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On the probability of reaching a barrier in an Erlang(2) risk process

M.M. Claramunt[1], M. Mármol[1] and R. Lacayo[2]

[1] Universitat de Barcelona, [2] Universitat Autònoma de Barcelona

Abstract

In this paper the process of aggregated claims in a non-life insurance portfolio as defined in the classical model of risk theory is modified. The Compound Poisson process is replaced with a more general renewal risk process with interoccurrence times of Erlangian type. We focus our analysis on the probability that the process of surplus reaches a certain level before ruin occurs, $\chi(u, b)$. Our main contribution is the generalization obtained in the computation of $\chi(u, b)$ for the case of interoccurrence time between claims distributed as Erlang(2, β) and the individual claim amount as Erlang (n, γ).

MSC: 91B30, 62P05

Keywords: risk theory, Erlang distribution, upper barrier, ordinary differential equation, boundary conditions.

1 Introduction

Ruin theory is concerned basically with the study of the insurer's solvency through the analysis of the level of reserves as a function of time and other important aspects such as the probability of ruin, the time of ruin and the severity of ruin.

Address for correspondence: M.M. Claramunt, M. Mármol. Departament de Matemàtica Econòmica, Financera i Actuarial. Universitat de Barcelona. Avda. Diagonal, 690. 08034 Barcelona. Tlf: 93.4035744. E-mail: mmclaramunt@ub.edu, mmarmol@ub.edu.

R. Lacayo. Departament de Matemàtiques. Universitat Autònoma de Barcelona. Edifici C. 08193 Bellaterra. Barcelona. Tlf. 93.5812534. E-mail: rlacayo@mat.uab.es.

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One of the most important probabilities related with ruin is the probability that the process of surplus reaches a certain level before ruin occurs (Dickson and Gray (1984), Dickson (1992) and Dickson and Egidio dos Reis (1994) analyzed this probability in the classical risk model). The aim of this paper is the study of this probability, $\chi(u,b)$.

In this paper the Poisson number process of the classical risk model is replaced with a more general renewal risk process with interoccurrence times of Erlangian type (see, e.g., Dickson and Hipp (1998, 2001), Dickson (1998), Cheng and Tang (2003), Sun and Yang (2004), Albrecher *et al.* (2005)). Dickson (1998) analyzed $\chi(u,b)$ for the particular case in which the interoccurrence times between claims are distributed as Erlang(2,2) and the individual claim amount has also an Erlang(2,2) distribution. Our main contribution in this paper is the generalization obtained in the computation of $\chi(u,b)$ for the case of interoccurrence time distributed as Erlang($2,\beta$) and the individual claim amount as Erlang (n,γ). Note that the Erlang distribution is a special case of the Gamma distribution where the shape parameter n is a positive integer.

The organization of the paper is as follows. In Section 2 we summarize the main results related to $\chi(u,b)$ in the classical risk model. In Section 3, we obtain an integro-differential equation for $\chi(u,b)$ in an ordinary Erlangian($2,\beta$) model, i.e. with interoccurrence time Erlang ($2,\beta$). In Section 3.1 we obtain and solve the corresponding differential equation for $\chi(u,b)$ assuming a general Erlang(n,γ) distribution for the individual claim amount. In Section 3.2 we provide numerical results for the particular case when the individual claim amount is distributed as an Erlang($2,\gamma$), and in Section 3.3 for the case of an Erlang ($1,\gamma$), i.e. exponential(γ) distribution. In Section 3.4 we analyze the influence of the individual claim amount distribution on $\chi(u,b)$ by comparing the numerical results.

2 Classical model

In the classical model of risk theory, the surplus, $R(t)$, at a given time $t \in [0,\infty)$ is defined as $R(t) = u + ct - \sum_{i=1}^{N(t)} X_i$, with $u = R(0)$ being the insurer's initial surplus. $N(t)$, the number of claims occurred until time t, follows a Poisson process with parameter λ, and X_i is the amount of the *i-th* claim and has density function $f(x)$ with mean μ. The instantaneous premium rate, c, is $c = \lambda\mu(1+\rho)$, where ρ, called the security loading, is a positive constant.

In this model, and in the more general ordinary renewal model, the interoccurrence time between claims, T_i, $i = 1,2,...$ is modeled as a sequence of independent and identically distributed random variables. T_1 denotes the time until the first claim and, in general, T_i denotes the time between the $i-1$-*th* and *i-th* claims. Note that in a Poisson process with parameter λ, $T_i, i \geq 1$ has an exponential distribution with mean $\frac{1}{\lambda}$.

Given that the time of the first claim, T_1, follows an exponential distribution with density function $f_{T_1}(t) = \lambda e^{-\lambda t}$, the probability $\chi(u,b)$ that the surplus process reaches the level $b > u$ before the time until ruin, defined as $\tau = \inf\{t : R(t) < 0\}$, can be obtained as

$$\chi(u,b) = \int_0^{t_0} \lambda e^{-\lambda t} \int_0^{u+ct} \chi(u + ct - x, b) f(x)\, dxdt + \int_{t_0}^{\infty} \lambda e^{-\lambda t} dt, \tag{1}$$

where $u + ct_0 = b$, so that the surplus process will reach b at time t_0 if no claims occur by time t_0 (Dickson and Gray, 1984).

The function $\chi(u,b)$ has also been related with ruin probabilities. The probability of ultimate ruin is defined as

$$\psi(u) = P[R(t) < 0 \text{ for some } t > 0],$$

and $\delta(u) = 1 - \psi(u)$ denotes the survival probability. It can be proved that (Dickson and Gray, 1984),

$$\delta(u) = \chi(u,b)\,\delta(b). \tag{2}$$

It is clear then that $\chi(u,b)$ can be computed as this ratio of survival probabilities as an alternative to using expression (1).

In a model with upper absorbing barrier b, such that when the reserve level reaches this barrier the process is finished, the quantity $1 - \chi(u,b)$ is also, by definition, the probability of ruin given that the initial reserve is u.

$\chi(u,b)$ plays also an important role in the model with a constant dividend barrier. In this model whenever the surplus reaches the level b, dividends are paid out in such amount that surplus stays at the barrier until the next claim. Obviously, the present value of the dividends paid out, $D(u,b)$, is a random variable that has a non-null probability at zero. This is the probability that dividends paid out are zero (Mármol *et al.*, 2003), i.e.,

$$P[D(u,b) = 0] = 1 - \chi(u,b).$$

3 Ordinary renewal model with $T_i \sim$ Erlang $(2,\beta)$

The classical Poisson risk model is an ordinary renewal process, with $T_i \sim Erlang(1,\lambda)$. In this section we assume that the number of claims is an ordinary renewal process in which the T_i are i.i.d. Erlang$(2,\beta)$ with density function,

$$k(t) = \beta^2 t e^{-\beta t}, \quad t > 0, \tag{3}$$

and distribution function

$$K(t) = 1 - e^{-\beta t}(\beta t + 1) \qquad \text{for } t \geq 0.$$

Then, as in expression (1), in Dickson (1998) it is obtained

$$\chi(u,b) = \int_0^{t_0} k(t) \int_0^{u+ct} \chi(u+ct-x,b) f(x)\,dx dt + \int_{t_0}^{\infty} k(t)\,dt. \tag{4}$$

Substituting $s = u + ct$ in (4) and differentiating twice with respect to u,

$$c^2\chi''(u,b) - 2\beta c\chi'(u,b) + \beta^2\chi(u,b) = \beta^2 \int_0^u \chi(u-x,b) f(x)\,dx. \tag{5}$$

Notice that equation (2), which in the classical model relates the survival probability with $\chi(u,b)$, is not true in the Ordinary Erlangian $(2,\beta)$ model because the lack of memory property is exclusive of the Exponencial distribution and does not hold for the general Erlang distribution (Dickson, 1998). As a result, $\chi(u,b)$ cannot be obtained as a ratio of survival probabilities, and for its calculation expression (5) must be used.

From (5) we obtain and solve the differential equation assuming that the individual claim amount is Erlang(n,γ), following the procedure presented by Dickson (1998).

3.1 Individual claim amount Erlang (n, γ)

In this section we assume that the individual claim amount follows an Erlang(n,γ) distribution with pdf

$$f(x) = \frac{\gamma^n x^{n-1} e^{-\gamma x}}{(n-1)!}. \tag{6}$$

To solve (5) let us define

$$h(u) = \beta^2 \int_0^u \chi(x,b) f(u-x)\,dx. \tag{7}$$

Substituting (6) in (7) yields

$$h(u) = \frac{\beta^2\gamma^n e^{-\gamma u}}{(n-1)!} \int_0^u \chi(x,b)(u-x)^{n-1} e^{\gamma x} dx.$$

Later on we will need an expression for the n-th derivative of the function above in terms of the lower order derivatives. This result is the essence of the following lemma.

Lemma 1 *The n-th derivative of the function $h(u)$ is given by*

$$h^{(n)}(u) = -\sum_{i=0}^{n-1} \binom{n}{i} h^{(i)}(u)\,\gamma^{n-i} + \beta^2\gamma^n\chi(u,b) \tag{8}$$

(see the proof in Appendix A)

After rewriting equation (5) in the form

$$c^2\chi''(u,b) - 2\beta c\chi'(u,b) + \beta^2\chi(u,b) = h(u),$$

it is clear that after differentiating i and n times, respectively, we obtain

$$c^2\chi^{(i+2)}(u,b) - 2\beta c\chi^{(i+1)}(u,b) + \beta^2\chi^{(i)}(u,b) = h^{(i)}(u), \tag{9}$$

and

$$c^2\chi^{(n+2)}(u,b) - 2\beta c\chi^{(n+1)}(u,b) + \beta^2\chi^{(n)}(u,b) = h^{(n)}(u). \tag{10}$$

Substitution in (10) of the value of $h^{(n)}(u)$ found in (8) and the value of $h^{(i)}(u)$ from (9) yields the following ordinary differential equation of order $(n+2)$ for $\chi(u,b)$:

$$a_{n+2}\chi^{(n+2)}(u,b) + a_{n+1}\chi^{(n+1)}(u,b) + a_n\chi^{(n)}(u,b) - \sum_{j=1}^{n-1} a_j\chi^{(j)}(u,b) = 0. \tag{11}$$

The value of the constant coefficients is given by

$$a_{n+2} = c^2$$

$$a_{n+1} = c^2\gamma n - 2\beta c$$

$$a_n = \beta^2 - 2\beta c\gamma n + \binom{n}{n-2}c^2\gamma^2$$

$$a_j = -c^2\binom{n}{j-2}\gamma^{n+2-j} + \binom{n}{j-1}2\beta c\gamma^{n+1-j} - \beta^2\binom{n}{j}\gamma^{n-j},\ \ j = 1,\dots,n-1.$$

If all the roots of the characteristic equation of (11), $\{r_i\}_{i=0}^{n+1}$, are different, it is a trivial matter to write down the solution for the ordinary differential equation above, namely:

$$\chi(u,b) = \sum_{i=0}^{n+1} \alpha_i e^{r_i u}, \tag{12}$$

where $\{r_i\}_{i=0}^{n+1}$ are functions of γ, β and c, while the $\{\alpha_i\}_{i=0}^{n+1}$ depend additionally on b. To obtain the values of $\{\alpha_i\}_{i=0}^{n+1}$ we need $(n+2)$ equations.

The first of them is obtained from the boundary condition $\chi(b, b) = 1$. Then,

$$\sum_{i=0}^{n+1} \alpha_i e^{r_i b} = 1. \tag{13}$$

From (12) we know $\chi'(u, b)$ and $\chi''(u, b)$. Substituting in (5), after rearranging terms, one easily obtains n equations, namely,

$$\sum_{i=0}^{n+1} \frac{\alpha_i}{(r_i + \gamma)^s} = 0 \quad , \ s = 1, \ldots, n. \tag{14}$$

From (4) , differentiating with respect to u, and considering (12) and its first and second derivatives, we obtain the last equation

$$1 = \alpha_0 + \frac{1}{\beta} \sum_{i=1}^{n+1} \alpha_i (\beta - c r_i) e^{r_i b}. \tag{15}$$

Consequently, after the combination of (13) , (14) and (15), we obtain the required set of $(n+2)$ equations from which to calculate the coefficients $\{\alpha_i\}_{i=0}^{n+1}$. They are

$$\begin{cases} \sum_{i=0}^{n+1} \alpha_i e^{r_i b} = 1 \\ \sum_{i=0}^{n+1} \frac{\alpha_i}{(r_i + \gamma)^s} = 0 \quad , s = 1, 2, \ldots, n \\ \frac{1}{\beta} \sum_{i=0}^{n+1} \alpha_i (\beta - c r_i) e^{r_i b} = 1. \end{cases} \tag{16}$$

3.2 $T_i \sim$ Erlang $(2, \beta)$ and $X \sim$ Erlang $(2, \gamma)$

In this section we study the case $n = 2$. The ODE can be obtained directly from (11) as

$$\begin{aligned} c^2 \chi''''(u, b) + \left(2\gamma c^2 - 2\beta c\right) \chi'''(u, b) + \\ \left(\beta^2 - 4\gamma\beta c + \gamma^2 c^2\right) \chi''(u, b) + \left(2\gamma\beta^2 - 2\gamma^2 \beta c\right) \chi'(u, b) = 0. \end{aligned} \tag{17}$$

This equation generalizes the one obtained by Dickson (1998) for the particular case $c = 1.1, \beta = 2$ and $\gamma = 2$.

The solution of (17) gives

$$\chi(u, b) = \sum_{i=0}^{3} \alpha_i e^{r_i u},$$

where $\{r_i\}_{i=0}^{3}$ are the roots of the characteristic equation of (17) .
From (16) , the system of equations required to find $\{\alpha_i\}_{i=0}^{3}$ is,

$$\begin{cases} \sum_{i=0}^{3} \alpha_i e^{r_i b} = 1 \\ \sum_{i=0}^{3} \frac{\alpha_i}{(r_i + \gamma)} = 0 \\ \sum_{i=0}^{3} \frac{\alpha_i}{(r_i + \gamma)^2} = 0 \\ \frac{1}{\beta} \sum_{i=0}^{3} \alpha_i (\beta - c r_i) e^{r_i b} = 1. \end{cases} \quad (18)$$

For $\gamma = 2, \beta = 2$, and $c = 1.1$, solving (18) and finding the roots of the characteristic equation (17) , we obtain in Table 1 the results for $\chi(u, b)$ for different values of u and b,

Table 1.

u/b	0	1	2	3	4	5
0	1	0.5802	0.3694	0.2805	0.2335	0.2049
1		1	0.7600	0.5828	0.4854	0.4258
2			1	0.8472	0.7096	0.6228
3				1	0.8939	0.7875
4					1	0.9224
5						1

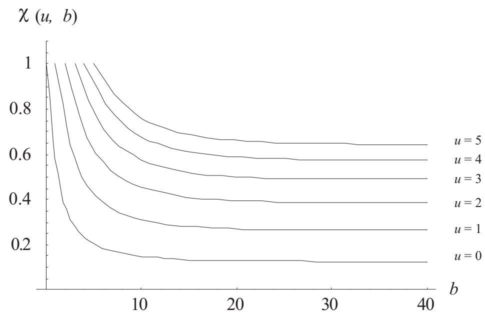

Figure 1: $\chi(u, b)$ *for* $u = 0, 1, 2, 3, 4, 5$.

Graphically we can represent the evolution of $\chi(u,b)$ with respect to b for different values of the initial surplus u. In Figure 1 $\chi(u,b)$ is plotted for $u = 0, 1, 2, 3, 4, 5$. For a given value of b, the probability of the reserves reaching that value before ruin, $\chi(u,b)$, is increasing in u. On the other hand, for a given value of u, the probability $\chi(u,b)$ is decreasing in b, and for each value of u tends toward a limiting value, as shown in Table 2 below.

Table 2.

u	0	1	2	3	4	5
$\lim_{b\to\infty}\chi(u,b)$	0.1268	0.2636	0.3855	0.4876	0.5727	0.6438

Obviously, as b tends to infinity, the probability $\chi(u,b)$ includes only those trajectories of the reserve process which do not lead to ruin. In other words,

$$\lim_{b\to\infty}\chi(u,b) = \delta(u). \tag{19}$$

Consequently, the limiting values for $\chi(u,b)$ just obtained are the values of the survival probability for the corresponding initial reserves u (they can be found in the discussion section written by De Vylder and Goovaerts in Dickson (1998)).

3.3 $T_i \sim$ Erlang (2,β) and $X \sim$ exp (γ)

Here we study the case $n = 1$. The corresponding ODE, from (11) is

$$c^2\chi'''(u,b) + \left(\gamma c^2 - 2\beta c\right)\chi''(u,b) + \left(\beta^2 - 2\beta\gamma c\right)\chi'(u,b) = 0,$$

with solution

$$\chi(u,b) = \alpha_0 + \sum_{i=1}^{2}\alpha_i e^{r_i u},$$

where r_1 and r_2 are the roots of

$$c^2r^2 + \left(\gamma c^2 - 2\beta c\right)r + \left(\beta^2 - 2\beta\gamma c\right) = 0.$$

In order to obtain $\{\alpha_i\}_{i=0}^{2}$, we put $n = 1$ in (16),

$$\begin{cases} \sum_{i=0}^{2} \alpha_i e^{r_i b} = 1 \\ \sum_{i=0}^{2} \dfrac{\alpha_i}{(r_i + \gamma)} = 0 \\ \frac{1}{\beta} \sum_{i=0}^{2} (\beta - c r_i) \cdot \alpha_i \cdot e^{r_i \cdot b} = 1. \end{cases}$$

For $\gamma = 1, \beta = 2$, and $c = 1.1$, we obtain in Table 3 the following results of $\chi(u, b)$

Table 3.

u/b	0	1	2	3	4	5	···	∞
0	1	0.6363	0.4318	0.3339	0.2779	0.2419	···	0.1199
1		1	0.7838	0.6106	0.5083	0.4425	···	0.2194
2			1	0.8518	0.7125	0.6204	···	0.3076
3				1	0.8906	0.7781	···	0.3858
4					1	0.9155	···	0.4552
5						1	···	0.5168

The behaviour of this probability in this case turns out to be the same as in Section 2.1 where the Erlang$(2, \gamma)$ distribution was assumed.

3.4 Numerical comparison

In order to study the influence of the distribution of the claim amount on the probability $\chi(u, b)$ we find the behaviour of the latter when the individual claim amount follows an Erlang$(n, \gamma), n = 1, 2, 3, 4, 5$ distribution. To ensure that the results can be compared to one another we set $n = \gamma$ and call the resulting distribution simply an Erlang(n). Note that in this case the mean of the claim n/γ is 1.

For $n = 1$ and $n = 2$ the probability $\chi(u, b)$ behaves as indicated in Sections 2.3 and 2.2, respectively, i.e., takes the value 1 for $u = b$, and for a fixed u, it is decreasing in b and tends to a limiting value which is the same as the survival probability in the model without a barrier.

The effect of the claim amount distribution on $\chi(u, b)$ depends, as expected, on the initial reserve and barrier levels u and b, and on the difference $u - b$ as well. The following three figures show the behavior of $\chi(u, b)$ as a function of b for initial reserve levels $u = 0, u = 1$ and $u = 2$, respectively. For the given u the graphs in each figure show the dependence of $\chi(u, b)$ on the Erlang parameter n for $n = 1, 2, 3, 4, 5$. These values have been chosen for illustration only, and carry no special significance.

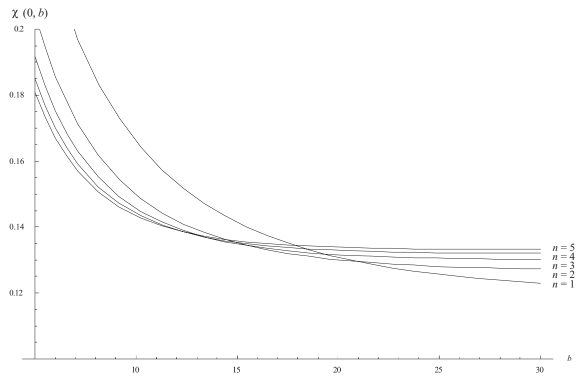

Figure 2: *$\chi(u, b)$ for $u = 0$, assuming $n = 1, 2, 3, 4, 5$.*

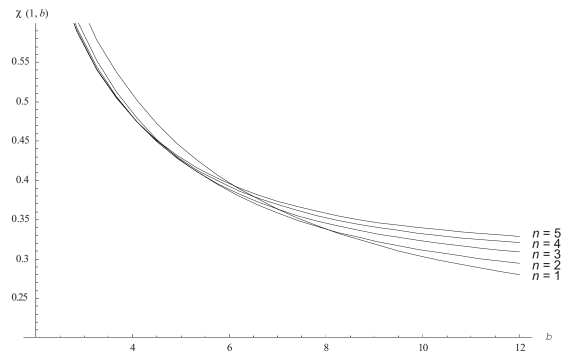

Figure 3: *$\chi(u, b)$ for $u = 1$, assuming $n = 1, 2, 3, 4, 5$.*

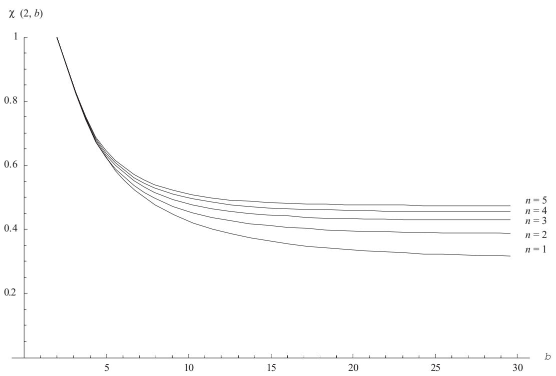

Figure 4: *$\chi(u, b)$ for $u = 2$, assuming $n = 1, 2, 3, 4, 5$.*

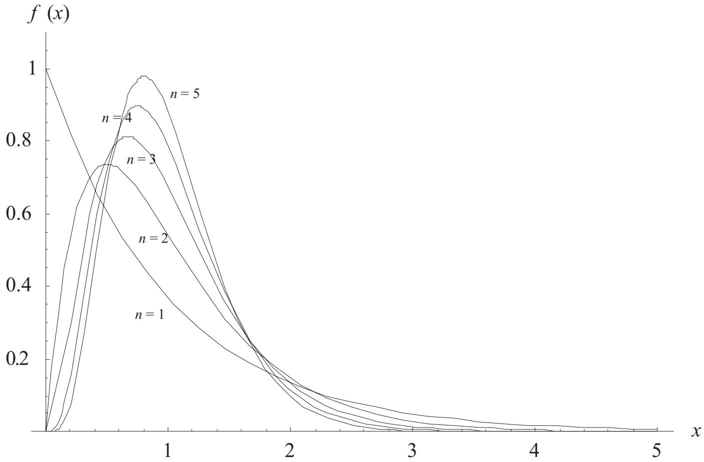

Figure 5: *pdf Erlang*(n) *for* $u = 1$*, assuming* $n = 1, 2, 3, 4, 5$.

Before analyzing our results further, in Figure 5 we provide the graphs of an Erlang(n) pdf with mean $E[\cdot] = 1$ and variance $Var[\cdot] = \frac{1}{n}$, also for $n = 1, 2, 3, 4, 5$. It is clear from the figure that with increasing n both the variance and the asymmetry decrease, and the pdf concentrates more and more around its mean 1.

Moreover, from Figures 2, 3 and 4, it follows that for values of u near zero and small b, $\chi(u, b)$ decreases as n increases. This behaviour is reversed as b grows larger (the graphs intersect at different points, and eventually those corresponding to larger n appear on top). A plausible explanation for this behaviour can be found in Figure 5, from which we see that for small n (recall that $n = 1$ coincides with the exponential case) the probability of occurrence of small and large claims is greater than that corresponding to large n. As a consequence, for values of u near zero and small b, the probability of reaching b before ruin occurs is greater for small n. For $b \gg u$, large claims take preponderance in reaching the ruin state and they are more likely for small n, thus $\chi(u, b)$ is smaller for small n.

Table 4.

$\delta(u)$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$\delta(0)$	0.1199	0.1268	0.1300	0.1319	0.1332
$\delta(1)$	0.2194	0.2636	0.2882	0.3041	0.3153
$\delta(2)$	0.3076	0.3855	0.4282	0.4552	0.4738
$\delta(3)$	0.3858	0.4876	0.5409	0.5736	0.5956
$\delta(4)$	0.4552	0.5727	0.6314	0.6663	0.6892
$\delta(5)$	0.5168	0.6438	0.7041	0.7388	0.7612

As u increases, the inversion process with increasing b disappears rapidly. In fact the graphs intersect very close to the initial abscissa b. This fact may be taken to mean that for initial reserves of substantial magnitude, the greater probability of small claims for small n loses relevance.

Below, in Table 4, we provide an additional table with the survival probabilities for all cases $n = 1, 2, 3, 4, 5$ (recall expression (19)). Note that they represent the survival probability in the absence of a barrier. In our case, as the table clearly shows, in the limit $\chi(u, b)$ decreases with increasing n in accordance with the results above.

Appendix A

Proof of Lemma 1.

Since the function $h(u)$ depends explicitly on n, for notational convenience we rewrite it as

$$\begin{aligned} h_n(u) &= \frac{\beta^2\gamma^n e^{-\gamma u}}{(n-1)!}\int_0^u \chi(x,b)(u-x)^{n-1} e^{\gamma x} dx \\ &= A_n e^{-\gamma u}\int_0^u B(u-x)^{n-1}\, dx, \end{aligned} \tag{20}$$

where, $A_n = \dfrac{\beta^2\gamma^n}{(n-1)!}$ and $B = \chi(x,b)e^{\gamma x}$.
For $n = 1$ we have

$$h_1(u) = \beta^2\gamma e^{-\gamma \upsilon}\int_0^u \chi(x,b)e^{\gamma x} dx,$$

and it readily follows through differentiation and substitution that

$$h_1'(u) = -\gamma h_1(u) + \beta^2\gamma\chi(u,b) \tag{21}$$

For $n > 1$, differentiating (20) once yields

$$h_n'(u) = A_n e^{-\gamma u}(n-1)\int_0^u B(u-x)^{n-2}\, dx - \gamma A_n e^{-\gamma u}\int_0^u B(u-x)^{n-1}\, dx,$$

which, after dropping the argument u, can be written as the recurrence relation

$$h_n' = -\gamma h_n + \gamma h_{n-1} \quad, n > 1 \tag{22}$$

Obviously we can obtain all the required derivatives of $h_n(u)$ by successive differentiation of (22).

From (22) we have

$$\begin{aligned} h_n'' &= -\gamma h_n' + \gamma h_{n-1}' = -\gamma h_n' + \gamma\left(-\gamma h_{n-1} + \gamma h_{n-2}\right) \\ &= -\gamma h_n' - \gamma\left(h_n' + \gamma h_n\right) + \gamma^2 h_{n-2} \\ &= -2\gamma h_n' - \gamma^2 h_n + \gamma^2 h_{n-2}. \end{aligned} \tag{23}$$

In the first line we used (22) to obtain h_{n-1}', and again in the second line to obtain γh_{n-1}.

In the same manner, from (23) we get

$$\begin{aligned} h_n''' &= -2\gamma h_n'' - \gamma^2 h_n' + \gamma^2 h_{n-2}' = -2\gamma h_n'' - \gamma^2 h_n' + \gamma^2\left(-\gamma h_{n-2} + \gamma h_{n-3}\right) \\ &= -2\gamma h_n'' - \gamma^2 h_n' - \gamma\left(h_n'' + 2\gamma h_n' + \gamma^2 h_n) + \gamma h_{n-3}\right) \\ &= -3\gamma h_n'' - 3\gamma^2 h_n' - \gamma^3 h_n + \gamma^3 h_{n-3}. \end{aligned} \tag{24}$$

Here we used in the first line (22) to obtain h_{n-2}', and in the second (23) to obtain $\gamma^2 h_{n-2}$.

It is clear that in the fourth derivative we would have to make use of (22) and of (24) and, in general, each derivative requires the use of (22) and of the previous one. Moreover, it follows easily from (22), (23) and (24), after transposing, that the rightmost term in the derivative of order k can be formally written as the k-th derivative of the product γh_n (this follows from Leibniz formula) provided the derivatives of γ are interpreted as regular powers. In other words, recalling that $h_n^{(0)} = h_n$,

$$\gamma^k h_{n-k} = (\gamma h_n)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} \gamma^j h_n^{(k-j)}, \; k < n. \tag{25}$$

In particular, for $k = n - 1$, (25) becomes

$$\gamma^{n-1} h_1 = (\gamma h_n)^{(n-1)} = \sum_{j=0}^{n-1} \binom{n-1}{j} \gamma^j h_n^{(n-1-j)}. \tag{26}$$

Note that in (21) h_1' is given in terms of h_1, that is, in terms of the summation appearing in (26). If we now differentiate (26) and make the appropriate substitutions, after transposing we obtain

$$\beta^2 \gamma^n \chi(u, b) = \sum_{j=0}^{n-1} \binom{n-1}{j} \gamma^j h_n^{(n-j)} + \sum_{j=0}^{n-1} \binom{n-1}{j} \gamma^{j+1} h_n^{(n-1-j)}.$$

In the first summation above we can safely change the upper limit to n because $\binom{n-1}{n} = 0$. In the second summation, by substitution of the dummy variable j with $j - 1$ the summation limits are changed from $j = 1$ to $j = n$. But since $\binom{n-1}{-1} = 0$, we put the lower limit at $j = 0$. Combining the resulting summations and given that $\binom{n-1}{j} + \binom{n-1}{j-1} = \binom{n}{j}$ we finally obtain

$$\beta^2\gamma^n\chi(u,b) = \sum_{j=0}^{n}\binom{n}{j}\gamma^j h_n^{(n-j)}.$$

The equivalence of this expression to (8) is evident. The proof of the Lemma is complete.

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