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System Identification using LQG-Balanced Model Reduction

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Abstract

System identification of linear multivariable dynamic models based on discrete-time data can be performed using a algorithm combining linear regression and LQG-balanced model reduction. The approach is applicable also to unstable system dynamics and it provides balanced models for optimal linear prediction and control.

Keywords: System Identification, Model Reduction

Introduction

Among many approaches to system identification, least-squares methods, maximum-likelihood methods, realization-based methods and subspace-based methods stand out as methods of choice in various contexts. However, certain weaknesses can be noticed in the capacity of algorithms to produce minimal model representation and to handle correlated noise or multi-input multi-output data. For example, a drawback with many implementations of (approximate) maximum-likelihood (ML) methods is that they rely on numerical optimization. Related problems appear in applications of these methods to multi-input multi-output systems where properties of uniqueness of parametrization become important. Another weak point in many system identification approaches to multivariable linear systems is how to find appropriate models for colored noise. The combination of these issues have inspired new efforts to improve pseudolinear regression and subspace-based models using singular value decomposition. Pseudolinear regression is often organized as a two-step method where the first step involves linear regression to find a high-order model and a second step in which the model order is reduced and where the disturbance model is found—*e.g.*, as an iterated Markov estimate. One alternative is to apply balanced model reduction in the second step. As balanced model reduction only can be applied to stable models, there is a limited application range for this method. However, Fuhrmann and Ober and more recently Salomon et al. have suggested a modified balanced model that exploited a modified balancing approach [3], [12]. Instead of solving for a pair of Gramians using Lyapunov function, it was suggested to be replaced by Riccati equation. An immediate application in the context of model reduction is that unstable systems may be objects for model reduction. The idea goes back at least to Desai and Pal [2], and Jonckheere and Silverman [9], who suggested LQG-like balanced realization for innovation models and Kalman filters obtained in covariance analysis [2, 9].

This important observation can also be exploited in the context of system identification. In the context of pseudolinear regression, the benefit is two-fold. Firstly, it permits the application of pseudolinear regression to unstable systems which, in, turn permits derivation of disturbance models. Secondly, by virtue of the LQG properties it permits the formulation of optimal linear model approximation to reduced-order models for application in LQG control and Kalman filtering. Important application is to be found in identification for control, Kalman filter design and spectrum analysis.

Preliminaries

Balanced Model Reduction

Given is a linear time-invariant m-inputs p outputs transfer matrix $G(s)$ with a realization given by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$; $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and A, B, C, D are matrices of appropriate dimensions. Denote the realization of $H(z)$ given in Eq. (1) by $\mathcal{S}\{A, B, C, D\}$ or

$$\mathcal{S} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2}$$

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The controllability and observability Gramians are defined as

$$P = \sum_{0}^{\infty} A^k B B^T (A^T)^k, \quad (3)$$

$$Q = \sum_{0}^{\infty} (A^T)^k C^T C A^k \quad (4)$$

Note that P, Q are also the solutions to the discrete Lyapunov equations (or Stein equations)

$$P = APA^T + BB^T \quad (5)$$

$$Q = A^T Q A + C^T C \quad (6)$$

In the case where (A, B) is controllable and (A, C) observable, there exists a linear transformation T such that $\mathcal{S}\{TAT^{-1}, TB, CT^{-1}, D\}$ is balanced—*i.e*, $TPT^T = (T^T)^{-1}QT^{-1} = \Sigma$ with $\Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. Now partition of the resulting transformed system matrix into

$$\mathcal{S} = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right], \quad \begin{array}{l} A_{11} \in \mathbb{R}^{r\times r}, \\ B_1 \in \mathbb{R}^{r\times m}, \; r < n \\ C_1 \in \mathbb{R}^{p\times r} \end{array} \quad (7)$$

Then, a reduced-order model $\mathcal{S}_r$ of order $r < n$ can be obtained as one of the following approximants

$$\mathcal{S}_r = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right], \sigma_{r+1} \le \|\mathcal{S} - \mathcal{S}_r\|_\infty \le 2 \sum_{k=r+1}^{n} \sigma_k$$

$$\mathcal{S}_r = \left[\begin{array}{c|c} A_{11} - A_{12}A_{22}^{-1}A_{21} & B_1 - A_{12}A_{22}^{-1}B_2 \\ \hline C_1 - C_2A_{22}^{-1}A_{21} & D - C_2A_{22}^{-1}B_2 \end{array}\right] \quad (8)$$

where the first one is known as balanced truncation whereas the second one is the singular perturbation approximation.

The discrete Lyapunov equation $APA^T + BB^T = P$ has a unique solution if and only if $\lambda_i(A)\lambda_j(A)^* \neq 1$, $\forall i, j$. Even for the unstable case, there may exist unique and symmetric solutions. A serious problem, however, is that the resulting P will be indefinite and useless for modell approximation purposes. Salomon *et al.* [12] showed that for (A, B) stabilizable, (A, C) detectable there are still relevant solutions obtained by replacing Lyapunov equations with the Riccati equations

$$\begin{aligned} P &= APA^T + BB^T \quad (9) \\ &- APB(R + B^T PB)^{-1} B^T PA^T \\ Q &= A^T QA + C^T C \quad (10) \\ &- A^T QC^T (R + CQC^T)^{-1} CQA \end{aligned}$$

The model reduction scheme obtained using this modification is called LQG-balanced model reduction.

System Identification Algorithm

Consider a discrete-time time-invariant system $\mathcal{S}_n(A, B, C, D)$ with system equations

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + v_k \\ y_k &= Cx_k + Du_k + e_k \end{aligned} \quad (11)$$

with input $u_k \in \mathbb{R}^m$, output $y_k \in \mathbb{R}^p$, state vector $x_k \in \mathbb{R}^n$ and zero-mean disturbance stochastic processes $v_k \in \mathbb{R}^n$, $e_k \in \mathbb{R}^p$ acting on the state dynamics and the output, respectively. The discrete-time system identification problem is to find estimates of system matrices A, B, C, D from finite sequences $\{u_k\}_{k=0}^N$ and $\{y_k\}_{k=0}^N$ of input-output data. Using a left matrix fraction description

$$A_L(z^{-1})^{-1}B_L(z^{-1}) = C(zI - A)^{-1}B + D$$
$$A_L(z^{-1})^{-1}C_L(z^{-1}) = C(zI - A)^{-1}K + I$$
$$A_L(z^{-1}) = I_p + A_1z^{-1} + \cdots + A_nz^{-n} \in \mathbb{R}^{p\times p}[z^{-1}]$$
$$B_L(z^{-1}) = B_0 + B_1z^{-1} + \cdots + B_nz^{-n} \in \mathbb{R}^{p\times m}[z^{-1}]$$
$$C_L(z^{-1}) = C_0 + C_1z^{-1} + \cdots + C_nz^{-n} \in \mathbb{R}^{p\times p}[z^{-1}]$$

To the purpose of linear regression for estimation, it is straightforward to formulate this model as counterpart to the autoregressive moving-average model with external input (ARMAX) used in time-series analysis

$$A_L(z^{-1})Y(z) = B_L(z^{-1})U(z) + C_L(z^{-1})W(z) \quad (12)$$

and the linear regresson model

$$\begin{aligned} Y(z) &= -\sum_{k=1}^{n} A_k z^{-k} Y(z) + \sum_{k=0}^{n} B_k [z^{-k}] U(z) \\ &+ \sum_{k=0}^{n} C_k z^{-k} W(z) \end{aligned} \quad (13)$$

Example 1—Spectral Ratio

Rational functions obtained from z-transformed input-output data

$$\widehat{H}(z) = \frac{Y(z)}{U(z)} \quad (14)$$

provide transfer function estimate that are highly nonminimal rational functions with many approximate pole-zero cancellations (Fig. 1). The pole-zero map with many poles and zeros within the unit circle as well as outside the unit circle is difficult to handle using standard balanced model reduction as the solutions for controllability and observability Gramians fail.

LQG-Balanced System Identification

Because $W(z)$ is not available to measurement, linear regression cannot be applied. As a substitute,

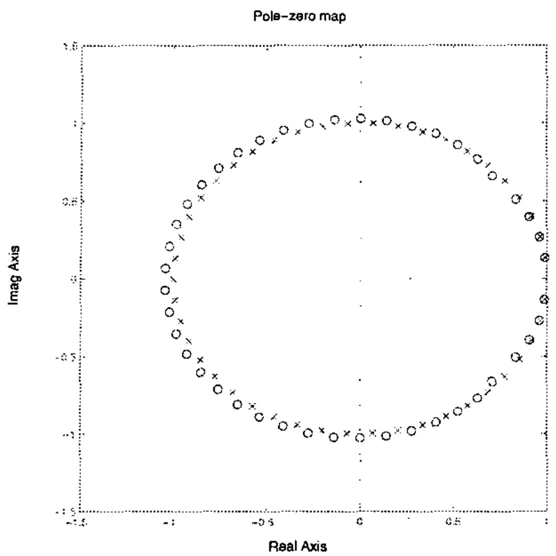

Figure 1: Pole-zero map of transfer function estimate obtained from spectral ratio model reduction exhibits many pole-zero cancellations inside and outside the unit circle. Pole pattern after LQG-balanced pole shifting shows a shift into stable region. Original poles are indicated by '×' and the LQG-shifted poles as '+'.

pseudolinear regression may be applied as an iterative procedure where the essential step is to find a pseudoregressor sequence to substitute the unknown regressor sequence. To the purpose of least-squares identification, then, it is suitable to organize model and data according to

1: Arrange for data sequences of discrete-time data $\{u_k\}$, $\{y_k\}$ with $\{t_k\}_{k=0}^N$ for $j = 0, 1, \ldots, q$, for some $q > n$.

2: Formulate the regression model

$$\begin{aligned} y_k &= -A_1 y_{k-1} - \cdots - A_n y_{k-n} \qquad (15) \\ &\quad + B_1 u_k^{(1)} + \cdots + B_n u_{k-n}, \quad y_k \in \mathbb{R}^p \\ \theta &= \left(A_1 \ldots A_n \quad B_1 \ldots B_n\right)^T, \quad \theta \in \mathbb{R}^{n(m+p)\times p} \end{aligned}$$

which suggests the linear regression model

$$\mathcal{M}_1: \quad \mathcal{Y}_N = \Phi_N \theta \qquad (16)$$

3: Arrange data sequences into matrices

$$\begin{aligned} \phi_k &= \left(-y_{k-1}^T \ldots - y_{k-n}^T \quad u_{k-1}^T \ldots u_{k-n}^T\right)^T \\ \mathcal{Y}_N &= \begin{pmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_N^T \end{pmatrix} \in \mathbb{R}^{N\times p}, \quad \Phi_N = \begin{pmatrix} \phi_1^T \\ \phi_2^T \\ \vdots \\ \phi_N^T \end{pmatrix} \in \mathbb{R}^{N\times n(m+p)} \end{aligned}$$

4: Compute the least-squares estimate $\widehat{\theta}$ and the residual sequence $E_N \in \mathbb{R}^{N\times p}$ with rows $\{\varepsilon_k^T\}_{k=1}^N$

$$\begin{aligned} \widehat{\theta}_N &= (\Phi_N^T \Phi_N)^{\dagger} \Phi_N^T \mathcal{Y}_N \qquad (17) \\ E_N(\widehat{\theta}) &= \mathcal{Y}_N - \widehat{\mathcal{Y}}_N = \mathcal{Y}_N - \Phi_N \widehat{\theta} \\ &= (I_N - \Phi_N (\Phi_N^T \Phi_N)^{\dagger} \Phi_N^T) \mathcal{Y}_N \qquad (18) \end{aligned}$$

5: Formulate a pseudoregression model using $\{\varepsilon_k\}_{k=1}^N$ to replace unknown disturbance $\{w_k\}$

$$\begin{aligned} y_k &= -A_1 y_{k-1} - \cdots - A_n y_{k-n}, \quad y_k \in \mathbb{R}^p, u_k \in \mathbb{R}^m \\ &\quad + B_1 u_{k-1} + \cdots + B_n u_{k-n} \\ &\quad + \varepsilon_k + C_1 \varepsilon_{k-1} + \cdots + C_n \varepsilon_{k-n} \qquad (19) \\ \theta &= \left(A_1 \ldots A_n \quad B_1 \ldots B_n \quad C_1 \ldots C_n\right)^T, \end{aligned}$$

which suggests the linear regression model

$$\mathcal{M}_2: \quad \mathcal{Y}_N = \Phi_N \theta, \quad \theta \in \mathbb{R}^{n(m+2p+1)\times p} \qquad (20)$$

As a result of the non-uniqueness of parameters, the normal equations of the associated least-squares estimation of θ will exhibit rank deficit in general. It is therefore natural to apply the least-squares solution

$$\widehat{\theta}_N = (\Phi_N^T \Phi_N)^{\dagger} \Phi_N^T \mathcal{Y}_N \qquad (21)$$

where $(\Phi_N^T \Phi_N)^{\dagger}$ denotes the matrix pseudo-inverse of $\Phi_N^T \Phi_N$. The associated least-squares estimate then obtained has the smallest 2-norm of all possible minimizers of the least-squares criterion.

Step 2—LQG-balanced Model Reduction: The regression models $\mathcal{M}_1$, $\mathcal{M}_2$ suggest nonminimal multivariable state-space models which may be objects for model reduction.

A nonminimal state-space model may be suggested as

$$\begin{aligned} A_x &= \begin{bmatrix} -A_1 & -A_2 & -A_3 & \cdots & -A_n \\ I_p & 0 & 0 & \cdots & 0 \\ 0 & I_p & 0 & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & I_p & 0 \end{bmatrix} \\ B_x &= \begin{bmatrix} I_m & 0 & 0 & \cdots & 0 \end{bmatrix}^T \qquad (22) \\ C_x &= \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix} \qquad (23) \end{aligned}$$

The intermediate high-order result

$$\begin{pmatrix} x_{k+1} \\ y_k \end{pmatrix} = \begin{pmatrix} A_x & B_x \\ C_x & 0 \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} \qquad (24)$$

permits application of LQG-balanced model reduction. Note that the state-space description of Eq. (24) is observable but not controllable.

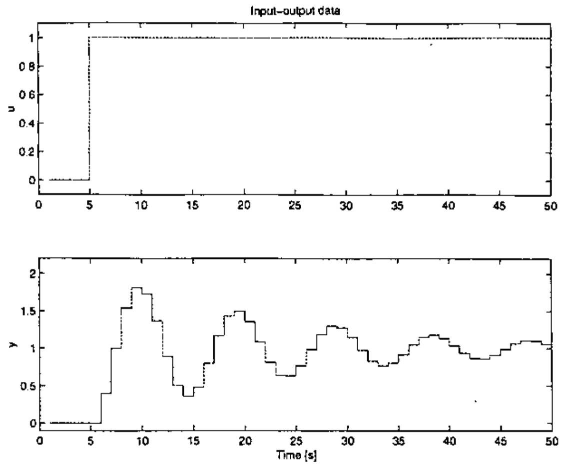

Figure 2: Input-output data of Example 2.

LQG-balancing in Spectrum Analysis

Based on N samples of input-output data, a transfer function estimate can be obtained as the spectral ratio

$$\widehat{H}(z) = S_{yu}(z)S_{uu}^{-1}(z) \quad \text{where} \tag{25}$$

$$S_{yu}(z) = \sum_{k=-N}^{N} C_{yu}(k)z^{-k}, \quad S_{yu} \in \mathbb{R}^{p\times m}(z) \tag{26}$$

$$S_{uu}(z) = \sum_{k=-N}^{N} C_{uu}(k)z^{-k}, \quad S_{uu} \in \mathbb{R}^{m\times m}(z) \tag{27}$$

The spectral ratio offers an interpretation as a non-minimal right matrix fraction description

$$A_R(z)\zeta(z) = U(z) \tag{28}$$

$$Y(z) = B_R(z)\zeta(z) \tag{29}$$

where

$$A_R(z) = z^N S_{uu}(z) \tag{30}$$

$$B_R(z) = z^N S_{yu}(z) \tag{31}$$

LQG-balanced model approximation serves to reduce the rational function to a coprime factorization.

Example 2—Spectrum Analysis
Step-response data ($N = 50$) obtained from the system

$$x_{k+1} = \begin{pmatrix} 1.5 & -0.9 \\ 1 & 0 \end{pmatrix} x_k + \begin{pmatrix} 0.4 \\ 0 \end{pmatrix} u_k \tag{32}$$

$$y_k = \begin{pmatrix} 1 & 0 \end{pmatrix} x_k \tag{33}$$

were used for transfer function estimation (Figs. 2-3). Singular values are shown in Fig. 4. Spectral

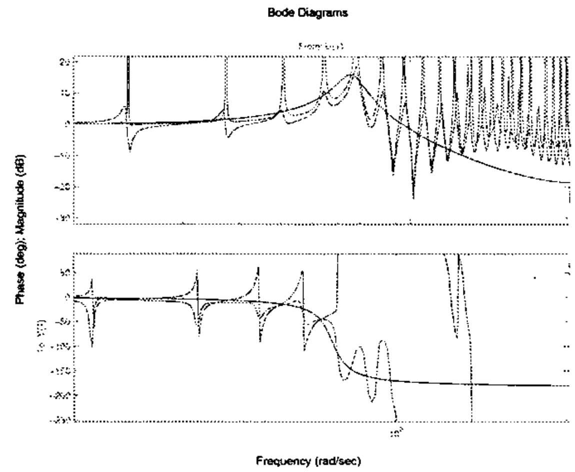

Figure 3: Transfer function estimates obtained from spectral ratio model reduction exhibit many pole-zero cancellations inside and outside the unit circle and transfer function estimate after LQG-balanced pole shifting shows a shift into stable region.

estimates, LQG-balanced spectral estimate and spectral estimate of the reduced-order model are shown in Fig. 3.

LQG-balancing for Frequency-domain Methods

Frequency response fitting based on least-squares identification in the complex frequency domain is a natural idea which also benefits from LQG balancing. Let the polynomial ratio

$$\widehat{G}(i\omega) = A^{-1}(i\omega)B(i\omega) \tag{34}$$

$$A(s) = s^n I_p + A_1 s^{n-1} + \cdots + A_{n-1}s + A_n$$

$$B(s) = B_1 s^{n-1} + \cdots + B_{n-1}s + B_n$$

denote a transfer function estimate to be fitted to the experimental data $G(i\omega_k)$ and known at the frequency points ω_k, $k = 1, 2, \ldots, N$. A natural goal of optimization is to minimize the error criterion

$$\min_{A,B} \sum_k \|A(s)Y(z) - B(s)U(z)\|^2_{s=i\omega_k,\ z=e^{i\omega_k}} \tag{35}$$

where $Y(z)$, $U(z)$ denote z-transformed input-output data. The linear regression problem with parameter vector θ takes on the format

$$\mathcal{Y}_N = \Phi_N\theta = \begin{pmatrix} \Phi_Y & \Phi_U \end{pmatrix} \theta \tag{36}$$

$$\theta = \begin{pmatrix} A_1 & \cdots & A_n & B_1 & \cdots & B_n \end{pmatrix}^T \tag{37}$$

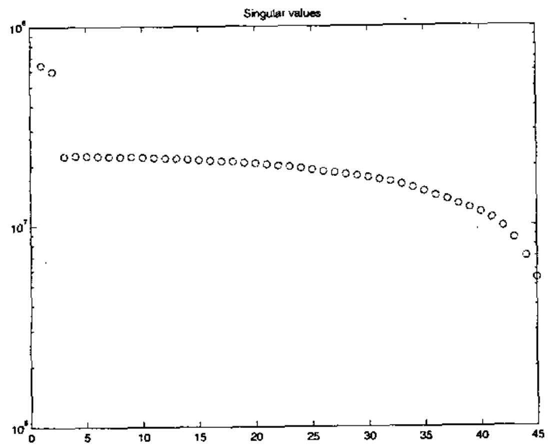

Figure 4: Singular values obtained from LQG-balancing of spectral ratio in Example 2. Model reduction exhibits many pole-zero cancellations inside and outside the unit circle and transfer function estimate after LQG-balanced pole shifting shows a shift into stable region.

with

$$\mathcal{Y}_N = \begin{pmatrix} (i\omega_1)^n Y^T(z^{i\omega_1}) \\ (i\omega_2)^n Y^T(z^{i\omega_2}) \\ \vdots \\ (i\omega_N)^n Y^T(z^{i\omega_N}) \end{pmatrix} \tag{38}$$

and the regressor matrices

$$\Phi_Y = \begin{pmatrix} -(i\omega_1)^{n-1} Y^T(e^{i\omega_1}) & \cdots i\omega_1 Y^T(z^{i\omega_1}) & Y^T(e^{i\omega_1}) \\ -(i\omega_2)^{n-1} Y^T(e^{i\omega_2}) & \cdots i\omega_2 Y^T(z^{i\omega_2}) & Y^T(e^{i\omega_2}) \\ \vdots & & \vdots \\ -(i\omega_N)^{n-1} Y^T(e^{i\omega_N}) & \cdots i\omega_N Y^T(z^{i\omega_N}) & Y^T(e^{i\omega_N}) \end{pmatrix}$$

$$\Phi_U = \begin{pmatrix} (i\omega_1)^{n-1} U^T(e^{i\omega_1}) & \cdots i\omega_1 U^T(z^{i\omega_1}) & U^T(e^{i\omega_1}) \\ (i\omega_2)^{n-1} U^T(e^{i\omega_2}) & \cdots i\omega_2 U^T(z^{i\omega_2}) & U^T(e^{i\omega_2}) \\ \vdots & & \vdots \\ (i\omega_N)^{n-1} U^T(e^{i\omega_N}) & \cdots i\omega_N U^T(z^{i\omega_N}) & U^T(e^{i\omega_N}) \end{pmatrix}$$

The least-squares solution minimizing is then

$$\widehat{\theta} = (\Phi^* \Phi)^{-1} \Phi^* \mathcal{Y} \tag{39}$$

where Φ^* denotes the transpose and complex conjugate of Φ.

LQG-balanced model reduction can be applied to the intermediate result

$$\begin{aligned} x_{k+1} &= \begin{pmatrix} -\widehat{A}_1 & -\widehat{A}_2 & \cdots & -\widehat{A}_n \\ I_p & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots 0 & I_p & 0 \end{pmatrix} x_k + \begin{pmatrix} I_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} u_k, \\ y_k &= \begin{pmatrix} \widehat{B}_1 & \widehat{B}_2 & \ldots & \widehat{B}_n \end{pmatrix} x_k \end{aligned}$$

Discussion and Conclusions

Assume that LQG-balanced model reduction exploits the Riccati equations

$$\begin{aligned} P &= A^T P A + B B^T - A^T P B (R + B^T P B)^{-1} B^T P A, \\ Q &= A Q A^T + C^T C - A Q C^T (R + C Q C^T)^{-1} C Q A^T \end{aligned}$$

or

$$\begin{aligned} P &= (A - BK)^T P (A - BK) + B B^T + K^T R K \\ K &= (R + B^T P B)^{-1} B^T P A, \\ Q &= (A - LC) Q (A - LC)^T + C^T C + L R L^T \\ L &= A Q C^T (R + C Q C^T)^{-1} \end{aligned} \tag{40}$$

Introduce the variables and factorizations

$$R = (R^{1/2})^T (R^{1/2}) \tag{41}$$

$$S_P = R + B^T P B = (S_P^{1/2})^T S_P^{1/2}, \tag{42}$$

$$S_Q = R + C Q C^T = S_Q^{1/2} (S_Q^{1/2})^T, \tag{43}$$

$$\mathcal{A}_P = \begin{pmatrix} A - BK & -B \\ S_P^{-1/2} (R^{1/2})^T K & 0 \end{pmatrix}, \tag{44}$$

$$\mathcal{A}_Q = \begin{pmatrix} A - LC & L R^{1/2} S_Q^{-1/2} \\ -C & 0 \end{pmatrix} \tag{45}$$

$$\mathcal{P}_P = \mathcal{P}_P^T = \begin{pmatrix} P & 0 \\ 0 & S_P \end{pmatrix} > 0, \tag{46}$$

$$\mathcal{P}_Q = \mathcal{P}_Q^T = \begin{pmatrix} Q & 0 \\ 0 & S_Q \end{pmatrix} > 0 \tag{47}$$

$$\mathcal{Q}_P = \begin{pmatrix} B B^T & 0 \\ 0 & R \end{pmatrix} \geq 0, \tag{48}$$

$$\mathcal{Q}_Q = \begin{pmatrix} C^T C & 0 \\ 0 & R \end{pmatrix} \geq 0 \tag{49}$$

Then, the Riccati equations of Eq. (40) may be represented by the Lyapunov equations

$$\mathcal{A}_P^T \mathcal{P}_P \mathcal{A}_P - \mathcal{P}_P = -\mathcal{Q}_P \tag{50}$$

$$\mathcal{A}_Q \mathcal{P}_Q \mathcal{A}_Q^T - \mathcal{P}_Q = -\mathcal{Q}_Q \tag{51}$$

where it can be concluded that all eigenvalues of the system matrices $\mathcal{A}_P$, $\mathcal{A}_Q$ have magnitude less than one. This formulation serves to express the formal similarity between Gramian-based balancing and that of LQG-balanced model reduction.

Moreover, the balancing and model reduction operations of Eqs. (7-8) can be represented by left and right matrix multiplications of the Lyapunov equations (50-51). For the case of balanced truncation, there are matrices $\mathcal{T}_P$ and $\mathcal{T}_Q$, respectively.

$$\begin{aligned} \mathcal{T}_P\mathcal{P}_P\mathcal{T}_P^T &= \mathcal{T}_P(\mathcal{A}_P^T\mathcal{P}_P\mathcal{A}_P + Q_P)\mathcal{T}_P^T \quad (52)\\ &= (\mathcal{T}_P^{T\dagger}\mathcal{A}_P\mathcal{T}_P^T)^T\cdot\mathcal{T}_P\mathcal{P}_P\mathcal{T}_P^T\cdot\mathcal{T}_P^{T\dagger}\mathcal{A}_P\mathcal{T}_P^T\\ &+ \mathcal{T}_PQ_P\mathcal{T}_P^T\\ \mathcal{T}_Q\mathcal{P}_Q\mathcal{T}_Q^T &= \mathcal{T}_Q(\mathcal{A}_Q\mathcal{P}_Q\mathcal{A}_Q^T + Q_Q)\mathcal{T}_Q^T \quad (53)\\ &= (\mathcal{T}_Q\mathcal{A}_Q\mathcal{T}_Q^\dagger)\cdot(\mathcal{T}_Q\mathcal{P}_Q\mathcal{T}_Q^T)\cdot(\mathcal{T}_Q\mathcal{A}_Q\mathcal{T}_Q^\dagger)^T\\ &+ \mathcal{T}_QQ_Q\mathcal{T}_Q^T \end{aligned}$$

Major application areas are to be found for optimal control and optimal prediction using reduced-order models. For a model $\mathcal{S}\{A_r, B_r, C_r, D_r\}$ of reduced order r, we have the reduced-order Riccati equations

$$\begin{aligned} P_r &= (A_r - B_rK_r)^TP_r(A_r - B_rK_r)\\ &\quad + B_rB_r^T + K_r^TRK_r + R_P\\ K_r &= (R + B_r^TP_rB_r)^{-1}B_r^TP_rA_r, \quad (54)\\ Q_r &= (A_r - L_rC_r)Q_r(A_r - L_rC_r)^T\\ &\quad + C_r^TC_r + L_rRL_r^T + R_Q \quad (55)\\ L_r &= A_rQ_rC_r^T(R + C_rQ_rC_r^T)^{-1} \end{aligned}$$

for some matrices R_P, R_Q which represent the difference between the higher-order model and the reduced-order model. An interpretation is that R_P, R_Q represent the approximation cost associated with the model approximation. For example, the observer

$$\begin{aligned} \widehat{\xi}_{k+1} &= (A_r - L_rC_r)\widehat{\xi}_k + (B_r - L_rD_r)u_k + L_ry_k,\\ \widehat{y}_k &= C_r\widehat{\xi}_k + D_ru_k, \quad (56) \end{aligned}$$

has an asymptotic covariance function from Q_r and its convergence rate described by

$$\begin{aligned} V(\widetilde{\xi}_k) &= \widetilde{\xi}_k^TQ_r\widetilde{\xi}_k, \quad \widetilde{\xi}_k = \widehat{\xi}_k - \xi_k \quad (57)\\ \Delta V(\xi_k) &= V(\widetilde{\xi}_{k+1}) - V(\widetilde{\xi}_k)\\ &= \widetilde{\xi}_k^T(-C_r^TC_r - L_rRL_r^T - R_Q)\widetilde{\xi}_k\\ &< 0, \quad \|\widetilde{\xi}_k\| \neq 0 \quad (58) \end{aligned}$$

Thus, the cost of optimal approximation can be quantified by R_Q in the context of optimal prediction [8].

Another interesting application which is opened up by the LQG-balanced model reduction is state-space model identification based on empirical transfer function estimates—*e.g.*, input-output spectrum ratios or cross-spectrum ratios. Previously, such approaches were hampered by the presence of unstable pole-zero cancellation in the rational functions obtained. A remaining problem, though, is how to treat cases with eigenvalues of A on the imaginary axis. Unlike regular balancing, LQG-balancing requires the solution of two Riccati equations and the computational cost for solving Riccati equations may be high. Hence, a serious concern is the order of the system to which LQG-balancing be applied. An interesting question for further investigation is how to exploit relationships to subspace-based identification and to the Krylov-Arnoldi methods—see [5], [1].

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