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# Statistical Modeling and Theoretical Analysis of the Influence of Laser Phase Fluctuations on Photon Echo Data Erasure and Stimulated Photon Echoes

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**Abstract**—The effect of random laser phase fluctuations on stimulated photon echoes (SPEs) and on coherently added SPEs was studied and evaluated both experimentally and analytically with statistical methods. The general concept of describing laser frequency fluctuations as a stationary stochastic process is presented and applied to three specific SPE configurations. The effect of phase fluctuations on erasing an SPE by coherently adding another SPE, phase-shifted by 180 degrees relative to the first, is presented.

## 1. INTRODUCTION

Presented in this paper is a theoretical analysis of how stochastic laser fluctuations affect the efficiency with which optical data, stored by photon echoes, can be selectively erased. It is compared with experimental results. In addition, experimental data on fluctuations of conventional stimulated photon echoes (SPEs) are analyzed in order to infer information about the statistical properties of the laser radiation.

The concept of data storage using SPEs has been investigated by several groups, for example [1–3] and attempts have been made to evaluate its potential for use in all-optical memories and for all-optical signal processing [4, 5]. SPEs in rare-earth-ion-doped crystals at liquid-helium temperatures can have decay times of up to several days. Theoretically, some of these crystals allow for storage of a million data bits per spatial point giving an areal density of >1 Tb/cm$^2$ [4]. SPEs use different frequency intervals within an inhomogeneously broadened material to store several bits of data per spatial point. As for any memory, memories using SPE techniques should preferably be rewritable. This requires a practical method for selectively erasing individual data bits. A method for both rewriting and selective erasure of data by coherent addition of echoes has been suggested [6] and investigated [7–9]. This method is sensitive to the phase noise of the light source. Theoretical calculations have been made to explain and predict the sensitivity of this method to phase changes [10, 11].

In the present paper, we present an analytical description of the phase noise influence on photon echo erasure using the theory for stochastic processes. In practice, the phase noise in the available light sources is a major obstacle for a practical implementation of operations using the coherent nature of SPEs. We have concentrated on the phase noise; however, several other factors also can affect or limit the functionality of coherent SPE operations. For example, amplitude noise, which has been neglected throughout this article to simplify the expressions derived for the data erasure efficiency, has to be taken into account in the general case. Decay processes occurring after excitation will also affect the efficiency of the erasure process. These have been neglected in the calculations since the experiments were designed to minimize such effects. We have also been working with low laser powers giving a near-linear response to the input pulses and thus have not taken into consideration nonlinear effects such as saturation.

We experimentally investigated the method of coherently erasing stimulated photon echoes and concluded that for the conditions prevailing in our experiments, the erasure efficiency is mainly limited by laser light phase fluctuations [9]. We have therefore developed a theoretical model that describes the effects of such fluctuations on the erasure efficiency, and compared it to our experimental data. In addition to describing the erasure efficiency, the theory can be used to model the effect of light source frequency fluctuations on the shot-to-shot stability of SPEs. Such pulse fluctuations have been measured and compared to the predictions from the model. Comparison of these two types of data implies that the shot-to-shot amplitude fluctuations of the SPEs are affected not only by the temporal coherence properties of the light source but also by other processes. Nevertheless, the model is still valuable for predicting the maximum achievable erasure efficiency, assuming a light source with known temporal coherence properties.

A short description of the mathematics behind stochastic processes applied to narrow-bandwidth light sources is given in the next section. In Section 3, an introduction to the physical description of photon echoes used here is given. Section 4 is split into a theoretical part (4.1), where the theory in Sections 2 and 3 are used to analytically calculate to what extent SPEs can be coherently erased as a function of the phase fluctuations of the light source, and an experimental part (4.2), where the analytical expressions found in subsection (4.1) are applied to experimental data. In Section 5, analytical expressions of three-pulse SPE area fluctuations due to frequency fluctuations for two cases are derived and experimental data are presented. In Section 6, we indicate how to calculate the power density spectrum of a laser using the methods used in the experimental part of Section 4, or how the results of such an experiment can be predicted assuming we have a knowledge of the power density spectrum. Section 7 contains a brief summary of our results and conclusions.

## 2. DESCRIPTION OF THE LASER PHASE NOISE AS A STOCHASTIC PROCESS

The basic idea of the model is to describe the instantaneous angular frequency, $\omega(t)$, by a stationary stochastic process. The process is stationary if its statistical properties are independent of time. We have here chosen to describe $\omega(t)$ by a normal process, i.e., a process where the probability function of a general stochastic variable $X$, $f_x(x)$, at every instant, $T_k$, is a Gaussian function with expectation value $\mu$ and variance $V$,

$$f_x(x) = \frac{1}{\sqrt{2\pi V}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{V}}. \quad (1)$$

In a more general case, the distribution function of the angular frequency at times $T_n$, where $n = 1, 2, \ldots, k$, is $k$-dimensional and has the form

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\sqrt{2\pi \det(\mathbf{\Sigma})}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})^T}, \quad (2)$$

where $\mathbf{x}$ is a $k$-dimensional vector of stochastic variables, $\boldsymbol{\mu}$ is the $k$-dimensional vector of expectation values and $\mathbf{\Sigma}$ is the $(n \times n)$-dimensional covariance matrix with elements $\mathbf{\Sigma}_{i,j} = C[X_i, X_j]$.

Here, $\mathbf{\Sigma}^{-1}$ is the inversion of $\mathbf{\Sigma}$ and $\det(\mathbf{\Sigma})$ is the determinant of $\mathbf{\Sigma}$ and $C[X(s), Y(t)]$ is the covariance function, which in the general case is defined by

$$C[X(s), Y(t)] \equiv E[\{X(s) - \mu_x\}\{Y(t) - \mu_Y\}], \quad (3)$$

where $E[X]$ is the expectation value of $X$ and $\mu_J = E[J]$.

The integral of a stationary process is not necessarily stationary. We therefore use the autocorrelation properties of $\omega(t)$ instead of the autocorrelation properties of the phase, $\phi(t)$, as the basic measure of the statistical properties of the laser light source. For calculations that do not directly take into consideration the stochastic properties of the phase process, the phase itself is the basic quantity describing phase fluctuations or any derivatives of this process. To evaluate further calculations analytically, the phase has been approximated by a Taylor series expansion around every $T_k$,

$$\phi(t) = \phi(T_k) + \phi'(T_k)(t - T_k) + \frac{1}{2}\phi''(T_k)(t - T_k)^2 + \ldots, \quad (4)$$

where $\phi'(T_k) = \left.\frac{\partial\phi(t)}{\partial t}\right|_{t = T_k}$ and $\phi''(T_k) = \left.\frac{\partial^2\phi(t)}{\partial t^2}\right|_{t = T_k}$.

Our light source is a frequency-stabilized continuous-wave laser and, due to the fact that the frequency is actively locked to a cavity, we have assumed that the instantaneous angular frequency, $\phi'(t) = \omega(t)$, is stationary. It is a general property of stationary processes that all their time derivatives are stationary processes as well. Using the assumption of stationarity for $\omega(t)$, we can here make use of the fact that the covariance for such a process is dependent on a general time difference $\tilde{\tau}$ only and use the simpler expression

$$r_\omega(\tilde{\tau}) \equiv r_{\omega,\omega}(\tilde{\tau}) = C[\omega(t), \omega(t + \tilde{\tau})], \quad (5)$$

which is the covariance function of the angular frequency process, $\omega(t)$. We will also use the general relationships [12, p. 133]

$$r_{\omega'}(\tilde{\tau}) = -\frac{\partial^2 r_\omega(\tilde{\tau})}{\partial \tilde{\tau}^2} \equiv -r_\omega''(\tilde{\tau}) \quad (6)$$

between the cross-correlation function for $\omega(t)$ and its derivative $\omega'(t)$ and [12, p. 138]

$$C\left[\int_a^b \omega(r)dr, \int_c^d \omega(s)ds\right] = \int_a^b\int_c^d r_\omega(r - s)drds \quad (7)$$

for the connection between the autocorrelation function for $\omega(t)$ and the autocorrelation function for $\phi(t)$. These hold for any stationary stochastic process. These basic definitions and approximations of the stochastic parameters will be used henceforth.

## 3. SIMPLE PHYSICAL DESCRIPTION OF SPEs

The electric field $E_{out}$ of a three-pulse SPE, is in the simplest case, described in terms of the input fields as [4]

$$E_{out}(t) = cE_1(t) \otimes E_2(t) * E_3(t). \quad (8)$$

The indices on $E_i$ indicate the time ordering of the input pulses, * denotes the convolution operator,

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(t)g(\tau - t)dt, \quad (9)$$

$\otimes$ denotes the cross-correlation operator,

$$f(t) \otimes g(t) = \int_{-\infty}^{\infty} f^*(t)g(\tau + t)dt, \quad (10)$$

and $c$ is a constant that among other things depends on the number density and electric dipole moment of the absorbers and the length of the active medium. It is constant for every fixed setup and will for simplicity be neglected. In the above expression, we have neglected the effects of homogeneous broadening and other decay processes. These decay processes could be included in the sum as convolutions/correlations with exponentially decaying functions. This would cause the echo size to decrease as a function of the time difference between the pulses and set a limit to the maximum length of possible pulse sequences. Alternatively, they could be included directly as multiplicative factors in the frequency domain. As the electric fields of the pulses are not directly measurable, we calculate the pulse energy of the output, defined as

$$A_{\text{out}} = 2n\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_S\int_{-\infty}^{\infty}|E_{\text{out}}(t)|^2 dt dS, \tag{11}$$

where $n$ is the index of refraction of the medium, $\varepsilon_0$ is the permittivity of vacuum, and $\mu_0$ is the permeability of vacuum. In the rest of this paper, we neglect the term $2n\sqrt{\varepsilon_0/\mu_0}$ and assume that the detector absorbs the whole spatial distribution of the $E$-field such that we can neglect the integration over the detector surface, $S$. To be able to find simple analytical expressions for the quantities we measure, we have chosen to model the light pulses as Gaussian-shaped pulses of duration $T$ (full width at $e^{-1}$ of maximum intensity) and equal intensity. Effects of intensity fluctuations do contribute to the outcome of the experiments, but since the fluctuations are only a few percent, this can be either neglected or measured and corrected for. The electric fields of these input pulses can therefore be described by

$$E_k(t) = E_0 e^{-2\left(\frac{t-T_k}{T}\right)^2 + i\phi(t)}. \tag{12}$$

## 4. COHERENT ERASURE OF SPEs

### 4.1. Theoretical Derivation of the Erasure Efficiency

One of the main objectives of this work is to investigate to what extent one can erase an SPE by coherently adding another SPE with a phase difference of $\pi$. The process of coherently adding two echoes is performed by means of a five-input-pulse sequence as that illustrated by Fig. 1. The output consists of two echoes that appear at the same instant of time, generated by the pulses 1a, 2a, 3, and 1b, 2b, 3, respectively. By shifting the phase of pulse 2b (or actually of any one of the pulses 1a/b or 2a/b) by $\pi$ radians, the two echoes interfere destructively and are effectively erased. This erasure process is not perfect if the phase fluctuations between pulses 1a and 2a and between pulses 1b and 2b are not identical. In this case, the output echo is not totally suppressed. We measured the area of the output with and without this $\pi$-radian phase shift, and will denote the different results below by $a+$ (no shift) and $a-$ ($\pi$-radian phase shift). The corresponding mathematical description of the input pulses is

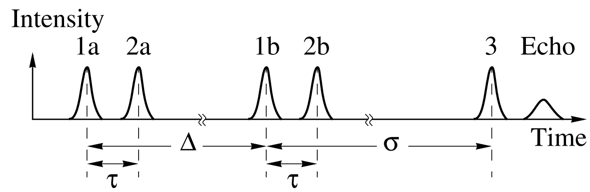


**Fig. 1.** The timing of the five input pulses for the erasure experiment.

$$\begin{cases} E1a(t) = E_0 e^{-2\left(\frac{t}{T}\right)^2 + i[\phi(0) + \omega_0 t]} \\ E2a(t) = E_0 e^{-2\left(\frac{t-\tau}{T}\right)^2 + i[\phi(\tau) + \omega_0(t-\tau)]} \\ E1b(t) = E_0 e^{-2\left(\frac{t-\Delta}{T}\right)^2 + i[\phi(\Delta) + \omega_0(t-\Delta)]} \\ E2b(t) = \mp E_0 e^{-2\left(\frac{t-\Delta-\tau}{T}\right)^2 + i[\phi(\Delta+\tau) + \omega_0(t-\Delta-\tau)]} \\ E3(t) = E_0 e^{-2\left(\frac{t-\sigma-\Delta}{T}\right)^2 + i[\phi(\sigma+\Delta) + \omega_0(t-\sigma-\Delta)]} \end{cases}. \tag{13}$$

We have made here a series expansion of the phase of the kind described in (4), and neglected derivatives of the phase of order one (angular frequency fluctuations) and higher, since one can show that these contribute very little to the final result. Here $\omega_0$ is the average angular frequency. Angular frequency fluctuations during the whole pulse sequence (pulses 1a to 3) give two echoes of slightly different frequency, i.e., the phase difference between them will vary linearly through the echo duration. If, however, this variation is small in comparison with the phase fluctuations (defined in [16]), i.e., $\Delta\omega T \ll \Delta\phi$, one can neglect this effect. This condition is fulfilled in this work, so the main contribution to fluctuations in the size of coherently added echoes are phase fluctuations. The size of these phase fluctuations is related partly to the size of the long-term frequency fluctuations, but mainly to the limited bandwidth of the servo loop in the frequency locking circuit of the laser. In terms of the correlation function $r_\omega(\tilde{\tau})$, the long-term fluctuations are described by its behavior around $\tau = 0$, while the bandwidth of the servo loop is related to the width of the $r_\omega(\tilde{\tau})$ curve. This bandwidth defines how much the phase fluctuates on short time scales and this primarily determines to what extent the echo can be suppressed. This also has the consequence that for the purposes of improving the erasure efficiency in our experiments, there is no point in limiting the long-term frequency fluctuations unless the fast, high-frequency components of the noise are also suppressed. Throughout our experiments, we used pulses with durations in the interval $80 < T < 500$ ns.

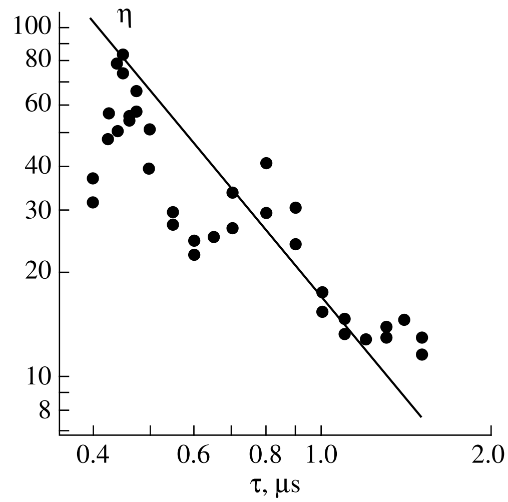


**Fig. 2.** Echo suppression as a function of τ. Experimental values (dots) and values adapted to experimental data (line), Δ = 3 μs, σ = 5 ms (parameters defined in the text). There are two measurements for each value of τ. The apparent increase of variance of the data with increasing echo suppression is due to the fact that the echoes get smaller with increasing echo suppression. The theoretical fit does not include the hyperfine modulation.

The long-term frequency deviations were measured to less than 170 kHz (Section 4.2). This means that the Taylor series expansion of the phase (4) can be neglected after the first term, giving [(8), (11), and (13)] echo areas of

$$A_{\text{out}}^{-} = \frac{(2\pi)^{5/2} T^5 E_0^6}{2\sqrt{3}} \sin^2\left(\frac{\Delta\phi}{2}\right) \tag{14}$$

for the erased echo and

$$A_{\text{out}}^{+} = \frac{(2\pi)^{5/2} T^5 E_0^6}{2\sqrt{3}} \cos^2\left(\frac{\Delta\phi}{2}\right) \tag{15}$$

for the non-erased echo, where we for simplicity have defined the variable

$$\Delta\phi = \phi(\tau) - \phi(0) - [\phi(\Delta + \tau) - \phi(\Delta)]. \tag{16}$$

The echo area is now a function of the laser phase and, therefore, also a function of the statistical properties of the phase, which is fully described by its autocorrelation function, providing earlier approximations are valid. To minimize the effect of disturbances not included in this model, we have chosen to measure the erasure efficiency, $\eta = E[A_{\text{out}}^{+}]/E[A_{\text{out}}^{-}]$. We use the general formula [14, p. 176] for the expectation value of a function $g(\mathbf{x})$ of a set of stochastic variables $\mathbf{x}$, with a probability function $f_{\mathbf{x}}(\mathbf{x})$,

$$E[g] = \int g(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x}, \tag{17}$$

to calculate $E[A_{\text{out}}^{\pm}]$. Assuming a Gaussian probability function, which here would be

$$f_{\Delta\Phi}(\Delta\phi) = \frac{1}{\sqrt{2\pi V}} e^{-\frac{1}{2}\frac{(\Delta\phi)^2}{V}}, \tag{18}$$

and using $g(\mathbf{x})$ calculated [(14) and (15)], this gives

$$E[A_{\text{out}}^{\pm}] = \frac{\pi^{5/2} T^5 E_0^6}{4\sqrt{3}} (1 \pm e^{-V/2}), \tag{19}$$

resulting in a very simple relationship between the variance of the laser phase fluctuations and the erasure efficiency,

$$\eta = \coth\left(\frac{V}{4}\right), \tag{20}$$

where

$$V = V[\Delta\phi] = C[\Delta\phi, \Delta\phi] \tag{21}$$
$$= C\left[\int_0^\tau \omega(r)dr - \int_\Delta^{\Delta+\tau} \omega(r)dr, \int_0^\tau \omega(s)ds - \int_\Delta^{\Delta+\tau} \omega(s)ds\right],$$

i.e., the variance of the difference of phase fluctuations during the two pulse pairs. From (20) and measurements of η, $V$ can be calculated and, by using (7), we get the expression

$$\begin{aligned} V = &\int_0^\tau\int_0^\tau r_\omega(s-r)drds + \int_\Delta^{\Delta+\tau}\int_\Delta^{\Delta+\tau} r_\omega(s-r)drds \\ &- \int_\Delta^{\Delta+\tau}\int_0^\tau r_\omega(s-r)drds - \int_0^\tau\int_\Delta^{\Delta+\tau} r_\omega(s-r)drds, \end{aligned} \tag{22}$$

which constitutes the relation between $V$ and $r_\omega(\tilde{\tau})$.

### *4.2. Application of the Model for Analysis of Experimental Data*

We excited SPEs from the $^3H_4$ to $^1D_2$ transition of $Pr^{3+}$ ions in a Pr-doped $YAlO_3$ crystal, with an excited state lifetime of 180 μs, although the theory should be equally applicable to any other similar system. Experimental data of the erasure efficiency vs. τ, the separation between the first and second pulse within each pulse pair, is shown in Fig. 2. Note that η is not a smoothly decreasing function of τ, rather it is strongly modulated. This is caused by the hyperfine structure of the $Pr^{3+}$ ions resulting in a variation of the echo size [15] as a function of both τ and Δ, the separation between pulse pairs. The standard deviation of the phase, $\sqrt{V}$, calculated by adapting (24) to the data (see below), is also shown in the diagram. In Fig. 3, the erasure efficiency vs. Δ is presented. The effect of decay between hyperfine levels, causing the echoes from the two-pulse

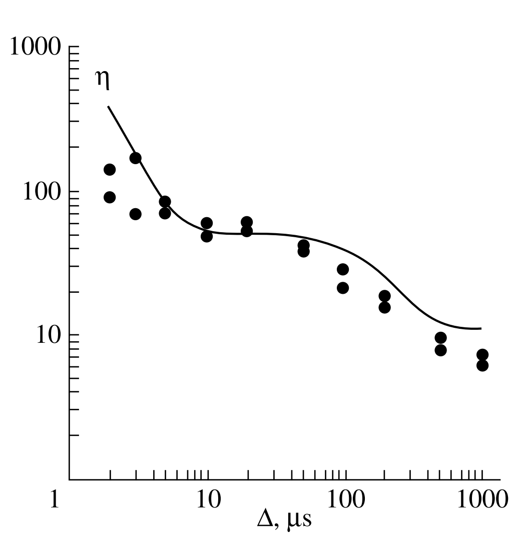


**Fig. 3.** Echo suppression as a function of Δ, the time separation between pulse pairs. Experimental values (dots) and adapted curve (line). Here, $T = 150$ ns, $\tau = 300$ ns and $\sigma = 0.5$ ms. (Definitions according to Fig. 1.)

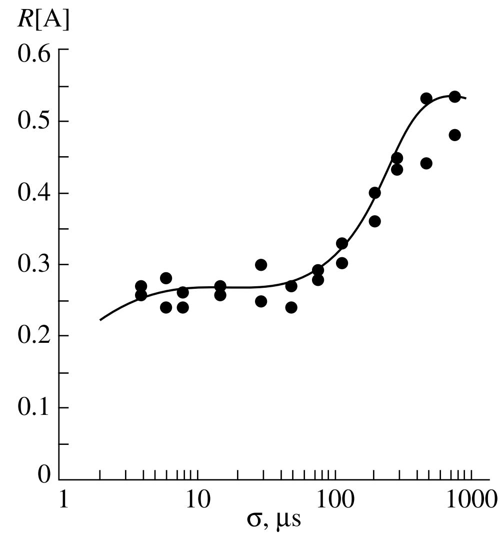


**Fig. 4.** Coefficient of variation $R[A]$ of three-pulse echoes as a function of σ. Experimental values (dots) and numerically adapted data (line), where a background equal to 0.2 has been added as described in (33).

pairs to be of different size, should here be negligible since the decay time [16] of 0.3–1 s for $Pr^{3+}$:$YAlO_3$ is much longer than any pulse separations used. Using the least squares method, an approximate $r_\omega(\tilde{\tau})$ with four free parameters $R$, $H$, $r$, and $h$ has been adapted to the data shown in Figs. 2, 3, and 4 (see Section 5), and the result is presented in the table. Here, $h$ and $H$ should be interpreted as how fast the corresponding part of the process is, while $r$ and $R$ correspond to the size of these fluctuations. $R + r$ is the variance of the long-term angular frequency fluctuations. This approximate $r_\omega(\tilde{\tau})$, which is described by

$$r_\omega(\tilde{\tau}) = re^{-\left(\frac{\tilde{\tau}}{h}\right)^2} + Re^{-\left(\frac{\tilde{\tau}}{H}\right)^2}, \qquad (23)$$

is the sum of two contributions. The first term is smaller and describes high-frequency fluctuations, which are beyond the bandwidth of the laser frequency stabilization electronics. The second, larger term describes long-term fluctuations possibly caused by temperature and low-frequency acoustical disturbances.

Using this $r_\omega(\tilde{\tau})$, we get from (22) the following expression for the variance of the phase fluctuations:

$$V = rh^2 Q\left(\frac{\Delta}{h}, \frac{\tau}{h}\right) + RH^2 Q\left(\frac{\Delta}{H}, \frac{\tau}{H}\right), \qquad (24)$$

where

$$Q(\Delta, \tau) = -2 + 2e^{-\tau^2} + 2e^{-\Delta^2} - e^{-(\Delta-\tau)^2} - e^{-(\Delta+\tau)^2} + \sqrt{\pi}\{2\tau\Phi(\tau) + 2\Delta\Phi(\Delta) - (\Delta-\tau)\Phi(\Delta-\tau) - (\Delta+\tau)\Phi(\Delta+\tau)\}. \qquad (25)$$

Here, $\Phi(y)$ is the error function,

$$\Phi(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-x^2} dx. \qquad (26)$$

## 5. ECHO AREA FLUCTUATIONS OF SPEs

The description of the influence on photon echoes from laser statistics can also be used to model the amplitude fluctuations of conventional SPEs. In this section, we will first describe how the SPE amplitude fluctuations can be determined from the correlation function for the laser frequency, $r_\omega(\tilde{\tau})$, and then compare this with the experimental data. We are therefore interested in calculating the coefficient of variation,

$$R[A_{out}] = \sqrt{V[A_{out}]}/E[A_{out}], \qquad (27)$$

of the echo area fluctuations of an ordinary SPE as a function of $r_\omega(\tilde{\tau})$. Using $r_\omega(\tilde{\tau})$ we have derived from experimental data in Section 4.2, we then compare these results with our calculations. Using a sufficiently short τ (here, τ is the time separation between pulses 1 and 2 and σ is the separation between pulses 2 and 3), we can assume that the frequency is approximately constant during and between pulses 1 and 2. When the size of the frequency shifts becomes comparable to the Fourier width of the pulses, the fluctuations of the echo areas

Values of the coefficients in (23)

| $r$, $4\pi^2 10^{12}$ s$^{-2}$ | $h$, μs | $R$, $4\pi^2 10^{12}$ s$^{-2}$ | $H$, μs |
|---|---|---|---|
| 0.15 | 3 | 1 | 280 |

should increase noticeably compared to the background noise level. As the fluctuations of the photon echoes, according to Fig. 4, are approximately constant for the first few tens of microseconds, we have assumed that $[r_\omega(0) - r_\omega(\tilde{\tau})]/r_\omega(0) \ll 1$ for $\tau \le 1$ μs. The experimental value of $\tau$ in these measurements is 1 μs, so assuming that the above approximation is correct, the frequency deviations between pulses 1 and 2 can be set equal to zero. For a conventional SPE, the first term in the Taylor expansion of the phase does not contribute to the output; adding an arbitrary, but constant, phase to any of the three input pulses still gives the same echo area. The next higher term is therefore included but we have made the assumption that terms of order two or higher in (4) can be truncated. The input pulses are consequently described by

$$\begin{cases} E_1(t) = E_0 e^{-2\left(\frac{t}{T}\right)^2 + i\phi(0) + i(\omega_0 + \omega_A)t} \\ E_2(t) = E_0 e^{-2\left(\frac{t-\tau}{T}\right)^2 + i\phi(\tau) + i(\omega_0 + \omega_A)(t-\tau)} \\ E_3(t) = E_0 e^{-2\left(\frac{t-\sigma-\tau}{T}\right)^2 + i\phi(\sigma+\tau) + i(\omega_0 + \omega_B)(t-\sigma-\tau)} \end{cases} \qquad (28)$$

From (8) and (11), the echo area is $A_{\text{out}} = \frac{E_0^6 T^5 \pi^{5/2}}{8\sqrt{3}} \times e^{-\frac{T^2}{6}(\omega_A - \omega_B)^2}$. To find $R[A_{\text{out}}]$, we now need $E[A_{\text{out}}]$ and $V[A_{\text{out}}]$. Here, we use the general formula [14, p. 177] [definitions as in (17)]

$$V[g] = \int g^2(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} - \left\{\int g(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x}\right\}^2. \qquad (29)$$

In this case, the distribution function (2) is two-dimensional and the covariance matrix $\mathbf{\Sigma}$ is

$$\mathbf{\Sigma} = \begin{bmatrix} C[\omega_A, \omega_A] & C[\omega_A, \omega_B] \\ C[\omega_B, \omega_A] & C[\omega_B, \omega_B] \end{bmatrix} = \begin{bmatrix} r_\omega(0) & r_\omega(\sigma) \\ r_\omega(\sigma) & r_\omega(0) \end{bmatrix}, \qquad (30)$$

which, using (2) and the fact that $(E[\omega_A], E[\omega_B]) = (0, 0)$, gives the distribution function of the vector of variables $\mathbf{x} = (\omega_A, \omega_B)$,

$$f_{\mathbf{x}}(\mathbf{x}) = f(\omega_A, \omega_B) = \frac{1}{2\pi\sqrt{r_\omega^2(0) - r_\omega^2(\sigma)}} \times \exp\left(-\frac{r_\omega(0)\omega_A^2 - 2r_\omega(0)r_\omega(\sigma)\omega_A\omega_B + r_\omega(0)\omega_B^2}{2(r_\omega^2(0) - r_\omega^2(\sigma))}\right), \qquad (31)$$

giving the final result

$$R[A_{\text{out}}] = \sqrt{\frac{3 + 2T^2(r_\omega(0) - r_\omega(\sigma))}{\sqrt{3}\sqrt{3 + 4T^2(r_\omega(0) - r_\omega(\sigma))}} - 1}. \qquad (32)$$

The erasure efficiency and the shot-to-shot fluctuations constitute two principally different ways of experimentally measuring frequency and phase fluctuations using SPEs. The results of the two different experiments give, with some deviations, consistent results. We have therefore concurrently adapted (32) using (23) to the data presented in Fig. 4, and (20) using (24) to the data presented in Figs. 2 and 3. Experimentally measured and numerically calculated values of the echo suppression are presented in Figs. 2 and 3. In Fig. 4, experimentally measured values of the coefficient of variation of the SPE areas and values of $R[A_{\text{out}}]$ adapted to the experimental data in Figs. 2–4 are presented. There is a clear difference between calculated values and experimental data, since $R[A_{\text{out}}]$ goes to zero as $\sigma$ goes to zero, as opposed to the measured data that have an offset from zero. The effect of higher derivatives of the phase was considered, but they are too small to explain this difference as shown below. If the difference was due to a decrease in the signal-to-noise ratio for large values of $\sigma$, $R[A_{\text{out}}]$ would increase with decreasing signal strength, but measurements where the echo signal strength has been kept constant by attenuating the excitation pulses as $\sigma$ is changed show the same behavior as in Fig. 4. One explanation could be that the constant $c$ in (8) includes some random factor due to processes such as bubbles in the cryostat, long-term laser power fluctuations or that the optical properties of the crystal are inhomogeneously distributed spatially, and the laser focus at the crystal moves slightly. We assume that these processes are stationary and independent of the phase diffusion process, and correct for them using the following equation for the coefficient of variation of the product of two independent stochastic variables $X$ and $Y$:

$$R^2[XY] = R^2[X] + R^2[Y] + R^2[X]R^2[Y]. \qquad (33)$$

We have assumed that the measured coefficient of variation presented in Fig. 4 is the result of both phase fluctuations and one or more stochastic processes fulfilling the requirements described above and corrected for the effect of this (these) unknown process(es) using (33). The values of $R[A_{\text{out}}]$ found by adapting the trial function chosen (23) to the data in Figs. 2–4 indicate that the effect of these unknown processes on the shot-to-shot variations of the SPE is bigger than the effect of the frequency shifts of the laser.

Stochastic fluctuations due to higher order terms in (3) could be an explanation to these unexplained fluctuations, but we will here show that these do not contribute substantially. Higher order terms would give a nonzero $R[A_{\text{out}}]$ for three-pulse SPEs when $\sigma \longrightarrow 0$ and $\tau \longrightarrow 0$, which would then explain why the experimental function does not go to zero when $\sigma$ and $\tau \longrightarrow 0$. In this case the three pulses have the same frequency, i.e. neither $\phi(T_k)$ nor $\phi'(T_k)$ can cause a nonzero variance of the echo output signal, so the next higher, second-order term is taken into account. The effect of terms of still higher orders should be smaller provided that the Fourier width of the pulses is larger than the linewidth of the laser, giving Taylor series expansion coefficients of decreasing size with increasing order. Terms of third or higher orders are therefore neglected. This gives the following description of the input pulses:

$$\begin{cases} E_1(t) = e^{-2\left(\frac{t}{T}\right)^2 + i\phi_0 + i\omega_0 t + iat^2} \\ E_2(t) = e^{-2\left(\frac{t}{T}\right)^2 + i\phi_0 + i\omega_0 t + iat^2} \\ E_3(t) = e^{-2\left(\frac{t}{T}\right)^2 + i\phi_0 + i\omega_0 t + iat^2}, \end{cases} \quad (34)$$

where $a = \left.\frac{\partial^2 \phi(t)}{\partial t^2}\right|_{t = T_k}$. This gives, using (8) and (11),

$$A_{\text{out}} = \frac{2\pi^{5/2}T^5}{\sqrt{3}(16 + a^2T^4)}. \quad (35)$$

Using (1), (17), (27), and (29), we get

$$R[A_{\text{out}}] = \sqrt{\frac{y^2 + \sqrt{\pi}\Phi_c(y)e^{y^2}(y/2 - y^3)}{\pi y^2 \Phi_c^2(y)e^{2y^2}} - 1}, \quad (36)$$

where $y = \frac{2}{T^2}\sqrt{\frac{2}{r_{\omega'}(0)}}$, i.e., a quantity related to the standard deviation of the fluctuations of the time derivative of the angular frequency and

$$\Phi_c(y) = \frac{2}{\sqrt{\pi}}\int_y^{\infty} e^{-t^2}dt \quad (37)$$

is the complementary error function. To be able to evaluate this expression numerically using (6), we need to know how $r_\omega(\tilde{\tau})$ behaves around $\tilde{\tau} = 0$. Here, we assume that the function we have chosen to adapt to our data [expression (23)] is valid even when we extrapolate it towards $\tilde{\tau} = 0$. Using this assumption, we get a value of 0.003 for $R[A_{\text{out}}]$ compared with the offset of 0.3 seen in the data in Fig. 4, which clearly indicates how small the contribution from higher order terms in (3) is (compare with Fig. 4).

## 6. ANALYZING THE POWER DENSITY SPECTRUM OF THE LIGHT SOURCE

The statistical properties of the frequency fluctuations of laser light are often expressed in terms of the power density spectrum, usually denoted as $P(\omega)$. The relation between $P(\omega)$ and the covariance function, $r_\omega(\tilde{\tau})$, is called the Wiener–Khintchine theorem. It states that

$$P(\omega) = F[r_\omega(\tilde{\tau})], \quad (38)$$

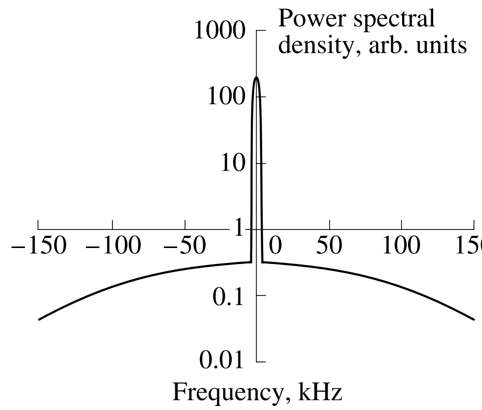


**Fig. 5.** Power spectral density of the laser light when fitting our four-parameter model to experimental data.

where $F$ denotes a Fourier transform. Using $r_\omega(\tilde{\tau})$ found in (23), ones finds the power density spectral function through a Fourier transformation. The result is presented in Fig. 5. Using the long-term storage property of SPE sequences in certain rare-earth-ion-doped crystals, we have a way of delaying a copy of the laser light for a very long time, possibly days, after which it then can be mixed with light from the same laser to perform homodyne detection of the frequency or phase deviations. This delay is longer than any regular delay line could be, for practical reasons, so this setup gives us a very high sensitivity to small long-term frequency and phase fluctuations.

Another way is to measure $P(\omega)$ of the laser light source, Fourier transform it utilizing the Wiener–Khintchine theorem to get $r_\omega(\tilde{\tau})$ and from this get information of how much the phase and frequency fluctuations will affect the echo. With this information, the effect of frequency fluctuations on the amplitude stability of a certain SPE system can be predicted. Assuming the system is used for storing data, and the method of coding the data is to define a digital "1" to be a light pulse of a certain minimum intensity and a digital "0" as a light pulse of a certain maximum intensity, these intensity fluctuations can be interpreted in terms of bit error rates (BERs). The BER is usually expressed in terms of the probability that a bit is misinterpreted. Assuming that the SPE area fluctuations can be approximately described by a Gaussian distribution, with a mean value of $\mu$ and a variance of $\sigma'$, which is a reasonable approximation for small values of the ratio $\sigma'/\mu$, the BER is easily calculated. If we define the limit between "0" and "1" to be $\mu/2$, then the probability that "1" is misinterpreted as "0" is

$$\Phi\left(\frac{\mu/2 - \mu}{\sigma'}\right) = \Phi\left(-\frac{1}{2}\frac{\mu}{\sigma'}\right) = \Phi\left(-\frac{1}{2}R^{-1}\right),$$

where $R$ is the coefficient of variation. The lowest $R$ achieved experimentally here is 0.25, which gives that the BER is approximately $2 \times 10^{-4}$, assuming that the probability for "0" to be misinterpreted as "1" is equal to the opposite process. This figure is very far from the BER rates needed in an operating computer, indicating that a better control of the processes that cause these BERs is needed, and since we do not know the reason for the high coefficient of variation in Fig. 4, also a better understanding of the effect of amplitude fluctuations on photon echoes is needed.

## 7. CONCLUSIONS

In this work, we have developed analytical methods to describe the effects of random laser phase and frequency fluctuations on SPEs. These have been used to explain the experimental SPE area fluctuations in three different cases. Random phase shifts between input pulse pairs exciting two coherently added echoes of opposite phase were used to explain why these do not give exactly a zero output. Frequency deviations from the nominal value were used to explain SPE area fluctuations for ordinary three-pulse echoes and finally the SPE area fluctuations due to the second time derivative of the phase were used to predict the SPE area fluctuations for temporally coinciding input pulses, i.e., conventional four-wave mixing. We have also shown how to use experimentally measured values of the statistical properties of SPEs to calculate the spectral power density of the laser used.

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