, ranging from the rule ignoring the command structure conveyed by the hierarchy, to the rule ignoring individual contributions to the joint proceeds. The rules are members of a one-parameter family with an interesting economic interpretation conveying a compromise between those two polar rules, allowing for certain upward transfers in the command structure.

The intermediate member of our family, obtained when the compromise between the polar rules is balanced, is a translation to our context of the so-called MIT strategy, which has shown to be an optimal mechanism in practice for social mobilization. We also show that the rule is optimal, within our family, if the aim is to maximize the expected revenues of the venture capitalist, i.e., the agent at the top of the hierarchy, and the process to get subordinates is probabilistic. In general, the rules within our family also exemplify usual practices in multi-level marketing, the marketing approach in which buyers are encouraged to take an active role

in promoting the product (offering them rewards for successful direct or indirect referrals of the product to other prospective buyers). Our results, therefore, provide normative foundations for such a type of strategies, formalizing the idea of 'bubbling up' revenues along the hierarchy.

Our analysis not only involves the benchmark case of linear hierarchies, but also more general hierarchical structures, including cases of joint ownership. Thereby, our results also provide new insights for the popular field of fair allocation in networks.

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