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# Some Distance Properties of Tailbiting Codes[1]

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*Abstract* — **The active tailbiting segment distance for convolutional codes is introduced. Together with the earlier defined active burst distance, it describes the error correcting capability of a tailbiting code encoded by a convolutional encoder. Lower bounds on the new active distance as well as an upper bound on the ratio between tailbiting length and memory of encoder such that its minimum distance $d_{\min}$ equals the free distance $d_{\text{free}}$ of the corresponding convolutional code are presented.**

## I. INTRODUCTION

Tailbiting codes can be obtained by terminating rate $R = b/c$ convolutional codes into block codes of length $l$ $c$-tuples [1]. For simplicity, we consider only binary codes. The error correcting capability of a block code is estimated via $d_{\min}$. There is no description which error patterns with more than $\lfloor \frac{d_{\min}-1}{2} \rfloor$ errors can be corrected. In order to describe the error correcting capability of a tailbiting code beyond the minimum distance argument, we define the active tailbiting segment distance $a_j^{tbs}$. We also give an upper bound on the length of a tailbiting code so that $d_{\min}$ equals $d_{\text{free}}$ of the corresponding convolutional code. This is useful when analyzing concatenated coding schemes containing tailbiting encoders.

## II. THE ACTIVE TAILBITING SEGMENT DISTANCE

Consider the convolutional code $\mathcal{C}$ encoded by a rate $R = b/c$ encoder. The binary matrix $\sigma_t$ denotes the encoder state at time $t$. Let $\mathcal{S}_{[t_1,t_2]}^{\sigma_s,\sigma_e}$, $0 \leq t_1 < t_2$, be the set of state sequences $\sigma_{[t_1,t_2]} = \sigma_{t_1} \ldots \sigma_{t_2}$ that start in state $\sigma_s$ and terminate in state $\sigma_e$, and do not have two consecutive zero states with zero input in between. Let $j_b$ be the smallest positive integer such that $\mathcal{S}_{[0,j_b+1]}^{0,0} \neq \emptyset$. The *$j$th order active burst distance* [2] is $a_j^b \triangleq \min_{\mathcal{S}_{[0,j+1]}^{0,0}} \{ w_H(v_{[0,j]}) \}$, $j \geq j_b$. For any code $\mathcal{C}$, $a_j^b$ is invariant over the set of its canonical encoders [2]. It is lower-bounded by a linearly increasing function $a_j^b \geq \alpha j + \beta^b$, $j \geq j_b$, where $\alpha$ is the *asymptotic slope*, and $\beta^b$ is chosen as large as possible. Let $j_r$ be the smallest number of steps that, starting in the allzero state, take us to any reachable state.

**Definition 1** *The $j$th order active tailbiting segment distance is* $a_j^{tbs} \triangleq \min_{\mathcal{S}_{[j_r,j_r+j+1]}^{\sigma,\sigma}} \{ w_H(v_{[j_r,j_r+j]}) \}$, *where $\sigma$ denotes any possible encoder state.*

**Theorem 1** *The active tailbiting segment distance is lower-bounded by* $a_j^{tbs} \geq \alpha(j+1)$, *for all $j \geq 0$.*

## III. PROPERTIES OF TAILBITING CODES VIA THE ACTIVE DISTANCES

We define an *incorrect path* at the receiver to be any trellis path differing from the *transmitted path*. For any $k_1, k_2 < l$, let $e_{[k_1,k_2]}$ denote the Hamming weight of the error pattern $e_{[k_1,k_2]} = e_{k_1} e_{k_1+1} \ldots e_{k_2}$, where $e_i$, $0 \leq i < l$, are $c$-tuples and all indices are evaluated modulo $l$. Then, we have

**Theorem 2** *Consider a tailbiting code $\mathcal{C}^{tb}$ of length $l$ $c$-tuples encoded by a convolutional encoder. Then, for $j_b < l$, a maximum likelihood (ML) decoder corrects all error patterns $e_{[0,l-1]}$ that satisfy $e_{[t,\ t+j \bmod l]} < \min\{a_j^b/2, a_{l-1}^{tbs}/2\}$ for $0 \leq t < l$, $j_b \leq j < l$. For $j_b \geq l$, all error patterns that satisfy $e_{[0,l-1]} < a_{l-1}^{tbs}/2$ are corrected.*

**Example 1** *A tailbiting code of length $l = 18$ 2-tuples encoded by $G(D) = (1 + D + D^2 \quad 1 + D^2)$ with $d_{\min} = 5$ is used on a binary symmetric channel. From Theorem 2 follows that*

$$e_{[0,17]} = 10\ 00\ 00\ 01\ 00\ 00\ 00\ 00\ 01\ 00\ 00\ 00\ 10\ 00\ 00\ 00\ 00\ 00$$

*is corrected by an ML decoder although the error pattern contains four channel errors which exceeds $\lfloor \frac{d_{\min}-1}{2} \rfloor$.*

Consider a tailbiting code of length $l$ $c$-tuples with minimum distance $d_{\min}$. The *free tailbiting length* $l_{\text{free}}$ is the shortest length $l$ for which $d_{\min}$ will remain equal to $d_{\text{free}}$ of the corresponding convolutional code for all tailbiting lengths greater than or equal to $l_{\text{free}}$. The free tailbiting length is upper-bounded by $l_{\text{free}} \leq \lfloor d_{\text{free}}/\alpha \rfloor$.

## IV. ENSEMBLE PROPERTIES OF THE ACTIVE TAILBITING SEGMENT DISTANCE

The concept of the active distances can be generalized to time-varying convolutional encoders.

**Theorem 3** *There exists a rate $R = b/c$ convolutional code $\mathcal{C}$ encoded by a time-varying encoder of memory $m$ such that* $a_j^{tbs} > \rho(j+1)c + O(\log m)$, *for $j = O(m) \geq j_o$ $m \to \infty$*

*where $\rho = h^{-1}(1-R)$ is the Gilbert-Varshamov parameter, $h()$ is the binary entropy function, and $j_o$ is the smallest integer satisfying $(1-R)(j+1)c \geq 4 \log m$.*

Using the Heller asymptotic bound we obtain

**Theorem 4** *There exists a tailbiting code $\mathcal{C}^{tb}$ encoded by a time-varying encoder of memory $m$, such that* $\lim_{m \to \infty} l_{\text{free}}/m \leq \frac{1}{2\rho}$.

## REFERENCES

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