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GRAU DE MATEMÀTIQUES

Facultat de Matemàtiques
Universitat de Barcelona

AMERICAN OPTIONS

Autor: Juan Raúl Quiles Petidier

Director: Dr. Josep Vives
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i Informàtica. (UB)

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Abstract

One of the most important things that rules the world, is the economy. And the science that explains better the economy, is maths.

When I was a child, I wanted to become an economist. So I decided to study maths because the background of the economy is maths, and knowing maths, you can understand the economy.

Studying maths, I have been so amazed on how from nothing, only using mathematical results, we can build real things.

This research work combines both things: a construction from nothing of an application to the economy, more precisely, applied to the financial markets.

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Contents

1 Introduction

Financial markets are a location where buyers and sellers meet to participate in the trade of assets at prices determined by the forces of supply and demand. This assets can be goods or services like equities, bonds, currencies, derivatives, options...etc. Depending on the asset negotiated, there are different types of financial markets.

In particular, one of the most popular financial markets is the "Future and Options Market". There, the buyers and sellers interchanges rights to sell or to buy a certain product at a fixed price at a stipulated time: A farmer who thinks that his harvest will not be so good, wants to make sure that he can sell his products to a good stock price in spite of being lower than the one set by the market, or an airline will not be so exposed to the volatility of the market and wants to ensure a fixed price of the fuel in spite of being higher than the one set by the market in the future.

As we can suppose, it exists different type of options depending on the clauses of the rights. This research work studies two type of options, the "European Options" and the "American Options". The European options are rights to buy or to sell a product at a certain price that only can be executed in the future. In this work, we will see what is the fair price that the buyer has to pay for this option and what does the seller have to do to face his obligation to the buyer in the future, called hedging strategy. American options are like European options but with the fact that the buyer can execute the option from the moment that has been negotiated, and the expiry time. So, apart from what is the fair price the buyer has to pay and what does the seller have to do to face his obligation, additionally the buyer always faces the decision to execute the option or to wait for a next time. This decision is not as easy as it could seem, because it is difficult to know if we will win more money executing the option later or executing now.

This work is composed by different parts:

- A first part dedicated to basic notions. Before to face the problem of determining the fair price to the buyer and the strategy to the seller for European options, we have to introduce some mathematical results to understand how we determine the fair price. Concepts about probability spaces like: what is a probability space, the conditional probability and independent probability with all their properties, expectation and conditional expectation are supposed to be known. We introduce some concepts about stochastic processes and martingales in discrete time and a formal definition of a financial market in discrete time.

  For the American options, for a better comprehension of how does they work, it is necessary to understand how to price and how to hedge European options before. So apart from the concepts described below, before to determine the fair price, the hedging strategy and the optimal stopping of the American options, we also need concepts about supermatingales, submartingales and optimal stopping.

- A second part dedicated to solve the problems of the fair price, hedging strategy and optimal stopping: For the European options, we define a model of financial market in discrete time in which we solve the problem of pricing and hedging European options, called CRR model. We also see how to solve these problems in two types of models derived from the CRR model: The single-step binomial model and the two-step binomial model. For the American options, as the model is used to price, to hedge and to determine the optimal stopping time is the same as the European options, we will see how to solve these problems in the two-step binomial model and in the CRR model, doing a dualism with the European options pricing.
- A final part for a program in C++ that solves the problem of pricing and which is the optimal stopping for American options.

2 Stochastic processes and martingales in discrete time

2.1 Basics elements of stochastic processes in discrete time

Definition 1. *A stochastic process* $X := \{X_n, n \in \mathbb{T}\}$ *is a sequence of random variables defined in a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and in a certain period of time* $\mathbb{T}$. *Usually* $\mathbb{T} = \{0, .., N\}$.

Definition 2. *An associated filtration in a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a sequence of* σ*-algebras* $\mathbb{F} := \{\mathcal{F}_n, n \in \mathbb{T}\}$ *that:*

- $\mathcal{F}_n \subseteq \mathcal{F}$, $\forall n \in \mathbb{T}$.
- $\mathcal{F}_{n-1} \subseteq \mathcal{F}_n$, $\forall n \in \mathbb{T}^* := \mathbb{T} - \{0\}$.

A probability space with an associate filtration $(\Omega, \mathcal{F}, \mathbb{F}.\mathbb{P})$ *is called filtered probability space.*

Definition 3. *Given a stochastic process* X, *we define its natural filtration as the sequence of* σ*-algebras:*

$$\mathcal{F}_n := \sigma\left\{X_k, k \leq n\right\}.$$

Definition 4. *We say that a stochastic process* X *defined in a filtered probability space is adapted if* $\forall n \in \mathbb{T}$, X_n *is* $\mathcal{F}_n$*-measurable.*

Remark 5. *Any stochastic process is adapted to his natural filtration.*

Definition 6. *A stochastic process* X *is predictable if:*

- X_0 *is* $\mathcal{F}_0$*-measurable.*
- X_n *is* $\mathcal{F}_{n-1}$*-measurable,* $\forall n \in \mathbb{T}^*$.

Remark 7. *Any predictable process is adapted.*

2.2 Martingales

Definition 8. *We say that a process $M := \{M_n, n \in \mathbb{T}\}$ is a martingale respect the filtration $\mathbb{F}$ if:*

1. *M is adapted to $\mathbb{F}$.*
2. *$\mathbb{E}(|M_n|) < \infty$, $\forall n \in \mathbb{T}$, it means that all the variables of the process are integrable.*
3. *$E[M_n|\mathcal{F}_{n-1}] = M_{n-1}$, q.s, $\forall n \in \mathbb{T}^*$.*

Example 9. *(Martingale process)*

- *Example 1: Given $X_1, X_2, .., X_n$ a sequence of random independent variables with $\mathbb{E}(|X_k|) < \infty$, $\forall k$ and:*

$$\mathbb{E}(X_k) = 0, \forall k.$$

  We define $S_0 := 0$ and:

$$S_n := X_1 + X_2 + .. + X_n.$$
$$\mathcal{F}_n := \sigma(X_1, X_2, ..., X_n),\ \mathcal{F}_0 = \{\emptyset, \Omega\}.$$

  We have that $\forall n \geq 1$:

$$\mathbb{E}(S_n|\mathcal{F}_{n-1}) = \mathbb{E}(S_{n-1}|\mathcal{F}_{n-1}) + \mathbb{E}(X_n|\mathcal{F}_{n-1}) = S_{n-1} + \mathbb{E}(X_n) = S_{n-1}.$$

  So, S_n is a martingale.

- *Example 2: Given $X_1, X_2, .., X_n$ a sequence of random independent variables with $\mathbb{E}(|X_k|) < \infty$, $\forall k$ and:*

$$\mathbb{E}(X_k) = 1, \forall k.$$

  We define $M_0 := 0$ and:

$$M_n := X_1 X_2 \cdots X_n.$$
$$\mathcal{F}_n := \sigma(X_1, X_2, ..., X_n), \mathcal{F}_0 = \{\emptyset, \Omega\}.$$

  We have that $\forall n \geq 1$:

$$\mathbb{E}(M_n|\mathcal{F}_{n-1}) = M_{n-1}\mathbb{E}(X_n) = M_{n-1}.$$

  So, M_n is a martingale.

Definition 10. *Let M be a martingale, H a predictable and bounded process respect to a filtration $\mathbb{F}$, $x_0 \in \mathbb{R}$ a constant. The transformation of the martingale M by the predictable process H is the process $X := \{X_n, n \in \mathbb{T}\}$ defined as:*

$$X_n := x_o + \sum_{k=1}^{n} H_k(M_k - M_{k-1}), n \in \mathbb{T}.$$

Proposition 11. *The transformation of a martingale is a martingale.*

Proof. *[3]: "Modelización estocástica, J.M Corcuera: pag.9".*

□

Proposition 12. *Given $M := \{M_n, n \geq 0\}$ an adapted and integrable process, we say that M is a martingale if and only if for all predictable and bounded process H and for all $n \geq 1$:*

$$\mathbb{E}(\sum_{i=1}^{n} H_i(M_i - M_{i-1})) = 0.$$

Proof. *[2]: "Introduction to Stochastic Calculus Applied to Finance 2nd Edition, D.Lamberton and B.Lapeyre: pag.20".* □

3 Discrete Financial Markets

3.1 Discrete-time formalism: Assets and strategies

Definition 13. *A discrete-time financial model is built on:*

1. *A finite probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *where:*
   (a) Ω *is a finite set of elements, where* $\omega \in \Omega$ *represents a possible evolution of the market.*
   (b) $\mathcal{F} = \mathcal{P}(\Omega)$ *where* $\mathcal{P}(\Omega)$ *denotes the collection of all subsets of the finite sample space* Ω.
   (c) $\mathbb{P}$ *is unknown and we assume* $\mathbb{P}(\omega) > 0$, $\forall \omega \in \Omega$.
2. *A filtration* $\mathbb{F} := \{\mathcal{F}_n, n \in \mathbb{T}\}$, *where* $\mathcal{F}_0 := \{\emptyset, \Omega\}$ *and* $\mathcal{F}_N := \mathcal{F}$. *The set* $\mathcal{F}_n$ *can be seen as the information available at time* n *and it can be called the* σ*-algebra of events up to time* n.
3. *A set of time* $\mathbb{T} := \{0, 1, 2, ..., N\}$ *where* N *is finite and fixed.* $\mathbb{T}^* := \mathbb{T} - \{0\}$.
4. *A deterministic process* $A := \{A_n, n \in \mathbb{T}\}$ *called riskless asset that represents a bank account. We set* $A_0 = 1$. *The return of the riskless asset over one period is constant and equal to* r, *so* $A_n = (1 + r)^n$.
5. *A finite number of risky assets* $S_n^1, ..., S_n^d$ *where* S_n^i *represents the price of the risky asset* i *at time* $n \in \mathbb{T}$. *We suppose that this assets are adapted to the filtration* $\mathcal{F}$ *given, so it is natural to choose the filtration:*

$$\mathcal{F}_n := \left\{S_k^i, 0 \leq k \leq n, 1 \leq i \leq d\right\}.$$

   This filtration is called natural filtration. The coefficient $\beta_n = 1/A_n$ *is interpreted as the discount factor and* $\tilde{S}_n^i := \frac{S_n^i}{A_n}$ *is the discounted price.*

   In our case $\tilde{S}_n^i := (1 + r)^{-n} S_n^i$, $\forall n \in \mathbb{T}$, *and* $\tilde{A}_n \equiv 1$.

Remark 14. *In the case that we have only one risky asset, its usual to denote* S_n *the price of the risky asset at time* $n \in \mathbb{T}$.

Remark 15. *Note that working with a finite probability space, all real-valued random variables are integrable.*

Definition 16. *A portfolio* V_n *is a set of risky and riskless assets.*

Definition 17. *A trading strategy is defined as a predictable stochastic process:*

$$\phi_n := \left\{(\phi_n^0, \phi_n^1, ..., \phi_n^d) \in \mathbb{R}^{d+1}, n \in \mathbb{T}^*\right\}$$

where ϕ_n^i denotes the number of shares of asset i held in the portfolio at time n. Recall that a trading strategy ϕ is predictable if $\forall i \in \{0, 1, ..., d\}$:

1. *ϕ_0^i is $\mathcal{F}_0$-measurable.*
2. *$\forall n \geq 1$, ϕ_n^i is $\mathcal{F}_{n-1}$-measurable.*

Definition 18. *The value of the portfolio at time n is the scalar product:*

$$V_n(\phi) = \phi_n S_n = \sum_{i=0}^{d} \phi_n^i S_n^i.$$

So, its discounted value is:

$$\tilde{V_n}(\phi) = \beta_n(\phi_n S_n) = \phi_n \tilde{S}_n.$$

Definition 19. *A strategy is called self-financing if the following equation is satisfied for all $n \in \{0, 1, ..., N-1\}$:*

$$\phi_n S_n = \phi_{n+1} S_n$$

Proposition 20. *The following results are equivalent:*

1. *The strategy ϕ is self-financing.*
2. *For any $n \in \{1, ..., N\}$,*

$$V_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \Delta S_j.$$

3. *For any $n \in \{1, ..., N\}$,*

$$\tilde{V_n}(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j \Delta \tilde{S}_j.$$

where $\Delta\tilde{S}_j$ is the vector $\tilde{S}_j - \tilde{S}_{j-1} = \beta_j S_j - \beta_{j-1} S_{j-1}$.

Proposition 21. *For any predictable process $\tilde{\phi} = \left\{(\phi_n^1, ..., \phi_n^d), 0 \leq n \leq N\right\}$ and for any $\mathcal{F}_0$-measurable variable V_0, there exists a unique predictable process $(\phi_n^0)_{0 \leq n \leq N}$ such that the strategy $\phi = (\phi^0, \phi^1, ..., \phi^d)$ is self-financing and its initial value $V_0(\phi) = V_0$.*

Proof. The proof of both propositions can be found in *[2]: "Introduction to Stochastic Calculus Applied to Finance 2nd Edition, D.Lamberton and B.Lapeyre: pag.17".* □

Definition 22. *A strategy ϕ is admissible if it is self-financing and $V_n(\phi) \geq 0$, for any $n \in \{0, 1, ..., N\}$.*

Definition 23. *An arbitrage strategy is an admissible and self-financing strategy that $V_0(\phi) = 0$, and $V_N(\phi) > 0$ with strictly positive probability.*

3.2 Viable and complete markets: First and second fundamental theorems of finance

Definition 24. *A market is viable if there are no arbitrage opportunities.*

Definition 25. *A probability $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ if for any $\mathcal{A} \in \mathcal{F}$:*

$$\mathbb{P}(\mathcal{A}) = 0 \Longleftrightarrow \mathbb{P}^*(\mathcal{A}) = 0.$$

It is written $\mathbb{P}^* \sim \mathbb{P}$.

Remark 26. *Note that in our finite probability space, this means simply that $\mathbb{P}^*$ satisfies also the condition $\mathbb{P}^*(\omega) > 0$, $\forall \omega \in \Omega$.*

The first Fundamental Theorem of Finance, characterizes viable markets in terms of the notion of equivalent probability measure and the notion of martingale. Note that this mean that we are relating the pure financial hypothesis that a market is viable with pure mathematical concepts.

Theorem 27 (First Fundamental Theorem of Finance or Fundamental Theorem of Asset Pricing)**.** *The market is viable if and only if it exists a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that the discounted prices of assets are $\mathbb{P}^*$-martingales. This probability is called probability risk neutral.*

Proof. [6]: "Mathematics of Financial Markets 2nd ed., R.J.Elliot and E.Kopp: pags.60-61". □

Definition 28. *Consider a $\mathcal{F}_N$-measurable and non-negative random variable H that can represent a payoff that can be obtained at time N. It is said that H is replicable if it exists a constant V_0 and a self-financing and admissible strategy ϕ such that $V_N(\phi) = H$.*

Remark 29. *Note that this is a pure financial definition and says that any quantity can be replicated with a good choose of V_0 and ϕ. Its validity in real markets is no so obvious as no arbitrage.*

Remark 30. *If a market is viable and it exists a self-financial strategy that replicates H, it is also admissible. Note that if it exists a risk neutral measure $\mathbb{P}^*$, processes $\tilde{S}$ are martingales and so it is $\tilde{V}_n(\phi)$ for any self-financing strategy ϕ because it is a transformation of a martingale. Then:*

$$\tilde{V}_n(\phi) = \mathbb{E}^*[\tilde{V}_N(\phi)|\mathcal{F}] = \mathbb{E}^*[\frac{H}{A_N}|\mathcal{F}_N] \geq 0.$$

because conditional expectation is a positive operator.

Definition 31. *The market is complete if every payoff is replicable.*

Theorem 32 (Second Fundamental Theorem of Finance)**.** *A viable market is complete if and only if there exists a unique probability measure* $\mathbb{P}^*$ *equivalent to* $\mathbb{P}$*, under which discounted prices are martingales.*

Proof. *[5]: "Discrete Models of Financial Markets, M.Capiński and E.Kopp: pags.39-40".* □

3.3 Pricing and hedging contingent claims in discrete and complete markets

It is assumed that the market is viable and complete. We denote by $\mathbb{P}^*$ the unique probability measure under which the discounted prices of financial assets are martingales.

Definition 33. *An European option (European contingent claim) is a contract that gives to you the right (not the obligation) to get a payoff G at maturity N, where G is a non-negative $\mathcal{F}_N$-measurable random variable. Its value depends on the market movement from $n = 0$ to $n = N$.*

Definition 34.

- *A call option is an European option that gives the right (not the obligation) to the owner to buy a specified amount of a good at a strike price K in the expiration date N. The payoff is $G = (S_N - K)^+$.*
- *A put option is an European option that gives the right (not the obligation) to the owner to sell a specified amount of a good at a strike price K in the expiration date N. The payoff is $G = (K - S_N)^+$.*

The good is called usually the underlying good and it can be a stock, an index, a currency, a commodity or any thing priced continuously in an open market.

Remark 35. *We notice that if at time N, $S_N \leq K$, to execute the call option has no sense because we would lose money, we can buy the good directly in the spot market at lower price S_N. Reciprocally, if $S_N \geq K$ and we execute the put option, we would also lose money, we are selling a good on a price lower than the price the market fixes.*

3.3.1 Pricing and hedge

Let's consider G a $\mathcal{F}_N$-measurable non-negative random variable that represents the payoff of an option.

Definition 36.

- *To price an option with payoff G is to determine the price at $n = 0$ of the right to receive the random quantity G at time $n = N$.*
- *To hedge an option with payoff G is to establish and investment strategy for the seller to cover his or her obligation to give G at $n = N$ to the buyer.*

Remark 37. *The completeness of the market guarantees the existence of a self-financing and admissible strategy ϕ that generates G in the sense that $\tilde{V}(\phi)$ is a martingale with $\tilde{V}_N(\phi) = G$ and $V_0 = \mathbb{E}^*(\tilde{G})$, so $\mathbb{E}^*(\tilde{G})$ can be a fair price for G. This expectation $\mathbb{E}^*$ is the expectation respect $\mathbb{P}^*$ where $\mathbb{P}^*$ is the unique probability* *measure under which the discounted prices of financial assets are martingales. This probability exists and is unique as a result of "Theorem 32 (Second Fundamental Theorem of Finance)".*

Moreover, ϕ is a possible hedging strategy and:

$$V_n(\phi) = \frac{S_n^0}{A_N}\mathbb{E}^*(G|\mathcal{F}_n)$$

is the price of the portfolio in each time n.

Remark 38. *It is important to notice that the computation of the option price only requires the knowledge of $\mathbb{P}^*$ and not $\mathbb{P}$. This means that is not necessary to know the real probability, is enough to consider the measurable space $(\Omega, \mathbb{F})$ equipped with the filtration $(\mathcal{F}_n)$ as the set of all possible states and the evolution of the information over time.*

Note that $\mathbb{P}$ is subjective and depends on your perspectives about the market. What this theory is saying is that the price of G is independent from the point of reference of the observer.

Definition 39. *According to the theory described below:*

- *The fair price of an European call is $C_0 := \mathbb{E}^*(\frac{(S_N-K)^+}{A_N})$.*
- *The fair price of an European put is $P_0 := \mathbb{E}^*(\frac{(K-S_N)^+}{A_N})$.*

Proposition 40 (Call-Put Parity)**.** *Lets consider the sequence of prices S_n. If C is the price of an European call on S_n with fair price K with expiration date N and P the price of the European put with the same data. If r is constant:*

$$C_0 - P_0 = S_0 - K(1+r)^{-N}.$$

Proof. *[2]: "Introduction to Stochastic Calculus Applied to Finance 2nd Edition, D.Lamberton and B.Lapeyre: pag.28".* □

4 Cox-Ross-Rubinstein model

Now, we will see a discrete financial model to study the assessment and hedge of Euorpean options in a small period of time.

The Cox-Ross-Rubinstein model give us the tools for the assessment and hedge of European options in a certain period of time $\mathbb{T} = \{0, 1, ..., N\}$.

This model was developed by Cox, Ross and Rubinstein about 1980. [1]

4.1 CRR model formalism

The Cox-Ross-Rubinstein model considers only two assets:

1. One risky asset whose price is S_n where $0 \leq n \leq N$.
2. One riskless asset whose return is $r > 0$ over one period of time and its value is $A_n = (1 + r)^n$.

Between two consecutive periods, the price changes by a factor that is either $1 + u$ (upwards) or $1+d$ (downwards) with $-1 < d < u$. So $S_n \in \{S_{n-1}(1 + d), S_{n-1}(1 + u)\}$. We suppose that S_0 is constant and $\mathbb{T} := \{0, 1, ..., N\}$.

We define the random variables $\{T_n, n \in \mathbb{T}\}$ such that:

$$T_n := \frac{S_n}{S_{n-1}}$$

With this notation, we also can write:

$$S_n = S_0 \prod_{i=1}^{n} T_i, n \in \mathbb{T}.$$

So, we can consider the following market model:

$$(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$$

where:

1. $\Omega = \{1 + d, 1 + u\}^N$.
2. $\mathbb{F} = \{\mathcal{F}_n, n \in \mathbb{T}\}$ with $\mathcal{F}_n := \sigma\{S_0, T_1, ..., T_N\} = \sigma\{S_0, S_1, ..., S_N\}$, in particular $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
3. $\mathcal{F} := \mathcal{F}_N$.
4. $\mathbb{P}$ is a probability over Ω such that:

$$\mathbb{P}(T_1 = x_1, ..., T_N = x_N) > 0, \forall (x_1, ..., x_N) \in \Omega.$$

4.2 Viability and completeness of the CRR model

We begin by the following characterization of viability in the CRR model:

Proposition 41. *$\tilde{S}$ is $\mathbb{P}$-martingale if and only if $E[T_n|\mathcal{F}_{n-1}] = 1+r$ for all $n \in \mathbb{T}^*$.*

Proposition 42. *The model CRR is viable if and only if $d < r < u$.*

Proposition 43. *We suppose $d < r < u$. If $\tilde{S}$ is $\mathbb{P}$-martingale, necessarily $\mathbb{P} = \mathbb{P}^*$ where $\mathbb{P}^*$ is the risky neutral probability of the proposition before.*

Proof. The proof of all this propositions can be found in: *[2]: "Introduction to Stochastic Calculus Applied to Finance 2nd Edition, D.Lamberton and B.Lapeyre: pag.28".* □

Corollary 44. *Considering the results given in "Proposition 41, 42 and 43", we can conclude that the CRR model is viable and complete.*

4.3 Pricing European options in CRR model

In this section, we are going to see how to calculate the value of the European options.

We consider the model CRR. Lets suppose that $\mathbb{P}^*$ is the only risky neutral probability such that the prices are martingales.

4.3.1 European call

Continuing with notation given before, C_n is the price of an European call at time n with strike price K with expiration date N, the payoff is $G = (S_N - K)^+$.
The price at time n of the European call is, as we have seen before:

$$C_n := \frac{A_n}{A_N}\mathbb{E}^*(G|\mathcal{F}_n).$$

Using that $A_n = (1+r)^n$ and $A_N = (1+r)^N$, we have that:

$$C_n = (1+r)^{-(N-n)}\mathbb{E}^*((S_N - K)_+|\mathcal{F}_n).$$

Also, in the CRR model $S_N = S_n \prod_{i=n+1}^{N} T_i$, so:

$$C_n = \frac{A_n}{A_N}\mathbb{E}^*[(S_n \prod_{i=n+1}^{N} T_i - K)_+|\mathcal{F}_n] =: c(n, S_n).$$

Changing S_n for x and considering that $\tilde{S_n}$ are $\mathcal{F}_n$-martingales:

$$c(n,x) = (1+r)^{-(N-n)}\mathbb{E}^*[(x \prod_{i=n+1}^{N} T_i - K)_+].$$

In CRR, $T_i = (1+u)$ or $T_i = (1+d)$. So we can suppose that we have:

- $j \in \mathbb{N}$ such that $T_1 = ... = T_j = (1+u)$.
- $N-n-j \in \mathbb{N}$ such that $T_{N-n-1} = ... = T_{N-n-j} = (1+d)$.

So:

$$\begin{aligned} c(n,x) =& (1+r)^{-(N-n)} \sum_{j=0}^{N-n} (x(1+d)^{N-n-j}(1+u)^j - K)_+ \binom{N-n}{j} (\frac{u-r}{u-d})^{N-n-j} (\frac{r-d}{u-d})^j \\ =& (1+r)^{-(N-n)} \sum_{j=j*(x)}^{N-n} x(1+d)^{N-n-j}(1+u)^j \binom{N-n}{j} (\frac{u-r}{u-d})^{N-n-j} (\frac{r-d}{u-d})^j \\ & -(1+r)^{-(N-n)} \sum_{j=j*(x)}^{N-n} K \binom{N-n}{j} (\frac{u-r}{u-d})^{N-n-j} (\frac{r-d}{u-d})^j \end{aligned}$$

where:

$$\begin{aligned} j*(x) &:= inf\left\{j : x(1+d)^{N-n-j}(1+u)^j > K\right\} \\ &= inf\left\{j : j > \frac{log(\frac{K}{x}) - (N-n)log(1+d)}{log(\frac{1+u}{1+d})}\right\}. \end{aligned}$$

We see that:

$$\frac{(u-r)(1+d)}{(u-d)(1+r)} + \frac{(r-d)(1+u)}{(u-d)(1+r)} = 1.$$

So, considering $p^* := \frac{r-d}{u-d}$, we can define:

$$\tilde{p} = \frac{p^*(1+u)}{1+r} = \frac{(r-d)(1+u)}{(u-d)(1+r)}.$$

We can rewrite:

$$c(n,x) = x \sum_{j=j*(x)}^{N-n} \binom{N-n}{j} \tilde{p}^j (1-\tilde{p})^{N-n-j} - \frac{K}{(1+r)^{N-n}} \sum_{j=j^*(x)}^{N} \binom{N-n}{j} (p^*)^j (1-p^*)^{N-n-j}.$$

Normally $n = 0$, so:

$$C_0 = S_0 \sum_{j=j^*(x)}^{N} \binom{N}{j} \tilde{p}^j (1-\tilde{p})^{N-j} - \frac{K}{(1+r)^N} \sum_{j=j^*(x)}^{N} \binom{N}{j} (p^*)^j (1-p^*)^{N-j}.$$

Finally we can write:

$$C_0 = S_0 \mathbb{P}(Bin(N,\tilde{p}) \geq j^*(x)) - \frac{K}{(1+r)^N} \mathbb{P}(Bin(N,p^*) \geq j^*(x)).$$

4.3.2 European put

Considering the call-put parity:

$$P_0 = C_0 - S_0 + \frac{K}{(1+r)^N}.$$

So, we have that:

$$P_0 = \frac{K}{(1+r)^N}\mathbb{P}(Bin(N,p^*) \le j^*(x)-1) - S_0\mathbb{P}(Bin(N,\tilde{p}) \le j^*(x)-1).$$

4.4 Hedging European options in CRR model

To find the strategy hedge ϕ we impose:

$$\phi_n^0(1+r)^n + \phi_n S_n = c(n, S_n).$$

So, we have the system:

$$\begin{cases} \phi_n^0(1+r)^n + \phi_n(1+d)S_{n-1} = c(n,(1+d)S_{n-1}). \\ \phi_n^0(1+r)^n + \phi_n(1+u)S_{n-1} = c(n,(1+u)S_{n-1}). \end{cases}$$

Solving the system we get $\phi_n = \Delta_n(n, S_{n-1})$ where:

$$\Delta(n,x) = \frac{c(n,(1+u)x) - c(n,(1+d)x)}{x(u-d)}.$$

Using the property of self-financing:

$$\phi_n^0 = C_0 + \sum_{l=1}^{n-1}\phi_l(\tilde{S}_l - \tilde{S}_{l-1}) - \phi_n\tilde{S}_{n-1} = \tilde{V}_{n-1}(\phi) - \phi_n\tilde{S}_{n-1}.$$

which is obviously $\mathcal{F}_{n-1}$-measurable. We notice that $\Delta_n(n, S_{n-1})$ is the quantity of risky assets that the seller of the option has on his portfolio at time n. We can also write:

$$\Delta_n(n, S_{n-1}) = \frac{C_n^u - C_n^d}{S_n^u - S_n^d}.$$

Where C^u is the value of the option at time n in the worst scenario and the same for the others. This strategy is called delta-neutral.

5 Single-step binomial model

We begin by examining the simplest possible setting from a CRR model.

5.1 Single-step binomial model formalism

We take time as discrete and reduced to just two time instants, the present and the future: $\mathbb{T} = \{0, 1\}$.

The market only has two possible scenarios: "up" or "down". At time 0 we assume we are given some asset S. The current price $S_0 > 0$ is known while its future price S_1 is not known, but we consider possible future prices and the probability of attaining them.

So, lets consider the following discrete time financial model:

1. $(\Omega, \mathcal{F}, \mathbb{P})$ where:
   (a) $\Omega = \{u, d\}$, $u :=$ up, $d :=$ down.
   (b) $\mathcal{F} = \mathcal{P}(\Omega)$.
   (c) $\mathbb{P}$ probability that $\mathbb{P}(u) = p$, $\mathbb{P}(d) = 1 - p$ where $p \in (0, 1)$.
2. $\mathbb{F} := \{\mathcal{F}_n, n \in \mathbb{T}\}$, where $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_N := \mathcal{F}$.
3. $\mathbb{T} := \{0, 1\}$, $0 :=$ present, $1 :=$ future.
4. $S^0 := \{S^0_0, S^0_1\}$ riskless asset which represent the money market account where $S^0_0 > 0$ and $S^0_1 = S^0_0(1 + r)$, for some $r > 0$.
5. $S := \{S_0, S_1\}$ risky asset that S_0 is known and $S_1 = \begin{cases} S^u = S_0(1 + u). \\ S^d = S_0(1 + d). \end{cases}$

5.2 Pricing European options in single-step binomial model

Now, we are going to define the initial price of an European option. Imagine we are a bank and we have to sell an option to a consumer with strike price "K" and the option to win a payoff "G". Which is the fair price we have to ask for?

As we have seen before, we are mainly working with "European call" and "European put".

The payoff at time 1 of the European call with strike price K is defined as:

$$C_1 = \begin{cases} S_1 - K & \text{if } S_1 > K \\ 0 & otherwise \end{cases}$$
$$= (S_1 - K)^+$$

The payoff at time 1 of the European put with strike price K is defined as:

$$P_1 = \begin{cases} K - S_1 & \text{if } S_1 < K \\ 0 & otherwise \end{cases}$$
$$= (K - S_1)^+$$

So, the problem consists on finding rational prices C_0 and P_0.

To avoid trivial cases, we assume that the strike price K satisfies:

$$S_0(1+d) \leq K \leq S_0(1+u).$$

Remark 45. *If $max_w S_n(w) < K$ the payoff of a call is $(S_n - K)^+ = 0$. The same if $max_w S_n(w) > K$ the payoff of a put is $(K - S_n)^+ = 0$.*

5.2.1 Pricing using risky neutral measure

As we are in a CRR model, we know that exists an unique risky neutral measure $\mathbb{P}^*$ under which the prices are martingales.

If we denote $\mathbb{E}^*$ the expectation respect this probability, we have that:

$$\mathbb{E}^*(\frac{S_1}{1+r}) = S_0.$$

We also know that the variable:

$$T_1 := \frac{S_1}{S_0}$$

has Bernoulli law with p^* parameter over the set $1+u, 1+d$.

So, by the definition of a discrete variable with Bernoulli law:

$$S_0(1+r) = p^* S_0(1+u) + (1-p^*)S_0(1+d).$$

Where, operating we get:

$$p^* = \frac{r-d}{u-d}$$
$$1 - p^* = \frac{u-r}{u-d}$$

Finally, using that prices are martingales and the theory we have seen about CRR European option pricing, for the call option its pricing is:

$$C_0 = \frac{1}{1+r}\mathbb{E}^*(C_1) = \frac{1}{1+r}(p^*(S_0(1+u)-K)^+ + (1-p^*)(S_0(1+d)-K)^+).$$

For the put option:

$$P_0 = \frac{1}{1+r}\mathbb{E}^*(P_1) = \frac{1}{1+r}(p^*(K-S_0(1+u))^+ + (1-p^*)(K-S_0(1+d))^+).$$

5.2.2 Pricing using non arbitrage principle

Firstly, we are going to do the pricing for a call option.

Lets consider a portfolio D composed by Δ units of risky asset and -1 units of the call option. So, at time 0:

$$D_0 := \Delta S_0 - C_0.$$

And, at time 1:

$$D_1 := \Delta S_1 - C_1.$$

Remark 46. *Note that $\Delta > 0$ means that we take a large position in assets, which means we buy.*

If our portfolio has -1 in a call option, means that we take a short position in this call option, so we sell.

We are looking for a portfolio that values the same in both cases at time 1. So we impose:

$$\Delta S_0(1+u) - C^u = \Delta S_0(1+d) - C^d.$$

Operating we get:

$$\Delta = \frac{C^u - C^d}{S_0(u-d)}.$$

Replacing Δ in D_1:

$$D_1 = \Delta S_0(1+u) - C^u = \frac{(C^u - C^d)}{u-d}(1+u) - C^u = \frac{(1+d)C^u - (1+u)C^d}{u-d}.$$

So, our portfolio worth:

$$D_0 = \frac{C^u - C^d}{u-d} - C_0$$

and

$$D_1 = \frac{(1+d)C^u - (1+u)C^d}{u-d}$$

in any scenario.

Using that our market is viable, the no arbitrage principle says that

$$D_0(1+r) = D_1.$$

Effectively, if $D_0(1+r) > D_1$, we can borrow D_0 money, sell the call option, deposit C_0 and buy Δ goods.

At expiration, we sell ΔS_1, pay C_1 and return the money borrowed $D_0(1+r)$. Totally, we have:

$$\Delta S_1 - C_1 - (1+r)D_0 = D_1 - (1+r)D_0 > 0.$$

We have an arbitrage, contradiction with the viability of the market.

If we suppose $D_0(1+r) < D_1$, we borrow Δ goods, we sell this goods and deposit ΔS_0. Moreover, we buy a call C_0. We deposit D_0 left in the bank. At expiration, we have $(\Delta S_0 - C_0)(1+r)$. We buy Δ goods with price S_1, we return the money borrowed, and execute the call, getting C_1. Finally we have:

$$(\Delta S_0 - C_0)(1+r) - \Delta S_1 + C_1 = D_0(1+r) - D_1 > 0.$$

Another time, we have an arbitrage, contradiction with the viability of the market.

So, we impose $D_0(1+r) = D_1$ and we get:

$$\frac{C^u - C^d}{u-d} - C_0 = \frac{(1+d)C^u - (1+u)C^d}{u-d}.$$

And, isolating C_0:

$$C_0 = \frac{(r-d)C^u + (u-r)C^d}{(1+r)(u-d)} = \frac{pC^u + (1-p)C^d}{1+r}.$$

For the European put, using the same argument:

$$P_0 = \frac{(r-d)P^u + (u-r)P^d}{(1+r)(u-d)} = \frac{pP^u + (1-p)P^d}{1+r}.$$

5.3 Hedging European options in single-step binomial model

In both cases, we build a portfolio (x, y) where x represents the amount of risky asset we have and y represents the amount of riskless asset. So, its initial value is:

$$V_0 = xS_0 + yS_0^0.$$

The final value is:

$$\begin{cases} V_1^u = xS_0(1+u) + yS_0^0(1+r). \\ V_1^d = xS_0(1+d) + yS_0^0(1+r). \end{cases}$$

Its discounted value:

$$\begin{cases} \tilde{V}_0 = V_0. \\ \tilde{V}_1 = V_1(1+r)^{-1} = x\tilde{S}_1 + yS_0^0. \end{cases}$$

We also suppose that:

- The market is frictionless: we do not impose any restrictions on the numbers "x" and "y", so unlimited short-selling is allowed.
- The assets are assumed to be arbitrarily divisible, meaning that x,y can take arbitrary real values.
- Any bound to x,y is imposed, assuming unlimited liquidity in the market.
- There are no transaction costs involved in trading, the same stock price applies to long (buy $x > 0$) and short (shell $x < 0$).
- Risk-free investment ($y > 0$) and borrowing ($y < 0$), both use the interest rate "r".

In this case, as we only have one period of time, "x" and "y" are constants ($\mathcal{F}_0$−measurable).

5.3.1 Hedging using risky neutral measure

Considering the pricing under the risk neutral measure, the problem of hedging consists of looking for a strategy (x,y) that satisfy:

$$xS_1 + y(1+r) = C_1.$$

Considering the two possible scenarios:

$$\begin{cases} xS_0(1+u) + y(1+r) = C^u. \\ xS_0(1+d) + y(1+r) = C^d. \end{cases}$$

Solving the system, we get:

$$\begin{cases} x = \Delta_1 = \frac{C^u - C^d}{S_0^u - S_0^d} = \frac{C^u - C^d}{S_0(u-d)}. \\ y = \frac{C^d - xS_0(1+d)}{(1+r)}. \end{cases}$$

So, our trading strategy consist on borrowing "y" money to bank, with "y" money plus the money we win selling the option C_0, we buy "x" goods in the market at price S_0. At expiration, our portfolio worth C^u if the market goes "up", and C^d if the market goes "down". In both cases, we can face the payment of the option.

For the put option, the method is the same but with a little difference, "x" will be negative so we have to sell "x" options at price S_0 and "y" will be positive meaning that we have to deposit "y" money to the bank.

5.3.2 Hedging using non arbitrage principle

Another way to hedge the European options using the non arbitrage principle of the CRR model is the following one.

Considering the pricing under non arbitrage principle, the strategy consist on: We sell the option, so we obtain C_0 money. We borrow $\Delta S_0 - C_0$ and with all this money we buy Δ goods.

At expiration, with the market goes "up" we have:

$$\Delta S_0(1+u) - C^u.$$

If the market goes "down", we have:

$$\Delta S_0(1+d) - C^d.$$

In both cases, with the value of our portfolio, we can return $C_0(1+r)$ money to the bank.

For the put option, the strategy is similar but our Δ will be negative, so we have to sell Δ goods and invest $\Delta S_0 + P_0$.

6 Two-step binomial model

6.1 Two-step binomial model formalism

Here we only take time to be $0, T, 2T$, we simplify the notation by just specifying the number of a step, ignoring its length.

The model is similar to the one-step model but with 1 more period of time.
So we have the following discrete-time financial model:

1. $(\Omega, \mathcal{F}, \mathbb{P})$ where:
   (a) $\Omega = \{uu, ud, du, dd\}$, $u :=$ up, $d :=$ down.
   (b) $\mathcal{F} = \mathcal{P}(\Omega)$.
   (c) $\mathbb{P}$ probability that $\mathbb{P}(uu) = p^2$, $\mathbb{P}(ud) = \mathbb{P}(du) = p(1-p)$, $\mathbb{P}(dd) = (1-p)^2$ where $p \in (0, 1)$.
2. $\mathbb{F} := \{\mathcal{F}_n, n \in \mathbb{T}\}$, where $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_N := \mathcal{F}$.
3. $\mathbb{T} := \{0, 1, 2\}$.
4. $S^0 := \{S^0_0, S^0_1, S^0_2\}$ riskless asset which represent the money market account where $S^0_0 > 0$ and $S^0_n = S^0_0(1 + r)^n$, for some $r > 0$.
5. $S := \{S_0, S_1, S_2\}$ risky asset that S_0 is known and:
$$S_1 = \begin{cases} S^u = S_0(1+u). \\ S^d = S_0(1+d). \end{cases} \qquad S_2 = \begin{cases} S^{uu} = S_0(1+u)^2. \\ S^{ud} = S^{du} = S_0(1+u)(1+d). \\ S^{dd} = S_0(1+d)^2. \end{cases}$$

6.2 Pricing and hedging European options in two-step binomial model

6.2.1 Pricing European options

The pricing of European options, as in the one-step model, the basis is the "No Arbitrage Principle" or the "Risky Neutral Measure". The idea is to move backwards in time, compute the prices f_1, where f is a put or a call, in the case "up" and the case "down", and repeat the same process as in one-step model getting the fair price f_0.

6.2.2 Hedging European options

The strategy to hedge, consist on to build a portfolio (x, y) where x represents the amount of risky asset we have and y represents the amount of riskless asset.
So, its initial value is:
$$V_0 = x_1 S_0 + y_1 S^0_1.$$

At time 1:

$$V_1 = x_1 S_1 + y_1 S_1^0.$$

At time 2:

$$V_2 = x_2 S_2 + y_2 S_2^0.$$

With the self-financing condition:

$$V_1 = x_2 S_1 + y_2 S_2^0.$$

We will see an example to illustrate the pricing and the hedging in the two-step binomial model.

Example 47. *Lets suppose that we have a risky asset which worth* 48€ *now. We know that in the following two quarters, its price can increase or decrease a* 10%. *Our riskless asset has an interest rate of* 5% *annual. So, our tree of prices is:*

$$S_1 = \begin{cases} S^u = 48(1+0.1) = 52.8€ \\ S^d = 48(1-0.1) = 43.2€ \end{cases}$$

$$S_2 = \begin{cases} S^{uu} = 48(1+0.1)^2 = 58.08€ \\ S^{ud} = S^{du} = 48(1+0.1)(1-0.1) = 47.52€ \\ S^{dd} = 48(1-0.01)^2 = 38.88€ \end{cases}$$

If our interest rate is 5% *annual* $\Rightarrow r = \frac{0.05}{4} = 0.0125$ *quarterly.*

Let's suppose that we have a call option with strike price 42€. *The risk neutral measure law says that the probability for the price goes up is* $p = \frac{r-d}{u-d} = 0.5625$ *and the probability for the price goes down is* $1 - p = 0.4375$.

We know that:

$$C_2 = \begin{cases} C^{uu} = (58.08 - 42) = 16.08€ \\ C^{ud} = C^{du} = (47.52 - 42) = 5.52€ \\ C^{dd} = 0€ \end{cases}$$

So, let's compute the fair price of the option using the risky neutral measure:

$$\begin{cases} C^u = \frac{pC^{uu}+(1-p)C^{ud}}{1+r} = \frac{9.045+2.415}{1.0125} = 11.32 \\ C^d = \frac{pC^{du}+(1-p)C^{dd}}{1+r} = \frac{3.105}{1.0125} = 3.06 \end{cases}$$

Finally:

$$C_0 = \frac{pC^u + (1-p)C^d}{1+r} = \frac{6.3675 + 1.3431}{1.0125} = 7.61$$

So, the fair price of the option is 7.61€.

Let's see how to construct the hedging strategy.

Firstly, we are looking for "x" and "y" such that:

$$x_1 S_1 + y_1(1+r) = C_1.$$

Remind that we have compute before C_1:

$$C_1 = \begin{cases} C^u = 11.32€ \\ C^d = 3.06€ \end{cases}$$

Considering the two possible scenarios, our portfolio has to value:

$$\begin{cases} x_1 S_0(1+u) + y_1(1+r) = C^u. \\ x_1 S_0(1+d) + y_1(1+r) = C^d. \end{cases}$$

Solving the system we get:

$$\begin{cases} x_1 = \Delta_1 = \frac{C^u - C^d}{S_0(u-d)} = \frac{11.32-3.07}{48 \cdot 0.2} = 0.86 \\ y_1 = \frac{C^d - xS_0(1+d)}{1+r} = \frac{3.06-43.2 \cdot 0.86}{1.0125} = -33.67 \end{cases}$$

So, our trading strategy consist on buying 0.86 goods. We need $0.86 \cdot 48 = 41.28€$. Considering that we have won 7.61€ selling the call, we need $41.28 - 7.61 = 33.67€$, so we borrow 33.67€ to the bank.

At time 1, in both cases, our portfolio will value:

$$\begin{cases} 0.86 \cdot S_0(1+u) - 33.67(1+r) = 0.86 \cdot 52.8 - 33.67 \cdot 1.0125 = 11.32 \\ 0.86 \cdot S_0(1+d) - 33.67(1+r) = 0.86 \cdot 43.2 - 33.67 \cdot 1.0125 = 3.06 \end{cases}$$

The next step is to build a portfolio that has to value:

$$x_2 S_2 + y_2(1+r) = C_2.$$

Now, we have to consider two situations:

- If the market is "up", $S_1 = S_0(1+u) = 52.8€$:
  The portfolio we are looking for has to value:

  $$\begin{cases} x_2 S_1(1+u) + y_2(1+r) = C^{uu} \\ x_2 S_1(1+d) + y_2(1+r) = C^{ud} \end{cases}$$

  We compute a new delta:

  $$\Delta_u = \frac{C^{uu} - C^{ud}}{S^u(u-d)} = \frac{16.08 - 5.52}{52.8 \cdot 0.2} = 1.$$

  So, we change from to have 0.86 goods to 1 units of goods, so we need to buy $1 - 0.86 = 0.14$ new ones. The price we have to pay for is $0.14 \cdot 52.8 = 7.39€$

(remember that $S^u = 52.8$). So we borrow 7.39€ to the bank in addition to the 33.67€ we have borrowed before.

In this case, our portfolio in the possible cases values:

$$1 \cdot S_0^{uu} - 33.67(1+r)^2 - 7.39(1+r) = 1 \cdot 58.08 - 33.67(1.0125)^2 - 7.39(1.0125) = 16.08$$

$$1 \cdot S_0^{ud} - 33.67(1+r)^2 - 7.39(1+r) = 1 \cdot 47.52 - 33.67(1.0125)^2 - 7.39(1.0125) = 5.527$$

Facing our obligation to pay, $C^{uu} = 16.08$€ in the "up" case and $C^{ud} = 5.52$€ in the "down" case, to the owner of the call option.

- If the market is "down", $S_1 = S_0(1+d) = 43.2$€:

  Our portfolio has to value:

$$\begin{cases} x_2 S_1(1+u) + y_2(1+r) = C^{uu}. \\ x_2 S_1(1+d) + y_2(1+r) = C^{ud}. \end{cases}$$

  Our delta values:

$$\Delta_d = \frac{C^{uu} - C^{ud}}{S^d(u-d)} = \frac{5.52 - 0}{43.2 \cdot 0.2} = 0.64$$

  As we can see, our new delta is higher than the delta at time 1, $0.64 < 0.86$, it means that we have to sell $0.86 - 0.64 = 0.22$ goods at price S^d, so we earn $0.22 \cdot 43.2 = 9.5$€ that we return to the bank (we continue owing $(33.67 - 9.5) \cdot 1.0125$ to the bank). So our portfolio, in both cases at time 2, will value:

$$\begin{cases} 0.64 \cdot S_0^{du} - 24.6(1+r) = 0.64 \cdot 47.52 - 24.6 \cdot 1.0125 = 5.52 \\ 0.64 \cdot S_0^{dd} - 24.6(1+r) = 0.64 \cdot 38.88 - 24.6 \cdot 1.0125 = 0 \end{cases}$$

  So, we can pay, $C^{du} = 5.52$€ in the "up" case and $C^{dd} = 0$€ in the "down" case, to the owner of the call option.

7 Supermartingales, submartingales and optimal stopping

Before to start the study of American options we need to introduce some basic notions, differents with the ones needed in the European options case, to understand how the American options works.

The basic properties about probability and martingales has been defined in section "Basic Notions", so it is not needed to be defined again.

7.1 Supermartingales and submartingales

Definition 48. *A process M is a supermartingale if M is an adapted and integrable process and*

$$E[M_n|F_{n-1}] \leq M_{n-1}, q.s, \forall n \in \mathbb{T}^*.$$

A process M is a submartingale if M is an adapted and integrable process and

$$E[M_n|F_{n-1}] \geq M_{n-1}, q.s, \forall n \in \mathbb{T}^*.$$

Proposition 49.
If M is a martingale, $\mathbb{E}(M_n) = \mathbb{E}(M_0)$,$\forall n \in \mathbb{T}$.

If M is a supermartingale, $\mathbb{E}(M_n) \leq \mathbb{E}(M_{n-1})$,$\forall n \in \mathbb{T}$.

If M is a submartingale, $\mathbb{E}(M_n) \geq \mathbb{E}(M_{n-1})$,$\forall n \in \mathbb{T}$.

Proof. We notice that if M is a martingale:

$$\mathbb{E}(M_n) = \mathbb{E}(E[M_{n+1}|\mathcal{F}_n]) = \mathbb{E}(M_{n+1}).$$

The first equality is because of the definition of martingale and the second due to the conditioned expectation properties.
In the case of M is a supermartingale or a submartingale, the proof is the same but with inequalities. □

Example 50. *(Supermartingale and submartingale)*

- *Example 1: Given (M_n) a $\mathcal{F}_n$-martingale. Lets consider the sequence $Y_n := |M_n|, n \geq 0$. Because of the monotonicity of the conditional expectation:*

  $$\mathbb{E}(Y_n|\mathcal{F}_{n-1}) = \mathbb{E}(|M_n||\mathcal{F}_{n-1}) \geq |\mathbb{E}(M_n|\mathcal{F}_{n-1})| = |M_{n-1}| = Y_{n-1}.$$

  So, Y is a submartingale.

- *Example 2: Given $(Z_n)_{0\leq n\leq N}$ an adapted sequence to $(\mathcal{F}_n)_{0\leq n\leq N}$ with finite expectation. We define X:*

- $X_N = Z_N$.
- $X_n = max(Z_n, \mathbb{E}(X_{n+1}|\mathcal{F}_n), 0 \leq n \leq N-1$.

By the definition:

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq X_n.$$

So, X is a $(\mathcal{F}_n)$-supermartingale.

Proposition 51. *If H is a bounded predictable and positive process and M is a supermartingale (resp. submartingale), the transformation of M for H is a supermartingale (resp. submartingale).*

Proof. Lets consider Y the transformation of M, we have that:

$$E[Y_{n+1} - Y_n|\mathcal{F}_n] = H_{n+1}E[M_{n+1} - M_n|\mathcal{F}_n].$$

If H is positive then $E[M_{n+1} - M_n|\mathcal{F}_n]$ preserves the character of supermartingale or submartingale. □

Theorem 52 (Doob Descomposition)**.** *Any supermartingale U admits the following unique decomposition, called Doob's decomposition:*

$$U_n = M_n - A_n$$

Where M is a martingale and A is a predictable and increasing process null at the origin.

Proof. Given U, we define:

$$M_0 := U_0, A_0 = 0$$
$$A_n := A_{n-1} + U_{n-1} - E[U_n|\mathcal{F}_{n-1}]$$

and:

$$M_n := M_{n-1} + U_n - E[U_n|\mathcal{F}_{n-1}].$$

Then we have:

$$\begin{aligned} M_n - A_n &= M_{n-1} + U_n - E[U_n|\mathcal{F}_{n-1}] - (A_{n-1} + U_{n-1} - E[U_n|\mathcal{F}_{n-1}]) \\ &= M_{n-1} + U_n - A_{n-1} - U_{n-1}. \end{aligned}$$

The process M is a martingale because:

$$E[M_n - M_{n-1}|\mathcal{F}_{n-1}] = E[U_n - E[U_n|\mathcal{F}_{n-1}]|\mathcal{F}_{n-1}] = 0.$$

The process A is predictable and null all the origin. Furthermore:

$$A_n - A_{n-1} = U_{n-1} - E[U_n|\mathcal{F}_{n-1}] \geq 0$$

due to U is a supermartingale. Let's see if its unique.

We suppose that:

$$M_n - A_n = M'_n - A'_n, \forall n \in \mathbb{T}.$$

Then:

$$M_n - M'_n = A_n - A'_n, \forall n \in \mathbb{T}.$$

As we know, $M_n - M'_n$ is a martingale and $A_n - A'_n$ is a predictible process, so we have:

$$\begin{aligned} A_{n-1} - A'_{n-1} =& M_{n-1} - M'_{n-1} \\ =& E[M_n - M'_n|\mathcal{F}_{n-1}] \\ =& E[A_n - A'_n|\mathcal{F}_{n-1}] \\ =& A_n - A'_n. \end{aligned}$$

So, $\forall n \in \mathbb{T}$:

$$A_n - A'_n = A_0 - A'_0 = 0.$$

Therefore, A is unique so M is unique. □

Definition 53. *A sequence Y_n of random variables is the Snell envelope of the sequence Z_n for $n = 1, ..., N$ adapted to $\mathcal{F}_n$ if:*

$$\begin{cases} Y_N = Z_N. \\ Y_{n-1} = max\left\{Z_{n-1}, \mathbb{E}(Y_n|\mathcal{F}_{n-1})\right\}, \forall n \leq N-1. \end{cases}$$

Remark 54. *As Y_{n-1} is $\mathcal{F}_{n-1}$-measurable and $Y_{n-1} \geq \mathbb{E}(Y_n|\mathcal{F}_{n-1})$ so the Snell envelope is a supermartingale.*

Theorem 55. *The Snell envelop Y of Z is the smallest supermartingale dominating Z.*

Proof. By definition, U is a supermartingale that satisfies $U_n \geq Z_n$, $\forall n \in \mathbb{T}$.

If V is another supermartingale such that $V_n \geq Z_n$, $\forall n \in \mathbb{T}$, its enough to prove that $V_n \geq U_n$, $\forall n \in \mathbb{T}$.

We will use inverse induction:

- For N is immediate.
- Assume that for n we have $V_n \geq U_n$. Lets see if its true for $n-1$:

  As V is a supermartingale, using the induction hypothesis we have:

  $$V_{n-1} \geq E[V_n|\mathcal{F}_{n-1}] \geq E[U_n|\mathcal{F}_{n-1}].$$

  So:

  $$V_{n-1} \geq max\{Z_{n-1}, E[U_n|\mathcal{F}_{n-1}]\} = U_{n-1}.$$

□

7.2 Stopping times

Definition 56. *A stopping time is a random variable:*

$$\tau : \omega \in \Omega \longrightarrow \tau(\omega) \in \mathbb{T} \cup \{\infty\}.$$

that for all $n \in \mathbb{T}$:

$$\{\tau \leq n\} \in \mathcal{F}_n.$$

Remark 57. *Note that $\{\tau = n\} \in \mathcal{F}_n, \forall n \in \mathbb{T}$ if and only if $\{\tau \leq n\} \in \mathcal{F}_n, \forall n \in \mathbb{T}$.*

This is an immediate consequence of the facts:

$$\{\tau = n\} = \{\tau \leq n\} - \{\tau \leq n-1\}.$$

And

$$\{\tau \leq n\} = \bigcup_{j=0}^{n} \{\tau = j\}.$$

Proposition 58. *If S and T are stopping times:*

1. *$S \vee T$ and $S \wedge T$ are stopping times.*
2. *The class:*
$$\mathcal{F}_T := \{A \in \mathcal{F} : A \cap \{T \leq n\} \in \mathcal{F}_n, \forall n \in \mathbb{T}\}$$
is a σ-algebra.
3. *If $S \leq T \Rightarrow \mathcal{F}_S \subseteq \mathcal{F}_T$.*
4. *If X is an adapted process, the variable X_T is $\mathcal{F}_T$-measurable.*

Proof.

1. We know that:
$$\{S \vee T \leq t\} = \{S \leq t\} \cap \{T \leq t\}.$$
And:
$$\{S \wedge T \leq t\} = \{S \leq t\} \cup \{T \leq t\}.$$
So, $S \vee T$ and $S \wedge T$ are stopping times.
2. Let's prove that $\mathcal{F}_T$ is a σ-algebra:
   - $\Omega \in \mathcal{F}_T$ because $\Omega \cap \{T \leq n\} = \{T \leq n\} \in \mathcal{F}_n$.
   - If $A \in \mathcal{F}_T$, A^c also, because:
   $$A^c \cap \{T \leq n\} = ((A \cap \{T \leq n\}) \cup \{T > n\})^c \in \mathcal{F}_n.$$
   - If $\{A_k, k \geq 1\}$ is a sequence of elements of F_T its union also because:
   $$(\cup_{k=1}^{\infty} A_k) \cap \{T \leq n\} = \cup_{k=1}^{\infty}(A_k \cap \{T \leq n\}).$$

3. If $A \in \mathcal{F}_S$, as $S \leq T$ q.s. we have:
$$A \cap \{T \leq n\} = A \cap \{S \leq n\} \cap \{T \leq n\} \in \mathcal{F}_n.$$
because $A \cap \{S \leq n\} \in \mathcal{F}_n$ due to $A \in \mathcal{F}_S$ and $\{T \leq n\} \in \mathcal{F}_n$ (T is a stopping time).

4. If B is a borelian of $\mathbb{R}$ we can write:
$$\{X_T \in B\} \cap \{T \leq n\} = \cup_{j=0}^{n} \{X_j \in B, T = j\} \in \mathcal{F}_n$$
because $\{X_j \in B, T = j\} \in \mathcal{F}_j, \forall j \leq n$.

□

Definition 59. *For any sequence of random variables X_n and any stopping time τ, the stopped process X_n^τ is:*
$$X_n^\tau(\omega) := X_{\tau(w)\wedge n}(\omega).$$
More explicitly:
$$X_n^\tau(\omega) = \begin{cases} X_n(\omega), & \text{if } \tau(w) \geq n. \\ X_\tau(\omega), & \text{if } \tau(w) \leq n. \end{cases}$$

Proposition 60. *If X is an adapted process and τ a stopping time, X^τ is also an adapted process. Moreover, if X is a martingale, supermartingale or submartingale and τ a stopping time, X^τ is also a martingale, supermartingale or submartingale.*

Proof. The fact that X^τ is adapted, is immediate.

On other hand, we can write:
$$X_n^\tau = X_0 + \sum_{j=1}^{n} 1\!\!1_{\{j \leq \tau\}}(X_j - X_{j-1}).$$
Note that:
$$\{j \leq \tau\} = \{\tau < j\}^c = \{\tau \leq j-1\}^c$$
and $\{\tau \leq j-1\}^c$ is $\mathcal{F}_{\{j-1\}}$-measurable.

So being $1\!\!1_{\{j \leq \tau\}}$ predictable, X^τ is the transformation of a martingale if X is a martingale.

In the case of a supermartingale we have:
$$\begin{aligned} E[X_n^\tau | \mathcal{F}_{n-1}] &= X_{n-1}^\tau + E[1\!\!1_{\{n \leq \tau\}}(X_n - X_{n-1}) | \mathcal{F}_{n-1}] \\ &= X_{n-1}^\tau + 1\!\!1_{\{n \leq \tau\}} E[X_n - X_{n-1} | \mathcal{F}_{n-1}] \leq X_{n-1}^\tau \end{aligned}$$
because, by the definition of a supermartingale:
$$E[X_n - X_{n-1} | \mathcal{F}_{n-1}] \leq 0.$$
For the case of a submartingale, the thinking is the same. □

Theorem 61. *(Doob theorem)*

- *If X is a martingale respect a filtration $\mathbb{F} := \{\mathcal{F}_n, n \geq 0\}$, S and T two stopping times that $S \leq T \leq c$, $c \in \mathbb{N}$ then $E(X_T|\mathcal{F}_S) = X_S$ q.s.*
- *If X is a submartingale, so $E(X_T|T_S) \geq X_S$.*
- *If X is a supermartingale, so $E(X_T|T_S) \leq X_S$.*

Proof. Firstly, we recall that T is bounded, so $|X_T| \leq \sum_{n=0}^{c} |X_n|$ has finite expectation.

The same happens with X_S.

We have to see that $\forall A \in \mathcal{F}_S$ we have:

$$\mathbb{E}(X_S \mathbb{1}_A) = \mathbb{E}(X_T \mathbb{1}_A) \Longleftrightarrow \mathbb{E}[(X_T - X_S)\mathbb{1}_A] = 0$$

We define $H_n := \mathbb{1}_{\{S \leq n \leq T\} \cap A}$ It is a bounded, positive and previsible process because:

$$\begin{aligned} \{S \leq n \leq T\} \cap A &= \{S \leq n\} \cap \{T \geq n\} \cap A \\ &= \{S \leq n-1\} \cap (\{T \leq n-1\})^c \cap A. \end{aligned}$$

In addition $\{S \leq n-1\} \cap (\{T \leq n-1\})^c \cap A \in \mathcal{F}_{n-1}$ because $\{S \leq n-1\} \cap A \in \mathcal{F}_{n-1}$ and $(\{T \leq n-1\})^c \in \mathcal{F}_{n-1}$.

So, we define:

$$W_n := X_0 + \sum_{i=1}^{n} \mathbb{1}_{\{\{S \leq i \leq T\} \cap A\}}(X_i - X_{i-1}).$$

W_n is a martingale, supermartingale or submartingale depending on what X is.

We can rewrite this expression in this form:

$$\begin{aligned} &X_0 + \sum_{i=1}^{n} \mathbb{1}_{\{\{S \leq i \leq T\} \cap A\}}(X_i - X_{i-1}) = X_0 + \sum_{i=S+1}^{T \wedge n} \mathbb{1}_A(X_i - X_{i-1}) \\ &= X_0 + \mathbb{1}_A(X_{T \wedge n} - X_{S \wedge n}), \end{aligned}$$

If X is a martingale $\Rightarrow X_0 + \mathbb{1}_A(X_{T \wedge n} - X_{S \wedge n})$ also is, so:

$$\mathbb{E}(X_0 + \mathbb{1}_A(X_{T \wedge n} - X_{S \wedge n})) = \mathbb{E}(X_0).$$

Then:

$$\mathbb{E}(\mathbb{1}_A(X_T - X_S)) = 0, \forall n \geq 0.$$

Choosing $n > c$, we have $T \wedge n = T$ and $S \wedge n = S$ so:

$$\mathbb{E}(\mathbb{1}_A(X_T - X_S)) = 0.$$

If X is a submartingale, analogously, $\forall n$:

$$\mathbb{E}(X_0 + \mathbb{1}_A(X_{T\wedge n} - X_{S\wedge n})) \geq \mathbb{E}(X_0).$$

And:

$$\mathbb{E}(\mathbb{1}_A(X_T - X_S)) \geq 0.$$

If X is a supermartingale, we have $\forall n$:

$$\mathbb{E}(X_0 + \mathbb{1}_A(X_{T\wedge n} - X_{S\wedge n})) \leq \mathbb{E}(X_0).$$

And:

$$\mathbb{E}(\mathbb{1}_A(X_T - X_S)) \leq 0.$$

□

8 American options

8.1 Introduction

As we have seen before, an European option confers the right to a random payoff G at an expired time N.

American options allow the holder to exercise the corresponding right Z at any time $n \leq N$.

We will work in a general discrete time model (see definition 13) with finitely many trading dates, where we assume that we are given a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where, as usual, Ω is taken in the form $\{u, d\}^N$, equipped with a filtration of fields $\mathcal{F}_n$ generated by some prices S_n (risky asset) representing the underlying security and P is the probability to get "u" or "d". We also denote by $\mathbb{P}^*$ the unique probability under which the discounted assets prices are martingales. As we are in a binomial model, we also have a riskless asset S^0 which represents an account in a bank.

The payoff depends on the values of S_n for all n up to the moment of exercise, so a representation of the payoff similar to the European case is not possible in general.

We need stopping times to describe optimal exercise of the option because at each time n we have to face the choice between exercising immediately and postponing this till later.

Definition 62. *We shall define the American option as a non-negative sequence Z_n adapted to a filtration $\mathcal{F}_n$, where Z_n is the immediate profit made by exercising the option at time n.*

- *In the case of an American call on the stock S_n with strike price K, $Z_n = (S_n - K)^+$.*
- *In the case of an American put on the stock S_n with strike price K, $Z_n = (K - S_n)^+$.*

We will see how to price and hedge American options and which is the optimal time of exercising throughout a binomial model example and finally the results that show us how to proceed in a general case.

8.2 Pricing American Options

First consider a binomial model and assume that the option holder's choice of exercise date is made in order to maximize the amount received. At each time she faces the choice between exercising immediately and postponing the execution later.

The sum of money given by the payoff can be seen at each time in each scenario, being a known function of the current stock price.

Valuing the alternative poses a problem and depends on assumptions about the future behavior of the stock. This makes it natural to seek to solve the pricing problem by means of backwards induction, while taking into account, at each node of the binomial tree, the additional choice of whether to exercise or not. The method is best illustrated through an example.

8.2.1 Example case

According to an example found in [5] *"Marek Capinski and Ekkehard Kopp (2012): Discrete Models of Financial Markets.Cambridge."*, let's consider a concrete example of a single-stock model in a 5-step binomial tree. Let's assume we have a risk-free return $r = 5\%$, an underlying security with initial price $S_0 = 100$, the two possible price movements are $u = 15\%$ and $d = -10\%$, so the tree of prices is the next one:

			n		
0	1	2	3	4	5
					201.14
				174.90	
			152.09		157.41
		132.25		136.88	
	115.00		119.03		123.19
100.00		103.50		107.12	
	90.00		93.15		96.41
		81.00		83.84	
			72.90		75.45
				65.61	
					59.05

Consider a put option with expiry date $N = 5$, exercise price $K = 100$ and payoff $Z_n = (K - S_n)^+$. Since the American option can be exercised at any time before to the expiry, it is necessary to compute the immediate payoff of the option Z_n at each node of the tree. The results are:

n					
0	1	2	3	4	5
					0.00
				0.00	
			0.00		0.00
		0.00		0.00	
	0.00		0.00		0.00
0.00		0.00		0.00	
	10.00		6.94		3.59
		19.00		16.17	
			27.10		24.55
				34.39	
					40.95

Pricing will be performed in a similar way as for European claims, starting from the expiry time and moving backwards. The value of the American put at time n is denoted by P_n so:

$$P_5 = Z_5 \Longrightarrow \tilde{P}_5 = \tilde{Z}_5$$

At time $n = 4$ the holder has a choice between exercising and waiting till the final moment. The decision about "waiting" now depends on the value of the European option with exercise date one step from now. This can easily be computed by using the non-risky neutral measure probability seen before. Recall that, in European Options $Q = (q, 1-q)$ where $q = \frac{r-d}{u-d} = \frac{0.05-(-0.10)}{0.15-(-0.10)} = 0.6$. At each node at time $n = 4$, we compute the discounted expected value of the payoff available after one further step, $\mathbb{E}^*(P_5|\mathcal{F}_4)$ following the formula that we have seen in the subsection "5.2.1 Pricing using risky neutral measure". We obtain the following numbers:

n	
4	5
	0.0
0.0	
	0.0
0.0	
	0.0
1.37	
	3.59
11.39	
	24.55
29.63	
	40.95

The value of the put at time 4 is the following random variable:

$$\frac{1}{1+r}\mathbb{E}_Q(Z_5|\mathcal{F}_4) = \frac{1}{1+r}\mathbb{E}_Q(P_5|\mathcal{F}_4).$$

Therefore, taking the benefit of an immediate exercise, if profitable:

$$P_4 = max\{Z_4, \frac{1}{1+r}\mathbb{E}_Q(P_5|\mathcal{F}_4)\} \Leftrightarrow \tilde{P}_4 = max\{\tilde{Z}_4, \mathbb{E}_Q(\tilde{P}_5|\mathcal{F}_4)\}.$$

As we see:

- At node (4,1): $\mathbb{E}^*(P_5|\mathcal{F}_4) = 0.00 = Z_4 = 0.00 \Longrightarrow P_4 = 0$.
- At node (4,2): $\mathbb{E}^*(P_5|\mathcal{F}_4) = 0.00 = Z_4 = 0.00 \Longrightarrow P_4 = 0$.
- At node (4,3): $\mathbb{E}^*(P_5|\mathcal{F}_4) = 1.37 > Z_4 = 0.00 \Longrightarrow P_4 = 1.37$.
- At node (4,4): $\mathbb{E}^*(P_5|\mathcal{F}_4) = 11.39 < Z_4 = 16.17 \Longrightarrow P_4 = 16.17$.
- At node (4,5): $\mathbb{E}^*(P_5|\mathcal{F}_4) = 29.63 < Z_4 = 34.39 \Longrightarrow P_4 = 34.39$.

So in the nodes (4,4) and (4,5) at time $n = 4$ it is better to exercise immediately rather than wait and the value of the option are the corresponding payoffs. In the others nodes, waiting is the best option. For the nodes (4,1) and (4,2) the immediate payoff is 0 and for the node (4,3), $\mathbb{E}^*(P_5|\mathcal{F}_4) > Z_4$.

Finally, at time $n = 4$ we have the following values for P_4 and P_5:

P_4	P_5
	0.0
0.0	
	0.0
0.0	
	0.0
1.37	
	3.59
16.17	
	24.55
34.39	
	40.95

Applying the same argument in the rest of steps:

$$\begin{array}{lcl} \tilde{P}_5 & = & \tilde{Z}_5. \\ \tilde{P}_{n-1} & = & max\{\tilde{Z}_{n-1}, \mathbb{E}_Q(\tilde{P}_n|\mathcal{F}_{n-1})\}, \forall n \leq 4. \end{array}$$

At time $n = 0$, we obtain the final table:

n					
0	1	2	3	4	5
					0.0
				0.0	
			0.0		0.0
		0.0		0.0	
	1.23		0.52		0.0
4.51		2.94		1.37	
	10.00		6.94		3.59
		19.00		16.17	
			27.10		24.55
				34.39	
					40.95

Where the current price at time 0 is $\tilde{P}_0 = 4.51$

Remark 63. *Note that for time* $n = 0, (1+r)^0 = 1$ *so* $P_0 = \tilde{P}_0$.

8.2.2 General case

In general, if we consider an American put with strike K (for the American call is the same, its only needed to change the payoff), its payoff at n is $Z_n = (K - S_n)^+$, where Z_n is an $\mathcal{F}_n$-measurable random variable.

Let's suppose that we are in a viable and complete market.

Denote by U_n the price of this option at n. Denote by $\tilde{Z}$ and $\tilde{U}$ the corresponding discounted processes.

At time N, the fair price is $U_N = Z_N$. At time $N-1$, the price U_{N-1} has to cover the payoff at $N-1$ and the current value of the call option at N, so it has to satisfy:

$$U_{N-1} = max\{Z_{N-1}, \frac{1}{1+r}E[U_N|\mathcal{F}_{N-1}]\} = max\{Z_{N-1}, S^0_{N-1}E[\tilde{U}_N|\mathcal{F}_{N-1}]\}.$$

So, thinking backward, for any $n \in \mathbb{T}^*$:

$$U_{n-1} = max\{Z_{n-1}, S^0_{n-1}E[\tilde{U}_n|\mathcal{F}_{n-1}]\}.$$

Equivalently:

$$\tilde{U}_{n-1} = max\{\tilde{Z}_{n-1}, E[\tilde{U}_n|\mathcal{F}_{n-1}]\}.$$

So, $\tilde{U}$ is the Snell envelope of $\tilde{Z}$.

Remark 64. *Note that in the European case we have:* $\tilde{U}_{n-1} = E[\tilde{U}_n|\mathcal{F}_{n-1}], \forall n \in \mathbb{T}^*$.

Clearly $\tilde{U}$ is a supermartingale, so it admits the Doob decomposition:

$$\tilde{U}_n = \tilde{M}_n - \tilde{A}_n$$

where $\tilde{M}$ is a martingale and $\tilde{A}$ is a predictable and increasing process null at the origin.

Lets consider a contingent claim $\tilde{M}_N$. The completeness of the market implies that it exists an unique admissible and self-financing strategy ϕ such that:

$$\tilde{V}_N(\phi) = \tilde{M}_N.$$

Where $\tilde{V}$ and $\tilde{M}$ are martingales, so:

$$\tilde{V}_n(\phi) = \tilde{M}_n, \forall n \in \mathbb{T}.$$

Therefore:

$$\tilde{U}_n = \tilde{V}_n(\phi) - \tilde{A}_n \leq \tilde{V}_n(\phi)$$

and

$$U_n = V_n(\phi) - A_n \leq V_n(\phi)$$

with

$$A_n := S_n^0 \tilde{A}_n.$$

So, with $V_0(\phi)$ the writer of the option is able to generate the quantity $V_n(\phi)$ at any $n \in \mathbb{T}^*$ and:

$$V_n(\phi) \geq U_n \geq Z_n.$$

So, the price of the American option can be:

$$V_0(\phi) = E[\tilde{V}_0(\phi)] = E[\tilde{M}_0].$$

8.3 Hedging American Options

8.3.1 Example case

In our example, lets suppose that we have written and sold the American put cashing the price, 4.51€.

To make our hedge strategy we have to construct a replicating strategy which is based on taking a position in the underlying and completing the portfolio with a position in the money market account.

- Time $n = 0$:

  Our position in the stock, as in the European Options case, is determined by:

  $$\Delta_0 = \frac{P_u - P_d}{S_u - S_d} = \frac{1.23 - 10}{115 - 90} = -0.35$$

  As the number is negative, it means short-selling the stock.

  The money market position is:

  $$4.51 + 0.35 \cdot 100 = 39.58€.$$

  We invest this amount in risk free for one period.

- Time $n = 1$:

  Consider the case where the stock has gone down, $S^d = 90€$. There are two cases: the holder of the option either exercises or not.

  - Suppose the option is exercised:

    We own the payoff 10 and we have to repurchase the fraction of the stock to close the short position. The stock is cheap so we only have to pay $0.35 \cdot 90 = 31.5€$.

    The risk-free investment exactly covers this cost since $r = 5\%$ and:

    $$-10 - 0.35 \cdot 90 = -41.56 = -39.58 \cdot (1.05)$$

  - Suppose the option is not exercised:

    We compute a new delta:

    $$\Delta_1 = \frac{2.94 - 19}{103.50 - 81} = -0.71$$

    This means that we have to increase our short position by short-selling additional $0.71 - 0.35 = 0.36$ shares, which will generate some money to be added to our risk-free investment:

    $$0.36 \cdot 90 + 39.58 \cdot (1 + 0.05) = 74.23€$$

However, we don't need all this money for further hedging since the value of waiting is:

$$\frac{1}{1.05}(0.6 \cdot 2.94 + 0.4 \cdot 19) = 8.92€$$

So, for replication such a value of our strategy is needed.

Therefore to cover our liabilities: Short position worth $-0.71 \cdot 90 = -64.4€$ and the option $-8.92€$, so we need 73.15€ that we could consume from our risk-free investment $74.23 - 73.15 = 1.08€$.
This means that our strategy would is not only self-financing, is also super-financing with means that we are superhedging strictly.

- Time $n = 2$:

  Let's suppose that the option is exercised:

  - Stock is up to $S^{du} = 103.50€$:

    We pay 2.94€ for the option and we have to buy back 0.71 units of a share and cash our savings:

    $$-2.94 - 0.71 \cdot 103.50 + 74.23 \cdot (1.05) = 1.135$$

  - Stock is down to $S^{dd} = 81€$:

    We pay 19 as the exercise pay-off. We buy back 0.71 of a share and we clear our money market account:

    $$-19 - 0.71 \cdot 81 + 74.23 \cdot (1 + 0.05) = 1.135$$

    In each case as a result of the sup-optimal policy of the option holder (the option should have been exercised at time $n = 1$) we win an extra money 1.08€ from the previous step increased by the risk-free return.

  If the option is not exercised, we compute a new delta and follow the same argument as in time 1.

The next times, we follow the same argument.

8.3.2 General case

In the general case, as we have seen before when we were pricing American Options, as we are in a complete and viable market, considering the Snell envelope under $\mathbb{P}^*$, $\tilde{U}$ of the sequence $(\tilde{Z}_n)$, where $\tilde{Z}_n$ is the discounted payoff of the American option, and considering the Doob decomposition of this sequence which is martingale, the completeness of the market implies that it exists an unique admissible and self-financing strategy ϕ and a portfolio V such that:

$$V_n(\phi) \geq U_n \geq Z_n.$$

Clearly, the writer of the option can hedge himself perfectly: once he receives the premium $U_0 = V_0(\phi)$, he can generate a wealth equal to $V_n(\phi)$ at time n which is bigger than U_n and Z_n.

Remark 65. *The strategy ϕ in each period is the same followed in subsection "5.3.1 Hedging using risky neutral measure".*

8.4 Optimal exercise

Another question related with American options, is to know what is the best moment to exercise the option.

8.4.1 Example case

Going back to our numerical example, suppose we bought the American put for $P_0 = 4.51$€. The problem we are facing at all times is the decision whether to exercise the option or not.

Consider the strategy of exercising at the earliest possible time when the option price is equal to the available payoff.

Of course, we do not exercise at time 0 since the payoff is 0. Let B_u (resp. B_d) be the set of all paths beginning with a u (resp. d) movement, and similarly define $B_{uu}, B_{ud}, B_{du}, B_{dd}$ and so on:

- Suppose the stock goes down in the first step, that is, consider $\omega \in B_d$. We exercise the option at time 1, cashing 10€, which as we saw, is higher than the expected profit from waiting. At node (2,2): $\mathbb{E}^*(P_2|\mathcal{F}_1) = 7.23 < 10$.
- Suppose the stock goes up in the first step, so let $\omega \in B_u$ Here we distinguish three cases:
  - If $\omega \in B_{uddd}$ we exercise at time 4, obtaining 16.17€.
  - If $\omega = uuddd$, $\omega = ududd$, $\omega = uddud$ we exercise at time 5 receiving 3.59€.
  - For other paths we do not exercise the option at all and receive zero. In other words we exercise at time 5 where the payoff is zero.

We have defined a random variable assigning to each ω the time when we exercise the option:

$$\tau_1(\omega) = \begin{cases} 1, & \text{if } \omega \in B_d \\ 4, & \text{if } \omega \in B_{uddd} \\ 5, & \text{otherwise} \end{cases}$$

We can define a natural modification, related to the early exercise, of the process of the option values. These values fluctuate with time when we observe them along various scenarios. For example, if $\omega = udddu$ we have the sequence:

$$P(n,\omega) = (4.51, 1.23, 2.94, 6.94, 16.17, 3.59).$$

In such a scenario our strategy tells us to exercise at time 4. Imagine that we keep the money we have cashed, so the sequence is modified to become:

$$P(n,\omega) = (4.51, 1.23, 2.94, 6.94, 16.17).$$

For this particular, $\tau(\omega) = 4$, and we left the sequence unchanged for $n \leq 4$, replacing the subsequent values by the value at time 4, so for $n \geq 4$ we have $P(n, \omega) = P(\tau(\omega), \omega)$.

Since the sum of money generated by such strategies can be random (in our special strategy the decision and the outcome depend on ω), their comparison is difficult. Random variables are functions and functions are rarely comparable. For this reason we need to associate a single number with each exercising strategy.

A natural candidate as optimality criterion is to maximize the mathematical expectation of the payoff obtained at the exercise time. If the moment at which we exercise is denoted by τ, the money received in a particular scenario ω is the payoff $G(\tau(\omega))$. These sums of money emerge at different time instants, so for economic reasons we should discount them to make them comparable.

To find the expected value of all discounted payments, note that we receive 10€ for $\omega \in B_d$, 16.17€ for $\omega \in B_{uddd}$ and 3.59€ for $\omega = uuddd, \omega = ududd$ or $\omega = uddud$ so that:

$$(1-q)\frac{10}{1+r} + q(1-q)^3\frac{16.17}{(1+r)^4} + 3q^2(1-q)^3\frac{3.59}{(1+r)^5} = 4.51$$

which, remarkably is the money we paid for the option.

Remark 66. *Analyzing the optimal strategy in our leading example, path by path, we can see that before we exercise, the prices follow a martingale scheme since in the Snell envelope, the maximum of the two is the discounted martingale expectation. After we exercise (i.e stop), the sequence becomes constant and so it is obviously a martingale*

8.4.2 General case

For the buyer of the option, there is no point in exercising at time n when $U_n > Z_n$ because he would trade an asset worth U_n (the option) for an amount Z_n (by exercising the option) so he would loose money.

On other hand, considering the Doob decomposition of U_n. Lets define:

$$\upsilon_0 := inf\left\{n \geq 0 : U_n = Z_n\right\}.$$

because at that time, selling the option provides the holder with a wealth $U_{\upsilon_{max}} = V_{\upsilon_{max}}(\phi)$ and, following the strategy ϕ from that time, he creates a portfolio whose value is strictly bigger than the option's at times $\upsilon_{max} + 1, \upsilon_{max} + 2, .., N$.

So, let's consider a second option, $\tau \leq \upsilon_{max}$, which allows us to say that U^{τ} is a martingale.

As a result, optimal dates of exercise are optimal stopping times for the sequence

$\tilde{Z}_n$ under probability $\mathbb{P}^*$.

From the writer's point of view, if he hedges himself using the strategy ϕ as defined in section "8.3 Hedging American Options", and if the buyer exercises at a non-optimal time τ, then $U_\tau > Z_\tau$ or $A_\tau > 0$.
In both cases, the writer makes a profit $V_\tau(\phi) - Z_\tau = U_\tau + A_\tau - Z_\tau > 0$ which is positive.

Let's see how calculate this optimal stopping time.

The following proposition relates stopping times with the Snell envelope.

Proposition 67. *Let U be the Snell envelope of a sequence Z. Lets define:*

$$\upsilon_0 := inf\left\{n \geq 0 : U_n = Z_n\right\}.$$

Then υ_0 is a stopping time and U^{υ_0} is a martingale.

Proof.

1. We have by definition $U_N = Z_N$, so $\upsilon_0 \leq N$.

   The set $\upsilon_0 = 0 = U_0 = Z_0 \in \mathcal{F}_0$, because U and Z are adapted processes.

   So, for $k \geq 1$ we have:

   $$\{\upsilon_0 = k\} = \{U_0 > Z_0\} \cap ... \cap \{U_{k-1} > Z_{k-1}\} \cap \{U_k = Z_k\} \in \mathcal{F}_k.$$

   Therefore, υ_0 is a stopping time.

2. Now, we want to see that U^{υ_0} is a martingale and not only a supermartingale as we know.

   We can write:

   $$U_n^{\upsilon_0} = U_0 + \sum_{j=1}^{n} \mathbb{1}_{\{\upsilon_0 \geq j\}} \Delta U_j.$$

   Then:

   $$U_n^{\upsilon_0} - U_{n-1}^{\upsilon_0} = \mathbb{1}_{\{\upsilon_0 \geq n\}}(U_n - U_{n-1})$$

   If $\upsilon_0 \geq n$ necessarily $U_{n-1} > Z_{n-1}$ and so $U_{n-1} = E[U_n|\mathcal{F}_{n-1}]$.

   Then:

   $$U_n^{\upsilon_0} - U_{n-1}^{\upsilon_0} = \mathbb{1}_{\{\upsilon_0 \geq n\}}(U_n - E[U_n|\mathcal{F}_{n-1}]).$$

   Finally:

   $$E[U_n^{\upsilon_0} - U_{n-1}^{\upsilon_0}|\mathcal{F}_{n-1}] = \mathbb{1}_{\{\upsilon_0 \geq n\}} \cdot 0 = 0$$

   because $\upsilon_0 \geq n$ is $\mathcal{F}_{n-1}$-measurable.

□

Corollary 68.
$$U_0 = E[Z_{\upsilon_0}|\mathcal{F}_0] = sup_\tau E[Z_\tau|\mathcal{F}_0]$$

Proof. The sequence U^{υ_0} is a martingale and so:

$$U_0 = E[U_N^{\upsilon_0}|\mathcal{F}_0] = E[U_{\upsilon_0}|\mathcal{F}_0] = E[Z_{\upsilon_0}|\mathcal{F}_0].$$

On the other hand, for any stopping time τ, U^τ is a supermartingale and:

$$U_0 \geq E[U_N^\tau|\mathcal{F}_0] = E[U_\tau|\mathcal{F}_0] = E[Z_\tau|\mathcal{F}_0].$$

□

Remark 69. *If τ_n^N is the set of stopping times taking values in $n, n+1, ..., N$ and $\upsilon_n = inf\{j \leq n : U_j = Z_j\} \in \tau_n^N$ we have:*

$$U_n = E[Z_{\upsilon_n}|\mathcal{F}_n] = sup_{\{\tau \in \tau_n^N\}} E[Z_\tau|\mathcal{F}_n].$$

Definition 70. *A stopping time υ is optimal respect to an adapted sequence Z if:*

$$E[Z_\upsilon|\mathcal{F}_0] = sup_\tau E[Z_n|\mathcal{F}_0].$$

Remark 71. *υ_0 is optimal with respect any adapted sequence.*

Theorem 72. *The following two statements are equivalent:*

1. *υ is optimal for Z.*
2. *$Z_\upsilon = U_\upsilon$ and U^υ is a martingale.*

Proof.

1. If υ is optimal, we have:

   $$E[Z_\upsilon] = sup_\tau E[Z_\tau].$$

   In particular, choosing $\tau = \upsilon_0$ we have $E[Z_\upsilon] \geq E[Z_{\upsilon_0}] = U_0$.

   On the other hand:
   $$E[Z_\upsilon] \leq E[U_\upsilon] \leq U_0.$$
   because U and U^υ are supermartingales.

   So:
   $$E[Z_\upsilon] \leq U_0^\upsilon = U_0.$$
   Since $Z_\upsilon \leq U_\upsilon$, we have $Z_\upsilon = U_\upsilon$.

To see that U^{υ} is a martingale, note that from the fact that U^{υ} is a supermartingale and the results we have just seen;

$$U_0 \geq E[U_n^{\upsilon}] \geq E[U_N^{\upsilon}] = E[U_{\upsilon}] = U_0.$$

So:

$$E[U_n^{\upsilon}] = E[E[U_{\upsilon}|\mathcal{F}_n]].$$

On the other hand:

$$U_n^{\upsilon} \geq E[U_N^{\upsilon}|\mathcal{F}_n] = E[U_{\upsilon}|\mathcal{F}_n].$$

So:

$$U_n^{\upsilon} = E[U_{\upsilon}|\mathcal{F}_n].$$

This is a martingale.

2. Assume that $Z_{\upsilon} = U_{\upsilon}$ and U^{υ} is a martingale.

   Being U^{υ} a martingale, we have:

$$U_0 = E[U_n^{\upsilon}] = E[U_{\upsilon}] = E[Z_{\upsilon}].$$

   But, if τ is any stopping time, U^{τ} is a supermartingale and:

$$U_0 \geq E[U_N^{\tau}] = E[U_{\tau}] \geq E[Z_{\tau}].$$

   So, υ is optimal.

□

Finally, we relate the concept of optimal stopping time with the increasing and predictable process related with Doob's decomposition of the Snell supermartingale.

Proposition 73. *The stopping time υ^* defined as $\upsilon^* = N$ if $A_N = 0$ and:*

$$\upsilon^* = inf\,\{n : A_{n+1} \neq 0\}$$

if $A_N \neq 0$, is the greater optimal stopping time associated to Z.

Proof.

1. Lets see that υ^* is a stopping time:

   Indeed, for any $k \in \mathbb{T}$:

$$\{\upsilon^* = k\} = \{A_1 = 0, ..., A_k = 0, A_{k+1} \neq 0\} \in \mathcal{F}_k.$$

   because A is predictable.

2. Lets see that υ^* is optimal:

   Indeed, $U_n = M_n - A_n$ and by definition, $A_j = 0$ for $j \leq \upsilon^*$.

   Therefore, $U^{\upsilon^*} = M^{\upsilon^*}$ and so U^{υ^*} is a martingale.

   On other hand:

$$\begin{aligned} U_{\upsilon^*} &= \sum_{j=0}^{N-1} \mathbb{1}_{\{\upsilon^*=j\}} U_j + \mathbb{1}_{\{\upsilon^*=N\}} U_N \\ &= \sum_{j=0}^{N-1} \mathbb{1}_{\{\upsilon^*=j\}} max\left\{Z_j, E[U_{j+1}|\mathcal{F}_j]\right\} + \mathbb{1}_{\{\upsilon^*=N\}} Z_N. \end{aligned}$$

   But if $\upsilon^* = j$, $A_{j+1} \neq 0$ and therefore:

$$E[U_{j+1}|\mathcal{F}_j] = E[M_{j+1} - A_{j+1}|\mathcal{F}_j] = M_j - A_{j+1} < M_j = U_j$$

   so $U_j = Z_j$. In general:

$$U_{\upsilon^*} = Z_{\upsilon^*}$$

   and by the "Theorem 72", we conclude that υ^* is optimal.

3. Let's see that υ^* is the greater optimal stopping, that is for any optimal stopping time υ, we have $\upsilon \leq \upsilon^*$.

   Assume that $\mathbb{P}(\upsilon > \upsilon^*) > 0$. Then:

$$E[U_\upsilon] = E[M_\upsilon] - E[A_\upsilon] = E[M_0] - E[A_\upsilon] = E[U_0] - E[A_\upsilon] < E[U_0].$$

   Because, $E[A_\upsilon] > 0$. This is contradictory with the fact that U^υ has to be a martingale by the "Theorem 72".

□

Corollary 74. *Being υ^* optimal, we have $U_{\upsilon^*} = Z_{\upsilon^*}$ and so $\upsilon_0 \leq \upsilon^*$.*
It means that any optimal stopping time υ will satisfy $\upsilon \leq \upsilon^*$ by the previous proposition and $\upsilon \geq \upsilon_0$ by the definition of υ_0.

Remark 75. *The time υ_0 indicate to us the first moment we can exercise in spite of not losing money if we wait. The time υ^* says to us the last moment to exercise without loosing money. In the situations in where the owner of the option win the same executing than waiting, $\upsilon_0 = \upsilon^*$.*

8.5 American and European Options

Consider an American option with intrinsic value $\{Z_n, n \in \mathbb{T}\}$ and the corresponding European option with payoff Z_N. Denote by U_n and u_n the corresponding price processes. We have the following proposition:

Proposition 76.

1. *$U_n \geq u_n$, $\forall n \in \mathbb{T}$.*
2. *If $u_n \geq Z_n$, $\forall n \in \mathbb{T} \Rightarrow u_n = U_n$ $\forall n \in \mathbb{T}$.*

Proof.

1. The sequence $\tilde{U}$ is a supermartingale. So:

$$\tilde{U}_n \geq E[\tilde{U}_N|\mathcal{F}_n] = E[\tilde{Z}_N|\mathcal{F}_n] = \tilde{u}_n.$$

2. The hypothesis $u_n \geq Z_n$ means $\tilde{u}_n = E[\tilde{Z}_N|\mathcal{F}_n]$ and so $\tilde{u}$ is a martingale and in particular a supermartingale such that $\tilde{u}_n \geq \tilde{Z}_n$. Therefore, necessarily $\tilde{u}_n \geq \tilde{U}_n$ because $\tilde{U}$ is the minimal supermartingale above Z. From (1) we proof the result that $u_n = U_n$, $\forall n \in \mathbb{T}$.

□

9 American Options in the CRR

9.1 Pricing American Options in the CRR model

Assume the CRR model given in section "Cox-Ross-Rubinstein Model" under the unique risk neutral measure.

- For the call price, following the same notation as before:

$$\begin{aligned}\tilde{u}_n &= (1+r)^{-N} E[(S_N - K)^+ | \mathcal{F}_n] \\ &\geq E[\tilde{S}_N - K(1+r)^{-N} | \mathcal{F}_n] \\ &= \tilde{S}_n - K(1+r)^{-N}.\end{aligned}$$

  So:

$$u_n \geq S_n - K(1+r)^{n-N} \geq S_n - K.$$

  On the other hand, $u_n \geq 0$, $\forall n \in \mathbb{T}$.

  Therefore, $u_n \geq (S_n - K)^+ = Z_n$ and by the previous proposition we have $u_n = U_n$, $\forall n \in \mathbb{T}$.

  This means that the price of the American and European call is the same.

- For the put option, the price is not the same.

  If we consider $Z_n := (K - S_n)^+$, we have $U_N = Z_N = (K - S_N)^+$ and

$$U_n := max\left\{Z_n, \frac{1}{1+r} E[U_{n+1} | \mathcal{F}_n]\right\}$$

Proposition 77. *We can write:*

$$U_n = P(n, S_n)$$

where:

$$P(N, x) := (K - x)^+$$

and:

$$P(n, x) := max\left\{(K - x)^+, \frac{1}{1+r} f(n+1, x(1+u)\right\}$$

with:

$$f(n+1, x) := (1-p)P(n+1, x(1+d)) + pP(n+1, x(1+u)).$$

Proof. We will proof the proposition by induction:

- For $n = N$ is clear.
- Assume the formula for $P(n, x)$ is valid for $n+1, n+2, ..., N$.

  Let's see if it is true for n. We have:

$$\begin{aligned} U_n &= max\left\{(K - S_n)^+, \tfrac{1}{1+r}E[P(n+1, S_{n+1})|\mathcal{F}_n\right\} \\ &= max\left\{(K - x)^+, \tfrac{1}{1+r}E[P(n+1, xT_{n+1})]\right\}(S_n) \\ &= max\left\{(K - x)^+, \tfrac{1}{1+r}f(n+1, x)\right\}(S_n) \\ &= P(n, S_n). \end{aligned}$$

□

Remark 78. *Note that $U_0 = P(0, S_0)$ is the initial price of the put option.*

10 Pricing American Options in C ++

In this section, according to [4]:*"Maciej J. Capiński and Tomasz Zastawniak (2012): Numerical Methods in Finance with C++. Cambridge."* we will see a programme in C++ that prices and determine the optimal stopping of American options.

We will compute and store the price of an American option not only at time 0, also for each time step n and node i in the binomial tree. In addition, we will compute the early exercise policy for an American option. The time steps n and nodes i at which the option should be exercised and characterized by the condition:

$$H(n,i) = h(S(n,i)) > 0.$$

Where:

- $h(S(n,i))$ is the payoff of the holder of the option at time step "n" and node "i" of the binomial tree. Where $S(n,i) = S(0)(1+U)^i(1+D)^{n-i}$.
- $H(n,i)$ is the price of the American option at time step "n" and node "i". Note that this prices can be computed by backward induction on n:
  - At expiry date N:
    $$H(N,i) = h(S(N,i))$$
    for each node $i = 0,1,...,N$
  - If $H(n+1,i)$ is already known at each node $i = 0,1,...,n+1$ for some $n = 0,...,N-1$ then:
    $$H(n,i) = max(\frac{qH(n+1,i+1)+(1-q)H(n+1,i)}{1+R}, h(S(n,i)))$$
    for each node $i = 0,1,...,n$. In particular, H(0) is the price of the American option at time 0. Also note that the discounted price process $(1+R)^{-n}H(n,i)$ is the Snell envelope of the discounted payoff process $(1+R)^{-n}h(S(n,i))$.

So, we are going to encode this information as data of type *bool*, taking just two possible values, 0 if the option should not be exercised at a given node or 1 otherwise, depending on whether the above condition is violated or not.

The binomial tree has the structure:

				...
			i=3	
		i=2		...
	i=1		i=2	
i=0		i=1		...
	i=0		i=1	
		i=0		...
			i=0	
				...
n=0	n=1	n=2	n=3	...

The "main" program is composed by 4 programmes that have differents functions:

- BinModel:
  - Ask for S0, U, D and R. Check that the values of S0, U, D and R are corrects and check that there is no arbitrage opportunity.
  - Compute the risk neutral measure and the stock price at node (n,i).
- BinLattince:
  - Creates a vector of vectors which provides the option price at each time n in node i (typename double).
  - It also creates a vector of vectors which provides if the option has to be executed at each time n in node i (typename bool).
- EurAmOptions:
  - Ask for an European and American option.
  - Prices the European Option using *PriceByCRR* and prices the AmericanOption using *PriceBySnell*.
  - Creates the *PriceTree* which has the price of the option in each time n and node i and creates the *StoppingTree* saying which time is optimal to exercise.
- MainPut:
  - Compute the prices and the optimal exercise of an American Put at each time n and node i.

10.1 BinModel

10.1.1 Code

BinModel.h

```
#ifndef BinModel_h
#define BinModel_h

class BinModel
{
   private:
         double s0;
         double U;
         double D;
         double R;
   public:
         double RiskNeutProb(); // Computing riskneutral probability
         double S(int n, int i); // Computing the stock price at node n,i
         int GetInputData(); // Displaying and checking model data
         double GetR();
   };

#endif
```

BinModel.cpp

```
#include "BinModel.h"
#include <iostream>
#include <cmath>
using namespace std;

double BinModel::RiskNeutProb()
{
        return (R-D)/(U-D);
}

double BinModel::S(int n, int i)
{
        return S0*pow(1+U,i)*pow(1+D,n-i);
}

int BinModel::GetInputData()
{
        // Entering data
        cout << "Enter S0: "; cin >> S0;
        cout << "Enter U: "; cin >> U;
        cout << "Enter D: "; cin >> D;
        cout << endl;
        // Making sure that 0<S0, -1<D<U, -1<R
        if (S0 <= 0.0 || U <= -1.0 || D <= -1.0 || U <= D || R <=-1..0)
        {
           cout << "Illegal data ranges" << endl;
           cout << "Terminating program" << endl;
           return 1;
        }
        // Checking for arbitrage
        if (R >= U || R <= D)
        {
           cout << "Arbitrage exists" << endl;
           cout << "Terminating program" << endl;
           return 1;
        }
```

```
        cout << "Input data checked " << endl;
        cout << There is no arbitrage" << endl << endl;
        return 0;
    }

double BinModel::GetR()
{
        return R;
}
```

10.1.2 Explanation

- class BinModel: Tells the compiler than a class named "BinModel" is to be defined.
- private S0, U, D, R: These variables will live together inside the class. They will be inaccessible on their own outside this class. Instead, main() and others parts of the program will only be able to access them via this class.
- public RiskNeutProb(), S(n,i), GetInputData(), GetR(): These variables will be accessible outside the class. We shall see calls to these functions made from other parts of the program.
  - RiskNeutProb(): From U, D, R compute the risk-neutral probability.
  - S(n,i): Knowing U and D, compute the prices of the stock.
  - GetInputData(): Ask for S0, U, D, R and check that 0<S0, -1<D<U and -1<R.
  - GetR(): Ask for R.
- BinModel::Function(): Shows that the function is a member of the "BinModel" class.

10.2 BinLattice

10.2.1 Code

BinLattice.h

```
#ifndef BinLattice_h
#define BinLattice_h

#include <iostream>
#include <iomanip>
#include <vector>
using name space std;

template<typename Type> class BinLattice
{
    private
        int N;
        vector < vector <Type> > Lattice;
    public:
        void SetN(int N_)
        {
            N=N_;
            Lattice.resize(N+1);
            for (int n=0; n<=N; n++) Lattice[n].resize(n+1);
        }
        void SetNode(int n, int i, Type x)
            {Lattice[n][i]=x;}
        Type GetNode(int n, int i)
            {return Lattice[n][i];}
        void Display()
        {
            cout << setiosflags(ios::fixed)
                  << setprecision(3);
            for (int n=0; n<=N; n++)
            {
                for(int i=0; i <=n; i ++)
                    cout << setw(7) << GetNode(n,i);
```

```
                cout << endl;
            }
            cout << endl;
        }
};

#endif
```

10.2.2 Explanation

The command *template <typename Type>* specifies that the "BinLattice"is no longer a class, it is a classe template with parameter *Type*.
We do this because to record the stopping policy it would be better to use class for data of type *bool*, so instead of duplicate the code, we parametrize the function.

We also notice that we dont need *.cpp* file, this is because the class template does not lead itself to separate compilation, a class template can only be compiled after an object has been declared using the template with a specific data type.

This class template contains:

- Two variables:
  - "N" to store the number of time steps in the binomial tree.
  - "Lattice" a vector of vectors to hold data of type *Type*.
- The following functions:
  - "SetN()": Function that takes a parameter of type *int*, assigns it to *N* and sets the size of the *Lattice* vector to *N+1*, the number of time instants *n* from 0 to *N*, and then for each *n* sets the size of the inner vector *Lattice[n]* to *n+1*, the number of nodes at time *n*.
  - "SetNode()": To set the value stored at step *n*, node *i*.
  - "GetNode()": To return the value stored at step *n*, node *i*.
  - "Display()": To print the values stored in the binomial tree lattice. The command *cout* fixed decimal points.

10.3 EurAmOptions

10.3.1 Code

EurAmOptions.h

```
#ifndef EurAmOptions_h
#define EurAmOptions_h

#include <BinLattice.h>
#include <BinModel.h>

class Option
{
    private:
        int N; //Steps to expiry
    public:
        void SetN(int N_){N=N_;}
        int GetN(){return N;}
        virtual double Payoff(double z)=0;
};

class EurOption: public virtual Option
{
    public:
        double PriceByCRR(BinModel Model); //Pricing European Option
};

class AmOption: public virtual Option
{
    public:
        double PriceBySnell(BinModel Model, //Pricing American Option
            BinLattice<double> & PriceTree,
            BinLattice<bool> & StoppingTree);
};

class Call: public EurOption, public AmOption
{
    private:
        double K; //Strike price
    public:
```

```
        void SetK(double K_){K=K_;}
        int GetInputData();
        double Payoff(double z);
};

class Put: public EurOption, public AmOption
{
    private:
        double K; //Strike price
    public:
        void SetK(double K_){K=K_;}
        int GetInputData();
        double Payoff(double z);
};

#endif
```

EurAmOptions.cpp

```
#include "EurAmOptions.h"
#include "BinModel.h"
#include "BinLattice.h"
#include <iostream>
#include <cmath>
using namespace std;

double EurOption::PriceByCRR(BinModel Model)
{
        double q=Model.RiskNeutProb();
        int N=GetN();
        vector<double> Price(N+1);
        for (int i=0; i<=N; i++)
        {
           Price[i]=Payoff(Model.S(N,i));
        }
        for (int n=N-1; n>=0; n–)
        {
          for (int i=0; i<=n; i++)
           {
              Price[i]=(q*Price[i+1]+(1-q)*Price[i])/(1+Model.GetR());
           }
        }
        return Price[0];
}

double AmOption::PriceBySnell(BinModel Model,
        BinLattice<double>& PriceTree,
        BinLattice<bool>& StoppingTree)
{
        double q=Model.RiskNeutrProb();
        int N=GetN();
        PriceTree.SetN(N);
        double ContVal;
        for (int i=0; i<=N; i++)
        {
```

```
            PriceTree.SetNode(N,i,Payoff(Model.S(N,i));
            StoppingTree.SetNode(N,i,1);
        }
        for (int n=N-1; n>=0; n–)
        {
            for (int i=0; i<=n; i++)
            {
                ContVal=(q*PriceTree.GetNode(n+1,i+1)
                    +(1-q)*PriceTree.GetNode(n+1,i))
                    /(1+Model.GetR());
                PriceTree.SetNode(n,i,Payoff(Model.S(n,i)));
                StoppingTree.SetNode(n,i,1);
                if (ContVal>PriceTree.GetNode(n,i))
                {
                    PriceTree.SetNode(n,i,ContVal);
                    StoppingTree.SetNode(n,i,0);
                }
                else if (PriceTree.GetNode(n,i)==0.0)
                {
                    StoppingTree.SetNode(n,i,0);
                }
            }
        }
        return PriceTree.GetNode(0,0);
}

int Call::GetInputData()
{
        cout << ”Enter call option data:” << endl;
        int N;
        cout << ”Enter steps to expiry N: ”; cin >> N;
        SetN(N);
        cout << ”Enter strike price K: ”; cin >> K;
        cout << endl;
        return 0;
}
```

```
double Call::Payoff(double z)
{
        if (z>K) return z-K;
        return 0.0;
}

int Put::GetInputData()
{
        cout << "Enter call option data:" << endl;
        int N;
        cout << "Enter steps to expiry N: "; cin >> N;
        SetN(N);
        cout << "Enter strike price K: "; cin >> K;
        cout << endl;
        return 0;
}

double Put::Payoff(double z)
{
        if (z<K) return K-z;
        return 0.0;
```

10.3.2 Explanation

- "Option": Gets the time to expiry N.Gets information about the option.
- "EurOption": Gives the fair price of the option pricing *PryceByCRR*.
- "AmOption": Gives the fair price of the American option pricing by *PriceBySnell*. Also computes the *PriceTree* and the *StoppingTree* of this option.
- "Call": Ask for the strike price of a call option.
- "Put": Ask for the strike price of a put option.
- "EurOption::PriceByCRR": Price by CRR an European option computing the price in each time n and node i in the vector *Price[i]* using the risk neutral measure method. The fair price will be *Price[0]*.
- "AmOption::PriceBySnell": Price by Snell the American option computing the price in the *PriceTree* and computes in the same time n and node i the *StoppingTree* using the condition that is better to execute, or to wait $ContVal > PriceTree.GetNode(n,i)$.
- "Call::GetInputData()": Gets the option data of a call, the steps to expiry N and the strike price K.
- "Call::Payoff()": Return the payoff of a call.
- "Put::GetInputData()": Obtain the information of a put option, the data time N, the steps to expiry N and the strike price K.
- "Put::Payoff()": Return the payoff of a put.

10.4 MainPut

10.4.1 Code

"Main.cpp"

```
#include "BinLattice.h"
#include "BinModel.h"
#include "EurAmOptions.h"
#include <iostream>
using namespace std;

int main()
{
        BinModel Model;
        if (Model.GetINputData()==1) return 1;
        Put Option;
        Option.GetInputData();
        BinLattice<double> PriceTree;
        BinLattice<bool> StoppingTree;
        Option.PriceBySnell(Model,PriceTree,StoppingTree);
        cout << "American put prices:" << endl << endl;
        PriceTree.Display();
        cout << "American put exercise policy:"
                << endl << endl;
        StoppingTree.Display();
        return 0;
}
```

10.4.2 Explanation

- "BinLattice<double> PriceTree;": Create object *PriceTree* with the information of prices.
- "BinLattice<bool> StoppingTree;": Create object *StoppingTree* with the information of the stopping times.
- "PriceTree.Display();": Display the prices for all nodes.
- "StoppingTree.Display();": Display the stopping policy that is 1s for the nodes where the American option should be exercised and 0s for the others.

11 Conclusions

After to have done this work, I have discovered an amazing world, the financial markets. Not only for the topic of this work, also for all the information that I have seen in the books I used for doing this work. I feel so curious to the way that mathematics are used in financial markets, in special the probability and the statistics. So difficult things like the hedging strategy or the optimal exercise of an American option, are easily explain using maths. So, I am very motivated to the use of maths in the Economy and I would like to continue studying and getting knowledge in this area. Also, I would like to continue studying the options not only in discrete time, also in continuous time in which the possibilities are higher and more interesting.

Another important point, is the knowledge I have got programming in C++. I am so satisfied with the informatics I have learn in the degree. Thanks to know how to program in C, I have not got any problem understanding how C++ works. The program also shows to me the importance of the computation in this area. All the mathematics results in Finance, need to be computed to carry out in real life.

I have to say, that the thing I like the less is the fact that all this results, are supposing a perfect situation of the market. Real life is not as good as we would like to expect, is more difficult. But for sure, nothing impossible to solve using maths.

Finally, another important point is the decision to make this research work in English. I have been able to learn a lot of specific vocabulary about financial markets that, for sure, will help me on the future a lot.

To sum up, I am very happy of the result of this work and I feel that do the research work, is one of the better ways to learn and help yourself to decide in which are apply our degree.

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