Higher Amalgamation and Finite Covers

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Abstract

Higher amalgamation is a model theoretic property. It was also studied under the name generalised independence theorem. This property is defined in stable, or more generally simple or rosy theories. In this thesis we study how higher amalgamation behaves under expansion by finite covers and algebraic covers.

We first show that finite and algebraic covers are mild expansions, in the sense that they preserve many model theoretic properties and behave well when imaginaries are added to them. Then we show that in pregeometric theories higher amalgamation over $\emptyset$ implies higher amalgamation over parameters. We also show that in general this is not true. In fact, for any stable theory with an algebraic closed set which is not a model we construct a finite cover which fails 4-Amalgamation. With some additional assumption we can also preserve higher amalgamation over the empty set. We apply this result to abelian groups and show that $(\mathbb{Z}/4\mathbb{Z})^\omega$ satisfies these assumptions.

Then we take the opposite direction: rather then investigating covers which have malicious properties towards amalgamation, we construct covers which will make higher amalgamation become true. First we give a new proof for the fact that there exists an algebraic cover of any stable $T^{\mathrm{eq}}_{\mathrm{acl}^{\mathrm{eq}}(\emptyset)}$ with higher amalgamation over $\emptyset$. A proof sketch of this was given by Hrushovski and a full proof appeared in an unpublished work by D. Evans. The new proof uses the notion of symmetric witness which was introduced by Goodrick, Kim and Kolesnikov. We also show with a similar approach that there exists an algebraic cover of any stable, omega-categorical theory with higher amalgamation over parameters.

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Chapter 1

Introduction

This introduction is split into three parts. The first part will be a summary of the thesis itself. The second part will put the thesis into context with earlier results. The third part will discuss what the thesis did not accomplish and where the author sees potential for future work.

1.1 Summary

This thesis is written for mathematicians with some background in first-order logic. The requirement for the reader is to have some knowledge in model theory. The knowledge of some introductory model theory lectures should be sufficient. It will be very helpful to have seen some variant of forking, for example the pregeometry of strongly minimal structures, the Morley rank of totally transcendental theories, the notion of (co)heirs, or the extension of definable types. In the thesis we work with an axiomatic approach and avoid the use of any explicit notion of forking when possible. For that we will introduce it as an abstract notion of independence instead. A reader who does not know any notion of independence should keep in mind that this notion naturally appears in many examples such as the algebraic independence in algebraically closed fields or the independence of subspaces of a vector space. But at some points we have to use its combinatorial description. Note that theories which have an abstract notion of independence (in T^{eq}) are rosy theories. The notion of independence can be combinatorially described as thorn-forking. The knowledge of NIP and NTP_2 is also assumed in some very minor parts.

Apart from the notion of independence, Chapter 2 contains several other things. We start with explaining the notation, which is used for multi-sorted theories. Then we define the monster model as a saturated model of class-size. We explain what imaginary elements of a theory are and also give the reason why they were introduced, namely to obtain stationarity of types over an algebraically closed set in stable theories (see Lemma 2.7.7). We also define rosy, stable and simple theories purely in terms of the abstract notion of independence. Finally we prove some very elementary things in category theory which will be used later.

In Chapter 3 we introduce the notion of algebraic and finite covers. An algebraic cover of some structure is an expansion of it with new sorts. In an algebraic cover the old part is (up to interdefinability) the same, and the new structure lies in the algebraic closure of the old structure. A finite cover is an algebraic cover which additionally requires that there is only one new sort and a finite-to-one function which maps the new elements to (some of) the old ones. Any algebraic cover of a theory which has its imaginaries added (T^{eq}) with only one new sort will automatically be a finite cover (see Corollary 3.2.4). If our language is countable we can also describe an algebraic cover as a sequence of finite covers. We then prove that adding imaginaries to the new theory does not destroy its property as an algebraic cover. Further we prove that going over to a finite cover preserves many model theoretic properties. For example categoricity, rosiness, stability and simplicity will be preserved. Moreover, the notion of independence will be preserved in a canonical way. Finally we give a condition which will help us in proving weak elimination of imaginaries of finite covers of ω-categorical theories.

Chapter 4 is about the central notion of this thesis, "Amalgamation" and contains only known results by other authors. We describe what an amalgamation problem is and what a solution to such a problem is. An n-amalgamation problem is a functor from $\mathcal{P}(\{1,\ldots,n\}) \setminus \{1,\ldots,n\}$ with inclusions as morphisms to the category of algebraically closed sets of a certain theory with partial elementary maps as inclusions. Additionally we have some independence condition about the images of the functor. A solution to an n-problem is an extension of this functor to one with domain $\mathcal{P}(\{1,\ldots,n\})$. We further discuss when a solution is considered unique, namely when any two solution functors are naturally isomorphic. This might fail even in stable theories (see Example 5.2.1). For the rest of chapter we reprove known results about amalgamation. We will see how the existence of a solution of n-amalgamation problem is connected to the uniqueness of a solution of an $(n-1)$-problem. Moreover, this notion can be used to define stable and simple theories. We (re)prove some things about the boundary property which is equivalent to the uniqueness of a solution in stable theories under the assumption that the solution of all lower problems is unique (see Proposition 4.4.7 and Corollary 4.5.9). To briefly describe the 3-boundary property we take 3 independent points of some model, say a, b, c. Any automorphism of the algebraic closure of ab which fixes the algebraic closure of a and the algebraic closure of b pointwise can be extended to an automorphism which fixes the algebraic closure of ac and bc. Finally we analyse why amalgamation over models in stable theories is always true.

In Chapter 5 we try to answer the question of whether amalgamation over parameters is related to amalgamation over the empty set. The first section gives an explicit description of the connection between the two if the independence notion is sufficiently nice. We develop for that the notion of separability of an independence notion. A notion of independence is separable whenever for any two sets (of some model) $A \subset B$ with B algebraically closed there exists some $C \subset B$ independent of A such that $\operatorname{acl}(AC) = B$ (in T^{eq}). This is always fulfilled in pregeometric theories with weak elimination of imaginaries.

Then we will take a long path to establish a counterexample: a theory which has amalgamation over $\emptyset$, but fails amalgamation over some parameters. For that we first generalise the construction done in Hrushovski's original example of failure of 3-uniqueness (which was a finite cover of the theory of an infinite set) to arbitrary theories. This construction will preserve weak elimination of imaginaries if our original theory was stable or ω-categorical.

We prove by using this construction that any theory (with an independence notion) which has some algebraically closed set A which is not a model of the theory has a finite cover which fails 3-uniqueness over A (see Lemma 5.5.1). We continue our road to a counterexample by establishing some theorems telling us that this construction preserves amalgamation over $\emptyset$ under certain conditions. We will then apply these theorems to abelian groups whose theory is closed under products. Finally we use all this to construct a finite cover of $(\mathbb{Z}/4\mathbb{Z})^\omega$ which has amalgamation over $\emptyset$, but fails it over some parameters. In a sense, this is the simplest example one can find, since any totally categorical structure which does not interpret an infinite group is in the algebraic closure of a disintegrated strongly minimal set. Hence the independence notion of such a theory is separable and therefore as we have seen amalgamation over $\emptyset$ implies amalgamation over parameters. We also have such an example where the original theory does not interpret any infinite group. It is uncountably categorical, but of course as discussed not ω-categorical. Finally we see that there is a "natural" example, namely the theory of compact complex manifolds, which has amalgamation over $\emptyset$ but fails it over some parameter.

In Chapter 6 we introduce the notion of an n-witness. We show that failure of amalgamation gives us the existence of such a witness. On the other hand the existence of such a witness gives us also the failure of amalgamation. We also show that if we look at amalgamation problems where the singletons satisfy the same type, then amalgamation is connected to a Morley witness.

Then in Chapter 7 we let a witness disappear. We do that by constructing a finite cover of $T^{\mathrm{eq}}_{\mathrm{acl}(\emptyset)}$ to which we add some generic element of the corresponding type of a witness. Then we add elements of the witness which contains this generic element in a canonical way (see "Construction of a Finite Cover"). Now in this finite cover the old witness will lose its ability to define an amalgamation problem which has no solution (see 7.2.2.) We then use this property for every witness and get some algebraic cover which has no witness of the old sort. We repeat this process countably many times to get some algebraic cover of $T^{\mathrm{eq}}_{\mathrm{acl}(\emptyset)}$ which has amalgamation over $\emptyset$.

In the final Chapter 8 we let all witnesses of an ω-categorical superstable structure disappear. Here we have the nice property that a witness is fully determined by some formula without parameter (every realisation of this formula is a witness). We define a finite cover of the theory (without adding the parameters $\mathrm{acl}(\emptyset)$ to our theory) for any witness (over parameters). With a similar reasoning (as in the chapter before) we then obtain an algebraic cover which has amalgamation (over parameters).

We summarise which chapters of this thesis are original. Chapter 2 has no new ideas. Chapter 3 is only new (for me) in the sense that I could not find reference for many results, but I expect that most of it is folklore. Also in Chapter 4 there is nothing new, except that some results are extended to rosy theories (which is straightforward) and that some proofs are given in greater detail. Chapter 5 is completely original (apart from what is cited). Chapter 6 contains only slight modifications of ideas of others. In Chapter 7 the results are not new, but the proof idea is. And finally in Chapter 8 the results are new, but they are closely related to the results of Chapter 7.

1.2 Related research

The development of good notions of independence started with strongly minimal theories, then Morley rank and the development of forking for stable and later for simple theories. Then it was continued beyond to the notion of thorn-forking in rosy theories. This is closing the development of the notion of independence for the following reason: this is the weakest of all notions, i.e. if there is a strict independence notion at all, then thorn-forking will be a strict independence notion and any elements being thorn-independent will be independent for any other strict independence notion (see Theorem 5.2 of [Adl09b]). Beyond that, recent work shows that forking still has nice properties in NTP_2 and its subclass NIP (see for example [CK12] and [BC14]). But this development does not play a crucial part in this thesis.

The notion of (finite, affine) covers originates from the analysis of totally/uncountable categorical structures (see Zilber's Ladder Theorem 5.0.1 of [Zil93]). Albrandt and Ziegler's [AZ91] also contains important work. Furthermore, in [EH93] finite covers of ω-categorical theories were analysed. The papers most relevant to our work are [Hru12] and [Eva09], where in the first the ideas for the relation between finite covers and amalgamation is started and the latter worked out some of that in greater detail.

The history of amalgamation (we will use) also goes along with the development of forking. Stationarity of types is nothing more than having uniqueness of 2-amalgamation of its independent realisations. Elimination of imaginaries were introduced by Shelah to obtain that any type over an algebraically closed set is stationary.[1] Shelah introduced higher amalgamation diagrams in [She83]. In [She90] (see Ch XII section 2 p 598) (unique) higher amalgamation for diagrams of models is proved. Hrushovski proved for pseudo-finite fields that 3-amalgamation holds. This was generalised to simple theories: the independence theorem by Kim and Pillay is nothing more then the proof of 3-amalgamation over models. The notion of higher amalgamation we use reappeared in [Hru98]. The generalised independence theorem, which says that a theory has n-amalgamation for every n was proved for algebraically closed fields with an automorphism (see Theorem after 1.9 in [CH99]). In [Kol05] simple theories were further analysed via the notion of amalgamation. This was then continued in [KKT08]. The assumption of 4-amalgamation was used in [PKM06] to construct a hyperdefinable group from the group configuration theorem for simple theories in [BTW04]. In the same paper [PKM06] it was also proved that (unique) n-amalgamation over a model in stable theories always holds.

The key paper is Hrushovski's [Hru12], which became available as a preprint in 2006. In there, it is shown that in stable theories uniqueness of 3-amalgamation, 4-amalgamation, generalised imaginaries and certain finite covers are all strongly connected (see Corollary 4.10 in [Hru12]). Of special interest is Proposition 4.11 (in [Hru12]), which tells us that for every stable theory there exists an algebraic cover such that n-amalgamation holds. Unfortunately the proof was only sketched. In the unpublished paper of David Evans [Eva09] this proof was worked out.

Research which continues the work of Hrushovski's paper (i.e. [Hru12]) and also heavily influenced this thesis is a series of publications by Goodrick,

[1] On page 186 in [Hru12] Hrushovski writes: "Elimination of imaginaries was introduced in [She78] precisely in order to obtain 2 -exactness for stable theories."

B. Kim and Kolesnikov. Namely the papers [GK10], [GKK14a], [GKK13b], [GKK13a], [GKK14b], [GKK15] and [Kim16]. In there many ideas of Hrushovski's paper (i.e. [Hru12]) are worked out and generalised. A model theoretic homology and homotopy theory is developed using the notion of amalgamation and then a sort of (higher) Hurewicz Theorem is proved for this (see Theorem 2.1 of [GKK14b]). It is also shown that the possible "homotopy" groups are abelian pro-finite groups and a series of examples is given with such "homotopy" groups. Also a higher analogue of these corresponding homotopy groupoids is discussed. The notion of polygroupoid is used for these "higher homotopy groupoids". Another version of homotopy theory of strong types with a not necessarily abelian fundamental group is developed in [KKL15].

Examples of stable theories (in fact totally categorical theories) which fail n-amalgamation (but guarantee lower uniqueness) have been constructed in [PS11]. The same author also showed (in [Pas12]) that, under certain assumptions, there is a connection between the Hrushovski construction and higher amalgamation. In [BHM15] it is shown that the theory of compact complex manifolds fails unique 3-Amalgamation over parameters. Then in [Pal16] the assumption of total amalgamation is used in some simple theory. Finally in [Kru15] a slightly different version of higher amalgamation is used (although they coincide under certain assumptions) to show that certain non-simple theories are pseudofinite.

1.3 Future work

It is left to be proven that an algebraic cover preserves other model theoretic notions such as NIP or NTP_2. The analysis of a connection between amalgamation and the uniqueness over a fixed set of parameters looks complete in the stable context. But outside that, one could probably try to find a counterexample with the property that n-amalgamation holds for every n but $(n-1)$-uniqueness fails. It would be interesting to see what the exact boundaries of this would be. For example, can uniqueness and amalgamation both fail and hold (for different n) infinitely often in the same theory? Then we question if there is a good dividing line between theories where amalgamation over the empty set is linked (i.e. there does not exist an algebraic cover such that only one of the two holds) to amalgamation over parameters (see Chapter 5.1) and theories where this not the case (see the $(\mathbb{Z}/4\mathbb{Z})^\omega$ example).

Another question is if we can generalise our cover $T_{2,\phi}$ (see Chapter 5) such that the "homotopy group" (in the amalgamation sense, see [GKK14b]) is any profinite abelian group (if the original theory has amalgamation over $\emptyset$). We can also ask if there is an algebraic cover for any (stable/simple) theory (or even in the rosy non-simple context) such that amalgamation over parameters holds for any n (outside the stable context 2-uniqueness would need to fail and outside the simple context 3-Amalgamation). Note that we only achieved this for superstable, ω-categorical theories. Now if we put all the ideas together then we can conjecture the following: For any stable theory and any profinite abelian group G there exists an algebraic cover such that the "homotopy group" (over $\emptyset$) of this cover is G. Is something similar also true for "higher homotopy groups" (in the sense of [GKK15])?

Moreover, we conjecture that, if in a stable theory forking is complicated enough (of course here the question is what the precise notion of complicated is) then for any two pro-finite abelian groups G, H, there exists an algebraic cover where the "homotopy group" over $\emptyset$ is G and the "homotopy group" over some parameter is H. Of course here we could also formulate some higher dimensional version of this. Finally we can ask if there is a good notion of (higher) amalgamation beyond theories with an independence notion. It is probably wise to try starting with forking in NIP or NTP_2, because there the Independence Theorem for NTP_2 theories holds (see Theorem 3.3 of [BC14]), which we could consider as suitable candidate for 3-amalgamation.

Chapter 2

Notation and Preliminaries

An apology: I will recurringly use the book [TZ12] to refer to third party results. So this book should not be viewed as an original source, but as something which serves my own convenience. A great many of the ideas around stability stem from [She90] and earlier publications of the same author.

2.1 Many-sorted language

We work in the first-order logic. The symbol T will normally represent some complete many-sorted first-order L-theory with infinite models. We use the same notation M for an L-structure and its domain. Also we will not distinguish between symbols in the language L and interpretation of them in some L-structure. Remember that a many sorted language L has some set S of sorts present. This means that variables and constants will be of a certain sort, relations will be of certain sorts $(s_1, \dots, s_n) \in S^n$ and function symbols will be of certain sorts, i.e. from some sorts $(s_1, \dots, s_n) \in S^n$ to some sort $s \in S$. For any many sorted L-structure M we write M_s to denote the elements of sort s. For more details about many-sorted languages, structures and theories see for example the "many-sorted" parts of chapter 1 and 2 in [TZ12].

Definition 2.1.1. Let T be an S-sorted L-theory. By T^{S_0} or $T \restriction S_0$ with $S_0 \subset S$, we mean the restriction of T to the language $L \restriction S_0$. The restriction of a language to some S_0 is done by removing function, relations, constants and sorts which are not of sorts in S_0^n. In similar fashion for an S-sorted L-structure M by M^{S_0} or $M \restriction S_0$ we mean the *reduct*. The reduct of M is constructed by removing all functions, relations and constants which are not of sorts in S_0^n and also all M_s with $s \in S - S_0$.

An L-formula $\phi(x_1, \dots, x_n)$ indicates that $x_1, \dots, x_n$ are all its free-variables with the possibility that some of them are dummies. We also allow that formulae have infinitely many variables, but require that all but finitely many of them are dummies. In classical notation this essentially means that if we have a formula $\phi(\overline{x}, \overline{y})$ and some potentially infinite set A, then by $\phi(\overline{x}, A)$ we mean a formula $\phi(\overline{x}, \overline{a})$ for some tuple $\overline{a}$ of A. Note that by this convention the following is true: Let S be the set of sorts of the fixed variable $\overline{y}$. Then for some formula $\phi(\overline{x}, \overline{y})$ we can plug in any set A for $\overline{y}$, as long as it contains at least one element for each sort of S.

2.2 Monster model

Definition 2.2.1. The *monster model* of a complete first-order theory T with infinite models is a **class-size** model of T which is κ-universal, strongly κ-homogeneous and κ-saturated for any cardinal κ. If the reader wants to work inside ZFC and there exists some saturated models of arbitrary large size, then they may take one saturated model which is of some large enough cardinal κ instead. If the reader wants to work inside ZFC and there do not exist saturated models (of arbitrary large size) the reader may take a large enough *special model* (of size κ) instead. A special model is a model M of cardinality λ which has a *specialising chain* $M_\kappa : \kappa < \lambda$. A specialising chain $M_\kappa : \kappa < \lambda$ (of M) is an elementary chain such that $M = \bigcup_{\kappa<\lambda} M_\kappa$ and such that for each κ we have that M_κ is κ^+-saturated.

Note that if the reader is working in one of the ZFC cases, then please make the following replacements in the rest of thesis: Whenever we mention a (definable) "class", except when we talk about equivalence classes, then replace the word "class" by the word "set" (of size $\leq \kappa$) and when we talk about a (definable) "set", then replace the word "set" by "small set" (i.e. of size $< \kappa$).

We denote by $\mathfrak{C}$ the monster model of T. For details on how to construct a monster model of class size and on the set-theory required[1] see Section 6.1 and the Appendix A of [TZ12]. Moreover, it is noted there that for a stable theory T there exists a *monster model of set-size* for every regular cardinal of size $\kappa^{|T|} = \kappa$. For the construction of special models a good exposition is Chapter 10.4 in [Hod93].

Now the notation will be that $A, B, C \ldots$ will normally denote subsets (not classes) (if you do not want to work with a class-size monster then consider them small sets) of the monster model $\mathfrak{C}$ (see 2.2.1), $a, b, c \ldots$ will be elements of the monster model and will often be confused with tuples. The same holds for variables x — they can also be considered tuples. As we often work in T^{eq} this will then be automatic (see next section). The notation $a \equiv_C b$ means $\mathrm{tp}(a/C) = \mathrm{tp}(a/C)$. By $S_x(A)$ we denote the type-space in the language $L(A)$ where the types have x as their variable(s) (hence they are of certain sort(s)). We may also write $S_{\overline{s}}(A)$ for $\overline{s}$ a tuple of sorts if we do not care about the concrete variable and just want them to be of sorts $\overline{s}$. We also omit x and write that $S(A)$ and hope that it is clear of which sort our types are (we require that the sort is fixed). By a *global type* we mean a type in the space $S(\mathfrak{C})$.

Recall the following:

Definition 2.2.2. An S-sorted theory T is *κ-stable* if for all subsets A of size $< \kappa$ of the monster model and all sorts $s \in S$ we have that $S_s(A)$ is of size $< \kappa$. We say that T is *stable* if it is κ-stable for some κ.

Note that if a theory is κ-stable, then for all subsets of the monster A of size $< \kappa$ and all tuples of sorts $\overline{s}$ of S we have that $S_{\overline{s}}(A)$ is of size $< \kappa$ (see Lemma 5.2.2 of [TZ12]).

[1] It will be the Bernays-Gödel set theory with global choice (in short BGC). See for example [Fel71] to note that the BGC set theory is a conservative extension of ZFC (so any set theoretical statement provable in BGC will be provable in ZFC).

Fact 2.2.3. (Many-sorted Ryll-Nardzewski Theorem) A many-sorted countable theory is ω-categorical if and only if for all $n \in \mathbb{N}$ and for every given n-tuple of sorts $\overline{s}$ we have that $S_{\overline{s}}(\emptyset)$ is finite.

This is just as in the single-sorted ω-categorical case. If there are only finitely many types (in a finite amount of sorts), then any countable structure will be ω-saturated (i.e. each type is realised). Then this direction follows by noting that any two countable ω-saturated model (of the same theory) are isomorphic. The other direction can be also proved by assuming that $S_{\overline{s}}(\emptyset)$ is infinite and then using Omitting Type Theorem.

The group $\mathrm{Aut}(A/B)$ contains all automorphisms of A which fix B pointwise. For A a subset of the monster, $\mathrm{dcl}(A)$ is the set of all A-definable elements or equivalently the elements which are fixed by all automorphisms which fix A. Further, $\mathrm{acl}(A)$ is the set of elements which are algebraic over A or equivalently the elements which have finite orbit under the automorphisms fixing A.

2.3 Imaginaries

An *imaginary* or *imaginary element* is a equivalence class of an 0-definable equivalence relation. We will usually add imaginary elements to our theory. We do that by adding a new sort for every 0-definable equivalence relation together with the canonical projection from the old sort to the new sort. If we add imaginaries to $M \models T$ we will call the new structure M^{eq} and the corresponding theory T^{eq}. We will refer to the elements of the old structure as real elements. $\mathrm{dcl}^{\mathrm{eq}}$ and $\mathrm{acl}^{\mathrm{eq}}$ will refer to the calculation of the definable and algebraic closure in T^{eq} (if it is ambiguous whether we are working in T or T^{eq}). For an explicit construction see Section 8.4 of [TZ12]. All of the following properties will hold for T^{eq}.

1. We say that a theory has *uniform elimination of imaginaries*, if every 0-definable equivalence relation E is the fibration of a 0-definable function f (i.e. aEb holds if and only if we have that $f(a) = f(b)$).

2. We say that a theory *eliminates imaginaries* or in short has *e.i.* if every imaginary is interdefinable with a real tuple.

3. We say that a theory *eliminates finite imaginaries* if for every finite set (of real elements) there exists a tuple such that the finite set is fixed by the same automorphisms as this tuple. We refer to such a tuple as the *code* of the finite set.

4. We say that a theory *weakly eliminates imaginaries* or in short has *wei* if for every imaginary e there exists a real tuple $d \in \mathrm{acl}(e)$ such that $e \in \mathrm{dcl}^{\mathrm{eq}}(d)$.

5. We say that a theory *geometrically eliminates imaginaries* or in short has *gei* if for every imaginary e there exists a real tuple $d \in \mathrm{acl}(e)$ such that $e \in \mathrm{acl}^{\mathrm{eq}}(d)$.

The only difference between point 1 and point 2 of the last definition is whether there are some constants for coding of functions. In fact, we have that a theory has elimination of imaginaries and has at least two 0-definable elements if and only if it has uniform elimination of imaginaries (see Lemma 8.4.7 of [TZ12]). By this we can easily see that T^{eq} has uniform elimination of imaginaries (the equivalence relation $x_1x_2Ey_1y_2$ defined by $x_1 \doteq x_2 \leftrightarrow y_1 \doteq y_2$ gives rise to two elements).

Fact 2.3.1. (8.4.10 in [TZ12]) A theory has weak elimination and elimination of finite imaginaries if and only if it has elimination of imaginaries.

Remark 2.3.2. Every automorphism of M extends uniquely to an automorphism of M^{eq}.

Fact 2.3.3. (8.4.3 in [TZ12].) If T eliminates imaginaries, then every definable class $\phi(\mathfrak{C})$ has a *canonical parameter* (that is a real tuple which is fixed by the same automorphisms which leave $\phi(\mathfrak{C})$ invariant).

Going over from T to T^{eq} will preserve many model theoretic properties. For example it does preserve the following:

Fact 2.3.4. (8.4.8 in [TZ12])

1. T is uncountably categorical if and only if T^{eq} is uncountably categorical.
2. T is κ-stable if and only if T^{eq} is κ-stable.
3. T is stable if and only if T^{eq} is stable.

2.4 Independence relation

We will now define a rather abstract concept. But the reader may verify that the notion which will be introduced coincidences in the theory of algebraically closed fields with algebraic independence.[2]

Definition 2.4.1. For a complete theory T, a ternary relation $a \mathop{\smile\!\!\!\!\!\!|}_B C$ between finite tuples a and sets B, C which is invariant under automorphisms is called an *independence relation* if it satisfies the following properties:

1. (Monotonicity and Transitivity) $a \mathop{\smile\!\!\!\!\!\!|}_A BC$ if and only if $a \mathop{\smile\!\!\!\!\!\!|}_A B$ and $a \mathop{\smile\!\!\!\!\!\!|}_{AB} C$.
2. (Symmetry) $a \mathop{\smile\!\!\!\!\!\!|}_A b$ if and only if $b \mathop{\smile\!\!\!\!\!\!|}_A a$, where we consider the tuples on the right side as finite sets.
3. (Finite Character) $a \mathop{\smile\!\!\!\!\!\!|}_A B$ if and only if $a \mathop{\smile\!\!\!\!\!\!|}_A B_0$ for all $B_0 \subset_{\text{finite}} B$.
4. (Local Character) There is a cardinal κ, such that for all a and B there is a $B_0 \subset B$ of cardinality less than κ such that $a \mathop{\smile\!\!\!\!\!\!|}_{B_0} B$.
5. (Existence) For all a, B, C there is a' such that $a \equiv_B a'$ and $a' \mathop{\smile\!\!\!\!\!\!|}_B C$.
6. (Anti-Reflexive) If $a \mathop{\smile\!\!\!\!\!\!|}_A a$, then $a \in \mathrm{acl}(A)$.

[2]Note that the theory of ACF_0 will be strongly minimal and by quantifier elimination the algebraic closure in the model-theoretic- and algebraic-sense coincide. From this it follows that the algebraic closure operator will be a pregeometry (which will give rise of some independence notion by for example Corollary 8.5.13 of [TZ12]).

Note that an independence relation $\downfree$ directly translates to (or rather defines) a special class of extensions of n-types $\sqsubset$: For $p \in S(A)$, $q \in S(B)$ with $A \subset B$ we have $p \sqsubset q$ if $p \subset q$ and there exists $a \models q$ such that $a \downfree_A B$. In simple theory this special class will coincide with the usual notion of non-forking extension.

The development of (abstract) notions of independence is an important part of modern model theory. It started with the notion of matroids (i.e. pregeometries) in [Wae30] ([Wae31]) and [Whi35][3], then continued with the Morley rank[4] and went on with the development of forking for stable[5] and later for simple theories[6]. Then continuing beyond to the notion of thorn-forking (defined first in [Ons02]), which is the weakest of all notions, i.e. if there is an independence notion at all, then thorn-forking will be an independence notion and any elements being thorn-independence will be independent for any other independence notion (see Theorem 5.2 of [Adl09b]).

Just requiring that T has an independence notion (and not T^{eq}) does not give any deep structure, as Example 4.5 of [Adl09a] shows. We will discuss more about independence notions in a more meaningful context, i.e. in rosy, simple and stable theories, later.

But first we will do more abstract nonsense.[7]

Lemma 2.4.2. *An independence notion always satisfies the following additional properties;*

1. *(Weak Existence)* $a \downfree_A A$ *holds for any* a, A,

[3] That the algebraic closure of strongly minimal defines an pregeometry was shown in [Mar66] (see [BL71] for an exposition of this work).

[4] Implicitly Morley rank is already defined in [Mor65] (see the Historic notes on page 52 in [Bal88]). An axiomatic approach can be found in [BB74] and [Lac76].

[5] Implicitly already defined by the rank in [She69]. An axiomatic approach can be found in [She78].

[6] This was done for weak forking in [She96] and for forking in [KP97] (see Claim 1.5 in the former and Theorem 4.2 in the latter).

[7] According to [Mac97] the term "abstract nonsense" dates at least back to 1942. Note that according to [Mon01] at least "[i]n algebra, the term "abstract nonsense" has a definite meaning without any pejorative connotation." I do use this term in a similar way just to tell the reader that the following proof will be purely axiomatic.

2. *(Right Monotonicity)* $a \mathop{\smile\!\!\!\!\!|}_A B$ *if and only if* $a \mathop{\smile\!\!\!\!\!|}_A AB$,

3. *(Base Monotonicity)* $a \mathop{\smile\!\!\!\!\!|}_A B$ *implies that* $a \mathop{\smile\!\!\!\!\!|}_{AB_0} B$ *for any* $B_0 \subset B$.

Proof. The first point is given since there exists $a' \equiv_A a$ with $a' \mathop{\smile\!\!\!\!\!|}_A A$ by Existence. Then as there is an automorphism fixing A and mapping a' to a, we have by invariance under automorphism that $a \mathop{\smile\!\!\!\!\!|}_A A$. For the second point note that $a \mathop{\smile\!\!\!\!\!|}_A AB$ implies $a \mathop{\smile\!\!\!\!\!|}_A B$ by Monotonicity and Transitivity. On the other hand if $a \mathop{\smile\!\!\!\!\!|}_A B$ holds then we get by Monotonicity and Transitivity again that $a \mathop{\smile\!\!\!\!\!|}_A AB$. The last point is true since $a \mathop{\smile\!\!\!\!\!|}_A B$ implies $a \mathop{\smile\!\!\!\!\!|}_A B_0 B$ which gives by Monotonicity and Transitivity that $a \mathop{\smile\!\!\!\!\!|}_{AB_0} B$ holds. □

Remark 2.4.3. An independence notion naturally extends so that it is a notion between triples A, B, C of sets. To see this note that by finite character and symmetry, we can define that $A \mathop{\smile\!\!\!\!\!|}_C B$ holds, if $a_0 \mathop{\smile\!\!\!\!\!|}_C B$ holds for some enumerations a_0 of all finite $A_0 \subset A$.

Lemma 2.4.4. *An independence notion is invariant under algebraic closure, i.e.* $A \mathop{\smile\!\!\!\!\!|}_B C$ *holds if and only if* $\mathrm{acl}(A) \mathop{\smile\!\!\!\!\!|}_B C$ *if and only if* $A \mathop{\smile\!\!\!\!\!|}_{\mathrm{acl}(B)} C$ *if and only if* $A \mathop{\smile\!\!\!\!\!|}_B \mathrm{acl}(C)$.

Proof. In this proof one should read the notation $A \mathop{\smile\!\!\!\!\!|}_B C$ as $a_0 \mathop{\smile\!\!\!\!\!|}_B c_0$ holds for all a_0 which do enumerate a finite subset of A and for all c_0 which do enumerate a finite subset of C. Then $A \equiv_B C$ means that there is an enumeration of A and an enumeration of C such that these enumerations have the same type over B.

If $A \mathop{\smile\!\!\!\!\!|}_B C$ holds, we can find an $A' \equiv_{BC} A$ (by compactness) with $A' \mathop{\smile\!\!\!\!\!|}_{BC} \mathrm{acl}(BC)$. Hence by the invariance under automorphism we have $A \mathop{\smile\!\!\!\!\!|}_{BC} \mathrm{acl}(BC)$. And therefore by Monotonicity and Transitivity we have $A \mathop{\smile\!\!\!\!\!|}_B C\mathrm{acl}(BC)$. Hence we have $A \mathop{\smile\!\!\!\!\!|}_B \mathrm{acl}(C)$.

If we assume that this point is true now, then we have $A \mathop{\smile\!\!\!\!\!|}_B B\mathrm{acl}(C)$ by Monotonicity and Transitivity. By the same argument as before we have $A \mathop{\smile\!\!\!\!\!|}_B \mathrm{acl}(BC)$. From here it follows that $A \mathop{\smile\!\!\!\!\!|}_{\mathrm{acl}(B)} \mathrm{acl}(BC)$ and therefore $A \mathop{\smile\!\!\!\!\!|}_{\mathrm{acl}(B)} C$.

If we assume that this point is true, then since $A \mathop{\smile\!\!\!\!\!|}_B B$ (and therefore $A \mathop{\smile\!\!\!\!\!|}_B \mathrm{acl}(B)$ as shown before), we have by Monotonicity and Transitivity that $A \mathop{\smile\!\!\!\!\!\!|}_{B} \mathrm{acl}(B)C$ and therefore $A \mathop{\smile\!\!\!\!\!\!|}_{B} C$. Now by Symmetry $\mathrm{acl}(A) \mathop{\smile\!\!\!\!\!\!|}_{B} C$ holds if and only if the other conditions hold. $\square$

Definition 2.4.5. We are now going to define several notions in a theory with an independence relation. A sequence $(a_i : i \in I)$ is called an *independence sequence* over A if we have $a_{<i} \mathop{\smile\!\!\!\!\!\!|}_{A} a_i$ where $a_{<i} = (a_j : j < i)$ for all $i \in I$. An indiscernible sequence which is also an independence sequence is called a *Morley sequence*. As the independence of a sequence does not depend on its ordering by symmetry, we say that $\{a_i \mid i \in I\}$ is an *independent set* over A if for any (all) ordering of I it is an independence sequence.

Lemma 2.4.6. *Let $(a_i : 2 \leq i \leq n)$ be an independence sequence over ba_1 and let $a_1 \mathop{\smile\!\!\!\!\!\!|}_{b} a_i$ for all i with $2 \leq i \leq n$. Then the sequence $(a_i : 1 \leq i \leq n)$ is independent over the parameter b.*

Proof. We prove this by induction, the basis $n = 2$ is given. Now by the induction hypothesis we have that $(a_i : 1 \leq i \leq n-1)$ is independent over b. Further we have that $a_1 \mathop{\smile\!\!\!\!\!\!|}_{b} a_n$ and $a_n \mathop{\smile\!\!\!\!\!\!|}_{ba_1} (a_i : 2 \leq i \leq n-1)$ by our assumptions. Hence by transitivity we have $a_n \mathop{\smile\!\!\!\!\!\!|}_{b} (a_i : 2 \leq i \leq n-1)$. $\square$

2.5 Rosy theories

Rosy theories are the most general class of theories which have an independence notion in T^{eq}:

Fact 2.5.1. (5.2 of [Adl09b]) T^{eq} has an independence notion if and only if T is rosy.

The reader not familiar with rosy theories may take this fact as a definition of a rosy theory. The independence notion will be the notion of thorn-forking. We will not care about the combinatorial description of thorn forking and just deal with it as an abstract notion. If the reader is more interested in this, then they could take a look at Adler's [Adl09a] and Ealy and Onshuus' [EO07] work.

Fact 2.5.2. (5.3 of [Adl09b]) Every reduct of a rosy theory is rosy.

Definition 2.5.3. We say that a rosy theory T is *superrosy*, if (in T^{eq}) we can pick the κ in the local character as ω. To phrase this in other terms, for every type p there exists a finite subset A_0 of its domain, such that $p \restriction A_0 \sqsubset p$. If a theory is superrosy and simple or stable we will call it *supersimple* or *superstable.*

Note that in general we can pick κ in the local character definition to be $(|T| + |A|)^+$ (see the proof of Theorem 1.6 of [Adl09b]).

2.6 Simple theories

Definition 2.6.1. An independence relation satisfies the *Independence Theorem over Models* if for every model M (of the same theory as the one the independence notion is associated with) and all tuples a, b, c, d with

$$a \equiv_M b,\ \ c \mathop{\smile\!\!\!\!\!\mid}_M d,\ \ a \mathop{\smile\!\!\!\!\!\mid}_M c \ \text{ and } \ b \mathop{\smile\!\!\!\!\!\mid}_M d,$$

there exists e such that $e \equiv_{Mc} a$, $e \equiv_{Md} b$ and $e \mathop{\smile\!\!\!\!\!\mid}_M cd$.

Fact 2.6.2. (Kim-Pillay)(7.3.13 of [TZ12]) A theory T has an independence notion $\mathop{\smile\!\!\!\!\!\mid}$ which satisfies the Independence Theorem over Models if and only if T is simple. Further $\mathop{\smile\!\!\!\!\!\mid}$ is the non-forking relation and the non-dividing relation (they coincide) (see Definition 2.6.3). The cardinal κ in the local character can be picked as $|T|^+$. In addition, if dividing or forking is an independence notion, then T is necessarily simple.

The reader not familiar with the context may take Fact 2.6.2 as definition for simple theories. Further, they should not worry too much about the non-forking (or non-dividing) relation $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{nf}}$ and they can use the abstract concept defined above most of the time. In addition, the reader should note that the notion of non-forking extension is just the class $\sqsubset$ (coming from the non-forking independence relation). In some proofs we do need the following combinatorial definition of dividing and forking:

Definition 2.6.3. (7.1.4 of [TZ12]) A formula $\phi(x, b)$ *divides* or we say that $\phi(x, b)$ is a dividing formula over A if there is a sequence of indiscernibles $(b_i)_{i<\omega}$ over A with $b_0 \equiv_A b$ and such that $\bigcup_{i<\omega} \phi(x, b_i)$ is inconsistent.

A complete type p divides over A if it contains a formula which divides over A. A partial type divides if all completions divide over A. Moreover, a formula $\phi(x, b)$ or a (partial) type p *forks* over A if it implies a disjunction of formulae $\bigvee_{i=1}^{l} \phi_i(x)$, such that for every i with $1 \leq i \leq l$ we have that ϕ_i divides over A. Finally we define the relation $a \mathop{\smile\!\!\!\!\!\!\mid\,}_B^{\mathrm{nf}} C$ to hold, if $\mathrm{tp}(a/C)$ does not fork over B.

Fact 2.6.4. (4.5 of [Adl09b])

1. Every reduct of a simple theory is simple.
2. T is simple if and only if T^{eq} is simple.

2.7 Stable theories

Definition 2.7.1. A type p is called *stationary* (with respect to some notion of independence) if for any two types q_1, q_2 over the same set of parameters with $p \sqsubset q_1, q_2$ we have $q_1 = q_2$.

Definition 2.7.2. An independence relation has *Weak Boundedness*, if for all a, A there is a cardinal κ such that for any $B \supset A$, there are at most κ $(a_i)_{i \in I}$ with $a_i \equiv_A a$, $a_i \mathop{\smile\!\!\!\!\!\!\mid\,}_A B$ and $a_j \not\equiv_B a_k$ for $k \neq j \in I$.

Fact 2.7.3. (8.5.10 of [TZ12]) A theory T has an independence notion $\mathop{\smile\!\!\!\!\!\!\mid\,}$ which satisfies Weak Boundedness if and only if T is stable. Further $\mathop{\smile\!\!\!\!\!\!\mid\,}$ is the non-forking relation. If T further has elimination of imaginaries then the cardinal κ in the Weak Boundedness definition can be set as $2^{|T|}$. We also have that any type over a model is stationary. If we further have that T has (weak) elimination of imaginaries (or work in T^{eq}), then any type over an algebraically closed set is stationary.

Again if the reader is not familiar with the context, they may take Fact 2.7.3 as definition for stable theories.

Fact 2.7.4. [Parameter Separation Theorem](see Theorem 12.31 of [Poi00] or Corollary 3.19 of [Sim15]) Let M be the model of some stable L-theory, let $\phi(\overline{x})$ be some L-formula and let $\psi(\overline{x}, \overline{m})$ be some $L(M)$-formula. Then there is some $L(\phi(M))$formula $\theta(\overline{x}, \overline{a})$ which is equivalent to $\phi(\overline{x}) \wedge \psi(\overline{x}, \overline{m})$ (in M).

Definition 2.7.5. A theory has *canonical bases*, if for every global type $p \in S(\mathfrak{C})$ there exists some set B which is fixed point-wise by the same automorphisms which leave p invariant.

Fact 2.7.6. (2.6 in [CF04]) A theory which has canonical bases is stable and has weak elimination of imaginaries. If on the other hand the theory is stable and has elimination of imaginaries then it has canonical bases.

Amalgamation diagrams have been around in model theory at least since the 1950ies.[8] The next lemma is also such an amalgamation diagram (this time of automorphisms). It is a corollary of the fact that in T^{eq} types over algebraically closed sets are stationary, which is due to Shelah (see 6.9(1) of [She90]). It first appeared in [Las91](see 3.3) together with the name amalgamation.

Lemma 2.7.7. *Let T be a stable theory with weak elimination of imaginaries. Let A be an algebraically closed set, a, b some tuples, α, β be two automorphisms with $\alpha \restriction A = \beta \restriction A$. Further suppose we have $a \mathop{\smile\!\!\!\!\!\!|}_A b$ and $\alpha(a) \mathop{\smile\!\!\!\!\!\!|}_{\beta(A)} \beta(b)$. Then $\alpha \restriction Aa \cup \beta \restriction Ab$ is an elementary map.*

Proof. Note that $\mathrm{tp}(a/A) \sqsubset \mathrm{tp}(a/Ab)$ and $\mathrm{tp}(\alpha(a)/\beta(A)) \sqsubset \mathrm{tp}(\alpha(a)/\beta(Ab))$. Now by stationarity (this by wei and since A is algebraic closed), we have $\mathrm{tp}(\alpha(a)/\beta(Ab)) = \mathrm{tp}(\beta(a)/\beta(Ab))$, therefore we have an automorphism γ fixing $\beta(Ab)$ and mapping $\beta(a)$ to $\alpha(a)$. Now we are finished since

$$\gamma \circ \beta \restriction Aab = \alpha \restriction Aa \cup \beta \restriction Ab.$$

$\square$

[8]In Chapter 6.4 of [Hod93] Hodges writes “It was Roland Fraïssé who first called attention to the [2-amalgamation] diagram”

Fact 2.7.8. (see Exercise 8.5.2 of [TZ12] or for a proof [Bal88]) (The Finite Equivalence Relation Theorem) Let T be stable, $A \subset B$ and let $q \neq p$ be types over B which do not fork over A. Then there is an A-definable equivalence relation E with finitely many classes such that $q(x) \cup p(y) \vdash \neg E(x, y)$.

Proposition 2.7.9. *Let T be stable and ω-categorical. Then any type p over a finite set A has only finitely many non-forking extensions. Hence there are only finitely many global types (over a fixed set of variables) which are non-forking over A.*

Proof. Let $(p_i : i \in I)$ be all global non-forking extensions of p. Now by the Finite Equivalence Relation Theorem, for every two types $p_i \neq p_j$ there exists an A-definable equivalence relation $E_{i,j}$ with $p_i(x) \cup p_j(y) \vdash \neg E_{i,j}(x, y)$. Now we use the Ryll-Nardzewski Theorem. Note that it states that in an ω-categorical theory there are only finitely many inequivalent L-formulae over some fixed finite tuple of variables. Since $T(A)$ is also ω-categorical, we have only finitely many different $E_{i,j}$. Hence we have only finitely many non-forking extensions. As there are only finitely many types over a finite set (by Ryll-Nardzewski) the second point automatically follows. $\square$

2.8 Category theory

We establish some elementary things in arbitrary categories:

Lemma 2.8.1. *Let S and C be arbitrary categories. Let $\mathfrak{a} : S \to C$ be a functor and let $(\mathfrak{a}(s) \xrightarrow{f_s} \mathfrak{b}(s))_{s \in S}$ be a system of isomorphisms in C. We define $\mathfrak{b} : S \to C$ by mapping the object s to $\mathfrak{b}(s)$ and a morphism $s \xrightarrow{f} t$ to*

$$\mathfrak{b}(f) = f_t \circ \mathfrak{a}(f) \circ f_s^{-1}.$$

Then $\mathfrak{b}$ is a functor, which is naturally isomorphic to $\mathfrak{a}$.

Proof. We check that $\mathfrak{b}$ is a functor. We have

$$\mathfrak{b}(\mathrm{id}_s) = f_s \circ \mathfrak{a}(\mathrm{id}_s) \circ f_s^{-1} = f_s \circ \mathrm{id}_{\mathfrak{a}(s)} \circ f_s^{-1} = f_s \circ f_s^{-1} = \mathrm{id}_{\mathfrak{b}(s)}$$

For arrows $s \xrightarrow{g} u$ and $u \xrightarrow{h} t$ in S we have

$$\begin{aligned}\mathfrak{b}(h) \circ \mathfrak{b}(g) = f_t \circ \mathfrak{a}(h) \circ f_u^{-1} \circ f_u \circ \mathfrak{a}(g) \circ f_s^{-1} \\ = f_t \circ \mathfrak{a}(h) \circ \mathfrak{a}(g) \circ f_s^{-1} = \mathfrak{b}(h \circ g).\end{aligned}$$

Hence $\mathfrak{b}$ is a functor. The natural isomorphism is given by $(\mathfrak{a}(s) \xrightarrow{f_s} \mathfrak{b}(s))_{s \in S}$, because we have $f_t \circ \mathfrak{a}(f) = \mathfrak{b}(f) \circ f_s$. □

Lemma 2.8.2. *Let S, U and C be categories such that S is a subcategory of U. Let $\mathfrak{b}^- : S \to C$ and $\mathfrak{a}^- : S \to C$ be two naturally isomorphic functors with natural isomorphism $(f_s : s \in S)$. Further let $\mathfrak{a} : U \to C$ be a functor, such that $\mathfrak{a} \restriction S = \mathfrak{a}^-$. Then there exists a functor $\mathfrak{b} : U \to C$ with $\mathfrak{b} \restriction S = \mathfrak{b}^-$, which is naturally isomorphic to $\mathfrak{a}$.*

Proof. Let the natural isomorphism between $\mathfrak{a}^-$ and $\mathfrak{b}^-$ be given by the isomorphisms $(f_s : s \in S)$. We use Lemma 2.8.1 with

$$\{f_s : s \in S\} \cup \{\mathrm{id}_{\mathfrak{a}(u)} : u \in \mathrm{ob}(U) - \mathrm{ob}(S)\}$$

as our system of isomorphisms to receive a functor $\mathfrak{b}$ which is naturally isomorphic to $\mathfrak{a}$. It is left to check that $\mathfrak{b} \restriction S = \mathfrak{b}^-$ holds. First note that it is clear that the objects are the right ones, i.e. $\mathfrak{b}^-(s) = \mathfrak{b}(s)$ for any $s \in \mathrm{ob}(S)$. So check that the image of any arrow $s \xrightarrow{h} s'$ in S is the same under $\mathfrak{b}^-$ and $\mathfrak{b}$. By definition of $\mathfrak{b}$ we have that $\mathfrak{b}(h) = f_{s'} \circ \mathfrak{a}(h) \circ f_s^{-1}$. Also as $(f_s : s \in S)$ gives a natural isomorphism between $\mathfrak{a}^-$ and $\mathfrak{b}^-$ we have

$$\mathfrak{b}^-(h) \circ f_s = f_{s'} \circ \mathfrak{a}^-(h)$$

and therefore

$$\mathfrak{b}^-(h) = f_{s'} \circ \mathfrak{a}^-(h) \circ f_s^{-1}.$$

As $\mathfrak{a} \restriction S = \mathfrak{a}^-$ this then gives

$$\mathfrak{b}^-(h) = f_{s'} \circ \mathfrak{a}(h) \circ f_s^{-1}.$$

Hence we have $\mathfrak{b}^-(h) = \mathfrak{b}(h)$ as required. □

Chapter 3

Algebraic and Finite Covers

In this chapter we will analyse some mild expansions of complete theories. We consider them mild because the algebraic closure (computed in the expansion) of any model of the original theory will be the domain of a model of the expansion. Further, the expansions do additionally preserve many model theoretic properties and also behave well when we add imaginary elements to them.

I have been unable to find precise references for the results in this section, though it is likely that many of them are in the realm of folklore (apart from Section 3.5). I became aware of them via personal communication with David Evans.

We use the following conventions throughout this chapter.

We let T be a complete S-sorted L-theory and let T' be a complete S'-sorted L'-theory with $T \subset T'$, $L \subset L'$ and $S \subset S'$. Moreover, we require that $L' \upharpoonright S = L$ (this means that any S-sorted function, relation or constant of L' is already part of L). Moreover (by that), we have $T' \upharpoonright S = T$ (meaning that we forget about any sentence in T' with relation and variables which are not S-sorted). We let $\mathfrak{C}$ be the monster model of T and M be a model of T. We let $\mathfrak{C}'$ be the monster model of T' and M' be a model of T'. Note that by these conventions (together with Definition 2.1.1) we have that $M' \upharpoonright S = M' \upharpoonright L = M' \upharpoonright (L' \upharpoonright S)$. We also use a complete S''-sorted

L''-theory T'' with the same properties as T' (in relation to T) and also set $\mathfrak{C}'', M''$ accordingly. Finally we refer to the sorts of S as the old sorts and to the sorts of $S' - S$ as the new sorts.

3.1 Definition

Definition 3.1.1. (The conventions are as defined above.) The model M is *embedded* in the model M', if we have that the 0-definable subsets of M' and powers of it are the same in the L and L' sense. This means that any S-sorted L'-definable set is definable in the language L. The theory T is *embedded* in the theory T', if for all models M' of T' the restriction M of M' to the sorts of L is embedded in M'. The model M is *stably embedded* in the model M', if we have that the definable subsets of M and powers of it are the same in the L and L' senses. This means that any S-sorted $L'(M')$-definable set is definable in the language $L(M)$. The theory T is *stably embedded* in the theory T', if for all models M' of T' the restriction M of M' to the sorts of L is stably embedded in M'. A model M or a theory T is said to be *fully embedded* in M' or T' respectively, if each of them are embedded and stably embedded.

Let M be some L-structure and ϕ some L-formula. If we consider $\phi(M')$ to be the *induced structure* of M (i.e. it has language $L_{\phi(M)} = \{R_\psi(x) \mid \psi \in L\}$ and the interpretation: for every $a \in \phi(M)$, $\phi(M) \models R_\psi(a) \iff M \models \psi(a)$) then we have that $\phi(M)$ is embedded in $M, \phi(M), \mathrm{id}_{\phi(M)}$. If M is moreover stable then we also have that it is stably embedded in M by the Parameter Separation Theorem (see Fact 2.7.4 and the next Remark 3.1.2)

Remark 3.1.2. Let M' be stable. Let M be embedded in M'. Then we have that M' is stably embedded M. Of course if we have additionally that $M' = \mathrm{acl}(M)$ then M' is an algebraic cover of M.

Proof. By the assumptions we only need to check that M is stably embedded in M'. So take some $\psi(\overline{x}, \overline{m'})$ formula with $\overline{m'} \in M'$ and with $\overline{x}$ of the old sorts (i.e. the sorts of M). Now by stability and as M is a restriction of sorts, we can apply Fact 2.7.4 with $\phi(\overline{x}) = \overline{x} \doteq \overline{x}$ and ψ. Hence we get some

$L'(M)$-formula $\theta(\overline{x}, \overline{m})$ which is equivalent to ψ (in M'). Now as $\theta(\overline{x}, \overline{y})$ is some L'-definable formula defining an subset of M we know by embeddedness that there is some $\theta'(\overline{x}, \overline{y}) \in L$ equivalent to $\theta(\overline{x}, \overline{y})$ (in M'). Which gives us stable embeddedness since we have then that $\theta'(x, \overline{m})$ is equivalent to $\psi(\overline{x}, \overline{m'})$. $\square$

The next result is part of Lemma 1 in the Appendix of [CH99].

Fact 3.1.3. If every automorphism of $\mathfrak{C}$ extends to one of $\mathfrak{C}'$, then T is fully embedded in T'. If L, L' are both countable languages, then the converse is also true.

We will later see that under the condition that T' is an algebraic cover of T, we have that the converse also holds for uncountable theories. For that see Lemma 3.1.11.

Definition 3.1.4. For M fully embedded in M', we say that M' is an *algebraic cover* of M, if M' is the algebraic closure of M (computed in M'). We further require that, if $S_1, \ldots, S_n$ are any number of new sorts of L' (compared to L), the restriction of M' to the sorts $S_1, \ldots, S_n$ together with the sorts of L is fully embedded in M'. We say that M' is a *finite cover* of M, if there is only a single new sort M_1 and there is a 0-definable function π from M_1 to M which is (boundedly) finite-to-one. For a natural number n we say that M' is a *n-cover* of M, if it is a finite cover and the definable function π is n-to-one, by which we mean that the fibre of each point has exactly n elements.

For T fully embedded in T', we say that T' is an *algebraic cover* of T, if whenever $M' \models T'$, we have that M' is an algebraic cover of the restriction M of M' to the sorts of L. The theory T' is a *finite cover* of T, if it is an algebraic cover of T and whenever $M' \models T'$, we have that M' is a finite cover of the restriction M of M' to the sorts of L. It is an *n-cover*, if it is a finite cover and whenever $M' \models T'$, we have that M' is an n-cover of the restriction M of M' to the sorts of L.

A more general definition of a finite cover is given in [Eva97b] (see Definition 1.1.2 in there). If one only cares about structure up to interdefinability, then our notion and the notion in [Eva97b] should coincide. But this proof is left to future work. We will only use the definition in here, which is suited for our needs.

Remark 3.1.5. Let T' be an algebraic cover of T. Then we have that for every $M \models T$ the structure $\mathrm{acl}^{T'}(M)$ is a model of T'.

Proof. So take some L'-formula $\phi(z, y)$ and some $m' \in \mathrm{acl}^{T'}(M)$. Then by Tarski's test $\mathrm{acl}^{T'}(M)$ will be a model of T' if and only if there exists $m'' \in \mathrm{acl}^{T'}(M)$ such that $T' \models \phi(m'', m')$ (assuming that $\phi(z, m')$ is satisfiable in $\mathfrak{C}'$). We take some formula $\psi_1(y, x_1)$ (with x_1 of the old sort) such that

$$T' \models \forall x \exists^{=n_1} y \psi_1(y, x_1)$$

and such that $T' \models \psi_1(m', m)$ for some $m \in M$ holds. Then by compactness and as $\mathrm{acl}(\mathfrak{C}) = \mathfrak{C}'$ there exist $\psi_2(z, x_2)$ (with x_1 of the old sort) such that

$$T' \models \forall x \exists^{<n_2} z \psi_2(z, x_2)$$

and such that $\exists x_2 \psi_2(z, x_2)$ covers

$$\exists y, x_1 \phi(z, y) \wedge \psi_1(y, x_1).$$

We set $\theta(x_1, x_2)$ to be following L'-formula

$$\theta(x_1, x_2) := \forall z \exists y \big((\phi(z, y) \wedge \psi_1(y, x_1)) \to \psi_2(z, x_2)\big).$$

Now there is some $c \in \mathfrak{C}$ such that $\mathfrak{C}' \models \theta(m, c)$. Hence by Tarski's test and as (by embeddedness) $\theta(x_1, x_2)$ is equivalent to an L-formula we have that there is some $m_0 \in M$ satisfying $\theta(m, x_2)$. But this means that $\phi(z, m')$ is satisfiable by an element algebraic over m_0 (more precisely some element satisfying $\psi_2(z, m_0)$). □

The next lemma tells us that we only need to check embeddedness (and not stable embeddedness) in case there is some finite-to-one map in order to verify that some extension is a finite cover.

Lemma 3.1.6. *Let $S' = S \cup \{s\}$. If M is embedded in M' and there is a 0-definable finite-to-one map π from s to the S-sorts, then M is stably embedded in the model M'.*

Proof. Fix some M'-definable subset $\phi(m', M)$ of M. Now let $\{(m_i)_{1 \leq i \leq k}\}$ be all the elements with $\pi(m') = \pi(m_i)$. For each $1 \leq i \leq k$ fix some

$$a_i \in \phi(m', M) - \phi(m_i, M)$$

and some

$$b_i \in \phi(m_i, M) - \phi(m', M)$$

(if the sets are non-empty). Let I_a be the set of all $1 \leq i \leq k$ for which there exists some a_i and let I_b be the set of all $1 \leq i \leq k$ for which there exists some b_i. Then by embeddedness we have that for the L'-formula

$$\exists z[\pi(z) \doteq x \wedge \phi(z, y) \wedge \bigwedge_{i \in I_a} \phi(z, y_i) \wedge \bigwedge_{i \in I_b} \neg\phi(z, y_{i+k}).$$

there exists an L-formula

$$\psi(x, y, (y_i)_{i \in I_a}, (y_{i+k})_{i \in I_b}),$$

with the same realisations. Now the $L(M)$-formula

$$\psi(\pi(m'), y, (a_i)_{i \in I_a}, (b_i)_{i \in I_b}) \in L(M)$$

has the same realisations as the $L'(M')$-formula $\phi(m', y)$. $\square$

We will show that asking whether M' *is a finite cover or an n-cover of* M for some models (of T' and T) is the same as asking if T' is a finite cover of T. Note that here we need to require that the finite-to-one function is bounded. Otherwise, by compactness, the function will not necessarily be finite-to-one in some bigger model. Moreover, if the models are saturated, then asking if M' *is an algebraic cover of* M is equivalent to asking whether T' is an algebraic cover of T. The idea to this lemma is due J. Kirby.

Lemma 3.1.7. *If M' is a finite cover of M, then T' is a finite cover of T. If M' is an n-cover of M, then T' is an n-cover of T. If M' is $|L'|^+$-saturated and if M' is an algebraic cover of M, then T' is an algebraic cover of T.*

Proof. First check that if M is embedded in M', then for any $M^* \equiv M'$ we have that $M^* \upharpoonright S$ is embedded in M^*. But this is just because for any variables x, which are of old sorts S, and for any L'-formula $\psi(x)$ there exists some L-formula $\phi(x)$ such that

$$M' \models \forall x(\psi(x) \leftrightarrow \phi(x))$$

and hence

$$M^* \models \forall x(\psi(x) \leftrightarrow \phi(x)),$$

which shows what we want.

Note that if there is some 0-definable bounded finite-to-one or n-to-one function from the new sort to the old in one pair of model (M', M), then this is of course true for any pair of models $(M^*, M^* \upharpoonright S)$ with $M^* \equiv M'$. Note that by Lemma 3.1.6 we have that for finite covers and for n-covers stable embeddedness follows. Hence we have shown the first part of the lemma.

Now take a pair (M', M) which is $|L'|^+$-saturated. Further we require that $M' \subset \mathrm{acl}^{M'}(M)$. Then for a variable z of some new sort s we can find $|L'|$-many formulas $(\phi_i(z, y_i))_{i \in I}$ such that $M' \models \exists^{\leq n_i} z \phi_i(z, y_i)$ and such that $\bigcup_{i \in I} \exists y_i \phi_i(z, y_i)$ covers all of M'_s. By compactness we can find a finite subset of I_0 of I such that $\bigcup_{i \in I_0} \exists y_i \phi_i(z, y_i)$ still covers all of M'_s. But this is first-order expressible, and hence we do have $M^* \subset \mathrm{acl}(M^* \upharpoonright S)$ for any $M^* \equiv M$.

Finally we need to check that stable embeddedness stays true. So take a pair M', M for which M is stably embedded in M'. So let x be variables of some old sorts and $\psi(x, z)$ be some L'-formula. Then for every $m' \in M'$ there exists $\phi_{m'}(x, m)$ some $L(M)$-formula such that $\phi_{m'}(x, m) \leftrightarrow \psi(x, m')$. Now we have that $\bigcup_{\phi \in L} \exists y \forall x(\phi(x, y) \leftrightarrow \psi(x, z))$ does cover all of the sort of z, and hence by compactness we can find an finite set X of L-formulas such that $\bigcup_{\phi \in X} \exists y \forall x(\phi(x, y) \leftrightarrow \psi(x, z))$ does cover all of the sort of z. This is expressible in an L'-sentence and hence stable embeddedness is preserved for different models of T'. $\square$

Remark 3.1.8. If T' is an algebraic cover of T note that then any automorphism of $\mathfrak{C}$ can be extended to one of $\mathfrak{C}'$. This is just because any automorphism of $\mathfrak{C}$ is by embeddedness a partial elementary map and this map can be extended to its algebraic closure which is all of $\mathfrak{C}'$.

A slight complication is given here. A priori the extension of a partial elementary map f to an automorphism of the monster can only be done if the domain of that map is a (small) set. Such an automorphism restricted to the algebraic closure of $\mathrm{dom}(f)$ is of course then an extension of f to the algebraic closure. But as we can do that for any model M (of T) of arbitrary size we may just neglect that issue. In fact, if we work in a countable language then we can also use stable embeddedness to conclude the above remark (see Fact 3.1.3).

Corollary 3.1.9. *Let T' be an algebraic cover of the theory T. Then for every set $A \subset \mathfrak{C}$ we have that algebraic and definable closure on the old sorts is the same in the T and T'-sense. This means that the following holds;*

$$\mathrm{dcl}^T(A) = \mathrm{dcl}^{T'}(A) \cap \mathfrak{C} \quad \textit{and} \quad \mathrm{acl}^T(A) = \mathrm{acl}^{T'}(A) \cap \mathfrak{C}.$$

Proof. Note that as said in the last remark every automorphism of $\mathfrak{C}$ extends to one of $\mathfrak{C}'$ and on the other hand we have that every automorphism of $\mathfrak{C}'$ restricted to the old sort is an automorphism of $\mathfrak{C}$. Hence for elements of $\mathfrak{C}$ having a finite orbit or being fixed over A by either an automorphism of $\mathfrak{C}$ or an automorphism of $\mathfrak{C}'$ is the same. $\square$

Lemma 3.1.10. *Let T' be an algebraic cover of T. Then any automorphism of $\mathfrak{C}$ which fixes M pointwise (some model of T) can be extended to an automorphism which fixes $\mathrm{acl}^{T'}(M)$ pointwise.*

Proof. Let α be an automorphism of $\mathfrak{C}$ which fixes M pointwise. We have that $M' = \mathrm{acl}^{T'}(M)$ is a model of T' (by Remark 3.1.5). Now we claim that $\alpha \cup \mathrm{id}_{M'}$ is an elementary map. For that note for any variable x of the old sort and any $\phi(x, m')$ $L'(M')$ by stable embeddedness there exists some $L(M)$-formula $\psi(x, m)$ which has the same realisations in $\mathfrak{C}$. Now this means that the truth of the $\phi(x, m')$ will be preserved under automorphisms of M. Hence the union $\alpha \cup \mathrm{id}_{M'}$ is an elementary map. $\square$

Lemma 3.1.11. *Let S' (the sorts of T') be $S \cup \{s\}$. Then we have that the following three points are equivalent;*

1. *T' is a finite cover of T.*
2. *The reduct $\mathfrak{C} = \mathfrak{C}' \restriction_S$ (of the monster model $\mathfrak{C}'$ of T') is a monster model of T; there is an L'-definable finite-to-one function from the new sort s to $\mathfrak{C}$ and every automorphism of $\mathfrak{C}$ can be extended to one of $\mathfrak{C}'$.*
3. *The reduct $\mathfrak{C} = \mathfrak{C}' \restriction_S$ (of the monster model $\mathfrak{C}'$ of T') is a monster model of T; there is an L'-definable finite-to-one function from any finite number of sorts of $\mathfrak{C}'$ to $\mathfrak{C}$ and every automorphism of $\mathfrak{C}$ can be extended to one of $\mathfrak{C}'$.*

Proof. We prove (1) to (2) first. The first point of (2) is clear as any type in $\mathfrak{C}$ is a partial type in $\mathfrak{C}'$ and therefore satisfiable. The second point is also given by definition. For the last point we need to check that automorphisms of $\mathfrak{C}$ are partial elementary maps of M'. If this is done we are finished, since we can extend any elementary map to the algebraic closure which is $\mathfrak{C}'$ (see Remark 3.1.8). But since $\mathfrak{C}$ is embedded in $\mathfrak{C}'$, we have that for any L'-definable class $\phi(\mathfrak{C})$ there exists some L-formula ψ satisfied by the same elements. Now for an automorphism α of $\mathfrak{C}$ we have $\mathfrak{C}' \models \phi(\overline{m})$ if and only if $\mathfrak{C} \models \psi(\overline{m})$ if and only if $\mathfrak{C} \models \psi(\alpha(\overline{m}))$ if and only if $\mathfrak{C}' \models \psi(\alpha(\overline{m}))$.

To prove that (2) implies (3) just note that we can extend the finite-one-map from the sort s to some finite number of sorts of $\mathfrak{C}'$ by defining $\pi(x) = x$ for x not of sort s. This the only thing in which the two points differ.

Finally for the direction (3) to (1) note that we need only to check that T is fully embedded in T'. But this follows directly from Fact 3.1.3. $\square$

The next definition gives us a way of combining two algebraic covers into a single algebraic cover. This notion will be used in Chapter 7 and Chapter 8.

Definition 3.1.12. Suppose T is fully embedded in T' and also fully embedded in T''. If M', M'' are models (of T', T'') with a common T-part $M \models T$ we denote by $M' \coprod_M M''$ the $L' \sqcup_L L''$-structure which is the disjoint union over M of M' and M'' and in which no new instances of atomic relations are added. Note that by $L' \sqcup_L L''$ we mean the union of L' and L'' such that any sort and any relation, function or constant symbol which is part of both L' and L'' is already part of L. This of course can always be achieved by renaming the languages. The theory of this does not depend on the choice of M', M'' and we denote it by $T' \times_T T''$. Clearly $\mathrm{Aut}(M' \coprod_M M'')$ is the fibre product

$$\begin{aligned}\mathrm{Aut}(M') \times_{\mathrm{Aut}(M)} \mathrm{Aut}(M'') = \\ \{(g', g'') \in \mathrm{Aut}(M') \times \mathrm{Aut}(M'') \mid g' \restriction M = g'' \restriction M\}.\end{aligned}$$

Note that T is fully embedded in $T' \times_T T''$ and this is an algebraic cover of T if T' and T'' are.

3.2 Relating algebraic covers with finite covers

Remember that we use the conventions defined on page 29.

Lemma 3.2.1. *Let S_1, S_2 be sorts of S such that $S_1^{\mathfrak{C}} \subset \mathrm{acl}(S_2^{\mathfrak{C}})$. Then there is some formula $\phi(x, \overline{y})$ (with $x \in S_1$ and $\overline{y} \in S_2$) and some natural number m such that*

$$\mathfrak{C} \models \forall \overline{y} \exists^{<m} x \phi(x, \overline{y}) \wedge \forall x \exists \overline{y} \phi(x, \overline{y}).$$

Proof. For every $a \in S_1^{\mathfrak{C}}$ we find an algebraic formula $\phi_a(x, \overline{b})$ with $\overline{b} \in S_2^{\mathfrak{C}}$ with $\mathfrak{C} \models \phi_a(a, \overline{b})$. Further we require that for every $\overline{b}$, we have that every $\phi_a(x, \overline{b})$ has strictly less realisations than some fixed number (say m_a). We can do that by replacing $\phi_a(x, \overline{y}_a)$ by $\phi_a(x, \overline{y}_a) \wedge \exists^{<m_a} x \phi_i(x, \overline{y}_a)$. Note that all the formulae $\exists \overline{y}_a \phi_a(x, \overline{y}_a)$ together cover $S_1^{\mathfrak{C}}$. Now by compactness there is a finite number of these formulae such that $\{\exists \overline{y}_{a_i} \phi_{a_i}(x, \overline{y}_{a_i}) : 1 \leq i \leq n\}$ cover $S_1^{\mathfrak{C}}$. We now set $\phi(x, \overline{y}) = \phi(x, \overline{y}_1 \ldots \overline{y}_n) = \bigvee_{1 \leq i \leq n} \phi_i(x, \overline{y}_i)$. Then $\exists \overline{y} \phi(x, \overline{y})$ is satisfied by every element of $S_1^{\mathfrak{C}}$ and $\models \forall \overline{y} \exists^{<m_1 + \cdots + m_n} x \phi(x, \overline{y})$ holds. $\square$

Lemma 3.2.2. *Let S_1, S_2 be sorts of S such that $S_1^{\mathfrak{C}} \subset \operatorname{acl}(S_2^{\mathfrak{C}})$ and such that $S_2^{\mathfrak{C}}$ is fully embedded in $\mathfrak{C}$. Then there is a* 0-eq-*definable finite-to-one function from $S_1^{\mathfrak{C}}$ to $(S_2^{\mathfrak{C}})^{\mathrm{eq}}$. This function is* 0-*definable in the language of* $\mathfrak{C} \cup (S_2^{\mathfrak{C}})^{\mathrm{eq}}$.

Proof. By Lemma 3.2.1 there is some formula $\phi(x, \overline{y})$ and some natural number m, such that $\models \forall \overline{y} \exists^{<m} x \phi(x, \overline{y})$ and $\exists \overline{y} \phi(x, \overline{y})$ covers $S_1^{\mathfrak{C}}$. We take the definable finite-to-one map f which maps $a \in S_1^{\mathfrak{C}}$ onto the imaginary a_E where the equivalence relation E is given by

$$E(x_1, x_2) = \forall \overline{y}(\phi(x_1, \overline{y}) \leftrightarrow \phi(x_2, \overline{y})).$$

This function is finite-to-one as there is a common b such that for any $a' \in a_E$ we have $\phi(a', b)$. Now each a_E is fixed by the same automorphisms as $\phi(a, \mathfrak{C})$. Furthermore, as $\phi(a, \mathfrak{C}) \subset S_2^{\mathfrak{C}}$, by stably embeddedness there is some formula with $\psi(z, \overline{y})$ in the language of the restriction of $\mathfrak{C}$ to the sort S_2 such that $\phi(a, \mathfrak{C}) = \psi(c, \mathfrak{C})$ for some $c \in S_2^{\mathfrak{C}}$. Hence there is a canonical parameter b_a of $\phi(a, \mathfrak{C})$ in $(S_2^{\mathfrak{C}})^{\mathrm{eq}}$. We may assume that for $\phi(a, \mathfrak{C}) \neq \phi(a', \mathfrak{C})$ we have that the corresponding canonical parameters b_a and $b_{a'}$ are not equal.

Now as all a_E and b_a are interdefinable in $\mathfrak{C}^{\mathrm{eq}}$, there is a 0-definable injective function g from $(a_E)_{a \in S_1^{\mathfrak{C}}}$ to the canonical parameter b_a of

$$(\phi(a, \mathfrak{C}))_{a \in S_1^{\mathfrak{C}}}.$$

To see this fix for every a_E and b_a a formula ψ with $\models \psi(a_E, b_a)$ and

$$\models \forall x \exists^{=1} \overline{y} \psi(x, \overline{y}) \wedge \forall \overline{y} \exists^{=1} x \psi(x, \overline{y}).$$

Now all the $\exists \overline{y} \psi(x, \overline{y})$ cover $S_1^{\mathfrak{C}}/E$, hence there is a finite number of them covering $S_1^{\mathfrak{C}}/E$, which we can assume to be disjoint. Hence $g \circ f$ is a 0-eq-definable finite-to-one function from $S_1^{\mathfrak{C}}$ to $(S_2^{\mathfrak{C}})^{\mathrm{eq}}$. Now note that since the graph of $(g \circ f)$ is in $\mathfrak{C} \cup (S_2^{\mathfrak{C}})^{\mathrm{eq}}$ and by embeddedness of S_2 we have that the function is 0-definable in the language of $\mathfrak{C} \cup (S_2^{\mathfrak{C}})^{\mathrm{eq}}$. $\square$

Corollary 3.2.3. *Let T' be an algebraic cover of $T = T^{\mathrm{eq}}$. Then from each new sort S_1 there is a* 0-*definable finite-to-one map to the old sorts S.*

Proof. Note that S_1 is covered by $(\exists y_i \phi_i(x, y_i) : i \in I)$ such that

$$T' \models \exists \forall y_i \exists^{<n_i} \phi_i(x, y_i).$$

By compactness we may take I to be finite. Hence we have S_1 is in the algebraic closure of all sorts of all y_i. As we work in T^{eq} we may assume that all the y_i belong to the same sort. Now we are finished as we can apply the last Lemma 3.2.2 $\square$

Corollary 3.2.4. *Suppose T uniformly eliminates imaginaries and let T' be an algebraic cover of T with only one new sort. Then T' is a finite cover of the theory T.*

Proof. Let $\mathfrak{C}$ be a monster model of T. By the Lemma 3.2.2 there is a 0-definable finite-to-one map f from the new sort to some sort of T^{eq} say S. As T eliminates imaginaries uniformly, there is a 0-definable function g from S to $\mathfrak{C}^m$ such that for every element of S there is a unique element in $\mathfrak{C}^m$. Now $g \circ f$ is an 0-eq-definable function from the new sort to $\mathfrak{C}$. This is a function whose graph is in the (home) sorts of T' and therefore definable in it. $\square$

Corollary 3.2.5. *Suppose T uniformly eliminates imaginaries. Further let T' be an algebraic cover of the theory T with only finitely many new sorts. Then we have that the theory T' is interdefinable with some theory T'', which is a finite cover of the theory T.*

Proof. The point is that we can interpret any finite number of sorts as a single sort: Let $S_1, \ldots, S_n$ be the new sorts of (T' compared to T). Interpret this as a single sort S' by adding relation symbols $R_1 \ldots R_n$ to it with $S_1, \ldots, S_n$ as its classes. Changing the language mildly such that the relations of T' are relations of sort S' (and sort of T) are of the right sort will finish the construction. Then this new structure is what we were looking for. The proof of Corollary 3.2.4 shows us that there is a 0-definable finite-to-one map from S to the sorts of T. Now because the $S_1, \ldots, S_n$ are definable by the new sort S, we have that T'' is a finite cover. $\square$

By the last Corollary we may start to confuse finite covers and algebraic covers (with finitely many new sorts) over T^{eq}.

3.3 Imaginaries and covers

Remember that we use the conventions defined on page 29.

The next lemma will help us to understand new imaginaries appearing in a finite cover.

Lemma 3.3.1. *Let M' be an algebraic cover of M and suppose M geometrically eliminates imaginaries. Then for every $e \in M'^{\mathrm{eq}}$ we have*

$$e \in \mathrm{acl}^{M'^{\mathrm{eq}}}(\mathrm{acl}^{M'^{\mathrm{eq}}}(e) \cap M).$$

Proof. Throughout this proof we let $\mathrm{acl}^{M'}$ be $\mathrm{acl}^{M'^{\mathrm{eq}}}$. Note that if the map $\pi : M' \to M$ is 0-definable and finite-to-one, then so is

$$\pi^{(n)} : M'^n \to M^n : (c_1, \ldots, c_n) \mapsto (\pi(c_1), \ldots, \pi(c_n)).$$

Hence we may assume that e is a class of a 0-definable equivalence relation $\sim$ on M' and that $e = E$ is a subset of M'. Now $F = \pi(E)$ is a definable subset of M (by stable embeddedness). Therefore by geometric elimination of imaginaries there exists $f \in \mathrm{acl}^{M'}(e) \cap M$ such that the canonical parameter f_1 of F is in $\mathrm{acl}^M(f)$.

Let $\alpha \in \mathrm{Aut}(M'/f)$, then as f_1 has finite orbit over f (say $f_1, \ldots, f_k$), we have $\alpha(f_1) = \alpha(f_i)$ for some $i \leq k$. Let F_i be the definable set which is corresponding to f_i. Then we have $\alpha(F) = F_i$ and we have that $E' = \alpha(E)$ is a $\sim$-class with $\pi(E') = F_i$ for some $1 \leq i \leq j$.

Pick now any $b_i \in F_i$. There is some $a' \in E'$ with $\pi(a') = b_i$ and with $E' = a'/\sim$. Thus the number of possibilities for E' is at most $|\bigcup_{i=1}^{k} \pi^{-1}(b_i)|$. But this is finite. So $e \in \mathrm{acl}^{M'}(f)$ and therefore $e \in \mathrm{acl}^{M'}(\mathrm{acl}^{M'}(e) \cap M)$. $\square$

Corollary 3.3.2. *Let M' be an algebraic cover of M. Further suppose M' is a saturated model. Then M'^{eq} is an algebraic cover of M^{eq}.*

Proof. First note that M' is an algebraic cover of M^{eq}: M' is in the algebraic closure of M^{eq} as it is in the algebraic closure of M. The structure M^{eq} also stays fully embedded in M' as this only depends on the definable sets on the home-sort of M^{eq} and M. We have clearly that $\mathrm{acl}^{M'^{\mathrm{eq}}}(M^{\mathrm{eq}}) = M'^{\mathrm{eq}}$ by Lemma 3.3.1. Now as every automorphism from M^{eq} extends to M', we have that every automorphism of M^{eq} extends to M'^{eq} as every automorphism of M' extends uniquely to one of $\mathfrak{C}'^{\mathrm{eq}}$. Hence by Fact 3.1.3 M^{eq} is fully embedded in M'^{eq} $\square$

Lemma 3.3.3. *Let M' be an algebraic cover of $M = M^{\mathrm{eq}}$. Further we require that the algebraic closure of the empty set is added as parameters to M, i.e we have $M^{\mathrm{eq}}_{\mathrm{acl}^{\mathrm{eq}}(\emptyset)} = M$. Then we have that the structure ${M'}_{\mathrm{acl}^{M'}(\emptyset)}$ (again this means we add the parameters $\mathrm{acl}^{M'}(\emptyset)$ to M') is an algebraic cover of M.*

Proof. We clearly have that ${M'}_{\mathrm{acl}(\emptyset)}$ is contained in $\mathrm{acl}(M)$. Hence we need to check that M is fully embedded in ${M'}_{\mathrm{acl}(\emptyset)}$. It is clear that the parameter definable sets of M with parameters from M' stay the same. So it is left to check embeddedness. For that take some definable set of M, say $\phi(M, \overline{c})$ with $\overline{c} \in \mathrm{acl}^{M'}(\emptyset)$. Now over M' the tuple $\overline{c}$ has finite orbit. Hence the canonical parameter e of $\phi(M, \overline{c})$ has finite orbit in M'. Hence e must have finite orbit in M. But as we required that any canonical parameter with finite orbit is already added as parameter in M we are finished. $\square$

3.4 A cover is a mild extension

Remember that we use the conventions defined on page 29.

If we look at the nature of algebraic and finite covers, one could be tempted to say that such expansion should preserve any serious model theoretic property. But this is slightly overconfident. As then one would need to take some fundamental model theoretic properties such as o-minimality as non-serious. For that first note that in o-minimal the definable and algebraic closure coincide (see 2.2 of [Mac00]). The point now is that in any theory in which the algebraic and definable closure coincidence we have that the 2-cover without any new relations (a two-to-one function from a new sort to the old) will have the property that definable closure and algebraic closure differ. Hence this kind of 2-cover of an o-minimal theory will be not o-minimal. But we can still show that many (stability-style) model-theoretic properties are preserved under extensions of algebraic covers (and finite covers).

Lemma 3.4.1. *Let T' be an algebraic cover of T. Let κ be a cardinal with $\kappa \geq |T'|^+$. Then a model M' (of T') is κ-saturated if and only if its reduct M of M' to the S-sorts is κ-saturated in the theory T.*

Proof. The direction left to right is clear, as any type in M is a partial type in M' and can therefore be satisfied by saturation of M'. Now assume M is κ-saturated. So take a type $p \in S_x(A)$ over some parameters $A \subset M'$ with $|A| = \lambda < \kappa$. By Lemma 3.2.2 there exists a definable (boundedly) finite-to-one π from the sort of p to some sorts of M.

Now note that the type

$$q = \{\exists x(\pi(x) \doteq y \wedge \psi(x) \mid \psi \in p\}$$

is satisfiable in M: It is finitely-satisfiable since

$$\exists x(\pi(x) \doteq y \wedge \bigwedge_i \psi_i(x)) \vdash \bigwedge_i \exists x(\pi(x) \doteq y \wedge \psi_i(x))$$

for any finite number of $\psi_i \in p$. To see that it is satisfiable, note first that as M is stably embedded in M' we can replace every formula in q by an $L(M)$-formula with parameters in M. Now this type will then have equal or less than $\lambda \cdot |T|$-many parameters. As $\lambda \cdot |T|$ is smaller then κ we know that q is satisfied.

We now fix m a realisation of q. And let $m_1, \dots, m_n$ be all elements of realising $\pi(x) \doteq m$. We claim that one of the m_i is a realisation of p. If not then for all m_i there would be a $\psi_i \in p$ with $m_i \models \neg\psi_i$. But since p is a complete type, we have $\bigwedge_i \psi_i \in p$ and hence $\exists x(\pi(x) \doteq y \wedge \bigwedge_i \psi_i(x)) \in q$. But this is impossible since m would not satisfy $\exists x(\pi(x) \doteq y \wedge \bigwedge_i \varphi_i(x)) \in q$. ∎

Corollary 3.4.2. *Let T' be an algebraic cover of T. Then T' is uncountably categorical if and only if T is uncountably categorical.*

Proof. If T' is uncountably categorical, then any uncountable model of T' is saturated. As we can extend any model of T to a model of T', we have that any model of T of the size of $|T'|^+$ is saturated by Lemma 3.4.1. Hence as two saturated models of T' of the same cardinality are isomorphic, we have that T is $|T'|^+$-categorical and therefore uncountably categorical.

If on the other hand T is uncountably categorical, then by Lemma 3.4.1 any model of T' of size $|T'|^+$ is saturated as any reduct to T is saturated. Hence again as two saturated models of T' of the same cardinality are isomorphic, we have that T' is $|T'|^+$-categorical. $\square$

Lemma 3.4.3. *Let T' be an algebraic cover of T. Then T' is rosy if and only if T is rosy. Further for all A, B, C of $(\mathfrak{C}')^{\mathrm{eq}}$ we have that $A \mathop{\smile\!\!\!\!\!\mid}_B C$ holds if and only if in T^{eq} we have that the following holds;*

$$\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathfrak{C}^{\mathrm{eq}} \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathrm{eq}}(B) \cap \mathfrak{C}^{\mathrm{eq}}} \mathrm{acl}^{\mathrm{eq}}(C) \cap \mathfrak{C}^{\mathrm{eq}}.$$

Proof. If T' is rosy then $(T')^{\mathrm{eq}}$ has an independence notion $\mathop{\smile\!\!\!\!\!\mid}^{T'}$. We check that $\mathop{\smile\!\!\!\!\!\mid}^{T'}$ restricted to subsets of $\mathfrak{C}^{\mathrm{eq}}$ is an independence notion of T^{eq} (this will give that T is rosy). It is invariant under automorphisms of T as we have seen that we can extend every automorphism of T to one of T'. The rest of the properties directly translate to T^{eq} as well (see Corollary 3.1.9 for the Anti-Reflexivity).

Let on the other hand T^{eq} have an independence notion. Now by Corollary 3.3.2 we know that $(T')^{\mathrm{eq}}$ is an algebraic cover of T^{eq} hence we may assume that $T = T^{\mathrm{eq}}$ and $T' = (T')^{\mathrm{eq}}$. We define that $a \mathop{\smile\!\!\!\!\!\mid}_B C$ is true in T' if the followings holds (with $\mathfrak{C}$ is the monster of T);

$$\mathrm{acl}(A) \cap \mathfrak{C} \mathop{\smile\!\!\!\!\!\mid}^{T}_{\mathrm{acl}^{\mathrm{eq}}(B) \cap \mathfrak{C}^{\mathrm{eq}}} \mathrm{acl}(C) \cap \mathfrak{C}.$$

We need to check that all the properties of an independence notion are satisfied. As every automorphism of T' if restricted to the sorts T is an automorphism of T, we have that $\mathop{\smile\!\!\!\!\!\!\mid}^{T'}$ is invariant of automorphism. We have that Monotonicity, Transitivity and Symmetry follows, because $\mathop{\smile\!\!\!\!\!\!\mid}^{T}$ has this property together with 2.4.4.

We check that Finite Character holds. We just need to check that $\operatorname{acl}(A) \cap \mathfrak{C} \mathop{\smile\!\!\!\!\!\!\mid}_{\operatorname{acl}(B)\cap\mathfrak{C}} \operatorname{acl}(C_0) \cap \mathfrak{C}$ for every finite $C_0 \subset C$ implies

$$\operatorname{acl}(A) \cap \mathfrak{C} \mathop{\smile\!\!\!\!\!\!\mid}_{\operatorname{acl}(B)\cap\mathfrak{C}} \operatorname{acl}(C) \cap \mathfrak{C}.$$

Then this implies of course that $\operatorname{acl}(A) \cap \mathfrak{C} \mathop{\smile\!\!\!\!\!\!\mid}_{\operatorname{acl}(B)\cap\mathfrak{C}} C'_0$ holds for any finite $C'_0 \subset \operatorname{acl}(C) \cap \mathfrak{C}$ hence by Finite Character of $\mathop{\smile\!\!\!\!\!\!\mid}^{T}$ we are finished.

For the Local Character we can take $\max(\kappa, \aleph_0)$ where κ is from the Local Character of $\mathop{\smile\!\!\!\!\!\!\mid}^{T}$. To see that, note first that (by the Local Character of T) if the following equation does hold

$$\operatorname{acl}(a) \cap \mathfrak{C} \mathop{\smile\!\!\!\!\!\!\mid}^{T}_{\operatorname{acl}(B)\cap\mathfrak{C}} \operatorname{acl}(B) \cap \mathfrak{C},$$

we can find $B_0 \subset (\operatorname{acl}(B) \cap \mathfrak{C})$ of cardinality κ such that

$$\operatorname{acl}(a) \cap \mathfrak{C} \mathop{\smile\!\!\!\!\!\!\mid}^{T}_{B_0} \operatorname{acl}(B) \cap \mathfrak{C}.$$

Now for each element b of B_0 there is some finite tuple c_b in B such that $b \in \operatorname{acl}(c_b) \cap \mathfrak{C}$. The set $\{c_b \mid b \in B_0\}$ has size at most $\max(\kappa, \aleph_0)$ and we have

$$\operatorname{acl}(a) \cap \mathfrak{C} \mathop{\smile\!\!\!\!\!\!\mid}^{T}_{\operatorname{acl}(\{c_b \mid b \in B_0\})\cap\mathfrak{C}} \operatorname{acl}(B) \cap \mathfrak{C}$$

since $B_0 \subset \operatorname{acl}(\{c_b \mid b \in B_0\}) \cap \mathfrak{C} \subset \operatorname{acl}(B) \cap \mathfrak{C}$ by Base Monotonicity (which follows from Monotonicity and Transitivity).

For the Existence Property for a, B, C we need to find $a' \equiv_B a$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}_B C$. Note that there is a 0-definable finite-to-one map π from the sort of a to the sort of T by Lemma 3.2.2. We can find $m \equiv_{\operatorname{acl}^{\mathrm{eq}}(B)\cap\mathfrak{C}} \pi(a)$ such that $m \mathop{\smile\!\!\!\!\!\!\mid}_{\operatorname{acl}^{\mathrm{eq}}(B)\cap\mathfrak{C}} \pi(C)$ by the Existence Property of T. We check that we have $a' \equiv_B a$ for some $a' \in \pi^{-1}(m)$. If we assume contrary, then there would be a formula $\phi(x, y) \in L'$ (the language of T') and a tuple $b \in B$, such that

$a \models \phi(x, b)$ but not that any $a' \in \pi^{-1}(m)$ satisfies $\phi(x, b)$. Hence we have $m \not\equiv_B \pi(a)$: we can see this via the formula $\psi(z, b) = \exists x \phi(x, b) \wedge \pi(x) \doteq z)$. There is again by Lemma 3.2.2 a 0-definable finite-to-one map from the sort of b to T. We will assume that this is also π. Now the canonical parameter of $\psi(z, b)$ has finite orbit over $\pi(b)$ (as it is definable over $b \in \mathrm{acl}^{\mathrm{eq}}(\pi(b))$). The canonical parameter of $\psi(z, b)$ can be picked in T by stable embeddedness and therefore it is in $\mathrm{acl}(B) \cap \mathfrak{C}$ (since it contains $\pi(b)$). Hence $m \not\equiv_{\mathrm{acl}(B)\cap\mathfrak{C}} \pi(a)$ which is impossible.

It is left to check Anti-Reflexivity for tuples a and sets A of $\mathfrak{C}'$. So assume that $a \mathop{\smile\!\!\!\!\!\!\perp}_A a$. We will show that $a \in \mathrm{acl}(A)$ then. We fix π some 0-definable finite-to-one map from the sort of a to some sorts T, which we can do by Lemma 3.2.2. We now claim that the following holds

$$\mathrm{acl}(a) \cap \mathfrak{C} \mathop{\smile\!\!\!\!\!\!\perp}^{T}_{\mathrm{acl}(A)\cap\mathfrak{C}} \mathrm{acl}(a) \cap \mathfrak{C}$$

if and only if

$$\pi(a) \mathop{\smile\!\!\!\!\!\!\perp}^{T}_{\mathrm{acl}(A)\cap\mathfrak{C}} \pi(a).$$

Note first that we clearly have $a \in \mathrm{acl}(\pi(a))$ and therefore

$$\mathrm{acl}(a) \cap \mathfrak{C} = \mathrm{acl}(\pi(a)) \cap \mathfrak{C} \subseteq \mathrm{acl}^T(\pi(a))$$

holds by Corollary 3.1.9. Hence the equivalence follows by Lemma 2.4.4. This gives us then that $\pi(a) \in \mathrm{acl}(A) \cap \mathfrak{C}$ holds. Therefore we have that $a \in \mathrm{acl}(\mathrm{acl}(A) \cap \mathfrak{C}) \subseteq \mathrm{acl}(A)$. □

Lemma 3.4.4. *Let T' be an algebraic cover of T with L, L' both countable languages. Then T is ω-categorical if and only if T' is ω-categorical.*

Proof. We will use the Ryll-Nardzewski Theorem in the proof without mentioning it. Let T be ω-categorical. Now for any given tuple of S'-sorts $\overline{s}$ by Lemma 3.2.2 we know that there is a 0-definable formula $\phi(\overline{x}, \overline{y})$ such that $\models \forall y \exists^{<m} x \phi(\overline{x}, \overline{y})$ and $\exists y \phi(\overline{x}, \overline{y})$ covers the $\overline{s}$-sorts. Now let $A_{\overline{b}}$ be the realisations of $\phi(\overline{x}, \overline{b})$. Note the b's only belong to one of finitely many types by ω-categoricity of T (say n-many). As we can extend every automorphism

of $\mathfrak{C}$, we can for $\bar{b} \equiv \bar{b}'$ find some automorphism which maps $A_{\bar{b}}$ onto $A_{\bar{b}'}$. Hence we know that there can be at most $n \cdot m$-many $\bar{s}$-sorted L'-types.

If T' is ω-categorical then it follows that there are only finitely many $\bar{s}$-sorted L'-definable sets. By embeddedness of T this directly gives finitely many $\bar{s}$-sorted L-definable sets. And hence there exists only finitely many $\bar{s}$-sorted L-formulae modulo T. $\square$

Proposition 3.4.5. *Let T' be an algebraic cover of T. Then T is κ-stable if and only if T' is κ-stable. Hence T is stable if and only if T' is stable.*

Proof. Note again that T'^{eq} is an algebraic cover of T^{eq} by Corollary 3.3.2. As a theory T is κ-stable if and only if T^{eq} is κ-stable, we may assume again that $T' = T'^{\mathrm{eq}}$ and $T = T^{\mathrm{eq}}$.

Let T be κ-stable. Now fix some B in the monster of T' with $|B| \leq \kappa$. What size does $S_x^{T'}(B)$ have, if x is still a variable of the old sorts? There are $|B| \cdot |T'| \leq \kappa$ many $L'(B)$ formulae $\phi(x, b)$. Now since T is stably embedded, we have that there is a $L(\mathfrak{C})$-formula $\psi(x, m)$ which has the same realisations as $\phi(x, b)$. Let A be the set of these parameters m. It has size $\leq \kappa$ as already noted. Hence $S_x^{T'}(B)$ has the same size as some $S_x^T(A)$ which has by the κ-stableness of T at most κ many elements.

Let π be 0-definable finite-to-one from some new sort of s' to the sort s. Now for a variable y of the new sort s' we will show, that for every $p \in S_x^{T'}(B)$ there are only finitely many $q \in S_y^{T'}(B)$ with $\pi(q) = p$. This will then finish the proof as then $|S_y^{T'}(B)| = |S_x^{T'}(B)|$. (Define $\pi(q)$ as in the last proof: take any $a \models q$ and set $p = \mathrm{tp}(\pi(a)/B)$, then note that it is well-defined via automorphisms).

We fix some $b \models p$. So take some realisation of $a \models q$ we know that there is an automorphism of T' fixing B which maps $\pi(a)$ to b, this of course also maps a to some element of $\pi^{-1}(b)$. Hence every element has the same type over B as some $\pi^{-1}(b)$. Hence as already discussed $|S^{T'}(B)| \leq \kappa$.

For the other direction note that any A-type in T is a partial type in T'. Hence there is an inclusion from $S_T(A)$ to $S_{T'}(A)$. Thus if T' is κ-stable, then T has to be κ-stable. $\square$

Lemma 3.4.6. *Let T' be an algebraic cover of T. Then T is simple if and only if T' is simple.*

Proof. The direction right to left follows from Fact 2.6.4, as we know then that if a theory is simple, then any reduct of it is simple as well.

We start showing the direction left to right. We assume that T is simple and T' is not simple (but we know that it is a least rosy by Lemma 3.4.3). Note again that T'^{eq} is an algebraic cover of T^{eq} by 3.3.2. By Fact 2.6.4 we know that any theory T'' is simple if and only if T''^{eq} is simple. Hence we may assume again that $T' = T'^{\mathrm{eq}}$ and $T = T^{\mathrm{eq}}$. Note that if an independence notion implies the non-dividing relation, then T is already simple (see 2.6.2). As T' is rosy non-simple we know that there exists a, B, C with $C = \mathrm{acl}(C)$ and $B = \mathrm{acl}(B)$ such that $a \mathop{\smile\!\!\!\!\!\!\;|}_B C$ and some $\phi(x, c) \in \mathrm{tp}_x(a/C)$ which divides over B. To finish the proof we will show that such conditions are not possible.

As $\phi(x, c)$ divides over B there is a Morley sequence $(c_i)_{i\in I}$of realisations of $\mathrm{tp}(c/B)$ such that $\bigwedge_{i\in I} \phi(x, c_i)$ is inconsistent. Let π be 0-definable finite-to-one from some sort of c to the some old sort (we find such a function by 3.2.2).

This implies that $\bigwedge_{i\in I} \exists x(\phi(x, c_i) \wedge \pi(x) \doteq y)$ is inconsistent: If it were not, fix a realisation of it (say b), then this implies that some $a' \in \pi^{-1}(b)$ (because this set is finite) realises infinitely many $\phi(x, c_i)$. But this is impossible since the sequence $(c_i)_{i\in I}$ is Morley and $\bigwedge_{i\in I} \phi(x, c_i)$ is inconsistent. Now let $(d_i)_{i\in I}$ be the canonical parameters of $\exists x(\phi(x, c_i) \wedge \pi(x) \doteq y)$. Note that they can be considered as part of T by stable embeddedness. And also note that they are a Morley sequence over B. And therefore they are a Morley sequence over $B \cap \mathfrak{C}$.

Moreover, if d is the canonical parameter of $\exists x(\phi(x, c) \wedge \pi(x) \doteq y)$, then $(d_i)_{i\in I}$ is a Morley sequence of $\mathrm{tp}(d/B \cap \mathfrak{C})$ (as $B \cap \mathfrak{C}$ is algebraically closed by 3.1.9). As $\bigwedge_i \exists x(\phi(x, c_i) \wedge \pi(x) \doteq y)$ is inconsistent, we have that the conjunction $\bigwedge_i$ "$y \in d_i$" is inconsistent. Hence "$y \in d$" divides over $B \cap \mathfrak{C}$. As $\mathfrak{C} \models \pi(a) \in d$ and $d \in \mathrm{acl}(\pi(c))$, we have $\mathrm{tp}(\pi(a)/\mathrm{acl}(C) \cap \mathfrak{C})$ divides over $B \cap \mathfrak{C}$ in (in T'). But as the 0-definable sets are the same in T and T', this implies that $\mathrm{tp}(\pi(a)/C \cap \mathfrak{C})$ divides over $B \cap \mathfrak{C}$ in T. But this is impossible as being independent implies non-dividing in T. $\square$

For more results about finite covers one may look at [EH93], chapter 2 of [Eva97b] and [Eva97a]. Another interesting publication in which finite covers are used is [Hru89].

3.5 WEI in ω-categorical theories

Remember that we use the conventions defined on page 29.

The following method for proving weak elimination of imaginaries of split finite covers of ω-categorical theories was provided by David Evans.

Notation: We denote by $\mathcal{A}(\mathfrak{C})$ the set of algebraic closures in $\mathfrak{C}$ of finite subsets of $\mathfrak{C}$. For subsets C_1, C_2 of some group G, then by $\langle C_1, C_2 \rangle$ we denote the subgroup generated by these two sets.

Fact 3.5.1. (16.17 of [Poi00]) Let T be some theory and $\mathfrak{C}$ its monster model. If there is no infinite strictly decreasing sequence $A_0 \supsetneq A_1 \ldots$ where each $A_i \in \mathcal{A}(\mathfrak{C})$) and for all $A, B \in \mathcal{A}(\mathfrak{C})$ we have

$$\langle \mathrm{Aut}(\mathfrak{C}/A), \mathrm{Aut}(\mathfrak{C}/B) \rangle = \mathrm{Aut}(\mathfrak{C}/A \cap B),$$

then T has weak elimination of imaginaries.

Lemma 3.5.2. *Let $\mathfrak{C}$ be ω-categorical. We have for all $A, B \in \mathcal{A}(\mathfrak{C})$ that*

$$\langle Aut(\mathfrak{C}/A), Aut(\mathfrak{C}/B) \rangle = Aut(\mathfrak{C}/A \cap B),$$

if and only if $\mathfrak{C}$ has weak elimination of imaginaries.

Proof. We show first, that in a countably categorical theory if for any sets $A, B \in \mathcal{A}(\mathfrak{C})$ we have $\langle \mathrm{Aut}(\mathfrak{C}/A), \mathrm{Aut}(\mathfrak{C}/B) \rangle = \mathrm{Aut}(\mathfrak{C}/A \cap B)$, then wei must be true. But we just need to note that due to ω-categoricity we have that algebraic closure of a finite set is finite and hence there is no infinite strictly decreasing sequence $A_0 \supsetneq A_1 \ldots$ where each A_i is the algebraic closure of a finite set of parameters (as A_0 is finite). Hence then weak elimination of imaginaries follows from the Fact 3.5.1.

For the other direction assume that $\mathfrak{C}$ has wei. Let $A, B \subset \mathfrak{C}$ be finite and algebraically closed, let $\overline{a}$ be enumeration of A and let D be the $\mathrm{Aut}(\mathfrak{C})$-orbit containing $\overline{a}$. By the Ryll-Nardzewski Theorem we have that D is 0-definable. Let H be the subgroup of $\mathrm{Aut}(\mathfrak{C})$ generated by $\mathrm{Aut}(\mathfrak{C}/A)$ and $\mathrm{Aut}(\mathfrak{C}/B)$ and let $E \subset D$ be the H-orbit containing $\overline{a}$. By the Ryll-Nardzewski Theorem we have that E is a class of a 0-definable equivalence relation R on D: To see that define the equivalence relation R as follows; $R(g(\overline{a}), f(\overline{a}))$ if $f^{-1} \circ g \in H$. Then this equivalence relation will be well defined and $\mathrm{Aut}(\mathfrak{C})$-invariant. By the $\mathrm{Aut}(\mathfrak{C})$-invariance and the Ryll-Nardzewski Theorem R is a 0-definable subset (of $\mathfrak{C}^{2n}$). Moreover, H is the stabiliser of E in $\mathrm{Aut}(\mathfrak{C})$. Let e be the corresponding imaginary of E. Let $X = \mathrm{acl}(e) \cap \mathfrak{C}$. By wei we obtain that $\mathrm{Aut}(\mathfrak{C}/X) \leq H$ and H stabilises X set wise. From the first, it follows that $X \subseteq A \cap B$; from the second we get that $X \subseteq A$ and $X \subseteq B$. This then gives us $H = \mathrm{Aut}(\mathfrak{C}/A \cap B)$ as required. $\square$

For the rest of this section we assume that $\pi : \mathfrak{C}' \to \mathfrak{C}$ is a finite cover such that $\mathfrak{C}$ has weak elimination of imaginaries The main point is that wei can be checked in the kernel $K = \mathrm{Aut}(\mathfrak{C}'/\mathfrak{C})$.

Assumption/ Notation: If $A \in \mathcal{A}(\mathfrak{C})$ then we define

$$\mathfrak{C}'(A) = \bigcup_{a \in A} \pi^{-1}(a) \subseteq \mathfrak{C}'.$$

Further let $K = \mathrm{Aut}(\mathfrak{C}'/\mathfrak{C})$ and $K_A = \mathrm{Aut}(\mathfrak{C}'/\mathfrak{C}, \mathfrak{C}'(A))$.

Definition 3.5.3. We say that a cover $\pi : \mathfrak{C}' \to \mathfrak{C}$ *splits* if there is a closed subgroup $H \leq \mathrm{Aut}(\mathfrak{C}')$ with $H \cap K = 1$ and $\mathrm{Aut}(\mathfrak{C}') = KH$.

The main consequence of this for us is the following:

Lemma 3.5.4. *(Lemma 2.2 of [Eva09]): Suppose that the cover $\pi : \mathfrak{C}' \to \mathfrak{C}$ is split and $X_1, \ldots, X_r \in \mathcal{A}(\mathfrak{C})$. Then we have that*

$$Aut(\bigcup_{i=1}^{r} \mathfrak{C}'(X_i) / \bigcup_{i=1}^{r} X_i) = Aut(\bigcup_{i=1}^{r} \mathfrak{C}'(X_i) / \mathfrak{C}).$$

Moreover, Aut$(\mathfrak{C}'/\mathfrak{C}'(X_1)) = K_{X_1}H_{X_1}$*, where* H_{X_1} *denotes the pointwise stabiliser in* H *of* X_1*.*

Proof. The containment $\supseteq$ is clear. For the converse, it is enough to prove the case $r = 1$ as $\mathrm{Aut}(\bigcup_{i=1}^{r} \mathfrak{C}'(X_i)/\mathfrak{C}) = \bigcap_{i=1}^{r} \mathrm{Aut}(\mathfrak{C}'(X_i)/\mathfrak{C})$. So suppose $f \in \mathrm{Aut}(\mathfrak{C}'/X_1)$. By assumption, there is is a closed subgroup $H \leq \mathrm{Aut}(\mathfrak{C}')$ with $H \cap K = 1$ and $\mathrm{Aut}(\mathfrak{C}') = KH$. Thus we have that $f = kh$ with $k \in K$ and $h \in H$. So clearly $h \in \mathrm{Aut}(\mathfrak{C}'/X_1)$.

Consider the restriction map from H_{X_1} to $\mathrm{Aut}(\mathrm{acl}^{\mathfrak{C}'}(X_1)/X_1)$. We claim that this has trivial image. To see this, note that this is a continuous map and the range is a profinite group. As X_1 is algebraically closed in $\mathfrak{C}$, it follows that $\mathrm{Aut}(\mathfrak{C}/X_1)$, has no proper open subgroup of finite index. Thus the only possible continuous image of H_{X_1} inside a profinite group is the trivial group. So h fixes each element of $\mathrm{acl}^{\mathfrak{C}'}(X_1)$ and therefore f and k agree on $\mathrm{acl}^{\mathfrak{C}}(X_1)$. As $k \in \mathrm{Aut}(\mathfrak{C}'/\mathfrak{C})$ we have shown that we can find for any $f \in \mathrm{Aut}(\mathrm{acl}^{\mathfrak{C}'}(X_1)/X_1)$ some element in $\mathrm{Aut}(\mathfrak{C}'(X_1)/\mathfrak{C})$. $\square$

Lemma 3.5.5. *For* $A, B \in \mathcal{A}(\mathfrak{C})$ *the following are equivalent:*

1. $K_A K_B = K_{A\cap B}$;
2. *Aut*$(\mathfrak{C}'(A)/\mathfrak{C}, \mathfrak{C}'(B)) =$ *Aut*$(\mathfrak{C}'(A)/\mathfrak{C}, \mathfrak{C}'(A \cap B))$ *and the same condition with* A *and* B *interchanged.*

Proof. (1) $\Rightarrow$ (2): Take $h \in \mathrm{Aut}(\mathfrak{C}'(A)/\mathfrak{C}, \mathfrak{C}'(A \cap B))$. Extend to $k \in K_{A\cap B}$. So there are $k_1 \in K_A$ and $k_2 \in K_B$ with $k = k_1k_2$. Then k_2 restricted to $\mathfrak{C}'(A)$ does what we want, as it fixes $\mathfrak{C}'(B)$ and as k_1 fixes $\mathfrak{C}'(A)$ it does have to be the same on $\mathfrak{C}'(A)$ as h.

(2) $\Rightarrow$ (1): Take $k \in K_{A\cap B}$ and let $h = k \restriction \mathfrak{C}'(A)$. There is $k' \in K_B$ which extends h. Then $k^{-1}k' \in K_A$ as required. $\square$

Lemma 3.5.6. *Suppose* $\pi : \mathfrak{C}' \to \mathfrak{C}$ *is split and* ω*-categorical and suppose that* $\mathfrak{C}$ *has weak elimination of imaginaries. If we have that* $K_A K_B = K_{A\cap B}$ *holds for all* $A, B \in \mathcal{A}(\mathfrak{C})$, *then* $\mathfrak{C}'$ *has weak elimination of imaginaries.*

Proof. We will use Lemma 3.5.2. So take a splitting $\mathrm{Aut}(\mathfrak{C}') = KH$. Let $X, Y \in \mathcal{A}(\mathfrak{C})$. By Lemma 3.5.4, we have $\mathrm{Aut}(\mathfrak{C}'/\mathfrak{C}'(X)) = K_X H_X$ and

$$\mathrm{Aut}(\mathfrak{C}'/\mathfrak{C}'(Y)) = K_Y H_Y.$$

We have to show that what they generate contains $\mathrm{Aut}(\mathfrak{C}'/\mathfrak{C}'(X \cap Y))$.

Now, by wei in $\mathfrak{C}$ (and Lemma 3.5.2) we have $\langle H_X, H_Y \rangle = H_{X \cap Y}$. By assumption $K_{X \cap Y} = K_X K_Y$, and this suffices. $\square$

Note that this and Lemma 3.5.5 gives us a way of checking weak elimination of imaginaries by the analysing the action of K on finite sets:

Corollary 3.5.7. *Suppose that $\pi : \mathfrak{C}' \to \mathfrak{C}$ is split and ω-categorical and suppose that $\mathfrak{C}$ has weak elimination of imaginaries. Suppose that whenever $A, B \in \mathcal{A}(\mathfrak{C})$, then*

$$Aut(\mathfrak{C}'(A)/\mathfrak{C}, \mathfrak{C}'(A \cap B)) = Aut(\mathfrak{C}'(A)/\mathfrak{C}, \mathfrak{C}'(B)).$$

Then $\mathfrak{C}'$ has weak elimination of imaginaries.

Chapter 4

Known results on Amalgamation

The nature of this chapter is purely recreational (in its original Latin sense). All the definitions and results in here (modulo some slight modifications and some corollaries) are results of other authors and have appeared somewhere else before. The main source of them are the publications [GKK13a] and [Hru12]. Some similar versions of some proofs in this chapter of results of others have already been in my Diplomarbeit (see [Zan12]). We will repeat the proofs for the convenience of the reader. Some readers may wonder why the categorical approach for defining higher amalgamation is taken and why we do not just work with the Property $B(n)$. One reason is that the categorical definition of higher amalgamation directly translates to the generalised independence theorem (see Definition 4.1.8 and the Remark thereafter). Another reason is that lower uniqueness of amalgamation has to be assumed such that the Property $B(n)$, $n+1$-amalgamation and unique n-amalgamation (called n-uniqueness) are all equivalent (see Fact 4.2.5). Moreover, to show this one cannot work over a fixed base but instead has to assume that the properties are true over all sets. That this makes a difference can be seen via the examples in Section 5.6 and Section 5.7. Also note that there is some sort of group configuration theorem for simple theories when assuming higher amalgamation (see [PKM06] or Chapter 9 of [Kim14]).

In this chapter we normally work in a complete rosy theory T with $\mathop{\smile\!\!\!\!\!\!\mid}\,$ as its independence notion. Note that some parts make sense outside the rosy context. We are now going to define the basic notion of amalgamation.

4.1 Definition of amalgamation

We let $\tilde{n}$ with $n \geq 2$ be the set $\{1, \dots, n\}$. By $[S]^k$ for some set S and some natural number k, we mean the set of all subsets of size k, i.e. it is a set of the following form $\{s \subset S \mid |s| = k\}$. Further we define $[n]^k$ to be $[\tilde{n}]^k$. By $u \subset_k s$ we mean that u has size k.

We view the power set $\mathcal{P}(\tilde{n})$, the set

$$\mathcal{P}^-(\tilde{n}) = \{s \mid s \subsetneq \tilde{n}\}$$

and more generally any $S \subset \mathcal{P}(\tilde{n})$ which is closed under subsets (i.e. for all $s \in S$ and all $t \subset s$ it follows that $t \in S$) as categories; where the objects are the elements of $\mathcal{P}(\tilde{n}), \mathcal{P}^-(\tilde{n}), S$ and the morphisms are the inclusion maps.

Let $\mathcal{C}$ be the category whose objects are the acl-closed subsets of $\mathfrak{C}$. Let $\mathcal{C}^{\mathrm{eq}}$ be the category whose objects are the $\mathrm{acl}^{\mathrm{eq}}$-closed subsets of $\mathfrak{C}^{\mathrm{eq}}$. In both categories, the morphisms are the partial elementary maps between the closed sets. If $\mathfrak{a}$ is a functor from S (a subset of $\mathcal{P}(\tilde{n})$ closed under subset) to $\mathcal{C}^{\mathrm{eq}}$, we write $\mathfrak{a}(s)$ for the image of the object $s \in S$ under $\mathfrak{a}$. For $s, t \in S$ with $s \subset t$ we write $\mathfrak{a}_s^t : \mathfrak{a}(s) \to \mathfrak{a}(t)$ for the partial elementary map which is the image of the morphism "$s \subset t$" under $\mathfrak{a}$. We will call the maps $\mathfrak{a}_s^t : \mathfrak{a}(s) \to \mathfrak{a}(t)$ *transition maps.* We further write $\mathfrak{a}_s^t(s)$ or $\mathfrak{a}^t(s)$ for $\mathfrak{a}_s^t(\mathfrak{a}(s))$ (that is the image of $\mathfrak{a}(s)$ under the function $\mathfrak{a}_s^t$).

Proposition 4.1.1. *If T weakly eliminates imaginaries then $\mathcal{C}^{\mathrm{eq}}$ and $\mathcal{C}$ are isomorphic. Moreover, we can pick the isomorphism in such a way that it preserves the independence relation.*

Proof. Let T weakly eliminate imaginaries. Now we are going to construct an isomorphism between the categories $\mathcal{C}^{\mathrm{eq}}$ and $\mathcal{C}$. For this it will be enough to construct two functors $F : \mathcal{C}^{\mathrm{eq}} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{C}^{\mathrm{eq}}$ with the property that the composition of them is the identity, i.e. $F \circ G = \mathrm{id}_{\mathcal{C}}$ and $G \circ F = \mathrm{id}_{\mathcal{C}^{\mathrm{eq}}}$.

We start by defining these functors. For any object B of $\mathcal{C}$ we set the functor G to be $G(B) = \mathrm{acl}^{\mathrm{eq}}(B)$. For an object of $\mathcal{C}^{\mathrm{eq}}$ set $F(B) = B \restriction \mathfrak{C}$. We clearly have that

$$F \circ G = G(B) \restriction \mathfrak{C} = \mathrm{acl}^{\mathrm{eq}}(B) \restriction \mathfrak{C} = B$$

because B is already algebraically closed (in $\mathfrak{C}$) and hence there are no new algebraic real elements. We also have that $G \circ F(B) = \mathrm{acl}^{\mathrm{eq}}(B \restriction \mathfrak{C}) = B$. This is because by weak elimination of imaginaries we have that any imaginary $e \in B$ has a real element a in its algebraic closure (hence it is in B) such that $e \in \mathrm{dcl}^{\mathrm{eq}}(a)$ and therefore $e \in \mathrm{acl}^{\mathrm{eq}}(B \restriction \mathfrak{C})$.

We let F map a morphism $f : B \to B'$ of $\mathcal{C}^{\mathrm{eq}}$ to $f \restriction (F(B))$ which is a morphism from $F(B)$ to $F(B')$. This also shows that F is a functor. Any morphism $f : B \to B'$ of $\mathcal{C}$ will uniquely extend to some elementary map f' from $\mathrm{dcl}^{\mathrm{eq}}(B)$ to $\mathrm{dcl}^{\mathrm{eq}}(B)$. But then $\mathrm{dcl}^{\mathrm{eq}}(B) = \mathrm{acl}^{\mathrm{eq}}(B)$ as $\mathrm{acl}^{\mathrm{eq}}(B)$ has no new real elements compared to B. Now set $G(f) = f'$. Now clearly $F \circ G(f) = f$. On the other hand, $G \circ F(f') = f'$ as we have seen that an elementary map on some $\mathrm{acl}^{\mathrm{eq}}$-closed set is already uniquely determined by its restriction to the real elements.

It is left to be checked that G is a functor. If an elementary map fixes some set, then any extension of it also fixes its definable closure. Hence identities are preserved. Also $G(f \circ g) = G(f) \circ G(g)$ holds as the maps on the real elements are the same and hence on the $\mathrm{dcl}^{\mathrm{eq}}$-closure which coincidences with $\mathrm{acl}^{\mathrm{eq}}$-closure as seen. The "Moreover"-part follows now easily by Lemma 2.4.4. $\square$

By the last Proposition if we work in a theory with weak elimination of imaginaries we can confuse the two categories $\mathcal{C}^{\mathrm{eq}}$ and $\mathcal{C}$.

Definition 4.1.2. Suppose that S is a subset of $\mathcal{P}(\tilde{n})$ closed under subsets and $\mathfrak{a} : S \to \mathcal{C}$ is a functor.

1. We say that $\mathfrak{a}$ is *independent* if for every nonempty $s \in S$ we have that $\{\mathfrak{a}^s(\{i\}) \mid i \in s\}$ is an $\mathfrak{a}^s(\emptyset)$-independent set.

2. We say that $\mathfrak{a}$ is *closed*, if for every non-empty $s \in S$ we have

$$\mathfrak{a}(s) = \operatorname{acl}(\bigcup_{i \in s} \mathfrak{a}^s(\{i\})).$$

Remark 4.1.3. Note that for two naturally isomorphic functors $\mathfrak{a}, \mathfrak{b} : S \to \mathcal{C}$ then if one is independent the other is as well. This is because independence is preserved under automorphisms (which every elementary map can be extended to). The same is true for closedness, that is either both isomorphic functors are closed or neither of the two functors is. To see this just note that any automorphism which maps $(\mathfrak{a}^s(\{i\})_{i \in s}$ onto $(\mathfrak{b}^s(\{i\})_{i \in s}$ does maps $\operatorname{acl}(\bigcup_{i \in s} \mathfrak{a}^s(\{i\}))$ onto $\operatorname{acl}(\bigcup_{i \in s} \mathfrak{b}^s(\{i\}))$.

Lemma 4.1.4. *(Remark 1.7. of [GKK13a]) Let S be closed under subsets. A closed functor $\mathfrak{a} : S \to \mathcal{C}$ is independent if and only if for every $u, t \subset v \in S$ the following is true*

$$\mathfrak{a}^v(t) \underset{\mathfrak{a}^v(t \cap u)}{\perp} \mathfrak{a}^v(u).$$

Proof. $\mathfrak{a}$ is independent if and only if

$$\bigcup_{i \in t} \mathfrak{a}^v(\{i\}) \underset{\bigcup_{j \in t \cap u} \mathfrak{a}^v(\{j\})}{\perp} \bigcup_{k \in u} \mathfrak{a}^v(\{k\}) \qquad \text{for all } u, t \subset v \in S.$$

Since independence is invariant of applying acl (see Lemma 2.4.4) that independence of $\mathfrak{a}$ is equivalent to the following holds

$$\operatorname{acl}(\bigcup_{i \in t} \mathfrak{a}^v(\{i\})) \underset{\operatorname{acl}(\bigcup_{j \in t \cap u} \mathfrak{a}^v(\{j\}))}{\perp} \operatorname{acl}(\bigcup_{k \in u} \mathfrak{a}^v(\{k\})) \quad \text{for all } \ u, t \subset v \in S.$$

Since $\mathfrak{a}$ is closed this is same as

$$\mathfrak{a}^v(t) \underset{\mathfrak{a}^v(t \cap u)}{\perp} \mathfrak{a}^v(u) \qquad \text{for all } u, t \subset v \in S.$$

□

Note that the assumption of closedness is necessary in Lemma 4.1.4. In order to see that, we set up some functor from $\mathcal{P}(3)$ to $\mathcal{C}$ with the following properties;

$$\operatorname{acl}(\mathfrak{a}^{\{1,2,3\}}(\{1\})\mathfrak{a}^{\{1,2,3\}}(\{2\})) \underset{\mathfrak{a}^{\{1,2,3\}}(\emptyset)}{\perp} \mathfrak{a}^{\{1,2,3\}}(\{3\}),$$

but with

$$\mathfrak{a}^{\{1,2,3\}}(\{1,2\}) \underset{\mathfrak{a}^{\{1,2,3\}}(\emptyset)}{\not\perp} \mathfrak{a}^{\{1,2,3\}}(\{3\}).$$

We can do that as it is not required that the functor is closed. Hence we can pick the object $\mathfrak{a}\{1,2\}$ big enough, such that then the following holds;

$$\mathfrak{a}^{\{1,2,3\}}(\{1,2\}) \underset{\mathfrak{a}^{\{1,2,3\}}(\emptyset)}{\not\perp} \mathfrak{a}^{\{1,2,3\}}(\{3\}).$$

Definition 4.1.5. An *n-amalgamation problem* over A is a closed independent functor $\mathfrak{a} : \mathcal{P}^{-}(\tilde{n}) \to \mathcal{C}$ such that $\operatorname{acl}(A) = \mathfrak{a}(\emptyset)$. A *solution* to an n-amalgamation problem $\mathfrak{a}$ is an extension of it to a closed independent functor $\mathfrak{a}' : \mathcal{P}(\tilde{n}) \to \mathcal{C}$.

Definition 4.1.6. For a natural number $n \geq 2$ we define the following:

1. T has *n-existence* or *n-amalgamation* over A (where A is a subset of $\mathfrak{C}^{\mathrm{eq}}$), if every n-amalgamation problem over A in $\mathcal{C}^{\mathrm{eq}}$ has a solution.

2. T has *n-uniqueness* over A (where A is a subset of $\mathfrak{C}^{\mathrm{eq}}$), if for every n-amalgamation problem over A in $\mathcal{C}^{\mathrm{eq}}$ any two solutions of it are naturally isomorphic.

3. T has *complete* n-amalgamation over A, if it has k-amalgamation over A for every $k \leq n$ and T has *total* amalgamation over A, if it has n-amalgamation over A for every n.

4. T has *complete* n-uniqueness over A if it has k-uniqueness over A for every $k \leq n$ and T has *total* uniqueness over A if it has n-uniqueness over A for every n.

If we omit the parameter A in the above definition, then this means that it should be true for any parameter set A (of $\mathfrak{C}^{\mathrm{eq}}$). We sometimes emphasise this by saying T has n-existence *over every set* (instead of omitting "over every set").

Any definition of the above points together with the suffix *over real parameters* is meant to be the same definition except that we work in $\mathcal{C}$ and $\mathfrak{C}$ (instead of $\mathcal{C}^{\text{eq}}$ and $\mathfrak{C}^{\text{eq}}$).

Remark 4.1.7. First note that any of the above definitions over A are true if and only if they are true over $\operatorname{acl}(A)$ (computed in either T or T^{eq}). Note that in a theory with weak elimination of imaginaries the above definitions without the suffix "over real parameters" are true if and only if their counterparts with the suffix "for real parameters" are true. This can be seen via Proposition 4.1.1. Of course in the case without the suffix "for real parameters" as A is a subset of $\mathfrak{C}^{\text{eq}}$ we have to replace this by some real elements A' such that $\operatorname{acl}^{\text{eq}}(A') = \operatorname{acl}^{\text{eq}}(A)$, which we can do because of weak elimination of imaginaries.

Definition 4.1.8 (Generalised Independence Theorem)**.** A theory T satisfies the *n'th Generalised Independence Theorem* over $A = \operatorname{acl}(A)$ if the following holds: Let for any $s \in \mathcal{P}^-(\tilde{n})$ $p_s(\overline{x}_s)$ be complete type over A such that the following point are satisfied;

1. For $s \subset s'$ we have $\overline{x}_s \subset \overline{x}_{s'}$ and $p_s \subset p_{s'}$.
2. For any $\overline{a}_s \models p_s$ we have that $\{\overline{a}_i \mid i \in s\}$ is independent over A (where $\overline{a}_i$ is the subtuple of a_s corresponding to the type $p_{\{i\}}$).
3. For any $\overline{a}_s \models p_s$ we have that every element of the tuple $\overline{a}_s$ is contained in $\operatorname{acl}(\bigcup a_i)$ (where $\overline{a}_i$ is the subtuple of a_s corresponding to the type $p_{\{i\}}$).

Then there exists some complete type $p_{\{1,\ldots,n\}}$ such that the points 1 3 remain true for all $s \in \mathcal{P}(\tilde{n})$.

Remark 4.1.9. A theory has n-existence over A if and only if the n'th Generalised Independence Theorem over $\operatorname{acl}(A)$ holds.

Proof. This is just a matter of translating the properties. Left to right one has to enumerate the sets appropriately and then take their type over $\mathrm{acl}(A)$. The other direction is just a matter of realising the types and then finding the appropriate elementary maps. See Proposition 2.10 of [KKT08] for a worked out proof. $\square$

Fact 4.1.10 (Generalised Independence Theorem of ACFA)**.** (see page 3009 of [CH99]) The ACFA (algebraic closed fields with automorphism) has total amalgamation.

Lemma 4.1.11. *(Remark 1.11 of [GKK13a]) T has n-uniqueness over A if and only if for every two naturally isomorphic n-amalgamation problems $\mathfrak{a}^-$ and $\mathfrak{b}^-$ over A we have that any two solutions $\mathfrak{a}^- \subset \mathfrak{a}$ and $\mathfrak{b}^- \subset \mathfrak{b}$ are naturally isomorphic.*

Proof. The direction right to left is clear as the cases one as to check for n-uniqueness are clearly contained in the right hand property. For the other direction note that by Lemma 2.8.2 we can go over from $\mathfrak{b}$ to some naturally isomorphic $\tilde{\mathfrak{b}}$ such that $\tilde{\mathfrak{b}}$ is a solution of $\mathfrak{a}^-$. Then by n-uniqueness over A $\tilde{\mathfrak{b}}$ is naturally isomorphic to $\mathfrak{a}$. Now of course this implies that $\mathfrak{b}$ is naturally isomorphic to $\mathfrak{a}$. $\square$

Definition 4.1.12. For a sequence $a_1, \dots, a_n$ by its *canonical S-functor over A* with $S \subset \mathcal{P}(\tilde{n})$ closed under subsets we mean the functor $\mathfrak{a} : S \to \mathcal{C}$ with $\mathfrak{a}(s) = \mathrm{acl}(\{a_i \mid i \in s\}A)$ and $\mathfrak{a}_s^t = \mathrm{id}\restriction_{\mathfrak{a}(s)}$.

For an independent sequence $a_1, \dots, a_n$ over A by its *canonical problem* we mean the canonical-$\mathcal{P}^-(\tilde{n})$-functor and by its *canonical solution* we mean the canonical-$\mathcal{P}(\tilde{n})$-functor.

Lemma 4.1.13. *(Claim 1.13 of [GKK13a]) Any closed independent functor $\mathfrak{a} : \mathcal{P}(\tilde{n}) \to \mathcal{C}$ is naturally isomorphic to a canonical solution of some independent sequence.*

Proof. For that take all the morphisms of $\mathfrak{a}_s^{\tilde{n}}$. They form a system of isomorphisms (if we think about them as arrows to $\mathfrak{a}_s^{\tilde{n}}(s)$) in the sense of Lemma 2.8.1.

Hence there is a naturally isomorphic functor $\tilde{\mathfrak{a}}$ with $\tilde{\mathfrak{a}}(s) = \mathfrak{a}_s^{\tilde{n}}(s)$ and

$$\tilde{\mathfrak{a}}_s^t = \mathfrak{a}_t^{\tilde{n}} \circ \mathfrak{a}_s^t \circ (\mathfrak{a}_s^{\tilde{n}})^{-1} = \mathfrak{a}_s^{\tilde{n}} \circ (\mathfrak{a}_s^{\tilde{n}})^{-1} = \mathrm{id}_{\mathfrak{a}_s^{\tilde{n}}(s)}.$$

This shows that $\tilde{\mathfrak{a}}$ is a canonical functor of $\tilde{\mathfrak{a}}(\{1\}), \ldots, \tilde{\mathfrak{a}}(\{n\})$ over $\tilde{\mathfrak{a}}(\emptyset)$ which is naturally isomorphic to $\mathfrak{a}$. $\square$

Lemma 4.1.14. *(Proposition 1.9 of [GKK13a]) The following are equivalent;*

1. *T has n-uniqueness over A*
2. *For any independent sequence $a_1, \ldots, a_n$ over A (in $\mathfrak{C}^{\mathrm{eq}}$) we have that every solution $\mathfrak{b}$ of the canonical problem $\mathfrak{a}^-$ of $a_1, \ldots, a_n, A$ is isomorphic to the canonical solution $\mathfrak{a}$ of $a_1, \ldots, a_n, A$.*
3. *For any independent sequence $a_1, \ldots, a_n$ over A (in $\mathfrak{C}^{\mathrm{eq}}$) we have that for every solution $\mathfrak{b}$ of the canonical problem $\mathfrak{a}^-$ of $a_1, \ldots, a_n, A$ the following map is elementary*

$$\bigcup_{\substack{s \in \mathcal{P}^-(\tilde{n}) \\ |s| = n-1}} \mathfrak{b}_s^{\tilde{n}}.$$

Proof. The direction "(1) implies (2)" is clear. For the direction "(2) implies (1)" is enough to show by Lemma 4.1.11 that every solution to an n-amalgamation problem is isomorphic to a canonical solution. But this has been done in Lemma 4.1.13.

To prove that (3) is implied by (2) note that if $\mathfrak{a}$ and $\mathfrak{b}$ are isomorphic, then there exists an elementary map $\sigma : \mathfrak{a}(\tilde{n}) \to \mathfrak{b}(\tilde{n})$, such that $\sigma \circ \mathfrak{a}_s^{\tilde{n}} = \mathfrak{b}_s^{\tilde{n}}$. Hence σ is an extension of

$$\bigcup_{|s|=n-1, s \in \mathcal{P}^-(\tilde{n})} \mathfrak{b}_s^{\tilde{n}}.$$

On the other hand if point (3) holds, then we can set $\sigma : \mathfrak{a}(\tilde{n}) \to \mathfrak{b}(\tilde{n})$ defined as an extension of $\bigcup_{|s|=n-1, s \in \mathcal{P}^-(\tilde{n})} \mathfrak{b}_s^{\tilde{n}}$ to the algebraic closure of its image and preimage. Clearly this then gives an isomorphism between $\mathfrak{a}$ and $\mathfrak{b}$ and therefore we have (2). $\square$

4.2 Relation between existence and uniqueness

Definition 4.2.1. For any $n \geq 2$ and any $k \geq 0$ we say that a theory T has $(n-1, n+k)$-*existence* over A, if any closed independent functor

$$\mathfrak{a} : \{u \subset \{1, \ldots, n+k\} \mid |u| < n\} \to \mathcal{C}^{\mathrm{eq}}$$

with $\mathrm{acl}(A) = \mathfrak{a}(\emptyset)$ has an extension to a closed independent functor

$$\mathfrak{a} : \mathcal{P}(\{1, \ldots, n+k\}) \to \mathcal{C}.$$

Note $(n-1, n)$-existence is by definition the same as n-existence.

Lemma 4.2.2 (4.1 of [Hru12])**.** *Let T be a theory with $(n-1, n)$-existence over every set. Then T has $(n-1, n+k)$-existence over every set for all $k \geq 0$.*

Proof. We will prove it by induction, the base case ($k = 0$) being given. So fix a partial problem $\mathfrak{a} : S \to \mathcal{C}$ with $S := \{u \subset \{1, \ldots, n+k\} \mid |u| < n\}$. Define $U = \{u \subset \{1, \ldots, n+k\} \mid |u| \leq n, (n+k) \in u\}$. Since we have n-existence, for all $u \in U$ there exists some closed independent ${}_u\mathfrak{a} : T_u \to \mathcal{C}$ extending $\mathfrak{a} \restriction T_u^-$ with $T_u = \{v \subset u\}$ and $T_u^- = \{v \subsetneq u\}$.

We claim that

$$\mathfrak{c} := \mathfrak{a} \cup \bigcup_{\substack{u \in U, \\ |u|=n}} {}^{u}_{\mathfrak{a}} : S \cup U \to \mathcal{C}$$

is a closed independent functor. To see this, note that there are no transition maps between any two distinct $u, u' \in U$ with $|u| = |u'| = n$ (as obviously we cannot have $n \subseteq n'$ for any such two sets). Hence $\mathfrak{c}$ is functor. For the same reason $\mathfrak{c}$ is closed and independent as all ${}_u\mathfrak{a}$ are.

We define a system $(\psi_s)_{s \in S \cup U}$ of elementary bijections. We set

$$\psi_s = \mathfrak{c}_s^{s \cup \{n+k\}}$$

for $|s| < n$ and $\psi_s = \mathrm{id} \restriction \mathfrak{c}(s)$ otherwise (i.e. $s \in U - S$). Then with this system of elementary bijections we go over from $\mathfrak{c}$ to a naturally isomorphic functor $\mathfrak{c}'$ as described in Lemma 2.8.1. Note that then $\mathfrak{c}_s'^{s\cup\{n+k\}} = id \restriction_{\mathfrak{c}'(s)}$ for $|s| < n$. Hence we have

$$\mathfrak{c}_s'^t = \mathrm{id} \restriction_{\mathfrak{c}'(t)} \circ \mathfrak{c}_s'^t \circ \mathrm{id} \restriction_{\mathfrak{c}'(s)} = \mathfrak{c}_{s\cup\{n+k\}}^{t\cup\{n+k\}} \restriction_{\mathfrak{c}'(s)} \text{ for all } s \subset t \in S \cup U.$$

Since U is isomorphic to the set to

$$V := \{s \subset \{1, \ldots, n + (k-1)\} \mid |s| < n\},$$

in order to apply $(n-1, n+k-1)$-existence (over $\mathfrak{c}(\{n+1\})$) on $\mathfrak{c}' \restriction U$ we need make sure that this functor is closed and independent (as a functor from V). But it is closed as $\mathfrak{c}'$ is closed. For independence use the independence of $\mathfrak{c}'$ together with Lemma 4.1.4 to see that the following holds;

$$\mathfrak{c}'^v(u) \underset{\mathfrak{c}'^v(u\cap t)}{\perp} \mathfrak{c}'^v(t) \text{ for all } u, t \subset v \in U \text{ with } u \cap t \supset \{n+k\},$$

which then by Lemma 4.1.4 again gives us independence of $\mathfrak{c} \restriction U$ (over $\mathfrak{c}(\{n+1\})$).

Let now

$$Q = \{u \subset \{1, \ldots, n+k\} \mid (n+k) \in u\}.$$

We get by $(n-1, n+k-1)$-existence a closed functor from $\mathfrak{b} : Q \to \mathcal{C}$ extending $\mathfrak{c}' \restriction U$, such that

$$(\mathfrak{b}^v(\{i, n+k\}) : i \in v)$$

is an independent sequence over $\mathfrak{b}^v(\{n+k\})$. We define now $\mathfrak{d} : Q \cup S \to \mathcal{C}$ by $\mathfrak{b} \cup \mathfrak{c}'$ together with

$$\mathfrak{d}_s^t := \mathfrak{b}_{s\cup\{n+k\}}^{t\cup\{n+k\}} \restriction_{\mathfrak{c}'(s)} = \mathfrak{c}_{s\cup\{n+k\}}^{t\cup\{n+k\}} \restriction_{\mathfrak{c}'(s)}$$

for $s \in S \cup U$, $t \in Q \cup S$.

It is left to define maps for $s \in S$ with $(n+1) \notin s$ and $t \in Q - U$, since we have for $s \in U$ that $\mathfrak{d}_s^t := \mathfrak{b}_{s\cup\{n+k\}}^{t\cup\{n+k\}} \restriction_{\mathfrak{c}'(s)} = \mathfrak{b}_s^t$ and for $t \in S \cup U$ that

$$\mathfrak{d}_s^t := \mathfrak{b}'^{t\cup n+k}_{s\cup\{n+k\}} \restriction_{\mathfrak{c}'(s)} = \mathfrak{c}'^{t\cup\{n+k\}}_t \circ \mathfrak{c}'^t_{s\cup\{n+k\}} \circ \mathfrak{c}'^{s\cup\{n+k\}}_s = \mathfrak{c}'^{t\cup\{n+k\}}_t \circ \mathfrak{c}'_s t = \mathfrak{c}'^t_s.$$

Note now that all transition maps of $\mathfrak{d}$, can be written as

$$\mathfrak{d}_s^t = \mathfrak{b}_{s\cup\{n+k\}}^{t\cup\{n+k\}} \restriction_{\mathfrak{d}(s)} \text{ for all } s, t \in S \cup Q.$$

Moreover, $\mathfrak{d}$ is obviously closed. Hence for $s, u, t \in S \cup Q$ with $s \subset u \subset t$

$$\mathfrak{d}_u^t \circ \mathfrak{d}_s^u = \mathfrak{b}_{u\cup\{n+k\}}^{t\cup\{n+k\}} \restriction_{\mathfrak{d}'(u)} \circ \mathfrak{b}_{s\cup\{n+k\}}^{u\cup\{n+k\}} \restriction_{\mathfrak{d}'(s)} = \mathfrak{b}_{s\cup\{n+k\}}^{t\cup\{n+k\}} \restriction_{\mathfrak{d}'(s)} = \mathfrak{d}_s^t.$$

Now we go over from $\mathfrak{d}$ to some natural isomorphic functor $\mathfrak{d}'$ as in Lemma 2.8.1 with $\psi_s = \mathfrak{d}_{s,\{1,\ldots,n+k\}}$. Note that

$$\mathfrak{d}'^t_s = \mathfrak{d}_t^{\{1,\ldots,n+k\}} \circ \mathfrak{d}_s^t \circ (\mathfrak{d}_t^{\{1,\ldots,n+k\}})^{-1} = \mathrm{id} \text{ for all } s, t \in Q \cup S.$$

Then extend $\mathfrak{d}'$ to

$$\mathfrak{e} : \mathcal{P}(\{1, \ldots, n + k\}) \to \mathcal{C},$$

by $\mathfrak{e}(v) = \mathrm{acl}(\bigcup_{i\in v} \mathfrak{d}'(\{i\}))$ and $\mathfrak{e}_s^t = \mathrm{id}$ for all $s, t, v \in \mathcal{P}(\{1, \ldots, n + k\})$. By closedness of $\mathfrak{d}'$, $\mathfrak{e}$ is obviously extending $\mathfrak{d}'$. So functoriality of $\mathfrak{e}$ is clear since we just need to check that $\mathfrak{e}(v) \subset \mathfrak{e}(v') \subset \mathfrak{e}(v'')$ for all

$$v \subset v' \subset v'' \in \mathcal{P}(\{1, \ldots, n + k\}),$$

which is clear. Closedness is also clear by the construction of the $\mathfrak{e}(v)$'s. Since we have that

$$\mathfrak{a} \subset \mathfrak{c} \cong \mathfrak{c}' \subset \mathfrak{d} \cong \mathfrak{d}' \subset \mathfrak{e},$$

we have that $\mathfrak{e} \restriction S \cong \mathfrak{a}$. Therefore we can find (by Lemma 2.8.2) some closed $\mathfrak{a}' : \mathcal{P}(\{1, \ldots n + k\}) \to \mathcal{C}$ which extends $\mathfrak{a}$.

It is now only left to check that $\mathfrak{e}$ (and therefore $\mathfrak{a}'$) is an independent functor. Since $\mathfrak{e} \restriction Q \cong \mathfrak{d} \restriction Q = \mathfrak{b}$ and all transition maps of $\mathfrak{e}$ are trivial, we have that

$$(\mathfrak{e}(\{i, n+k\}) : i \in \{1, \dots, n+k\})$$

is an independent sequence over $\mathfrak{e}(\{n+k\})$, therefore we have that

$$(\mathfrak{e}(\{i\}) : i \in \{1, \dots, n+k\})$$

is an independent sequence over $\mathfrak{e}(\{n+k\})$. Moreover, since $\mathfrak{e} \restriction U \cup S \cong \mathfrak{c}$ we have that

$$\mathfrak{e}(\{i\}) \perp_{\mathfrak{e}(\emptyset)} \mathfrak{e}(\{n+k\}) \, \text{for all} \, i \in \{1, \dots, n+k-1\}.$$

So Lemma 2.4.6 gives us that $(\mathfrak{e}(\{i\}) : i \in \{1, \dots, n+k\})$ is an independent sequence over $\mathfrak{e}(\emptyset)$. □

Proposition 4.2.3 (4.1 of [Hru12]). *Assume the theory T has n-existence over every set and assume that T has n-uniqueness over A. Then we have that T has $(n+1)$-existence over A.*

Proof. Take an $(n+1)$-problem $\mathfrak{a}$ over A. By the Lemma 4.2.2 we can find for $\mathfrak{a} \restriction_S$ with $S := \{u \subset \{1, \dots, n+1\} \mid |u| < n\}$ a closed independent extension

$$\mathfrak{b} : \mathcal{P}(\{1, \dots, n+1\}) \to \mathcal{C}.$$

Now look at the sets $T_t = \{s \subset t\}$ for $t \in \mathcal{P}(\{1, \dots, n+1\})$ with $|t| = n$. By n-uniqueness $\mathfrak{a} \restriction T_t$ is isomorphic to $\mathfrak{b} \restriction T_t$ (since $\{s \subsetneq t\}$ is isomorphic to $\mathcal{P}^-(\tilde{n})$). We claim that there is a natural isomorphism between $\mathfrak{a}$ and the functor $\mathfrak{b} \restriction_{\mathcal{P}(\{1,\dots,n+k\})^-}$: First note that $\mathfrak{a} \restriction S = \mathfrak{b} \restriction S$. Now $(n-1)$-uniqueness there exists isomorphisms $\phi_t : \mathfrak{a}(t) \to \mathfrak{b}(t)$. Hence all these maps together are a natural isomorphism. So by Lemma 2.8.2 we have found an extension of $\mathfrak{a}$. □

We can improve the above result for theories with complete n-uniqueness over some set. For that, note that any rosy theory has automatically 2-existence over any set by the Existence property of the independence notion.

Lemma 4.2.4. *Let T be rosy. Suppose that T has m-uniqueness over A for every $2 \leq m < n$. Then T has n-existence over A.*

Proof. Let $\mathfrak{a} : \mathcal{P}^{-}(\tilde{n}) \to \mathcal{C}$ be an amalgamation problem. Now restrict this functor to $\{\{i\} \mid 1 \leq i \leq n\}\} \subset \mathcal{P}^{-}(\tilde{n})$. Since we have 2-existence over every set, we can apply $(1, n)$-existence to this restricted functor by Lemma 4.2.2. Now the restriction of this functor to $\mathcal{P}^{-}(\tilde{n})$ is isomorphic to $\mathfrak{a}$ since we have m-uniqueness over A for every $m < n$. Hence we have also found an extension of $\mathfrak{a}$ by Lemma 2.8.2. $\square$

Note that at least in simple theories (I suspect this easily extends to rosy theories and any $l < n$) we have that the other direction is true, but one has to assume n-existence/uniqueness over all sets:

Fact 4.2.5. (Corollary 3.17 of [GKK13a]) Let a simple theory T have k-uniqueness over all sets be true for all $l \leq k < n$ with $2 \leq l \leq 3$, then we have that the following is equivalent:

1. T has n-uniqueness.
2. T has $n + 1$-existence.
3. T has Property $B(n)$ (see later this chapter for the definition).

Note that in stable theories by this and Proposition 4.3.1 we have that 3-uniqueness and 4-amalgamation are equivalent.

4.3 Amalgamation in stable and simple theories

The next two results are folklore in the sense that they essentially reduce to theorems of Shelah and Kim-Pillay. They show that we can define stable and simple theories only in terms of an abstract independence notion and an amalgamation condition.

The next proposition reduces to the fact that all types over models (or algebraically closed sets in the case of elimination of imaginaries) are stationary if and only if the theory is stable.

Proposition 4.3.1. *Let T be a rosy theory. The following are equivalent;*

1. *The theory T is stable,*

2. *T has 2-uniqueness for real parameters over every model of T,*

3. *T has 2-uniqueness (over every parameter set).*

Proof. We prove "(1) implies (2)" first. In a stable theory any independence notion is the non-forking independence notion (see Fact 2.7.3). Since non-forking extension over models are unique in stable theories (see 8.5.4 of [TZ12]), we can re-prove Lemma 2.7.7 over models without the assumption of weak elimination of imaginaries. Now we get 2-uniqueness for real parameters over every model of T: Fix two solutions $\mathfrak{a}', \mathfrak{b}$ of the same 2-amalgamation problem $\mathfrak{a}$. Then we can use our re-proved Lemma 2.7.7 to note that the following map is elementary

$$\mathfrak{a}'^{\{1,2\}}_{\{1\}} \circ (\mathfrak{b}^{\{1,2\}}_{\{1\}})^{-1} \cup \mathfrak{a}'^{\{1,2\}}_{\{2\}} \circ (\mathfrak{b}^{\{1,2\}}_{\{2\}})^{-1}.$$

Now any extension of this map to the algebraic closure of its image and preimage shows us that $\mathfrak{a}, \mathfrak{b}$ are isomorphic.

We check that "(2) implies (1)". For that we show that a theory with 2-uniqueness for real parameters over models then the independence notion has weak boundedness (see Fact 2.7.3). So for a type $p = \mathrm{tp}(a/A)$ and any $B \supset A$ show that there is some κ such that there are only κ-many non-forking extension of p to B. To see that fix a model M which contains A. There are only boundedly many say κ extension to this model (and hence at most κ non-forking extension to M). Now by 2-uniqueness over every model we have that $(p_i : i \in \kappa)$ are all these types extended over MB which are non-forking over A. Now each non-forking extension of p to B is contained in one of these types.

The proof that (3) holds if and only if (1) is essentially the same as the one we see for (1) and (2). This is as T is stable if and only if T^{eq} is stable. The only thing which is new is that in stable theories types are stationary over $\mathrm{acl}^{\mathrm{eq}}$-closed sets (see 8.5.3 of [TZ12]). Hence we get that 2-uniqueness over any parameter set holds. □

The following results essentially breaks down to the Independence theorem of simple theories.

Proposition 4.3.2. *We have that T is simple if and only if 3-existence over models holds.*

Proof. This is a matter of translating 3-existence to the independence theorem of simple theories (see Fact 2.6.2) and vice versa. So assume that T is simple. And let $\mathfrak{a} : \mathcal{P}^-(3) \to \mathcal{C}$ be an amalgamation problem over some M.

We may assume that $\mathfrak{a}(\emptyset) = \mathfrak{a}_s(\emptyset)$ for any $s \in \mathcal{P}^-(3)$

$$\mathfrak{a}(\{1\}) = \mathfrak{a}_{\{1,2\}}(\{1\}) = \mathfrak{a}_{\{1,3\}}(\{1\})$$

and $\mathfrak{a}(\{2\}) = \mathfrak{a}_{\{1,2\}}(\{2\}) = \mathfrak{a}_{\{2,3\}}(\{2\})$ by going over to an isomorphic problem and solving this instead (which we can do by Lemma 2.8.2): for that replace $\mathfrak{a}(\emptyset)$ by $\mathfrak{a}_{\{1,2\}}(\emptyset)$ $\mathfrak{a}(\{1\})$ by $\mathfrak{a}_{\{1,2\}}(\{1\})$ and $\mathfrak{a}(\{2\})$ by $\mathfrak{a}_{\{1,2\}}(\{2\})$. Set the transition maps of $\mathfrak{a}_{\emptyset}^{\{1,2\}}$, $\mathfrak{a}_{\{1\}}^{\{1,2\}}$ and $\mathfrak{a}_{\{2\}}^{\{1,2\}}$ as the identity. Then extend the transition maps $\mathfrak{a}_s^t$ (which are elementary maps) to some automorphisms f_s^t for $s \subset t \in \mathcal{P}^-(3)$. Set

$$\mathfrak{a}(\{1,3\}) := f_{\{1\}}^{\{1,2\}} \circ (f_{\{1\}}^{\{1,3\}})^{-1}(\mathfrak{a}(\{1,3\})),$$

$$\mathfrak{a}(\{2,3\}) := f_{\{2\}}^{\{1,2\}} \circ (f_{\{2\}}^{\{2,3\}})^{-1}(\mathfrak{a}(\{2,3\})).$$

Then set $\mathfrak{a}_{\{3\}}^{\{1,3\}}$ as $f_{\{1\}}^{\{1,2\}} \restriction (\mathfrak{a}(\{3\}))$ and $\mathfrak{a}_{\{3\}}^{\{2,3\}}$ as $f_{\{2\}}^{\{2,3\}} \restriction (\mathfrak{a}(\{3\}))$.

We can work in T_M and apply the independence theorem to $c = \mathfrak{a}(\{1\})$, $d = \mathfrak{a}(\{2\})$, $a = \mathfrak{a}_{\{1\}}^{\{1,3\}}(\mathfrak{a}(\{3\}))$ and $b = \mathfrak{a}_{\{2\}}^{\{2,3\}}(\mathfrak{a}(\{3\}))$. Therefore there exists e with $e \equiv_c a$, $e \equiv_d b$ and $e \mathop{\smile\!\!\!\!\!\!\mid}\, cd$. Hence we have some automorphism g_1 which fixes c and maps a to e and automorphism g_2 which fixes d and maps b to e. We set then $\mathfrak{a}_{\{1,2,3\}}(\{3\})$ as e. We need to set the transition maps accordingly by $\mathfrak{a}_{\{1,3\}}^{\{1,2,3\}} = g_1 \restriction \mathfrak{a}(\{1,3\})$, $\mathfrak{a}_{\{2,3\}}^{\{1,2,3\}} = g_2 \restriction \mathfrak{a}(\{2,3\})$ and $\mathfrak{a}_{\{3\}}^{\{1,2,3\}} = g_1 \circ \mathfrak{a}_{\{3\}}^{\{1,3\}}$. We set the rest of the transition maps to be the identity. This is then a solution to the amalgamation problem.

For the other direction we translate the setup (same notation as in Definition 2.6.1) of the independence theorem to an amalgamation functor over M. Set $\mathfrak{a}(\{1\}) = \mathrm{acl}(c)$,

$$\mathfrak{a}(\{2\}) = \mathrm{acl}(d), \mathfrak{a}\{3\} = \mathfrak{a}_{\{1,3\}}(\{3\}) = \mathrm{acl}(a) \text{ and } \mathfrak{a}_{\{2,3\}}(\{3\}) = \mathrm{acl}(b)$$

and all transition maps except $\mathfrak{a}^{\{2,3\}}_{\{3\}}$ to be the identity. We set $\mathfrak{a}^{\{2,3\}}_{\{3\}}$ to be some elementary map which maps $\mathrm{acl}(a)$ to $\mathrm{acl}(b)$ (there is one since $a \equiv_M b$). Now a solution $\mathfrak{a}'$ to this problem gives also the independence theorem over models: To make sure that c and d stay the same, extend $\mathfrak{a}'^{\{1,2,3\}}_{\{1,2\}}$ to some automorphism and apply the inverse of it to $\mathfrak{a}'(\{1,2,3\})$. If we set e to be the image of $\mathfrak{a}'^{\{1,2,3\}}_{\{3\}}$ we are done. □

4.4 Property $B(n)$

The following two definitions we will give are somehow more model theoretically accessible (compared to higher amalgamation). We will see that they do coincide with higher uniqueness under some additional assumptions. We can also think about this new property as a condition that a certain amalgamation functor is isomorphic to some canonical solution functor (for that see the proof of Proposition 4.4.7).

Notation: *For any tuple* $(a_1, \ldots, a_n)$ *by* $(a_1, \ldots \hat{a}_i \ldots, a_n)$ *we denote the subtuple* $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$.

Definition 4.4.1. Let T be an arbitrary complete theory and let A be some subset of the monster model of T. A sequence $a_1 \ldots a_n$ is said to have *Property $B(n)$ over A*, if every $c \in \mathrm{acl}(a_1, \ldots, a_{n-1}A)$ which is in the definable closure of

$$\bigcup_{i=1}^{n-1} \mathrm{acl}(a_1 \ldots \hat{a}_i \ldots a_{n-1} a_n A),$$

is in the definable closure of $\bigcup_{i=1}^{n-1} \mathrm{acl}(a_1 \ldots \hat{a}_i \ldots a_{n-1} A)$. A rosy theory is said to have *Property $B(n)$ over A*, if in T^{eq} every independent sequence over A has Property $B(n)$ over A. A rosy theory is said to have *Property $B(n)$*, if it has *Property $B(n)$ over A* for every parameter set A.

We can restate the same definition (see Lemma 4.4.6) with automorphisms:

Definition 4.4.2. Let T be an arbitrary complete theory and let A be some subset of the monster model of T. A sequence $a_1, \ldots, a_n$ is said to have *Property $B_{Aut}(n)$ over A*, if we have that the following holds;

$$\operatorname{Aut}\big(\operatorname{acl}(a_1 \ldots a_{n-1}A)/\bigcup_{i=1}^{n-1} \operatorname{acl}(a_1 \ldots \hat{a}_i \ldots a_{n-1}a_nA)\big) = \operatorname{Aut}\big(\operatorname{acl}(a_1 \ldots a_{n-1}A)/\bigcup_{i=1}^{n-1} \operatorname{acl}(a_1 \ldots \hat{a}_i \ldots a_{n-1}A)\big).$$

A rosy theory is said to have *Property $B_{Aut}(n)$ over A*, if in T^{eq} every independent sequence over A has Property $B_{\mathrm{Aut}}(n)$ over A. A rosy theory is said to have *Property $B_{Aut}(n)$*, if it has Property $B_{\mathrm{Aut}}(n)$ over A for every parameter set A.

Again as in the definition of amalgamation and uniqueness by *Property $B_{Aut}(n)$ for real parameters over A* and *Property $B(n)$ for real parameters over A* we mean the same as the above definition but in T (instead of T^{eq}). By a similar argument as done in the proof of Proposition 4.1.1 we can show that these notions are equivalent (to the versions without the suffix) in case weak elimination of imaginaries holds.

Remark 4.4.3. $B(2)$ (over real parameters) is true in any rosy theory. In a stable theory $B_{\mathrm{Aut}}(2)$ is always true.

Proof. Just write down the definitions. $B(2)$ over A means that

$$c \in \operatorname{acl}(a_1A) \cap \operatorname{dcl}(a_2A)$$

is also in $\operatorname{acl}(A)$ for any two independent a_1, a_2 over A. But this is trivially true by Anti-Reflexivity of the independence notion.

In order to see the second claim, take some f, which is in the automorphism group $\operatorname{Aut}(\operatorname{acl}(a_1A)/\operatorname{acl}(A))$. Then note that by Lemma 2.7.7, we have that the union $f \cup \operatorname{id}\restriction_{\operatorname{acl}(a_2A)}$ is a partial elementary map □

Remark 4.4.4. In a theory where the algebraic closure coincidences with the definable closure, then Property $B_{\mathrm{Aut}}(n)$ (and $B(n)$ (see next lemma)) holds over any set as the automorphism groups are trivial in this case.

The next lemma tells us that $B(n)$ and $B_{\mathrm{Aut}}(n)$ are the same in T^{eq}. But I suspect that the following question can be answered positively.

Question 4.4.5. Does there exist a theory with weak elimination of imaginaries such that $B(n)$ (over A) holds for real parameters but $B_{\mathrm{Aut}}(n)$ (over A) fails for real parameters?

Lemma 4.4.6. *Let T be any complete theory. If a sequence $a_1, \dots, a_n$ has Property $B_{Aut}(n)$ over A (for real parameters), then it has Property $B(n)$ over A (for real parameters). If our theory has additionally elimination of finite imaginaries and we assume that $n \geq 3$, then the converse is true, i.e. a sequence $a_1, \dots, a_n$ has Property $B(n)$ over A (for real parameters), then it has Property $B_{Aut}(n)$ over A (for real parameters).*

Proof. Assume that $B_{\mathrm{Aut}}(n)$ holds for some sequence $a_1 \dots a_n$. Any

$$c \in \mathrm{acl}(a_1 \dots a_{n-1}A) \cap \mathrm{dcl}\big(\bigcup_{i=1}^{n-1} \mathrm{acl}(a_1 \dots \hat{a}_i \dots a_{n-1}a_nA)\big),$$

will be fixed by all automorphisms in

$$\mathrm{Aut}\big(\mathrm{acl}(a_1 \dots a_{n-1}A)/\bigcup_{i=1}^{n-1} \mathrm{acl}(a_1 \dots \hat{a}_i \dots a_{n-1}a_nA)\big).$$

By $B_{\mathrm{Aut}}(n)$ this means that c will also be fixed by all of

$$\mathrm{Aut}\big(\mathrm{acl}(a_1 \dots a_{n-1}A)/\bigcup_{i=1}^{n-1} \mathrm{acl}(a_1 \dots \hat{a}_i \dots a_{n-1}A)\big).$$

Hence we have that $c \in \mathrm{dcl}(\bigcup_{i=1}^{n-1} \mathrm{acl}(a_1 \dots \hat{a}_i \dots a_{n-1}A))$.

For the second point c be some tuple in $\mathrm{acl}(a_1 \dots a_{n-1}A)$. Let X_c be the orbit set of c under

$$\operatorname{Aut}\big(\operatorname{acl}(a_1 \dots a_{n-1}A)/\bigcup_{i=1}^{n-1}\operatorname{acl}(a_1 \dots \hat{a}_i \dots a_n A)\big).$$

Now as X_c has finite size, by elimination of finite imaginaries there is a code for it in $\operatorname{acl}(a_1 \dots a_{n-1}A)$. Also as X_c is fixed by the automorphism group, it is in $\operatorname{dcl}\big(\bigcup_{i=1}^{n-1}\operatorname{acl}(a_1 \dots \hat{a}_i \dots a_n A)\big)$. Now by $B(n)$ of this sequence this code is then also in $\operatorname{dcl}\big(\bigcup_{i=1}^{n-1}\operatorname{acl}(a_1 \dots \hat{a}_i \dots a_{n-1}A)\big)$. But this means that any such tuple c has the same orbit under the two automorphism groups, so the groups coincide. □

The next proposition is essentially Lemma 3.3 of [GKK13a].

Proposition 4.4.7. *Let T be some rosy theory. If T has n-uniqueness over A (over real parameters), then it has Property $B_{Aut}(n)$ over A (over real parameters).*

Proof. In order to prove this, we will translate the automorphism property of $B_{\mathrm{Aut}}(n)$ to an amalgamation functor. So take any independent sequence $a_1, \dots, a_n$ over A. Name the sequence's canonical problem functor $\mathfrak{a}$ and name its canonical solution functor $\mathfrak{a}'$. If we can extend

$$f \in \operatorname{Aut}\big(\operatorname{acl}(a_1 \dots a_{n-1}A)/\bigcup_{i=1}^{n-1}\operatorname{acl}(a_1 \dots \hat{a}_i \dots a_{n-1}A)\big)$$

to some

$$g \in \operatorname{Aut}(\operatorname{acl}(a_1 \dots a_{n-1}A)/\bigcup_{i=1}^{n-1}\operatorname{acl}(a_1 \dots \hat{a}_i \dots a_{n-1}a_n A)),$$

we are finished, since the first automorphism group is bigger than the second.

For that define another solution $\mathfrak{a}''$ of $\mathfrak{a}$. The object stays the same, i.e.

$$\mathfrak{a}''(\{1, \dots, n\}) = \mathfrak{a}'(\{1, \dots, n\}).$$

The transition maps are inclusions apart from the transition map

$$\mathfrak{a}^{\{1,\dots,n\}}_{\{1,\dots,n-1\}} = f.$$

By n-uniqueness these two solutions are isomorphic. Hence there exists some elementary map

$$g : \mathfrak{a}'(\{1, \ldots, n\}) \to \mathfrak{a}''(\{1, \ldots, n\}),$$

such that $g \circ \mathfrak{a}'_{s,\{1,\ldots,n\}} = \mathfrak{a}''_{s,\{1,\ldots,n\}}$ for any $s \subsetneq \{1, \ldots, n\}$. This then gives that $g \restriction_{\mathrm{acl}(a_1 \ldots a_{n-1})} = f$. As all the other transition maps are identities, we have that the g is the elementary map we were looking for. $\square$

4.5 Total amalgamation and Property $B(n)$

In this section we show that the Property $B(n)$ over A coincidences with n-uniqueness over A, if we assume that lower uniqueness over A is also true.

Definition 4.5.1. Suppose $2 \leq k \leq n$. We say that T has *relative* (k, n)-*uniqueness* (over A), if in T^{eq} for every independent sequence $a_1, \ldots, a_n$ over A with $\mathfrak{a}$ its canonical solution functor and elementary maps $(\sigma_u : u \in [\tilde{n}]^{k-1})$ with $\sigma_u \in \mathrm{Aut}(\mathfrak{a}(u))$ and $\sigma_u \restriction \mathfrak{a}^u(v) = \mathrm{id}$ for all $v \subset u$, we have that the union $\bigcup_{u \in [\tilde{n}]^{k-1}} \sigma_u$ is an elementary map.

We already have seen a similar property in Lemma 4.1.14, but there it was not required that $\sigma_u \restriction \mathfrak{a}^u(v) = \mathrm{id}$.

Lemma 4.5.2. *([GKK13a], Lemma 3.10) Suppose $n \geq 2$. Then we have that the theory T has property $B_{Aut}(n)$ over A if and only if the theory T has relative (n, n)-uniqueness over A.*

Proof. Assume $B_{\mathrm{Aut}}(n)$ over A fails. Hence there exists some automorphism

$$f \in \mathrm{Aut}\big(\mathrm{acl}(a_1 \ldots a_{n-1}A) / \bigcup_{i=1}^{n-1} \mathrm{acl}(a_1 \ldots \hat{a}_i \ldots a_{n-1}A)\big) - \mathrm{Aut}\big(\mathrm{acl}(a_1 \ldots a_{n-1}A) / \bigcup_{i=1}^{n-1} \mathrm{acl}(a_1 \ldots \hat{a}_i \ldots a_n A)\big).$$

This gives us that the following map is not elementary

$$f \cup \bigcup_{i=1}^{n-1} \mathrm{id}_{\mathrm{acl}(a_1 \ldots \hat{a}_i \ldots a_n A)}.$$

For the other direction set

$$A_i = \mathrm{acl}(a_1 \ldots \hat{a}_i \ldots a_n A),$$

$$B_i = \bigcup_{j \neq i} \mathrm{acl}(a_1 \ldots \hat{a}_j \ldots \hat{a}_i \ldots a_n A),$$

$$C_i = \bigcup_{j \neq i} \mathrm{acl}(a_1 \ldots \hat{a}_j \ldots a_n A).$$

Since $B_{\mathrm{Aut}}(n)$ holds, we can extend every $\sigma_i \in \mathrm{Aut}(A_i/B_i)$ to some

$$\sigma_i' \in \mathrm{Aut}(A_i/C_i).$$

Extend σ_i' to some $\sigma_i'' \in \mathrm{Aut}(\mathfrak{C}/C_i)$. Now set $\sigma = \sigma_n'' \circ \ldots \circ \sigma_1''$. Now as every σ_j'' with $j \neq i$ fixes A_i we have $\sigma \restriction A_i = \sigma_i$. Hence $\sigma_1 \cup \ldots \cup \sigma_n$ is elementary as it is the same as σ restricted to $A_1 \cup \ldots \cup A_n$. $\square$

Lemma 4.5.3. *([GKK13a], Lemma 4.4) Suppose $n \geq k \geq 2$ and T has property $B_{Aut}(k)$ over A. Then T has relative (k, n)-uniqueness over A.*

Proof. Use induction for the proof. We already know by Lemma 4.5.2 that (k, k)-uniqueness does hold. Hence the induction basis is given. We assume now that $(k, n-1)$-uniqueness holds. Let $a_1, \ldots, a_n$ be an independent sequence (over A), let $\mathfrak{a}$ be its canonical solution functor and let $(\sigma_u : u \in [\tilde{n}]^{k-1})$ be such that $\sigma_u \in \mathrm{Aut}(\mathfrak{a}(u))$ and $\sigma_u \restriction \mathfrak{a}^u(v) = \mathrm{id}$ for all $v \subset u$.

We need to show that $\bigcup_{u \in [\tilde{n}]^{k-1}} \sigma_u$ is elementary. We set

$$V_i = \{v \subset_k \tilde{n} \mid \{i, i+1\} \subset v\}.$$

Now for any $v \in V_i$ and any $u \subset_{k-1} v$, if i is the smallest number such that $\{i, i+1\}$ is not contained in u we set f_u to be σ_u, otherwise $\mathrm{id} \restriction \mathfrak{a}(u)$. Then by $B_{\mathrm{Aut}}(k)$ over A we know that $\bigcup_{u \subset v} f_u$ is elementary.

We extend this map to the algebraic closure of its domain (and image), i.e. $\mathfrak{a}(v)$ and call it f_v. We claim that we can apply $(k, n-1)$-uniqueness to the sequence $\mathfrak{a}(1) \dots \mathfrak{a}(\{i, i+1\}) \dots \mathfrak{a}(n)$ together with maps $f_v : v \in V_i$. For that check that for $u \subsetneq v$, with either $\{i, i+1\} \subset u$ and $|u| = k-1$ or $\{i, i+1\} \cap u = \emptyset$ and $|u| = k-2$, we have that $f_v \restriction \mathfrak{a}(u)$ is the identity. But this is clear by the construction of f_v. Hence we have that $f_{V_i} := \bigcup_{v \in V_i} f_v$ is an elementary map.

Now we claim that $f_{V_n} \circ \dots \circ f_{V_1}$ is some extension of σ. To see that note that each σ_u only does appear in exactly one f_{V_i}: for every $u \subsetneq \tilde{n}$ there is exactly one smallest i such that $\{i, i+1\}$ not contained in u, hence it is in some f_{V_i}. And then note that for $u \subset_{k-1} \widetilde{n+1}$, if i is not the smallest such that $\{i, i+1\}$ not contained in u or if $\{i, i+1)\}$ is contained in u, we have $f_{V_i} \restriction \mathfrak{a}(u) = \mathrm{id}$. Hence we have that $f_{V_n} \circ \dots \circ f_{V_1} \restriction \mathfrak{a}(u) = \sigma_u$. □

Corollary 4.5.4. *Suppose $n \geq 2$ and suppose that T has property $B_{Aut}(\ell)$ over $\emptyset$ for all ℓ with $2 \leq \ell \leq n$. Then T has relative (k, n)-uniqueness over $\emptyset$ for all k with $2 \leq k \leq n$.*

Proof. Use the last lemma. □

Corollary 4.5.5. *Let T has Property $B_{Aut}(\ell)$ over $\emptyset$ for all ℓ with $2 \leq \ell \leq n$. Suppose $1 \leq r \leq n$ and $a_1, \dots, a_n$ is a sequence independent over $\emptyset$. Further let $\mathfrak{a}$ be its canonical solution functor. Let $\{\sigma_u \mid u \in [n]^r\}$ be such that $\sigma_u \in Aut(\mathfrak{a}(u))$ and $\sigma_u(x) = \sigma_v(x)$ whenever $x \in \mathfrak{a}(u) \cap \mathfrak{a}(v)$. Then*

$$\bigcup\{\sigma_u \mid u \in [\tilde{n}]^r\}$$

is an elementary map.

Proof. We prove this by induction on r, the case $r = 1$ being straightforward. Consider the compatible system of elementary maps $\{\tau_v \mid v \in [n]^{r-1}\}$ given by $\tau_v = \sigma_u \restriction \mathrm{acl}(a_i : i \in v)$ whenever $v \subset u$. By the induction hypothesis, the union of these is an elementary map. So it extends to an automorphism τ. By the previous result, relative $(r+1, n)$-uniqueness holds. We can apply this to $\{\tau^{-1}\sigma_u \mid u \in [n]^r\}$ to obtain an elementary map ρ extending all $\tau^{-1}\sigma_u$. Then $\tau\rho$ is an elementary map extending all σ_u, as required. □

The following result is similar to 3.5 of [Hru12], but works for a fixed set. The idea for this is due to David Evans.

Corollary 4.5.6. *A rosy theory T with 2-uniqueness over A, which has Property $B(k)$ over A for every k with $3 \leq k \leq n$, has n-uniqueness over A.*

Proof. Fix two solutions $\mathfrak{a}', \mathfrak{b}$ of the same n-amalgamation problem $\mathfrak{a}$. Without loss we may assume that $\mathfrak{a}'$ is a canonical functor (see Lemma 4.1.14). Note that by applying 2-uniqueness over A $n-1$-many times, we know that the following union of elementary maps is elementary

$$\bigcup_{i=1}^{n} \mathfrak{a}'^{\{1,\ldots,n\}}_{\{1\}} \circ (\mathfrak{b}^{\{1,\ldots n\}}_{\{1\}})^{-1}.$$

Hence by extending this map to some automorphism and moving $\mathfrak{b}$ we may assume that $\mathfrak{b}(\{1,\ldots,n\}) = \mathfrak{a}'(\{1,\ldots,n\})$ and $\mathfrak{b}^{\{1,\ldots,n\}}_{\{i\}} = \mathrm{id}$. Now we can apply Corollary 4.5.5 to note that the following is elementary

$$\bigcup \{\mathfrak{b}^{\{1,\ldots,n\}}_{u} \mid u \in [\tilde{n}]^2\}.$$

Hence by moving $\mathfrak{b}$ with this map we may assume that $\mathfrak{b}^{\{1,\ldots,n\}}_{\{i,j\}} = \mathrm{id}$. Then repeating this argument for 3 and so on, we finally get that $\mathfrak{a}'$ and $\mathfrak{b}$ are the same. This then of course shows that these maps were already naturally isomorphic in the beginning. □

Corollary 4.5.7. *A rosy theory T with complete i-uniqueness over A with $i \geq 2$, which has Property $B(k)$ over A for every k with $n \geq k > i$, has complete $n+1$-amalgamation and complete n-uniqueness over A.*

Proof. Note that complete i-uniqueness over A implies $B(l)$ over for every l with $l \leq i$ (this is by Proposition 4.4.7 and Lemma 4.4.6). Now complete n-uniqueness over A is obvious by the last corollary. Then complete $n+1$-amalgamation follows from Lemma 4.2.4. □

Corollary 4.5.8. *A rosy theory T with weak elimination of imaginaries and i-uniqueness over A for real parameters with $i \geq 2$, which has Property*

$B_{Aut}(k)$ over A for real parameters for every k with $n \geq k > i$, has complete $n+1$-amalgamation over A and complete n-uniqueness over A.

Proof. As already noted if a theory has weak elimination of imaginaries we get that 2-uniqueness over A for real parameters implies 2-uniqueness over A. Also we have that complete i-uniqueness over A for real parameters implies $B_{\mathrm{Aut}}(l)$ over for every l for real parameters with $l \leq i$ (this is by Proposition 4.4.7). Then by weak elimination of imaginaries we have that Property $B_{\mathrm{Aut}}(n)$ over A for real parameters implies Property $B_{\mathrm{Aut}}(n)$ over A. As $B_{\mathrm{Aut}}(n)$ over A implies $B(n)$ over A (see Lemma 4.4.6), we can then use Corollary 4.5.7 to finish the proof. $\square$

Corollary 4.5.9. *A stable theory T with elimination of imaginaries which has Property $B(k)$ over A for every k with $n \geq k > 2$ (or equivalently complete i-uniqueness for some $i > 2$ and Property $B(k)$ for every $n \geq k > i$), has complete $n+1$-amalgamation over A and complete n-uniqueness over A. A stable theory T with weak elimination of imaginaries and Property $B_{Aut}(n)$ over A for real parameters for every k with $n \geq k > 2$ (or equivalently complete i-uniqueness over real parameters for some $i > 2$ and Property $B_{Aut}(k)$ for real parameters for every k with $n \geq k > i$) has also complete $n+1$-amalgamation over A and complete n-uniqueness over A.*

Proof. For both parts note that a stable theory has 2-uniqueness over every set and in the second case this is true for the suffix "over real parameters". Hence we can use Corollary 4.5.7 for the first part and Corollary 4.5.8 for the second part. $\square$

4.6 Amalgamation of Morley sequences

We will show that complete n-uniqueness of Morley sequence implies complete n-uniqueness.

Fact 4.6.1. (3.11 in [GKK15]) Let $T = T^{\mathrm{eq}}$ be stable and with k-uniqueness for every $k < n$. Suppose n-uniqueness fails (over A). Then there exists some

Morley sequence $a_1, \ldots, a_n$ over some A such that Property $B(n)$ (over A) for this sequence fails.

In fact the following is part of the proof of the above fact. We will reprove it for the convenience of the reader.

Lemma 4.6.2. *We fix some stable theory T with elimination of imaginaries. Let $b_1, \ldots, b_{n+1}$ be an independent sequence over A which fails Property $B(n+1)$ over A and let k-uniqueness with $k \leq n$ be true over $b_1, \ldots, b_{n-1}, A$ and over $b_n A$. Then there exists some Morley sequence (of $(n+1)$-tuples) $a_1, \ldots, a_{n+1}$ over A with*

$$a_i = (a_{1,i}, \ldots, a_{n+1,i}) \equiv_A (b_1, \ldots, b_{n+1})$$

where $a_{i,i} = b_i$ such that $a_1, \ldots, a_{n+1}$ fails $B(n+1)$ over A.

Proof. We pick in $\mathrm{tp}(b_1, \ldots, b_{n+1}/\mathrm{acl}(A))$ the a_i for i with $1 \leq i \leq n+1$ successively such that

$$a_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}_{Ab_i} a_1, \ldots, a_{i-1}, b_{i+1}, \ldots, b_{n+1}.$$

Now assume by induction that $a_1, \ldots a_{i-1}, b_i, \ldots, b_{n+1}$ is independent over A Hence we have

$$b_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}_A a_1, \ldots, a_{i-1}, b_{i+1}, \ldots, b_{n+1}$$

and we have that the following is true by construction

$$a_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}_{Ab_i} a_1, \ldots, a_{i-1}, b_{i+1}, \ldots, b_{n+1}.$$

This together gives us by Transitivity that $a_1, \ldots, a_i, b_{i+1}, \ldots, b_{n+1}$ is independent over A. Hence $a_1, \ldots, a_{n+1}$ is an independent sequence over A where all elements satisfy the same stationary type. This gives use that the sequence $(a_i : 1 \leq i \leq n+1)$ is a Morley sequence.

We now need to check that this sequence fails Property $B(n+1)$ over A.

For that fix some

$$f \in \operatorname{Aut}\big(\operatorname{acl}(b_1 \dots b_n A) / \bigcup_{i=1}^{n} \operatorname{acl}(b_1 \dots \hat{b}_i \dots b_n A)\big)$$

which does not extend to some automorphism in

$$\operatorname{Aut}\big(\operatorname{acl}(b_1 \dots b_n A) / \bigcup_{i=1}^{n} \operatorname{acl}(b_1 \dots \hat{b}_i \dots b_{n+1} A)\big).$$

Note also that $a_1, \dots, a_{n-1}, b_n$ is an independent sequence over $b_1, \dots, b_{n-1}, A$. We set $B = \operatorname{acl}(b_1 \dots b_{n-1} A)$. We build inductively (by the size of s) elementary f_s for $s \subset \{1, \dots, n\}$ with $f_s \subset f_t$ for $s \subset t$ and the following additional properties,

1. if $s \subset \{1, \dots, n-1\}$ then $f_s = \mathrm{id}_{\operatorname{acl}(a_i B : i \in s)}$,
2. if $s = s_0 \cup \{n\}$ with $s_0 \subset \{1, \dots, n-1\}$ is an elementary map which extends f with $\operatorname{acl}(a_i b_n B : i \in s_0)$ as its image and preimage.

To see that this is possible first note that by our assumption of k-uniqueness for $k \leq n$ over B, Lemma 4.5.2 and Proposition 4.4.7, we have relative (k, k)-uniqueness for $k \leq n$ over $b_1, \dots, b_{n-1}, A$. Now if all f_t with $|t| < k$ are constructed, then for $|s| = k$ construct f_s by extending $\bigcup_{t \subsetneq s} f_t$ (which is elementary by relative (k, k)-uniqueness) to the algebraic closure of its image and preimage.

Now for the partial elementary map $f_{1,\dots,n}$ we have that the following two equations holds for all $i \leq n-1$;

$$f_{1 \dots n} \restriction_{\operatorname{acl}(b_1 \dots b_n A)} = f \quad \text{and} \quad f_{1 \dots n} \restriction_{\operatorname{acl}(a_1 \dots a_i \dots a_{n-1} b_n A)} = \mathrm{id} \restriction_{\operatorname{acl}(a_1 \dots a_i \dots a_{n-1} b_n A)} .$$

Now use (n, n)-uniqueness over $b_n A$ on

$$f_{1 \dots n} \text{ and } \mathrm{id}_{\operatorname{acl}(a_1 \dots \hat{a}_i \dots a_n)} : 1 \leq i \leq n-1$$

to obtain the following elementary map;

$$f^{+} \in \mathrm{Aut}\big(\mathrm{acl}(a_1 \dots a_n A) / \bigcup_{i=1}^{n} \mathrm{acl}(a_1 \dots \hat{a}_i \dots a_n A)\big).$$

Now if we could extend f^{+} to some element of

$$\mathrm{Aut}\big(\mathrm{acl}(a_1 \dots a_n A) / \bigcup_{i=1}^{n} \mathrm{acl}(a_1 \dots \hat{a}_i \dots a_{n+1} A)\big),$$

then we would also be able to extended f to some element of

$$\mathrm{Aut}\big(\mathrm{acl}(b_1 \dots b_n A) / \bigcup_{i=1}^{n} \mathrm{acl}(b_1 \dots \hat{b}_i \dots b_{n+1} A)\big).$$

This of course by the choice of f is impossible. □

Corollary 4.6.3. *A stable theory which has Property $B(n)$ for Morley sequences over all sets, has complete $n+1$-amalgamation over all sets and complete n-uniqueness over all sets. Moreover, a stable theory which has Property $B(n)$ for Morley sequences over all finite sets, has complete $n+1$-amalgamation over all finite sets and complete n-uniqueness over all finite sets.*

Proof. Assume that n-uniqueness over some (finite) set A fails. Further assume that k-uniqueness holds for all $k \leq n-1$ over all (finite) sets. By Corollary 4.5.9 we know that then $B(n)$ fails over A. Now use Lemma 4.6.2 to note that then $B(n)$ fails for some Morley sequence over A, contradicting our assumptions. □

4.7 Amalgamation over models

The following result is Proposition 1.6 of [PKM06].[1]

Fact 4.7.1. We have total n-amalgamation and n-uniqueness over models in any stable theory.

[1] If our amalgamation diagram contains only models (of T) then this result can be already found in [She90](see Ch XII section 2 p 598).

The author cannot resist to make the following silly remark. This result gives us a (true) class of stable theories with total amalgamation over all sets. To see that this has actually class-size, note that $\{T_A \mid A \subset_{\text{set-size}} \mathfrak{C}\}$, where $\mathfrak{C}$ is the monster-model of $T = \text{ACF}_0$, has class size and is a subclass of all stable theories which have total amalgamation over any set.

Differentially closed fields contain an algebraically closed set which is not a model of its theory. But I suspect that in DCF higher amalgamation is still true. The theories of compact complex manifolds with or without a fixed automorphism (CCM and CCMA) also contain an algebraically closed set which is not a model of their theory. However in CCM the algebraic closure of the empty set is a model of the theory (for this see Section 5.7).

To see why the Fact 4.7.1 is true and why its proof cannot be used outside of the stable context note that the following is proved in [PKM06] (in order to prove Fact 4.7.1). Remember that a global type q which is the extension of some type $p \in S(M)$ is a *coheir*, if q is finitely satisfiable in M. Moreover, in stable theories the notion of a coheir coincides with that of non-forking extension. See Section 8.1 of [TZ12] for more information.

Lemma 4.7.2. *Let T be some arbitrary complete theory and let M be some model of T. Further let $a_1, \ldots, a_n$ be a sequence of tuples. Now if we have that the type $\text{tp}(a_n/\text{acl}(Ma_1 \ldots a_{n-1}))$ is a coheir of $\text{tp}(a_n/M)$, then $a_1, \ldots, a_n$ has Property $B(n)$ over M.*

Proof. This proof is essentially a copy-paste of the proof of Lemma 1.5(2) of [PKM06]. We will work in T_M. Take some

$$c \in \text{acl}(a_1 \ldots a_{n-1}) \cap \text{dcl}(\bigcup_{i=1}^{n-1} \text{acl}(a_1 \ldots \hat{a}_i \ldots a_{n-1}a_n)).$$

We can fix $f_1, \ldots, f_{n-1}$ such that $(\text{tp}(f_i/a_1 \ldots \hat{a}_i \ldots a_n))_{1 \leq i < n}$ are algebraic types, $c \in \text{dcl}(f_1 \ldots f_{n-1})$. Then we fix some formula $\phi_i(y, a_1, \ldots \hat{a}_i \ldots, a_n)$ such that it isolates $\text{tp}(f_i/a_1, \ldots \hat{a}_i \ldots, a_n)$. Further we fix some formula $\psi(x, f_1, \ldots, f_{n-1})$ such that it isolates $\text{tp}(c/f_1 \ldots f_{n-1})$ (and has therefore at most one realisation).

It is easy to see that the formula

$$\exists y_1 \dots y_{n-1}(\psi(c, y_1 \dots y_{n-1}) \wedge \bigwedge_{1 \leq i < n} \phi_i(a_1, a_2 \dots \hat{a}_i \dots a_{n-1}, x, y_i))$$

is part of the type $\mathrm{tp}(a_n/ca_1 \dots a_{n-1})$. Now as $\mathrm{tp}(a_n/\mathrm{acl}(a_1 \dots a_{n-1}))$ is a coheir of $\mathrm{tp}(a_n/\emptyset)$ and $c \in \mathrm{acl}(a_1 \dots a_n)$ we have that there exists an $m \in M$ (as we worked in T_M) such that

$$\models \exists y_1 \dots y_{n-1}(\psi(c, y_1 \dots y_{n-1}) \wedge \bigwedge_{1 \leq i < n} \phi_i(a_1, a_2 \dots \hat{a}_i \dots a_{n-1}, m, y_i)).$$

Hence we have that $c \in \mathrm{dcl}(\bigcup_{i<n} \mathrm{acl}(a_1 \dots \hat{a}_i \dots a_{n-1} M))$ as required. $\square$

Now from this result the Fact 4.7.1 follows as a corollary. To see that, note first that in a stable theory every non-forking extension of a type over a model is in fact a coheir extension. Therefore Property $B(n)$ over models holds. Note also that in a stable theory any type over a model is stationary and therefore 2-uniqueness over models holds (see Proposition 4.3.1). Hence we are finished as we apply Corollary 4.5.9 now, which tells us then that total amalgamation and total uniqueness over any model is true.

Chapter 5

Amalgamation over Parameters

We will see in this chapter that amalgamation can fail in finite covers, while in the original structure it may hold. Moreover, we construct finite covers such that amalgamation and uniqueness over the empty set hold, but 3-uniqueness over some parameter set fails. But first we investigate a case where amalgamation with and without parameters is connected.

5.1 Separable independence notion

We are going to establish results which will show that under some conditions amalgamation problems over parameters can be translated to amalgamation problems over $\emptyset$. This then shows that in this case total uniqueness over the empty set implies total uniqueness over all sets.

Definition 5.1.1. We say that a theory T has a *separable independence notion* if it is rosy and if in T^{eq} for all sets $A \subset B$ there exists some C such that $A \mathop{\smile\!\!\!\!\!\mid}_{\emptyset} C$ and $\mathrm{acl}(AC) = \mathrm{acl}(B)$. We say that a theory has *separable forking* if it is simple and non-forking is a separable independence notion.

We will analyse which theories have a separable independence notion. There are also examples in this thesis which do not have a separable independence notion. These are the example in Section 5.6 and the two examples in Section 5.7. In order to see that this is true, note that these examples do not satisfy Theorem 5.1.10.

Remark 5.1.2. We have that theories with a separable independence notion and 2-uniqueness over $\emptyset$ are stable.

Proof. We work in T^{eq}. By Fact 2.7.3 it will be enough to show that the independence notion has weak boundedness. So take some sequence $(a_i : i \in I)$, some set A and some set $B \supset A$ such that we have $a_i \equiv_A a_j$ for all $i, j \in I$ and $a_i \mathop{\smile\!\!\!\!\!\!|}\,_A B$ for all $i \in I$. We will show that the set $\{\mathrm{tp}(a_i/B) \mid i \in I\}$ has at most size $2^{|T|}$ to finish the proof. Note that we may assume that $a_i \equiv_{\mathrm{acl}(\emptyset)} a_j$ for all $i, j \in I$ as there are no more then $2^{|T|}$ different types in $S(\mathrm{acl}(\emptyset))$. Hence we may work in $T^{\mathrm{eq}}_{\mathrm{acl}^{\mathrm{eq}}(\emptyset)}$.

Since our theory has a separable independence notion we know that there exists $a'_i \mathop{\smile\!\!\!\!\!\!|}\, A$ s.t. $\mathrm{acl}(a'_i A) = \mathrm{acl}(a_i A)$. Moreover we may assume that $a'_i \equiv a'_j$ for all $i, j \in I$ (as the a_i have that property). We also fix some formula $\phi(x, y, \overline{z})$ such that for all $a, \overline{b}$ we have $\phi(x, a, \overline{b})$ has less then n-many solutions and some $\overline{a} \in A$ such that $\phi(x, a'_i, \overline{a})$ has a_i in its solution set. We claim that we can use $2^{|T|}$ as bound.

Now we have $a_i \mathop{\smile\!\!\!\!\!\!|}\,_A B$ and therefore $\mathrm{acl}(a_i A) \mathop{\smile\!\!\!\!\!\!|}\,_A B$. Hence $a'_i \mathop{\smile\!\!\!\!\!\!|}\,_A B$ and therefore by transitivity $a'_i \mathop{\smile\!\!\!\!\!\!|}\, B$. Now by 2-uniqueness we know that $a'_i \equiv_B a'_j$. As each a_i realises $\phi(x, a'_i, \overline{a})$ with $\overline{a} \in A$ (which solution set is smaller then n), we know that there no more then n-many types $p_1, \ldots, p_n \in S(B)$ such that for any i we can find some j with $1 \leq j \leq n$ such that we have $\mathrm{tp}(a_i/B) = p_j$. $\square$

Lemma 5.1.3. *If T has a separable independence notion, then any algebraic cover of T has a separable independence notion.*

Proof. Let T_1 be some algebraic cover of T. Then T_1 will be rosy because of Lemma 3.4.3. By Corollary 3.3.2 and as separability has to be checked in T_1^{eq} we may assume that $T = T^{\mathrm{eq}}$ and $T_1 = T_1^{\mathrm{eq}}$. Now let $A \subset B$ be subsets of the monster model of T_1. Let $\{S_i : i \in I\}$ be all sorts in which B has elements. By Lemma 3.2.2 we have that for any finite set of sorts $\{S_i : i \in I_0\}$ (where $I_0 \subset I$), there is a definable finite-to-one map π_{I_0} to some sorts of T. Use the separability of T on $A_0 \subset B_0$ defined as $A_0 := \bigcup_{I_0 \subset I} \pi_{I_0}(A)$ and $B_0 := \bigcup_{I_0 \subset I} \pi_{I_0}(B)$. Now find some C such that we have $A_0 \mathop{\smile\!\!\!\!\!\!|}\,_{\emptyset} C$ and

$\operatorname{acl}(A_0C) = \operatorname{acl}(B_0)$ in T. We claim that $A \mathop{\smile\!\!\!\!\!\!\mid}_{\emptyset} C$ and $\operatorname{acl}(AC) = \operatorname{acl}(B)$ in T_1. The first point follows, as by Lemma 3.4.3 we have $A_0 \mathop{\smile\!\!\!\!\!\!\mid}_{\emptyset} C$ in T_1 and then as $A \subset \operatorname{acl}(A_0)$ in T_1 we can use Lemma 2.4.4 to show this. The second point is also true, since we have additionally that $B \subset \operatorname{acl}(B_0) = \operatorname{acl}(A_0C)$ in T_1 and therefore we have that $\operatorname{acl}(A_0C) = \operatorname{acl}(AC) = \operatorname{acl}(B)$ holds in the theory T_1. □

Definition 5.1.4. A theory T is said to be *pregeometric* if the algebraic closure operator is a pregeometry on $\mathfrak{C}$ (see Appendix C.1 of [TZ12]).

Lemma 5.1.5. *We let T be a single-sorted pregeometric theory with geometric elimination of imaginaries. Then T has a separable independence notion.*

Note that o-minimal and strongly minimal theories are pregeometric. See 2.2.4 of [Mac00] for the former and 5.7.5 of [TZ12] for the latter.

Proof. First note that the notion of independence of a pregeometry defines some independence relation in the sense of Definition 2.4.1. Then note that it is enough to check, by geometric elimination of imaginaries, that the independence notion is separable in the real elements. Let $\{a_i \mid i < \alpha\}$ be a basis (in the sense of the pregeometry) of $\operatorname{acl}(B)$ such that for some $\lambda < \alpha$, $(a_i : i < \lambda)$ forms a basis of $\operatorname{acl}(A)$. Now $C = \{a_i \mid \lambda \leq i < \alpha\}$ is independent of A and $\operatorname{acl}(AC) = \operatorname{acl}(B)$. □

More about pregeometric theories which do have geometric elimination of imaginaries can be found in [Gag05] and [Ele05].

Definition 5.1.6. An L-theory T is called *almost strongly minimal*, if there exists some strongly minimal (in T) formula $\phi(x) \in L(\mathfrak{C})$ such that $\mathfrak{C} = \operatorname{acl}(\phi(\mathfrak{C}))$.

Lemma 5.1.7. *Let T be an almost strongly minimal L-theory with geometric elimination of imaginaries which has an 0-definable (i.e. L-definable) strongly minimal formula ϕ with $\mathfrak{C} = \operatorname{acl}(\phi(\mathfrak{C}))$ for the monster model $\mathfrak{C}$ of T. Then we have that T has a separable independence notion.*

Proof. First note that T is uncountably categorical (see Theorem 4.7.3 of [Hod93]), therefore is stable and hence has an independence notion. By Lemma 5.1.3, Lemma 5.1.5 and Lemma 3.1.7 it will be enough to show that $\mathfrak{C}$ is an algebraic cover of $\phi(\mathfrak{C})$ (with the induced structure L-structure of $\mathfrak{C}$). To make sense of it in our setup that we replace $\mathfrak{C}$ by the two sorted structure $(\mathfrak{C}, \phi(\mathfrak{C}), \mathrm{id}_{\phi(\mathfrak{C})})$. We know that $\phi(\mathfrak{C}))$ (with the induced L-structure) will be embedded in this two sorted structure. Now since our theory is stable we get then stable embeddedness by Remark 3.1.2. Now as we have $\mathfrak{C} = \mathrm{acl}(\phi(\mathfrak{C}))$ by our assumptions. $\square$

By the last lemma we know that if we allow the naming of a finite number of elements of the monster (the elements defining the strongly minimal set) then each almost strongly minimal theory with gei has separable forking. But is this true without naming a finite number of elements or without assuming gei, i.e. have all almost strongly minimal theories separable forking? Hence then the next fact would give us lot of examples of theories with separable forking.

Fact 5.1.8. (see 10.2.6 of [TZ12]) Any uncountably categorical structure, with no definable infinite group in T^{eq}, is almost strongly minimal.

Question 5.1.9. Is the separable independence notion equivalent to any other model theoretic notion?

The following result is the main use of the separable independence notion.

Theorem 5.1.10. *Let T be a rosy theory with a separable independence notion. Further suppose that T has n and $(n+1)$-uniqueness over $\emptyset$. Then T has n-uniqueness over any set.*

Proof. We work in T^{eq}. We will prove the second condition of Lemma 4.1.14 to get n-uniqueness over any set. So take a sequence $a_1, \ldots, a_n$ which is independent over some set A. Then fix the canonical problem $\mathfrak{a}^*$ (of this sequence) and its canonical solution $\mathfrak{a}$ and fix also some other solution $\mathfrak{b}$. We are going to construct an isomorphism between $\mathfrak{a}$ and $\mathfrak{b}$.

By separability of the independence notion we can find c_i with $c_i \perp_{\emptyset} A$ such that $\operatorname{acl}(c_i A) = \operatorname{acl}(a_i A)$. Now because $c_i \perp_{\emptyset} A$ and $c_1, \ldots, c_n$ is an independent sequence over A, we have by Lemma 2.4.6 that the sequence

$$c_1 \ldots c_n c_{n+1} \text{ with } c_{n+1} = A$$

is independent over $\emptyset$. We take the canonical $n+1$-amalgamation problem $\tilde{\mathfrak{a}}^-$ (over $\emptyset$) of $c_1, \ldots, c_{n+1}$ and its canonical solution to $\tilde{\mathfrak{a}}$.

We will extend $\mathfrak{b}$ now to say $\tilde{\mathfrak{b}}$ such that this is solving the problem $\tilde{\mathfrak{a}}^-$. We define $\tilde{\mathfrak{b}}$ to be equal to $\tilde{\mathfrak{a}}$ on $\mathcal{P}(\{1, \ldots, n+1\})^-$. We set the object $\tilde{\mathfrak{b}}(\{1, \ldots, n+1\})$ as $\mathfrak{b}(\{1, \ldots, n\})$. Then we define the transition maps $\tilde{\mathfrak{b}}_s^{\{1,\ldots,n+1\}}$:

1. We set $\tilde{\mathfrak{b}}_s^{\{1,\ldots,n+1\}}$ as $\mathfrak{b}_s^{\{1,\ldots,n\}}$ for $s \subsetneq \{1, \ldots, n+1\}$ with $(n+1) \in s$.
2. We set $\tilde{\mathfrak{b}}_s^{\{1,\ldots,n+1\}}$ as $(\mathfrak{b}_s^{\{1,\ldots,n\}} \restriction \operatorname{acl}(c_i : i \in s))$ for $s \subsetneq \{1, \ldots, n+1\}$ with $(n+1) \notin s$ and $s \neq \{1, \ldots, n\}$.
3. We set $\tilde{\mathfrak{b}}_{\{1,\ldots,n\}}^{\{1,\ldots,n+1\}}$ as some (arbitrary) elementary extension to the algebraic closure (of its image and preimage) of the following map

$$\bigcup_{1 \le i \le n} (\mathfrak{b}_{\{1,\ldots\hat{i}\ldots,n\}}^{\{1,\ldots,n\}} \restriction_{\operatorname{acl}(c_1 \ldots \hat{c}_i \ldots c_n)}).$$

We check that $\tilde{\mathfrak{b}}$ is a functor. We use the fact that $\mathfrak{b}$ is a functor. Because of that it is only left to check that $\tilde{\mathfrak{b}}_{\{1,\ldots,n\}}^{\{1,\ldots,n+1\}}$ is well-defined. For that first note that the maps

$$\mathfrak{b}_{\{1,\ldots\hat{i}\ldots,n\}}^{\{1,\ldots,n\}} \restriction_{\operatorname{acl}^{\mathrm{eq}}(c_1 \ldots \hat{c}_i \ldots c_n)}$$

are giving a solution to the canonical problem (over $\emptyset$) of the sequence $c_1, \ldots, c_n$. Hence by the third point of Lemma 4.1.14 we can see that

$$f := \bigcup_{1 \le i \le n} (\mathfrak{b}_{\{1,\ldots\hat{i}\ldots,n\}}^{\{1,\ldots,n\}} \restriction_{\operatorname{acl}(c_1 \ldots \hat{c}_i \ldots c_n)})$$

is an elementary map. Hence $\tilde{\mathfrak{b}}_{\{1,\ldots,n\}}^{\{1,\ldots,n+1\}}$ can be picked as an elementary extension of f to the algebraic closure (of its preimage and image).

Now $\tilde{\mathfrak{b}}$ is isomorphic to the canonical solution $\tilde{\mathfrak{a}}$ by $n+1$-uniqueness over $\emptyset$. Let us take the natural isomorphism $(\tilde{\mathfrak{a}} \xrightarrow{f_s} \tilde{\mathfrak{b}})_{s \in \mathcal{P}(n \tilde{+} 1)}$ from $\tilde{\mathfrak{a}}$ to $\tilde{\mathfrak{b}}$. Further we define the following

$$S := \{s \in \mathcal{P}(n \tilde{+} 1) \mid (n+1) \in s\}.$$

Of course the category S is isomorphic to $\mathcal{P}(\tilde{n})$. So if we think about our original solution functor $\mathfrak{a}$ and $\mathfrak{b}$ as functors from S to $\mathcal{C}$, we have that $\tilde{\mathfrak{a}} \restriction_S = \mathfrak{a}$ and $\tilde{\mathfrak{b}} \restriction_S = \mathfrak{b}$ and hence $(\tilde{\mathfrak{a}} \xrightarrow{f_s} \tilde{\mathfrak{b}})_{s \in S}$ gives a natural isomorphism between them. $\square$

Remark 5.1.11. Note that the proof of this proposition does not use the fact that we work in T^{eq}, hence it also gives some results for n-uniqueness over real parameters in any pregeometric theory:

Let T be a theory with a separable independence notion in the real parameters (same definition but in T instead of T^{eq}). Further suppose that T has n and $n+1$-uniqueness over $\emptyset$ for real parameters. Then T has n-uniqueness over any set for real parameters.

5.2 The standard example of non-3-uniqueness

We will see later in this chapter that, after adding a finite cover, every theory which has an algebraically closed set which is not a model can fail amalgamation. Then we use this construction to establish examples where amalgamation over some non-empty set will fail, but amalgamation over the empty set holds.

But first we remember the easiest example, which fails 3-uniqueness over the empty set. This example is in fact totally categorical and is due to Hrushovski. It appeared as Example 1.7 in [PKM06].

Example 5.2.1. Let A be an infinite set, $[A]^2$ the set of all subset of A of size 2. Let $B = [A]^2 \times \{0,1\}$ (the double cover of $[A]^2$). Let $E \subset A \times [A]^2$ be the membership relation, i.e. $(a_1, \{a_2, a_3\}) \in E$ if and only if $a_1 = a_2$ or $a_1 = a_3$.

Further let the relation P be the subset of B^3 such that

$$((w_1, i_1), (w_2, i_2)(w_3, i_3)) \in P$$

if and only if there exists distinct $a_1, a_2, a_3 \in A$ such that for

$$\{i, j, k\} = \{1, 2, 3\},$$

we have $w_i = \{a_j, a_k\}$ and also we have that $i_1 + i_2 + i_3 = 0 \mod 2$. Also let $\pi : B \to [A]^2$ be the projection map, i.e. $\pi((w, i)) = w$.

Now we let M be 3-sorted structure $(A, [A]^2, B, E, P, \pi)$. This structure has weak elimination of imaginaries (see Lemma 5.4.6). We check that for pairwise non-equal $a_1, a_2, a_3 \in A$ Property $B(3)$ over $\emptyset$ fails. This then of course implies that 3-uniqueness fails over $\emptyset$ (see Lemma 4.4.7 and Lemma 4.4.6). Note that for $a \in A$ we have $\mathrm{acl}(a) = \{a\}$. Hence for $a_1, a_2 \in A$ we have that

$$(\{a_1, a_2\}, 0) \notin \mathrm{dcl}(\mathrm{acl}(a_1)\mathrm{acl}(a_2)),$$

as there is an automorphism taking $(\{a_1, a_2\}, 0)$ to $(\{a_1, a_2\}, 1)$ and fixing a_1, a_2. But as $(\{a_1, a_3\}, 0)$ is in $\mathrm{acl}(a_1, a_3)$ and $(\{a_2, a_3\}, 0)$ is in $\mathrm{acl}(a_2, a_3)$, and as $P((\{a_1, a_3\}, 0), (\{a_2, a_3\}, 0), x)$ defines $(\{a_1, a_2\}, 0)$, we can see that

$$(\{a_1, a_2\}, 0) \in \mathrm{dcl}(\mathrm{acl}(a_1, a_3), \mathrm{acl}(a_2, a_3)).$$

Hence we have shown that Property $B(3)$ over $\emptyset$ fails.

Now this construction is actually a finite cover of subsets of size 2 of the theory of an infinite set. The construction (see the next section) done in this example can be done in a very general way on the 2-sets of a definable subset of an arbitrary structures. Later we will then see that, if we choose the definable set right, we can have failure of $B(3)$ for any independent realisations of this definable set. Also under some additional conditions we can preserve total amalgamation over $\emptyset$. And later this will then gives us theories where $B(3)$ over some parameters fail, but total amalgamation over the empty set will hold.

Also note that this example has inspired many others. In [PS11] a stable totally categorical theory (of similar type) was constructed with complete n-amalgamation and complete $n-1$-uniqueness but failure of n-uniqueness over $\emptyset$. In similar fashion in [GKK13b] (see Remark 2.29) it was shown that there are totally categorical theories with failure of 3-uniqueness over $\emptyset$ with a more complicated "homology" group (see that same paper to know what this means).

5.3 A double cover of a definable set

The structure

We start constructing a 2-cover for some arbitrary L-theory T and some L-formula $\phi(x, y)$. For that we fix some L-theory T and some L-formula $\phi(x, y)$ for the rest of this section. Set

$$\psi(x_1, x_2, y) = \phi(x_1, y) \wedge \phi(x_2, y) \wedge \neg(x_1 \doteq x_2).$$

Let S_1 be the sort of x_1 (and therefore of x_2) and let S_3 be the sort of y. Now let S_2 be the class which consists of all subsets of S_1 of size 2. The sort S_2 is of course a sort in T^{eq}, but if S_2 is not already contained, we add it to our theory T together with the membership relation. Further we set $D = \{(\{a_1, a_2\}, b) \in S_2 \times S_3 \mid (a_1, a_2, b) \models \psi\}$ (a definable subset of $S_2 \times S_3$). We say that d_1, d_2, d_3 are *compatible sets* if there exist elements $a_1, a_2, a_3 \in S_1$ pairwise non equal and some $b \in S_3$ such that the following elements are part of the definable class D;

$$d_1 = (\{a_1, a_2\}, b),\ d_2 = (\{a_2, a_3\}, b)\ \text{ and }\ d_3 = (\{a_1, a_3\}, b).$$

We fix a new sort S_ψ, a new function π from S_ψ to $S_2 \times S_3$, and some ternary relation symbol R on S_ψ. Define S_ψ to be a double cover ($D \times \{0, 1\}$) of D, namely S_ψ contains $(d, 0)$ and $(d, 1)$ for every $d \in D$. For readability reasons we will write $(d,i) \in S_\psi$ as i_d. Further let

$$\pi : S_\psi \to D : \pi(i_d) = d.$$

For $i_{d_1}, j_{d_2}, k_{d_3} \in S_\psi$ (as defined $i,j,k \in \{0,1\}$ and $d_1, d_2, d_3 \in D$) we define that $R(i_{d_1}, j_{d_2}, k_{d_3})$ holds if and only if $i + j + k = 0 \mod 2$ and d_1, d_2, d_3 are compatible. Now this $L \cup \{S_\psi, \pi, R\}$-structure is an extension of the monster model $\mathfrak{C}$ of T and we will call it $\mathfrak{C}^+$. If we do this construction for some model M, we call it M^+. Note that $\mathrm{Th}(\mathfrak{C}^+) = \mathrm{Th}(M^+)$.

The axiomatisation

We give an axiomatisation of the $L \cup \{S_\psi, \pi, R\}$-theory $\mathrm{Th}(\mathfrak{C}^+)$. We will prove in Lemma 5.3.3 that this axiomatisation is complete (and defines the same theory as $Th(\mathfrak{C}^+)$). But for now we call it $T_{2,\phi}$. The L-part will be the same as T. The new axioms are the following:

1. π is two-to-one (with range S_ψ) and has image D.

$$\forall x_1 x_2 z \big(\psi(x_1, x_2, z) \to \exists^{=2} x (\pi(x) \doteq (\{x_1, x_2\}, z))\big)$$

$$\forall x \forall y \big(\pi(x) \doteq y \to \exists z_1 z_2 z (\psi(z_1, z_2, z) \wedge (\{z_1, z_2\}, z) \doteq y)\big)$$

2. If $R(x_1, x_2, x_3)$ holds, then the images $\pi(x_1), \pi(x_2), \pi(x_3)$ have to be compatible sets.

$$\forall x_1 x_2 x_3 \exists y_1 y_2 y_3 z \big(R(x_1, x_2, x_3) \to \bigwedge_{1 \le i \le 3} \pi(x_i) \doteq (\{y_i, y_{(i+1 \mod 3)}\}, z)\big)$$

3. For all x_1, x_2 for which there exists some x_3 such $\pi(x_1)\pi(x_2)\pi(x_3)$ is compatible, there exists exactly one x_3 such that $R(x_1, x_2, x_3)$ holds.

$$\forall x_1 x_2 \big(\exists z_1 z_2 z_3 z \bigwedge_{1 \le i \le 2} \pi(x_i) \doteq (\{z_i, z_{(i+1)}\}, z) \to \exists^{=1} x_3 R(x_1, x_2, x_3)\big)$$

4. For all elements x_1, x_2, x_2 for which $R x_1 x_2 x_3$ holds, we have that for

any permutation of x_1, x_2, x_3 the relation R holds.

$$\forall x_1 x_2 x_3 \big(R(x_1, x_2, x_3) \to \bigwedge_{\sigma \in S(3)} R(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \big)$$

5. (*tetrahedron-like*) For all z, for all x_1, x_2, x_3, x_4 non-equal points in $\phi(x, z)$ and for all

$$y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34}$$

(of sort S_ψ) such that $\pi(y_{ij}) = (\{x_i, x_j\}, z)$ and we have that if either all or exactly one of

$$R(y_{12}, y_{13}, y_{23}), R(y_{13}, y_{14}, y_{34}) \text{ and } R(y_{12}, y_{14}, y_{24})$$

holds, then $R(y_{23}, y_{34}, y_{24})$ holds. And if of

$$R(y_{12}, y_{13}, y_{23}), R(y_{13}, y_{14}, y_{34}) \text{ and } R(y_{12}, y_{14}, y_{24})$$

either 0 or 2 are true, then $R(y_{23}, y_{34}, y_{24})$ fails.

In formulae with $\neg$ set accordingly, we can write it as follow:

$$\begin{aligned}
&\forall z \forall x_1 x_2 x_3 x_4 \forall y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34} \\
&\Big[\Big(\bigwedge_{1 \le i < j \le 4} \neg (x_i \doteq x_j) \wedge \bigwedge_{1 \le i \le 4} \phi(x_i, z) \wedge \bigwedge_{1 \le i < j \le 4} \pi(y_{ij}) \doteq (\{x_i, x_j\}, z) \wedge \\
&(\neg) R(y_{12}, y_{13}, y_{23}) \wedge (\neg) R(y_{13}, y_{14}, y_{34}) \wedge (\neg) R(y_{12}, y_{14}, y_{24}) \Big) \\
&\to (\neg) R(y_{23}, y_{34}, y_{24}) \Big]
\end{aligned}$$

Verification

We check that these axioms are true in $\mathfrak{C}^+$. All but the last point following directly from the construction. Hence it is left to check that the structure is tetrahedron-like. For that we set

$$
\begin{aligned}
y_{12} &= i_{(\{x_1,x_2\},z)}, \quad y_{13} = j_{(\{x_1,x_3\},z)},\\
y_{14} &= k_{(\{x_1,x_4\},z)}, \quad y_{23} = l_{(\{x_2,x_3\},z)},\\
y_{24} &= m_{(\{x_2,x_4\},z)}, \quad y_{34} = n_{(\{x_3,x_4\},z)}.
\end{aligned}
$$

We have then that

$$l = i + j(+1) \mod 2,$$

$$m = j + k(+1) \mod 2,$$

$$n = i + k(+1) \mod 2$$

(+1 is there, in the case when the corresponding elements are not in the relation R). In order to fulfil tetrahedron-like, we have that $l + m + n = 0 \mod 2$ if either 0 or 2 "+1" are present and $l + m + n = 1 \mod 2$ if either 1 or 3 "+1" are present. But this can be seen by the equation

$$
\begin{aligned}
l + m + n &= i + j(+1) + j + k(+1) + i + k(+1) \mod 2\\
&= 2i + 2j + 2k(+1)(+1)(+1) \mod 2\\
&= (+1)(+1)(+1) \mod 2.
\end{aligned}
$$

Now a good way to imagine this, is to think about the set $D \cap S_2 \times \{b\}$ as the lines between elements in $\phi(\mathfrak{C}, b)$. Then S_ψ is the double cover of these lines. Think about $0_{\{a_1,a_2\}}$ as normal line and $1_{\{a_1,a_2\}}$ as a dotted line. Then a "triangle" is in R, whenever the triangle consists of either 0 or 2 dotted lines. (see Figure 5.1 on page 92)

Remark 5.3.1. I do not see any obstruction, why the construction in this section could not be extended to more than a double cover (namely an n-cover with more complicated group structure than $\mathrm{Sym}(n)$). Instead of covering 2-sets we could surely also cover $(n-1)$-sets and replace the tetrahedron-like property with some polyhedron-like property. Then if we assume lower amalgamation and uniqueness we should be able to prove a similar result

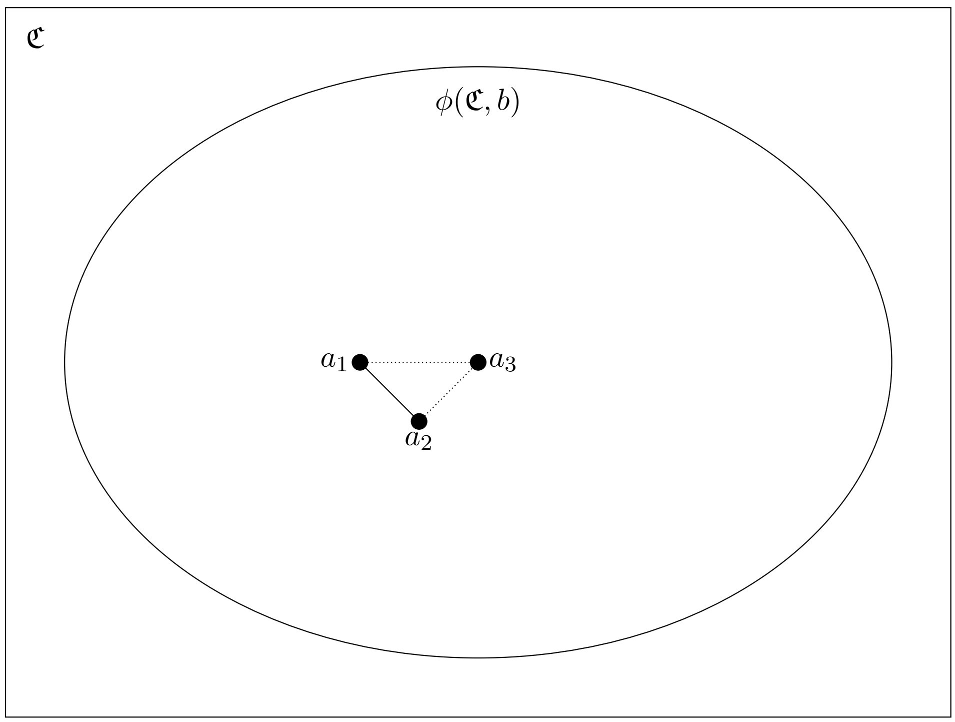

Figure 5.1: Double cover

as Lemma 5.5.1 with Property $B(n)$ (and hopefully also similar results to Proposition 5.5.15). Moreover, then in this case if T is the theory of infinite set, then $T_{2,\phi}$ should be the same as the example of [PS11].

Having positively answered the last remark one could then ask:

Question 5.3.2. Does there exists some algebraic cover which has Property $B(k)$ (over empty set) for infinite many k's and also fails Property $B(l)$ (over empty set) for infinite many l's?

Lemma 5.3.3. *$T_{2,\phi}$ is a complete $L \cup \{S_\psi, \pi, R\}$-theory. Moreover, $\mathfrak{C}^+$ is the monster model of this theory.*

Proof. Fix some model M^* (of some completion) of $T_{2,\phi}$. Since $M^* \restriction L \models T$ we can extend $M^* \restriction L = M$ to M^+ (as described in the beginning of this section). We will show that M^* is isomorphic to M^+ to finish the proof. As we have $\mathrm{Th}(M^+) = \mathrm{Th}(\mathfrak{C}^+)$, this will also give us that the monster $\mathfrak{C}^*$ of $T_{2,\phi}$ is isomorphic to $\mathfrak{C}^+$.

We define an isomorphism from M^+ to M^* which fixes M. Because we require that M is fixed, we may assume that $\phi(x, b)$ has some realisations for only one b (and ignore it). We enumerate the class $\phi(M, b)$ (in any way). Without loss we may assume that $\{a_\alpha \mid \alpha \in \mathrm{Ord}\} = \phi(M, b)$ where Ord are the ordinal numbers.

Now we will describe the automorphism say f. Note that Axiom 4 will be used without noting it. By Axiom 1 we can map $0_{\{a_0,a_\alpha\}}$ for any α to any element in $\pi^{-1}(\{a_0, a_\alpha\})$ (of M^*). Then by Axiom 2 and 3 for every α, β there exists a realisation a of $R(x, f(0_{\{a_0,a_\alpha\}}), f(0_{\{a_0,a_\beta\}}))$ with $\pi(a) = (\{a_\beta, a_\alpha\})$. We set the automorphism f on $0_{\{a_\alpha,a_\beta\}}$ to be this a. By Axiom 1 we make this map then a bijection between M^+ and M^* by mapping $1_{\{a_\alpha,a_\beta\}}$ to the other element of $\pi^{-1}(\{a_0, a_\alpha\})$.

It is left to check that the relation R is preserved under f. So we need to check that $i + j + k = 0 \mod 2$ if and only if

$$M^* \models R\big(f(i_{\{a_\alpha,a_\beta\}}), f(j_{\{a_\beta,a_\gamma\}}), f(k_{\{a_\alpha,a_\gamma\}})\big).$$

For that note that we have that

$$M^* \models R\big(f(i_{\{a_\alpha,a_\beta\}}), f(0_{\{a_0,a_\alpha\}}), f(0_{\{a_0,a_\beta\}})\big) \text{ iff } i = 0$$

$$M^* \models R\big(f(j_{\{a_\alpha,a_\gamma\}}), f(0_{\{a_0,a_\alpha\}}), f(0_{\{a_0,a_\gamma\}})\big) \text{ iff } j = 0$$

$$M^* \models R\big(f(k_{\{a_\beta,a_\gamma\}}), f(0_{\{a_0,a_\beta\}}), f(0_{\{a_0,a_\gamma\}})\big) \text{ iff } k = 0.$$

Now by our assumption that R is tetrahedron-like (Axiom 5) these points give exactly what we want. □

Lemma 5.3.4. *$T_{2,\phi}$ is a finite cover of T.*

Proof. By Lemma 5.3.3 we may assume that $\mathfrak{C}^+$ is the monster model of $T_{2,\phi}$. Clearly π is a 0-definable finite-to-one map from the new sort of $\mathfrak{C}^+$ to $\mathfrak{C}$. We will show that every L-automorphism of $\mathfrak{C}$ extends to an L^+-automorphism of $\mathfrak{C}^+$. This will finish the proof as by Fact 3.1.3 will know then, that $\mathfrak{C}$ is fully embedded in $\mathfrak{C}^+$. For that fix some L-automorphism f of $\mathfrak{C}$. We define $h_f \in \mathrm{Aut}(\mathfrak{C}^+)$ the extension of f. We set $h_f(i_d) = i_{f(d)}$ for all $d \in D$. For that it is enough to check that $R(i_{d_1}, j_{d_2}, k_{d_3})$ holds if and only if $R(i_{\alpha(d_1)}, j_{\alpha(d_2)}, k_{\alpha(d_3)})$ for all $d_1, d_2, d_3 \in D$. Hence we need to check that compatibility is preserved, but this is clear since an automorphism of $\mathfrak{C}$ is a bijection on the old sorts. Hence $\mathfrak{C}^+$ is a finite cover of $\mathfrak{C}$. □

Remark 5.3.5. We have seen that we can extend any automorphism of $\mathfrak{C}$ which fixes $A \subset \mathfrak{C}$ to one of $\mathfrak{C}^+$ which fixes $\pi^{-1}(A)$.

5.4 General properties of the cover $T_{2,\phi}$

Lemma 5.4.1. *Let $\mathfrak{C}$ be the monster of T and $\mathfrak{C}^+$ be the above described extension (and hence the monster model of $T_{2,\phi}$). We have that $G = Aut(\mathfrak{C}^+)$ is the inner semidirect product of the following closed subgroup*

$$H = \{\alpha \in Aut(\mathfrak{C}^+) \mid \alpha(i_d) = i_{\alpha(d)} \, \textit{for all } \; d \in D\}$$

and the normal, closed subgroup $N = Aut(\mathfrak{C}^+/\mathfrak{C})$.

Proof. It is enough to check that there exists some homomorphism

$$F : G \to H$$

which is the identity on H with kernel N. First note that any automorphism f of $\mathrm{Aut}(\mathfrak{C}^+)$ is in $\mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C})$ if and only if $f \restriction_{\mathfrak{C}} = \mathrm{id}$. Note that $\restriction_{\mathfrak{C}}$ is a homomorphism and its kernel N is a normal subgroup. Then note that $F' : \mathrm{Aut}(\mathfrak{C}) \to H$ defined by extending each $\alpha \in \mathrm{Aut}(\mathfrak{C})$ to some $h_\alpha \in H$ by setting $h_\alpha(i_d) = i_{\alpha(d)}$ for all $d \in D$ is an isomorphism. In the proof of Lemma 5.3.4 we already see that this is well-defined.

Then F' is a homomorphism because for $\alpha, \beta \in \mathrm{Aut}(\mathfrak{C})$, we have

$$h_\beta \circ h_\alpha(i_d) = i_{\beta\circ\alpha(d)} = h_{\beta\circ\alpha}(i_d).$$

It is surjective, as for each $h \in H$ we have $F'(h \restriction_{\mathfrak{C}}) = h$. It is injective as for $\beta \neq \alpha \in \mathrm{Aut}(\mathfrak{C})$ we have $h_\beta \neq h_\alpha$ (as both are extensions of the former). As already noted we know that the homomorphism $\restriction_{\mathfrak{C}}$ has N as its kernel and hence the kernel of $F := F' \circ \restriction_{\mathfrak{C}}$ is also N. Hence F is the homomorphism with the desired properties and therefore we are finished. □

Lemma 5.4.2. *A bijective map f on $\mathfrak{C}^+$ with $f \restriction_{\mathfrak{C}} = \mathrm{id}_{\mathfrak{C}}$ is in Aut$(\mathfrak{C}^+/\mathfrak{C})$ if and only if the following is true*

$$R(f(0_{d_1}), f(0_{d_2}), f(0_{d_3}))$$

for all compatible elements d_1, d_2, d_3 in the set D.

Proof. Left to right is clear by definition. For the other direction since compatibility will preserved by f assume that there exists $i_{d_1}, j_{d_2}, k_{d_3} \in S_\psi$ with compatible $d_1, d_2, d_3 \in D$ such that

$$\models R(i_{d_1}, j_{d_2}, k_{d_3}) \text{ and } \models \neg R(f(i_{d_1}), f(j_{d_2}), f(k_{d_3}))$$

or vice versa. We may assume without loss that the former holds, as otherwise the latter holds, i.e. $\models \neg R(i_{d_1}, j_{d_2}, k_{d_3})$ and $\models R(f(i_{d_1}), f(j_{d_2}), f(k_{d_3}))$ holds, but then it is easy to see that the former holds for the elements ${i'}_{d_1}, j_{d_2}$ and k_{d_3} with $i' = i + 1 \mod 2$.

Hence this implies that $i + j + k = 0 \mod 2$, but

$$f(i) + f(j) + f(k) = 1 \mod 2.$$

This means that f cannot be the identity on 1 or 3 of them, as then $f(i) + f(j) + f(k) = 0 \mod 2$ would hold. Hence either f can only be the identity on either 0 or 2 of then $i_{d_1}, j_{d_2}, k_{d_3}$. But then

$$R(f(0_{d_1}), f(0_{d_2}), f(0_{d_3}))$$

can not hold, as either 1 or 3 of 0's will be 1's after applying f. $\square$

Lemma 5.4.3. *Let $\mathfrak{C}$ be the monster of T and $\mathfrak{C}^+$ be the above described extension (and hence the monster model of $T_{2,\phi}$). Let b be an element of sort S_3. Then for any subclasses A_1, A_2, C of $\phi(\mathfrak{C}, b)$ with $A_1 \cap A_2 = C$ and any $f, g \in N(= Aut(\mathfrak{C}^+/\mathfrak{C}))$ with*

$$f \restriction \pi^{-1}([C]^2 \times \{b\}) = g \restriction \pi^{-1}([C]^2 \times \{b\}),$$

we have that

$$f \restriction \pi^{-1}([A_1]^2 \times \{b\}) \cup g \restriction \pi^{-1}([A_2]^2 \times \{b\})$$

is an elementary map which can be extended to an automorphism of N.

Further if $A_1 \cap A_2 = \emptyset$ holds, then for each $a_1 \in A_1$ and $a_2 \in A_2$ there exists automorphisms h, h' in N both extending

$$f \restriction \pi^{-1}([A_1]^2 \times \{b\}) \cup g \restriction \pi^{-1}([A_2]^2 \times \{b\})$$

with $h(0_{(\{a_1,a_2\},b)}) = 0_{(\{a_1,a_2\},b)}$ and $h'(0_{(\{a_1,a_2\},b)}) = 1_{(\{a_1,a_2\},b)}$.

Proof. We will show that

$$f \restriction \pi^{-1}([A_1]^2 \times \{b\}) \cup g \restriction \pi^{-1}([A_2]^2 \times \{b\})$$

can be extended to an automorphism. As $\mathfrak{C}$ will be fixed and there is no interaction between elements of S_ψ with different canonical projections to S_3, we may assume that S_3 consists only of one b (set the automorphism as the identity on the rest) and forget about this parameter. This essentially means that we assume that $D \subset S_2$.

We are going to define a map h with preimage and image $\pi^{-1}([A_1 \cup A_2]^2)$ which does extend the following map

$$f \restriction \pi^{-1}([A_1]^2) \cup g \restriction \pi^{-1}([A_2]^2)$$

and moreover fulfils the properties of Lemma 5.4.2, i.e. we need to check that $R\big(h(0_{d_1}), h(0_{d_2}), h(0_{d_3})\big)$ holds for all compatible elements

$$d_1, d_2, d_3 \in [A_1 \cup A_2]^2.$$

In order to do that we need to define the map h for any element $a_1 \in A_1 - A_2$ and any element $a_2 \in A_2 - A_1$. We may assume that there exist such elements, because otherwise we would be finished.

To make notation easier we now will start confusing elements 0_d with 0 and 1_d with 1, hence writing something like $i_d + j_{d'} = k_{d''} \mod 2$, which is meant to be $i + j = k \mod 2$. If $A_1 \cap A_2 = C$ is non-empty take an element c of it. Then $h(0_{\{a_1,a_2\}})$ will map to $g(0_{\{a_1,c\}}) + f(0_{\{a_2,c\}}) \mod 2$. We check that this value does not depend on the choice of c. For that take another $c' \in A_1 \cap A_2$. Note that we have that

$$f(0_{\{c,c'\}}) + f(0_{\{a_1,c'\}}) = f(0_{\{a_1,c\}}) \mod 2$$

and we also have that

$$g(0_{\{c,c'\}}) + g(0_{\{a_2,c'\}}) = g(0_{\{a_2,c\}}) \mod 2$$

since both f and g are automorphisms (see Lemma 5.4.2). Since f, g are the same on their common preimage part, this give us that $f(0_{\{c,c'\}}) = g(0_{\{c,c'\}})$

holds. Hence we have that the following equation is true;

$$g(0_{\{a_2,c\}}) + f(0_{\{a_1,c\}}) = 2f(0_{\{c,c'\}}) + g(0_{\{a_2,c'\}}) + f(0_{\{a_1,c'\}}) \pmod 2.$$

So we need check that for $a_1, a_1' \in A_1$ and $a_2 \in A_2$ that

$$(\mathfrak{C}^+) \models R\big(f(0_{\{a_1,a_1'\}}), h(0_{\{a_1,a_2\}}), h(0_{\{a_1',a_2\}})\big).$$

But that can be seen by the following equation:

$$\begin{aligned}
&f(0_{\{a_1,a_1'\}}) + h(0_{\{a_1,a_2\}}) + h(0_{\{a_1',a_2\}})\\
&= f(0_{\{a_1,a_1'\}}) + (f(0_{\{a_1,c\}}) + g(0_{\{a_2,c\}})) + (f(0_{\{a_1',c\}}) + g(0_{\{a_2,c\}}))\\
&= f(0_{\{a_1,a_1'\}}) + f(0_{\{a_1,c\}}) + f(0_{\{a_1',c\}}) + 2g(0_{\{a_2,c\}}) =\\
&= f(0_{\{a_1,a_1'\}}) + f(0_{\{a_1,c\}}) + f(0_{\{a_1',c\}})\\
&= 0 \pmod 2.
\end{aligned}$$

The same holds for $a_1 \in A$ and $a_2, a_2' \in B$ as we can exchange the roles of A_1 and A_2 in the calculation above. Hence we have $R(h(0_{d_1}), h(0_{d_2}), h(0_{d_3}))$ holds for all compatible $d_1, d_2, d_3 \in [A_1 \cup A_2]^2$.

If $A_1 \cap A_2$ is empty we fix any $a_1 \in A_1$ and $a_2 \in A_2$, and set

$$h(0_{\{a_1,a_2\}}) = 0_{\{a_1,a_2\}}.$$

Here we have a free choice, we could also pick $h(0_{\{a_1,a_2\}}) = 1_{\{a_1,a_2\}}$. We will put this in square brackets in our calculations to keep track of it, as this will show the last part of this lemma. For $a_1' \in A_1$ and $a_2' \in A_2$ we define

$$\begin{aligned}
h(0_{\{a_1,a_2'\}}) &= g(0_{\{a_1,a_2'\}})[+h(0_{\{a_1,a_2\}})] && \pmod 2,\\
h(0_{\{a_1',a_2\}}) &= f(0_{\{a_1,a_1'\}})[+h(0_{\{a_1,a_2\}})] && \pmod 2,\\
h(0_{\{a_1',a_2'\}}) &= g(0_{\{a_2,a_2'\}}) + f(0_{\{a_1,a_1'\}})[+h(0_{\{a_1,a_2\}})] && \pmod 2.
\end{aligned}$$

We need to check that for $a_1', a_1'' \in A_1$ and $a_2' \in A_2$

$$\models R\big(f(0_{\{a_1',a_1''\}}), h(0_{\{a_1',a_2'\}}), h(0_{\{a_1'',a_2'\}})\big).$$

But the following easy calculation shows it:

$$
\begin{aligned}
& f(0_{\{a_1',a_1''\}}) + h(0_{\{a_1',a_2'\}}) + h(0_{\{a_1'',a_2'\}}) \\
&= f(0_{\{a_1',a_1''\}}) + g(0_{\{a_2,a_2'\}}) + f(0_{\{a_1,a_1'\}})[+h(0_{\{a_1,a_2\}})] \\
&\qquad\qquad + g(0_{\{a_2,a_2'\}}) + f(0_{\{a_1,a_1''\}})[+h(0_{\{a_1,a_2\}})] \\
&= f(0_{\{a_1',a_1''\}}) + f(0_{\{a_1,a_1'\}}) + f(0_{\{a_1,a_1''\}}) + 2g(0_{\{a_2,a_2'\}})[+2h(0_{\{a_1,a_2\}})] \\
&= f(0_{\{a_1',a_1''\}}) + f(0_{\{a_1,a_1'\}}) + f(0_{\{a_1,a_1''\}}) \\
&= 0 \pmod 2.
\end{aligned}
$$

Therefore we have again, that $R(h(0_{d_1}), h(0_{d_2}), h(0_{d_3}))$ is true for all compatible $d_1, d_2, d_3 \in [A_1 \cup A_2]^2$.

Now suppose we take the identity automorphism of $\mathfrak{C}^+$. Then by the same argument as in the "$A_1 \cap A_2 = \emptyset$"-case but for $A_3 = \phi(\mathfrak{C}, b) - (A_1 \cup A_2)$ and $(A_1 \cup A_2)$, we can extend the following map $h \restriction_{\pi^{-1}([A_1 \cup A_2]^2)} \cup \mathrm{id}_{\pi^{-1}([A_3]^2)}$ to some h^* such that $R(h^*(0_{d_1}), h^*(0_{d_2}), h^*(0_{d_3}))$ holds for all compatible d_1, d_2, d_3 from $D (= [\phi(\mathfrak{C}, b)]^2)$. Hence by Lemma 5.4.2 again, we have that h^* together with $\mathrm{id}_{\mathfrak{C}}$ is an automorphism in N. $\square$

Remark 5.4.4. Let the assumption be the same as in Lemma 5.4.3. Then for any element $a_2 \in \phi(\mathfrak{C}, b) - A_1$ and for every $a \in \phi(\mathfrak{C}, b)$ with $a \neq a_2$, we have that $0_{(\{a,a_2\},b)}$ is not in the definable closure of $\mathfrak{C} \cup \pi^{-1}([A_1]^2 \times \{b\})$. To see this take the identity elementary map on $\pi^{-1}(A_1' \times \{b\})$ with $A_1' = A_1 \cup \{a\}$ and take the identity elementary map $\pi^{-1}(\mathfrak{C} - A_{1'} \times \{b\})$, then apply the last part of Lemma 5.4.3.

Corollary 5.4.5. *Let T be a countable ω-categorical theory. If T has weak elimination of imaginaries, then $T_{2,\phi}$ has weak elimination of imaginaries.*

Proof. We check the conditions of Fact 3.5.7. So take some

$$f \in \mathrm{Aut}(C(X)/\mathfrak{C}, C(X \cap Y))$$

for $\mathrm{acl}^{\mathfrak{C}}$-closed sets X, Y in $\mathfrak{C}$. We may set $C(X) = \bigcup_{x \in X \cap D} \pi^{-1}(x)$. As there is no interaction between elements $(\{a_1, a_2\}, b)$ and $(\{a_1, a_2\}, b')$ for

$b \neq b'$, we may assume that the sort S_3 contains only one element. Note that by algebraic closedness of X, Y, we have that if $(\{a_1, a_2\}, b) \in D$ is in X (or Y) then so is a_1, a_2, b. Hence in order to prove the corollary we note we have $C(X) = \pi^{-1}([X \cap S_1]^2 \times \{b\})$, $C(Y) = \pi^{-1}([Y \cap S_1]^2 \times \{b\})$ and

$$C(X \cap Y) = \pi^{-1}([X \cap Y \cap S_1]^2 \times \{b\}).$$

Now we are finished since

$$f \restriction C(X \cap Y) = \mathrm{id}_{C(X \cap Y)}$$

and therefore by Lemma 5.4.3 we have that $f \cup \mathrm{id}_{C(Y)}$ is in

$$\mathrm{Aut}(C(X)/\mathfrak{C}, C(Y)). \qquad \square$$

Proposition 5.4.6. *Let T be stable with elimination of imaginaries. Then $T_{2,\phi}$ has weak elimination of imaginaries.*

Proof. The idea is to use Fact 2.7.6. Because our theory is stable we can fix some set-size monster model $M \models T$ (see Section 2.2) and denote by M^+ its extension to $T_{2,\phi}$ (it will be a set-size monster model as well).

Fix some $p \in S_n(M^+)$ (in $T_{2,\phi}$) and fix some $\bar{c}_0 \models p$. Then by elimination of imaginaries and stability $\mathrm{tp}(\pi(\bar{c}_0)/M)$ has a canonical base B_0. We claim that B_0 together with finitely many elements of $S_\psi^{M^+}$ is a canonical base of p. For that we add any of $a_m, b \in M$ which is part of $d_{(\{a,a_m\},b)} \in \bar{c}_0$ to B_0. We call this new set B. Note that this will be independent of the choice of $\bar{c}_0$. Now for any $a_m, a'_m, b \in B$ with $(a_m, b) \models \phi(x, y)$ and $(a'_m, b) \models \phi(x, y)$, we add $0_{\{a_m,a'_m\},b}$ and $1_{\{a_m,a'_m\},b}$ to B.

We claim that this B is a canonical base of p. So for $f \in \mathrm{Aut}(\mathfrak{C}^+/B)$ which fixes M^+ set-wise, we must show that if $\bar{c} \models p$ then $f(\bar{c}) \models p$. We have that $\mathrm{tp}(\pi(\bar{c})/M) = \mathrm{tp}(f(\pi(\bar{c}))/M)$ since the set B contains a canonical base of $\mathrm{tp}(\pi(\bar{c})/M)$. Hence there is some $g \in \mathrm{Aut}(\mathfrak{C}/M)$ such that we have $g(\pi(\bar{c})) = f(\pi(\bar{c}))$. Now this g can be extended to an element $h \in H$, such that h fixes M^+ (see Remark 5.3.5). If we can show that there is some $f' \in \mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C}) \cap \mathrm{Aut}(\mathfrak{C}^+/M^+)$ such that $f'(h(\bar{c}))) = f(\bar{c})$, then we are finished as $f' \circ h \in \mathrm{Aut}(\mathfrak{C}^+/M^+)$ which maps $\bar{c}$ to $f(\bar{c})$. This means that it is enough to show that the orbit of any $\bar{c} \models p$ in $\mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C}, M^+)$ and in $\mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C}, B)$ is the same. Note as $h^{-1} \circ f$ is in $\mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C}, B)$, we can then pick $f'' \in \mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C}, M^+)$ with $f''(\bar{c}) = h^{-1} \circ f(\bar{c})$ and hence then $h \circ f''(\bar{c}) = f(\bar{c})$.

$$\begin{tikzcd}
h^{-1} \circ f(\bar{c}) & f(\bar{c}) \arrow[l, "h^{-1}"'] \arrow[dd, bend left, "\pi"] \\
\bar{c} \arrow[u, "f''"'] \arrow[ur, "f"'] \arrow[r, "h"'] \arrow[d, "\pi"'] & h(\bar{c}) \arrow[d, "\pi"'] \\
\pi(\bar{c}) \arrow[r, "g"] & f(\pi(\bar{c}))
\end{tikzcd}$$

Figure 5.2: Canonical base maps

We may assume that there is only one such b, as there is no interaction between elements $d_{(\{a,a'\},b)}$, $d_{(\{a'',a'''\},b')}$ for different b', and absorb the parameter to our theory. Note that by the choice of B, we have that for any $d_{\{a,a_m\}} \in \bar{c}$ with $a_m \in M$ we have $a_m \in B$. We let A be the set of all $a_m \in M$ which are part of some $d_{\{a,a_m\}} \in \bar{c}$. Note that $A \cap B = A = A \cap M$. Use M, $A \cup (\mathfrak{C} - M)$ and $\pi(B)$, $A \cup (\mathfrak{C} - M)$ both each time with Lemma 5.4.3 to note that the automorphisms of $\mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C}, B)$ and $\mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C}, M^+)$ are the same on $\pi^{-1}(A \cup (\mathfrak{C} - M))$: For $f \in \mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C}, B)$ we have by Lemma 5.4.3 that $f \restriction_{\pi^{-1}(A \cup (\mathfrak{C} - M))} \cup \mathrm{id} \restriction_M$ is elementary and (as $f \restriction_{M \cap A} = f \restriction_{B \cap A} = \mathrm{id}_A$) can be extended to an automorphism of $\mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C}, M^+)$. Now since $\bar{c}$ is a tuple of elements in $\pi^{-1}(A \cup (\mathfrak{C} - M))$ we are done. $\square$

Question 5.4.7. Does $(T^{\mathrm{eq}})_{2,\phi}$ always have weak elimination of imaginaries?

5.5 Amalgamation properties of $T_{2,\phi}$

Now we are ready to use our construction $T_{2,\phi}$, to establish examples of failure of 3-uniqueness over algebraically closed sets which are not the domain of a model of theory T.

Force failure of $B(3)$

Proposition 5.5.1. *Let T be a complete rosy L-theory and let A be some algebraically closed set which is not a model of T. Then T has a finite cover T' which fails Property $B(3)$ over A for real parameters. In fact, if $\phi(x,y)$ is some L-formulas such that $\phi(x,a)$ with $a \in A$ is not satisfiable in A, then $T_{2,\phi}$ fails Property $B(3)$ over A for real parameters. Moreover, we can pick A to be the algebraic closure of some finite set.*

Proof. Fix an algebraically closed set A which is not a model of T. Then by Tarski's Test there exists some L-formula $\phi(x,\overline{y})$ and some $\overline{b} \in A$ such that $\phi(x,\overline{b})$ is satisfiable but not satisfied in A. Moreover, $\phi(x,\overline{b})$ has to be realised by infinitely many elements.

We construct for this $\phi(x,y)$ the finite cover $\models T_{2,\phi}$. We check that this finite cover $\models T_{2,\phi}$ fails $B(3)$ over any $A' \subset A$ which contains $\overline{b}$ in order to finish the proof. Note that this will then also show the "Moreover"-part. Therefore fix any type p in $S(A')$ which contains $\phi(x,\overline{b})$. Let a_1, a_2, a_3 be independent realisations of p. We check that $B(3)$ for a_1, a_2, a_3 over A' fails. Hence we have to check that $\mathrm{dcl}(\mathrm{acl}(a_1A')\mathrm{acl}(a_2A'))$ is a proper subset of

$$\mathrm{acl}(a_1a_2A') \cap \mathrm{dcl}(\mathrm{acl}(a_1a_3A'), \mathrm{acl}(a_2a_3A')). \tag{5.1}$$

Now note that we have $\mathrm{acl}(a_1A') \mathop{\smile\!\!\!\!\!\!|}_{A'} \mathrm{acl}(a_2A')$ (this is true by Symmetry, Monotonicity and Lemma 2.4.4 or one could have picked independent elements in the type $\mathrm{tp}(\mathrm{acl}(a_1A')/\mathrm{acl}(A'))$.). By this and by Anti-Reflexivity we have then that the following is true

$$\mathrm{acl}(a_1A') \cap \mathrm{acl}(a_2A') \subset \mathrm{acl}(A').$$

Further there is no element in $\mathrm{acl}(A)$ and therefore in $\mathrm{acl}(A')$ which satisfies $\phi(x,\overline{b})$. Hence by our choice of ψ (see page 88), we have for all $a' \in \mathrm{acl}(a_1A')$ and all $a'' \in \mathrm{acl}(a_2A')$ that there is no algebraic element over A' which satisfies both $\psi(a', y_2, \overline{b})$ and $\psi(a'', y_2, \overline{b})$. Hence by Lemma 5.4.3 we can find an automorphism fixing $\mathrm{acl}(a_1A')$, $\mathrm{acl}(a_2A')$ and $\mathfrak{C}$ and mapping $0_{(\{a_1,a_2\},\overline{b})}$ to $1_{(\{a_1,a_2\},\overline{b})}$. Clearly this shows the element $0_{(\{a_1,a_2\},\overline{b})}$ is not part of the set $\mathrm{dcl}(\mathrm{acl}(a_1A'), \mathrm{acl}(a_2A'))$.

But we have that $0_{(\{a_1,a_3\},\bar{b})}$ is part of $\operatorname{acl}(a_1a_3A')$ and $0_{(\{a_2,a_3\},\bar{b})}$ is part of $\operatorname{acl}(a_2a_3A')$. The new relation R gives us then, that the $0_{(\{a_1,a_2\},\bar{b})}$ is the only element which satisfies $R(x, 0_{(\{a_1,a_3\},\bar{b})}, 0_{(\{a_2,a_3\},\bar{b})})$. Hence of course we have that

$$0_{(\{a_1,a_2\},\bar{b})} \in \operatorname{dcl}(\operatorname{acl}(a_1a_3A'), \operatorname{acl}(a_2a_3A')).$$

But now we are finished. Since we have shown, that the inequality 5.1 is fulfilled. This then give us, as we already discussed, that the constructed finite cover fails the Property $B(3)$ over the set A'. □

Remark 5.5.2. Note that the last proof did only use the properties of forking which are true in any theory. Hence the following is true as well: Let T be a complete L-theory such that there exists some acl-closed A which is not a model of T. Then T has a finite cover T' which fails $B(3)$ for some Morley sequence over A.

Corollary 5.5.3. *Let T be ω-categorical rosy and let A be some algebraically closed set which is not a model of T. Then T^{eq} has a finite cover T' which fails $B(3)$ over A.*

Note that if T is an ω-categorical theory which does have a sort with infinitely many elements, then T does have some algebraically closed set which is not a model of T. This is as due to the Ryll-Nardzewski theorem the algebraic closure of any finite set of parameters is finite in each sort.

Proof. This follows from Proposition 5.5.1 and Corollary 5.4.5. □

Corollary 5.5.4. *Let T be stable and let A be some algebraically closed which is not a model of T. Then T^{eq} has a finite cover T' which fails $B(3)$ over A.*

Proof. This follows from Proposition 5.5.1 and Proposition 5.4.6. □

So this corollary together with Fact 4.7.1 shows that in stable theories there is no abstract model theoretic reason (i.e. reason based on properties preserved under finite cover) why 3-uniqueness should be true, except if the algebraic closure only produces models of the theory.

$B_{\mathbf{Aut}}(n)$ is $B(n)$ in $T_{2,\phi}$ modulo T

In this subsection we proof a rather technical lemma which shows us that in $T_{2,\phi}$ Property $B_{\mathrm{Aut}}(n)$ and B_n are more or less the same. For the previous Subsection (i.e. the first subsection of Section 5.5) this will give us no new insight. But it will have some use later: we will preserve the Property $B_{\mathrm{Aut}}(n)$ over some (fixed) set (and therefore n-uniqueness in stable theories with lower uniqueness) in the extension $T_{2\,phi}$. After we have done that, we will in some examples combine this result with Proposition 5.5.1 to get examples with total uniqueness over $\emptyset$ and failure of 3-uniqueness over some non-empty set.

The main assumption in the technical lemma we have to take about the cover $T_{2,\phi}$ will be the following definition.

Definition 5.5.5. If $b_1 \dots b_n$ is an independent sequence over $\emptyset$, we set

$$B = \bigcup_{i=1}^{n-1} \mathrm{acl}^{\mathfrak{C}^+}(b_1 \dots \hat{b}_i \dots b_{n-1} b_n) \text{ and } B^- = \bigcup_{i=1}^{n-1} \mathrm{acl}^{\mathfrak{C}^+}(b_1 \dots \hat{b}_i \dots b_{n-1}).$$

We say that $T_{2,\phi}$ has *Property $B(n)$ in Aut($\mathfrak{C}^+/\mathfrak{C}$)* if for every independent sequence in $b_1 \dots b_n$ over $\emptyset$ and every element e in $\mathrm{acl}^{\mathfrak{C}^+}(b_1 \dots b_{n-1})$ we have that the orbit of e over $\mathrm{Aut}(\mathfrak{C}^+/B\mathfrak{C})$ and over $\mathrm{Aut}(\mathfrak{C}^+/B^-\mathfrak{C})$ is the same (i.e. we have $e \in \mathrm{dcl}(\mathfrak{C}B)$ if and only if we have $e \in \mathrm{dcl}(\mathfrak{C}B^-)$).

Lemma 5.5.6. *Let T be a rosy theory with Property $B_{Aut}(n)$ over $\emptyset$ for real parameters. Let $\phi(x, y)$ be some formula such that for any $a, b \models \phi(x, y)$ we have that $b \in \mathrm{acl}(a)$ and such that $T_{2,\phi}$ has Property $B(n)$ in Aut($\mathfrak{C}^+/\mathfrak{C}$). Then we have that $T_{2,\phi}$ has Property $B_{Aut}(n)$ over $\emptyset$ for real parameters. If T has $B_{Aut}(n)$ over $\emptyset$ (in T^{eq}) and $T_{2,\phi}$ has weak elimination of imaginaries then we can omit the term for real parameter.*

Proof. Let $\mathfrak{C}$ be the monster of T and $\mathfrak{C}^+$ the monster of $T_{2,\phi}$. Fix an independent sequence (of tuples) $b_1, \dots, b_n$ over $\emptyset$ in $\mathfrak{C}$ (and not $\mathfrak{C}^+$). Note that by Corollary 3.1.9 it is indeed enough take elements of $\mathfrak{C}$ (and not of the cover). Now we have, since Property $B_{\mathrm{Aut}}(n)$ over the empty set holds in the theory T, that for the sets

$$B_{\mathfrak{C}} = \bigcup_{i=1}^{n-1} \operatorname{acl}^{\mathfrak{C}}(b_1 \dots \hat{b}_i \dots b_{n-1} b_n) \text{ and } B_{\mathfrak{C}}^{-} = \bigcup_{i=1}^{n-1} \operatorname{acl}^{\mathfrak{C}}(b_1 \dots \hat{b}_i \dots b_{n-1})$$

the following automorphism groups are the same;

$$\operatorname{Aut}(\operatorname{acl}^{\mathfrak{C}}(b_1 \dots b_{n-1})/B_{\mathfrak{C}}) = \operatorname{Aut}(\operatorname{acl}^{\mathfrak{C}}(b_1 \dots b_{n-1})/B_{\mathfrak{C}}^{-}).$$

By Remark 5.3.5 this then gives that for the same sets in $\mathfrak{C}^{+}$, i.e.

$$B = \bigcup_{i=1}^{n-1} \operatorname{acl}^{\mathfrak{C}^{+}}(b_1 \dots \hat{b}_i \dots b_{n-1} b_n) \text{ and } B^{-} = \bigcup_{i=1}^{n-1} \operatorname{acl}^{\mathfrak{C}^{+}}(b_1 \dots \hat{b}_i \dots b_{n-1})$$

the following corresponding automorphism groups are the same;

$$\operatorname{Aut}(\operatorname{acl}^{\mathfrak{C}}(b_1 \dots b_{n-1})/B) = \operatorname{Aut}(\operatorname{acl}^{\mathfrak{C}}(b_1 \dots b_{n-1})/B^{-}). \tag{5.2}$$

Also by Lemma 5.4.1 we know that we can write each $f \in \operatorname{Aut}(\mathfrak{C}^{+}/B^{-})$ as a composition $f = f_h \circ f_n$ with $f_n \in \operatorname{Aut}(\mathfrak{C}^{+}/\mathfrak{C})$ and $f_h \in H$ (see Lemma 5.4.1 for the definition of H). By Equation 5.2 we know that $f_h \restriction \operatorname{acl}^{\mathfrak{C}}(b_1, \dots, b_{n-1})$ can be extended such that it fixes B and this then extends uniquely (in H) to $\operatorname{acl}^{\mathfrak{C}^{+}}(b_1, \dots, b_{n-1})$. Hence we only need to check that $f_n \restriction \operatorname{acl}^{\mathfrak{C}^{+}}(b_1, \dots, b_{n-1})$ can be extended such that it fixes B.

Because of this, it is left to show that for any tuple $\overline{e} \in \operatorname{acl}^{\mathfrak{C}^{+}}(b_1, \dots, b_{n-1})$ the orbit of it over $\mathfrak{C}B^{-}$ is the same as over $\mathfrak{C}B$. As the orbit on the old sort is trivial in $\operatorname{Aut}(\mathfrak{C}^{+}/\mathfrak{C})$, we may assume that $\overline{e}$ is purely of sort S_ψ and as there is no interaction between elements of S_ψ with different canonical projections to S_3, we may assume that S_3 consists only of one element and forget about this parameter, i.e. we may assume $D \subset S_2$.

Claim. We may assume that $\overline{e}$ is a singleton.

Proof(Claim): We assume that the orbit of each singleton of

$$S_\psi \cap \operatorname{acl}^{\mathfrak{C}^{+}}(b_1 \dots b_{n-1})$$

is the same over $\mathfrak{C}B$ and $\mathfrak{C}B^-$. Then we will prove by induction of the length of the tuple, that the tuple $(e_1, \ldots, e_n)$ has the same orbit over $\mathfrak{C}B$ and $\mathfrak{C}B^-$. Hence assume that the orbit of tuples of length $n-1$ is the same over $\mathfrak{C}B$ and $\mathfrak{C}B^-$. Take some $f \in \mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C}B^-)$. Then there exists $f', f'' \in \mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C}B)$ with

$$f'(e_1 \ldots e_{n-1}) = f(e_1 \ldots e_{n-1})$$

and $f''(e_n) = f(e_n)$.

We may assume that $e_n \notin \mathrm{dcl}(e_1 \ldots e_{n-1})$, because otherwise we would be finished. Let A be the projection of $e_1, \ldots, e_{n-1}$ onto S_1. Let $e_n = 0_{\{a_1,a_2\}}$. Now we have to consider several cases: If a_1 and a_n are both not in A, then we can use the last part of Lemma 5.4.3 with $A_1 = A$ and $A_2 = \{a_1, a_2\}$ and the claim follows. If a_1 is in and a_2 is not in A or vice versa, then we can use the last part of Lemma 5.4.3 with $A_1 = A$ and $A_2 = \{a_2\}$ (or vice versa) and the claim follows. For a_1 and a_2 are both in A and as we have that $e_n \notin \mathrm{dcl}(e_1 \ldots e_{n-1})$, we know that there cannot be a sequence $i_{\{a_1,a'_1\}}, j_{\{a'_1,a'_2\}} \ldots k_{\{a'_{m-1},a'_m\}}, l_{\{a'_m,a_2\}}$ with each element of this sequence be one of the e_i's with $i < n$. Hence we can split A into disjoint A_1 and A_2, such that $a_1 \in A_1$ and $a_2 \in A_2$ and each e_i for $i < n$ is either in $\pi^{-1}([A_1]^2)$ and $\pi^{-1}([A_2]^2)$. Hence applying the last part of Lemma 5.4.3 to this sets gives us the claim. $\square_{\text{Claim}}$

Now by Property $B(n)$ in $\mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C})$ we have that any singleton

$$e \in \mathrm{acl}^{\mathfrak{C}^+}(b_1, \ldots, b_{n-1})$$

has the same orbit in $\mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C}B)$ and in $\mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C}B^-)$. But this shows what we wanted. $\square$

Keep $B(3)$ over $\emptyset$ true

We now examine the situation in which Property $B(3)$ over the empty set and Property $B(3)$ over parameters can differ.

Definition 5.5.7. Fix some rosy L-theory T. An L-formula $\phi(x, y)$ is said to have *only dependent realisations in x over $\emptyset$* (in T), if for all a_1, a_2, b with

$a_1, b \models \phi(x,y)$ and $a_2, b \models \phi(x,y)$, we have $a_1 \not\downfree_{\emptyset} a_2$ and $b \in \operatorname{acl}(a_1)$.

Remark 5.5.8. Let T be some rosy L-theory. Then a formula $\phi(x,y)$ such that for any $\models \phi(a,b)$ we have $b \in \operatorname{acl}(a)$ and $b \notin \operatorname{acl}(\emptyset)$ has only dependent realisations in x over $\emptyset$

Proof. Take any a_1, b and a_2, b both satisfying $\phi(x,y)$, then we have that $b \in \operatorname{acl}(a_1) \cap \operatorname{acl}(a_2)$. As we have $b \notin \operatorname{acl}(\emptyset)$ we get that by Anti-Reflexivity that $a_1 \not\downfree_{\emptyset} a_2$. $\square$

By the last remark we can easily see that the following is true

Remark 5.5.9. Let T be a rosy L-theory with $\operatorname{acl}(\emptyset)$ finite. Then the L-formula $\phi(x_1, x_2; y)$ defined as $\neg(x_1 \doteq x_2) \wedge x_2 \doteq y \wedge y \notin \operatorname{acl}(\emptyset)$ has only dependent realisations in (x_1, x_2) over the empty set.

Proposition 5.5.10. *Let T be rosy theory with $B_{Aut}(3)$ over $\emptyset$ for real parameters and let $\phi(x,y)$ be a formula with only dependent realisations in x over $\emptyset$. Then $T_{2,\phi}$ has $B_{Aut}(3)$ over $\emptyset$ for real parameters.*

Proof. By the Lemma 5.5.6 it is enough to show that $B(3)$ in $\operatorname{Aut}(\mathfrak{C}^+/\mathfrak{C})$ holds. So fix an independent sequence (of tuples) b_1, b_2, b_3 over $\emptyset$ in $\mathfrak{C}$ (and not $\mathfrak{C}^+$). We will show $B(3)$ in $\operatorname{Aut}(\mathfrak{C}^+/\mathfrak{C})$ (for this sequence) by showing the following; if there is an automorphism α of $\mathfrak{C}^+$ moving $0_{(\{a_1,a_2\},b)}$ to $1_{(\{a_1,a_2\},b)}$ and fixing $\mathfrak{C}B^-$ (with $B^- = \operatorname{acl}^{\mathfrak{C}^+}(b_1) \cup \operatorname{acl}^{\mathfrak{C}^+}(b_2)$, then there is an automorphism moving $0_{(\{a_1,a_2\},b)}$ to $1_{(\{a_1,a_2\},b)}$ and fixing $\mathfrak{C}B$ (with $B^- = \operatorname{acl}^{\mathfrak{C}^+}(b_1 b_3) \cup \operatorname{acl}^{\mathfrak{C}^+}(b_2 b_3)$).

Claim. We may assume that a_1, a_2 are in B^-.

Proof(Claim): Note that if a_1, a_2 are in B, then as a_1, a_2 are in $\operatorname{acl}(b_1, b_2)$ by independence they will be in B^-. But by Remark 5.4.4 we have that if either one of a_1, a_2 would not be in B, then it follow that $0_{(\{a_1,a_2\},b)}$ would not be in $\operatorname{dcl}(B)$. But then we would be finished as there is an automorphism moving $0_{(\{a_1,a_2\},b)}$ to $1_{(\{a_1,a_2\},b)}$ and fixing $\mathfrak{C}B$. $\square_{\text{Claim}}$

Note that we cannot have $a_1 \in \operatorname{acl}^{\mathfrak{C}^+}(b_1)$ and $a_2 \in \operatorname{acl}^{\mathfrak{C}^+}(b_2)$ as $a_1 \not\downfree a_2$ holds since ϕ has only dependent realisations. But then as a_1, a_2 are part of B^- we have that they are either both part of $\mathrm{acl}^{\mathfrak{C}^+}(b_1)$ or else they are both part of $\mathrm{acl}^{\mathfrak{C}^+}(b_1)$. This then of course gives that $0_{(\{a_1,a_2\},b)} \in \mathrm{dcl}(B^-)$ as required. $\square$

Corollary 5.5.11. *Let T be a rosy theory having $B_{Aut}(3)$ over $\emptyset$ for real parameters, let $\mathrm{acl}(\emptyset)$ be finite and let $A \neq \mathrm{acl}(\emptyset)$ be some algebraically closed set which is not a model of T. Then there exists some formula $\phi(x,y)$ such that $T_{2,\phi}$ has $B_{Aut}(3)$ over $\emptyset$ for real parameters and fails $B(3)$ over A for real parameters.*

Proof. We pick the same formula (for A) as in Proposition 5.5.1. Further we may assume that for $\overline{a}, \overline{b} \models \phi(x,y)$ we have $\overline{b} \in \mathrm{dcl}(\overline{a})$ and $\mathrm{acl}(\overline{b}) \not\subset \mathrm{acl}(\emptyset)$. The proof of Proposition 5.5.1 will still work and by Remark 5.5.8 we know that the formula will also have only dependent realisations. Hence we can apply Proposition 5.5.10 to finish the proof. $\square$

Corollary 5.5.12. *Let $T(= T^{\mathrm{eq}})$ be a stable theory with 3-uniqueness over $\emptyset$, let $\mathrm{acl}(\emptyset)$ be finite and let $A \neq \mathrm{acl}(\emptyset)$ be some algebraically closed set which is not a model of T. Then there exists some formula $\phi(x,y)$ such that $T_{2,\phi}$ has 3-uniqueness over $\emptyset$ and fails $B(3)$ over A.*

Proof. First note that $T_{2,\phi}$ has weak elimination of imaginaries (see Proposition 5.4.6). By Corollary 5.5.11 (by weak elimination of imaginaries we can omit the suffix "over real parameters") we have as $B_{\mathrm{Aut}}(3)$ over $\emptyset$. Now Proposition 4.4.7 gives us 3-uniqueness over $\emptyset$. $\square$

Corollary 5.5.13. *Let T be a rosy ω-categorical theory with weak elimination of imaginaries such that there exists at least one sort of infinite size. Let Property $B_{Aut}(3)$ over $\emptyset$ be true as well. Then there exists some formula $\phi(x,y)$ such that $T_{2,\phi}$ has $B_{Aut}(3)$ over $\emptyset$ and fails $B(3)$ over some non-empty set.*

Proof. This is just due to the fact that the algebraic closure of a finite set is finite (in each sort) and hence not a model of our theory. We can then apply Corollary 5.5.11. Since we have weak elimination of imaginaries by Corollary 5.4.5, we can omit the suffix "over real parameters". $\square$

Keep $B(n)$ over $\emptyset$ true for all n

Theorem 5.1.10 shows that if our theory has a separable independence notion, then there is a connection between total uniqueness over the empty set and 3-uniqueness over parameters (in fact n-uniqueness). We will see that under certain assumptions there is no connection between these two notions.

Definition 5.5.14. Fix some rosy L-theory T. An L-formula $\phi(x, y)$ is said to *split in x (over $\emptyset$)* (in T), if for any two subsets of the monster A_1, A_2 with $A_1 \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} A_2$ and any $a, b \models \phi(x, y)$ with $a \in \operatorname{acl}(A_1 A_2)$ and $b \in \operatorname{acl}(A_1)$ we have that there exists $a' \in \operatorname{acl}(A_1)$ with $a' \models \phi(x, b)$.

We will later see examples of split formulae. But first we examine the main use of this notion.

Proposition 5.5.15. *Let $T(= T^{\mathrm{eq}})$ be a rosy theory with Property $B_{Aut}(n)$ over $\emptyset$ for real parameters and let $\phi(x, y)$ be a formula which splits in x over $\emptyset$ and such that for any $a, b \models \phi(x, y)$ we have that $b \in \operatorname{dcl}(a)$. Then $T_{2,\phi}$ has Property $B_{Aut}(n)$ over $\emptyset$ for real parameters.*

Again by Corollary 5.4.5 and Proposition 5.4.6 as before we can omit "for real parameters" in the above proposition if we additionally assume either stability or ω-categoricity. Unfortunately the notion of a split formula is a bit mysterious to me, as I could not prove an equivalent to the Corollaries 5.5.12 and 5.5.13. Probably the mystery comes from the fact that this notion was reverse-engineered from the example $(\mathbb{Z}/4\mathbb{Z})^{\omega}$.

Proof. By the Lemma 5.5.6 it is enough to show that $B(n)$ in $\operatorname{Aut}(\mathfrak{C}^+/\mathfrak{C})$. So fix an independent sequence (of tuples) $b_1, \ldots, b_n$ over $\emptyset$ in $\mathfrak{C}$ (and not $\mathfrak{C}^+$). We will show $B(n)$ in $\operatorname{Aut}(\mathfrak{C}^+/\mathfrak{C})$ (for this sequence) by showing the following; if there is an automorphism α of $\mathfrak{C}^+$ moving $0_{(\{a_1,a_2\},b)}$ to $1_{(\{a_1,a_2\},b)}$ and fixing $\mathfrak{C}B^-$ (with $B^- = \bigcup_{i=1}^{n-1} \operatorname{acl}^{\mathfrak{C}^+}(b_1 \ldots \hat{b}_i \ldots b_{n-1})$), then there is an automorphism moving $0_{(\{a_1,a_2\},b)}$ to $1_{(\{a_1,a_2\},b)}$ and fixing $\mathfrak{C}B$ (with

$$B = \bigcup_{i=1}^{n-1} \operatorname{acl}^{\mathfrak{C}^+}(b_1 \ldots \hat{b}_i \ldots b_{n-1} b_n)).$$

Claim. We may assume that $a_1, a_2 \in B^-$.

Proof(Claim): Note that if $a_1, a_2 \in B$ then as $a_1, a_2 \in \mathrm{acl}^{\mathfrak{C}^+}(b_1, \ldots, b_{n-1})$, by independence of the b_i's, they are in B^-. Hence it is enough to check that $a_1, a_2 \in B$. By Remark 5.4.4 we know that if a_1 is not in B then we have that $0_{(\{a_1,a_2\},b)} \notin \mathrm{dcl}(\mathfrak{C}B)$. But then we would be finished as there is an automorphism moving $0_{(\{a_1,a_2\},b)}$ to $1_{(\{a_1,a_2\},b)}$ and fixing $\mathfrak{C}B$, which is impossible. $\square_{\text{Claim}}$

We may assume that

$$a_1 \in \mathrm{acl}^{\mathfrak{C}^+}(b_2, \ldots, b_{n-1}) \ \text{ and } \ a_2 \in \mathrm{acl}^{\mathfrak{C}^+}(b_1, \ldots, b_{n-2}).$$

As we have that $b \in \mathrm{dcl}(a_i)$ for $i \in \{1, 2\}$ by the assumptions, we have that b is contained in $\mathrm{acl}(b_2 \ldots b_{n-2})$: this follows from the independence of the b_i's and the Anti-Reflexivity axiom, as b is in both $\mathrm{acl}^{\mathfrak{C}^+}(b_2 \ldots b_{n-1})$ and $\mathrm{acl}^{\mathfrak{C}^+}(b_1 \ldots b_{n-2})$.

By our assumptions we have that $\phi(x, y)$ splits in x over $\emptyset$. We apply the splitting of ϕ to the sets $A_1 = \mathrm{acl}(b_2 \ldots b_{n-2})$ and $A_2 = b_1 b_{n-1} b_n$. Note that we have shown that $b \in A_1$. As we have that $a_1 \in \mathrm{acl}(A_1 A_2)$ and get therefore some

$$a_1' \in \mathrm{acl}(A_1) \ \text{ with } \ a_1' \models \phi(x, b).$$

Note that this is already enough as now $0_{\{a_1,a_1'\}}$ is in $\mathrm{acl}(a_2 \ldots a_{n-1})$ and $0_{\{a_1',a_2\}}$ is in $\mathrm{acl}(a_1 \ldots a_{n-2})$. As then $0_{\{a_1,a_2\}}$ is the only solution of

$$R(x, 0_{\{a_1',a_2\}}, 0_{\{a_1,a_1'\}})$$

and hence it is in $\mathrm{dcl}(\mathfrak{C}B^-)$. This gives us finally $B(n)$ in $\mathrm{Aut}(\mathfrak{C}^+/\mathfrak{C})$. $\square$

5.6 $T_{2,\phi}$ of an abelian group

We now will work with modules. For that, fix some ring R and an R-module M. A module will be of language $(0, +, -, r)_{r \in R}$ where $r : M \to M$ has the obvious interpretation (left multiplication with the ring element). Any module is stable, this is due to Fisher, E. [Fis72] (implicitly) and independently Baur,

W. [Bau75] (see Theorem 2.1 of [Zie84] or Theorem 3.1 of [Pre03] for a proof and the paragraph above the latter for the historic references). And we know that every formula in such a theory is equivalent to a Boolean combination of positive primitive formulae. A *pp-formula* is of the form $\exists \overline{y}(\bigwedge_i \gamma_i)$ where $\gamma_i(\overline{y}, \overline{x})$ are equations ($\gamma_i(\overline{z}) = r_1 z_1 + \cdots + r_l z_l \doteq 0$).

Fact 5.6.1. Let T be the complete theory of M. The following are equivalent;

1. A complete theory T of some module M is closed under products (i.e. $M \equiv M^\omega$).

2. The class of models of T is closed under products.

Moreover, if these points are true then there are no pairs $\phi > \psi$ of pp-formulae such that $\phi(M)/\psi(M)$ is a non-trivial finite group.

Proof. That the first two points are equivalent is by Exercise 4 after Proposition 2.29 in [Pre88]. The "Moreover"-part is by Exercise 2(i) after Theorem 2.12 also in [Pre88]. $\square$

Fact 5.6.2. (5.36 of [Pre88]) In a complete theory T of modules with quantifier elimination which is also closed under products then following are equivalent;

1. We have $A \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} B$ for $A, B \subset M$ (a model of T).

2. There exists a direct summand $C_1 \bigoplus C_2$ of M with the property that $A \subset C_1$ and $B \subset C_2$.

Note that an abelian group can be considered a $\mathbb{Z}$-module (in the above described language) since $n.x$ can be written as $x + \cdots + x$.

Lemma 5.6.3. *Let T be a $\{+, -, 0\}$-theory of an infinite abelian group with quantifier elimination which is also closed under products. Further let*

$$\phi(x, y) = n.x \doteq y \wedge \neg(y \doteq 0)$$

with $n \in \mathbb{N} - \{0\}$. Then the formula $\phi(x, y)$ has only dependent realisations in x over $\emptyset$, and splits in x over $\emptyset$.

Proof. Since the theory is closed under products we have that $n.x = 0$ has either only the trivial or infinitely many solutions for every $n > 1$ (see last point of Fact 5.6.1). By this and by quantifier elimination we have that $\mathrm{acl}(\emptyset) = \{0\}$. Clearly ϕ satisfies the conditions of Remark 5.5.8 and hence has only dependent realisations in x over $\emptyset$.

It is left to check that the formula splits in x. We take A_1, A_2 (subsets of the monster) with $A_1 \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} A_2$ and take $a, b \models \phi(x, y)$ with $a \in \mathrm{acl}(A_1A_2)$ and $b \in \mathrm{acl}(A_1)$. Now by Fact 5.6.2 we know that

$$\mathrm{acl}(A_1A_2) = \mathrm{acl}(A_1) \oplus \mathrm{acl}(A_2).$$

Hence we have $a = a_1 + a_2$ with $a_1 \in \mathrm{acl}(A_1)$ and $a_2 \in \mathrm{acl}(A_2)$. We have then $b = n.a = n.a_1 + n.a_2$. As $a_2 \in \mathrm{acl}(A_2)$ and $-n.a_1, b \in \mathrm{acl}(A_1)$ holds and as $\mathrm{acl}(A_1) \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} \mathrm{acl}(A_2)$ also holds, we get $-n.a_1, b \mathop{\smile\!\!\!\!\!\!\;|}_{\emptyset} n.a_2$. But as we have that $-n.a_1 + b = n.a_2$ and as $\mathrm{acl}(A_1) \cap \mathrm{acl}(A_2) = \{0\}$ it follows that $n.a_2 = 0$. Hence we have $n.a_1 = b$ and therefore we have shown that the formula is split. $\square$

We will now see an example. $(\mathbb{Z}/4\mathbb{Z})^{\omega}$ satisfies all conditions of the last lemma. More generally one could use the following fact to produce more structures satisfying the assumptions of Lemma 5.6.3.

Fact 5.6.4. (16.3 of [Pre88]) Let T be a theory of an abelian group. Then this theory has quantifier elimination if and only if some model is the direct product of elements of either $\{\mathbb{Q}\} \cup \{\mathbb{Z}_{p^\infty} \mid p \text{ prime}\}$ or of $\{\mathbb{Z}_{p_1^{n_1}}, \dots, \mathbb{Z}_{p_k^{n_k}}\}$ with $p_1, \dots, p_k$ are distinct primes.

Also note that any structure satisfying the assumption of Lemma 5.6.3 does have weak elimination of imaginaries. In fact the following is true;

Fact 5.6.5. (3.10 of [KP92]) Let T be a complete theory of modules with quantifier elimination. Then T weakly eliminates imaginaries.

Example: a finite cover of $(\mathbb{Z}/4\mathbb{Z})^{\omega}$

Now construct a finite cover of $T = Th((\mathbb{Z}/4\mathbb{Z})^{\omega}, +, -, 0)$ which fails amalgamation over some non-empty set, but has total amalgamation over $\emptyset$.

Fix $\mathfrak{C}$ the monster model of T. Note T is totally categorical (see point 5 on page 132 in [Har00]), has elimination of quantifiers (see point 2 on page 134 of [Har00]), has Morley and U-rank 2 (see 1.32 in [Har00]). We have also weak elimination of imaginaries by Fact 5.6.5. Note that elimination of quantifiers implies that every formula $\phi(x_1, \ldots, x_n)$ is equivalent to a Boolean combination of formulae $\sum_{i=1}^{n} z_i x_i \doteq 0$ with $z_i \in \{-2, -1, 0, 1, 2\}$. Moreover, the theory is closed under products as it is the theory of a structure which is an ω-product.

The only positive basic formulae with a single free variable (i.e. atomic formulas and there negations, here equations and there negations) which have non trivial realisations are the formulae of the form $(-)2x \doteq a$. They will have infinitely many solutions if a is of order 2 or 0. As the sets of form $2x \doteq a$ are the cosets of $2x \doteq 0$, we know that any two distinct such sets have empty intersection. Hence any definable (by a conjunction of basic formulae) set with more then one solution will contain a cofinite part of some coset and is therefore infinite. Hence we have "dcl = acl". Also we have that $\mathrm{acl}(\emptyset) = \{0\}$ (this was for example shown in the proof of Lemma 5.6.3). Note that $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$ if and only if $2\mathfrak{C} \cap \mathrm{dcl}(A) \cap \mathrm{dcl}(C) \subset \mathrm{dcl}(B)$. For any $a, b \neq 0$ with $2a = b$ the type $\mathrm{tp}(a/b)$ forks over the empty set.

Proposition 5.6.6. *Let $T = Th((\mathbb{Z}/4\mathbb{Z})^{\omega})$ and let $\phi(x, y)$ be the formula saying "x has order 4 and $2x \doteq y$". Then $T_{2,\phi}$ has total uniqueness over $\emptyset$ and total amalgamation over $\emptyset$, but fails 3-uniqueness over every element of order 2 (in fact over every subset of "$2x \doteq 0$").*

Proof. First note that $T_{2,\phi}$ has weak elimination of imaginaries by Corollary 5.4.5. Since "dcl = acl" as already noted, T has total amalgamation and total uniqueness over every set (see Remark 4.4.4 and Corollary 4.5.7). By Lemma 5.6.3 we know that the formula "x has order 4 and $2x \doteq y$" is split. Hence we can apply the Proposition 5.5.15 and then use Corollary 4.5.9 to get that that $T_{2,\phi}$ has total uniqueness and amalgamation over $\emptyset$. Finally by the "in fact"-part of Proposition 5.5.1 we see that the cover has failure of 3-uniqueness over elements of order 2. $\square$

This finite cover of $(\mathbb{Z}/4\mathbb{Z})^{\omega}$ which has amalgamation over $\emptyset$, but fails it over some parameters, could be the easiest (totally categorical) example we can find (although probably we could take any $(\mathbb{Z}/p^2\mathbb{Z})^\omega$ by Lemma 5.6.3, Fact 5.6.4 and Fact 5.6.5). It should be the easiest as any uncountably categorical structure, which does not interpret an infinite group, is a almost strongly minimal set (for that see Fact 5.1.8). Now as Lemma 5.1.7 suggests there has to be at least some complication if our structure is totally categorical and does not interpret an infinite group. Maybe it is not possible at all to find an almost strongly minimal theory which does not have total uniqueness over the empty set, but fails 3-uniqueness over some set.

5.7 More examples

We will now see an example, which will not interpret an infinite group. By the last paragraph we know then that this example will therefore not be uncountably categorical. But note that this can also be easily seen by a direct argument.

A finite cover of a relational structure

Take the theory T of an equivalence relation E with infinitely many classes of infinite size which has an extra sort added for the equivalence classes, and a map π mapping each element to its corresponding equivalence class.

Take a monster model $\mathfrak{C} = (\mathfrak{C}_n, \mathfrak{C}_e, E, f)$ where $\mathfrak{C}_n$ are the normal elements and $\mathfrak{C}_e$ the corresponding equivalence classes. This theory is ω-categorical and of Morley rank 2. To see ω-categoricity fix two countable models M, N, and enumerate all equivalence classes of each model. Then fix a bijection f_i between the i'th equivalence classes of the two models. Then $\bigcup_{i\in\mathbb{N}} f_i$ induces an isomorphism between the two models. This can be used to show that the assumptions of Lemma 3.5.2 hold and hence we have weak elimination of imaginaries.

Note that "acl = dcl" in this structure, and hence by Remark 4.4.4 and Corollary 4.5.7 it has total amalgamation and total uniqueness over every set of parameters. Also note that $\mathrm{acl}(\emptyset) = \emptyset$. Note that $\mathrm{dcl}(A) = A \cup f(A)$.

The independence notion $A \perp_C B$ is given by

$$\mathrm{dcl}(A) \cap \mathrm{dcl}(B) \subseteq \mathrm{dcl}(C).$$

Therefore any two sets A, B are independent over $\emptyset$ if and only if their definable closures are disjoint in $\mathfrak{C}_e$.

We define $\phi(x, y)$ as $f(x) \doteq y$ and construct the finite cover $T_{2,\phi}$. This theory has weak elimination of imaginaries by Corollary 5.4.5. By looking at the proof of Lemma 5.5.1, we see that it has failure of 3-uniqueness over any element in $\mathfrak{C}_e$. We check that the assumptions of Lemma 5.5.15 are fulfilled. Of course for $\models \phi(a, b)$, we have $b \in \mathrm{dcl}(a)$ and $b \notin \mathrm{dcl}(\emptyset) = \emptyset$. But this already proves that ϕ has only dependent realisations by Anti-Reflexivity of forking.

So it is left to show that ϕ splits. Let A_1, A_2 be two independent sets over $\emptyset$ and let $a, b \models \phi(x, y)$ with $a \in \mathrm{dcl}(A_1A_2)$ and $b \in \mathrm{dcl}(A_1)$. We claim that $a \in \mathrm{dcl}(A_1)$. First note that $a \in A_1 \cup A_2$ as it cannot be in $f(A_1A_2)$ by definition of ψ. If $a \in A_2$, then b would be $\mathrm{dcl}(A_2)$ and hence we would have the following by independence $\mathrm{dcl}(A_1) \cap \mathrm{dcl}(A_2) = \mathrm{dcl}(\emptyset) = \emptyset$. This of course is impossible. Hence we have shown that the formula $\phi(x, y)$ splits in x. So we have found a finite cover with total amalgamation and uniqueness over the empty set, but failure of 3-uniqueness over some parameters, which does moreover not interpret an infinite group.

Compact complex manifold

We will now see a "natural" example where total amalgamation and total uniqueness over the empty set are true, but 3-uniqueness over some parameter set fails.

Definition 5.7.1. Let A be the structure with a sort for each compact complex manifold X, and a relation for each complex analytic subset of a product of sorts. The theory CCM is defined as the first-order theory of A.

The following can be found in [Moo05].

Fact 5.7.2. CCM is a theory that has quantifier elimination, eliminates imaginaries and each sort has finite Morley rank (hence CCM is stable).

Remark 5.7.3. $\mathrm{acl}(\emptyset)$ of CCM is a model of it. This implies that CCM has total amalgamation over $\emptyset$ and total uniqueness over $\emptyset$.

Proof. By definition every tuple $\overline{c}$ of complex numbers has its own sort. Hence by definition for every sort all points of the model A will be named in $\mathrm{acl}(\emptyset)$. For the second point we can use Fact 4.7.1 by stability. $\square$

Fact 5.7.4. (Theorem 2.1 of [BHM15]) The theory of Compact Complex Manifolds (CCM) fails 3-uniqueness over some non-empty set.

5.8 Amalgamation over models in $T_{2,\phi}$

In this section we will investigate if it is possible to get failure of amalgamation over a model in $T_{2,\phi}$. We know by Fact 4.7.1 and Proposition 3.4.5 that it is necessary that a theory which has such a property is not stable. The question if amalgamation over models can fail outside the stable context is already positively answered: see section 5.2 of [GKK13a] for an example of a simple theory which has failure of $B(3)$ over a model.

One possibility to find many examples of this type, could be to use theories which eliminate the quantifier $\exists^{\infty}$. If T is such a theory, we can add a generic predicate P to it (hence work in the theory T_p) as described in [CP98]. Note that in general T_p is not stable even if T was. It should be possible to check that $(T_p)_{2,p}$ fails $B(3)$ over some model. Note that this would even make sense outside the simple context, for example for a geometric T with NTP_2 (see 7.3 of [Che14]).

But this is left to future work, and instead we will work out another approach.

Lemma 5.8.1. *Let T be a rosy theory and M be a model of it. If there exists a formula $\phi(x, b)$ not satisfiable in M, which has some realisation a, such that $a \mathop{\smile\!\!\!\!\!\!|}_M b$, then in $T_{2,\phi}$ Property $B(4)$ fails over M for real parameters. In* *fact, it fails for a_1, a_2, b, a_3 where a_1, a_2, a_3 are any independent realisations* $\text{tp}(a/bM)$.

Proof. We will confuse tuples with singletons. We fix $\phi(x, b)$ not satisfied in some (fixed) M and a realisation a of it with $a \mathop{\smile\!\!\!\!\!\!\,|}_M b$. We construct $T_{2,\phi(x,y)}$ and work in it. We fix a_1, a_2, a_3 some independent realisations $\text{tp}(a/bM)$.

Now we claim that $B(4)$ fails over M for a_1, a_2, b, a_3. Hence we claim that there exists some element $0_{(\{a_1,a_2\},b)}$ which is part of

$$\text{acl}(a_1a_2bM) \cap \text{dcl}(\text{acl}(a_1a_2a_3M), \text{acl}(a_1ba_3M), \text{acl}(a_2ba_3M)) - \\ \text{dcl}(\text{acl}(a_1a_2M), \text{acl}(a_1bM), \text{acl}(a_2bM)).$$

It is clear that the following formula is true in our finite cover

$$R(0_{(\{a_1,a_2\},b)}, 0_{(\{a_1,a_3\},b)}, 0_{(\{a_2,a_3\},b)}).$$

Further as $0_{(\{a_1,a_3\},b)} \in \text{acl}(a_1ba_3)$ and $0_{(\{a_2,a_3\},b)} \in \text{acl}(a_2ba_3)$ we have that $0_{(\{a_1,a_2\},b)} \in \text{dcl}\big(\text{acl}(a_1a_2a_3), \text{acl}(a_1ba_3), \text{acl}(a_2ba_3)\big)$.

Note that as $a_1a_2M \mathop{\smile\!\!\!\!\!\!\,|}_M b$ we have that $b \notin \text{acl}(a_1a_2M)$. This is because we would have $b \in \text{acl}(M)$ and therefore $b \in M$. But this is of course impossible as otherwise $\phi(x, b)$ would have a realisation in M by Tarski's Test. Hence there are no elements of the form $0_{(*,b)}$ (where $*$ represents any 2-set) in $\text{acl}(a_1a_2M)$. Note that as $a_1 \mathop{\smile\!\!\!\!\!\!\,|}_M a_2bM$ we have $a_1 \notin \text{acl}(a_2bM)$. For the same reason as $a_2 \mathop{\smile\!\!\!\!\!\!\,|}_M a_1bM$ we have $a_2 \notin \text{acl}(a_1bM)$. Hence by Lemma 5.4.3 there is some automorphism fixing $\text{acl}(a_1bM)$, $\text{acl}(a_2bM)$ and $\text{acl}(a_1a_2M)$ and mapping $0_{(\{a_1,a_2\},b)}$ to $1_{(\{a_1,a_2\},b)}$. □

Fact 5.8.2. In all NTP_2 theories with the independence property (not NIP) there is a global type $\mathfrak{p}$ non-forking over some model M which is not finitely satisfiable in it (see Section 4 of [CKS16]). There are TP_2 theories such that all global types $\mathfrak{p}$ non-forking over some model M are finitely satisfiable in it (see Section 5.3 of [CKS16]). A simple theory is non-stable if and only if it has the independence property (see for example Remark 2.25 [Cas11]). Every simple theory is NTP_2 (see III.7.11 in [She90]).

Corollary 5.8.3. *In any simple non-stable theory T there exists a formula $\phi(x, y)$ such that $T_{2,\phi}$ fails $B(4)$ over some model for real parameters.*

Proof. As our theory T is simple non-stable, by Fact 5.8.2 there is a global type $\mathfrak{p}$ non-forking over some model M which is not finitely satisfiable in it. We fix some formula $\phi(x, b) \in \mathfrak{p}$ which is not satisfiable in M. Now we have that $\mathfrak{p} \restriction_{Mb}$ does not fork over M and hence for any $a \models \mathfrak{p} \restriction_{Mb}$ we have $a \mathop{\smile\!\!\!\!\!\!\mid}_M b$. Therefore we can apply Lemma 5.8.1 and get that $T_{2,\phi}$ fails $B(4)$ over M. From this it follows by definition of $B(4)$, that $T_{2,\phi}$ fails $B(4)$ over $\mathrm{acl}^{T_{2,\phi}}(M)$. Now we are finished as $T_{2,\phi}$ is a finite cover of T, and hence $\mathrm{acl}^{T_{2,\phi}}(M)$ is a model of $T_{2,\phi}$. $\square$

We have already noted that $B(3)$ can fail for a Morley sequence over some algebraically closed set which is not a model even when the theory is not simple (see Remark 5.5.2). Then next the question is asking in similar fashion if the last corollary is true for forking outside the simple context.

Question 5.8.4. Let T be a theory with a global type $\mathfrak{p}$ non-forking over some model M but not finitely satisfiable in it. Does T have a formula $\phi(x, y)$ such that $T_{2,\phi}$ fails $B(4)$ for some Morley sequence over a model?

For that we first remember Lemma 4.7.2. It says that if types are finitely satisfiable over some model, then $B(n)$ holds for that type. The assertion in Question 5.8.4 cannot be true in some cases; in particular in some cases where the assumption fails. The second point of the Fact 5.8.2 tells us that outside NTP_2 this does not need to work. Also it cannot be pseudofinite NIP. This is due to Exercise 6.19 of [Sim15] which says that in any pseudofinite NIP theory: a global type is non-forking over a model if and only if it is finitely satisfiable in it. But by the first point of the Fact 5.8.2 we know that this is at least true for all NTP_2-theories with the independence property.

Chapter 6

Witnesses of Failure

In this chapter we further analyse the situation when Property $B(n)$ is failing. The notion of n-witness is nothing special, it is just defining some formulae and elements which "witness" the failure of the Property $B(n)$. The notion of a Morley witness is a bit more advanced. It uses the fact that amalgamation over Morley sequences implies general amalgamation. But these two notions are less sophisticated than the notions which inspired them, namely the notion of symmetric witnesses to non-3-uniqueness which appeared in [GK10] and its generalised version called a symmetric witness to the failure of n-uniqueness (see 3.12 of [GKK15]). The versions we use have only the minimal amount of properties needed in this thesis. They will later be used in Chapter 7 and Chapter 8. There we use them to define some algebraic cover with total amalgamation. The more advanced notions in [GK10] and [GKK15] were used to get some definable algebraic structures which were groupoids in the former and polygroupoids in the latter case.

6.1 Localise failure of $B(n)$

The next definition is only some re-writing of Property $B(n)$ in a way which will be more suitable for our later needs in Chapter 7.

Remember by $\phi(\overline{x}, A)$ we mean a formula $\phi(\overline{x}, \overline{a})$ for some tuple $\overline{a}$ of A.

Definition 6.1.1. We fix some theory $T = T^{\mathrm{eq}}$ and some $n \geq 3$. Let the tuple $a_1, \dots, a_n, f_1, \dots, f_n$ consist of elements of the monster model of T. We say that this tuple $a_1, \dots, a_n, f_1, \dots, f_n$ is an *n-witness* over A, if we have that it satisfies the following four properties;

1. $a_1, \dots, a_n$ are independent over A,
2. $f_n \in \mathrm{acl}(a_1 \dots a_{n-1}A) - \mathrm{dcl}\big(\bigcup_{i=1}^{n-1} \mathrm{acl}(a_1 \dots \hat{a}_i \dots a_{n-1}A)\big)$,
3. There exists some formulae

$$\phi_i(x_1, \dots \hat{x}_i \dots, x_n, y; z_i) : 1 \leq i \leq n-1$$

   with some natural number m_i such that we have

$$\models \phi_i(a_1, \dots \hat{a}_i \dots, a_n, A; f_i) \text{ for every } 1 \leq i \leq n-1,$$

   and we have that

$$\models \forall x_1, \dots \hat{x}_i \dots, x_n, y \exists^{<m_i} z \phi_i(x_1, \dots \hat{x}_i \dots, x_{n-1}, y; z_i),$$

4. $f_n \in \mathrm{dcl}(f_1 \dots f_{n-1})$.

Lemma 6.1.2. *Property $B(n)$ over A fails if and only if there exists an n-witness over A.*

Proof. Assume that $B(n)$ over A fails. So this means that there exists some independent $a_1, \dots, a_n$ such that

$$B = \Big(\mathrm{acl}(a_1 \dots a_{n-1}A) \cap \mathrm{dcl}\big(\bigcup_{i=1}^{n-1} \mathrm{acl}(a_1 \dots \hat{a}_i \dots a_n A)\big)\Big) - \mathrm{dcl}\big(\bigcup_{i=1}^{n-1} \mathrm{acl}(a_1 \dots \hat{a}_i \dots a_{n-1}A)\big)$$

is non-empty. Then by definition of dcl and acl there exists

$$f_i \in \mathrm{acl}(a_1 \dots \hat{a}_i \dots a_n A)$$

such that $a_1, \dots, a_n, f_1, \dots, f_n$ is an n-witness over A: Pick any f_n in B and then find the $f_i \in \operatorname{acl}(a_1 \dots \hat{a}_i \dots a_n A)$ accordingly such that $f_n \in \operatorname{dcl}(f_1 \dots f_{n-1})$. Then of course we can find formulae ϕ_i such that the condition 3 is satisfied. Of course if there is an n-witness, then $B(n)$ fails as well, as we can easily reverse the above process. □

6.2 Definition of a Morley n-Witness

Definition 6.2.1. We fix some L-theory $T = T^{\mathrm{eq}}$ and some $n \geq 3$. A *Morley n-witness* over A is a tuple $a_1, \dots, a_n, f_1, \dots, f_n$ (of elements of the monster of T) with the following properties:

1. $a_1, \dots, a_n$ are independent realisations of some type $p \in S(A)$,
2. $f_n \in \operatorname{acl}(a_1 \dots a_{n-1} A) - \operatorname{dcl}\big(\bigcup_{i=1}^{n-1} \operatorname{acl}(a_1 \dots \hat{a}_i \dots a_{n-1} A)\big)$,
3. There exists some L-formula $\phi(x_1, \dots, x_{n-1}, y; z)$ and some natural number m such that we have

$$\models \forall x_1, \dots, x_{n-1}, y \exists^{<m} z \phi(x_1, \dots, x_{n-1}, y; z)$$

   and we have that

$$\models \phi(a_1, \dots \hat{a}_i \dots, a_n, A; f_i) \text{ for every } i \leq n-1,$$

4. $f_n \in \operatorname{dcl}(f_1 \dots f_{n-1})$.

We use the term "Morley" in the last definition, as we have that $a_1, \dots, a_n$ is a finite Morley sequence whenever the theory is stable and A is an $\operatorname{acl}^{\mathrm{eq}}$-closed set.

Remark 6.2.2. A Morley n-witness is an n-witness. To see that just note that we can set $\phi_i = \phi$ for all i with $1 \leq i \leq n-1$.

Remark 6.2.3. Let $a_1, \dots a_n, f_1, \dots, f_n$ be a Morley n-witness over A. Then for any elements $b_1, \dots, b_n, g_1, \dots, g_n$ and B with

$$b_1, \dots, b_n, g_1, \dots, g_n, B \equiv a_1, \dots, a_n, f_1, \dots, f_n, A,$$

we have that $b_1, \ldots, b_n, g_1, \ldots, g_n$ is a Morley n-witness over B. Moreover, we can omit the term "Morley" and the statement still remains valid.

Proof. We need to check that all the properties are preserved under applying some automorphisms. That the b_i's are an independent sequence is clear by invariance of independence under automorphisms. It is also clear that the b_i's satisfy the same type over B. Also the third point is clear as we can use the same ϕ_i's. The fourth point is true as we can find some ψ such that $\psi(f_1, \ldots, f_{n-1}, y)$ define f_n and such that for any $h_1, \ldots, h_{n-1}$ we have that $\psi(h_1, \ldots, h_{n-1}, y)$ it has at most one realisation.

Now it is left to check that the second point remains valid. First note that f_n is in $\mathrm{acl}(a_1 \ldots a_{n-1}A)$ is witnessed by some formula similar to ϕ and hence stays true under automorphisms. It is also true that

$$f_n \notin \mathrm{dcl}\Big(\bigcup_{i=1, i\neq k}^{n} \mathrm{acl}(a_1 \ldots \hat{a}_i \ldots a_{n-1}A)\Big)$$

is preserved under automorphisms. For that note that being algebraic over $a_1, \ldots \hat{a}_i \ldots, a_{n-1}, A$ is preserved under automorphisms. Then for any elements

$$c_1 \ldots c_l \in \Big(\bigcup_{i=1, i\neq k}^{n} \mathrm{acl}(a_1 \ldots \hat{a}_i \ldots a_{n-1}A)\Big)$$

and any formula τ, we have

$$\models \tau(f_n, c_1 \ldots c_l) \to \exists y(\neg(y \doteq f_n) \wedge \tau(y, c_1 \ldots c_l)). \qquad \square$$

6.3 The existence of Morley witnesses

Lemma 6.3.1. *Let $p \in S(A)$ for some set A and let $(a_i : i \leq n)$ be a Morley sequence of realisations of p. Suppose d is in*

$$\Big(\mathrm{acl}(a_1, \ldots, a_{n-1}A) \cap \mathrm{dcl}\big(\bigcup_{i=1}^{n-1} \mathrm{acl}(a_1 \ldots \hat{a}_i \ldots a_n A)\big)\Big) - \mathrm{dcl}\big(\bigcup_{i=1}^{n-1} \mathrm{acl}(a_1 \ldots \hat{a}_i \ldots a_{n-1}A)\big).$$

Then there exists $f_1, \dots, f_{n-1}$ such that $a_1, \dots, a_n, f_1, \dots, f_n$ with $f_n = d$ is a Morley n-witness over A.

Proof. Let $a_1, \dots, a_n, d(= f_n)$ be as given in the assumptions. So point one and two of the witness definition are true. We need to find the elements $f_1, \dots, f_{n-1}$. Note that we can find $e_i \in \operatorname{acl}(a_1 \dots \hat{a}_i \dots a_n A)$ for $(i \leq n-1)$ such that $f_n \in \operatorname{dcl}(e_1 \dots e_{n-1})$.

Further we can find formulae $\chi_i(x_1, \dots, x_{n-1}, y, z_i)$ with $i \leq n-1$ having the following property:

- $\models \forall \overline{x}, y \exists^{<m_i} z_i \chi_i(x_1, \dots, x_{n-1}, y, z_1)$
- $e_i \models \chi_i(a_1 \dots \hat{a}_i \dots a_n, A, z_i)$

We now set $$\phi(x_1, \dots, x_{n-1}, y, z_1 \dots z_{n-1}) = \bigwedge_{i=1}^{n-1} \chi_i(x_1 \dots x_{n-1}, y, z_i).$$

Also set $\overline{f}_i = (f_{(1,i)}, \dots, f_{(n-1,i)})$ to be some realisation tuple of

$$\phi(a_1, \dots \hat{a}_i \dots, a_n, A, z_1, \dots, z_{n-1}) \text{ with } f_{(i,i)} = e_i.$$

So ϕ fulfils the third point of the witness-definition. And as e_i is part of (the tuple) f_i we also have that f_n is in $\operatorname{dcl}(f_1 \dots f_{n-1})$. □

Corollary 6.3.2. *Let T be a stable theory with weak elimination of imaginaries and complete $(n-1)$-uniqueness over all finite sets. Let A be some finite set. Then Property $B(n)$ over A fails if and only if there exists a Morley n-witness over A. Moreover, the statement stays true if we omit the word finite.*

Proof. We have already seen in Lemma 6.1.2 that the existence of an n-witness over A implies failure of $B(n)$ over A. As a Morley n-witness is an n-witness we are done with this direction. If on the other hand the theory fails Property $B(n)$ over A, then by Lemma 4.6.2 we can find a Morley sequence which fails property $B(n)$ over A. Now we can use Lemma 6.3.1 to deduce that there is a Morley n-witness over A. □

Corollary 6.3.3. *Let T be a stable theory with elimination of imaginaries and complete $n-1$-uniqueness over all finite sets. Let A be some finite set. Then n-uniqueness over A fails if and only if there exists a Morley n-witness over A. Moreover, the statement remains valid if we omit the term "finite".*

Proof. We have seen in the proof in Corollary 6.3.2 that there exists a Morley n-witness over A if and only if Property $B(n)$ over A fails. Now by elimination of imaginaries and by Lemma 4.4.6 we have that Property $B(n)$ over A fails if and only if Property $B_{\text{Aut}}(n)$ over A fails. Now by Corollary 4.5.9 we have that Property $B_{\text{Aut}}(n)$ over A fails if and only if n-uniqueness over A fails. $\square$

6.4 Superstable n-witness

Lemma 6.4.1. *Let T be a superstable theory (in fact superrosy is sufficient) and let $a_1, \ldots, a_n, f_1, \ldots, f_n$ be an n-witness over $A (= \operatorname{acl}(A))$. Then there exists some finite $A_0 \subset A$ such that $a_1, \ldots, a_n, f_1, \ldots, f_n$ is an n-witness over A_0.*

Proof. First of all since our theory is superstable, there is some finite $A_0 \subset A$ such that $a_1, \ldots, a_n$ is an independent sequence over A_0. We can add (finitely many) parameters from A to A_0 such that points 2 and 3 of the definition of an n-witness is true for $a_1, \ldots, a_n, f_1, \ldots, f_n$ and A_0. The independence of the sequence over this new A_0 is provided by base monotonicity. The fourth point stays also true (as we did not change f_i). $\square$

Corollary 6.4.2. *A superstable theory with total uniqueness over all finite sets has total uniqueness and amalgamation over all sets.*

Proof. If we have total uniqueness over all sets, then total amalgamation over all sets will be automatic by Lemma 4.2.4. Assume that n-uniqueness fails over some set A with n minimal. Note that we have $n > 2$ by Proposition 4.3.1. Then by Corollary 6.3.3 we know that there exists a Morley n-witness over A. Hence by Lemma 6.4.1 we know that there exists an n-witness over some finite A_0. Now by Lemma 6.1.2 we know that Property $B(n)$ over A_0 fails.

Hence by Lemma 4.4.6 we know that $B_{\mathrm{Aut}}(n)$ over A_0 fails. This then gives failure of n-uniqueness over A_0 by Proposition 4.4.7. But this is of course impossible by total uniqueness over all finite sets. $\square$

6.5 An open question about witnesses

Question 6.5.1. Is it enough, in order to obtain total uniqueness, to check uniqueness over finite sets? Or more precisely, is the following true: a theory that has n-uniqueness over all finite sets, has n-uniqueness over any set.

We have seen that this is true in superstable theories (see Corollary 6.4.2). But what about strictly stable theories? First we establish a lemma to further analyse the situation.

Lemma 6.5.2. *Let T be stable with elimination of imaginaries. Let A be an algebraically closed set, $\phi(x_1, \ldots, x_n)$ be an $L(A)$-formula, $a_1, \ldots, a_n$ a Morley sequence over A such that $\models \phi(a_1, \ldots, a_n)$. Then for any finite subset $B_0 \subset A$, there is some finite subset $A_0 \supset B_0$ of A and a Morley sequence $b_1, \ldots b_n$ over $\mathrm{acl}(A_0)$ such that $\models \phi(b_1, \ldots, b_n)$ and $b_1 \equiv_{\mathrm{acl} A_0} a_1$.*

Proof. We prove this by induction on number of variables x_i. The case $n = 1$ is obvious. Now assume we have proved the result for $n-1$. So take $a_1, \ldots, a_n$ a Morley sequence over some algebraically closed A such that $\models \phi(a_1, \ldots, a_n)$. Also fix some finite $B_0 \subset A$. Now as $\mathrm{tp}(a_n/A, a_1, \ldots, a_{n-1})$ is non-forking over A, it ϕ-types are all definable over $\mathrm{acl}^{\mathrm{eq}}(A)$ and therefore definable over A by elimination of imaginaries (see for example Theorem 8.5.1 of [TZ12]). Hence let $\psi(x_1, \ldots, x_{n-1}) \in L(A)$ be the definition of the $\phi(x_1, \ldots, x_{n-1}; x_n)$-type of the type $\mathrm{tp}(a_n/Aa_1 \ldots a_{n-1})$. Moreover, let C_0 be set of all parameters of ψ. Now by induction for $\psi(x_1, \ldots, x_{n-1})$ there exists a Morley sequence $b_1 \ldots b_{n-1}$ over $\mathrm{acl}(A_0)$ for some finite $A_0 \supset B_0C_0$ such that $\mathrm{tp}(a_1/\mathrm{acl}(A_0)) = \mathrm{tp}(b_1/\mathrm{acl}(A_0))$ and $\models \psi(b_1, \ldots, b_{n-1})$ holds.

Now take the unique global non-forking extension p of $\mathrm{tp}(b_1/\mathrm{acl}(A_0))$. If we show that the global $\phi(x_1, \ldots, x_{n-1}; x_n)$-type q^ϕ of $\mathrm{tp}(a_n/A, a_1, \ldots, a_{n-1})$ is contained in p we are finished. This is because by $\models \psi(b_1, \ldots, b_n)$ we have that $\phi(b_1, \ldots, b_{n-1}, x_n) \in p$ and hence any $b_n \models p \restriction \mathrm{acl}(A_0), b_1, \ldots, b_{n-1}$ will give us the desired Morley sequence $b_1, \dots, b_n$ over $\mathrm{acl}(A_0)$. To see that this ϕ-type is contained in p, it is enough to check that $\mathrm{tp}(a_1/\mathrm{acl}(A_0)) \cup q^\phi$ is non-forking over A_0. But this is clear since it is definable over $\mathrm{acl}(A_0)$, hence non-forking over $\mathrm{acl}(A_0)$ and therefore non-forking over A_0. $\square$

Now we try to show that from a witness over an arbitrary set, we can construct a witness over a finite set. Let $(a_1, \dots, a_n, f_1, \dots, f_n, A)$ be a Morley n-witness. Note that by a slight modification of the Morley witness, we may assume that

$$(a_1, \dots, a_{n-1}, f_n) \equiv_A (a_1, \dots \hat{a}_i \dots, a_n, f_i)$$

for any $1 \leq i \leq n$ (see the Definition 3.12 of [GKK15] for that (this is the so called symmetric witness to the failure of n-uniqueness)). We may By the definition of the witness we may replace our f_i's by the tuples

$$(a_1, \dots \hat{a}_i \dots, a_n, f_i).$$

We now fix $\phi(x_1, \dots, x_{n-1}, y)$ some $L(A)$-formulas as in Point 3 of the Morley n-witness such that $\models \phi(a_1, \dots \hat{a}_i \dots, a_n, f_i)$ holds. Then we require that for all $b_1, \dots, b_n$ and $1 \leq i \leq n$ we have

$$\forall y_1, \dots \hat{y}_i \dots, y_n \exists^{=1} y_i [\phi(b_1, \dots, b_{n-1}, y_n) \wedge \dots \wedge \phi(b_2, \dots, b_n, y_1) \rightarrow \psi(y_1, \dots, y_n)],$$

where $\psi(y_1, \dots, y_n)$ has the following properties: $\models \psi(g_1, \dots, g_n)$, for all elements $g_1, \dots, g_{n-1}$ we have that $\psi(g_1, \dots, x_i, \dots, g_{n-1})$ has either exactly one or no realisations. Now apply the previous lemma to the Morley sequence $a_1, \dots, a_n$ over $\mathrm{acl}(A)$. Let B_0 be the parameters appearing in the formula ϕ. Hence we find $b_1, \dots, b_{n-1}$ a Morley sequence over $\mathrm{acl}(A_0)$ such that $\models \exists y \phi(b_1, \dots, b_{n-1}, y)$ where $A_0 \supset B_0$ is a finite subset of A. Now extend the Morley sequence $b_1, \dots, b_{n-1}$ by b_n. Now it is left to check that $b_1, \dots, b_n, g_1, \dots, g_n, A_0$ for $\models \phi(b_1, \dots \hat{b}_i, \dots, b_n, g_i)$ is a Morley witness. Everything to be checked is quite straightforward except for the following remaining question: Does $\mathrm{tp}(g_n/\mathrm{acl}(b_1 A_0) \dots \mathrm{acl}(b_{n-1} A_0))$ have more than one realisation (at least after adding a finite number of parameters to A_0)? I was not able to answer this question.

Chapter 7

An Algebraic Cover with Total Amalgamation over $\emptyset$

We will now eliminate what Hrushovski named generalised imaginaries in his paper [Hru12]. In our case we will construct a finite cover such that an n-witness will lose property 2 of its definition in this cover. This means it loses its property which corresponds to the failure of Property $B(n)$. For any witness we will take every such cover and add them all to our theory. We repeat this ω-many times in order to get an algebraic cover which does have total amalgamation and total uniqueness over $\emptyset$. Although the result is already known, for that see 4.3 of [Hru12] and the proof of 4.11 of [Hru12] done in [Eva09], the method there used differs from the proof in here. The method used, does establish that certain finite, internal covers split in order to show total amalgamation and total uniqueness over $\emptyset$. In [Hru12] only canonical-2-amalgamation (2-uniqueness) over $\emptyset$ was assumed. Here we will assume that the theory is stable, i.e. has 2-uniqueness over any (algebraically closed) set (see Proposition 4.3.1). However I do not see any obstruction as to why we could not do this kind of construction in any arbitrary first-order theory.

Note: All mentioned in the next two sections is either directly taken from chapter 3 of D. Evans' [Eva09] or is a slight modification of it.

7.1 Construction of a finite cover

The following construction is taken from the proof of ([Hru12], 4.3).

We work in a monster model of a complete stable L-theory T. For our purposes we can assume that L is relational and T has quantifier elimination. Suppose $(\theta_i(x, y_i, z_i) : 1 \leq i \leq m)$ are L-formulae with the property that $\theta_i(a, b, z_i)$ is algebraic with the same amount of realisations for all a, b. If y_i, z_i of $\theta_i(x, y_i, z_i)$ and y_j, z_j of $\theta_j(x, y_j, z_j)$ are of the same sort we replace these formulae by $\theta' = \theta_i \vee \theta_j$. (Note that if $\theta_0(x, y, z)$ is any L-formula and a_0, b_0 are such that $\theta_0(a_0, b_0, z)$ is algebraic with m realisations, realised by c_0, then there is an L-formula $\theta(x, y, z)$ such that $\forall x, y \exists^{=m} z \theta(x, y, z)$, $\models \theta(a_0, b_0, c_0)$ and $\models \theta(x, y, z) \rightarrow \theta_0(x, y, z)$.)

Suppose $p(x)$ is a complete stationary type over $\emptyset$. Fix θ_i (for $1 \leq i \leq m$) as above. Let $M \models T$ and M^* be a sufficiently saturated elementary extension of M (as we work in a stable theory we may assume that both are set-size monster models). Let $a^* \in M^*$ realise $p|M$, the definable extension of p to M.

Define

$$C_i = \Theta(a^*, M) = \{(b, c^*) \mid c^* \in M^*, b \in M \text{ and } M^* \models \theta_i(a^*, b, c^*)\}.$$

Note that by the algebraicity, this only depends on the choice of a^* and not on the choice of M^*. We make the disjoint union $M \cup \bigcup_{i=1}^{m} C_i \cup \{a^*\}$ into a structure $M^+ = C(M, a^*)$ by giving it the induced structure from (M^*, a^*). We define T^+ to be the following theory $\mathrm{Th}(M^+)$.

More formally we let $L^+ \supset L$ be a language with new sorts NC_i for i with $1 \leq i \leq m$, function symbols π_i from NC_i to some L-sorts, a new constant symbol $*$, and for each L-formula R a new relation symbol NR. To make M^+ into an L^+-structure we give M its L-structure, take $NC_i(M^+) = C_i$, define $\pi_i((b, c^*)) = b$, interpret the new constant symbol as a^*, and for a new n-ary relation symbol NR and $e_1, \ldots, e_n \in M^+$ we define that the following holds;

$$M^+ \models NR(e_1, \ldots, e_n) \Leftrightarrow M^* \models R(e_1, \ldots, e_n).$$

We do think about NR as a union of relation $\bigcup_{S \in \mathcal{S}} NR_{\psi,S}$. Where $\mathcal{S}$ is constructed inductively as follows. $\mathcal{S}_0$ contains the sort of R say $S_1, \ldots, S_n$. If we have $(S'_1, \ldots, S'_n) \in \mathcal{S}_i$, and $S'_j = S_j$ has the same sort as y_i, z_i (of θ_i) then replace S_j by NC_j and add this new tuple to $\mathcal{S}_i$. We then repeat this process until $\mathcal{S}$ does not grow anymore. Note that this $\mathcal{S}$ will be finite.

It is clear that if $a^*, a^{**} \models p|M$ then $C(M, a^*)$ and $C(M, a^{**})$ are isomorphic over M (assume M^* is sufficiently homogeneous, and use an automorphism over M which takes a^* to a^{**}). By construction, the map π is finite-to-one.

Lemma 7.1.1. *With the above notation, T^+ is a finite cover of T.*

Proof. We check that the equivalent conditions of 3.1.11 hold. First note that for any saturated model M^+ of T^+, the reduct of M^+ to its L-sorts is a saturated model of T. The functions π_i are finite-to-one and have the new sorts NC_i as its preimage.

Now fix a saturated model M^+ of T^+, name its L-part M. It is left to prove that every L-automorphism of M extends to an L^+-automorphism of M^+. For that we view M^+ as an L-structure $\tilde{M}^+$; first we have deal with the new constant and sort a^* and NC_i. We do think about them as if they were part of the old sorts (so they are of the same sorts as x,y_i and z_i of $\theta_i(x, y_i, z_i)$ are). We interpret each L-formula ψ by interpreting it as the corresponding $\bigcup_S NR_{\psi,S}$.

By definition of T^+ the quantifier-free diagram of $\tilde{M}^+$ is consistent with T, and therefore we can consider $\tilde{M}^+$ as a substructure of the monster model of T. Furthermore, we have that $a_{\tilde{M}^+} \equiv_M a^*$ in T. Now because of that we can extend any automorphism of M to one of the monster model of T and fix $a_{\tilde{M}^+}$. This stabilises the set of M^+ and further preserves the L^+-structure. Hence it can be considered an L^+-automorphism of M^+. □

7.2 The finite cover eliminates a witness

Definition 7.2.1. If T, p and θ_i for $1 \leq i \leq m$ are as above, we denote by $T_{p,\theta_i:1\leq i\leq m}$ the L^+-theory of $C(M, a^*)$, where M is an ω-saturated model of

T and $a^* \models p|M$. We will also refer to this as a *definable finite cover* of the theory T.

In the last section we proved that we can construct a certain finite cover. The following proposition combines the above construction and the n-witness of failure of amalgamation. Hence here our proof starts to differ from [Hru12] and [Eva09].

Proposition 7.2.2. *Let $T(= T^{\mathrm{eq}})$ be a stable theory. Let A be some algebraically closed set, let $a_1, \ldots, a_n$ be an independent sequence over A and let d be any element of the set*

$$\Big(\mathrm{acl}(a_1 \ldots a_{n-1}A) \cap \mathrm{dcl}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}(a_1 \ldots \hat{a}_j \ldots a_n A)\big)\Big) - \mathrm{dcl}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}(a_1 \ldots \hat{a}_j \ldots a_{n-1})\big).$$

Then there exists a finite cover $(T_A)^+$ of T_A such that

$$d \in \mathrm{dcl}^{(T_A)^+}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}^{(T_A)^+}(a_1 \ldots \hat{a}_j \ldots a_{n-1})\big).$$

By $\mathrm{dcl}^{(T_A)^+}$ and $\mathrm{acl}^{(T_A)^+}$ we mean the evaluation of acl *and* dcl *in the theory $(T_A)^+$.*

Proof. We will work in T_A^{eq}. By Lemma 6.1.2 we can fix an n-witness $a_1, \ldots, a_n,\ f_1, \ldots, f_n$ over $\emptyset$ with $d = f_n$. Let $\phi_i(x_1, \ldots \hat{x}_i \ldots, x_{n-1}; z_i)$ for all i with $1 \le i \le n-1$ be the formulae satisfying condition 3 of the witness definition. Now extend T to the finite cover $T_{p,\phi_i:1\le i\le n-1}$ as defined in 7.2.1 with $p = \mathrm{tp}(a_n)$. We may assume that $a_n = a^*$ by Remark 6.2.3, where a^* is the new generic constant of the finite cover.

Then because $T \models \phi_i(a_1, \ldots \hat{a}_i \ldots, a_n; f_i)$, we have that f_i is in the finite set $\pi^{-1}(a_1, \ldots \hat{a}_i \ldots, a_{n-1})$. This then gives that

$$f_i \in \mathrm{acl}^{M^+}(a_1, \ldots \hat{a}_i \ldots, a_{n-1}, A).$$

Now there is a formula $\psi(y_1, \ldots, y_{n-1}; y_n)$ such that $T \models \psi(f_1, \ldots, f_{n-1}; f_n)$ and $T \models \exists^{=1} y \psi(f_1, \ldots, f_{n-1}; y)$. Note $NR_{\phi, NC_1 \ldots NC_{n-1}, S'}$ denotes the new relation symbol (coming from ϕ) of sorts $(NC_1, \ldots, NC_{n-1}, S')$. Hence if S' is the sort of f_n, we have that

$$NR_{\phi, NC_1 \ldots NC_{n-1}, S'}(f_1, \ldots, f_{n-1}; y)$$

isolates f_n and therefore shows that $f_n \in \mathrm{dcl}^{T^+}(f_j : 1 \leq j \leq n-1)$. This shows that

$$d \in \mathrm{dcl}^{M^+}\big(\bigcup_{i=1}^{n-1} \mathrm{acl}^{M^+}(a_1 \ldots \hat{a}_i \ldots a_{n-1})\big).$$

Hence $T_{p, \phi_i : 1 \leq i \leq n-1}$ is the finite cover we are looking for. $\square$

7.3 Eliminate all witnesses

The next theorem will be proved by adding every possible finite cover (as described above in this chapter) and then repeat this process ω-many times.

Theorem 7.3.1. *Let T be stable. There exists an algebraic cover T^* of $T^{\mathrm{eq}}_{\mathrm{acl}^{\mathrm{eq}}(\emptyset)}$ which has total amalgamation and uniqueness over $\emptyset$.*

Proof. First a brief description of the proof. Add any possible finite cover constructed in the first section of this chapter to our theory. Then use the Proposition 7.2.2 to note that $B(n)$ over $\emptyset$ is true for any independent sequence of old sort. Then repeat this process ω-many times to eliminate any malicious behaviour (in terms of failure of $B(n)$) in any of these new algebraic covers.

We start the real proof. We may assume that $T = T^{\mathrm{eq}}_{\mathrm{acl}^{\mathrm{eq}}(\emptyset)}$. For that first note that we only need to prove total uniqueness over $\emptyset$ by Proposition 4.2.3. Denote the language of T by L_0. Fix a set-size monster-model M of T. We will construct a chain $(M_i : i \in \omega)$ of algebraic covers of M with language L_i such that $M_i = (M_i^{\mathrm{eq}})_{\mathrm{acl}^{\mathrm{eq}}(\emptyset)}$, $M_0 = M$ and M_i is a monster model. Note that it will be enough to prove that M_i is an algebraic cover of M_{i-1} as then by induction it follows that M_i is an algebraic cover of M. Note that the requirement $M_i = (M_i^{\mathrm{eq}})_{\mathrm{acl}^{\mathrm{eq}}(\emptyset)}$ can be made true as it holds for M_0 and we can just go over to $(M_i^{\mathrm{eq}})_{\mathrm{acl}^{\mathrm{eq}}(\emptyset)}$ (and preserve that it is an algebraic cover) inductively by Lemma 3.3.1 and Lemma 3.3.3.

We construct all finite covers $(\hat{M}_i)_{p,\phi_i:1\leq i\leq m}$ for some type p in $S_{M_i}(\emptyset)$ and $\phi_i(x,y,z) \in L_i$ some formulae such that there is a $k_i \in \mathbb{N}$ such that for any $b, a \in M_i$ we have $M_i' \models \exists^{=k}\phi(b,a,z)$. Then join all these covers $(M_i)_{p,\phi_i:1\leq i\leq m}$ together to a new structure M_{i+1}, by making the languages disjoint over L_i', i.e. use $\coprod_M$ of Definition 3.1.12. Let T_{i+1} be the theory of M_{i+1}. This structure is a monster model as it is saturated by Lemma 3.4.1. By Proposition 7.2.2 we have that for any M_i the following holds (*): for any independent sequence $a_i : 1 \leq i \leq n$ in M_i (which is from $S_{M_i'}(\emptyset)$) and any d in the set

$$\Big(\mathrm{acl}^{M_i}(a_1 \dots a_{n-1}) \cap \mathrm{dcl}^{M_i}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}^{M_i}(a_1 \dots \hat{a}_j \dots a_n)\big)\Big) - \mathrm{dcl}^{M_i}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}^{M_i}(a_1 \dots \hat{a}_j \dots a_{n-1})\big),$$

we have

$$d \in \mathrm{dcl}^{M_{i+1}}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}^{M_{i+1}}(a_1 \dots \hat{a}_j \dots a_{n-1})\big).$$

Take $M^* = \bigcup_{i\in\mathbb{N}} M_i$ and T^* its theory. M^* is an algebraic cover of M. It has elimination of imaginaries (in fact $M^* = (M^*)^{\mathrm{eq}}$), as any imaginary of M^* is already an imaginary of some M_i and hence an element of M_{i+1}. We claim that T^* has complete amalgamation for every n. It will be enough to prove it for the model M^* since it is saturated by Lemma 3.4.1.

By Corollary 4.5.7 it is left to check that Property $B(n)$ over $\emptyset$ holds for every n to finish the proof. So take d in

$$\Big(\mathrm{acl}^{M^*}(a_1 \dots a_{n-1}) \cap \mathrm{dcl}^{M^*}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}^{M^*}(a_1 \dots \hat{a}_j \dots a_n)\big)\Big)$$

for $a_1, \dots, a_n$ be some independent sequence in M^*. Now we have that these

$a_1, \dots, a_n$ are part of some M_j. We can then find some M_i (with $i \geq j$) such that d is in

$$\Big(\mathrm{acl}^{M_i}(a_1 \dots a_{n-1}) \cap \mathrm{dcl}^{M_i}\big(\bigcup_{j \neq n} \mathrm{acl}^{M_i}(a_1 \dots \hat{a}_j \dots a_n)\big)\Big)$$

but then as already noted we have

$$d \in \mathrm{dcl}^{M_{i+1}}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}^{M_{i+1}}(a_1 \dots \hat{a}_j \dots a_{n-1})\big). \qquad \square$$

Corollary 7.3.2. *Let T be a stable theory with a separable forking. Then $T^{\mathrm{eq}}_{\mathrm{acl}^{\mathrm{eq}}(\emptyset)}$ has an algebraic cover which has total amalgamation and uniqueness over every set.*

Proof. Use Theorem 7.3.1 and Theorem 5.1.10 to conclude that the theory has an algebraic cover with total uniqueness over every set. Now total amalgamation follows from Proposition 4.2.3. $\square$

Remark 7.3.3. Let T be rosy with 2-uniqueness over the empty set. Then the above construction should work as well in there. If this theory does not have 2-uniqueness we still should be able to use it in order to construct an algebraic cover with Property $B(n)$ over the empty set.

Chapter 8

ω-categorical Algebraic Cover with Total Amalgamation

In the last chapter we constructed an algebraic cover with total amalgamation (and uniqueness) over $\emptyset$. Denote this cover as T^{geq} as for generalised imaginaries sorts (see [Hru12]). We have seen in Chapter 6 that this construction does not necessarily guarantee that total amalgamation (and uniqueness) over arbitrary set holds. In fact, by looking at the cover of $(\mathbb{Z}/4\mathbb{Z})^{\omega}$, i.e. Example 5.6, we can see that this is not true. The problem here is that realisations of the global non-forking (over $\emptyset$) types will not produce an element a with $2a = c$ with c non generic and failure of 3-uniqueness persists. But at least 2-uniqueness of stable theories suggests that this should be true: It is easy to see that $(T^{\mathrm{eq}})_A$ is essentially the same as $(T_A)^{\mathrm{eq}}$. So the natural question is if we can, by altering the construction of the algebraic cover in the last chapter, have some T^{geq} such that $(T^{geq})_A$ and $(T_A)^{geq}$ are the same. In this chapter we will produce an algebraic cover of an ω-categorical stable theory such that $(T^{geq})_A$ and $(T_A)^{geq}$ have the same higher amalgamation properties. To see what we need to do, we take a look at Proposition 7.2.2. In that Proposition we can see, that the construction of our finite cover was dependent on the type p_0 and the formulae ϕ_i. We will get rid of the dependency on p_0 and make our construction only dependent on the formula ϕ.

8.1 Construction of a cover

The construction in this section is very similar to that of the last chapter.

We fix a stable L-theory $T(= T^{\mathrm{eq}})$. We further fix a saturated $M \models T$ of size $|M| \geq |T|^+$. This is in fact a set size monster model. Further, fix a formula $\phi(x, y, z)$ such that for some fixed $m \in \mathbb{N}$ we have that for any b, c the formula $\phi(b, c, z)$ has either m realisations or is not satisfiable at all. For any b which has the same sort as the variable x in ϕ, we define

$$C_b = \{q \in S_y(M) \mid \exists^{=m} z \phi(b, y, z) \in q, q \text{ non-forking over } b\}.$$

Further let

$$C_M = \dot{\bigcup}_{b \in M} C_b.$$

Now fix one realisation a_p for each $p \in C_M$ such that they form an independent set over M. We further fix the set A consisting of the pairs (b, a_p) where a_p is the generic (over M) element and b the corresponding parameter where the type p came from. Then we define

$$C_A = \{(b, a, c) \mid (b, a) \in A, c \models \phi(b, a, z)\}.$$

We are ready to define an extension M^+ of M. Extend L to L^+ with a new sort NA, a function symbol π from NA to an L-sort, a new sort NC and a function symbol ρ from NC to NA. Further for every L-formula $\psi(x_1, \ldots, x_n)$ and a tuple of sorts $S \in \mathcal{S}_\psi$ (we define $\mathcal{S}_\psi$ in the next paragraph) a new relation symbol $NR_{\psi,S}$.

We define $\mathcal{S}_\psi$ inductively as follows. Let the variables of ψ be $S_1, \ldots, S_n$. We set $\mathcal{S}_\psi = \{(S_1, \ldots, S_n)\}$ in the beginning. If S_i is of the same sort as x, y (or x, y, z) of $\phi(x, y, z)$, then S_i will be replaced by the new sort NA (or NC). Add this new tuple to $\mathcal{S}$. Repeat until $\mathcal{S}_\psi$ does not grow any more.

Make M^+ a L^+ model with M its L-part by setting $NA(M^+) = A$, $\pi(b, a) = b$, $NC(M^+) = C_A$, $\rho(b, a, c) = (b, a)$, and the $NR_{\psi,S}$ on M^+ will be interpreted as ψ would be in the monster model of T, i.e. for

$$e_1, \ldots, e_n \in (M^+)^n \cap (S_1, \ldots, S_n)$$

let $M^+ \models NR_{\psi,S}(e_1, \ldots, e_n)$ if and only if $\mathfrak{C} \models \psi(e_1, \ldots, e_n)$ where $\mathfrak{C}$ is the monster of T. We name the corresponding theory T^+.

If we are given a model M and the set A, then C is uniquely determined by them. We write $M^+ = C(M, A)$, if it is constructed out of M and A. Further the interesting part of our construction depends only on the formula $\phi(x, y, z)$. To indicate that the construction was done using ϕ, we denote T^+ by T^+_ϕ and M^+ as M^+_ϕ.

Let A, M be fixed set-wise by some automorphism f of the monster of T. Then its values on A are determined by the values of the automorphism on M. Obviously this will also fix the set C set-wise.

Lemma 8.1.1. *We can extend every automorphism $f_0 \in Aut(M)$, to some automorphism of the monster of T which fixes A set-wise.*

Proof. We take some automorphism f_0 of M. We enumerate the set A. Without loss let (b_α, a_α) with $\alpha < \kappa)$ be the enumeration. We inductively construct elementary maps f_α with $\alpha \leq \kappa$ (starting with f_0) such that the domain of f_α is $M \cup \{a_\beta \mid \beta < \alpha\}$ and the image of f_α is part of $M \cup A$. Moreover we require that for any ordinals $\alpha' < \alpha \leq kappa$ we have

$$f_\alpha \upharpoonright M \cup \{a_\beta \mid \beta < \alpha'\} = f_{\alpha'}.$$

So let all f_β such that $\beta < \alpha$ be constructed, where α is some ordinal less or equal to κ. If α is a limit ordinal, then set $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. So take α with $\alpha = \beta + 1$. We can extend f_β to some automorphism of the monster model say f. By invariance under automorphisms now we have that $f(A)$ is also an independent set over M. Let $f((b_\beta, a_\beta)) = (b', a')$. By invariance under automorphisms $a_\beta \perp_{b_\beta} M$ implies $a' \perp_{b'} M$. Moreover we have $a_\beta \perp_M (a_\gamma : \gamma < \beta)$ and hence $f(a_\beta) \perp_M f(a_\gamma : \gamma < \beta)$ by invariance under automorphism. Hence by transitivity of forking there is some $p \in S(M \cup f_\beta(\{a_\gamma \mid \gamma < \beta\}))$ which is non forking over over the set b' and contains $\exists z\phi(b', y, z)$ with $a' \models p$. Hence by construction of A there is some $(b', a'') \in A$ with $(b', a'') \equiv_{M \cup f_\beta(\{a_\gamma \mid \gamma < \beta\})} (b', a')$. This means that there is an automorphism f'_α mapping (b', a') to (b', a'') and fixing

$$M \cup f_\beta(\{a_\gamma \mid \gamma < \beta\}).$$

Now we set f_α to be the restriction of the automorphism $f \circ f'_\alpha$ to

$$M \cup \{a_\beta \mid \beta < \alpha\}. \qquad \square$$

8.2 Verifying the finiteness of the cover

In general stable theories it will not be true that this construction is a finite cover. The problem is that the set C_b will not be finite and therefore π will not be a finite-to-one function. So to analyse the situation in the general context could be to leave the first order context. Another solution would be to work with this non-saturated structure. If we want to have a finite cover in general, one has to make a choice in C_b. Then of course we need some additional assumptions to have something similar to Lemma 8.1.1. Maybe additional symmetry assumptions of the formula ϕ (coming from a more sophisticated Morley witness) could be the solution. But this is all left to future work, and instead we make additional assumptions:

From now on we assume that T is countable, ω-categorical and stable for the rest of this chapter.

Hence by Proposition 2.7.9 we have that C_b is finite for every b. The reader may keep in mind that this is the only time where ω-categoricity is used, hence we may take any theory such that all C_b are (boundedly) finite. This then gives us that π and therefore $\pi \circ \rho$ are finite-to-one maps.

Remark 8.2.1. Because the maps are bounded finite-to-one the following trivially holds in T^+: For any element (b, a, c) in the NC-sort, we have $(b, a, c) \in \mathrm{acl}^{T^+}(\pi \circ \rho((b, a, c)))$.

Lemma 8.2.2. *T^+ is a finite cover of T.*

Proof. We do similar things as in Lemma 7.1.1. We check that the equivalent conditions of Lemma 3.1.11 hold. First note that for any saturated model M^+ of T^+, the reduct of M^+ to its L-sorts is a saturated model of T. The function π and $\rho \circ \pi$ are both finite-to-one and have the new sorts as their preimage. Now fix a saturated model M^+ of T^+, name its L-part M. It is left to prove that every L-automorphism of M extends to an L^+-automorphism of M^+.

For that we view $M^+ = C(M, A)$ as an L-structure $\tilde{M}^+$: first we view the new sorts NA and NC, as part of the old sorts, i.e. the sort of x, y and the sort of x, y, z of $\phi(x, y, z)$. We interpret each L-formula ψ by interpreting as the corresponding $\bigcup_S NR_{\psi,S}$. By definition of T^+ the quantifier-free diagram of $\tilde{M}^+$ is consistent with T, and therefore we can consider $\tilde{M}^+$ as a substructure of the monster of T. Furthermore, we have that $A_{\tilde{M}^+} \equiv_M A$ in T. Now because of that we can extend any automorphism of M to one of the monster of T and fix $A_{\tilde{M}^+}$ point-wise (see Lemma 8.1.1). This stabilises the set of M^+ then and further preserves the L^+-structure and is hence an automorphism of M^+. $\square$

8.3 The finite cover eliminates witnesses

In similar fashion as in Proposition 7.2.2 we will combine now the finite cover above and the Morley n-witness.

Theorem 8.3.1. *Let T be a stable and ω-categorical L-theory. Let b be some finite tuple, let $(a_i : i \leq n)$ be an independent sequence of $p \in S(b)$ and let the element d be in the following set,*

$$\Big(\mathrm{acl}(a_1 \dots a_{n-1} b) \cap \mathrm{dcl}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}(a_1 \dots \hat{a}_j \dots a_n b)\big)\Big) - \mathrm{dcl}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}(a_1 \dots \hat{a}_j \dots a_{n-1} b)\big).$$

Then we can find a finite cover T^+ of T such that

$$d \in \mathrm{dcl}^{T^+}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}^{T^+}(a_1 \ldots \hat{a}_j \ldots a_{n-1}b)\big).$$

Proof. Use Lemma 6.3.1 to find a Morley witness $a_1, \ldots, a_n, f_1, \ldots, f_n$ over b such that $d = f_n$. For our purposes we may assume that a_n is generic over M, i.e. a_n realises some non-forking extension of p with parameter set M. For the $\phi(x_1, \ldots, x_{n-2}; x_{n-1}, y_1; y)$, which are given by condition 3 of the Morley witness, construct a finite cover $C(M, A)$ of a model M which contains $a_1, \ldots, a_{n-1}$. We can construct the cover such that $(b, a_n) \in A$ as we already assumed that a_n is generic.

Therefore we have $f_i \in \mathrm{acl}^{T^+}(a_1, \ldots \hat{a}_i \ldots, a_{n-1}, b)$: to see this let S be the old sort of a_i. Then as $NR_{\phi,S\ldots S,NA,NC}$ is interpreted as ϕ in T, we have that $(a_1, \ldots \hat{a}_i \ldots, a_{n-1}; a_n, b; f_i) \models NR_{\phi,S\ldots S,NA,NC}$ and also that the following formula is algebraic

$$NR_{\phi,S\ldots S,NA,NC}(a_1, \ldots \hat{a}_i \ldots, a_{n-1}; a_n, b; y).$$

Now there is a formula $\psi(y_1, \ldots, y_{n-1}; y_n)$ in L such that we have

$$T \models \psi(f_1, \ldots, f_{n-1}; f_n) \text{ and } T \models \exists^{=1} y \psi(f_1, \ldots, f_{n-1}; y).$$

Hence if S' is the sort of f_n we have that

$$NC_{\psi,NC\ldots NC,S'}(f_1, \ldots, f_{n-1}; y)$$

showing that $f_n \in \mathrm{dcl}^{T^+}(f_j : j \neq n)$. We have already shown that

$$f_i \in \mathrm{acl}^{T^+}(a_1 \ldots \hat{a}_i \ldots a_{n-1}b).$$

Hence this gives use what we required, namely that

$$f_n \in \mathrm{dcl}^{T^+}\big(\bigcup_j \mathrm{acl}^{T^+}(a_1 \ldots \hat{a}_j \ldots a_{n-1}b)\big). \qquad \square$$

8.4 Eliminate all witnesses

Similar to Theorem 7.3.1 the next theorem will be proved by adding every possible finite cover (of the covers described above in this chapter) and then iterating this process ω-many times.

Theorem 8.4.1. *Let T be a stable and ω-categorical theory. There exists an algebraic cover T^* which has total amalgamation and total uniqueness over all finite sets.*

Proof. Denote the language of $T(= T^{\mathrm{eq}})$ by L. Fix a set-size monster-model M of T. We will construct an increasing chain (in terms of "$\subset$") of L_i-structures $(M_i : i \in \omega)$ of algebraic covers (with countable language) with $M = M_0$. We may assume that $M_i = M_i^{\mathrm{eq}}$ by Lemma 3.3.1.

So let M_i be constructed. Construct all algebraic covers $(\hat{M_i})_\phi$ for all formulas $\phi(x, y, z) \in L_i$ for which there exists some $k \in \mathbb{N}$ such that for any $b, a \in M_i$ we have

$$M_i \models \exists z \phi(b, a, z) \rightarrow \exists^{=k} z \phi(b, a, z).$$

Then join all these covers $(\hat{M_i})_\phi$ together to a new structure M_{i+1}, by making the languages disjoint over L, i.e. use the $\coprod_M$ operator of Definition 3.1.12. Let T_{i+1} be the theory of M_{i+1}.

By Proposition 8.2.2 it is clear that for any M_i the following holds (*): for any finite set of parameters B, for any independent sequence $(a_i : i \leq n)$ of realisations of a type $p \in S(B)$ and any d in the set

$$\Big(\mathrm{acl}^{M_i}(a_1 \dots a_{n-1} B) \cap \mathrm{dcl}^{M_i}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}^{M_i}(a_1 \dots \hat{a}_j \dots a_n B)\big)\Big) - \mathrm{dcl}^{M_i}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}^{M_i}(a_1 \dots \hat{a}_j \dots a_{n-1} B)\big),$$

we have that this d is in the following set

$$\mathrm{dcl}^{M_{i+1}}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}^{M_{i+1}}(a_1 \dots \hat{a_j} \dots a_{n-1} B)\big).$$

We define $M^* = \bigcup_{i\in\mathbb{N}} M_i$ and let T^* be its theory. This M^* is an algebraic cover of M. Note that it has already elimination of imaginaries, as any imaginary of M^* is already an imaginary of some $M_i(= M_i^{\mathrm{eq}})$. We claim that T^* has total amalgamation and total uniqueness over all finite set. It will be enough to prove it for the model M^* since it is saturated by Lemma 3.4.1.

To finish the proof note that by Corollary 4.6.3 it is enough to check that for every Morley sequence $a_1, \dots, a_n$ over any finite set B Property B_n holds. So let $a_1, \dots, a_n$ be some independent sequence (over B) in M^* such that the following set is non-empty;

$$\mathrm{acl}^{M^*}(a_1 \dots a_{n-1}B) \cap \mathrm{dcl}^{M^*}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}^{M^*}(a_1 \dots \hat{a}_j \dots a_n B)\big).$$

We fix some element d of this set. Now we have that these $a_1 \dots a_n, B$ are part of some M_j. We can then find some M_i (with $i \geq j$) such that d is in

$$\mathrm{acl}^{M_i}(a_1 \dots a_{n-1}B) \cap \mathrm{dcl}^{M_i}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}^{M_i}(a_1 \dots \hat{a}_j \dots a_n B)\big).$$

But then as already noted in Theorem 8.3.1 we have

$$d \in \mathrm{dcl}^{M_{i+1}}\big(\bigcup_{j=1}^{n-1} \mathrm{acl}^{M_{i+1}}(a_1 \dots \hat{a}_j \dots a_{n-1}B)\big). \qquad \square$$

Corollary 8.4.2. *Let T be a superstable (in fact ω-stable) and ω-categorical theory. There exists an algebraic cover T^* which has total amalgamation over all sets and total uniqueness over all sets.*

Proof. Take the algebraic cover of Theorem 8.4.1 to get total uniqueness over all finite sets. Then apply Corollary 6.4.2 to finish the proof. $\square$

Of course we should ask whether the last result extends to general superstable or even strictly stable theories.

Question 8.4.3. Does every (super)stable theory have an algebraic cover with total amalgamation and total uniqueness over all (finite) sets?

Outside the stable context, formulating a good question becomes more difficult. We remember for example Corollary 5.1.2. From there we see that we cannot find a non-stable theory, which has a separable independence notion and 2-uniqueness over the empty set. This is due to the fact that an algebraic cover is stable if and only if the old theory is stable. Hence in this situation in which we lack 2-uniqueness, we cannot hope to apply our techniques in order to construct an algebraic cover with 3-uniqueness, at least, not without seriously modifying our construction.

We could still ask positively;

Question 8.4.4. For any theory T with k-uniqueness over the empty set, does there exist an algebraic cover of T such that it has n-uniqueness over the empty set for all $n \geq k$?

We could ask this in terms k-uniqueness over parameters.

Question 8.4.5. For any theory T with k-uniqueness over some set A, does there exist an algebraic cover of T such that it has n-uniqueness over A for all $n \geq k$?

Maybe it is not true for the very first of such k, but instead from some point on. Also we could also try to answer these questions above, instead of the case of n-uniqueness, for the Property $B(n)$. See also Remark 7.3.3 for this. On the other hand if one is a pessimist by nature, then one would probably ask for a malicious structure.

Question 8.4.6. Does there exist a structure which fails k-uniqueness (over the empty set) and such that any algebraic cover of this structure fails k-uniqueness as well?

Or an infinite version of this question:

Question 8.4.7. Does there exist some structure which fails l-uniqueness (over empty set) for infinite many l's and such that any algebraic cover of it still fails l-uniqueness (over empty set) infinitely many times?

And then finally something more similar to Question 5.3.2.

Question 8.4.8. Does there exist some structure which has k-uniqueness (over empty set) for infinite many k's and also fails l-uniqueness (over empty set) for infinite many l's and such that any algebraic cover of it still fails l-uniqueness (over empty set) infinitely many times?

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