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SAMPLING OF A SYSTEM WITH A TIME DELAY

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SAMPLING OF A SYSTEM WITH A TIME DELAY

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Abstract

This report deals with the problem of sampling a continuous time system which includes a time delay. It is shown that the infinite dimensional continuous time system can be represented by a finite dimensional sampled data system. The expressions for the sampled data representations are also derived.

1. INTRODUCTION

A continuous time linear system with a time delay is an infinite dimensional system. To model the delay it is necessary to store a function of time over a time interval equal to the time delay. In this report it will be shown that the sampled data representation of such a system is finite dimensional. This is due to the sampling mechanism and to the fact that the input signal is assumed to be constant over periods of time equal to the sampling period. The expressions for the sampled data representations will be derived.

There has been a false folklore in digital control that the time delay should be on integer times the sampling period. Statements such that "the time delay is a multiple of the sampling period" is often seen.

Pulse transfer functions of systems with a delay that is not a fraction of the sampling period is easily obtained by using the modified z-transform. See Jury (1964) and Jury (1977) for thorough discussions of the modified z-transform. State space systems with time delays have not been analyzed to the same extent. In Franklin and Powell (1980) and Åström and Wittenmark (1984) it is shown that a time delay followed by a finite dimensional system easily can be sampled. The sampled data representation is finite dimensional. The result in Åström and Wittenmark (1984) was derived independently from the result given in Franklin and Powell (1980) and has been used in courses for

2. PROBLEM FORMULATION

Consider the system in Fig. 1. The system consists of two finite dimensional linear subsystems described by the state equations

$$S_1 : \begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1 u \\ y_1 &= C_1 x_1 + D_1 u \end{aligned} \tag{2.1}$$

$$S_2 : \begin{aligned} \dot{x}_2 &= A_2 x_2 + B_2 u_2 \\ y &= C_2 x_2 + D_2 u_2 \end{aligned} \tag{2.2}$$

The orders of the systems are n_1 and n_2 respectively. It is assumed that there is one input and one output of each subsystem. Further it is assumed that

$$u_2(t) = y_1(t-\tau) \tag{2.3}$$

i.e. that there is a time delay between the two subsystems. Let the time delay be

$$\tau = (d-1)h + \tau' \tag{2.4}$$

where d is an integer, h is the sampling period and τ' is a fraction of the sampling interval i.e.

equations for $x_1(kh)$ and $x_2(kh)$ should be derived. Three cases will be considered:

Case 1 - Time delay before the system. I.e. $A_1 = B_1 = C_1 = D_2 = 0$, $D_1 = 1$.

Case 2 - Time delay after the system. I.e. $A_2 = B_2 = C_2 = D_1 = 0$, $D_2 = 1$.

Case 3 - Time delay between the subsystems. It is then assumed that $D_1 = D_2 = 0$.

Case 1 is solved in Franklin and Powell (1980) and Åström and Wittenmark (1984), but the result is repeated in the following section together with the derivation of the sampled data representations for the other two cases.

3. SAMPLING OF A SYSTEM WITH A TIME DELAY

The three problems formulated in the previous section are now going to be solved.

<u>Case 1 - Time delay before the system</u>

defined in (2.4). This gives

$$x_2(kh+h) = e^{A_2h}x_2(kh) + \int_{kh}^{kh+\tau'} e^{A_2(kh+h-s)}B_2ds\cdot u(kh-dh)$$

$$+ \int_{kh+\tau'}^{kh+h} e^{A_2(kh+h-s)}B_2ds\cdot u(kh-(d-1)h)$$

$$= e^{A_2h}x_2(kh) + \int_0^{\tau'} e^{A_2(h-\tau')}e^{A_2s'}B_2ds'\cdot u(kh-dh)$$

$$+ \int_0^{h-\tau'} e^{A_2s'}B_2ds'\cdot u(kh-(d-1)h)$$

Introduce the notations

$$\Phi_2(t) = e^{A_2t} \tag{3.1}$$

$$\Gamma_2(t) = \int_0^t e^{A_2s}B_2ds \tag{3.2}$$

then

$$\det[zI-\Phi_2(h)] = 0$$

The system with time delay has the same poles as the system without time delay and d additional poles at the origin. I.e. the order of the operator (3.4) is $n_2 + d$. The sampled data representation is of finite order despite the fact that the continuous time system is infinite dimensional. The reason is that the input signal is constant over the sampling periods. It is thus only necessary to store these values.

Notice that only the zeros and not the poles of (3.4) will vary when τ' is changed. The integer d is the pole excess of the sampled data system.

Case 2 - Time delay after the system

It is now assumed that the subsystem S_2 has a transfer function that is unity, i.e. that

$$A_2 = B_2 = C_2 = D_1 = 0 \qquad D_2 = 1$$

The sampled representation of (2.1) is

$$x_1(kh+h) = e^{A_1 h} x_1(kh) + \int_0^h e^{A_1 s} B_1 ds\, u(kh)$$

$$= e^{A_1(h-\tau')} x_1(kh-dh) + \int_0^{h-\tau'} e^{A_1 s'} B_1 ds' u(kh-dh)$$

$$= \Phi_1(h-\tau') x_1(kh-dh) + \Gamma_1(h-\tau') u(kh-dh) \qquad (3.6)$$

The pulse transfer operator is now

$$H_2(q) = C_1 \left[\Phi_1(h-\tau')(qI-\Phi_1(h))^{-1}\Gamma_1(h) + \Gamma_1(h-\tau')\right] q^{-d} \qquad (3.7)$$

The sampled representation is also in this case finite dimensional.

THEOREM 3.1

The pulse transfer operator $H_2(q)$ given by (3.7) is equal to $H_1(q)$ given by (3.4) if the systems S_1 and S_2 are the same.

<u>Proof</u>

To show the theorem it is first shown that

$$\Gamma(h) = \Phi(\tau')\Gamma(h-\tau') + \Gamma(\tau')$$

For convenience the indices have been dropped. Using the definitions of Φ and Γ it is found that

$$\Phi(s+t) = \Phi(s)\Phi(t)$$

it follows that

$$H_2(q) = C\left[\Phi(h-\tau')(qI-\Phi(h))^{-1}\Gamma(h) + \Gamma(h-\tau')\right]q^{-d}$$

$$= C(qI-\Phi(h))^{-1}\left[\Phi(h-\tau')\Gamma(h) + q\Gamma(h-\tau') - \right.$$

$$\left. - \Phi(h)\Gamma(h-\tau')\right]q^{-d}$$

$$= C(qI-\Phi(h))^{-1}\left[q\Gamma(h-\tau') + \right.$$

$$\left. \Phi(h-\tau')[\Gamma(h)-\Phi(\tau')\Gamma(h-\tau')]\right]q^{-d}$$

$$= C(qI-\Phi(h))^{-1}\left[q\Gamma(h-\tau') + \Phi(h-\tau')\Gamma(\tau')\right]q^{-d}$$

$$= H_1(q)$$

□

Case 3 - Time delay between the subsystems

This more complicated case can be solved using the same ideas as for the previous cases. To be able to make a comparison and to introduce some

The sampled representation of (3.8) is obtained by integrating the differential equations

$$x_1(kh+h) = e^{A_1 h} x_1(kh) + \int_{kh}^{kh+h} e^{A_1(kh+h-s)} B_1 u(s)ds$$

$$= e^{A_1 h} x_1(kh) + \int_{0}^{h} e^{A_1 s} B_1 ds\ u(kh)$$

$$= \Phi_1(h)x_1(kh) + \Gamma_1(h)u(kh)$$

$$x_2(kh+h) = e^{A_2 h} x_2(kh) + \int_{kh}^{kh+h} e^{A_2(kh+h-s)} A_{21} x_1(s)ds \quad (3.9)$$

In the interval $kh \le s \le kh + h$

$$x_1(s) = e^{A_1(s-kh)} x_1(kh) + \int_{kh}^{s} e^{A_1(s-s')} B_1 ds'\ u(kh) \quad (3.10)$$

Thus

$$\int^{kh+h} \ldots A_1(kh+h-s) \ldots A_1(s-kh) \ldots$$

$$\Phi_{21}(h) = \int_{kh}^{kh+h} e^{A_2(kh+h-s)} A_{21} e^{A_1(s-kh)} ds$$

$$= \int_{0}^{h} e^{A_2 s'} A_{21} e^{A_1(h-s')} ds'$$

$$\Gamma_2'(h) = \int_{kh}^{kh+h} e^{A_2(kh+h-s)} A_{21} \int_{kh}^{s} e^{A_1(s-s')} B_1 ds' \, ds$$

$$= \int_{0}^{h} e^{A_2 s''} A_{21} \Gamma_1(h-s'') ds''$$

Γ_1 is defined as in (3.2). The sampled version of (3.8) is thus

$$\begin{bmatrix} x_1(kh+h) \\ x_2(kh+h) \end{bmatrix} = \begin{bmatrix} \Phi_1(h) & 0 \\ \Phi_{12}(h) & \Phi_2(h) \end{bmatrix} \begin{bmatrix} x_1(kh) \\ x_2(kh) \end{bmatrix} + \begin{bmatrix} \Gamma_1(h) \\ \Gamma_2'(h) \end{bmatrix} u(kh) \tag{3.12}$$

$$y(kh) = \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1(kh) \\ x_2(kh) \end{bmatrix}$$

It is clear from the derivation that if there is a delay in the system then problem to the two problems treated above. Consider the system in Fig. 2. Compared with Fig. 1 the subsystem S_1 and the delay are interchanged. The states of S_1 in Fig. 2 are denoted x'_1 and are related to the states of S_1 in Fig. 1 through

$$x_1'(t) = x_1(t-\tau).$$

The states of S_2 in Fig. 2 are identical to the states of S_2 in Fig. 1.

Sampling the system in Fig. 2 is the same problem as treated in Case 1. S_1 in series with S_2 is given by (3.8). Using (3.3) and (3.12) gives

$$\begin{bmatrix} x_1'(kh+h) \\ x_2(kh+h) \end{bmatrix} = \begin{bmatrix} \Phi_1(h) & 0 \\ \Phi_{21}(h) & \Phi_2(h) \end{bmatrix} \begin{bmatrix} x_1'(kh) \\ x_2(kh) \end{bmatrix} + $$

$$+ \begin{bmatrix} \Phi_1(h-\tau) & 0 \\ \Phi_{21}(h-\tau) & \Phi_2(h-\tau) \end{bmatrix} \begin{bmatrix} \Gamma_1(\tau) \\ \Gamma_2'(\tau) \end{bmatrix} u(kh-h)$$

$$+ \begin{bmatrix} \Gamma_1(h-\tau) \\ \Gamma_2'(h-\tau) \end{bmatrix} u(kh).$$

The problem is that $x_2(kh)$ is not a function of $x_1(kh)$ but of

$$x_1'(kh) = x_1(kh-\tau).$$

Case 2 and (3.6) gives

$$x_1(kh-\tau) = \Phi_1(h-\tau)x_1(kh-h) + \Gamma_1(h-\tau)u(kh-h).$$

Finally

$$\begin{aligned} x_2(kh+h) &= \Phi_{21}(h)\left[\Phi_1(h-\tau)x_1(kh-h) + \Gamma_1(h-\tau)u(kh-h)\right] \\ &\quad + \Phi_2(h)x_2(kh) + \Gamma_2'(h-\tau)u(kh) \\ &\quad + \left[\Phi_{21}(h-\tau)\Gamma_1(\tau) + \Phi_2(h-\tau)\Gamma_2'(\tau)\right]u(kh-h) \\ &= \Phi_{21}(h)\Phi_1(h-\tau)x_1(kh-h) + \Phi_2(h)x_2(kh) \\ &\quad + \left[\Phi_{21}(h)\Gamma_1(h-\tau) + \Phi_{21}(h-\tau)\Gamma_1(\tau) \right. \\ &\quad \left. + \Phi_2(h-\tau)\Gamma_2'(\tau)\right]u(kh-h) + \Gamma_2'(h-\tau)u(kh). \end{aligned}$$

The derivation is summarized in the following theorem.

THEOREM 3.2

Periodic sampling of the system (2.1) - (2.3) with the sampling interval h and with

$$0 < \tau < h$$

$$\Phi_i(t) = e^{A_i t} \qquad i = 1,2$$

$$\Gamma_1(t) = \int_0^t e^{A_1 s} B_1 ds$$

$$\Phi_{21}(t) = \int_0^t e^{A_2 s} A_{21} e^{A_1(t-s)} ds$$

$$\Gamma_2'(t) = \int_0^t e^{A_2 s} A_{21} \Gamma_1(t-s) ds$$

□

<u>Remark</u>

The sampled data representation is obtained by sampling the system without time delay for the sampling periods h, h-τ, and τ. Compare (3.12). This implies that the computations are easily made using standard programs for sampling a system.

Finally the order of the sampled data system will be discussed. One state space representation of (3.13) is

$$\begin{bmatrix} x_1(kh+h) \end{bmatrix} \quad \begin{bmatrix} \Phi_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(kh \end{bmatrix} \quad \begin{bmatrix} \Gamma_1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(kh) \\ x_2(kh+h) \\ u(kh) \end{bmatrix} = \begin{bmatrix} \Phi_1 & 0 & \Gamma_1 \\ \bar{\Phi}_{21} & \Phi_2 & \bar{\Gamma}_2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(kh-h) \\ x_2(kh) \\ u(kh-h) \end{bmatrix} + \begin{bmatrix} 0 \\ \Gamma_2' \\ 1 \end{bmatrix} u(kh)$$

$$\begin{bmatrix} x_1(kh) \\ y(kh) \end{bmatrix} = \begin{bmatrix} \Phi_1 & 0 & \Gamma_1 \\ 0 & C_2 & 0 \end{bmatrix} \begin{bmatrix} x_1(kh-h) \\ x_2(kh) \\ u(kh-h) \end{bmatrix} \tag{3.14}$$

In the general case d old values of u has to be stored. The sampled data system thus has the order

$$n_1 + n_2 + d$$

if $\tau > 0$. That this is the minimum order is easily understood from the derivation of (3.13). We have the following theorem:

THEOREM 3.3 - The order of the sampled data system
Periodic sampling of the infinite dimensional continuous time system (2.1) - (2.3) with the sampling interval h and with $\tau > 0$ gives a sampled data representation of order

$$n_1 + n_2 + d$$

4. AN EXAMPLE

Consider the system in Fig. 3. Let the sampling interval be $h = 1$. The continuous time system without time delay is described by, compare (3.8),

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 1]x.$$

Sampling this system with sampling period h gives

$$x(kh+h) = \begin{bmatrix} e^{-h} & 0 \\ 1-e^{-h} & 1 \end{bmatrix} x(kh) + \begin{bmatrix} 1-e^{-h} \\ h-1+e^{-h} \end{bmatrix} u(kh)$$

Thus

$$\Phi_1(h) = e^{-h} \qquad \Phi_2(h) = 1$$

$$\Gamma_1(h) = 1 - e^{-h} \qquad \Gamma_2'(h) = h - 1 + e^{-h}$$

$$\Phi_{21}(h) = 1 - e^{-h}$$

This gives

$$\Phi_{21}^{-} = (1-e^{-h})e^{-(h+\tau)}$$

$$n_1 + n_2 + d$$

where $n_1 + n_2$ is the dimension of the finite dimensional part of the continuous time system. The integer d is defined in (2.4).

The sampled data representation is given in Theorem 3.2. The expressions in (3.13) are determined by sampling the continuous time system without the time delay for the sampling periods h, $h-\tau'$ and τ'. This implies that the sampled data representation is easy to compute using standard programs for sampling a system.

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