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Loop Numbers of Knots and Links

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LOOP NUMBERS OF KNOTS AND LINKS

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LOOP NUMBERS OF KNOTS AND LINKS

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To Claus, my mentor.

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Directed by: Dr. Claus Ernst, Dr. Uta Ziegler, and Dr. Lan Nguyen

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This thesis introduces a new quantity called loop number, and shows the conditions in which loop numbers become knot invariants.

For a given knot diagram D, one can traverse the knot diagram and count the number of loops created by the traversal. The number of loops recorded depends on the starting point in the diagram D and on the traversal direction. Looking at the minimum or maximum number of loops over all starting points and directions, one can define two positive integers as loop numbers of the diagram D. In this thesis, the conditions under which these loop numbers become knot invariants are identified. In particular, the thesis answers the question when these numbers are invariant under flypes in the diagram D.

Chapter 1

Motivating Problem

Though not the first mathematician formally working on knot theory, Peter Tait tabulated the first knot table and established many backbone results based on which modern knot theory has been built upon. One of Tait's works, proven in 1990 but still historically called the Tait Flyping Conjecture, states that any two diagrams of the same alternating link are related by a sequence of flypes [**10**]. As the number of knots and links increases exponentially as the function of their crossings number n [**6**], deriving whether two knots K_1 and K_2 are equivalent or actually distinct becomes more and more difficult with increasing crossing number. To aid with the task of distinguishing knots, knot invariants are introduced.

A knot invariant is a quantity that can be computed for any two knots K_1 and K_2 and if the computation of the invariant yields different results, then K_1 and K_2 are different knots. This thesis introduces a new quantity called loop numbers.

If one shows that loop numbers are invariant under flypes for some families of knots and links, then loop numbers become knot invariants by the Tait flyping theorem. Thus, loop numbers can be used to tell these knots and links apart.

Chapter 2

Background and Definitions

Many of the definitions and results in this chapter can be found in any knot theory textbook, such as [**2, 4, 7, 8**]. However, to make this thesis self contained, most terms used throughout the thesis are defined in this chapter.

2.1. Knot theory basics

Definition 2.1. *A link L of n components is a subset of $\mathbb{R}^3$ that consists of n disjoint, piecewise linear, simple closed curves. A link of one component is a knot.*

Conventionally, by a link L, we mean a knot or link. And by a knot K, we mean only a knot.

Definition 2.2. *Two links L_1 and L_2 are equivalent in $\mathbb{R}^3$ if there exists an isotopy H deforming L_1 into L_2.*

In particular, given L_1 and $L_2 \in \mathbb{R}^3$, there exists a function H, such that

$$H : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3.$$

We denote h_t as H restricted to the domain $\mathbb{R}^3 \times \{t\}$, for $t \in [0,1]$.

h_t is an orientation preserving homeomorphism of $\mathbb{R}^3$, wherein

$$h_0(L_1) = L_1$$

$$\text{and } h_1(L_1) = L_2.$$

This signifies a continuous deformation of a link L_1 into L_2. At $t = 0$, h_t is the identity function. At $t = 1$, L_1 has been deformed into L_2.

Definition 2.3. *Each equivalence class of links is called a link type.*

Definition 2.4. *A link invariant I is defined as a quantity $I(L)$ assigned to a link L (or a link diagram D) such that if L_1 and L_2 are of the same link type, then $I(L_1) = I(L_2)$.*

Definition 2.5. *A link diagram D is a projection of the link L into $\mathbb{R}^2$. In the diagram, we use the under- and over- information at a crossing to refer to the relative heights above $\mathbb{R}^2$ of the arcs belonging to a crossing in the inverse image.*

Definition 2.6. *A diagram D is regular if no three points on the link project to the same point on D.*

Then a regular diagram D can be viewed as a 4-regular planar graph.

Definition 2.7. *A stereographic projection is a mapping that projects a sphere onto a plane. The mapping is defined on the entire sphere, except at the projection point.*

Definition 2.8. *An alternating diagram of link is a regular projection whose crossings alternate between under- and overpasses when traversing along the diagram. Links which have alternating diagrams are called alternating links.*

Definition 2.9. *A crossing C in an alternating diagram D is called nugatory if D can be drawn as in Figure 2.1.1.*

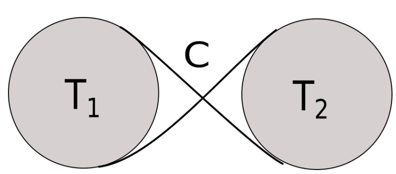

Figure 2.1.1. The diagram D has a nugatory crossing, and thus, D is not reduced.

An alternating diagram D is reduced if it contains no nugatory crossing.

Not all links are alternating. This thesis focuses on alternating diagrams of knots. For the rest of the thesis, we assume all diagrams are 4-regular and alternating. In particular, we often do not indicate the over and under passes at crossings in our diagrams, since there are only two ways to change such a picture into a knot diagram that is alternating, and which of the two is irrelevant for our consideration.

DEFINITION 2.10. *The crossing number of a link L is the minimum number of crossings needed in a regular projection of L.*

DEFINITION 2.11. *The orientation of a link diagram is a direction on each component indicated by a small arrow in the diagram. A group of half twists in an oriented diagram can be categorized as parallel or antiparallel.*

Figure 2.1.2.a shows an antiparallel orientation of a group of 3 half twists, and Figure 2.1.2.b shows a parallel orientation of D containing only half twists.

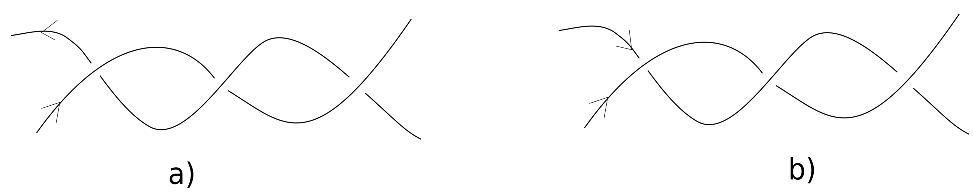

FIGURE 2.1.2. Orientations of a diagram: a) antiparallel, b) parallel.

Link diagrams can be manipulated by Reidemeister moves. There are three types of Reidemeister moves, as shown in Figure 2.1.3.

It has been shown that two links L_1 and L_2 are isotopic if there exists a sequence of Reidemeister moves that changes a diagram of L_1 to a diagram of L_2, without changing their link type [**13**].

DEFINITION 2.12. *A 2-tangle T is a proper embedding of the disjoint union of 2 arcs t_1 and t_2 into a 3-ball $\mathcal{D}^3$. We denote this $(\mathcal{D}^3, T(t_1, t_2))$, or $(\mathcal{D}^3, T)$.*

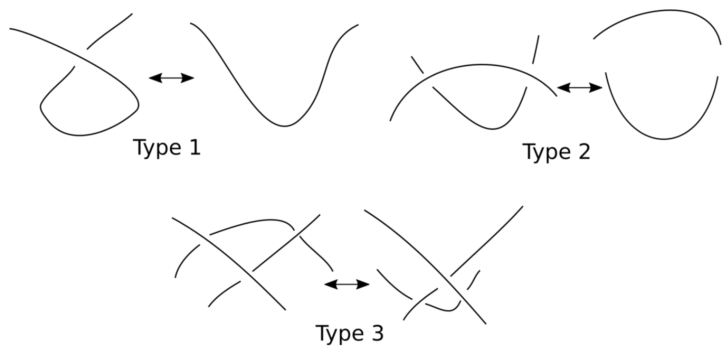

FIGURE 2.1.3. Three types of Reidemeister moves.

Two tangles $(\mathcal{D}^3, T)$ and $(\mathcal{D}^3, T')$ are equivalent if there is an isotopy from $(\mathcal{D}^3, T)$ to $(\mathcal{D}^3, T')$ that keeps the end points of T pointwise fixed.

A tangle diagram is a projection of the tangle $(\mathcal{D}^3, T)$ into a plane. A tangle diagram follows the same conventions as a link diagram. That means it must be regular and we indicate under- and over- information at each crossing. In a tangle diagram, we project the boundary of the ball $\mathcal{D}^3$ into a circle. Often, we say tangle T to mean the diagram of $(\mathcal{D}^3, T)$, and within this thesis, we always view T as a 2-tangle.

The four endpoints of the two arcs in a tangle diagram can be labelled as Northwest (NW), Northeast (NE), Southwest (SW), and Southeast (SE). These orientations determine parities of a tangle, as shown in Figure 2.1.4

- Parity (∞): one arc connects the SW to the NW, the other SE to NE.
- Parity (1): one arc connects the NW to SE, and the other NE to SW.
- Parity (0): one arc connects the SW to the SE, the other NW to NE.

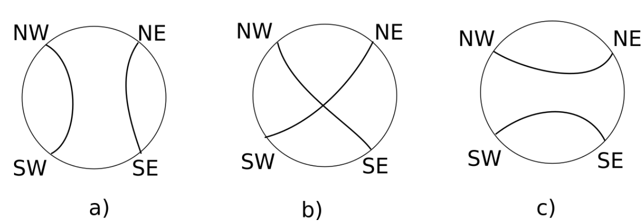

FIGURE 2.1.4. a) Parity (∞), b) Parity (1), and c) Parity (0).

Tangles can be added. Figure 2.1.5 illustrates a sum $(A+B)$ of two tangles A and B, where the NE endpoint of A is connected with the NW endpoint of B, and the SE endpoint of A is connected by the SW endpoint of B. We note that $(A+B)$ can contain in addition to the two arcs a simple closed loops. This happens if both A and B have parity (∞).

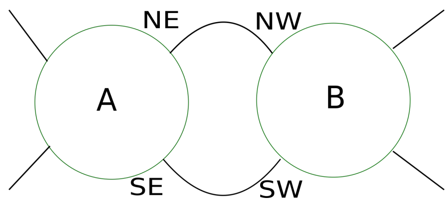

FIGURE 2.1.5. Two tangles A and B can be added by joining their endpoints as labelled.

DEFINITION 2.13. *Given a tangle A, the numerator of A, denoted $N(A)$, is defined by connecting the four ends of A in pairs NW with NE, and SW with SE. The denominator of A, denoted $D(A)$, is defined by connecting the four ends in pairs NW with SW, and NE with SE. See Figure 2.1.6.*

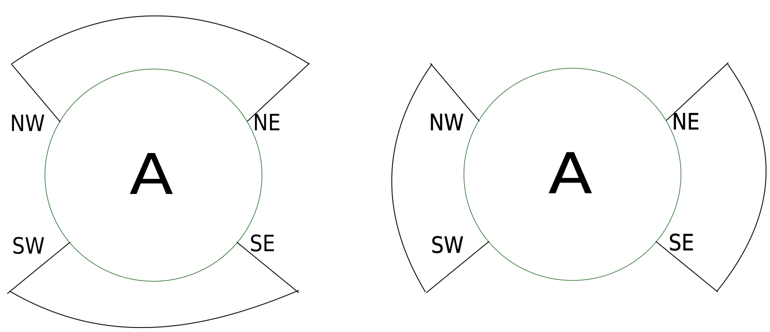

FIGURE 2.1.6. The figure on the left is the numerator $N(A)$. The figure on the right is the denominator $D(A)$ the tangle A.

DEFINITION 2.14. *All 2-tangles are classified, as follows.*

- *A 2-string tangle is rational if it is isotopic to the tangle shown in Figure 2.1.7(iv). The shaded disc separates the two strings of the tangle T in the* *ball $\mathcal{D}^3$. This isotopy is allowed to move the endpoints of T along the boundary of the ball $\mathcal{D}^3$.*

- *A 2-string tangle $(\mathcal{D}^3, T)$ is locally knotted if there exists a sphere in $\mathcal{D}^3$ meeting T tranversely in two points such that the 2-ball bounded by the sphere intersect one of the arcs in a knotted spanning arc.*
- *A 2-string tangle $(\mathcal{D}^3, T)$ is prime if is not rational and not locally knotted*

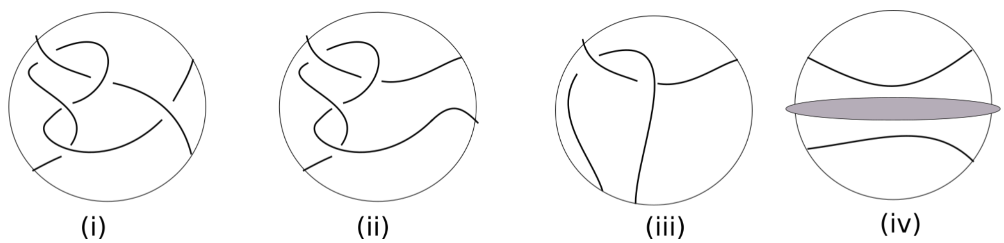

FIGURE 2.1.7. The 2 strings of this tangle can be unwound while its endpoints move along the boundary $\mathcal{D}^3$.

Figure 2.1.7(i) shows a rational tangle, which can be unwound by a deformation. Figure 2.1.8 shows a locally knotted tangle, and Figure 2.1.9 shows a prime tangle.

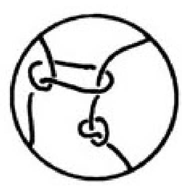

FIGURE 2.1.8. Diagram of a locally knotted tangle.

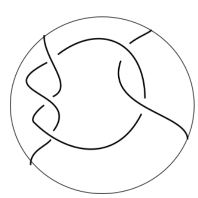

FIGURE 2.1.9. Diagram of a prime tangle.

DEFINITION 2.15. *The closure of an arc α in the ball $(\mathcal{D}^3, \alpha)$ is the arc α itself joining an arc β, such that the following are satisfied:*

$\alpha \cup \beta$ is a closed curve in $\mathbb{R}^3$,

α is all contained in the ball $\mathcal{D}^3$,

and β runs along the boundary $\mathcal{D}^3$ of the tangle.

DEFINITION 2.16. *A flype in a link diagram is a 180 degree rotation of a tangle T such that a single crossing is flipped from one side of T to the other. See Figure 2.1.10.*

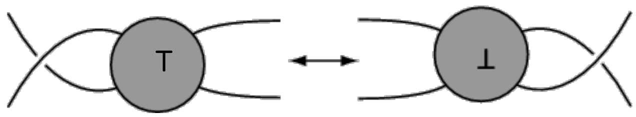

FIGURE 2.1.10. A flype rotates the tangle T, and moves the crossing from the left to the right of T.

DEFINITION 2.17. *Let c be a crossing in an alternating diagram D. The flyping circuit of c is the decomposition of D into group of half-twists $c_1, c_2, ..., c_r$ for $r \geq 1$ and tangles $T_1, T_2, ...T_r$* [**5**]. *D can be drawn as shown in Figure 2.1.11, where groups of half twists, indicated by the c_i.*

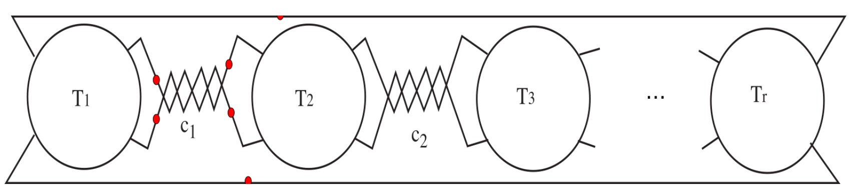

FIGURE 2.1.11. A flyping circuit.

Note that we do not restrict the decomposition of D to be unique, thus it is not necessary that r is maximal with respect to the pattern. The name flyping circuit arises from the fact that the half twists between the tangles can be flyped to be between any two tangles in the decomposition.

2.2. Rational tangle diagrams

Among the three types of tangles, rational tangles play a key role in establishing the invariance of loop numbers. This section describes rational tangles in detail. It also introduces a square diagram of the rational tangle [**3**], and a special class of knots and links formed by rational tangles.

The components of the square diagram of rational tangles, such as types of layers, types of crossings, will be defined exclusively in this thesis.

2.2.1. Continued fractions.

DEFINITION 2.18. *A continued fraction for the rational number $\frac{a}{b}$, denoted by $C(a_1, a_2, ..., a_n)$, is a finite expression of the form:*

$$\frac{a}{b} = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \cfrac{1}{\cdots + \cfrac{1}{a_1}}}},$$

where all the a_i are in $\mathbb{Z}$ and $a_1 \neq 0$.

A rational tangle can be represented by a continued fraction, and a continued fraction can be evaluated to form a rational tangle. Thus we can use a sequence of integers to denote a rational tangle. Figure 2.2.2 shows the rational tangle $T = \frac{37}{26}$ (or $-\frac{37}{26}$) defined by the sequence $T < 3, 1, 2, 2, 1 >$. In drawing the diagam of this sequence, one starts first with 3 crossings horizontally, then 1 vertically, 2 horizontally, another 2 vertically, and finally 1 horizontally.

The negative sign in $-\frac{37}{26}$ is defined by convention, when the first crossing encountered when entering from the NW starts as an underpass, like illustrated in Figure 2.2.1.

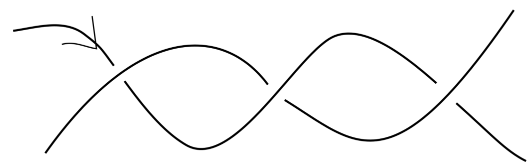

FIGURE 2.2.1. A group of 3 half twists that starts with an underpass.

A rational tangle admits several different minimal diagrams. The crossings of a tangle T can be rearranged to become a square diagram [**3**], as seen in Figure 2.2.3. In this thesis, we call this square diagram a rational square.

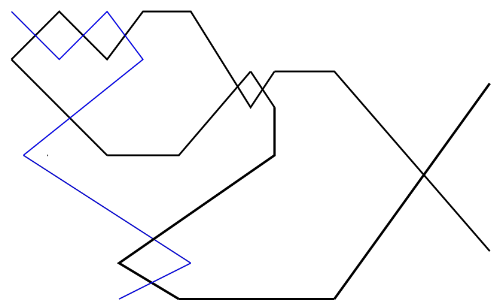

FIGURE 2.2.2. A standard diagram of the rational tangle $T < 3, 1, 2, 2, 1 >$.

2.2.2. Square diagrams of a rational tangles. Figure 2.2.2 shows an example of a standard rational tangle diagram. Figure 2.2.3 is a square diagram for the standard rational tangle in Figure 2.2.2.

If $T < a_n, a_{n-1}, .., a_1 >$ represents a rational tangle, then for any partition of the a_i into two non-negative integers, we have:

$$a_n = a_n;$$

$$a_{n-1} = a'_{n-1} + a''_{n-1};$$

$$a_{n-2} = a'_{n-2} + a''_{n-2};$$

$$...$$

$$a_1 = a'_1 + a''_1.$$

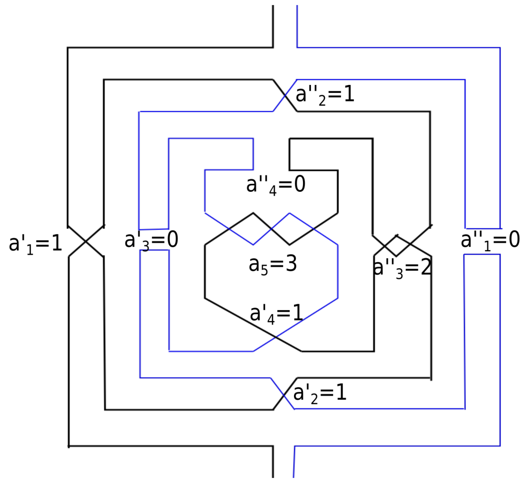

FIGURE 2.2.3. A square diagram of the rational tangle in Figure 2.2.2.

Crossings of a rational tangle diagram can be flyped alternately between horizontal and vertical orientations. Notice that a rational square makes it easy to see all possible flypes.

DEFINITION 2.19. *A layer is a component of a rational square, consisting of two groups of half twists:* $a_i = a'_i + a''_i$ *with the exception of the innermost layer. The innermost layer consists of one group of half twists* a_n*. To be a layer, at least one of the two group of crossings must be non-zero.*

DEFINITION 2.20. *A layer in the rational square is called one-sided if it has zero crossing on one side.*

We assume that the strings of the 2-tangle diagram T are 2-colored in blue and black. This gives rise to the categorization of layers in terms of color composition.

DEFINITION 2.21. *There are two types of layers:*

a) pure layer, denoted by P-layer, is a layer where each crossing contains either all blue or all black arcs.

b) mixed layer, denoted by M-layer, is a layer where each crossing contains 2 colors, blue and black.

Consequently, a crossing of one color is called a P-crossing, and a crossing of 2 colors is called an M-crossing.

Figure 2.2.4 illustrates a P-layer and an M-layer.

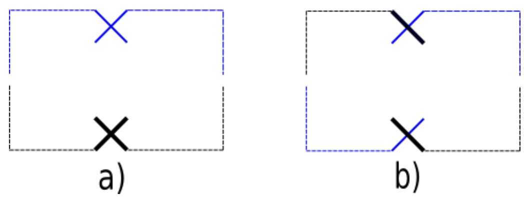

FIGURE 2.2.4. Types of layers in a rational square: a) P-layer, b) M-layer.

Layers in a rational square are enumerated based on the rational tangle continued fractions. For an h layer square, the outermost layer is denoted by l_1, and the innermost layer is l_h.

2.3. Links formed by rational tangles

DEFINITION 2.22. *L is a Montesinos link if $L = N(A_1 + A_2 + ... + A_k)$ for $k \geq 1$, where each A_i is rational.*

Figure 2.3.1 shows a Montesinos link. There can be multiple tangles between A_2 and A_{k-1}, as long as all k tangles are rational.

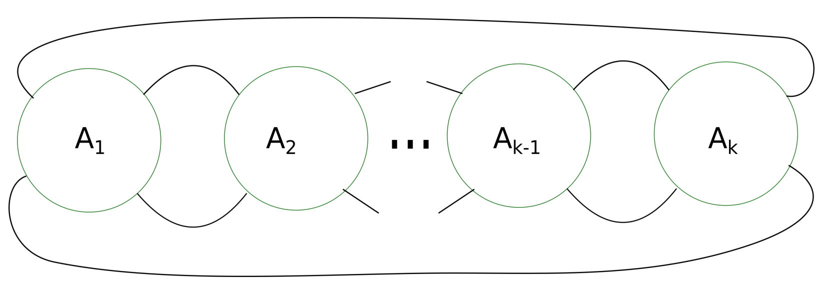

FIGURE 2.3.1. A Montesinos link.

For $k = 2$, the numerator of a two component sum $A + B$ creates a special type of Montesinos knot, called a 4-plat or a two-bridge knot; see Figure 2.3.2 and 2.3.3.

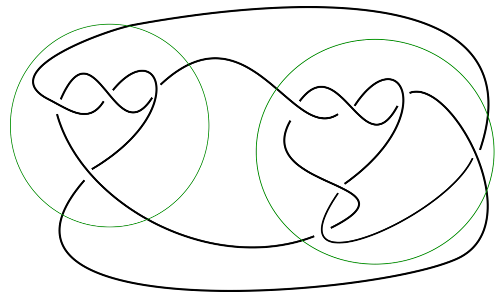

FIGURE 2.3.2. A 4-plat presented as a sum of 2 rational tangles.

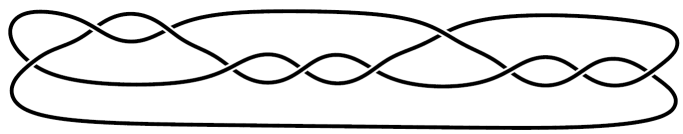

FIGURE 2.3.3. The same 4-plat as Figure 2.3.2, in its canonical diagram.

2.4. Meander Diagrams

The result in this thesis will be used to resolve an open question about Meander diagrams of knots, introduced by Jablan in 2013 [**12**]. Thus, we need to define the concepts involving it.

Definition 2.23. *An open meander is a configuration consisting of an oriented simple curve, and a line in the plane, called the axis of the meander, that crosses a finite number of times and intersects the simple curve only transversally* [**12**].

A valid open meander has no nugatory crossing. The intersections along the axis are enumerated by $1, 2, 3, ..., n$ for every open meander. The order of these n numbers, when tracing the simple curve, makes a permutation of order n. Figure 2.4.1 shows an example of an open meander by permuting the numbers from 1 to 6 as $< 1, 2, 5, 4, 3, 6 >$. Another permutation of order 6 that makes an open meander is shown in Figure 2.4.2.

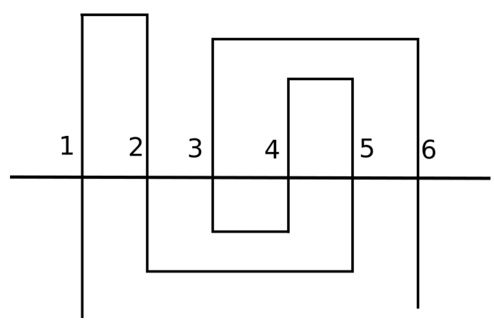

Figure 2.4.1. An open meander: $< 1, 2, 5, 4, 3, 6 >$.

Figure 2.4.3 shows that the ends can be on opposite sides of the axis. This can happen only when the number of crossings is odd.

The closure of the open meander is a join of the loose ends together so that they do not intersect other arcs and do not create a cycle. Figure 2.4.4 shows the closure of the open meander $< 1, 6, 3, 4, 5, 2, 7 >$.

Definition 2.24. *A meandric shadow is the closure of an open meander.*

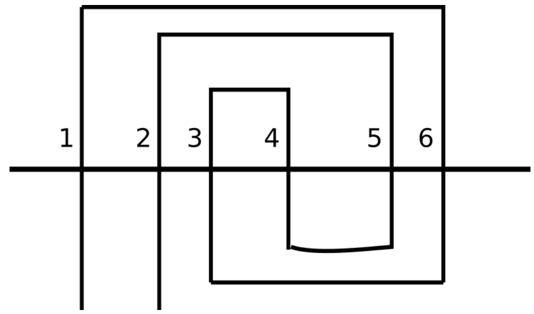

FIGURE 2.4.2. An open meander: $< 1, 6, 4, 3, 5, 2 >$, another permutation of order 6.

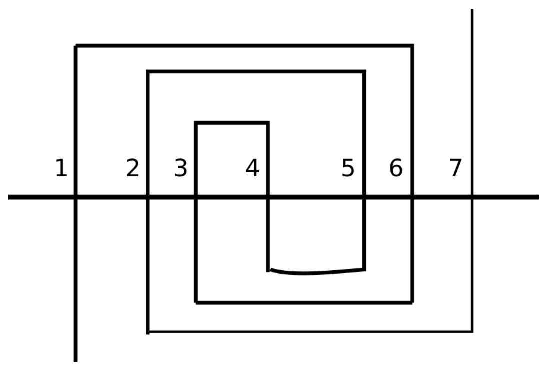

FIGURE 2.4.3. An open meander: $< 1, 6, 3, 4, 5, 2, 7 >$, whose ends are on opposite sides of the axis.

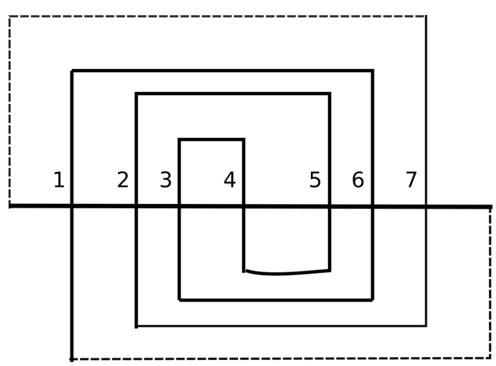

FIGURE 2.4.4. The closure of the open meander: $< 1, 6, 3, 4, 5, 2, 7 >$.

DEFINITION 2.25. *Meander diagram of a knot is created by assigning over- and underpasses on the crossings of the meandric shadow.*

Figure 2.4.5 is an example of a meander diagram of the figure-eight knot by the permutation $< 5, 4, 1, 2, 3 >$ and the assignment of under- and overpasses to its meandric shadow.

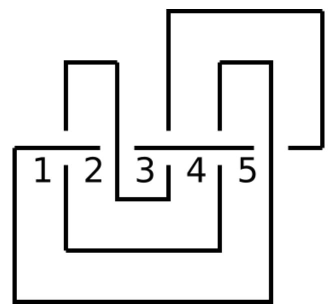

FIGURE 2.4.5. A meander diagram of the figure-eight knot.

The open question in Jablan's paper is whether every alternating knot has a meander diagram. It has been confirmed by computation that the statement is true for knots with at most 9 crossings [**12**].

Chapter 3

Loop numbers basics

This chapter defines concepts related to loop numbers in a link diagram, such as a loop, extreme values of loop numbers, and conditional loop numbers. It also describes some basic properties of loop numbers which are used in this thesis. The two important properties are shown in the theorems below.

- Theorem 3.9 asserts that two loop numbers obtained from specified starting positions are, in fact, equal.

- Theorem 3.13 describes the relationship of loop numbers and crossing numbers of a link diagram.

3.1. Define loops in knot diagrams

Let D be a diagram of a knot K. D can be viewed as a 4-regular planar graph G, whose vertices are the crossings of K where the under and over information are ignored. In G we pick an arbitrary starting point S on one of the edges of G and choose a direction d. Starting at S, we move along G with the rule that if we encounter a vertex (or crossing) of D, then we continue with the opposite edge just like we do when we traverse the knot diagram D. Eventually we will reach a vertex v of G for the second time. Let L be the loop between the first and second encounter of v. Delete the loop L from G by deleting all the edges of the loop. This creates vertices (at least one) of degree two, which are ignored in the proof that follows. Using the reduced graph we start over, that is we start at S walking in the same direction d until we encounter a first loop which we again delete. Eventually after deleting one or several loops, G will be reduced to a single cycle, which will be the last loop we encounter in the process. Figure 3.1.1 illustrates this process for a diagram of a figure-eight knot.

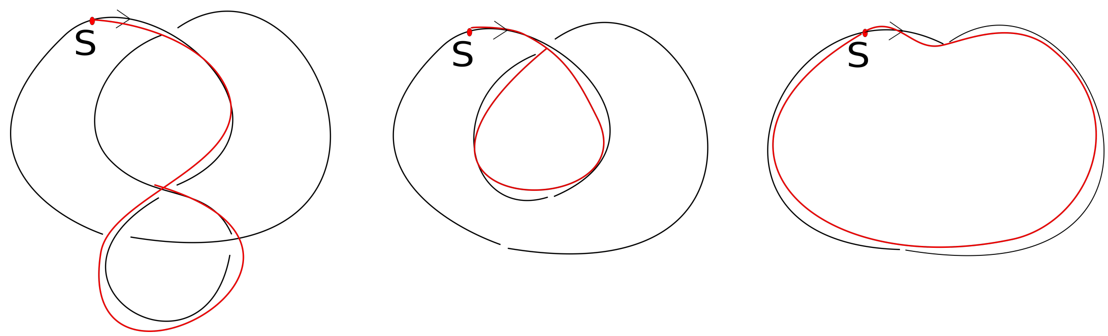

Figure 3.1.1. Traversing this figure-eight knot starting at S yields 3 loops.

Definition 3.1. *For a diagram D, the process yields a finite number of loops, depending on the nature of D, the position of the initial point S, and the chosen direction d. We denote this number by $lp(D, S, d)$, and refer to it as a loop number.*

Given an oriented arc a, we define $lp(a, P, d)$ as the number of loops obtained when starting at an end-point P of the arc and traversing in the given direction d on this arc. The loop process is repeated on arc a until there is no self-intersection left.

Often we omit the the direction d or the starting point S and write $lp(D, S)$ or $lp(D)$, when the omitted parameters are implied in the context.

Definition 3.2. *Given a diagram obtained on the numerator $N(T_1 + T_2 + t(\pm 1))$ of two tangles T_1, T_2, and a single crossing $t(\pm 1)$, we define outside loop numbers as the loop numbers collected in the diagram D by starting at the positions outside both of the tangles, denoted $\mathbb{S} = \{S_i\}$. Figure 3.1.2 shows these six positions. Starting points that are not in the set of arcs shown in Figure 3.1.2 give rise to inside loop numbers.*

Definition 3.3. *An outer loop α of the diagram D is formed by left-over arcs inside its tangles with the arcs outside these tangles, connecting them together.*

The remaining concepts of loops will become useful in Chapter 7.

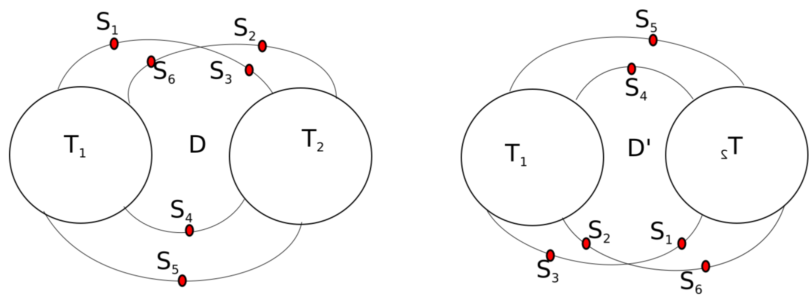

FIGURE 3.1.2. Outside values recorded at the six positions in the diagram D repeat in the flyped diagram D'.

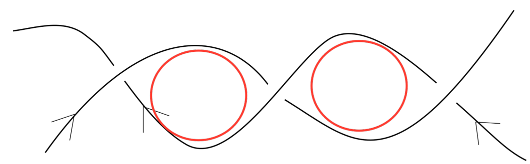

FIGURE 3.1.3. 2 small loops formed between 3 half twists.

DEFINITION 3.4. *A small loop is a loop formed between half twists that are antiparallel.*

Figure 3.1.3 shows an example of 2 small loops in 3 antiparallel half twists, where the red arrows specify the orientation of this tangle diagram.

3.2. Conditional loop numbers

The loop numbers generated by a tangle in a diagram D depend on the choice of the starting point S and the orientation d in D. These choices of S and d tell us the order of how one traverses along the arcs of the tangle. To express the values of these loop numbers symbolically, we define conditional loop numbers.

DEFINITION 3.5. *The left-over of a string diagram is the piece of arc left when loops have been removed while traversing the string diagram. Figure 3.2.2 illustrates the left-over of arc t_2 when one traverses t_2 first and removes one loop; $lp(t_2) = 1$.*

If the tangle $T(t_1, t_2)$ contains 2 strings such that one does not intersect the left-over of the other, the fact that t_1 has been traversed does not affect loop numbers obtained when traversing t_2.

DEFINITION 3.6. *In a 2-tangle $T(t_1, t_2)$ with 2 oriented arcs t_1 and t_2, we define $lp(t_1|t_2)$ as the number of loops obtained when one traverses along t_1 subject to the condition that t_2 has already been traversed. Note that $lp(t_1|t_2)$ does not count the outside loops that may be created when traversing t_1.*

When a point P splits exactly one of the two arcs of T into 2 parts, t'_1 and t''_1, a possible traversing order for T is t'_1, then t_2, and finally t''_1. The loop numbers obtained from this traversing order are $lp(t'_1)$, $lp(t_2|t'_1)$, and $lp(t''_1|t_2, t'_1)$.

Figure 3.2.1 gives an example of a conditional loop number, $lp(t_2|t_1) = 1$. This loop number is obtained by traversing t_2 subject to the condition that t_1 has already been traversed. When the traversal order changes in a diagram, the possible loop numbers may also change. Figure 3.2.2 illustrates that $lp(t_1|t_2) = 0 \neq lp(t_2|t_1)$.

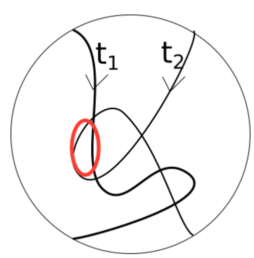

FIGURE 3.2.1. The loop counted as $lp(t_2|t_1)$ is red highlighted in the figure. $lp(t_1)$ is not accounted in $lp(t_2|t_1)$.

In a 2-tangle T, $lp(T, S_i)$, where $S_i \in \mathbb{S}$, denotes the number of loops the tangle T contributes to the whole diagram when starting at the end points in $\mathbb{S}$. If t_1 is traversed before t_2 in T in a chosen direction, then

$$lp(T, S_i) = lp(t_1) + lp(t_2|t_1).$$

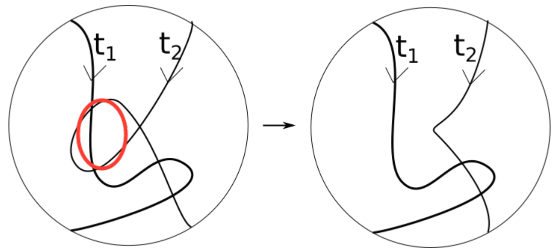

FIGURE 3.2.2. If arc t_2 is traversed first, 1 loop is removed, and the left-over of t_2 is shown with arc t_1. Thus, $lp(t_2) = 1$ and $lp(t_1|t_2) = 0$.

When t_2 is traversed before t_1 in T in the same direction, then

$$lp(T, S_i) = lp(t_2) + lp(t_1|t_2).$$

In Figure 3.2.3, the arrows also stand for the possible positions of S_i. If t_1 is traversed first, we obtain a different value for $lp(T, S_i)$ from when t_2 is traversed first. In particular,

- traversing t_1 first yields no loop on t_1 itself, and two loops later on t_2 (red highlighted). Thus, $lp(T) = 2$.

- traversing t_2 first yields one loop on t_2 itself (green highlighted), and no loop later on t_1. Thus, for the same tangle T, $lp(T) = 1$.

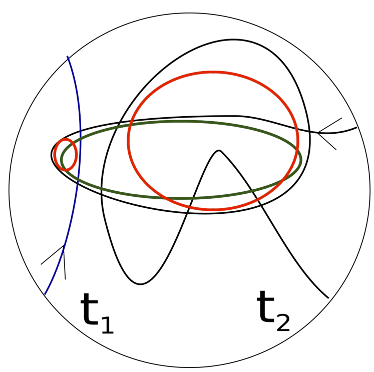

FIGURE 3.2.3. $lp(t_1) + lp(t_2|t_1) = 0 + 2 = 2$; $lp(t_2) + lp(t_1|t_2) = 1 + 0 = 1$.

In fact, it is possible for a tangle to have several different loop numbers that are dependent on starting end-points and orientations. In Figure 3.2.4, suppose m and n are even integers representing a nonzero number of half twists, and direction d is given by the arrows. We obtain different loop number values as follows.

$$lp(t_1) = 1,$$

$$lp(t_2) = 2,$$

$$lp(t_1|t_2) = n - 1,$$

$$lp(t_2|t_1) = m - 1 + 1 = m.$$

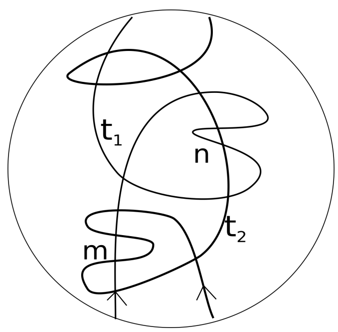

FIGURE 3.2.4. A tangle with four different loop numbers $lp(t_1)$, $lp(t_2)$, $lp(t_1|t_2)$ and $lp(t_2|t_1)$.

In Figure 3.2.5, $lp(t_1''|t_2, t_1')$ denotes the order in which t_1' is traversed first by starting at P. The next arc in this tangle to be traversed is t_2, which does not intersect any left-over arc of t_1'. That is to say, t_2 has property (NI) with respect to t_1' (Definition 3.7). When one traverses t_1'', t_2 and t_1' have already been traversed, and t_1'' does intersect the left-over arc of t_1' and t_2.

Regardless if an arc intersects itself or not to form loops, if the left-over of this arc intersects the other arc, we denote this property by (I).

DEFINITION 3.7. *An arc t_1 has property (I) if t_1 intersects the left-over of t_2. If the arc t_1 does not intersect the left-over of the other arc t_2 inside the tangle, we say that t_1 has property (NI).*

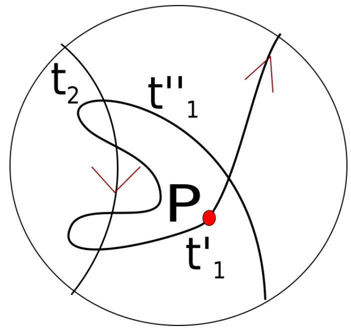

FIGURE 3.2.5. $lp(t_1''|t_2, t_1')$ denotes the following traversing order: t_1', then t_2, and finally t_1''.

The arc t_2 being traversed first in this order can be referred to as "with respect to t_2". So, if t_1 has property (NI) with respect to t_2, then t_1 does not intersect the left-over of t_2, and $lp(t_1|t_2) = lp(t_1)$. Note that the traversal starts at an end-point, as illustrated with arrows in Figure 3.2.6.

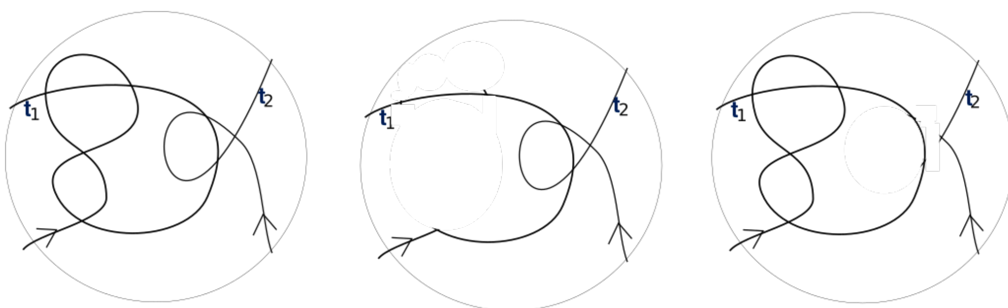

FIGURE 3.2.6. The arc t_1 has property (NI) with respect to t_2, and t_2 has property (I) with respect to t_1.

Consider a tangle of parity (0) or (∞) in a given direction. The following are possible combinations for 2 arcs regarding the properties (I) or (NI):

- Both arcs have property (I), thus tangle T has property $(2I)$;
- Both arcs have property (NI), thus tangle T has property $(2NI)$;
- One arc self-intersects, and the other does not, denoted by (INI).

For a 2-tangle T of parity (1), both arcs must have property (I). Thus, T always has property (2I).

Definition 3.8. *For a knot diagram D, define $minLp(D, d)$ as the minimum of all possible values $lp(D, M_j, d)$ obtained by traversing D starting at different choices of M_j. Similarly, $maxLp(D, d)$ denotes the maximum loop numbers of D, and $aveLp(D, d)$ the average value of loop numbers obtained on D in both directions. Thus, we have the following:*

$$minLp(D, d) = \min_{M_j \in D} lp(D, M_j, d);$$

$$maxLp(D, d) = \max_{M_j \in D} lp(D, M_j, d);$$

$$aveLp(D, d) = \text{ave}_{M_j \in D} lp(D, M_j, d).$$

In the definition above, the minimum and the maximum loop numbers represent extreme values of a given direction. If it can be proven for the pair $minLp(D, d)$ and $maxLp(D, d)$ to be invariant for every direction, then the result holds for both directions in D. We define the minimum over minimum loop numbers, the maximum over maximum loop numbers, and the average of average loop numbers of both directions as follows.

$$minLp(D) = \min\{lp(D, d), lp(D, d')\},$$

$$maxLp(D) = \max\{lp(D, d), lp(D, d')\},$$

$$aveLp(D) = \frac{1}{2}(aveLp(D, d) + aveLp(D, d')).$$

We note that there are are only finitely many choices of starting points S which need to be considered . A knot diagram of n crossings has n vertices and $2n$ edges when considered as a 4-regular graph. Since it does not matter where we put S on a given edge, there are only $2n$ choices of a starting point S for a given direction. Thus, using both directions, we only have to consider $4n$ possible configurations of $lp(D, M, d)$. In the above definition, we only need to take the minimum or maximum over finitely many values $lp(D, M, d)$. Moreover we consider the average $aveLp(D, d)$ as an average over exactly $4n$ configurations of $lp(D, M, d)$. Thus,

$$aveLp(D) = \frac{1}{4n} \sum lp(D, M).$$

However, the results presented later in the next section allow us to reduce the number of different starting points M to be checked by at least a half.

3.3. Basic properties of loop numbers

THEOREM 3.9. *Given an oriented knot diagram D with a crossing O, the two loop numbers obtained by choosing starting points S and S' on the two edges pointing toward the crossing O in D are equal.*

The situation is illustrated in Figure 3.3.1, $lp(S, d) = lp(S', d)$. An example of this is shown in Figure 3.3.2 for an 8-crossing knot diagram.

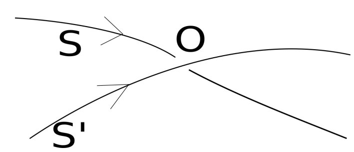

FIGURE 3.3.1. The two starting points in D pointing towards the same crossing O.

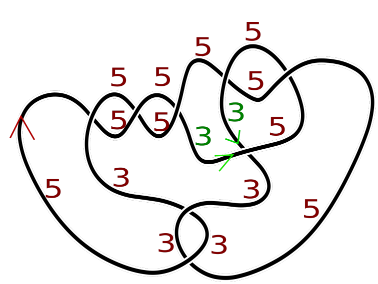

FIGURE 3.3.2. Loop numbers starting at two points towards the same crossing (for example, given by the green arrows) are the same.

PROOF. Suppose that the crossing O with two starting points S and S' in Figure 3.3.3 are a part of a diagram D. Starting at S in the given direction d, one passes through a long arc a and collects all the loops along a as well as leaving some left-over arcs just before hitting O the second time. The number of loops obtained at this point is $lp(a)$. At the crossing O, a big loop made of joined arcs from the left-over of a is removed. Thus the entire arc a is removed completely before one passes through the point S'. From S', in the same direction one collect all loops along an arc, denoted b. Thus the number of loops by traversing the long arc b is $lp(b)$ while coming back to S to complete the cycle.

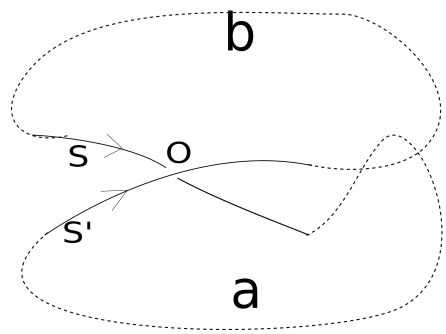

FIGURE 3.3.3. The dashed curves in D denote that a and b may intersect itself and one another at multiple crossings.

We then have the following equations:

$$lp(D,S) = lp(a) + 1 + lp(b) + 1;$$

$$lp(D,S') = lp(b) + 1 + lp(a) + 1.$$

The key for these equations to hold is that all the left-over arcs of either a or b are removed as a loop, before the other arc is traversed. Thus,

$$lp(D,S) = lp(D,S').$$

□

Theorem 3.9 gives us other useful information about loop numbers, as formalized in the corollaries below.

Corollary 3.10. *Let V_i be a possible starting position in a knot diagram D with a given orientation d. Each possible value of $lp(D, V_i, d)$ occurs an even number of times.*

Proof. Suppose D has n crossings, then D has $2n$ edges. For the direction d, at every crossing, we have 2 edges pointing towards the same crossing. By Theorem 3.9, the loop numbers obtained in this direction starting on one edge of the pair, or on the other edge, are the same. Thus, each loop number that appears generates an even frequency at each crossing. □

Corollary 3.11. *For any group of k half twists, we have $2k$ starting points with the same loop number.*

Proof. Let D be a link diagram with an orientation d. Assume that the group of k half twists makes up a tangle T.

If $k = 1$, it is trivial that one traverses the single crossing and comes back to the other arc pointing towards the same crossing. Thus Theorem 3.9 applies directly.

Consider $k > 1$. Let P_1, P_2, ..., P_{2k} the starting points as labelled on the diagram D in Figure 3.3.4, for $k = 3$ as an illustration. Between k half twists, there are $(k - 1)$ distinct arcs on the same string in T. Starting at a point P_i in the direction d, one traverses T through $k - i + 1$ or $2k - i + 1$ remaining crossings of T, then collects self intersections of other tangles in D. Suppose that the loops collected generate a sequence $b_1, b_2, b_3, ..., b_j$ of loops.

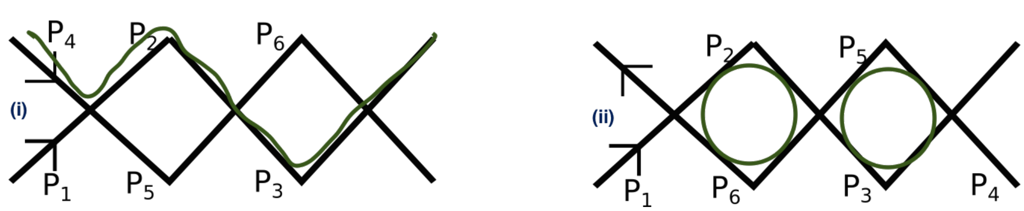

Figure 3.3.4. The starting points on (I) parallel, and (ii) antiparallel cases for $k = 3$.

When re-entering T, there are 2 possible orientations: a) parallel, b) antiparallel.

Case a: The returning orientation in T is antiparallel. At this point, one obtains $k-i-1$ small loops between k crossings in T, then exits T again and collects another sequence of loops, $b_{j+1}, b_{j+2}, ...b_m$. Finally, one comes back to the starting points P_i and collect i more small loops. The loops collected outside T are independent with the small loops in T. As the result, for any i, the total number of loops from T is always $k-1$, and the sequence of loop numbers outside T is always $b_1, ..., b_m$.

Case b: The resulting orientation is parallel. Starting on any P_i, one collects the same number of loops outside T as before, for any i. When re-entering T, a big loop is formed with the arcs outside this group of k half twists, and an arc that contains all crossings with the tangle T. Figure 3.3.4 on the left shows this case with a starting point P_1. Thus, we have the same loop number at these positions from P_1 to P_{2k}

In both cases, starting on P_i for $i = 1, .., 2k$ yields $2k$ loop numbers of the same value.

□

Due to these results, it suffices to check only 4 starting points for the 8-crossing knots example in Figure 3.3.2, instead of $8 * 4 = 32$ starting points.

PROPOSITION 3.12. *In a regular diagram of a knot or link, a single loop contains an odd number of crossings.*

PROOF. We consider a regular diagram of a knot. The most trivial loop is formed by a single half-twist, which contains 1 crossing. See Figure 3.3.5a.

Non trivial loops are formed first by a trivial loop, plus one or more additional arcs intersecting it. We suppose that a loop forms a disk and the boundary divides its space into 2 regions: inside the disk, and outside of it (See Figure 3.3.5). If it is a non-trivial loop, there must be one or more arcs cutting this loop. For each arc that cuts the disk, it has to traverse from outside of the disk boundary into the disk interior, and then re-enter the outside region, crossing the loop exactly twice. The number of crossings formed by all (if any) of these arcs is even with a single loop, plus 1 for the vertex where a starting point originates from.

Thus, it follows that a single loop includes an odd number of crossings. □

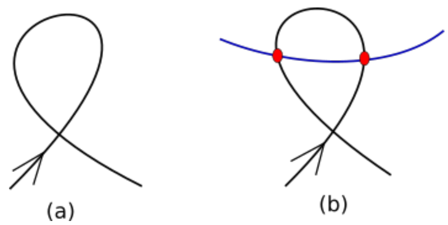

FIGURE 3.3.5. (a) A trivial loop containing 1 crossing, (b) a non-trivial loop having another arc intersecting it.

THEOREM 3.13. *Let n be the number of crossings in diagram D, S and d are any starting point and direction in D. Then,*

$$n \not\equiv lp(D,S,d) \ mod\ (2).$$

PROOF. Suppose for a chosen starting point S and a direction d, we obtain $lp(D,S,d) = a$. Let $X = Cr(D) + a$. When a loop L_1 is removed from diagram D, it must contain an odd number of crossings as the result of Corollary 3.12. Then, we have $(a-1)$ loops left to remove. Notice that we have reduced an even value X by the sum of 1 loop and an odd number of crossings. Repeat this loop removal process until we have 1 loop left. Then for every loop removal not containing the starting point S, an even value of (1 loop and an odd crossing) is subtracted from X. At the final loop counting process, we have 0 crossing left and 1 loop (the one containing S). Thus, X must be odd and the theorem follows. □

Theorem 3.13 helps avoid errors in loop number calculations. For the example in 3.3.2, the number of crossing in the knot diagram is 8, thus all loop numbers must be odd (which are 3 and 5).

Chapter 4

Statements of main results

The preliminary properties in Chapter 3 enable the results shown in the rest of this thesis. The goal is to investigate whether loop numbers are invariant under flypes.

We will first consider the simplest flyping circuit containing two tangles, and show that the loop numbers obtained from starting points outside these tangles remain invariant under a flype. Then, we generalize the result for a flyping circuit with any number of tangles. Chapter 5 shows these results for outside loop numbers.

Chapter 6 outlines conditions where inside loop numbers are invariant under flypes. These conditions depend on the structure of the flyping circuit being considered. However, our result only holds true if certain restrictions are made on the flyping circuits.

This chapter will introduce the tools necessary for the proofs in succeeding chapters.

4.1. Summary of main theorems

In a diagram D containing only parallel flyping circuits, where the tangles are virtually unknotted, then the set of distinct loop numbers remains unchanged under flypes. Theorem 5.4 states this result formally for outside loop numbers.

Theorem (5.4). *In a flyping circuit where the decomposition of tangles is maximal, the set of distinct outside loop numbers are invariant under flypes.*

Theorem 6.2 states the result for loop numbers in general, in a parallel flyping circuit. Note that the concept of virtually unknotted is defined in the next subsection.

Theorem (6.2). *In a diagram D containing only parallel flyping circuits and the tangles of D are virtually unknotted, $minLp(D)$ and $maxLp(D)$ are invariant under flypes.*

In a diagram D containing an antiparallel flyping circuit, the set of outside loop numbers does not change after a flype.

There are two scenarios involving the inside loop numbers in an antiparallel flyping circuit:

- If the circuit contains only rational tangles, then the set of both outside and inside loop numbers are invariant under flypes. Theorem 7.6 is a formal version of this result:

THEOREM (7.6). *Let D be an alternating diagram of a Montesinos knot, then the minimum and maximum loop numbers are invariant under flypes*

- If the circuit contains virtually unknotted tangles, then a restricted set of outside and inside loop numbers are invariant under flypes. This set include loop numbers starting inside a parity (∞) tangle and outside loop numbers.

4.2. The virtually unknotted property

Lemma 4.5 introduced in this section is the key in establishing the relative value of loop numbers which a tangle contributes to the whole diagram.

DEFINITION 4.1. *The diagram of an arc t in a ball $\mathcal{D}^3$ contains a subknot K if a diagram of knot type K can be obtained from the closure of the arc t after crossing changes on t.*

Millett showed that every minimal diagram of a non-trivial knot K contains a trefoil subknot [**11**].

DEFINITION 4.2. *An arc t_i in tangle $T(t_1, t_2)$ is virtually unknotted if its diagram contains no subknot. The arc t_i is virtually knotted if it contains a subknot. The tangle T is virtually unknotted if each strand in its diagram is virtually unknotted.*

In Figure 4.2.1, the diagram of arc t contains a subknot. Crossing changes are made on t so that the closure of resulting diagram is a diagram of the trefoil. Therefore, it is virtually knotted by Definition 4.2.

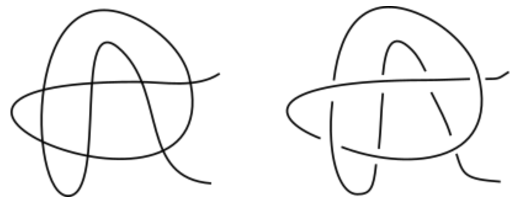

FIGURE 4.2.1. The arc t on the left contains a subknot.

If we make crossing changes on the closure of the arc in Figure 4.2.1 to obtain an alternating projection, then we also obtain a knot. This gives rise to an alternatingly unknotted arc.

DEFINITION 4.3. *The diagram of an arc t in a ball $\mathcal{D}^3$ is alternatingly unknotted if the knot formed by closing t using a simple arc on the boundary of $\mathcal{D}^3$ and then making the diagram alternating is the unknot*

Since we use loop numbers in alternating projections, and the virtually unknotted property does not trivially induce such projections, we prove that the notion of a virtually unknotted arc and an alternatingly unknotted arc are equivalent in the Proposition 4.4.

PROPOSITION 4.4. *An arc is virtually unknotted if and only if it is alternatingly unknotted.*

PROOF. Virtually unknotted clearly implies alternatingly unknotted. If a diagram D contains no subknot, turning it into an alternating projection does not make a subknot appear.

Suppose that the converse is false. Then, there must be an alternatingly unknotted arc that contains a subknot. Let $(\mathcal{D}^3, t)$ be the example of such an arc with the smallest number of crossings. Thus, $(\mathcal{D}^3, t)$ contains no nugatory crossing. Since $(\mathcal{D}^3, t)$ is virtually knotted and alternatingly unknotted, the closure of the alternating projection of $(\mathcal{D}^3, t)$ must contain a trivial crossing. We remove this nugatory crossing in $(\mathcal{D}^3, t)$ by a rotation. Let the resulting arc be $(\mathcal{D}^3, t')$.

At this point, $(\mathcal{D}^3, t')$ is still alternatingly unknotted, but it must contain a subknot since the removal of a nugatory crossing cannot cancel the subknot, and $(\mathcal{D}^3, t')$ contains fewer crossings than $(\mathcal{D}^3, t)$. This contradicts the minimality assumption on $(\mathcal{D}^3, t)$. Thus, the alternatingly unknotted arc t is also virtually unknotted. □

LEMMA 4.5. *Let D be a diagram of the closure of an arc t. If arc t is virtually unknotted, then the loop number of the diagram D starting at any point S is $Cr(t) + 1$. Moreover, the generated loops will always be the same regardless of the starting point S.*

PROOF. We will use induction on the number of crossings of t. If $Cr(t) = 1$ then the diagram D looks like a figure 8. Traversing the closure of t by starting at any S yields the same loop number: 2. Thus, in the case $Cr(t) = 1$, the lemma is true. Suppose it is true for $Cr(t) = k$. We must show that it is true for $Cr(t) = k + 1$.

It is known that a reduced alternating diagram of a knot is non-trivial [**9**]. In other words, in a reduced alternating diagram of a non-trivial knot or link, there exists no nugatory crossing. Let D be a diagram which is the closure of an virtually unknotted arc t with $Cr(t) = k + 1$. There exists a nugatory crossing shown in Figure 4.2.2 in D.

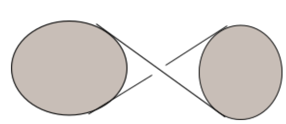

FIGURE 4.2.2. Virtually unknotted arcs can be transformed into the displayed structure.

Using a rotation, we can remove the crossing C and change D to a diagram D' as shown in Figure 4.2.3. By induction, the lemma holds in D' with

$$lp(D', S, d) = Cr(t) = k + 1,$$

regardless of the direction d and starting point S. Thus,

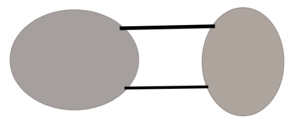

FIGURE 4.2.3. One crossing on the tangle can be undone to obtain a k crossing arc.

$$lp(D, S, d) = Cr(t) + 1 = k + 2,$$

since a nugatory crossing always increase the loop number by one regardless of the starting position. □

COROLLARY 4.6. *Given a virtually unknotted arc t and a starting point P that divides t into t' and t'', then we have:*

$$lp(t') + lp(t''|t') = lp(t).$$

This corollary directly follows from Lemma 4.5, regardless how a starting position in t divides the arc t into t' and t'', the loop number t contributes to the whole diagram is constant.

It is important to note that if A and B are virtually unknotted tangles of parity (0) and (1), then $(A + B)$ is also a virtually unknotted tangle. If either A or B has parity (∞), then the tangle sum may not carry forward the virtually unknotted property. We will often refer to this result when proving how inside loop numbers change in flyping circuits in Chapter 6.

Chapter 5

Outside loop numbers

We suppose that a diagram D contains multiple tangles (at least 2). The loop numbers obtained on arcs outside of all these tangles are called outside loop numbers (Definition 3.2). This chapter shows that the set of distinct values of outside loop numbers are invariant under flypes. This is a part of Theorem 5.2.1, which states that the outside and and inside loop numbers are invariant when the flyping circuit contains only parallel flyping circuits.

In section 5.1, we consider a simple flyping circuit containing two tangles A and B, and a single crossing O. The strategy of this chapter is to go case by case, depending on the structures of flypable diagrams of alternating links. A flyping circuit can be either parallel or antiparallel, which result in two cases that are dealt with separately.

Section 5.2 proves the invariance of outside loop numbers in a general flyping circuit with an arbitrary number of tangles in parallel orientation, and section 5.3 shows the invariance of outside loop numbers generally in the antiparallel orientation.

These theorems will be proven for a given direction d. Similarly, the result will also hold for the other direction d'. Definition 3.8 for $minLp(D)$ and $maxLp(D)$ induces the desired results for both directions.

5.1. The simplest flyping circuit

The simplest and non-trivial flyping circuit contains 2 tangles and one single crossing. Let D represent the diagram of this circuit. Figure 5.1.1 illustrates this simplest structure. To prove the invariance of outside loop numbers, we consider six possible cases based on the (I) or (NI) properties of the two tangles in the simplest flyping circuit. In each case, we determine the distinct loop numbers collected by starting at any arc outside the two tangles respectively, before the flype on a diagram D, and afterwards on the resulting diagram D'.

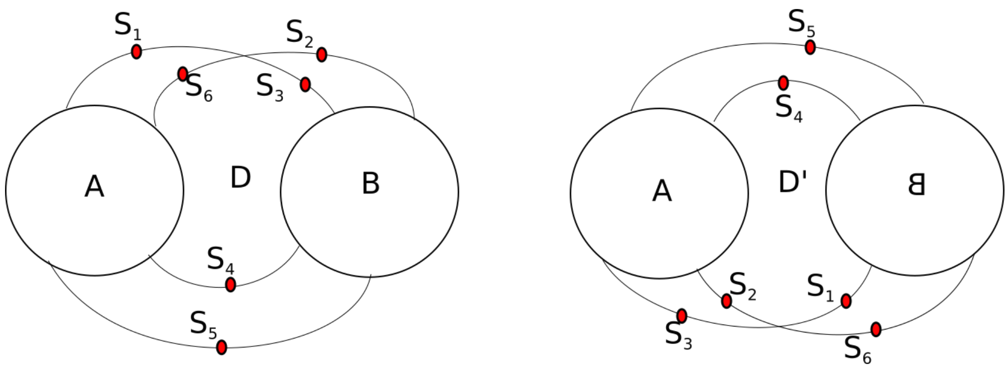

FIGURE 5.1.1. D contains 2 tangles and one crossing. D' is the result of flyping the tangle B in D.

In Figure 5.1.1, given an oriented diagram D of a flyping circuit containing two tangles A and B and a single crossing, let S_i for $i = 1, 2, .., 5, 6$ be the six different starting positions on the six arcs outside of the tangles A and B in the diagram D. We also let the diagram D' denote the resulting diagram when flyping the tangle B in D.

THEOREM 5.1. *Given the two diagrams D and D' related by a flype as shown in Figure 5.1.1, then the set of distinct values of $\{lp(S_i)\}$ is equal to the set of distinct values $\{lp(S_i')\}$.*

In particular

$$\max(lp(S_i)) = \max(lp(S_i'))\,,\ \textit{and}$$

$$\min(lp(S_i)) = \min(lp(S_i')).$$

Moreover, for $i = 1, \ldots, 6$ we have $lp(D, S_i, d) = lp(D', S_j', d)$ for some $j \in \{1, 2, \ldots, 6\}$.

Once this is proven in the simplest flyping circuit, we will generalize the result with Theorem 5.2. Properties of the tangles A and B include the tangle parity and the properties (I) or (NI) of their arcs. We prove Theorem 5.1 by considering all combinations of these properties in D and D'. Subsection 5.1.1 shows the cases where the diagram D contains a parallel flyping circuit, and subsection 5.1.2 shows when D contains an antiparallel flyping circuit. Moreover, the proof involves only one direction d. As the proof of Chapter 5 is long, only Case I in Section 5.1.1 is shown with all details. The other cases are shortened, leaving similar ideas to the reader.

5.1.1. Parallel orientation. Orientating the diagram D will make a parallel flyping circuit in the following cases:

- Case I: Both A and B have parity (0);
- Case II: Both A and B have parity (1).

We do not consider the combination of both tangles having different parities (0) and (1) because this results in a link of 2 components, which we do not deal with in this thesis.

Proof. **Case I:** Both A and B have parity (0).

Consider a flyping circuit of 2 tangles $A(a_1, b_2)$, and $B(b_1, b_2)$. Figure 5.1.2 conceptually illustrates this case. The dashed segments inside the tangles A and B represent long arcs, which means they can intersect themselves and one another in a complex way, as long as the endpoints are paired by NW with SW, and NE with SE. All conceptual figures in other cases will use this notation. By Theorem 3.9, it follows that

$$lp(D, S_1, d) = lp(D, S_6, d).$$

Flyping the tangle B of diagram D results in the diagram D' as shown in Figure 5.1.3. All cases that follow refer to the same notations on these diagrams D and D'. We notice the same equation in D': $lp(D', S'_1, d) = lp(D', S'_6, d)$.

There are different scenarios for the tangle $A(a_1, a_2)$ and the tangle $B(b_1, b_2)$. In the subcases below, $(2NI)$ denotes the property of a tangle whose both arcs have property

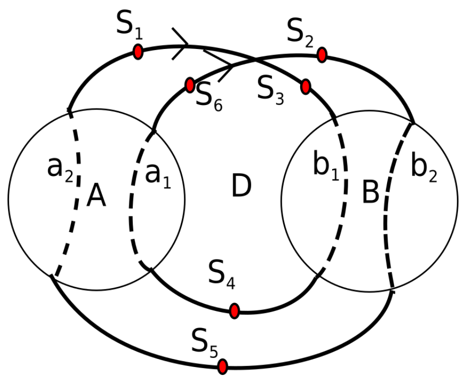

FIGURE 5.1.2. The diagram D contains two parity (0) tangles and a single crossing.

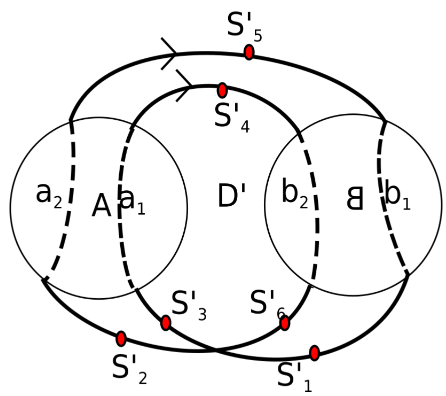

FIGURE 5.1.3. The diagram D' is the result of flyping B in D.

(NI) with respect to the other. (INI) denotes the property of a tangle whose one arc has the property (I), and the other (NI). $(2I)$ denotes the property of a tangle whose both arcs have property (I) with respect to the other.

- We show that the three cases below have similar behaviors in the way that the distribution of (NI) or (I) does not change, even under a flype on one tangle.
  (1) Subcase 1: A has property $(2NI)$ and B has property $(2NI)$.
  (2) Subcase 2: A has property $(2I)$ and B has property $(2I)$.
  (3) Subcase 3: A has property $(2NI)$ and B has property $(2I)$.
- The two cases below have similar behaviors.
  (1) Subcase 4: A has property $(2NI)$ and B has property (INI).

(2) Subcase 5: A has property $(2I)$ and B has property (INI).

- The case below stands alone. We need to check its sub-cases because traversing each tangle through its (I) arc first, or (NI) arc first potentially creates different scenarios. For instant, we do not know how different the case where a_1 and b_1 have property (I) can be different to the one where a_2 and b_1 have this property. Details of this case will be discussed in the analysis below.

(1) Subcase 6: A has property (INI) and B has property (INI).

The cases below do not need to be considered because of the symmetry in the diagram D. Rotating D by 180 degrees switches the roles of A and B, making the resulting structure identical to the cases already outlined above.

- A has property $(2I)$, B has property $(2NI)$;
- A has property (INI), B has property $(2NI)$;
- A has property (INI), B has property $(2I)$.

Subcase I.1: Both tangles have property (2NI)

Figure 5.1.4 is an example of this situation.

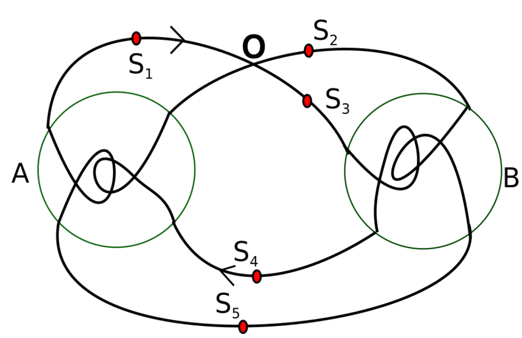

FIGURE 5.1.4. Both A and B have property (2NI).

Starting at S_1, one traverses through crossing O, enters tangle B at b_1 and picks up $lp(b_1)$ loops. We write $S_1 \to b_1$ to denote that we enter the tangle B via arc b_1, then pick up $lp(b_1)$ loops. Using this short-hand notation, we obtain the following route:

$$\begin{aligned} S_1 &\to O : 0 \\ &\to b_1 : lp(b_1) \\ &\to a_1 : lp(a_1) \\ &\to O : 1 \\ &\to b_2 : lp(b_2) \\ &\to a_2 : lp(a_2) \\ &\to S_1 : 1. \end{aligned}$$

When exiting tangle A the second time, one creates an outer loop, and regardless the nature of the tangles A and B, an additional loop will be picked up; we denote this $\to O : 1$. Thus, when one enters the tangle B the second time, $lp(b_2)$ is collected independently with b_1. The loop number picked up if starting from S_1 for this diagram construction is:

$$lp(D, S_1, d) = lp(b_1) + lp(a_1) + lp(b_2) + lp(a_2) + 2.$$

As this equation holds regardless of the nature of the tangles A and B, in all the cases in a parallel flyping circuit, we skip the details and conclude that $lp(D, S_1, d) = a$.

Next, we consider the other starting points: S_2, S_3, S_4, S_5.

Starting at S_2, one enters first tangle B, through the arc b_2, then enters a_2, and hits crossing O. The nature of tangles A and B, both $(2NI)$, decides the loop number collected the second time one traverses these tangles. When one hits crossing O the first time, no outer loop is obtained, given the starting position S_2. Thus, when entering the tangle B the second time, as B has both arcs non-intersecting, we obtain $lp(b_1)$.

Entering A through the arc a_1, with property (NI), we obtain $lp(a_1)$. At this point, when hitting the crossing O the 2nd time, the left-over arcs of b_1 and of a_1 form with the band outside A and B an outer loop, we denote this +1. When completing the cycle at S_2, an additional outer loop is obtained from the circuit band with the left-over of b_2 and of a_2. The whole route starting at S_2 in this sub-case is:

$$
\begin{aligned}
S_2 &\to b_2 : lp(b_2) \\
&\to a_2 : lp(a_2) \\
&\to O : 0 \\
&\to b_1 : lp(b_1) \\
&\to a_1 : lp(a_1) \\
&\to O : 1 \\
&\to S_2 : 1.
\end{aligned}
$$

Thus tangle arcs property (2NI) results in the following loop number:

$$lp(D, S_2, d) = lp(b_2) + lp(a_2) + lp(b_1) + lp(a_1) + 2.$$

Starting at S_3, one enters tangle B at b_1, then tangle A at a_1. The loop number obtained at this point is the sum of $lp(b_1) + lp(a_1)$. When one enters tangle B the second time, $lp(b_2)$ is collected, due to property (2NI). Similarly, $lp(a_2)$ is collected. One outer loop was counted when one visits crossing O the second time, and the other when one last hits S_3 to complete the cycle. The route obtained by starting at S_3 is denoted below:

$$\begin{aligned} S_3 &\to b_1 : lp(b_1) \\ &\to a_1 : lp(a_1) \\ &\to O : 0 \\ &\to b_2 : lp(b_2) \\ &\to a_2 : lp(a_2) \\ &\to O : 1 \\ &\to S_3 : 1. \end{aligned}$$

Thus, starting at S_3 in this diagram yields the same loop number as at S_1 or S_2:

$$lp(D, S_3, d) = lp(b_1) + lp(a_1) + lp(b_2) + lp(a_2) + 2.$$

Starting at S_4 and then S_5, one traverses in a very similar manner as at S_3. From this point, routes are not shown if they are similar to those of the cases already presented. S_4 yields the following loop number:

$$lp(D, S_4, d) = lp(a_1) + lp(b_2) + lp(a_2) + lp(a_1) + 2.$$

S_5 yields the following loop number:

$$lp(D, S_5, d) = lp(a_2) + lp(b_1) + lp(a_1) + lp(b_2) + 2.$$

In summary, due to property (2NI) of both tangles A and B, we obtain the same loop number at all 5 locations.

For $i = 1, 2, 3, 4, 5$, we have:

$$lp(D, S_i, d) = lp(a_2) + lp(b_1) + lp(a_1) + lp(b_2) + 2 := a.$$

We use $a, b, c, ...$ to denote different integer values when locally comparing loop numbers of a diagram D and its flyped diagram D'.

Let Lp be the collection of loop numbers for a given case, where we order of the loop numbers corresponds to the starting point order $\{S_1, S_2, S_3, S_4, S_5\}$. In a parallel flyping circuit, these six outside loop numbers can make at most five distinct values. Therefore, we do not need to consider S_6 in the given traversal direction.

For this case, we have:

$$Lp = \{a, a, a, a, a\}.$$

Let us flype the tangle B in D and denote the resulting diagram D'. The distinct 5 choices of the starting points outside the tangles are shown in Figure 5.1.3. The traversal routes in the diagram D' depend on starting points are listed below.

$$\begin{aligned} S_1' &\to O : 0 \\ &\to a_1 : lp(a_1) \\ &\to b_2 : lp(b_2) \\ &\to O : 1 \\ &\to a_2 : lp(a_2) \\ &\to b_1 : lp(b_1) \\ &\to S_1' : 1. \end{aligned}$$

The resulting loop number of D' starting at S'_1 is:

$$lp(D', S'_1, d) = lp(a_1) + lp(b_2) + lp(a_2) + lp(b_1) + 2.$$

Similarly, starting at other positions S'_i, for $i = 2, 3, 4, 5$, yields the same loop number:

$$lp(D', S'_2, d) = lp(a_2) + lp(b_1) + lp(a_1) + lp(b_2) + 2;$$

$$lp(D', S'_3, d) = lp(a_1) + lp(b_2) + lp(a_2) + lp(b_1) + 2;$$

$$lp(D', S'_4, d) = lp(b_2) + lp(a_2) + lp(b_1) + lp(a_1) + 2;$$

$$lp(D', S'_5, d) = lp(b_1) + lp(a_1) + lp(b_2) + lp(a_2) + 2.$$

In summary of Subcase I.1, the collection of loop numbers Lp from these 5 starting points remains invariant in Lp' by a flype:

$$Lp = Lp' = \{a, a, a, a, a\}.$$

Subcase I.2**:** A **has property** $(2NI)$ **and** B **has property** $(2I)$

Figure 5.1.5 is an example of this situation.

Let a denote the same value as previous case. Starting at S_1 yields the same loop number.

$$lp(D, S_1, d) = lp(b_1) + lp(a_1) + lp(b_2) + lp(a_2) + 2 := a.$$

Starting at S_2, one enters B and collects $lp(b_2)$ loops, then enters A to collect $lp(a_2)$ loops before hitting crossing O. When entering B the second time, as the tangle B has

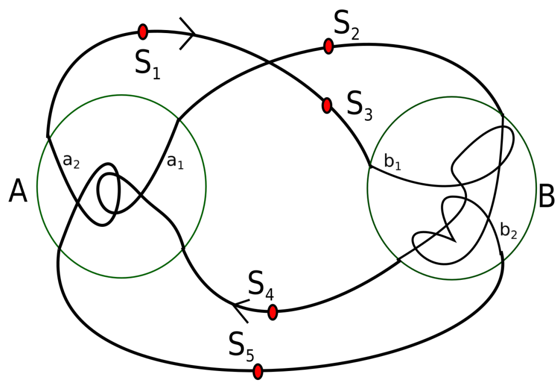

FIGURE 5.1.5. A has property $(2NI)$, and B has property $(2I)$.

property $(2I)$, the fact that b_2 has been traversed and its left-over does intersect b_1 yields $lp(b_1|b_2)$, and one additional loop outside B that connects the left-over arc of b_1 with the arcs outside the tangles; because of this additional loop involving the conditional loop number, when passing O the second time, no outer loop is obtained. Then one enters tangle A and collects $lp(a_1)$ due to property $(2NI)$ of A. Finally, the other outer loop is the one connecting the left-over of b_2 and the rest of the outside arcs.

$$
\begin{aligned}
S_2 &\to b_2 : lp(b_2) \\
&\to a_2 : lp(a_2) \\
&\to O : 0 \\
&\to b_1 : lp(b_1|b_2) + 1 \\
&\to a_1 : lp(a_1) \\
&\to O : 0 \\
&\to S_2 : 1.
\end{aligned}
$$

Starting at S_2 yields the following loop number:

$$lp(D, S_2, d) = lp(b_2) + lp(a_2) + lp(b_1|b_2) + lp(a_1) + 2 := b.$$

Starting at S_3 yields the following loop number:

$$lp(D, S_3, d) = lp(b_1) + lp(a_1) + lp(b_2|b_1) + lp(a_2) + 2 := c.$$

Starting at S_4: As tangle A is (2NI), the second time one enters A, a_1 was already traversed and its left-over arc does not intersect a_2. Thus, we get an independent loop number $lp(a_2)$. Also, the left-over arc of b_2 became a part of the loop removed when O was passed the second time; thus the second time one enters B, only b_1 is there.

$$\begin{aligned} S_4 &\rightarrow a_1 : lp(a_1) \\ &\rightarrow O : 0 \\ &\rightarrow b_2 : lp(b_2) \\ &\rightarrow a_2 : lp(a_2) \\ &\rightarrow O : 1 \\ &\rightarrow b_1 : lp(b_1) \\ &\rightarrow S_4 : 1. \end{aligned}$$

Starting at S_4 yields the following loop number:

$$lp(D, S_4, d) = lp(a_1) + lp(b_2) + lp(a_2) + lp(b_1) + 2 := a.$$

Starting at S_5 yields the following loop number:

$$lp(D, S_5, d) = lp(a_2) + lp(b_1) + lp(a_1) + lp(b_2) + 2 := a.$$

Let Lp be the collection of loop numbers obtained from D of tangle A (2NI), and tangle B (2I), then

$$Lp = \{a, b, c, a, a\}.$$

Consider a flype of diagram D, into diagram D' as shown in Figure 5.1.3.

Starting at S'_1 in D', we get the following route:

$$\begin{aligned} S'_1 &\to O : 0 \\ &\to a_1 : lp(a_1) \\ &\to b_2 : lp(b_2) \\ &\to O : 1 \\ &\to a_2 : lp(a_2) \\ &\to b_1 : lp(b_1) \\ &\to S'_1 : 1. \end{aligned}$$

As passing the crossing O the second time, an outer-loop is formed from the left-over arcs of a_1 and b_2. This outer-loop is then is removed from the diagram D'. Thus, the arc properties of A and B do not influence the loop formation in D'. The total loop number obtained after flyping B is given below:

$$lp(D', S'_1, d) = lp(a_2) + lp(b_1) + lp(a_1) + lp(b_2) + 2 := a.$$

Starting at S'_2 in D', we get the following result:

$$lp(D', S'_2, d) = lp(a_2) + lp(b_1) + lp(a_1) + lp(b_2|b_1) + 2 := c.$$

The route in this diagram is below:

$$\begin{aligned} S_2' &\to a_2 : lp(a_2) \\ &\to b_1 : lp(b_1) \\ &\to O : 0 \\ &\to a_1 : lp(a_1) \\ &\to b_2 : lp(b_2|b_1) + 1 \\ &\to O : 0 \\ &\to S_2' : 1. \end{aligned}$$

Similarly, starting at S_3', S_4', and then S_5', one get the following loop numbers:

$$lp(D', S_3', d) = lp(a_1) + lp(b_2) + lp(a_2) + lp(b_1|b_2) + 2 := b;$$

$$lp(D', S_4', d) = lp(b_2) + lp(a_2) + lp(a_1) + lp(b_1|b_2) + 2 := b;$$

$$lp(D', S_5', d) = lp(b_1) + lp(a_1) + lp(b_2|b_1) + lp(a_2) + 2 := c.$$

Let Lp' be the collection of loop numbers obtained in D', then

$$Lp' = \{a, c, b, b, c\}.$$

We observe that the frequencies of the numbers a, b, and c change after the flype, but the two sets still include the same 3 possibly distinct values a, b, and c.

Subcase I.3: Both A and B have property (2I)

Figure 5.1.6 shows an example of this case. We consider the following scenarios:

As every arc in a tangle intersects the left-over of the other arc in the same tangle, the traversing order of each arc in a tangle results in conditional loop numbers. Starting

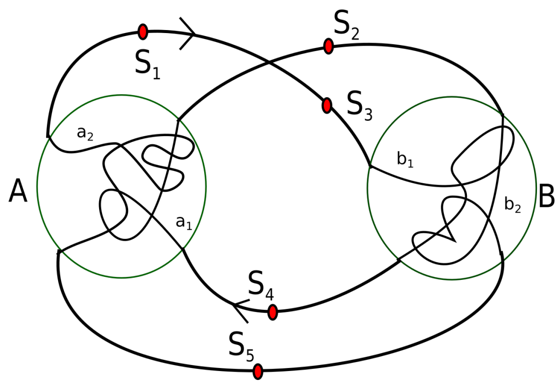

FIGURE 5.1.6. Both of the tangles A and B have property $(2I)$.

at S_2 yields the following loop number:

$$lp(D, S_2, d) = lp(b_2) + lp(a_2) + lp(b_1|b_2) + lp(a_1) + 2 := d.$$

Starting at S_3, we get the following loop number:

$$lp(D, S_3, d) = lp(b_1) + lp(a_1) + lp(b_2|b_1) + lp(a_2) + 2 := e.$$

Starting at S_4 yields the following loop number:

$$lp(D, S_4, d) = lp(a_1) + lp(b_2) + lp(a_2|a_1) + lp(b_1) + 2 := f.$$

Starting at S_5 yields the loop number:

$$lp(D, S_5, d) = lp(a_2) + lp(b_1) + lp(a_1|a_2) + lp(b_2) + 2 := g.$$

In summary, the collection of loop numbers we obtain for this case is

$$Lp_3 = \{a, d, e, f, g\}$$

where all the values a, d, e, f, g could be distinct from one another.

We now consider loop numbers of D' in the same direction d.

Starting at $S'_1 \in D'$, we obtain the same loop number

$$lp(D', S'_1, d) = a.$$

Starting at $S'_2 \in D'$ yields the following loop number:

$$lp(D', S'_2, d) = lp(a_2) + lp(b_1) + lp(a_1|a_2) + lp(b_2) + 2 := g.$$

Starting at $S'_3 \in D'$ yields the following loop number:

$$lp(D', S'_3, d) = lp(a_1) + lp(b_2) + lp(a_2|a_1) + lp(b_1) + 2 := f.$$

Starting at $S'_4 \in D'$ yields the following loop number:

$$lp(D', S'_4, d) = lp(b_2) + lp(a_2) + lp(b_1|b_2) + lp(a_1) + 2 := d.$$

Starting at $S'_5 \in D'$ yields the following loop number:

$$lp(D', S'_5, d) = lp(b_1) + lp(a_1) + lp(b_2|b_1) + lp(a_2) + 2 := e.$$

Let Lp be the collection of outside loop numbers of D, and Lp' be the collection of outside loop numbers of D'. For the Subcase I.3, we get the following values:

$$Lp = \{a, d, e, f, g\}$$

and

$$Lp' = \{a, g, f, d, e\}.$$

Subcase I.4: A has property (2NI), B has property (INI).

The intersecting arc in B can be b_1 or b_2. Assume that arc b_1 has property (I) with respect to b_2, and b_2 has property (NI) with respect to b_1. Figure 5.1.7 shows an example of this case. Starting at S_1, due to the order of outer-loop formation, the

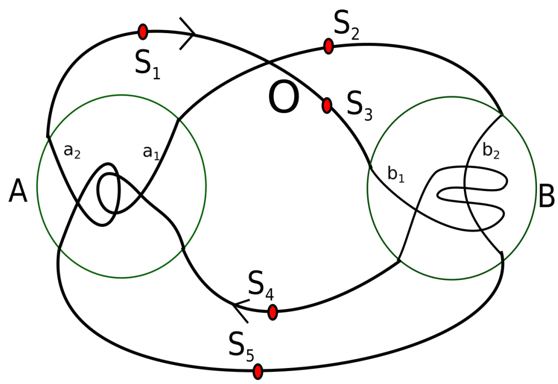

FIGURE 5.1.7. A has property $(2NI)$; B has property (INI).

removal of the first outer-loop made from the left-over arc of b_1 and of a_1 implies that loop numbers collected from tangle B and A later becomes independent with the left-over arcs of each. This yields a loop number below:

$$lp(D, S_1, d) = lp(a_2) + lp(b_1) + lp(a_1) + lp(b_2) + 2 := a.$$

Starting at S_2, when one enters B the second time, b_2 has been already traversed, and its left-over arc intersects b_1. Thus, we get $lp(b_1|b_2)$ when exiting tangle B.

$$\begin{aligned} S_2 &\to b_2 : lp(b_2) \\ &\to a_2 : lp(a_2) \\ &\to O : 0 \\ &\to b_1 : lp(b_1|b_2) + 1 \\ &\to a_1 : lp(a_1) \\ &\to O : 0 \\ &\to S_2 : 1. \end{aligned}$$

Thus, the loop number obtained from this route is

$$lp(D, S_2, d) = lp(b_2) + lp(a_2) + lp(b_1|b_2) + lp(a_1) + 2 := d.$$

Next, we consider starting at S_3. When traversing tangle B the second time, one goes through b_2 when b_1 has already been traversed. As the left-over arc of b_1 does intersect b_2, this might change the value of loop number of tangle B. We then get $lp(b_2|b_1)$ in this case. A similar argument works for traversing the tangle A the second time, but because of property (2NI), $lp(a_2)$ is obtained the second time one traverses tangle A.

Thus, the route of this case is summarized below:

$$\begin{aligned} S_3 &\to b_1 : lp(b_1) \\ &\to a_1 : lp(a_1) \\ &\to O : 0 \\ &\to b_2 : lp(b_2) \\ &\to a_2 : lp(a_2) \\ &\to O : 1 \\ &\to S_3 : 1. \end{aligned}$$

The loop number for the starting point S_3 is:

$$lp(D, S_3, d) = lp(b_1) + lp(a_1) + lp(b_2) + lp(a_2) + 2 := a.$$

Starting at S_4 yields the following loop number:

$$lp(D, S_4, d) = lp(a_1) + lp(b_2) + lp(a_2) + lp(b_1) + 2 := a.$$

Starting at S_5 yields the following loop number:

$$lp(D, S_5, d) = lp(a_2) + lp(b_1) + lp(a_1) + lp(b_2) + 2 := a.$$

Let Lp be the collection of outer loop numbers in D.

$$Lp = \{a, d, a, a, a\}$$

We now consider a flype on tangle B of diagram D, into D', as shown in Figure 5.1.3.

Starting at $S'_1 \in D'$, we obtain the following route:

$$\begin{aligned} S'_1 &\to O : 0 \\ &\to a_1 : lp(a_1) \\ &\to b_2 : lp(b_2) \\ &\to O : 1 \\ &\to a_2 : lp(a_2) \\ &\to b_1 : lp(b_1) \\ &\to S'_1 : 1. \end{aligned}$$

The loop number of D' starting at S'_1 is:

$$lp(D', S'_1, d) = lp(a_1) + lp(b_2) + lp(a_2) + lp(b_1) + 2 := a.$$

Similarly, the loop number of D' starting at S'_2 is:

$$lp(D', S'_2, d) = lp(a_2) + lp(b_1) + lp(a_1) + lp(b_2) + 2 := a.$$

The loop number of D' starting at S_3' is:

$$lp(D', S_3', d) = lp(a_1) + lp(b_2) + lp(a_2) + lp(b_1|b_2) + 2 := d.$$

The loop number of D' starting at S_4' is:

$$lp(D', S_4', d) = lp(b_2) + lp(a_2) + lp(b_1|b_2) + lp(a_1) + 2 := d.$$

The loop number of D' starting at S_5' is:

$$lp(D', S_5', d) = lp(b_1) + lp(a_1) + lp(b_2) + lp(a_2) + 2 := a.$$

Let Lp be the collection of loop numbers obtained.

$$Lp = \{a, d, a, a, a\}.$$

The set of loop numbers obtained from flyping D into D' is given by Lp' below:

$$Lp' = \{a, a, d, d, a\}.$$

We now consider the diagrams D and D' with no change on A, but with b_1 having property (I), and b_2 having property (NI). Whenever we enter B the second time in either D or D', if the arc to be traversed has the (NI) property, then we pick up a conditional loop number, i.e. $lp(b_2|b_1)$, and one outer loop. This is similar to what occurs in the previous case, but with a the roles of (I) and (NI) switched. Thus, it is not necessary to consider this combination in detail.

Subcase I.5**:** A **has property (2I), and** B **has property (INI).**

The arc b_2 has property (I) with respect to b_1. And the arc b_1 has property (NI) with respect to b_2. Figure 5.1.8 illustrates this case with an example.

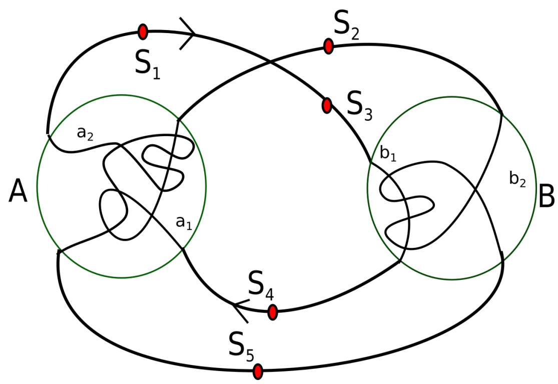

FIGURE 5.1.8. A has property $(2I)$. B has property (INI).

Starting at S_1, we always obtain $lp(D, S_1, d) = a$.

Starting at S_2, we obtain the following route:

$$\begin{aligned} S_2 &\to b_2 : lp(b_2) \\ &\to a_2 : lp(a_2) \\ &\to O : 0 \\ &\to b_1 : lp(b_1) \\ &\to a_1 : lp(a_1|a_2) + 1 \\ &\to O : 0 \\ &\to S_2 : 1. \end{aligned}$$

Thus, the loop number obtained from this route is

$$lp(D, S_2, d) = lp(b_2) + lp(a_2) + lp(b_1) + lp(a_1|a_2) + 2 := j.$$

Similarly, the loop number for D is starting at S_3 is:

$$lp(D, S_3, d) = lp(b_1) + lp(a_1) + lp(b_2|b_1) + lp(a_2) + 2 := e.$$

Starting at S_4, we obtain the following loop number:

$$lp(D, S_4, d) = lp(a_1) + lp(b_2) + lp(a_2|a_1) + lp(b_1) + 2 := f.$$

Starting at S_5 yields the following loop number:

$$lp(D, S_5, d) = lp(a_2) + lp(b_1) + lp(a_1|a_2) + lp(b_2) + 2 := j.$$

In this subcase, the collection of outside loop numbers obtained from D is

$$Lp = \{a, j, e, f, j\}.$$

We now consider a flype on tangle B, which yields diagram D', as simplified in Figure 5.1.3.

Starting at $S_1' \in D'$ yields the same loop number as starting at S_1 in other cases:

$$lp(D', S_1', d) = a.$$

Starting at $S_2' \in D'$ yields the following loop number:

$$lp(D', S_2', d) = lp(a_2) + lp(b_1) + lp(a_1|a_2) + lp(b_2) + 2 := j.$$

Similarly, starting at $S_3' \in D'$ yields the following loop number:

$$lp(D', S_3', d) = lp(a_1) + lp(b_2) + lp(a_2|a_1) + lp(b_1) + 2 := f.$$

Starting at $S_4' \in D'$ yields the following loop number:

$$lp(D', S_4', d) = lp(b_2) + lp(a_2) + lp(b_1) + lp(a_1) + 2 := a.$$

When arriving at crossing O the second time, the left-over of a_2, b_1 connect with the outside arcs to make an outer-loop, which follows that when entering tangle A the second time, one collects only $lp(a_1)$.

Starting at $S_5' \in D'$ yields the following loop number:

$$lp(D', S_5', d) = lp(b_1) + lp(a_1) + lp(b_2|b_1) + lp(a_2) + 2 := e.$$

Let Lp' be the collection of outside loop numbers in D'. Then we have:

$$Lp = \{a, j, e, f, j\} \text{ and}$$

$$Lp' = \{a, j, f, a, e\}.$$

Subcase I.6: Both tangles have property (INI).

The combination of (I) and (NI) arcs in each tangle generates 2 possibilities:

- Subcase a: The arcs a_2 and b_2 have property (I). See Figure 5.1.9.
- Subcase b: The arcs a_1 and b_2 have property (I). See Figure 5.1.10.

It suffices to consider the two cases above by symmetry, because in each case we will then consider the flype on B, which switches the property (I) for b_1 and b_2 An arc a_1 in a tangle $A(a_1, a_2)$ has property (I) means that it intersects the left-over of a_2.

<u>Subcase a:</u>

Starting at S_1 yields the same loop number as other cases when starting at S_1:

$$lp(D, S_1, d) = lp(a_1) + lp(b_2) + lp(a_2) + lp(b_1) + 2 := a.$$

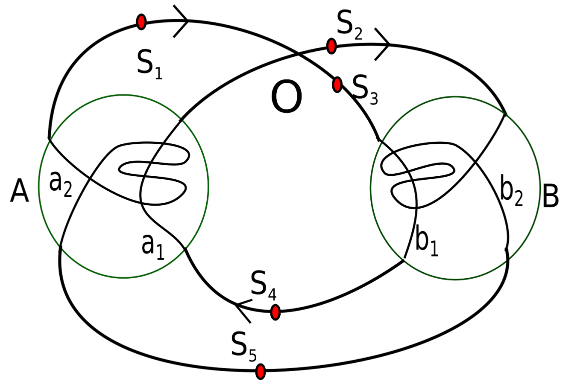

FIGURE 5.1.9. Both A and B have property (INI). The arcs a_2 and b_2 have property (I).

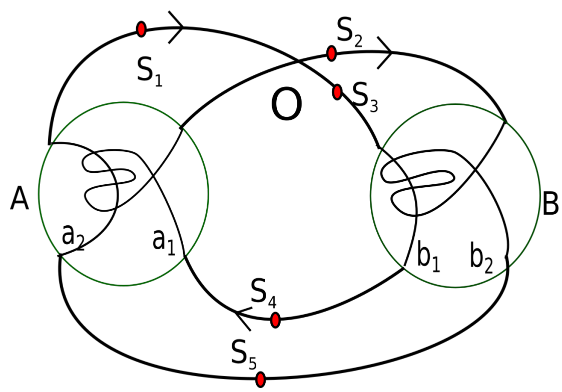

FIGURE 5.1.10. . Both A and B have property (INI) and the arcs a_1 and b_2 have property (I).

Using the same method to traverse the conceptual diagram and its flyped, we obtain other symbolic loop numbers starting at other positions in D, as shown below.

$$lp(D, S_2, d) = lp(b_2) + lp(a_2) + lp(b_1|b_2) + lp(a_1) + 2 := d;$$

$$lp(D, S_3, d) = lp(b_1) + lp(a_1) + lp(b_2) + lp(a_2) + 2 := a;$$

$$lp(D, S_4, d) = lp(a_1) + lp(b_2) + lp(a_2) + lp(b_1|b_2) + 2 := d;$$

$$lp(D, S_5, d) = lp(a_2) + lp(b_1) + lp(a_1|a_2) + lp(b_2) + 2 := j.$$

The collection of loop numbers obtained in this case is

$$Lp_a = \{a, d, a, d, j\}.$$

In the flyped diagram D', very similarly, we obtain the following outside loop numbers:

$$lp(D', S'_1, d) = lp(b_1) + lp(a_1) + lp(b_2) + lp(a_2) + 2 := a;$$

$$lp(D', S'_2, d) = lp(a_2) + lp(b_1) + lp(a_1|a_2) + lp(b_2) + 2 := j;$$

$$lp(D', S'_3, d) = lp(a_1) + lp(b_2) + lp(a_2) + lp(b_1|b_2) + 2 := d;$$

$$lp(D', S'_4, d) = lp(b_2) + lp(a_2) + lp(a_1) + lp(b_1|b_2) + 2 := d;$$

$$lp(D', S'_5, d) = lp(b_1) + lp(a_1) + lp(b_2) + lp(a_2) + 2 := a.$$

Let Lp'_a denote the collection of outside loop numbers in D', then

$$Lp'_a = \{a, j, d, d, a\}.$$

Subcase b: The following are outside loop numbers obtained in the diagram D:

$$lp(D, S_1, d) = lp(a_1) + lp(b_2) + lp(a_2) + lp(b_1) + 2 := a;$$

$$lp(D, S_2, d) = lp(b_2) + lp(a_2) + lp(b_1) + lp(a_1|a_2) + 2 := j;$$

$$lp(D, S_3, d) = lp(b_1) + lp(a_1) + lp(b_2|b_1) + lp(a_2) + 2 := e;$$

$$lp(D, S_4, d) = lp(a_1) + lp(b_2) + lp(a_2) + lp(b_1) + 2 := a;$$

$$Lp(D, S_5, d) = lp(a_2) + lp(b_1) + lp(a_1|a_2) + lp(b_2) + 2 := j.$$

Let Lp_b denote the collection of outside loop numbers in this case, then

$$Lp_b = \{a, j, e, a, j\}.$$

Consider a flype on tangle B of D in this sub-case. The resulted diagram is D'. See Figure 5.1.3.

We get the following outside loop numbers in D':

$$lp(D', S'_1, d) = lp(a_1) + lp(a_2) + lp(b_1) + lp(b_2) + 2 =: a;$$

$$lp(D', S'_2, d) = lp(a_2) + lp(b_1) + lp(a_1|a_2) + lp(b_2) + 2 := j;$$

$$lp(D', S'_3, d) = lp(a_1) + lp(a_2) + lp(b_1) + lp(b_2) + 2 := a;$$

$$lp(D', S'_4, d) = lp(b_1) + lp(b_2) + lp(a_2) + lp(a_1|a_2) + 2 := j;$$

$$lp(D', S'_5, d) = lp(b_1) + lp(a_1) + lp(b_2|b_1) + lp(a_2) + 2 := e.$$

The flyped diagram D' in this case produces a collection of loop numbers as follows:

$$Lp'_b = \{a, j, a, j, e\}.$$

Based on this results, we conclude that when both tangles have parity (0), the collection of distinct outside loop numbers in the direction d does not change under a flype.

Case II: Both tangles A and B have parity (1)

The method used to evaluate loop numbers is similar to Case I. Therefore, we will report only the results at various starting positions in D and D', leaving the details to the readers.

In a parity (1) tangle $A(a_1, a_2)$, both arcs a_1 and a_2 intersect the left-over arc of the other. As both A and B have parity (1), they both have property $(2I)$.

Figure 5.1.11 shows an example of a diagram of parity (1) tangles in parallel orientation.

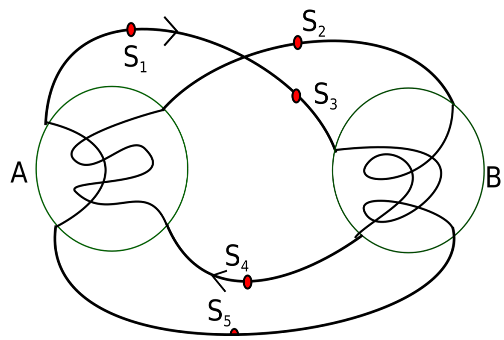

FIGURE 5.1.11. An example where both tangles have parity (1).

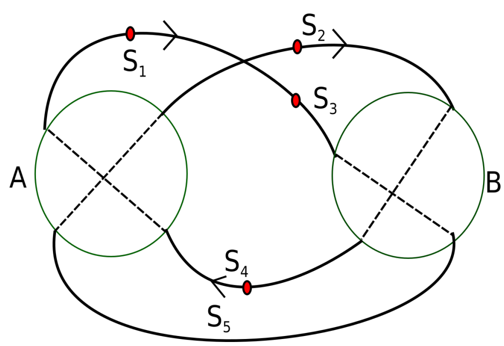

FIGURE 5.1.12. A flyping circuit of all parity (1) tangles.

Starting at S_1 on D, we obtain the following loop number:

$$lp(S_1, D, d) = lp(b_1) + lp(a_1) + 1 + lp(b_2) + lp(a_2) + 1 := a.$$

Starting at S_2, we obtain the loop number below:

$$lp(S_2, D, d) = lp(a_1) + lp(a_2) + lp(b_2) + lp(b_1|b_2) + 2 := d.$$

Similarly, we obtain the following loop numbers by starting at S_3, S_4, and S_5:

$$lp(S_3, D, d) = lp(a_1) + lp(a_2) + lp(b_1) + lp(b_2|b_1) + 2 := e.$$

$$lp(S_4, D, d) = lp(a_2) + lp(b_1) + lp(b_2) + lp(a_1|a_2) + 2 := j.$$

$$lp(S_5, D, d) = lp(a_1) + lp(b_1) + lp(b_2) + lp(a_2|a_1) + 2 := f.$$

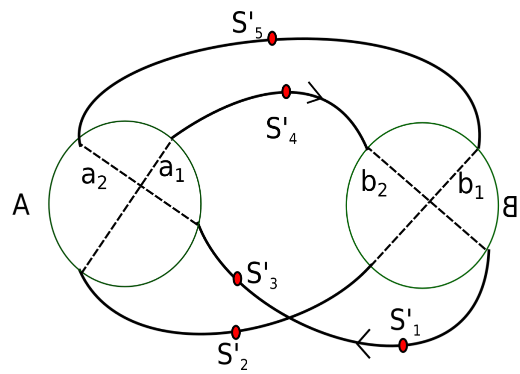

FIGURE 5.1.13. The flyped diagram D' of D in Figure 5.1.12.

Figure 5.1.13 shows a flyped diagram of D. In D', we obtain the following loop numbers dependent on starting positions:

$$lp(S'_1, D', d) = lp(a_2) + lp(b_1) + 1 + lp(a_1) + lp(b_2) + 1 := a;$$

$$lp(S'_2, D', d) = lp(a_1) + lp(b_1) + lp(b_2) + lp(a_1|a_2) + 2 := f;$$

$$lp(S'_3, D', d) = lp(a_2) + lp(b_1) + lp(b_2) + lp(a_1|a_2) + 2 := j;$$

$$lp(S'_4, D', d) = lp(a_1) + lp(a_2) + lp(b_2) + lp(b_1|b_2) + 2 := d;$$

$$lp(S'_5, D', d) = lp(a_1) + lp(a_2) + lp(b_1) + lp(b_2|b_1) + 2 := e.$$

From diagram D, we obtain a collection Lp of loop numbers starting at S_i, for $i = 1, 2, 3, 4, 5$:

$$Lp = \{a, d, e, j, f\};$$

and of D':

$$Lp' = \{a, f, j, d, e\}.$$

This implies that the collection of outside loop numbers does not change after a flype.

5.1.2. Antiparallel orientation. A flyping circuit is antiparallelly oriented if it contains an (∞) tangle. Let D be a diagram of a flyping circuit formed by $N(A+B+t(\pm 1))$, we have the following cases:

- Case I: One tangle has has parity (∞), and the other has parity (1).

- Case II: One tangle has has parity (∞), and the other has parity (0).

If both tangles have parity (∞), then the resulted structure is a link, which is not considered in this proof.

Case I: One tangle has has parity (∞), and the other has parity (1).

There are two situations, depending on whether A or B has parity (∞):

- The tangle B has parity (1), and A has parity (∞).

- The tangle A has parity (1), and B has parity (∞).

The tangle with parity (1) must also have property (I) in terms of its arc intersection.

(1) The tangle B has parity (1), and A has parity (∞). Therefore, there are 3 possible situations for D, depending on the arcs of A.
   (a) Subcase I.1: A has property (2I).
   (b) Subcase I.2: A has property (2NI).
   (c) Subcase I.3: A has property (INI).

   The diagram D in Figure 5.1.14 conceptually illustrates three subcases above. Note that the only actual arcs of the flyping circuit are in black. The shaded areas define regions R_1, R_2, R_3. By Theorem 3.9, we always have $lp(D, S_1) = lp(D, S_6)$. Thus, we will not consider the position S_6.

   Subcase I.1: A has property (2I)

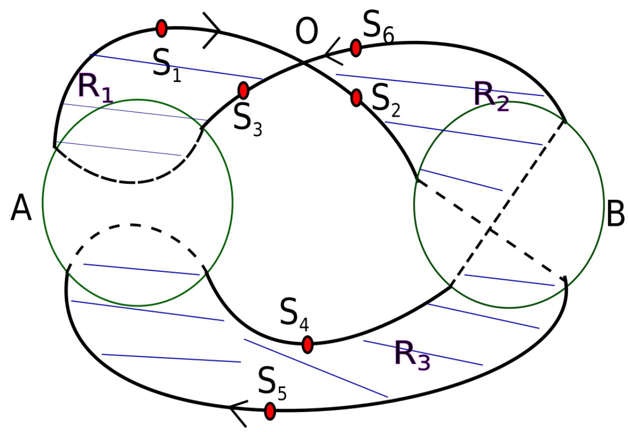

FIGURE 5.1.14. Flyping circuit of a parity (1) tangle and a (∞) tangle.

Starting at S_1 or S_2, we obtain a loop number which is independent with the property $(2I)$ of A.

$$lp(S_1, D, d) = lp(b_1) + lp(a_2) + lp(b_2|b_1) + lp(a_1) + 3 := a, \text{ and}$$

$$lp(S_2, D, d) = lp(b_1) + lp(a_2) + lp(b_2|b_1) + lp(a_1) + 3 := a.$$

The loop number starting at S_3 is affected by the property $(2I)$ of A. When traversing A the second time through a_2, one picks up a conditional loop number. The region R_2 is merged with R_3, leaving only 2 outer loops.

$$lp(S_3, D, d) = lp(a_1) + lp(b_1) + lp(a_2|a_1) + lp(b_2) + 2 = b.$$

Starting at S_4 is independent with the $(2I)$ property of A:

$$lp(S_4, D, d) = lp(b_2) + lp(a_1) + lp(b_1|b_2) + lp(a_2) + 3 = c.$$

Starting at S_5, we have a conditional loop number on A. There are 2 outer loops as region R_1 is merged with R_3 by the left-over arcs of a_2, b_2, and a_1 passing through O.

$$lp(S_5, D, d) = lp(a_2) + lp(b_2) + lp(a_1|a_2) + lp(b_1) + 2 = d.$$

The collection of loop numbers we obtain from D is then,

$$Lp_1 = \{a, a, b, c, d\}$$

Consider a flype on tangle B, into diagram D' as in Figure 5.1.15. The diagram D' also serves 2 other cases that follow. Using similar techniques, we obtain the following loop numbers:

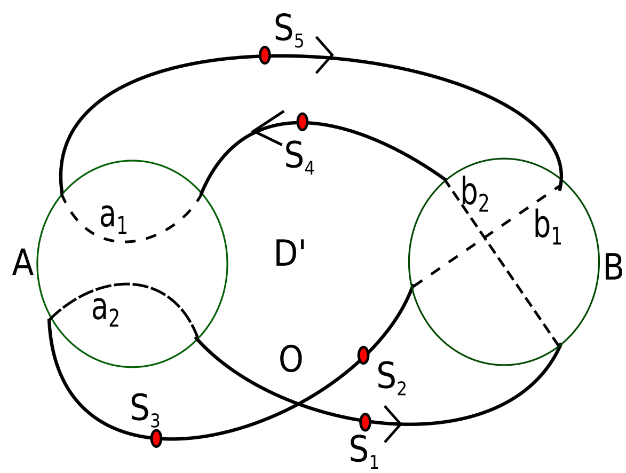

FIGURE 5.1.15. The flyped diagram D' where A has property $(2I)$, and B has parity (1).

$$lp(S'_1, D, d) = lp(b_2) + lp(a_1) + lp(b_1|b_2) + lp(a_2) + 3 = c;$$

$$lp(S'_2, D, d) = lp(a_2) + lp(b_2) + lp(a_1) + lp(b_1|b_2) + 3 = c;$$

$$lp(S'_3, D, d) = lp(a_2) + lp(b_2) + lp(a_1|a_2) + lp(b_1) + 2 = d;$$

$$lp(S'_4, D, d) = lp(a_1) + lp(b_1) + lp(a_2|a_1) + lp(b_2) + 2 = b;$$

$$lp(S'_5, D, d) = lp(b_1) + lp(a_2) + lp(b_2|b_1) + lp(a_1) + 3 = a.$$

Let Lp' denote the collection of loop numbers in D', then

$$Lp' = \{c, c, d, b, a\}.$$

Subcase I.2: A has property (2NI)

Figure 5.1.16 is a particular example for this case.

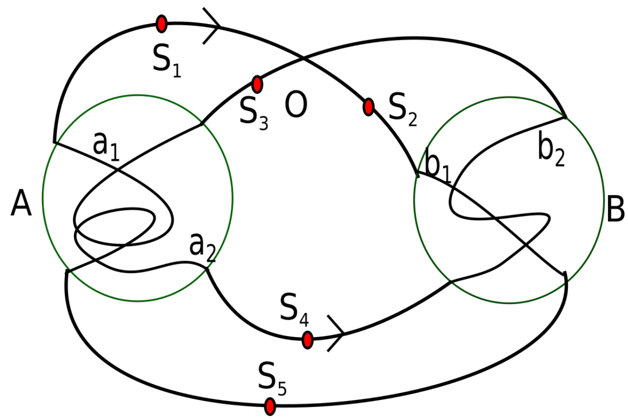

FIGURE 5.1.16. An example where A has parity (∞) and its arcs have property ($2NI$).

Starting at any of the three positions S_1, S_2, or S_3, we obtain the same loop number:

$$lp(D, S_1, d) = lp(a_1) + lp(a_2) + lp(b_1) + lp(b_2|b_1) + 3 = a;$$

$$lp(D, S_2, d) = lp(a_1) + lp(a_2) + lp(b_1) + lp(b_2|b_1) + 3 = a;$$

$$lp(D, S_3, d) = lp(a_1) + lp(a_2) + lp(b_1) + lp(b_2|b_1) + 3 = a;$$

Starting at S_4 or S_5, we also obtain identical loop numbers.

$$lp(D, S_4, d) = lp(a_1) + lp(a_2) + lp(b_2) + lp(b_1|b_2) + 3 = c;$$

$$lp(D, S_5, d) = lp(a_2) + lp(b_2) + lp(a_1) + lp(b_1|b_2) + 3 = c.$$

In summary, when A has property ($2NI$) and B has property ($2I$), D has the following collection of outer loop numbers:

$$Lp = \{a, a, a, c, c\}.$$

When we flype tangle B, the resulting diagram D' has the following loop numbers:

$$lp(D', S_1', d) = lp(b_2) + lp(a_1) + lp(b_1|b_2) + lp(a_2) + 3 = c;$$

$$lp(D', S_2', d) = lp(a_2) + lp(b_2) + lp(a_1) + lp(b_1|b_2) + 3 = c;$$

$$lp(D', S_3', d) = lp(a_2) + lp(b_2) + lp(a_1) + lp(B1|b_2) + 3 = c;$$

$$lp(D', S_4', d) = lp(a_1) + lp(b_1) + lp(a_2) + lp(b_2|b_1) + 3 = a;$$

$$lp(D', S_5', d) = lp(a_1) + lp(b_1) + lp(a_2) + lp(b_2|b_1) + 3 = a.$$

In summary, we obtain the following collection of outer loop numbers for D':

$$Lp' = \{c, c, c, a, a\}.$$

Subcase I.3: A has property (INI)

Figure 5.1.17 is an example of D in this case, where a_2 has property (I). We will consider the case when a_1 has property (I) later.

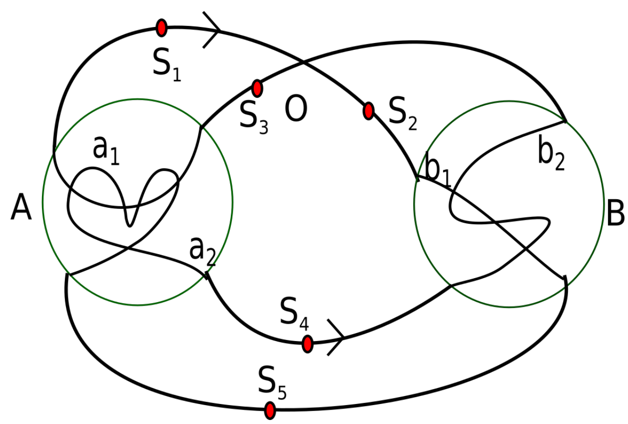

FIGURE 5.1.17. A has property (INI), where the left-over arc of a_1 intersects a_2. B is a parity (1) tangle.

We obtain the following loop numbers for diagram D:

$$lp(D, S_1, d) = lp(b_1) + lp(a_2) + lp(b_2|b_1) + lp(a_1) + 3 = a;$$

$$lp(D, S_2, d) = lp(b_1) + lp(a_2) + lp(b_2|b_1) + lp(a_1) + 3 = a;$$

$$lp(D, S_1, d) = lp(a_1) + lp(b_1) + lp(a_2|a_1) + lp(b_2) + 2 = b;$$

$$lp(D, S_4, d) = lp(a_2) + lp(b_2) + lp(a_1) + lp(b_1|b_2) + 3 = c;$$

$$lp(D, S_5, d) = lp(b_2) + lp(a_1) + lp(b_1|b_2) + lp(a_2) + 3 = c.$$

Let Lp_a be the collection of loop numbers in D for this case. Then,

$$Lp_a = \{a, a, b, c, c\}.$$

Now we consider a flype on tangle B. The resulting diagram D' has the following loop numbers:

$$lp(D', S_1', d) = lp(b_2) + lp(a_1) + lp(b_1|b_2) + lp(a_2) + 3 = c;$$

$$lp(D', S_2', d) = lp(a_2) + lp(b_2) + lp(a_1) + lp(b_1|b_2) + 3 = c;$$

$$lp(D', S_3', d) = lp(a_2) + lp(b_2) + lp(a_1) + lp(b_1|b_2) + 3 = c;$$

$$lp(D', S_4', d) = lp(a_1) + lp(b_1) + lp(a_2|a_1) + lp(b_2) + 2 = b;$$

$$lp(D', S_5', d) = lp(b_1) + lp(a_2) + lp(b_2|b_1) + lp(a_1) + 3 = a.$$

The collection of loop numbers obtained in D' is as follows:

$$Lp_a' = \{c, c, c, b, a\}.$$

We now consider the case when a_2 has property (I). See Figure 5.1.18 for an example. Using the similar technique, we get the following for outside loop

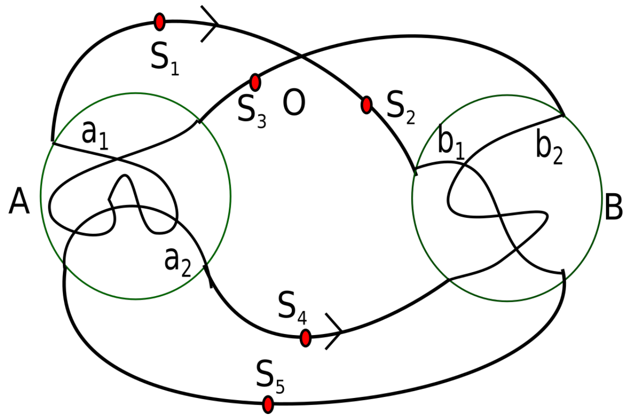

FIGURE 5.1.18. The left-over arc of a_2 intersects a_1.

numbers of D and D' accordingly:

$$lp(S_1) = lp(b_1) + lp(a_2) + lp(b_2|b_1) + lp(a_1) + 3 := a;$$

$$lp(S_2) = lp(b_1) + lp(a_2) + lp(b_2|b_1) + lp(a_1) + 3 := a;$$

$$lp(S_3) = lp(b_1) + lp(a_2) + lp(b_2|b_1) + lp(a_1) + 3 := a;$$

$$lp(S_4) = lp(b_2) + lp(a_1) + lp(b_1|b_2) + lp(a_2) + 3 := c;$$

$$lp(S_5) = lp(a_2) + lp(b_2) + lp(a_1|a_2) + lp(b_1) + 2 := b.$$

In the flyped diagram D', we obtain the following outside loop numbers:

$$lp(S_1') = lp(b_2) + lp(a_1) + lp(b_1|b_2) + lp(a_2) + 3 := c;$$

$$lp(S_2') = lp(a_2) + lp(b_2) + lp(a_1|a_2) + lp(b_1) + 2 := b;$$

$$lp(S_3') = lp(a_2) + lp(b_2) + lp(b_1|b_2) + lp(a_1) + 3 := c;$$

$$lp(S_4') = lp(a_1) + lp(b_1) + lp(a_2) + lp(b_2|b_1) + 3 := a;$$

$$lp(S_5') = lp(b_1) + lp(a_2) + lp(b_2|b_1) + lp(a_1) + 3 := a.$$

Let Lp_b and Lp'_b be the collections of outside loop numbers in D and D' accordingly, then

$$Lp_b = \{a, a, a, c, b\} \text{ and}$$

$$Lp'_b = \{c, b, c, a, a\}.$$

(2) Finally, we consider the case when the tangle A has parity (1), and B has parity (∞). Figure 5.1.19 illustrates the situation with an example of D. The flyped diagram D' in Figure 5.1.20 has the same structure as the diagram D in previous case where we flype a parity (1) tangle. Thus, we obtain similar results as the previous case and leave details to the reader. In this case, the values a, b, c represent the same symbolic loop numbers as the case right before. The following collections of outside loop numbers conclude Case I.

$$Lp = \{c, b, c, a, a\};$$

$$Lp' = \{a, a, a, c, b\}.$$

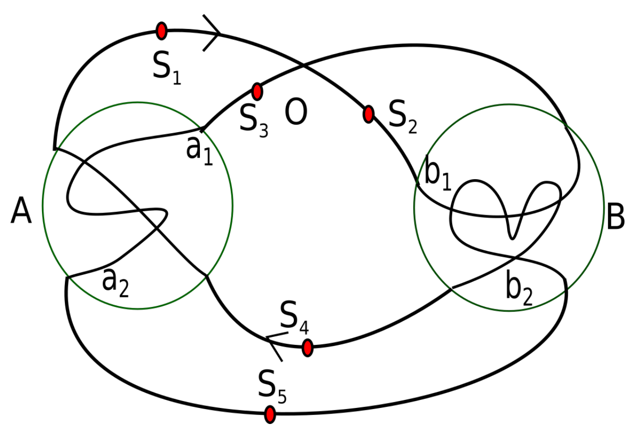

FIGURE 5.1.19. D has a parity (1) tangle A, and a parity (∞) tangle B.

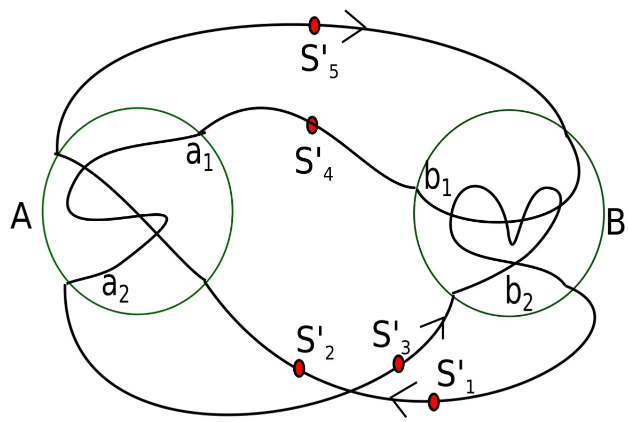

FIGURE 5.1.20. D' resembles the structure of D in previous situation in Figure 5.1.18.

Case II: One tangle has property (∞) and the other has property (0).

The final case consists of an anti-parallel flyping circuit where the two tangles have parities (∞) and (0). As in previous cases, $a, b, c, ...$ denote integer values when comparing the outside loop numbers of D and D'.

(1) Suppose the tangle A has parity (0), and B has parity (∞). As the arcs of each tangle may have one of the three properties: $(2I)$, $(2NI)$, and (INI), we have a combination of 6 cases, since some combinations can be eliminated by symmetry. In all sub-cases that follow, we refer to the diagram D shown in

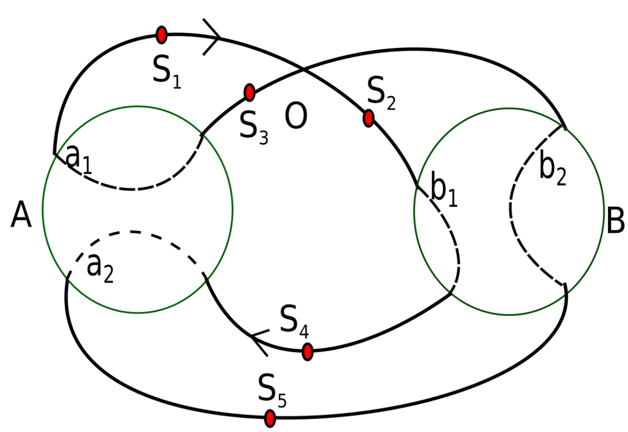

FIGURE 5.1.21. A conceptual diagram where A has parity (0), and B has property (∞).

Figure 5.1.21, and a diagram D' shown in Figure 5.1.22.

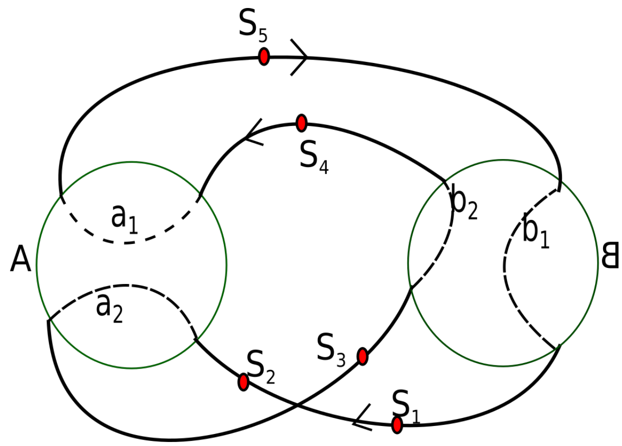

FIGURE 5.1.22. A flype on B from Figure 5.1.21.

Using the same techniques as one does for the parallel orientation and parity (0), we get the following subcases.

(a) Both tangles have property $(2NI)$ We obtain a single value for all starting positions in both D and D':

$$For\ i = 1, 2, .., 5:\ lp(D, S_i, d) = lp(a_1) + lp(a_2) + lp(b_1) + lp(b_2) + 2 := a$$

$$Lp = \{a, a, a, a, a\}, \text{ and}$$

$$Lp' = \{a, a, a, a, a\}.$$

(b) When the tangle B has property $(2I)$, and A has either property $(2NI)$ or (INI), the situation is identical to the case when B has parity (1). We will only summarize the outcome and leave the details to the reader.

$$lp(D, S_1, d) = lp(a_1) + lp(a_2) + lp(b_1) + lp(b_2|b_1) + 3 = a;$$

$$lp(D, S_2, d) = lp(a_1) + lp(a_2) + lp(b_1) + lp(b_2|b_1) + 3 = a;$$

$$lp(D, S_3, d) = lp(a_1) + lp(b_2) + lp(b_1) + lp(a_2|a_1) + 3 = b;$$

$$lp(D, S_4, d) = lp(a_2) + lp(b_2) + lp(b_1) + lp(a_1|a_2) + 3 = d;$$

$$lp(D, S_5, d) = lp(a_1) + lp(a_2) + lp(b_2) + lp(b_1|b_2) + 3 = c.$$

When both A and B have property $(2I)$, the collections of outside loop numbers are:

$$Lp = \{a, a, b, d, c\};$$

$$Lp' = \{a, d, c, b, a\}.$$

When A has property $(2NI)$, the collections of of outside loop numbers are:

$$Lp = \{a, a, a, c, c\};$$

$$Lp' = \{a, c, c, c, a\}.$$

(c) A has property $(2I)$, and B has property (INI).

See Figure 5.1.23 for an example. We obtain the following loop numbers

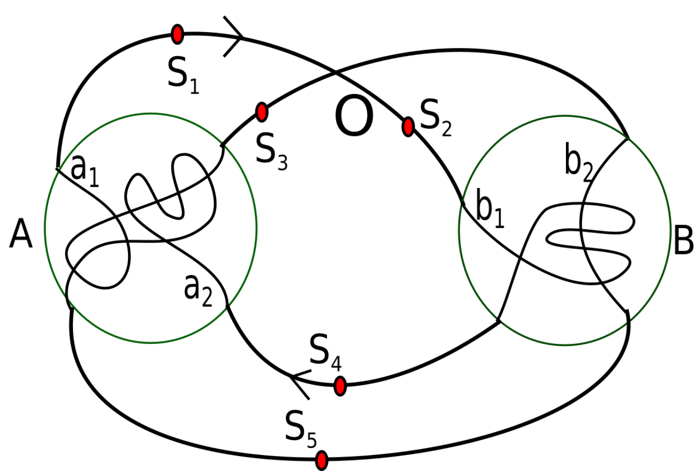

FIGURE 5.1.23. A has property $(2I)$. B has property (INI).

in D:

$$lp(D, S_1, d) = lp(a_1) + lp(a_2) + lp(b_1) + lp(b_2|b_1) + 3 = g;$$

$$lp(D, S_2, d) = lp(a_1) + lp(a_2) + lp(b_1) + lp(b_2|b_1) + 3 = g.$$

Starting at S_3, or S_4, unlike at other starting points, we have 2 outer-loops. The value conditional loop number containing arcs of A is counted for inside A:

$$lp(D, S_3, d) = lp(a_1) + lp(b_1) + lp(b_2) + lp(a_2|a_1) + 2 = e;$$

$$lp(D, S_4, d) = lp(a_2) + lp(b_2) + lp(b_1) + lp(a_1|a_2) + 2 = d.$$

Starting at S_5, as we traverse the NI arc b_2 first, we merge the two outer-loops containing b_2, a_1 connecting with O into one.

$$lp(D, S_5, d) = lp(a_1) + lp(a_2) + lp(b_2) + lp(b_1) + 2 = a.$$

The collection of outside loop numbers in D is:

$$Lp = \{g, g, e, d, a\}.$$

In the flyped diagram, the collection of outside loop numbers becomes

$$Lp' = \{g, d, e, a, a\}.$$

(d) A has property $(2NI)$ and B has property (INI). Figure 5.1.24 is an example where the arc with (I) property is b_2. Starting at S_i, for $i = 1, 2, 3$, we obtain the following outside loop numbers:

$$lp(D, S_i, d) = lp(a_1) + lp(a_2) + lp(b_1) + lp(b_2|b_1) + 3 = g.$$

Starting at S_j, for $j = 4, 5$, as we traverse the (NI) arc b_2 first, the arc b_2 connects with a_2 and O to make a single outerloop:

$$lp(D, S_j, d) = lp(a_2) + lp(a_1) + lp(b_2) + lp(b_1) + 2 = a.$$

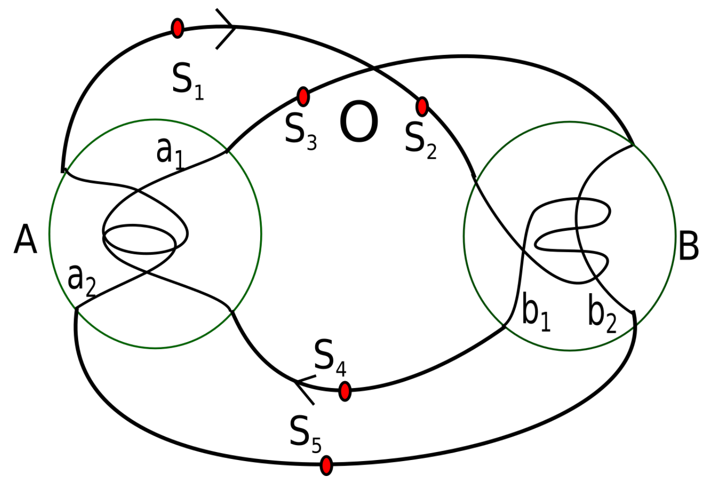

FIGURE 5.1.24. A has property $(2NI)$, B has property (INI), and b_1 intersects the left-over arc of b_2.

The collection of outside loop numbers of D is:

$$Lp = \{g, g, g, a, a\}.$$

When we flype B, the collection of outside loop numbers becomes

$$Lp' = \{a, a, g, g, a\}.$$

Figure 5.1.25 is an example where the arc with property (I) is b_1. By symmetry, we obtain the same collection of loop numbers as the previous case.

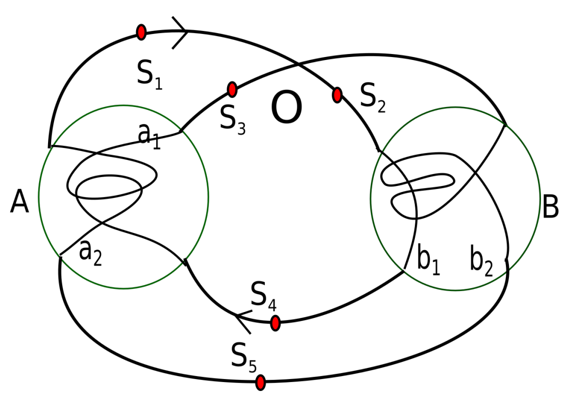

FIGURE 5.1.25. A has property $(2NI)$, B has property (INI), and b_2 intersects the left-over arc of b_1.

(e) A and B both have property (INI). We check subcases where the order of (I) property changes on A. Then by flyping B, we make sure to cover all possible situations by symmetry.

(i) Subcase a: The left-over arc of a_2 intersects a_1. See Figure 5.1.26 The outside loop numbers of D are as follows:

$$lp(D,S_1) = lp(b_1) + lp(a_2) + lp(b_2|b_1) + lp(a_1) + 2 = lp(D,S_2) = lp(D,S_3) := c;$$

$$lp(D,S_4) = lp(a_2) + lp(b_2) + lp(a_1|a_2) + lp(b_1) + 2 := d;$$

$$lp(D,S_5) = lp(b_2) + lp(a_1) + lp(b_1) + lp(a_2) + 2 := a.$$

The collections of outside loop numbers for D and D' accordingly are:

$$Lp = \{c, c, c, d, a\}, \text{ and}$$

$$Lp' = \{a, d, a, c, c\}.$$

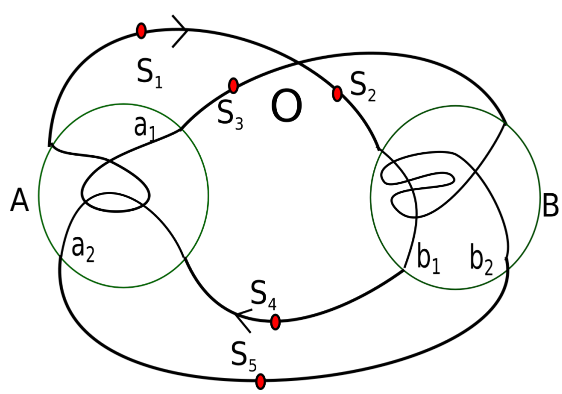

FIGURE 5.1.26. Both tangles have property (INI), and the left-over arc of a_2 intersects a_1.

(ii) Subcase b: When the left-over arc of a_1 intersects a_2, we obtain the following collections of outside loop numbers in D:

$$lpD,S_1) = lp(b_1) + lp(a_2) + lp(b_2|b_1) + lp(a_1) + 3 := c;$$

$$lp(D, S_2) = lp(b_1) + lp(a_2) + lp(b_2|b_1) + lp(a_1) + 3 := c;$$

$$lp(D, S_3) = lp(a_1) + lp(b_1) + lp(a_2|a_1) + lp(b_2) + 2 := e;$$

$$lp(D, S_4) = lp(a_2) + lp(b_2) + lp(a_1) + lp(b_1) + 2 := a;$$

$$lp(D, S_5) = lp(b_2) + lp(a_1) + lp(b_1) + lp(a_2) + 2 := a.$$

We compute outside loop numbers for D' similarly, and obtain the following collections:

$$Lp = \{c, c, e, a, a\}, \text{ and}$$

$$Lp' = \{a, a, a, e, c\}.$$

Figure 5.1.27 is an example of this subcase.

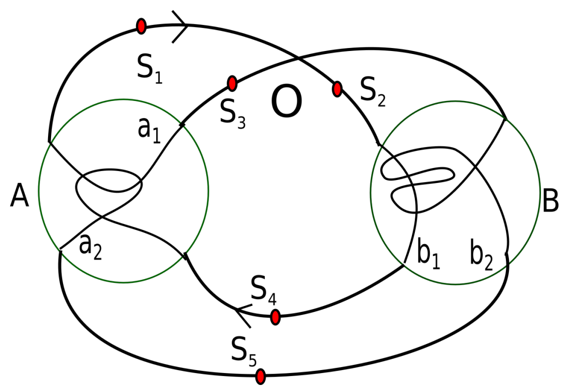

FIGURE 5.1.27. Both tangles have property (INI), and the left-over arc of a_1 intersects a_2.

(2) Tangle A has parity (∞) and tangle B has parity (0). The flype happens on a parity (0) tangle, resulting in D' that has the same situation as the diagram D in the first case. Thus, by symmetry, this situation will result in the same collections of loop numbers as the subcase II.1 and can be skipped.

By considering tangle properties and all different combinations of outside starting positions, we have shown that a flype results in the collection of outside loop numbers having the same distinct values with different frequencies. Therefore, with the given diagram D and starting positions S_i, we generate D' and conclude that:

$$maxlp(D, S_i, d) = maxlp(D', S_i, d), \text{ and}$$

$$minlp(D, S_i, d) = minlp(D', S_i, d).$$

□

To show that the pair $minLp(D)$ and $maxLp(D)$ is an invariant, we need to complete the argument by considering loop numbers obtained by starting on arcs inside the tangles in the given flyping circuit. Chapter 6 deals with inside loop numbers. In the remaining sections of this chapter, we show the invariance of loop numbers more generally.

5.2. The general parallel flyping circuit

After considering how outside loop numbers appear in a simple flyping circuit of both orientations, we are ready to show Theorem 5.2, stated in Chapter 4.

We suppose that the flyping circuit C contains k tangles, one of which is a single crossing, at the jth position. We assume that the flyping circuit decomposition is maximal, that is no further tangle decomposition can be made in it. With this assumption, each half-twist can be considered a one-crossing tangle.

Outside loop numbers denote the loop numbers obtained from traversing the circuit C starting at an arc not within the boundary of any tangle.

THEOREM 5.2. *Let D be a diagram of a parallel flyping circuit containing k tangles, in which the j-th tangle contains only one crossing as shown in Figure 5.2.1, then the collection of outside loop numbers stays invariant under flypes.*

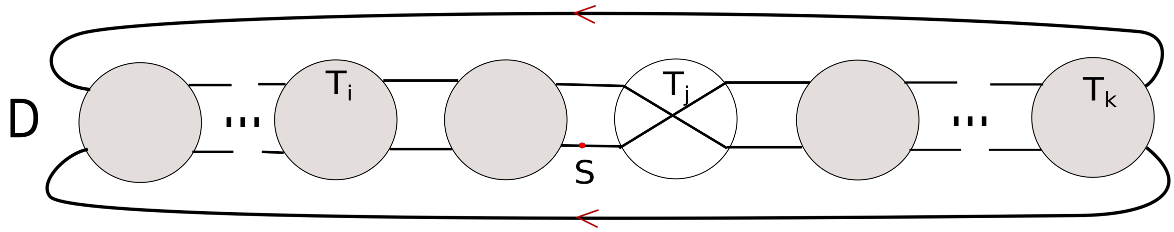

FIGURE 5.2.1. The single crossing tangle T_j always exists in the general parallel flyping circuit of k tangles.

PROOF. We notice that a flype can move T_j from one position to another in the circuit. A number of such flypes only changes the index of this T_j tangle. Let D' denote the resulting diagram after any number of such flypes. All tangles of D include: $T_1, T_2, ..., T_j, ..., T_k$.

We pick a traversal direction d and a starting point S on an outside arc. For each tangle T_i we enter, let t_{i_1} be the arc we traverse first and t_{i_2} be the arc we traverse last. Since T_j has property (I), there is at least one tangle $T_i(t_{i_1}, t_{i_2})$ so that t_{i_2} has property (I) with respect to t_{i_1}. If there are several tangles with the property (I), then we pick T_i to the first tangle with this property (I) encountered when traversing D starting at S. Note that it is possible that $i = j$. Then , the loop number obtained from the diagram D, starting at S in the chosen direction is:

$$lp(D,S) = 2 + lp(t_{i_1}) + lp(t_{i_2}|t_{i_1}) + \sum_{s \neq i} (lp(t_{s_1}) + lp(t_{s_2})).$$

The value 2 comes from the following two outer-loops:

- One forms at the intersection of t_{i_1} and t_{i_2} due to property (I) above.

- The other forms at the end of the process when the loop collection process ends.

The sum $\sum_{s \neq i} (lp(t_{s_1}) + lp(t_{s_2}))$ denotes the number of loops obtained from each of the $k-1$ tangles. As the first big loop removed contains one of the strands t_{i_1}, no other conditional loop number appears.

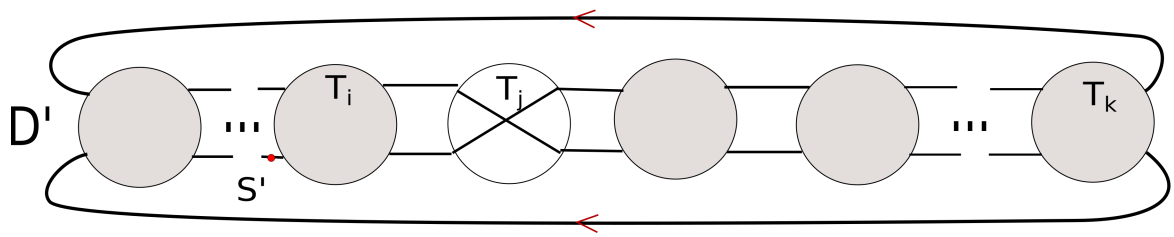

FIGURE 5.2.2. A flype moves T_j from one portion to an adjacent portion (compare to Figure 5.2.1).

A single flype moves T_j from one portion in the flyping circuit to an adjacent portion. See Figure 5.2.2. A number of flypes moves this single crossing tangle to any other portion. Regardless of the number of flypes, pick S' on the diagram D' so that in the same direction d, S' immediately leads to the arc t_{i_1}. Then, we have the following loop number:

$$lp(D', S') = 2 + lp(t_{i_1}) + lp(t_{i_2}|t_{i_1}) + \sum_{s \neq i} (lp(t_{s_1}) + lp(t_{s_2}))$$

Since the starting position S is arbitrary, regardless where the starting position is on the general flyping cicuit, the loop number obtained from D repeats in the flyped diagram D'. This result applies under no additional assumption on the knotting nature of the k tangles. □

5.3. The general antiparallel flyping circuit

This section addresses the invariance of outside loop numbers in an antiparallel flyping circuit similarly to Theorem 5.1. The strategy to show for a general case is, as done previously, to pick an arbitrary starting position in D, then shows that there exists a position in D' which yields the same loop number.

THEOREM 5.3. *In a diagram D containing an antiparallel flyping circuit, the set of distinct outside loop numbers remains invariant under flypes.*

PROOF. As before, we assume that the flyping circuit contains k tangles: $T_1, T_2, ..., T_k$ and it has a maximal tangle decomposition. Figure 5.3.1 illustrates this structure.

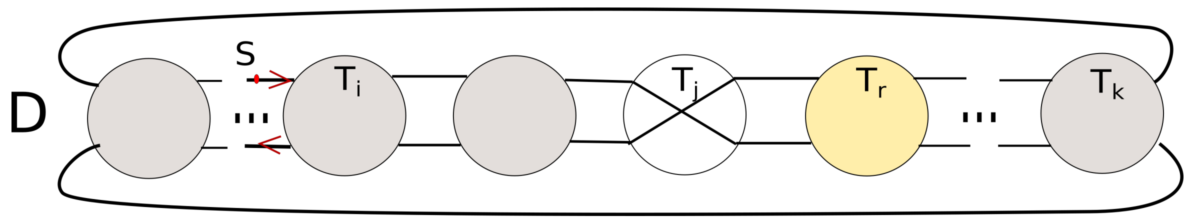

FIGURE 5.3.1. An antiparallel flyping circuit contains k tangles.

For a flyping circuit to be anti-parallel, there exists exactly one parity (∞) tangle. If there is more than one (∞) tangle, then we have a link. Without the loss of generality, let this tangle be T_r. We denote the single crossing tangle by T_j. Pick any starting point S leading into some tangle at the $i-th$ position in D. As previously done, Let t_{i_1} be the arc of T_i encountered first and t_{i_2} be the arc encountered later in T_i when we traverse D starting at S and in the direction d.

Case I: Assuming t_{r_2} has property (NI) with t_{r_1}, then the loop number obtained from D starting at S is:

$$lp(D,S) = \sum_{i \neq r}^{k} lp(t_{i_1}) + \sum_{i \neq r}^{k} lp(t_{i_2}|t_{i_1}) + lp(r_1) + lp(r_2) + n,$$

where n is the total number of outer loops, that is $n-1$ equals to the number of arcs t_{i_2} has property (I) with respect to t_{i_1}.

Note that if t_{i_2} has property (NI) with t_{i_1}, then, $lp(t_{i_2}|t_{i_1}) = lp(t_{i_2})$, and no outer loop is formed when traversing t_{i_2}.

Thus, the value of n is the number of tangles which t_{i_2} has property (I) with respect to t_{i_1} plus 1. Trivially, $n \leq k+1$. We can verify this with the simplest case when an infinity tangle is involved, the maximum number of outer-loop obtained is 3. See the cases in Subsection 5.1.2 when it is possible to pick up 3 outerloops.

We consider the loop number of a corresponding position S' in the flyped diagram D'. As shown before, the single crossing tangle moves to some other position. If S was previously chosen in D to lead first to t_{i_1}, then locate S' so that it leads first to this same strand t_{i_1} in D'. With such choice, we obtain

$$lp(D', S') = \sum_{i \neq r}^{k} lp(t_{i_1}) + \sum_{i \neq r}^{k} lp(t_{i_2}|t_{i1}) + lp(r_1) + lp(r_2) + n,$$

for the same n corresponding in D. Thus,

$$lp(D', S') = lp(D, S).$$

Case II: If we assume t_{r_2} has property (I) with t_{r_1}, then the loop number obtained from D starting at S depends on the relative position of S in the flyping circuit. Assume that the first tangle encountered when starting at S in direction d is T_u, so that we start to traverse the arc t_{u_1}. We have two sub-cases as follows.

Subcase 1:

All the tangles T_i encountered between S and entering T_r for the first time have the property that t_{i_1} has property (NI) with respect to t_{i_2}. Then,

$$lp(D, S) = \sum_{i \neq r}^{k} (lp(t_{i_1}) + lp(t_{i_2})) + lp(r_1) + lp(r_2|r_1) + 2.$$

If we choose $S \in D$ so that it leads immediately to T_r, then

$$lp(D, S) = lp(D', S').$$

Subcase 2:

There exists a tangle T_i encoutered between S and entering T_r for the first time so that t_{i_2} has property (I) with respect to t_{i_1}. So the first outer loop includes all of t_{r_1}, so that T_r always contributes no conditional loop number. Then, we obtain:

$$lp(D,S) = \sum_{i \neq r}^{k} lp(t_{i_1}) + \sum_{i \neq r}^{k} lp(t_{i_2}|t_{i_1}) + lp(r_1) + lp(r_2) + n;$$

where n is the number of outerloops. Again, there are two possibilities in the flyped diagram D':

- Start at the same position S' in D', that is we start by traversing the arc t_{u_1}. See Figure 5.3.2. If there still exists a tangle T_i encountered between S' and entering T_r for the first time so that t_{i_2} has property (I) with respect to t_{i_1}, then we get exactly the same loop number:

$$lp(D,S') = lp(D,S).$$

- The other case is that there does not exists a tangle T_i encountered between S' and entering T_r for the first time so that t_{i_2} has property (I) with respect to t_{i_1}. This is only possible if that tangle T_i consists of the single crossing tangle and in D' this single crossing tangle was flyped in a position that is on the other side of T_u. In this case we chose S' to be on the arc that leads out of T_u, that is if we traverse t_{u_2} using the direction d then t_{u_2} ends on an outside arc and we chose S' on this outside arc.

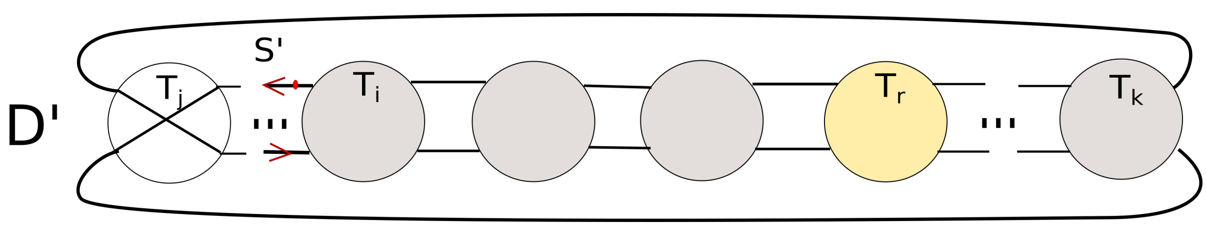

FIGURE 5.3.2. Two flypes have moved the single crossing tangle.

If we now traverse the diagram D' starting at S' then we will encountered $lp(r_2)$ first, however the the single crossing tangle must be traversed twice before we entered back into T_r and thus r_2 will be completely deleted before we entered T_r a second time. Thus no conditional loop number will appear for T_r. Further more for all other tangles (besides the one tangle) we entered the same arc first that we could have entered first when starting at S in the diagram D. Thus we obtain

$$lp(D, S') = lp(D, S).$$

In both assumptions, since the choice of S was arbitrary, we have shown that regardless where the outside starting position is in D, one can find a corresponding position S' in D' to obtain the same loop number. Thus, for an anti-parallel flyping circuit of k maximal tangles, the collection of outside loop numbers remains unchanged after any number of flypes. $\square$

Combing Theorem 5.2 and 5.3, we have the following result:

THEOREM 5.4. *In a flyping circuit where the decomposition of tangles is maximal, the set of distinct outside loop numbers are invariant under flypes.*

Chapter 6

Inside loop numbers

We now look into loop numbers of the diagram D given a starting point $P \in \{P_i\}$ inside a tangle $T(t_1, t_2)$, as conceptually illustrated in Figure 6.0.1. We want to compare the collection of loop numbers obtained from D with the one obtained from D', the resulting diagram when the tangle B is flyped. The combination of the parities of the two tangles makes 6 cases, which are grouped into Sections 6.1 and 6.2 with regards to the orientation of the flyping circuit the two tangles form.

- Section 6.1 includes parallel flyping circuits.
  (1) Case 1: Both tangles T and B have parity (0).
  (2) Case 2: Both tangles have parity (1).
- Section 6.2 includes antiparallel flyping circuits.
  (1) Case 3: T has parity (∞) and B has parity (0).
  (2) Case 4: T has parity (∞) and B has parity (1).
  (3) Case 5: T has parity (0) and B has parity (∞).
  (4) Case 6: T has parity (1) and B has parity (∞).

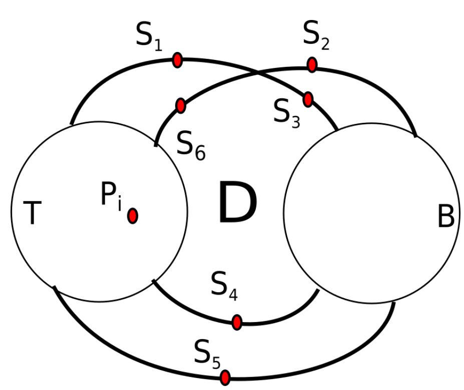

Figure 6.0.1. The six positions $\{S_1, S_2, S_3, S_4, S_5, S_6\}$ for outside loop numbers, $\{P_i\}$ for a number of inside starting points.

From Figure 6.0.1, we recall that S_i for $i = 1, 2, ...6$ stand for the six starting positions outside the two tangles in D and D'. Let $\{P_i\}$ denote the set of distinct starting points inside T (not the end-points, which are $S_i''s$) of the diagram D and D'.

Looking into each scenario allows us to generalize the appearance of loop numbers when starting points include the positions P_i inside the tangle T by Theorem 6.1.

THEOREM 6.1. *Given a diagram D of $N(T + B + t(\pm 1))$ where T and B are both rational, and $t(\pm 1)$ is a single crossing, then for P a starting point in $T \in D$, either $lp(P) = lp(P')$ or if $lp(P) \neq lp(P')$, then both $lp(P)$ and $lp(P')$ are outside loop numbers.*

Without the loss of generality, let $P_1 \in \{P_i\}$ be a starting point on t_1. P_1 divides t_1 into t_1' and t_1'' as shown in each of the following diagrams. In the proof of Theorem 6.1, the first two cases (in Section 6.1) require the tangles to be virtually unknotted. The last four cases require a stronger condition that the tangles are rational.

6.1. Inside loop numbers in a parallel flyping circuit

Case 1: Both tangles have parity (0). Figure 6.1.1 conceptually illustrates this case. In D, starting at P_1 in the given direction, recall that we say t_2 has property (I)

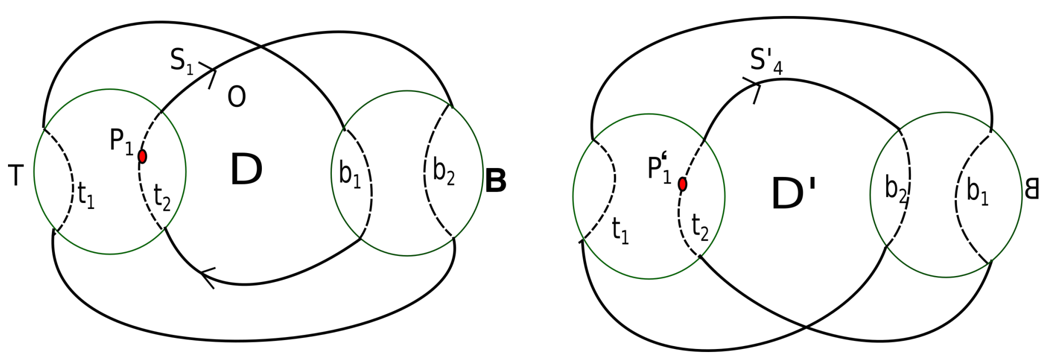

FIGURE 6.1.1. $P \in t_1$ inside the (0) tangle T.

with respect to t_1' when the left-over arc of t_1' intersects t_2. Assume that t_2 has property

(I) with respect to t'_1, then

$$lp(D,P_1) = lp(t'_1) + lp(b_2) + lp(t_2|t'_1) + lp(b_1|b_2) + lp(t''_1|t_2,t'_1) + 2.$$

When the $lp(t_2|t'_1)$ loops are obtained, an outer-loop made of the left-over of b_2 joined with some pieces of t_2 and t'_1 is removed. For this reason, when b_1 is traversed, b_2 has already been removed. Thus,

$$lp(b_1|b_2) = lp(b_1).$$

The second outer-loop has the left-over of b_1 union some pieces oft_2 and t''_1. We denote the final value of loop number obtained below,

$$lp(D,P_1) = lp(t'_1) + lp(b_2) + lp(t_2|t'_1) + lp(b_1) + lp(t''_1|t_2,t'_1) + 2.$$

In D', starting at P_1, given the same intersection relations, we get,

$$lp(D',P_1) = lp(t'_1) + lp(b_2) + lp(t_2|t'_1) + lp(b_1) + lp(t''_1|t_2,t'_1) + 2.$$

This holds because one traverses b_2 when the rest of b_1 has already been removed.

We finally get the equation,

$$lp(D,P_1) = lp(D',P_1).$$

What happens when t_2 has property (NI) with t'_1? If b_1 also has property (NI) with b_2, then we get the following:

$$lp(D,P_1) = lp(t'_1) + lp(b_2) + lp(t_2) + lp(b_1) + lp(t''_1|t_2,t'_1) + 2,$$

and

$$lp(D',P_1) = lp(t'_1) + lp(b_2) + lp(t_2) + lp(b_1) + lp(t''_1|t_2,t'_1) + 2$$

$$\Rightarrow lp(D,P_1) = lp(D',P_1).$$

If b_1 has property (I) with b_2, then we get the following loop numbers:

$$lp(D, P_1) = lp(t_1') + lp(b_2) + lp(t_2) + lp(b_1) + lp(t_1''|t_1') + 2.$$

The value $lp(b_1)$ is obtained because at this point, all the left-over arc of b_1 has been removed with an outer-loop. The same reason applies to the conditional loop number $lp(t_1''|t_2, t_1') = lp(t_1''|t_1')$, as t_2 has no bearing on this loop since it is removed before. Recall that the Corollary 4.6 of Lemma 4.5 tells us that the loops obtained from t_1 are constant if its traversing order is preserved. Then, we have: $lp(t_1') + lp(t_1''|t_1') = lp(t_1)$. The value of $lp(D, P_1)$ can be written, and in fact is equal to one of the outside loop numbers, S_1 as shown in Figure 6.1.1 (See Case I of Section 5.1.1). We explored the value of $lp(D, S_1)$ as an outside loop number in Case I of Section 5.1.1.

$$lp(D, P_1) = lp(t_1) + lp(t_2) + lp(_1) + lp(b_2) + 2 = lp(D, S_1).$$

From the flyped diagram D', we get:

$$lp(D', P_1) = lp(t_1') + lp(b_2) + lp(t_2) + lp(b_1|b_2) + lp(t_1''|t_1') + 2.$$

Notice that the conditional loop number $lp(t_1''|t_2, t_1') = lp(t_1''|t_1')$ because the left-over of t_2 has already been removed when t_1'' is traversed. Apply Lemma 4.5, we get

$$lp(D', P_1) = lp(t_1) + lp(t_2) + lp(b_2) + lp(b_1|b_2) + 2 = lp(D', S_4').$$

See Figure 6.0.1 for the position of S_4' in D'.

Thus in this case, though the loop number $lp(D, P)$ can change from to a different value $lp(D', P')$, both of these values appear as outside loop numbers.

In summary of the first case, the collection of loop numbers regardless starting inside or outside the tangle T remain the same after the flype on the tangle B.

Case 2: Both tangles have parity (1), shown conceptually in Figure 6.1.2, which implies b_1 and b_2 have property (I) with each other.

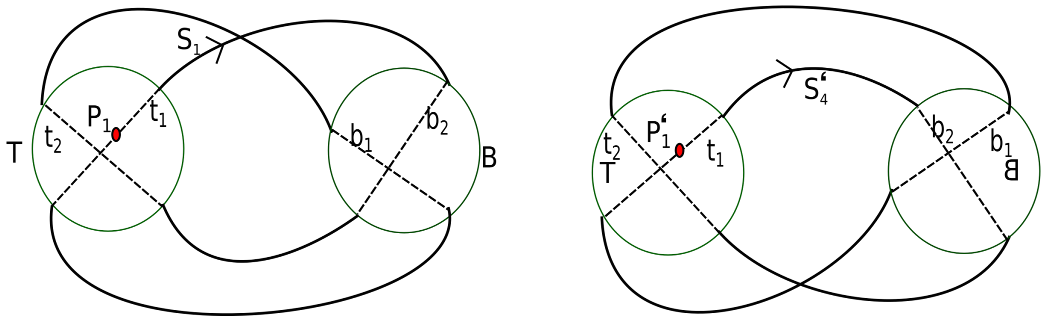

FIGURE 6.1.2. Both T and B have parity (1).

Assume t_2 has property (I) with t_1', then:

$$lp(D, P_1) = lp(t_1') + lp(b_2) + lp(t_2|t_1') + lp(b_1) + lp(t_1''|t_2, t_1') + 2.$$

In this given direction, the first outer loop is made of the left-over arc of b_2 and the intersection of t_2 with t_1', which contains some left-over arcs of t_1' and t_2. The second outer loop arises from the left-over arcs of $b_1 \cup t_1''$ and some of t_2.

In the flyped diagram D', we obtain the same value:

$$lp(D', P_1) = lp(t_1') + lp(b_2) + lp(t_2|t_1') + lp(b_1) + lp(t_1''|t_2, t_1') + 2.$$

This shows that,

$$p(D, P_1) = lp(D', P_1).$$

If t_2 has property (NI) with t_1', then

$$lp(D, P_1) = lp(t_1') + lp(b_2) + lp(t_2) + lp(b_1) + lp(t_1''|t_1') + 2$$

$$= lp(t_1) + lp(t_2) + lp(b_1) + lp(b_2) + 2 = lp(D, S_1),$$

where $lp(D, S_1)$ is an outer loop number. See Figure 6.1.2.

The value $lp(b_1|b_2) = lp(b_1)$ and $lp(t_1''|t_2, t_1') = lp(t_1''|t_1')$ occur due to the fact that b_2 and t_2 have been completely removed as in an outerloop before b_1 and t_1'' are traversed correspondingly.

In D', we get:

$$lp(D', P_1) = lp(t_1') + lp(b_2) + lp(t_2) + lp(b_1|b_2) + lp(t_1''|t_1') + 2$$

$$= lp(t_1) + lp(t_2) + lp(b_2) + lp(b_1|b_2) + 2 = lp(D', S_4'),$$

where $lp(D', S_4')$ is an outer loop. See Figure 6.1.2.

The values $lp(D, S_1)$ and $lp(D', S_4')$ were considered in Case II of Section 5.1.1. In this case, $lp(D, P_1)$ changes under a flype, but both values are outside loop numbers. Thus the collection of loop numbers of D and D' remain invariant under a flype.

Note that in Cases 1 and 2, we use the assumption that the tangle T is virtually unknotted. We did not use the stronger assumption that T is rational. We also note that no assumption on B were made. We will pick this at the end of this chapter when all inside loop numbers are evaluated.

6.2. Inside loop numbers in an antiparallel flyping circuit

Case 3: T has parity (∞) and B has parity (0), as in Figure 6.2.1.

Assume that t_2 has property (I) with t_1', and b_1 can have property (I) or (NI) with b_2:

$$lp(D, P_1) = lp(t_1') + lp(b_2) + lp(t_2|t_1') + lp(b_1) + lp(t_1''|t_2, t_1') + 2.$$

As the loop containing the left-over of b_2 is removed before b_1 is traversed, we get $lp(b_1)$ instead of a conditional loop number. The conditional loop number $lp(t_1''|t_2, t_1')$ suggests that the left-over arcs of t_1 may intersect with t_2 on both t_1' and t_1''. Regardless, we

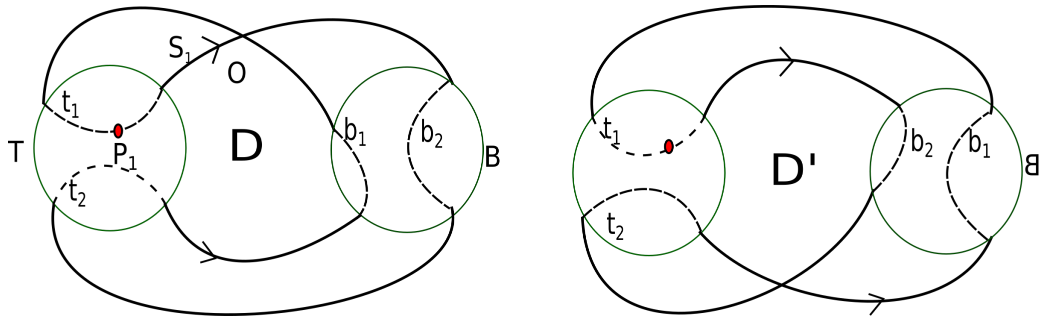

FIGURE 6.2.1. Consider the inside loop number when starting at $P_1 \in$ a parity (∞) tangle.

obtain the same loop number in D':

$$lp(D', P_1) = lp(t_1') + lp(b_2) + lp(t_2|t_1') + lp(b_1) + lp(t_1''|t_2, t_1') + 2.$$

Thus,

$$lp(D, P_1) = lp(D', P_1).$$

If t_2 has property (NI) with t_1', and b_1 has property (I) with b_2, then

$$lp(D, P_1) = lp(t_1') + lp(b_2) + lp(t_2) + lp(b_1|b_2) + lp(t_1''|t_1') + 3.$$

By Lemma 4.5,

$$lp(t_1') + lp(t_1''|t_1') = lp(t_1),$$

and

$$lp(D, S_1) = lp(t_1) + lp(t_2) + lp(b_2) + lp(b_1|b_2) + 3.$$

We see that the value of $lp(D, P_1)$ occurs on outside loop number starting at S_1. This was considered in Case II of Section 5.1.2. Regardless, a flype on B does not change this loop number:

$$lp(D', P_1) = lp(t_1') + lp(b_2) + lp(t_2) + lp(b_1|b_2) + lp(t_1''|t_1') + 3.$$

Finally, we assume that t_2 has property (NI) with t_1', and b_1 has property (NI) with b_2:

$$lp(D, P_1) = lp(t_1') + lp(b_2) + lp(t_2) + lp(b_1) + lp(t_1''|t_1') + 2;$$

which is equal to $lp(D', P_1)$.

In summary, the collection of loop numbers in D stays unchanged after a flype on B.

Case 4: T has parity (∞) and B has parity (1), as conceptually shown in Figure 6.2.2. Here, b_1 and b_2 always have property (I) to each other. Assuming that t_2 has

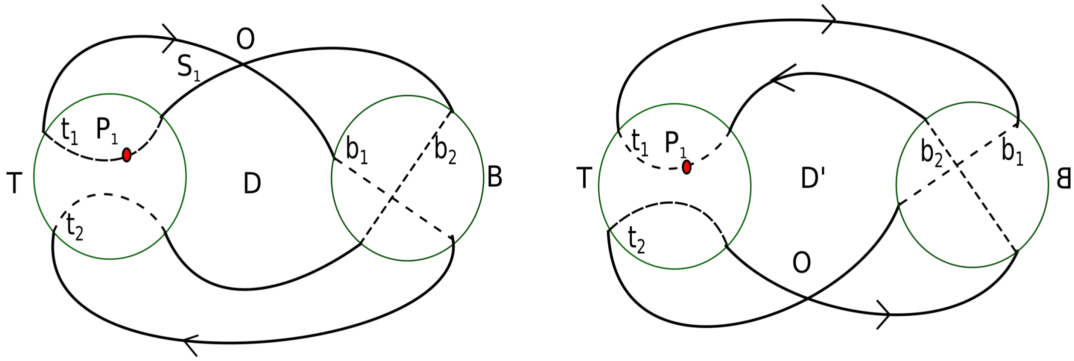

FIGURE 6.2.2. The position of $P_1 \in t_1$ when T has parity (∞) and B has parity (0).

property (I) with t_1' yields:

$$lp(D, P_1) = lp(t_1') + lp(b_2) + lp(t_2|t_1') + lp(b_1) + lp(t_1''|t_2, t_1') + 2 = lp(D', P_1).$$

Assume t_2 has property (NI) with t_1':

$$lp(D, P_1) = lp(t_1') + lp(b_2) + lp(t_2) + lp(b_1|b_2) + lp(t_1''|t_1') + 3$$

$$\Rightarrow lp(D, P_1) = lp(t_1) + lp(t_2) + lp(b_2) + lp(b_1|b_2) + 3 = lp(D, S_1)$$

In the flyped diagram D', we get:

$$lp(D', P_1) = lp(t_1') + lp(b_2) + lp(t_2) + lp(b_1|b_2) + lp(t_1''|t_1') + 3 = lp(D, P_1).$$

Note that this case does not require the virtually unknotted property of either tangle.

Case 5: T has parity (0) and B has parity (∞), as conceptually shown in Figure 6.2.3.

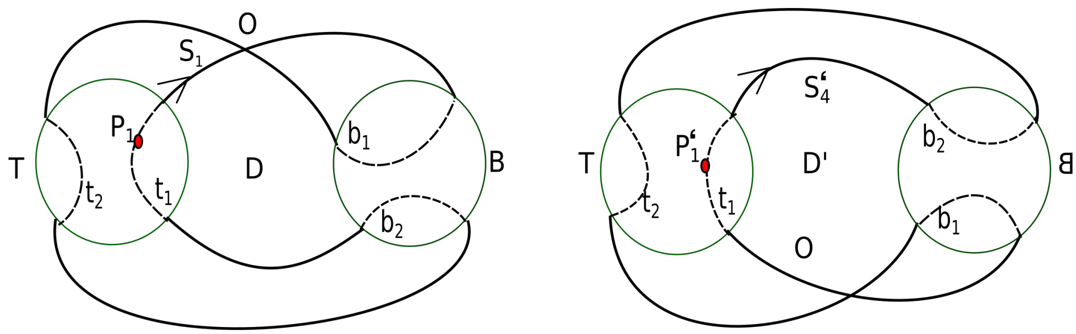

FIGURE 6.2.3. Consider inside loop number when T has parity (0) and B has parity (∞).

Assuming t_2 has property (I) with t_1', and b_2 has property (I) or (NI) with b_1 yields:

$$lp(D, P_1) = lp(t_1') + lp(b_1) + lp(t_2|t_1') + lp(b_2) + lp(t_1''|t_2, t_1') + 3.$$

On the flyped diagram D', this value does not change:

$$lp(D', P_1) = lp(t_1') + lp(b_1) + lp(t_2|t_1') + lp(b_2) + lp(t_1''|t_2, t_1') + 3.$$

If t_2 has property (NI) with t_1', and b_1 has property (NI) or (I) with b_2, then:

$$lp(D, P_1) = lp(t_1') + lp(b_1) + lp(t_2) + lp(b_2) + lp(t_1''|t_2, t_1') + j.$$

The number of outerloops j in this case may vary from 2 to 3, as follows:

- If t_1'' does not intersect the left-over arc of t_2 before intersecting t_1', then:

$$lp(D, P_1) = lp(t_1') + lp(b_1) + lp(t_2) + lp(b_2) + lp(t_1''|t_2, t_1') + 2,$$

  and $lp(t_1''|t_2, t_1') = lp(t_1''|t_1')$, because the left-over of t_2 does not affect the conditional loop number $lp(t_1''|t_1')$. Then by Lemma 4.5,

$$lp(D, P_1) = lp(t_1) + lp(t_2) + lp(b_1) + lp(b_2) + 2.$$

  This value is equal to the loop number obtained by starting at S_4 in D. See Figure 6.2.3. In particular,

$$lp(D, S_4) = lp(t_1) + lp(b_1) + lp(t_2) + lp(b_2) + 2.$$

  So the value of inside loop number $lp(D, P_1)$ is in this case an outside loop number.

- If t_1'' intersects the left-over arc of t_2 before intersecting t_1', then

$$lp(D, P_1) = lp(t_1') + lp(b_1) + lp(t_2) + lp(b_2) + lp(t_1''|t_2, t_1') + 3.$$

  This value does not appear in D'. In this specific case, we will need additional assumptions on T to guarantee the invariance of loop numbers in general. Chapter 7 will continue this case under the assumption that T is rational.

Case 6: T has parity (1) and B has parity (∞). Figure 6.2.4 conceptually illustrates this case. Assume t_2 has property (I) with t_1', and b_1 has property (NI) or (I) with b_2:

$$lp(D, P_1) = lp(t_1') + lp(b_2) + lp(t_2|t_1') + lp(b_1) + lp(t_1''|t_2, t_1') + 3.$$

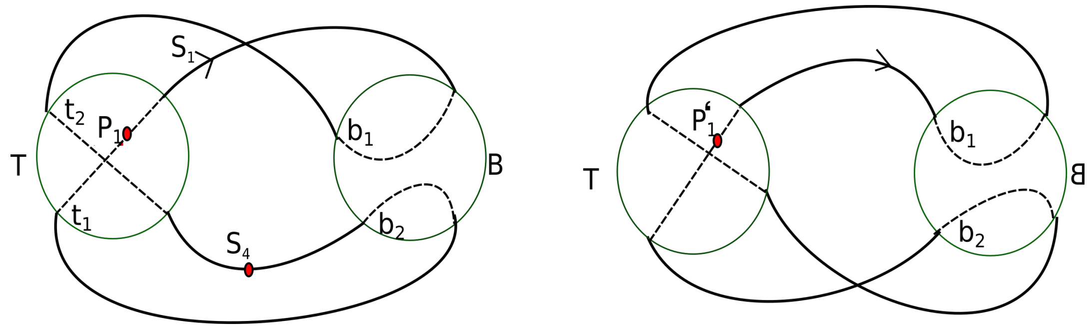

FIGURE 6.2.4. Consider the inside loop number when T has parity (1), and B has parity (∞).

In the flyped diagram, we get the same loop number:

$$lp(D', P_1) = lp(t_1') + lp(b_2) + lp(t_2|t_1') + lp(b_1) + lp(t_1''|t_2, t_1') + 3.$$

What changes if t_2 has property (NI) with t_1', and b_1 has property (I) with b_2? As T is a parity (1) tangle, t_1'' must intersect the left-over of t_2 before it can intersect the left-over of t_1', we obtain:

$$lp(D, P_1) = lp(t_1') + lp(b_2) + lp(t_2) + lp(b_1) + lp(t_1''|t_2, t_1') + 3.$$

On the flyped diagram D', we get a different loop number:

$$lp(D', P_1) = lp(t_1') + lp(b_2) + lp(t_2) + lp(b_1|b_2) + lp(t_1''|t_1') + 2.$$

This value may not be equal to $lp(D, P_1)$. However, using Lemma 4.5, we see that this value occurs in one of the outside positions, S_4':

$$lp(D', S_4') = lp(t_1) + lp(t_2) + lp(b_2) + lp(b_1|b_2) + 2 = lp(D', P_1).$$

Thus, only the value $lp(D, P_1)$ is new.

$$lp(D, P_1) = lp(t_1') + lp(b_2) + lp(t_2) + lp(b_1) + lp(t_1''|t_2, t_1') + 3.$$

We will continue the discussion of the unresolved situations from cases 5 and 6 in Chapter 7, where we need the additional assumption that T is rational.

However, by showing cases 1 and 2, together with the results on outside loop numbers in a parallel flyping circuit, we have proven Theorem 6.2 summarized in Section 4.1.

THEOREM 6.2. *In a diagram D containing only parallel flyping circuits with virtually unknotted tangles, the values $minLp(D)$ and $maxLp(D)$ are invariant under flypes.*

Chapter 7

Loop numbers in Montesinos knots

We recall that Montesinos knots contain only rational tangles (Section 2.2). These knot diagrams can form parallel or antiparallel flyping circuits. Chapter 5 shows that the outside loop numbers are invariant under flypes in general, without a restriction on the knot type. Chapter 6.2 shows when the inside loop numbers repeat under flypes, provided that the tangles are virtually unknotted. To shed light on the last 2 cases of Chapter 6.2, we make a stronger assumption on the antiparallel flyping circuit in the diagram D, namely that the tangle T is rational.

In this chapter, the following conventions are used:

Recall that the tangle T admits a square diagram, described in Section 2.2.2. Suppose that T has $h \geq 1$ layers (see Definition 2.19). Let $\mathbb{P} = \{P_j\}$ denote the set of starting point inside the rational tangle T so that every P_j divides t_1 into 2 parts, called sub-arcs, and t_2 has property (NI) with respect to the first sub-arc. For simplicity, when T has a single eligible starting point P_k, for a fixed value of k, we call the sub-arcs t_1' and t_1''. t_1' is the sub-arc being traversed first in the tangle T. When considering different starting points P_i which t_2 has the property (NI) with in a rational square, P_i belongs to layer l_i, and the sub-arcs are called t_{1_i}' and t_{1_i}''. Figure 7.0.1 illustrates these conventions with one eligible starting point $P_1 \in l_1$ on the left, and 2 eligible starting points $P_3 \in l_3$ and $P_1 \in l_1$ on the right.

Q is an endpoint of t_1, where a traversal into T starts. R is an endpoint of t_2, on the opposite side of Q. The notation $lp(T, P)$ can be shortened to $lp(P)$, which denotes the loop number T contributes to a whole diagram D by starting the traversal at P. In this chapter, the left-over arcs of T, if any, will be taken into account, though not counted as loops in $lp(T)$. The terms defined below are specific to this chapter.

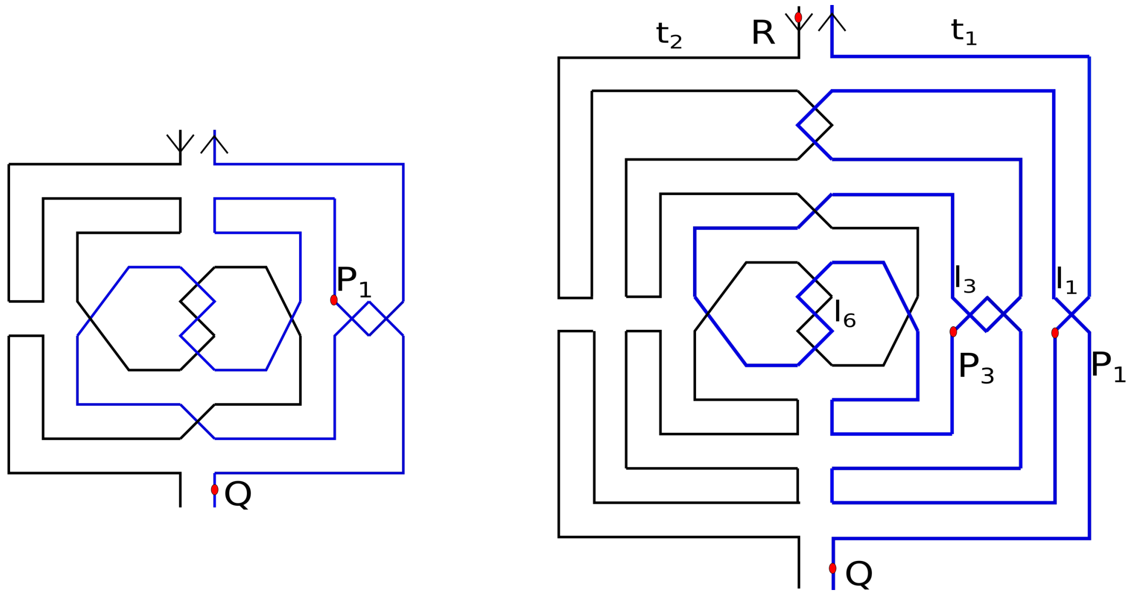

FIGURE 7.0.1. Two rational squares which yield different sets $\mathbb{P}$ of eligible starting points. The blue arc represents t_1. The tangle on the left has 4 layers. The one on the right has 6 layers.

DEFINITION 7.1. *For any 2 points P_i and P_k in $\mathbb{P}$, the equation $lp(P_i) \triangleq lp(P_k)$ means that $lp(P_i) = lp(P_k)$, and that the actual loops generated are exactly the same.*

DEFINITION 7.2. *Intermediate loops are closed curves formed across different layers in a square diagram of a rational tangle.*

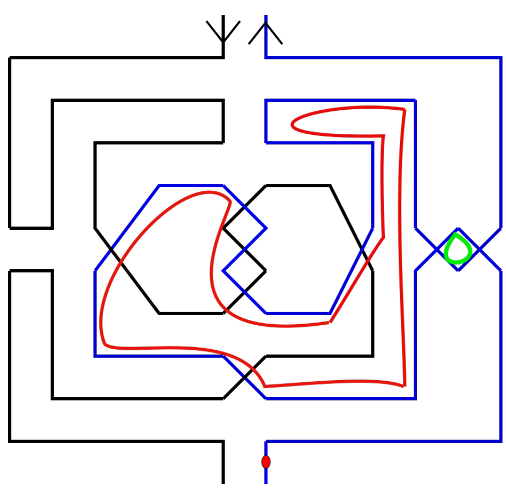

FIGURE 7.0.2. The red highlighted loop is an intermediate loop. The green highlighted loop is a miniloop.

DEFINITION 7.3. *In an oriented rational square, shown in Figure 7.0.2, a miniloop is a small loop in a single P-layer, formed between antiparallel half-twists.*

DEFINITION 7.4. *A pure loop is a loop made of one color in a rational square. Pure loops include intermediate loops and miniloops.*

7.1. Structural conditions on layers

In a rational square, if all layers are M-layers, any inside starting point of T yields property (I) for t_2 with respect to t'_{1_i}. In order for any starting positions in P_i in $\mathbb{P}$ where t_2 has property (NI) with respect to t'_{1_i} to exist, the following restrictions are implied:

(i) There exists a P-layer. More specifically, a potential starting position $P \in \mathbb{P}$ may exist only on a side that has a nonzero number of crossings of the P-layer. We choose P_i to be at the entrance of the first pure crossing on this layer l_i in the direction pointing towards the crossing.

(ii) If the layer l_1 is mixed with both groups of nonzero crossings, then starting anywhere inside T yields property (I) for t_2 with respect to t'_{1_1}. Thus, tracing for legitimate positions P_i requires every M-layer l_j with $j < i$ of T to have no crossing on the side that involves t'_{1_i} (See Definition 2.20 for one sided layers). Assuming that all P_i are separated by a group of half-twists, there are finitely many potential points P_i that need to be considered.

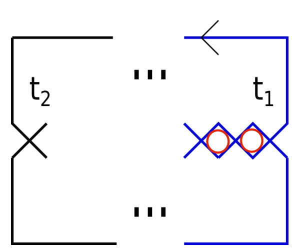

FIGURE 7.1.1. Two miniloops on t_1, and none on t_2.

(iii) In the cases we consider, miniloops (Definition 7.3) appear only on a P-layer that has 2 or more crossings on a side with an antiparallel orientation. For example, in Figure 7.1.1, there are 2 miniloops on t_1, and none on t_2 as shown in Figure 7.1.1. These loops automatically account for any group of half-twists in a P-layer with antiparallel orientation. In other words, the number of miniloops is often independent of the starting positions. Thus, when comparing $lp(P_i)$ and $lp(P_k)$, we can ignore the miniloops. Furthermore, it only matters whether the number of crossings in a group of half-twists is odd or even. So in all figures in this section, we only draw a group of half-twists with one crossing for an odd number of crossings, and with 2 crossings for an even number of crossings.

7.2. Identifying the eligible set $\mathbb{P}$ in rational squares

LEMMA 7.5. *Let $P_j \in \mathbb{P}$ be starting point in a rational tangle $T(t_1, t_2)$ with parity (0) or (1), such that t_2 has property (NI) with respect to the sub-arc t'_{1_i} of T. Then we have $lp(T,P) = lp(S_i)$ for some $S_i \in \mathbb{S}$, where $\mathbb{S}$ is defined in Definition 3.2.*

We consider the following cases to show Lemma 7.6:

Section 7.2.1: T has a single P-layer.

Section 7.2.2: T has two or more P-layers.

(1) Either l_1 is an M-layer or l_1 is a P-layer with crossings on both sides.
(2) The layer l_1 is a P-layer without crossing on one side.

7.2.1. T has a single P-layer.

In this case, there can be at most a single $P \in \mathbb{P}$. Let the only P-layer be l_i. Since $i < h$, the relative location of i creates two sub-cases: $i > 1$ and $i = 1$.

Subcase 1: $i > 1$**.** This means l_1 is an M-layer, and $i \in \{2, ..., h-1\}$.

Figure 7.2.1 conceptually describes this situation. The dashed arcs represent arcs whose exact shape is not determined by the figure. In particular the dashed arcs in the layers l_j for $j > i$ can intersect themselves or each other. The innermost layer l_h is an M-layer by default. Between l_i and l_1 are a number of M-layers. T must be a parity (1) tangle in this case, because the black arc in t_1 from l_i must reach the M-crossing C and exit T on the vertically opposite side with the other end point of t_1. Furthermore for P_i to yield property (NI), each M-layer outside l_i must be have zero crossings on the side that contains a part of t_1'.

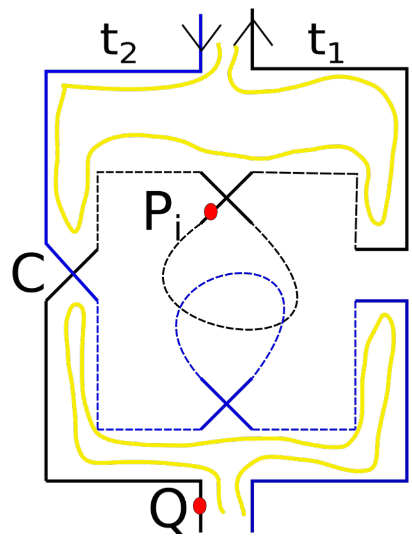

FIGURE 7.2.1. The P-layer is ith, for $h > i > 1$. Let Q be the entrance endpoint of t_1.

Claim: Traversing T starting at P_i yields the same loops as starting at Q.

This is due to the M-crossing C on l_1 as shown in Figure 7.2.1. Starting at Q, one will eventually enter l_i at the pure crossing, and come back to it to remove all pure black loops (This is symbolized by the single dashed black loop in the figure) connecting pieces of black arcs from l_h to l_i. When continuing the traversal from P_i, all the black loops have been removed. Thus, when reentering T on the blue arc t_2, we hit the crossing C. One collects pure blue loops made of pieces of arcs from l_h to l_i as though the black arc has no bearing on the blue loop (symbolized by the dashed blue loop in this figure).

Finally, there is one arc left in the tangle that will be a part of a loop containing Q and C. This is indicated by the yellow highlight in Figure 7.2.1.

If one starts at P_i instead, the order of traversing the black arc changes, but the same pure blue loops are removed first, then the same pure black loops. Thus, due to property (NI) between t_2 and t_1', the traversal leaves the same trace in T. In both cases (starting at Q or at P_i), exactly the same loops are created. The two arcs that join l_i with l_1 are highlighted. These two arcs will be a part of loops that are not contained in the tangle T. Moreover, any miniloops will account for $lp(T)$ regardless of starting at P_i or at Q. Thus, the starting point P_i creates the same loops as the starting point Q. We conclude that:

$$lp(P_i) \triangleq lp(Q).$$

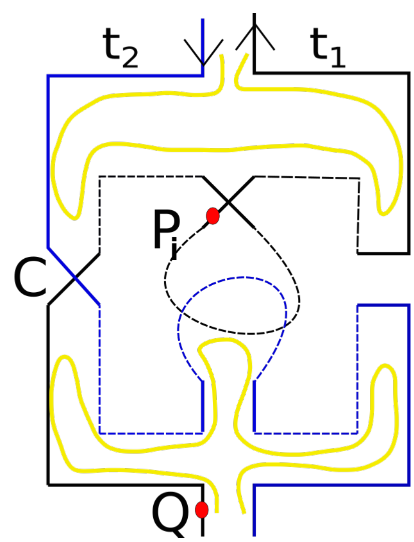

FIGURE 7.2.2. The P-layer is ith, for $h > i > 1$. Let Q be the endpoint of t_1. Layer l_i has zero crossing on one side.

This result holds even when t_2 has zero crossing on the P-layer l_i. Figure 7.2.2 illustrates this situation conceptually. The zero crossings on the blue reduces the number of intermediate blue loops by 1 regardless whether one starts at Q or at P_i. The key reason for this is the fact that the black dashed loop is either removed completely before the blue arc is traversed, or does not intersect the blue arc due to property (NI). The argument is identical to the previous case. Thus, we also get:

$$lp(P_i) \triangleq lp(Q).$$

As $Q \in \{S_i\}$, this value is invariant under flypes, based on the results of Chapter 5.

Subcase 2: $i = 1$**.** This means the only P-layer is l_1.

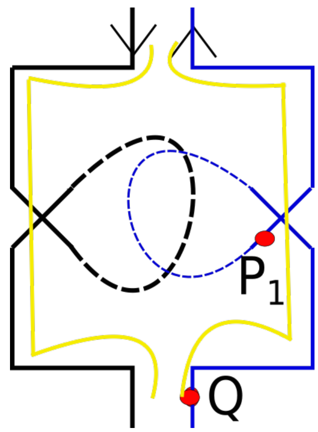

FIGURE 7.2.3. The only P-layer is l_1.

On each side (left or right), the arcs have the same color vertically, then T must be a parity (0) tangle. Starting the traversal at Q results in $lp(t_1) + lp(t_2)$ as the contribution of the tangle T to the total loop number. All arcs of the same color in layers l_2 to l_h are removed at the pure crossings on l_1. Starting at $P_1 \in t_1$, one exits T and re-enters it through t_2, leaving the same trace as starting at Q due to the reason that t_2 has the property (NI) with respect to t_1'. When starting at P_1, one will exit T and enter it through t_2. On t_2, all the inside blue arcs are removed, before one re-enters T at Q.

As a result, at both positions P_1 and Q, we collect the same outer loop, indicated by the highlighted arcs in Figure 7.2.3, and the tangle T contributes $lp(t_1) + lp(t_2)$ to the loop number in both cases.

This result holds even when l_1 has zero crossing on t_2. Figure 7.2.4 conceptually shows how the outer loop is affected by the zero crossing on t_2. Starting at Q, one

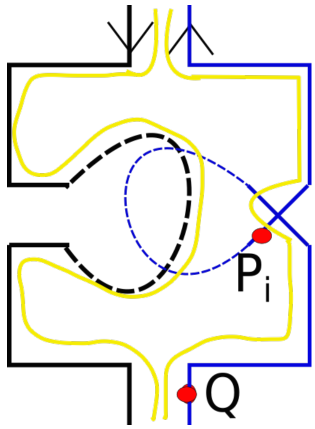

FIGURE 7.2.4. The P-layer is l_1 and one side has zero crossing.

removes completely all parts of t_1 except the highlighted arc on the right in Figure 7.2.3. This is indicated as the dashed black loop in the figure. Once this is removed, traversing t_2 will not intersect any pieces of t_1. This creates the same situation as starting at P_1 due to the property (NI) to t_1. The only difference the zero crossing makes on $lp(T)$ is that $lp(t_2)$ is reduced by one loop.

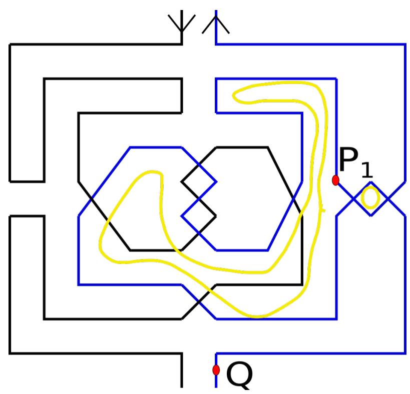

FIGURE 7.2.5. The P-layer is l_1, with crossings on t_1 side.

Figure 7.2.5 is an example of this case: T consists of 4 layers, where l_1 is a P-layer with crossings on t_1. The loops of T are highlighted.

7.2.2. T has two or more P-layers. If l_1 is a P-layer with crossings on both sides, the P-crossing on l_1 gives rise to an eligile starting point $P_1 \in \mathbb{P}$. This was previously proven in Section 7.2.1. Other P_i on the layers l_i, for $1 < i < h$, are only candidates for the set $\mathbb{P}$, if they yield the (NI) property between t_2 with respect to t'_{1_i}. The property (NI) is met the only crossing on t'_1 are P-crossings. This can happen only if all M-layers between l_1 and l_i are one sided and have zero crossings on the side involving t'_{1_i}. If l_j was an M-layer and it had crossings on both sides, then all candidates P_k, for $k > j$, would yield property (I), thus we would not consider them. If l_1 is a one sided zero P-layer, the outcome for $\mathbb{P}$ differs from the rest of other cases. Thus, we separate this situation into Section 7.2.2.2 only to distinguish its outcome. Section 7.2.2.1 operates under the assumption that l_1 is not a one sided zero P-layer.

The structure of the M-layers between two P-layers that have crossings on both sides falls into three subcases as follows.

- Subcase 1: There is exactly one M-layer.

- Subcase 2: There are exactly two M-layers.

- Subcase 3: There are k M-layers, for $k \geq 3$.

7.2.2.1. *Either l_1 is an M-layer or l_1 is a P-layer with crossings on both sides.* Consider the structure between 2 P-layers of T to see how the loop generation varies. As the focus is what occurs between these 2 P-layers, we draw the outermost P-layer on the layer l_1 in the figures that follow. In general, there can be multiple layers of any type outside the more outer P-layer, and inside the more inner P-layer.

Subcase 1: There is only one M-layer between 2 P-layers.

To give rise to potential starting points in $\mathbb{P}$, this M-layer must have zero crossing on the side involving t'_{1_i}, and an even number of crossing on the side involving t''_{1_i}.

Figure 7.2.6 illustrates this conceptually. Let the more inner P-layer be l_{i+2}, then l_{i+1} is an M-layer, and l_i is the P-layer. From l_1 to l_i, and from l_{i+3} to l_h, there can be multiple layers of either type. Let t_1 be the strand so that backing up on it yields

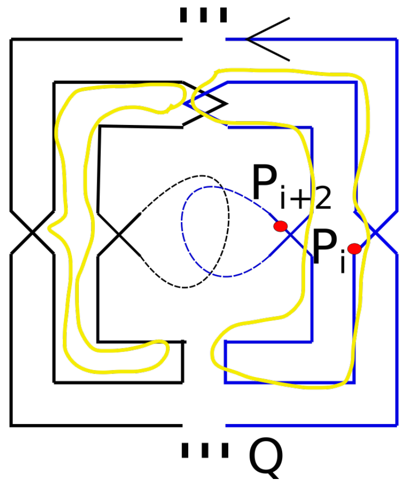

FIGURE 7.2.6. Between 2 P-layers is a single M-player.

two potential P positions with the property (NI). Starting at either P_{i+2} or P_i gives the same two intermediate loops: one blue and one black, which connect a P-crossing on l_{i+2} with a P-crossing l_i on each strand. The two intermediate loops are highlighted in Figure 7.2.6. The miniloops are automatically accounted for independently with a starting positions in an antiparallel flyping circuit. The reader needs to convince herself or himself that regardless whether one starts at P_i or P_{i+2}, the loops generated that are outside the structure shown in Figure 7.2.6 are the same. This yields the result: $lp(P_{i+2}) \triangleq lp(P_i)$

If l_{i+2} has zero crossing on t_2, then there is no intermediate loop on t_2 connecting l_i with l_{i+2}, instead, there is an arc. So, the number of intermediate loop connecting 2 P-crossings is reduced by 1 on that side. Figure 7.2.7 illustrates this. The layers l_1 to l_i have zero crossing on one side so that the potential starting points P_i yield property (NI). This does not change the fact that the outer loops being removed are the same as when l_{i+2} has crossings on both sides. If the zero side on l_{i+2} is on t_1, then we again eliminate one intermediate loop from t_1 and the number of loops is reduced by one. In general, the zero crossing on the more inner P-layer only extends the boundary of the intermediate loop to deeper layers of that color.

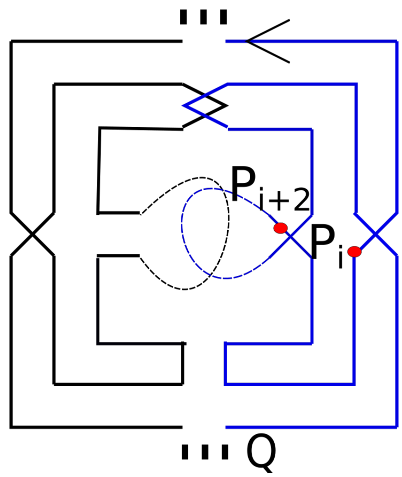

FIGURE 7.2.7. Between 2 P-layers is a single M-player, but the more inner P-layer has no crossing on t_2.

Subcase 2: There are only two M-layers between 2 P-layers.

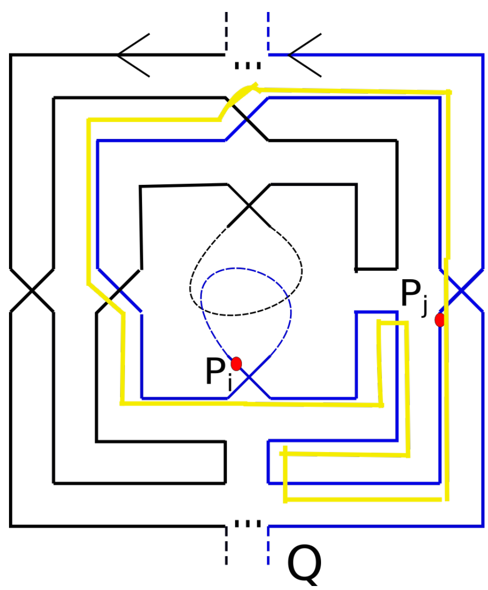

FIGURE 7.2.8. Between 2 P-layers are two M-layers. $j = i + 3$.

The layers l_1 to l_i have zero crossing on one side so that the potential starting points P_i yield property (NI). The two P-layers are l_{i+3} and l_i. As before, in order for both P_i and P_{i+3} to have the (NI) property, the two M-layers l_{i+1} and l_{i+2} must have zero crossings on the side involving t_1', and subsequently an odd number of crossings on the other side. There can be many other layers of any type inside l_{i+3}, and many outside l_i. If the outside layers of l_i are M-layers, then they all must have zero crossing on the side for P_i and P_{i+3} to yield the (NI) property for t_2 with respect to t'_{1_i} and $t'_{1_{i+3}}$. The split of t_1 into sub-arcs are defined at the beginning of this chapter. There are two intermediate loops between l_{i+3} and l_i regardless of starting at P_i or P_{i+3}. The number of intermediate loop for each color is one, regardless whether one starts at P_i or P_{i+3}. The intermediate loops connecting P_i and P_{i+3} is highlighted in Figure 7.2.8.

If l_{i+3} or l_i ($l_i \neq l_1$) has zero crossing on t_1, it reduces the number of legitimate starting positions in $\mathbb{P}$. As before, a zero crossing on a P-layer only extend the boundary of the intermediate loop involving it. Similarly, if the zero side is on t_2, the loops outside the structure shown in Figure 7.2.8 will be the same, regardless of the starting position P_i or P_{i+3}. In addition, the two intermediate loops are also the same. Thus, we have the equation $lp(P_i) \triangleq lp(P_{i+3})$.

Starting the traversal at Q, one removes the blue loops completely before reentering T. Thus, the fact that these blue loops have been traversed or not yet (as starting at P_i) will generate the same loops. Thus, we also have

$$lp(P_{i+3}) = lp(P_i) = lp(Q).$$

Figure 7.2.8 illustrates this case with an example, where $i = 4$ and $j = 1$. As noted before, the example can extend to more general cases (thus the dashed arcs) as long as the (NI) property is met for t'_{1_i}.

Subcase 3: There are more than two M-layers between 2 P-layers l_i and l_j.

It implies that,

$$h > i > j > 1.$$

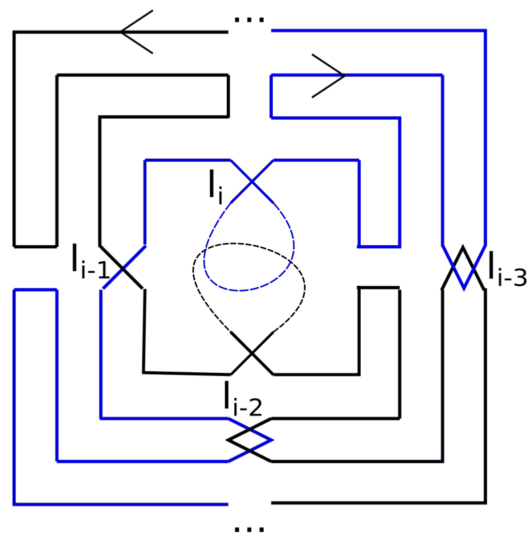

FIGURE 7.2.9. As long as the M-layer l_{i-2}, l_{i-3}, l_{i-4}, ... have an even number of crossings, there can be more M-layers.

Two potential points in $\mathcal{P}$ exist on l_i and l_j. In order for both P_i and P_j to have the (NI) property, the M-layers in between i and j, and from l_1 to l_{j-1} must have zero crossing on one side.

Claim:

The M-layers between l_i and l_j must have a number of crossings following the pattern: $odd, even, ..., even, odd$.

PROOF. Let l_{i-1} have an odd crossing on one side, and l_{i-2} have an even crossing. If l_{i-3} has an odd number of crossing, then $j = i - 4$. If l_{i-3} has an even number of crossing, then l_{i-4} is still an M-layer. This confirms the pattern of any number k of M-layers between l_i and l_j, for $k > 2$. Figure 7.2.9 illustrates this pattern on M-layers.

There are two intermediate loops generated by connecting arcs from l_i to l_j regardless of the starting point P_i or P_j. In Figure 7.2.10, the pattern of M-layer crossings between the two P-layer is "odd, even, odd". However, in general we note that the number of even crossing M-layer can extend arbitrarily. As before, two intermediate loops are generated and one of the intermediate loops is highlighted in Figure 7.2.10. Starting

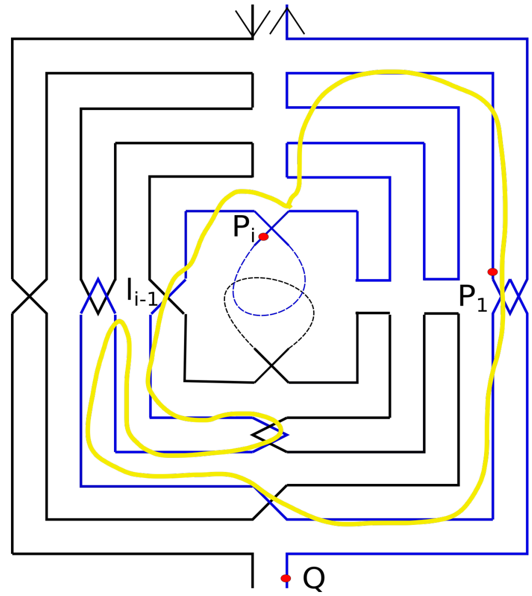

FIGURE 7.2.10. Between 2 P-layers are more than two M-layers.

the traversal at P_i or P_j, due to the property (NI) of t'_{1_i} and t'_{1_j} with respect to t_2, one obtains the same loops as starting at Q. The difference is the order of when the loops are obtained. For example, starting at Q, one obtains and removes the corresponding intermediate loops and miniloops (if any) before reaching P_i or P_j. We leave the rest of the details to the reader, and note that:

$$lp(P_i) \triangleq lp(P_j) \triangleq lp(Q).$$

If the layer l_i has zero crossing on any strand, the intermediate loop extends on this layer, which reduces by 1 from the total number of intermediate loops. However, the outer-loops do not change regardless starting at Q or in $\mathbb{P}$. □

All three subcases may happen simultaneously in a rational tangle T. Figure 7.2.11 illustrates the eligible starting points $P_i \in t_1$ for $1 \leq i \leq k$, and conceptually Q is an endpoint of t_1. The set $\mathbb{P}$ is located entirely on t_1.

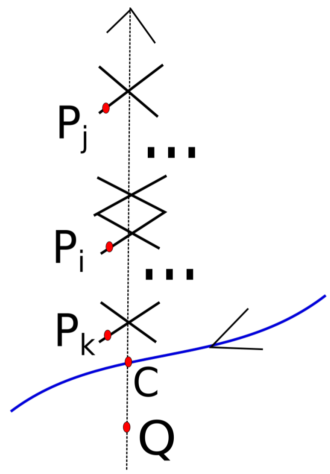

FIGURE 7.2.11. Potential starting points in $\mathbb{P}$ when there are 2 or more P-layers. $j < i < k$. C is a crossing on the innermost layer, and no further points in $\mathbb{P}$ can appear.

Claim:

$$lp(P_s) \triangleq lp(Q),$$

where $s \in \{j, i, ..., k\}$ so that $P_s \in \mathbb{P}$.

PROOF. This happens because the layers between the outermost layer containing a point of $\mathbb{P}$ and the innermost layer containing a point of $\mathbb{P}$ define a unique set of intermediate loops and miniloops, that will always be obtained from any starting position $P \in \mathbb{P}$. Furthermore, regardless of the structure, the loops outside of the structure defined by the points in $\mathbb{P}$ will always be the same. Furthermore, $lp(P) = lp(Q)$ for all $P \in \mathbb{P}$. The details are left to the reader. □

7.2.2.2. *The layer l_1 is a P-layer with zero crossings on one side.* Let l_i be a P-layer with crossings on both sides. In order for the potential starting position P_i on l_i to yield property (NI), all the sides with no crossing on layers l_{i-1} to l_1 must be arranged as illustrated in Figure 7.2.12. R is an endpoint of t_2 as shown in Figure 7.2.12.

Consider the loops generated by starting the traversal at P_i: First, one exits the tangle T and re-enters on t_2 without intersecting t_1'. Subsequently, one removes all the

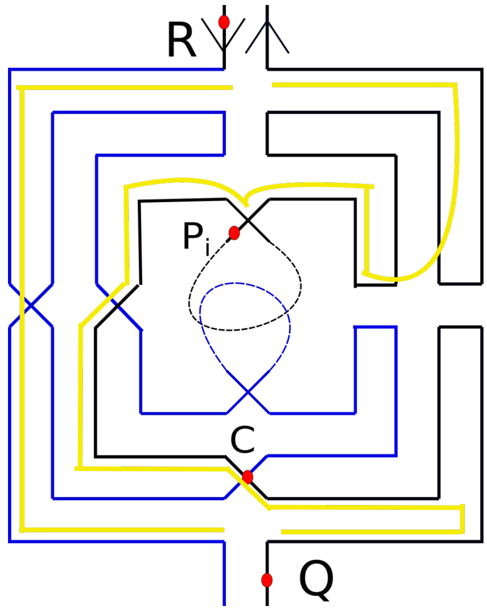

FIGURE 7.2.12. $lp(P_i) = lp(R)$. Starting at P_i or at R leaves the same left-over arcs, as highlighted.

pure loops on t_2. Due to the single P-layer on l_1 and the non-zero side being on t_2, t_1'' has property (NI) with respect to t_2. Thus, when the recycle completes at P_i, all the pure loops of t_1 are collected. As the result,

$$lp(P_i) = lp(t_1) + lp(t_2).$$

The loops generated by starting the traversal at Q are not the same, due to the M-crossing C on layer l_2. When re-entering the tangle T on t_2, one does not collect all the pure loops of t_2 due to the intersection at C. Thus, the outer loops this traversal generates differ from starting at P_i. As the result, due to different loops generated, it is possible that $lp(P_i) \neq lp(Q)$. Let R be an endpoint of t_2 on the opposite side of Q. Starting at R generates identical loops as starting at P_i. This is due to the fact that either at P_i or R, one removes the same pure loops on t_2 first, so the P-layer being on l_1 yields the same outer loop before one collects pure loops on t_1. As the result,

$$lp(P_s) \triangleq lp(R),$$

where $s \in \{i, ..., k\}$ so that $P_s \in \mathbb{P}$ and $R \in \{S_i\} \Rightarrow lp(P_s)$ repeats in the flyped diagram.

Under the assumption that P_i is on a one sided pure layer, Figure 7.2.12 illustrates conceptually the combinations of possible positions of starting points in $\mathbb{P}$ and indicates that the exact same loops would be obtained by starting at the endpoint $R \in t_2$.

In general, all 3 subcases between 2 P-layers can happen simultaneously. The details of this special case are left to the reader.

By proving that the inside loop numbers either do not change or repeat in the flyped diagrams as outside loop numbers, we have proven the following theorem.

THEOREM 7.6. *Let D be an alternating diagram of a Montesinos knot, then the minimum and maximum loop numbers are invariant under flypes.*

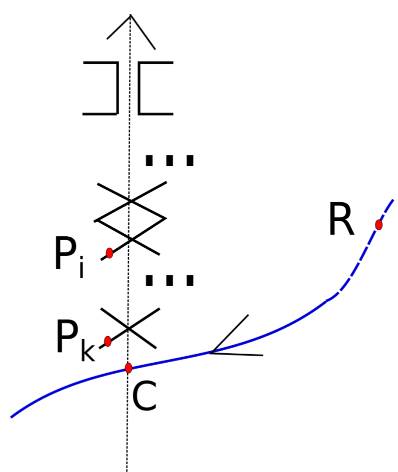

FIGURE 7.2.13. $lp(P_j)$ may be different from $lp(Q)$, but $lp(P_j) = lp(R)$. $i \le j \le k$.

Chapter 8

The existence of Meander diagrams for knots

This chapter answers the following question: If all k diagrams D_i, including highly non-minimal ones, are allowed, is there a bound on the $minLp(D_i)_{i=1}^{k}$? The answer to this question will be applied to address a conjecture, which states that every knot has a meander diagram (as introduced in Section 2.4).

8.1. A diagrammatic algorithm

Definition 8.1. *A knot diagram admits a valid 2-coloring if each strand of the knot diagram can be colored one of the two given colors, subject to the following rules:*

- Exactly 2 colors must be used.

- At each crossing, the 2 adjacent strands have different colors.

Adams et al introduced an algorithm to make a valid 2-coloring on any knot diagrams. This can be done because every knot has a projection that can be decomposed into sub-arcs such that each sub-arc never crosses itself [**1**]. Figure 8.1.1 shows that the valid 2-coloring can be done on a highly non-minimal diagram of the figure-eight knot (on the far right).

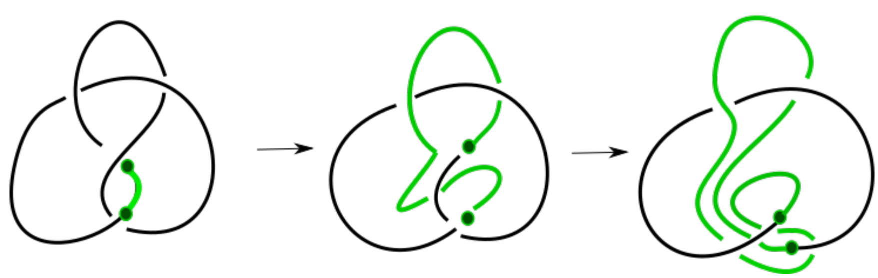

Figure 8.1.1. A highly non-minimal diagram of the knot 4-1 admits a valid 2-coloring (on the far right).

The following concepts were introduced with this algorithm.

Definition 8.2. *Transition vertices are vertices on a valid 2-coloring diagram, such that the colors switch from one to the other on the same strand of the knot diagram.*

Here, a strand of a knot diagram starts and ends at crossings.

Definition 8.3. *A face is a closed region on the plane, bounded by pieces of arcs of a knot diagram.*

The number of pieces of arcs determines the number of sides needed for form a face. For example, zooming in the red circled region in Figure 8.1.2, one will see that this is an odd-sided face (the number of sides is 3).

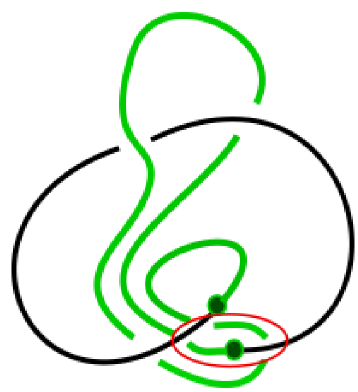

Figure 8.1.2. An odd-sided face is circled in red.

The following result was shown regarding the types (odd or even) of faces in a knot diagram [**1**].

Theorem 8.4 (Adams et al, 2008). *Every knot has a diagram D with exactly two odd-sided faces, which can be made to be triangles.*

Figure 8.1.3 shows an odd-sided face with one transition vertex.

The proof of this theorem, not the result itself, helps deriving the existence of a four-sided face, which is applied in showing Theorem 8.5. A sketch of the proof goes as follows:

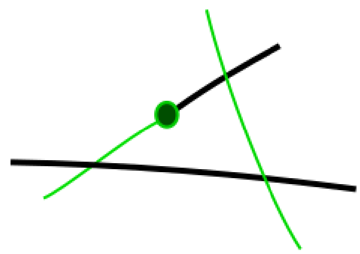

FIGURE 8.1.3. A triangle face has 1 transition vertex.

- The resulting diagram on the far right of Figure 8.1.1 has all even-sided faces, except for those that have transition vertices on their edges.

- Perform a connected sum of the knot K and the unknot. Adams showed that the there is always a four-sided face with two transition vertices.

This result is applied to show the minimum loop number among all knot diagrams in Section 8.2.

8.2. The exact value of the minimum number of loops in all diagrams

Assume that we allow highly non-minimal diagrams in determining the minimum loop numbers. Theorem 8.4 asserts that any knot diagram can be made into a connected sum with an unknot. This technique guarantees there is a four-sided face with two transition vertices. Figure 8.2.1 shows a four-sided face with the under- and over- passes ignored.

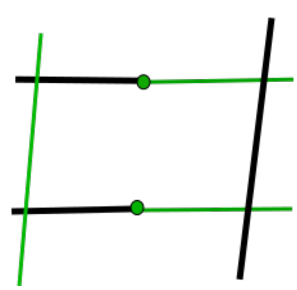

FIGURE 8.2.1. The 4-side face with 2 transition vertices in every highly non-minimal knot diagram induced from the 2-coloring algorithm above.

Theorem 8.5. *Any knot K has a diagram D such that there exists a starting point S on D with $lp(D, S, d) = 2$, where d can be any direction.*

Proof. Make any knot diagram given into D as a connected sum of itself and an unknot in a way that it admits a valid 2-coloring, as introduced in Section 8.1. Perform a Reidemeister I move, and a Reidemeister II move on this face as shown in Figure 8.2.2. Now it can be seen that starting at the point S in the figure (in either direction) achieves a loop number of two. □

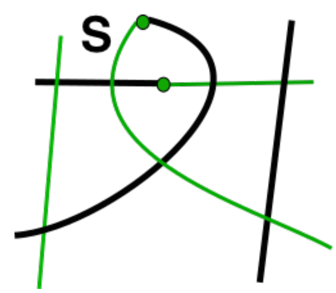

Figure 8.2.2. Perform a Reidemeister move type I and II on the 4-sided face on Figure 8.2.1.

8.2.1. A proof of the Meander diagram conjecture. Recall that an open meander has four endpoint that can be by two arcs that do not intersect the meander (except at their endpoints) and that do not intersect each other. A diagram obtained this way is called a meander diagram (Definition 2.25).

Conjecture 1 (Jablan, 2014). Every knot has a meander diagram

Proof. Every knot has a diagram D that has a four-sided face as introduced in Section 8.4. Consider the a starting point S in D that realizes the loop number two for a given direction d. The diagram can be split into two simple closed curves x_1 and x_2 whose intersections create the crossings of D. All but one of these intersections are transversal except for one crossing C that separates the two loops when the diagram is traversed. The splitting is done by performing a stereographic projection (Definition 2.7) of the whole knot diagram induced by the connected sum with an unknot, starting at the new face in Figure 8.2.2. Place the new face on the sphere S^2, so that the transition vertex lie on the north pole of S^2. The projection point is S in the stereographic projection. The number of crossings in the knot diagram is preserved by the projection. The two transition vertices become the endpoints of a curve, that happens to be a straight line under the projection. (This happens because a stereographic projection maps a sphere onto a plane; thus, curves on the sphere are straight lines on the plane).This straight line is the axis of the meander diagram, followed by a curved arc around the axis that closes the cycle. The new diagram is now a meander diagram. $\square$

Chapter 9

Conclusion and future work

9.1. Conclusion

The thesis introduces loop numbers of minimal alternating knot diagrams. It shows the conditions under which these numbers become invariant under flypes in the simplest flyping circuits, and in general flyping circuits. The family of knots that satisfies these conditions is the Montesinos knots. The proof involves one direction in a diagram. By symmetry, the invariance holds for the opposite direction, and since

$$minLp(D) = min\{minLp(D,d), minLp(D,d')\}$$

and

$$maxLp(D) = max\{maxLp(D,d), maxLp(D,d')\},$$

the invariance also holds in general.

Finally, it tackles the Meander conjecture [**12**] by responding to the following question: if we allow the link diagram D to be highly non-minimal, is there a lower bound on the loop number of all diagrams D?

9.2. Future work

The classes of alternating links include (I) single component links (knots) and (ii) multiple component links. The thesis has not address the second case (ii). Future studies may address the following questions.

(1) Can the results of loop numbers on Montesinos knots be extended to other families of knots and links?

    Other families of interest are, for example, the almost-alternating links, non-alternating links up to a number of crossings.

(2) Given a diagram D, and two directions d and d', are the following true?

$$maxLp(D,d) = maxLp(D,d');$$

$$minLp(D,d) = minLp(D,d');$$

$$aveLp(D,d) = aveLp(D,d').$$

What is the percentage of equality for a fixed link type and fixed number of crossings in the equations above?

(3) Can examples where the $minLp(D)$ changes under flypes be constructed?

We have a counter example when $maxLp(D)$ changes under a flype. One of the tangles in the diagram D in this case is virtually knotted. However, similar conditions do not make the value $minLp(D)$ change under flypes.

(4) Is the rational condition stronger than needed to have the loop numbers invariant under flypes?

CONJECTURE 2. Given a minimal diagram D of an alternating link, so that D is flypable and contains a single crossing C and virtually unknotted tangles, then the set of distinct loop numbers do not change under flypes.

(5) Given a diagram with a fixed crossing number n, what is the maximum number of different loop numbers for this diagram?

For example, the knot diagram in Figure 3.3.2 has 8 crossings, and 2 loop numbers on both directions: 3 and 5. Can a minimal diagram of n crossings, where n is odd, have the values $2, 4, .., n-1$ as loop numbers?

(6) How should loop numbers be defined in links of more than one components? Once the theory on loop numbers are established for knots, can they be carried to links?

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